Projective Metrics for Coding Theory

Gabor Riccardi

University of Pavia (UniPv)

with **Hugo Sauerbier Couvée** , Technical University of Munich (TUM)

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Introduction to Coding Theory

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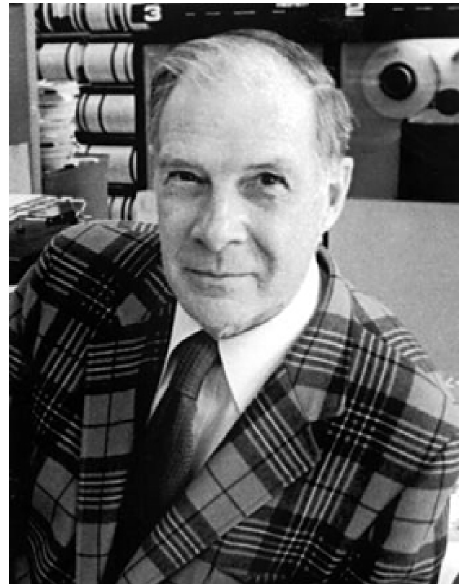
  "Damn it, if the machine can detect an error, why can't it locate the position of the error and correct it?"



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Btw the message says SOS in Morse, so maybe go seek help.

A better way to do it

We rearrange the original message $\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{0}$ in a 4 by 4 grid:

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Since $2^4 =$ total number of bits $+$ (case in which there is no error) $= 15 + 1$ and if there is up to one error, every redundant bit halvens the number the possible locations of where the error might be, we can always correct up to one error in the message.

	0	0	1
1	0	0	1
0	1	1	0
0	0	0	0

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We observe that if we sum two Hamming codes, it remains an Hamming code (that is the parity checks remain valid also for the result of the sum):

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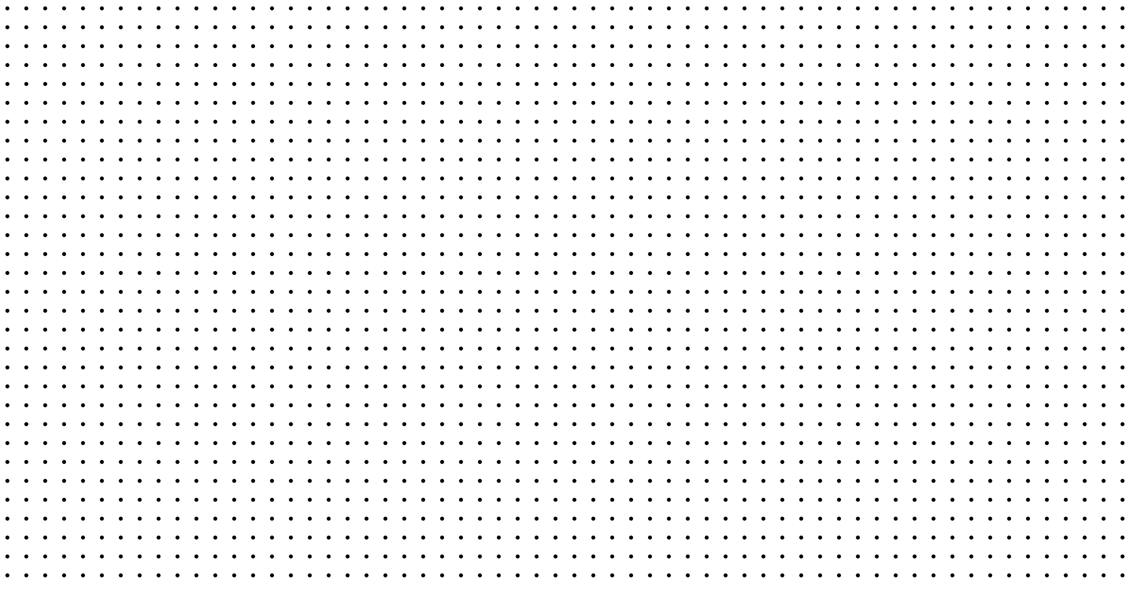
Thus, since we can choose the numbers inside the 11 blue cells arbitrarily they form a 11 dimensional linear subspace of $\mathbb{F}_2^{15}$. For this reason these codes are referred to as [15,11] Hamming Codes.

A step back.



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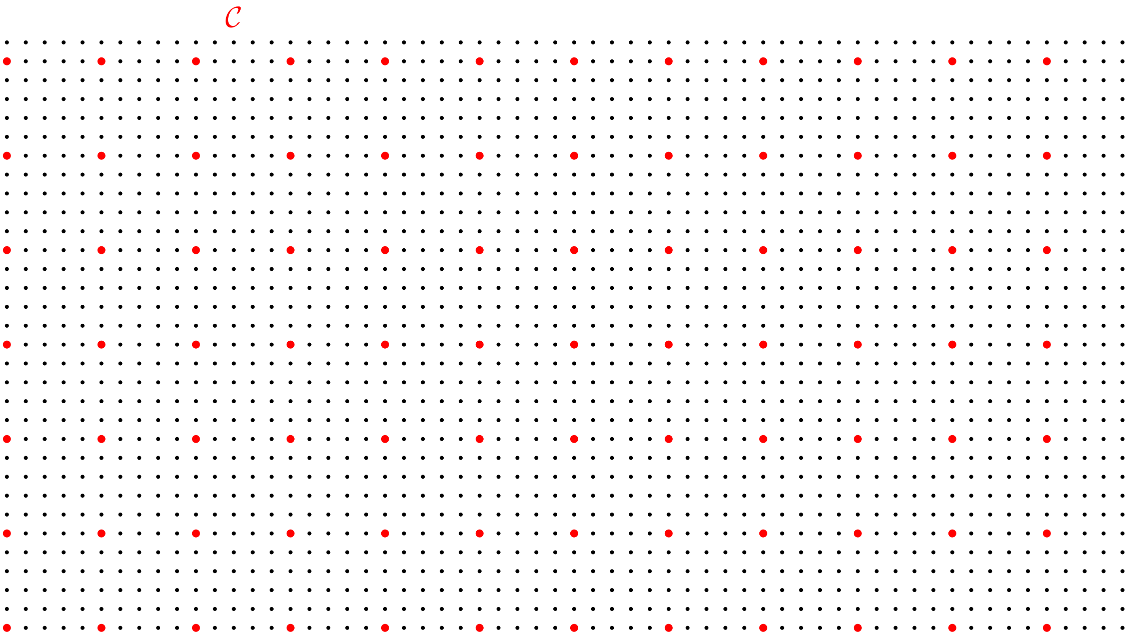

