Projective Metrics for Coding Theory

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Introduction to Coding Theory

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Btw the message says SOS in Morse, so maybe go seek help.

A better way to do it

We rearrange the original message $\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{0}$ in a 4 by 4 grid:

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Since $2^4 =$ total number of bits $+$ (case in which there is no error) $= 15 + 1$ and if there is up to one error, every redundant bit halvens the number the possible locations of where the error might be, we can always correct up to one error in the message.

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Thus, since we can choose the numbers inside the 11 blue cells arbitrarily they form a 11 dimensional linear subspace of $\mathbb{F}_2^{15}$. For this reason these codes are referred to as [15,11] Hamming Codes.

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- The minimum distance d is a simple measure of the goodness of a code.

Classical Sphere Packing Results

- Given a message length n and a minimum distance d, we want to find the largest code with minimum distance d.
- This can also be seen as the sphere packing problem for spheres of radius $\lfloor \frac{d-1}{2} \rfloor$.

Theorem 1 (Hamming Bound)

Let $\mathcal{C} < \mathbb{F}_q^n$ be a code with $d_H(\mathcal{C}) = d$ then:

$$|\mathcal{C}| \leq \frac{q^n}{\sum_{i=o}^{t} \binom{n}{i} (q-1)^i}$$

Where $t := \lfloor \frac{d-1}{2} \rfloor$.

Theorem 2 (Singleton Bound)

Let $\mathcal{C} < \mathbb{F}_q^n$ be a code with $d_H(\mathcal{C}) = t$ then:

$$|\mathcal{C}| \leq q^{n-d+1}.$$

The codes satisfying the Hamming bound or the Singleton bound are called respectively **Perfect codes** and **MDS codes** (maximum distance separable codes).

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Remark: 1) Codes are perfect if the balls of size t centered on the codewords completely fill up $\mathcal{V}$
2)The Hamming codes are Perfect codes, while the "send the same message multiple times"-codes are not perfect.

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- A different approach is to try to get analogous results on a family of metrics.
- In this presentation we show our results on the family of projective metrics.

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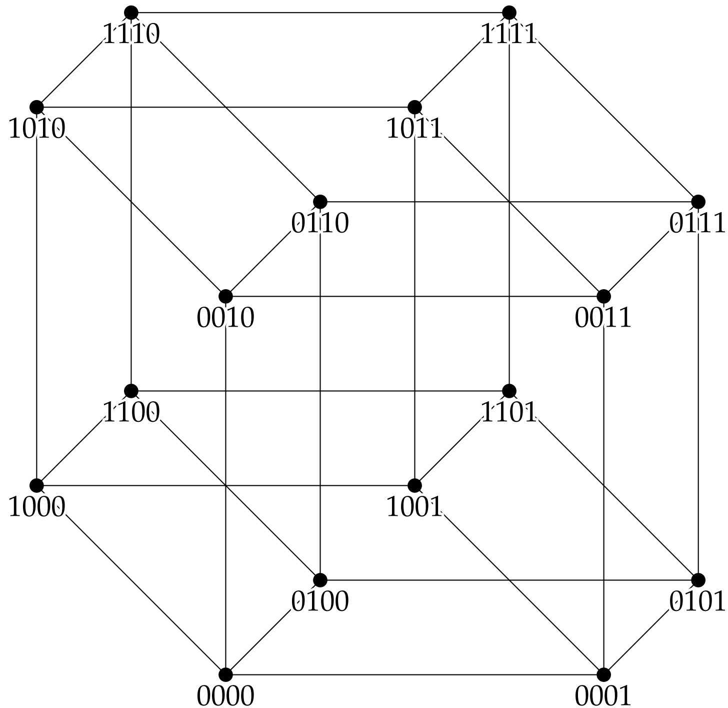
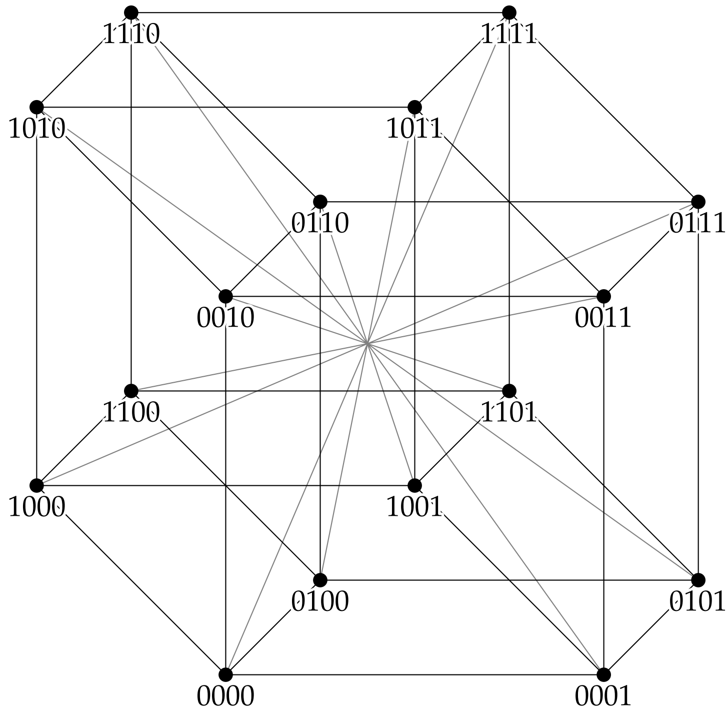
The **projective metric** $d_{\mathcal{F}}(\cdot, \cdot) : V \times V \to \mathbb{N}_{\geq 0}$ corresponding to $\mathcal{F}$ is