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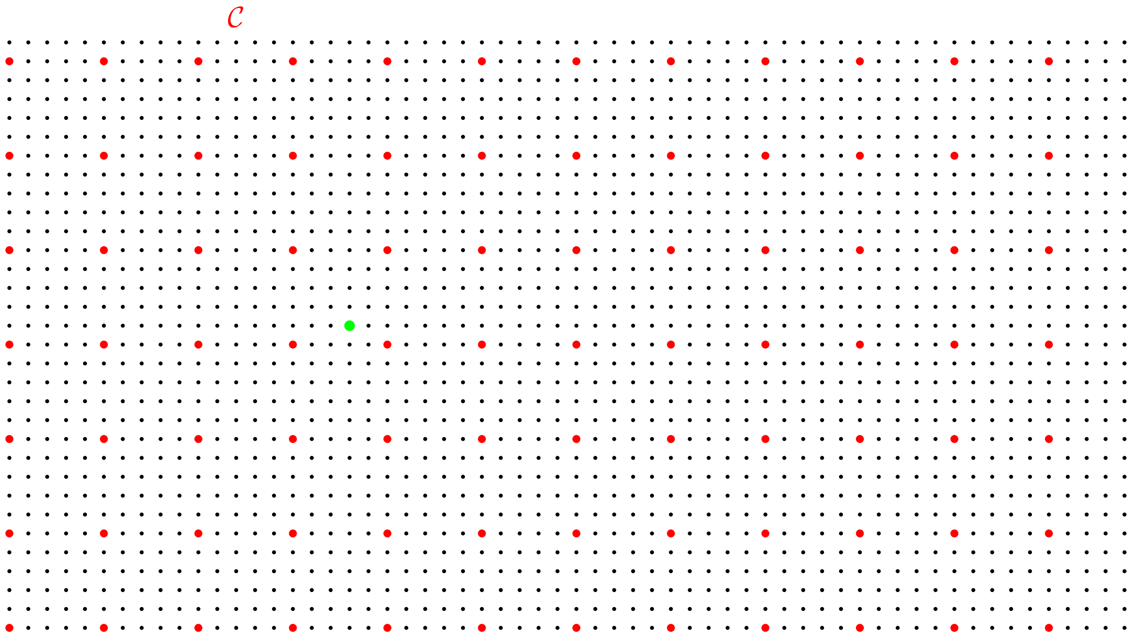

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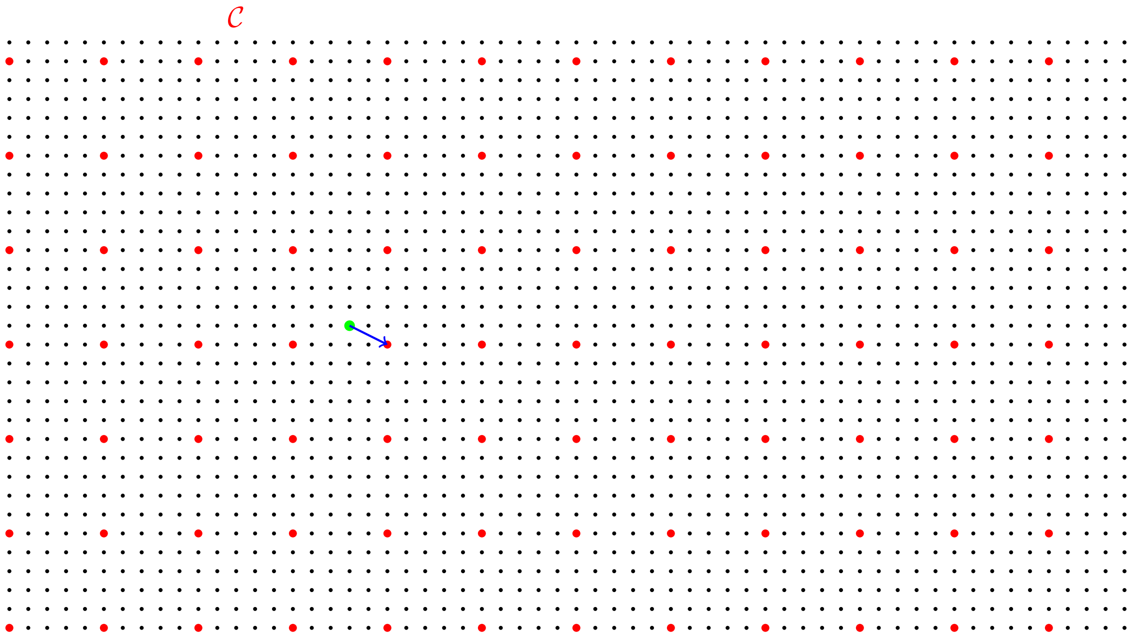

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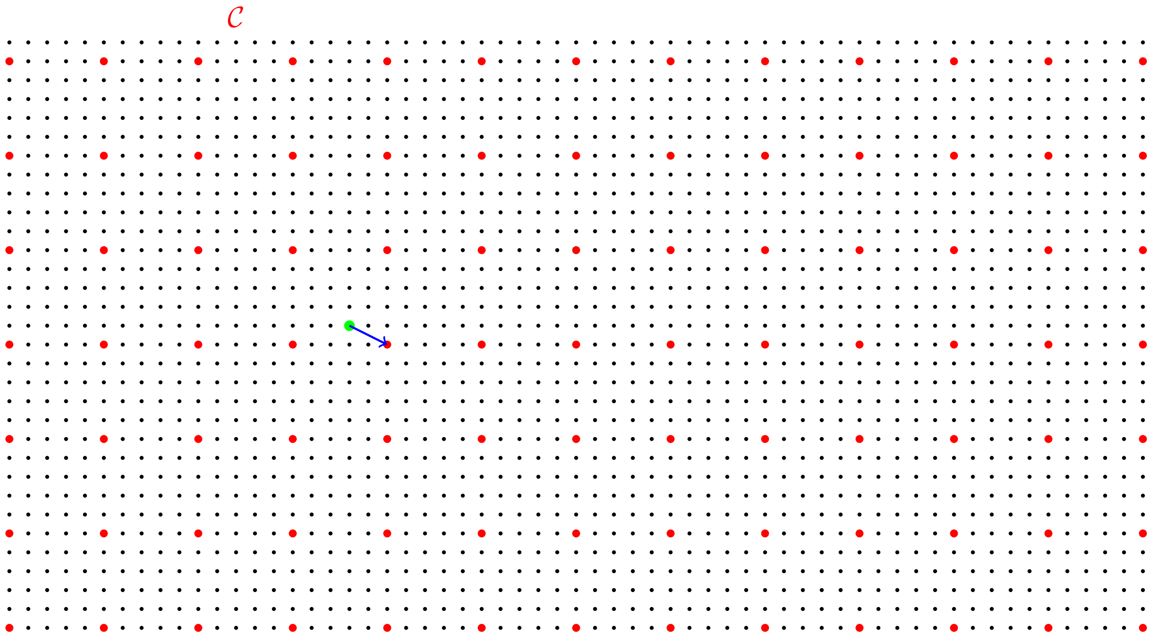

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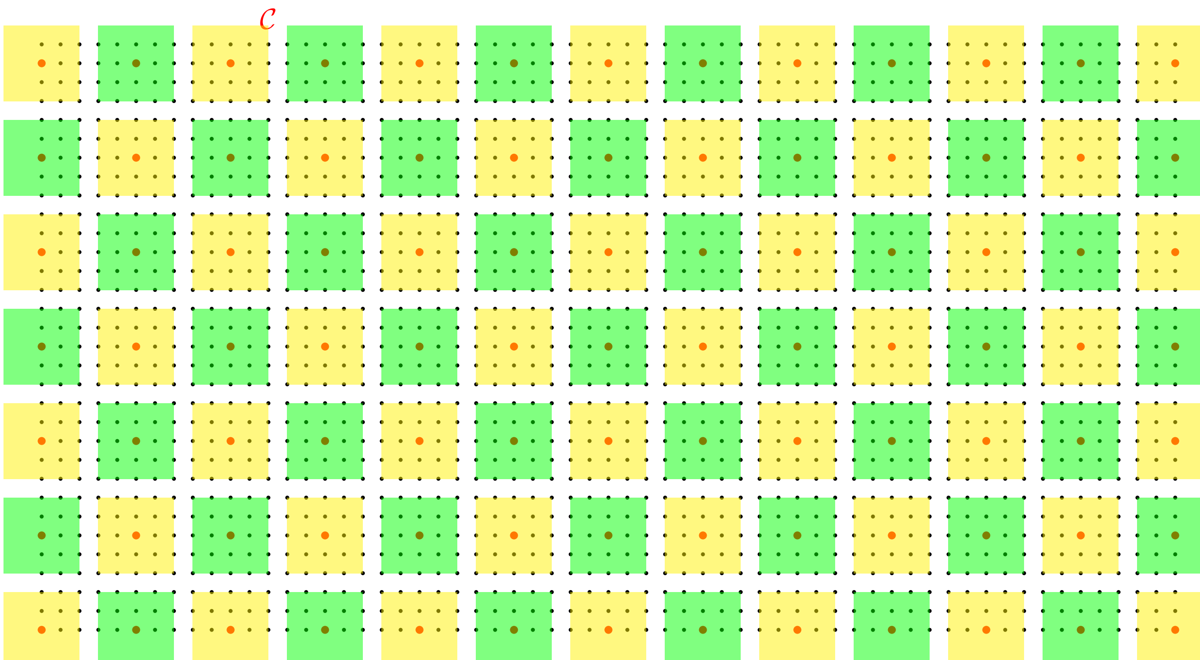