$$d_{\mathcal{F}}(x, y) := \mathrm{wt}_{\mathcal{F}}(y - x).$$

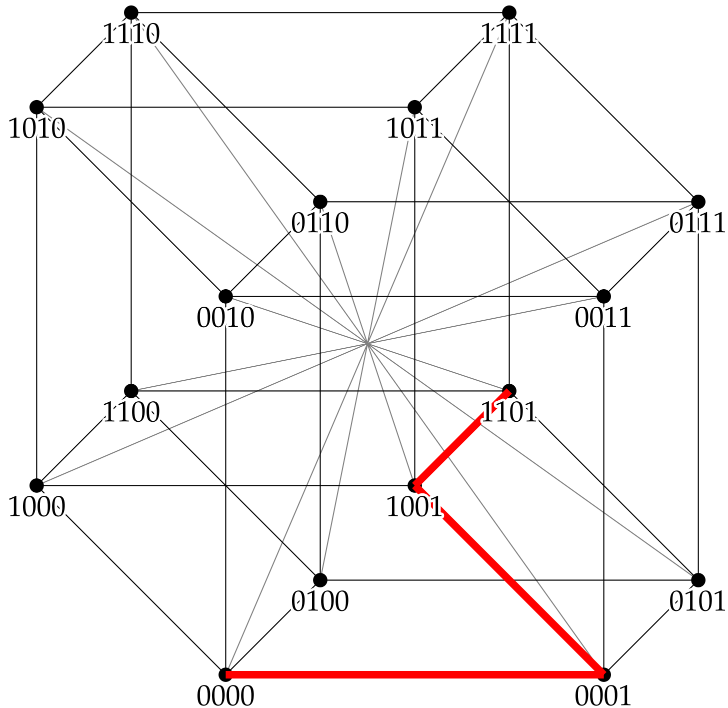
Vertices: vectors of $\mathbb{F}_2^4$



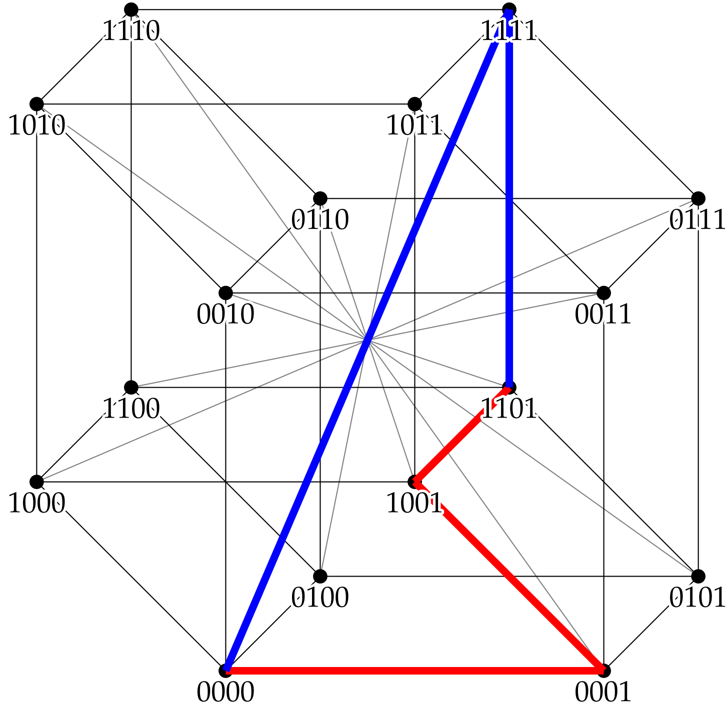