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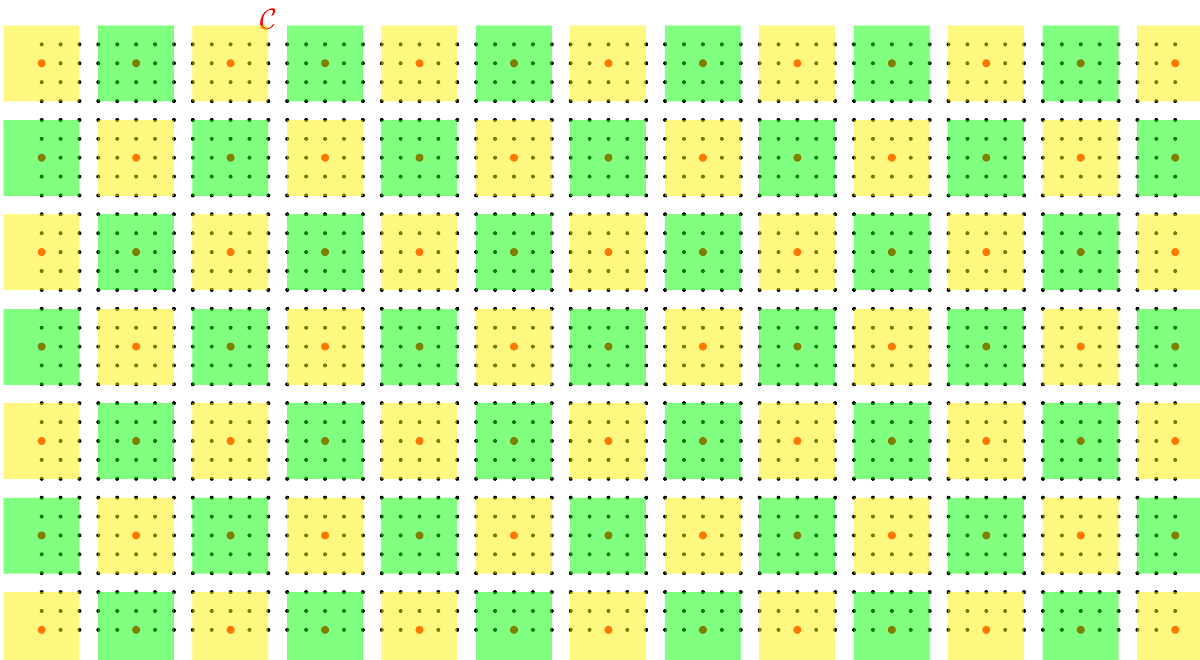

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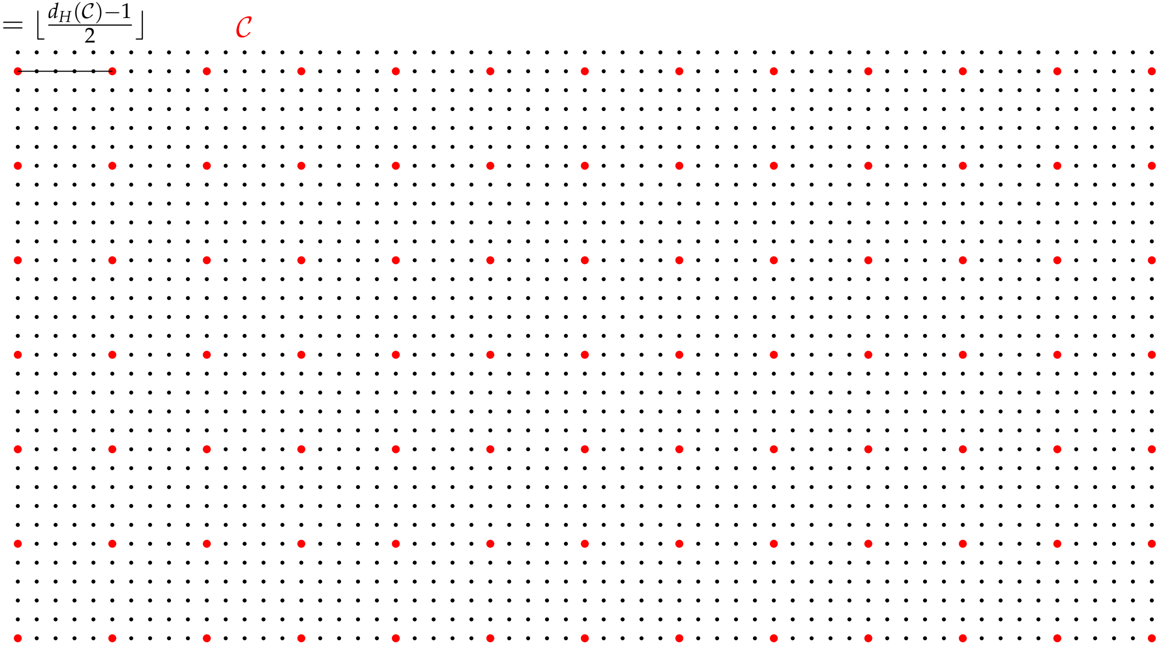
$r = \lfloor \frac{d_H(\mathcal{C})-1}{2} \rfloor$



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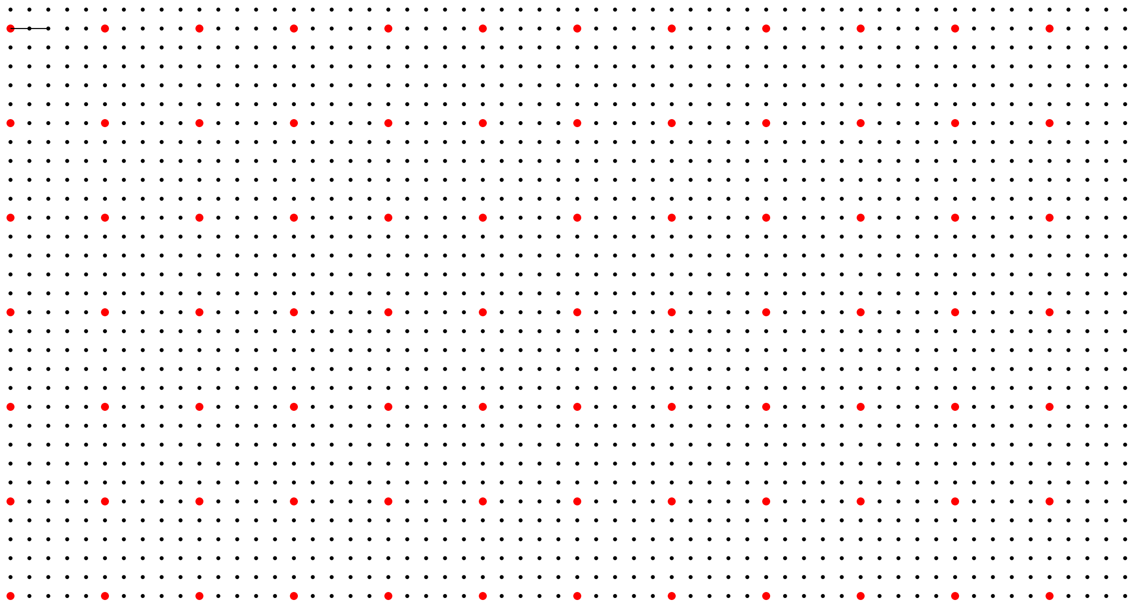
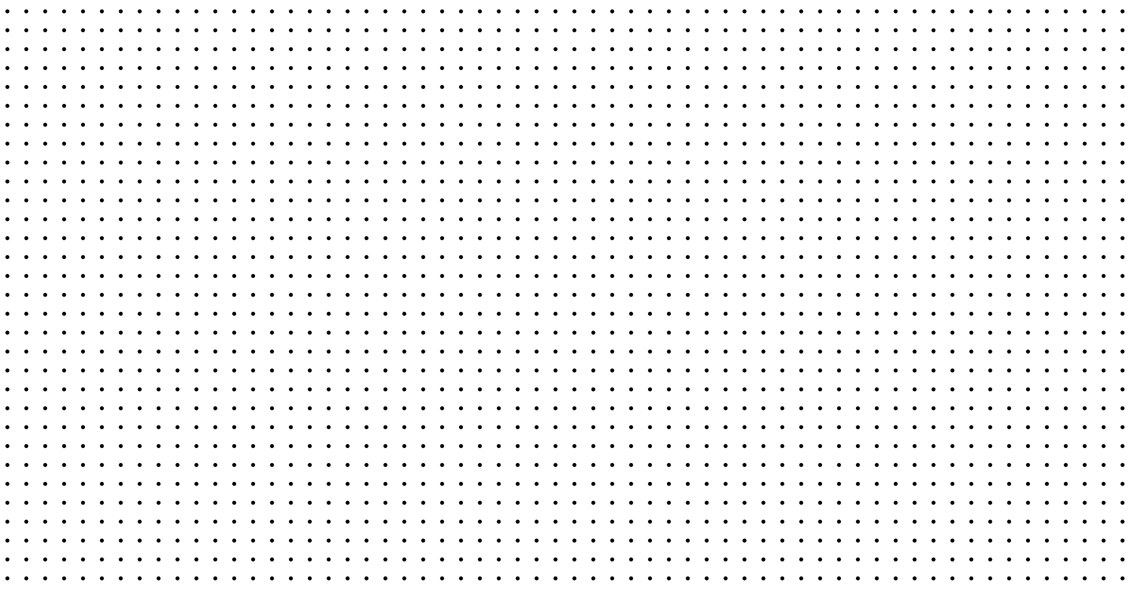
$d_H(\mathcal{C}) := \min_{x \neq y \in \mathcal{C}} d_H(x, y)$ $\mathcal{C}$



Sphere Packing



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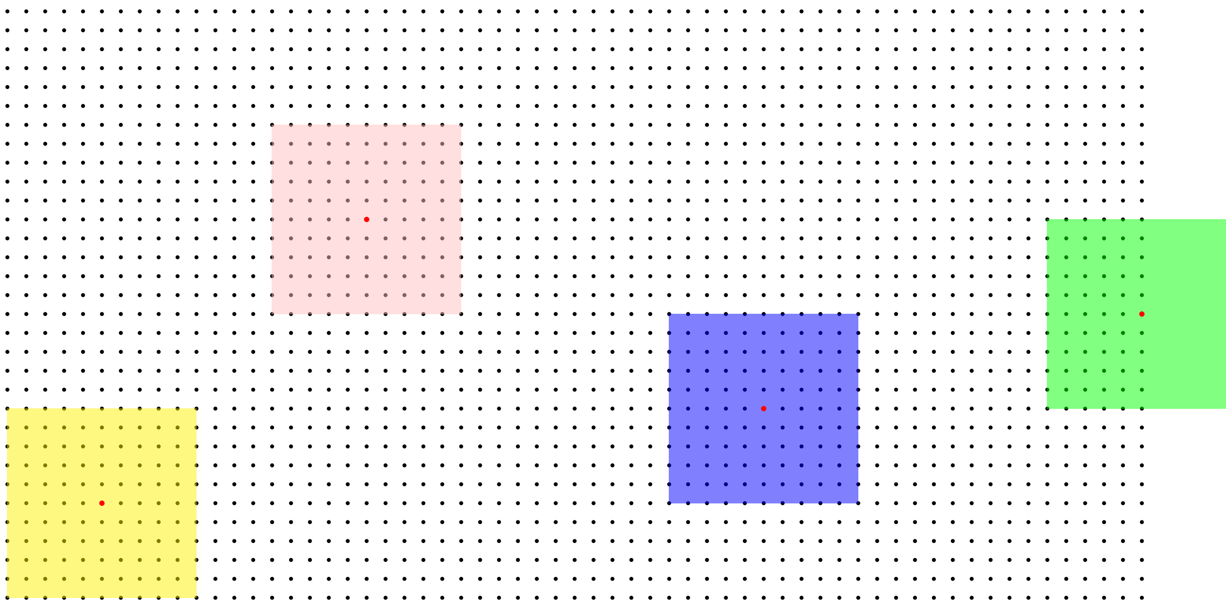