Distance from 0000 to 1101:



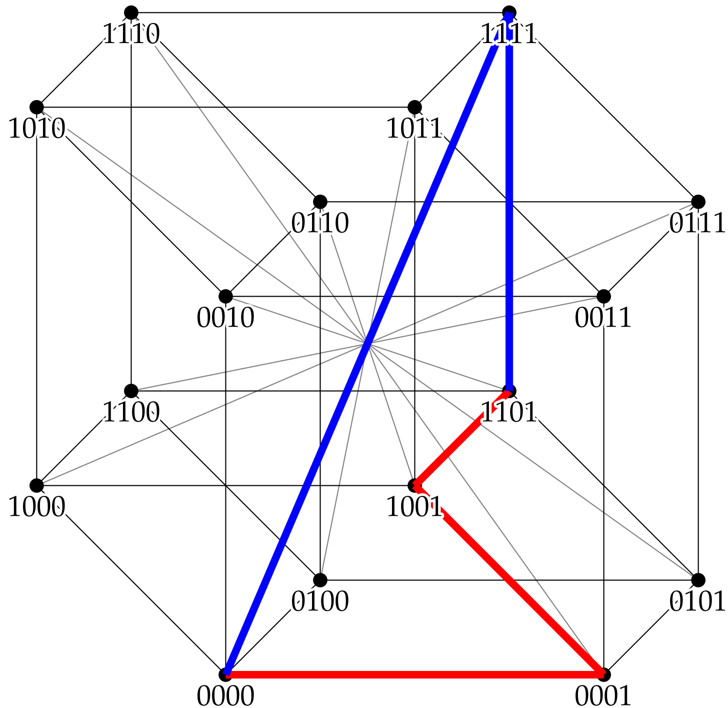
Distance from 0000 to 1101: red: 3,



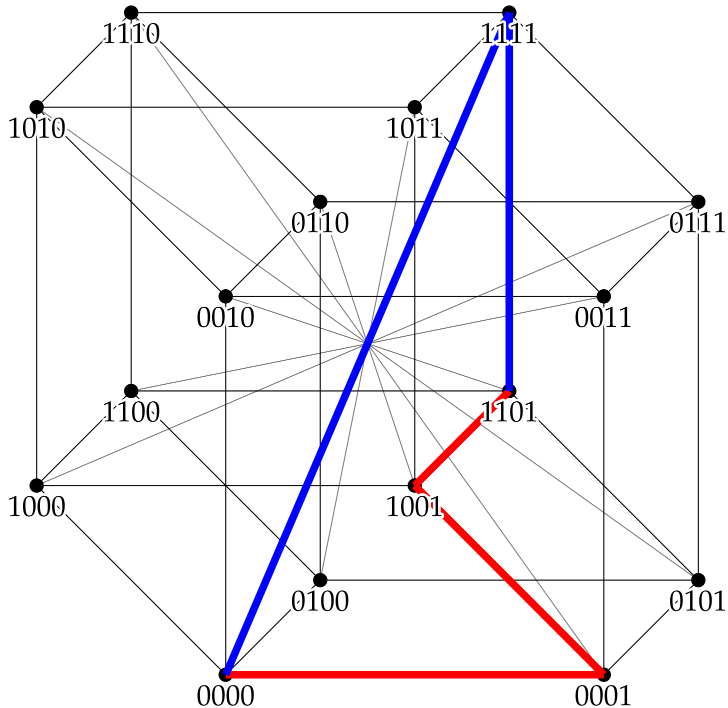
Distance from 0000 to 1101: red: 3, blue: 2