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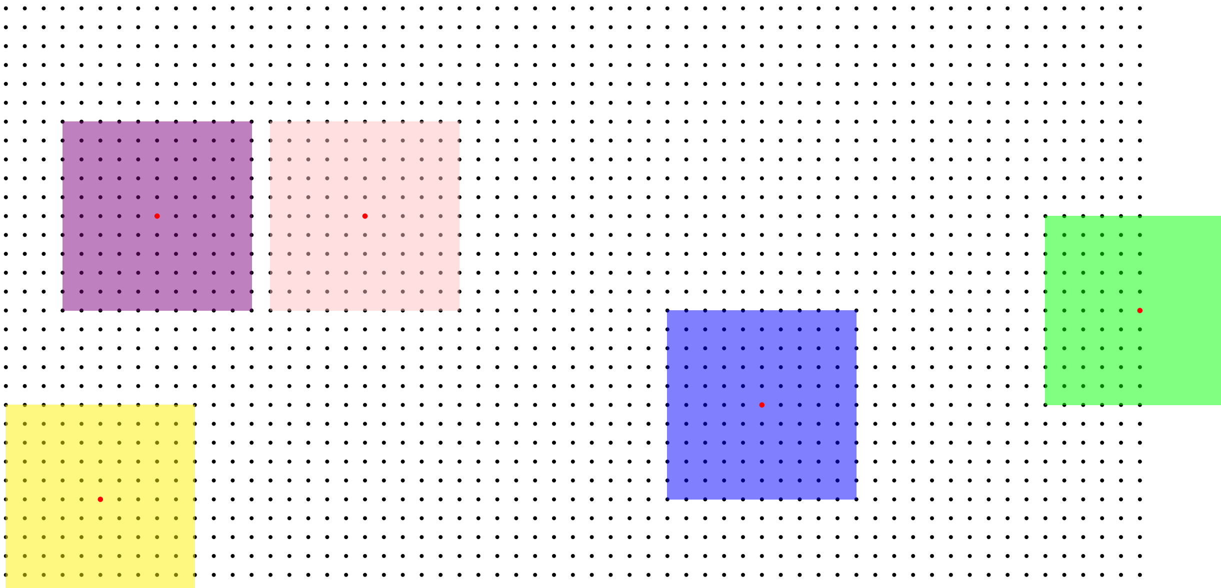

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Theorem 1 (Hamming Bound)

Let $\mathcal{C} < \mathbb{F}_q^N$ be a code with $d_H(\mathcal{C}) = d$ then:

$$|\mathcal{C}| \leq \frac{q^N}{\sum_{i=o}^{t} \binom{N}{i} (q-1)^i}$$

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: The codes satisfying the Hamming bound or the Singleton bound are called respectively **Perfect codes** and **MDS codes** (maximum distance separable codes).

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2)The Hamming codes are Perfect codes, while the "send the same message multiple times"-codes are not perfect.

Other Metrics

Hamming metric

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & \boxed{1} \end{pmatrix} \quad \to \quad \mathrm{wt}_{\mathrm{Hamming}} = 3$$

Other Metrics

Hamming metric

$$\begin{pmatrix} 0 & \colorbox{pink}{1} & 0 & 0 & \colorbox{pink}{1} & 0 & \colorbox{pink}{1} \end{pmatrix} \quad \rightarrow \; \mathrm{wt}_{\mathrm{Hamming}} = 3$$

Rank metric

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Cover metric (rows and columns)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \rightarrow \mathrm{wt}_{\mathrm{Cover}} = 3$$

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Phase-rotation metric

$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \colorbox{pink}{1} & 0 \end{pmatrix} \quad \to \quad \mathrm{wt}_{\mathrm{Phase\text{-}Rot}} = 1 + 1 = 2$$

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Phase-rotation metric

More: burst metric, tensor metric, combinatorical metrics, etc.

Projective Metrics

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Motivation

- Introduced by Gabidulin and Simonis (1997)
- A **generalization** of many metrics in coding theory

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The **projective weight function** $\mathrm{wt}_{\mathcal{F}}(\cdot) : V \to \mathbb{N}_{\geq 0}$ corresponding to $\mathcal{F}$ is

$$\mathrm{wt}_{\mathcal{F}}(x) := \min\{t \in \mathbb{N}_{\geq 0} \mid x \text{ is in the linear span of } t \text{ projective points } \langle f_i \rangle \in \mathcal{F}\}$$

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The **projective metric** $d_{\mathcal{F}}(\cdot, \cdot) : V \times V \to \mathbb{N}_{\geq 0}$ corresponding to $\mathcal{F}$ is

$$d_{\mathcal{F}}(x, y) := \mathrm{wt}_{\mathcal{F}}(y - x).$$

Hamming metric

$$\begin{pmatrix} 0 & \colorbox{pink}{1} & 0 & 0 & \colorbox{pink}{1} & 0 & \colorbox{pink}{1} \end{pmatrix}$$

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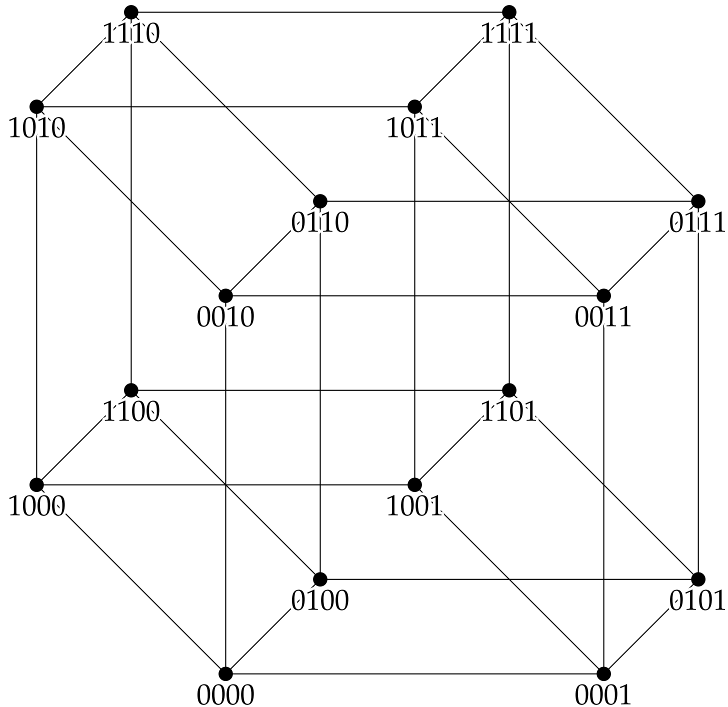
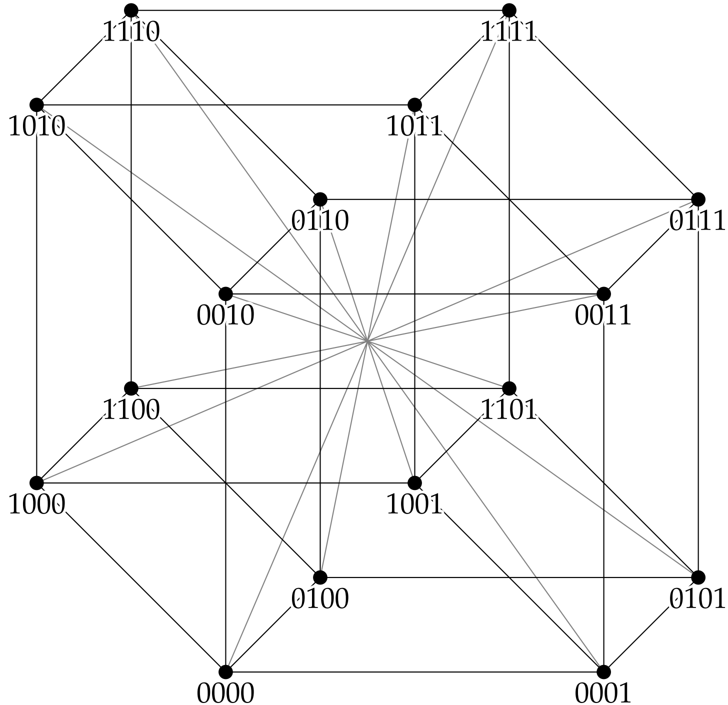
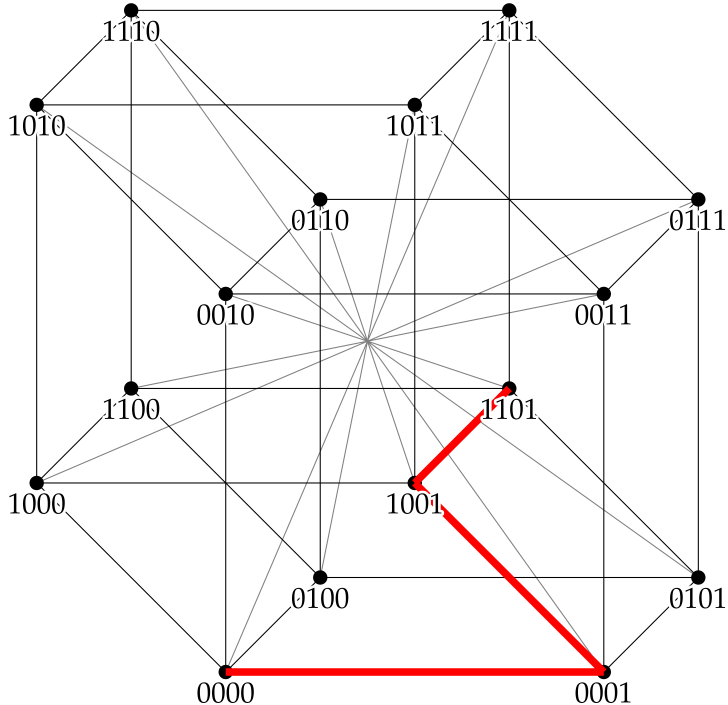
Vertices: vectors of $\mathbb{F}_2^4$

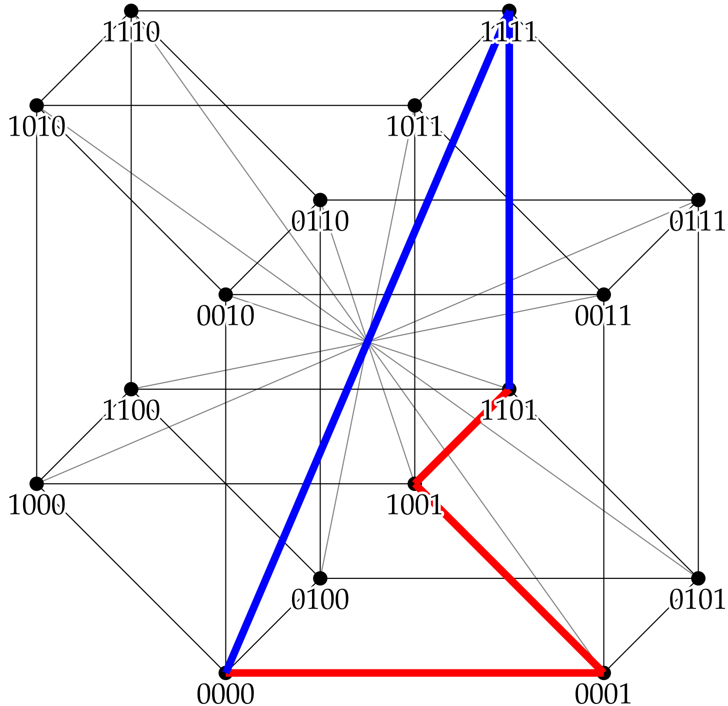


Distance from 0000 to 1101:



Distance from 0000 to 1101: red: 3,




1110
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Let $\mathcal{C} \subseteq V$ be a subset and let $d = \min\{d_{\mathcal{F}}(x, y) \mid x \neq y \in \mathcal{C})\}$. Then

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Coincides with Singleton bounds for specific projective metrics!