Graph distance on Clebsch graph $=$ **Phase-rotation metric/distance** on $\mathbb{F}_2^4$



Graph distance on Clebsch graph $=$ **Phase-rotation metric/distance** on $\mathbb{F}_2^4$
An edge is a Hamming error or the all-bits-flip error



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Sum-Rank metric

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

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Cover metric (rows and columns)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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Sum-Rank metric

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \quad \rightarrow \ \mathcal{F} = \Big\{\text{spans of } \Big(\text{ some 0 blocks } \Big| \text{ rank 1 matrix } \Big| \text{ some 0 blocks } \Big)\Big\}$$

Cover metric (rows and columns)

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Theorem 3 (General Singleton-type bound)

Let $\mathcal{C} \subseteq V$ be a subset and let $d = \min\{d_{\mathcal{F}}(x, y) \mid x \neq y \in \mathcal{C})\}$. Then

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Coincides with Singleton bounds for specific projective metrics!

Where two codes are equivalent if there exists a linear Hamming isometry sending one onto the other. The following result tells us that a projective metric is univocally determined by it's parent code.

Theorem 4

Let $\bar{P}r_{\mathbf{N}}(V)$ be the set containing the equivalence classes of projective metrics on V of size $\mathbf{N}$ and $\bar{G}r_{\mathbf{N}-N}(\mathbb{F}_q^{\mathbf{N}})$ be the set containing the equivalence classes of subspaces of $\mathbb{F}_q^{\mathbf{N}}$ of dimension $\mathbf{N}-N$. Then there exists a bijection:

$$\Psi : \bar{P}r_{\mathbf{N}}(V) \to \bar{G}r_{\mathbf{N}-N}(\mathbb{F}_q^{\mathbf{N}})$$
$$\bar{w}_{\mathcal{F}} \mapsto \bar{\mathcal{C}}_{\mathcal{F}}$$

Where $\bar{\mathcal{C}}_{\mathcal{F}}$ is the parent code of $\mathcal{F}$.

Characterization of projective isometries

What can we do?

Definition 0.1

An $\mathbb{F}$-isometry is a linear isomorphism $L : V \to V$ such that $L(\mathcal{F}) = \mathcal{F}$. The set of $\mathbb{F}$-isometries, with the operation of composition, forms a group denoted as $isom_{\mathbb{F}}(V)$.

Theorem 5

Let $stab_H(\mathcal{C})$ be the stabilizer of the parent code $\mathcal{C}$ respect to the Hamming isometries, then $isom_{\mathcal{F}}(V) \cong stab_H(\mathcal{C})$

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- Please let me know if you know a (partial) answer in any of these contexts!

Current research

- Algorithms for calculating $\mathrm{wt}_{\mathcal{F}}(v)$ for $v \in V$
- Are there general methods for obtaining sphere sizes $|\{v \in V \mid \mathrm{wt}_{\mathcal{F}}(v) = t\}|$ for $t \in \mathbb{N}$?
- Is there a natural way do generilize other concepts of coding theory? Dual Codes? Perfect Codes? ecc...
- Approach?: using poset lattice of projective metrics, where $\mathrm{wt}_{\mathcal{F}} \preccurlyeq \mathrm{wt}_{\mathcal{G}}$ iff $\mathcal{F} \subseteq \mathcal{G}$