Sketch of Proof

Write $\mu := \mu_{\mathcal{F}}(d-1)$ and let $\mathcal{G} = \{g_1, \ldots, g_\mu\} \subseteq \mathcal{F}$ be the linearly independent subset of size μ from the definition of $\mu_{\mathcal{F}}(t)$.

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Consider the projection:

$$\begin{aligned} \mathrm{Proj} : \mathcal{C} &\to \mathbb{F}_q^{N-\mu} \\ \sum_{i=1}^{N} c_i g_i &\mapsto (c_{\mu+1}, \ldots, c_N) \end{aligned}$$

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Let $a, b \in \mathcal{C}$ and suppose $\mathrm{Proj}(a) = \mathrm{Proj}(b)$. Then $\mathrm{Proj}(a-b) = 0$, so $a - b \in \langle \mathcal{G} \rangle$. But this means that

$$d_{\mathcal{F}}(a, b) = \mathrm{wt}_{\mathcal{F}}(a-b) \leq d-1 < d,$$

so $a = b$.

A deep connection to Classical Coding

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Theorem 4

$$\Psi : \bar{P}r_{\mathbf{N}}(V) \to \bar{G}r_{\mathbf{N}-N}(\mathbb{F}_q^{\mathbf{N}})$$
$$\bar{w}_{\mathcal{F}} \mapsto \bar{\mathcal{P}}_{\mathcal{F}}$$

Where $\bar{P}r_{\mathbf{N}}(V)$ is the set containing the equivalence classes of projective metrics on V of size $\mathbf{N}$ and $\bar{G}r_{\mathbf{N}-N}(\mathbb{F}_q^{\mathbf{N}})$ be the set containing the equivalence classes of subspaces of $\mathbb{F}_q^{\mathbf{N}}$ of dimension $\mathbf{N} - N$.

A connection to Classical Coding Theory

Definition 0.1 (Perfect codes)

Given a distance function d on V and a code $\mathcal{C} \subseteq V$, we say that $\mathcal{C}$ is ***perfect*** *with respect to d if there exists a $t \in \mathbb{N}$ such that:*

$$\bigsqcup_{c \in \mathcal{C}} B_t(c) = V \tag{3}$$

where $B_t(c) := \{x \in V : d(x, c) \leq t\}$ are disjoint balls of radius t centered around codewords.

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Which is kind of fun, because you are using one specific metric to get results on a class of metrics, containing the metric itself.

Characterization of projective isometries

What can we do?

Definition 0.2

An $\mathcal{F}$-isometry is a linear isomorphism $L : V \to V$ such that $d_{\mathcal{F}}(Lx, Ly) = d_{\mathcal{F}}(x, y)$ for all $x, y \in V$.

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Theorem 5

Let $stab_H(\mathcal{P})$ be the stabilizer of the parent code $\mathcal{P}$ respect to the Hamming isometries, then $isom_{\mathcal{F}}(V) \cong stab_H(\mathcal{P})$

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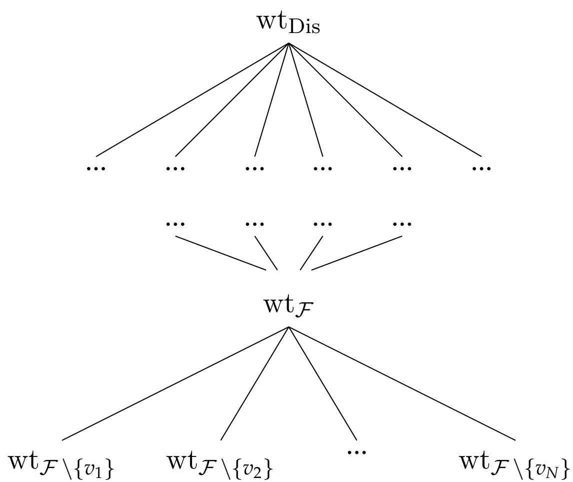

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- **Subspace Arrangements:**
  A ball of size t in projective metric corresponds to the size of the **subspace arrangement** generated by subsets of $\mathcal{F}$ of size less than t.

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  pause And $\mathrm{wt}_{\mathcal{F}}$-isometries are simply homographies fixing $\mathcal{F}$.
- **Subspace Arrangements:**
  A ball of size t in projective metric corresponds to the size of the **subspace arrangement** generated by subsets of $\mathcal{F}$ of size less than t.
- **Matroid Theory:**
  We can observe the matroid associated with the family $\mathcal{F}$.

Connections to Other Fields

Connections to Other Fields
(Of mathematics, not algebraic fields.)

- **Graph Theory:**
  The Cayley graph of $\mathbb{F}_q^N$ with generating set $\mathcal{F}$; $\mathrm{wt}_{\mathcal{F}}(v)$ is the graph distance between v and 0.
- **Classical Coding Theory:**
  The parent code $\mathcal{P}$ of $\mathrm{wt}_{\mathcal{F}}$.
- **Projective Geometry:**
  The projective weights are constant on the projective classes of $\mathbf{P}(V)$.
  pause And $\mathrm{wt}_{\mathcal{F}}$-isometries are simply homographies fixing $\mathcal{F}$.
- **Subspace Arrangements:**
  A ball of size t in projective metric corresponds to the size of the **subspace arrangement** generated by subsets of $\mathcal{F}$ of size less than t.
- **Matroid Theory:**
  We can observe the matroid associated with the family $\mathcal{F}$. This seems to describe well the first two layers of the intersection lattice.

Thank you for your attention.