Projective Metrics

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Introduction to Coding Theory

Coding theory is the study of the properties of codes and their respective fitness for specific applications. Why?

- Retrieve information from corrupted messages
- Store data (which is just sending a message to your future self)
- Data compression
- Cryptography

How?

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Let's repeat each row of a message three times. If we obtain the message:

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Let's repeat each row of a message three times. If we obtain the message:

0	0	0	1	0	1	0	0	0
0	0	0	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0

Since "1" appears twice and "0" once, we may assume more likely that the original message was:

0	0	0	1	1	1	0	0	0

But this way, we took three times the length of the message to correct one error!

Btw the message says SOS in Morse, so maybe go seek help?

A BETTER WAY TO DO IT

We rearrange the original message

0	0	0	1	1	1	0	0	0

in a 4 by 4 grid:

	0	0	0
1	0	0	1
0	1	1	0
0	0	0	0

Where the red cells are the four redundant bits which encode the parity of subsets of columns or rows in the following way.

	0	0	0
1	0	0	1
0	1	1	0
0	0	0	0

$$\boxed{0} = \boxed{1} + \boxed{1}$$

Since 2^4 = total number of bits + (case in which there is no error) $= 15 + 1$ and if there is up to one error, every redundant bit halvens the number the possible locations of where the error might be, we can always correct up to one error in the message.

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1 = 1

Since $2^4 =$ total number of bits $+$ (case in which there is no error) $= 15 + 1$ and if there is up to one error, every redundant bit halvens the number the possible locations of where the error might be, we can always correct up to one error in the message.

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Hamming Codes

The subset of codes in $\mathbb{F}_2^{15}$ constructed the same way are called Hamming Codes. We observe that if we sum two Hamming codes, it remains an Hamming code (that is the parity checks remain valid also for the result of the sum):

	0	0	1
0	0	0	1
1	0	1	1
0	1	0	0

+

	0	0	0
1	0	0	1
0	1	1	0
0	0	0	0

=

	0	0	1
1	0	0	0
1	1	0	1
0	1	0	0

Thus, since we can choose the numbers inside the 11 blue cells arbitrarily they form a 11 dimensional linear subspace of $\mathbb{F}_2^{15}$. For this reason these codes are referred to as [15,11] Hamming Codes.

- Note given a code in $\mathbb{F}_2^{15}$ by changing one bit we can always recover an Hamming Code. Thus what we are doing to correct a message is simply taking the closest valid message!
- Using **nnd** we partition the message space into balls centered on the codewords. If we receive a message, then we simply look at what ball it is in and then the center of that ball is the most likely correct message (This is called nearest neighbour decoder (nnd)).
- To decode as many messages as possible, we take the largest radius such that the balls remain disjoint.
- This radius equals to $\lfloor \frac{d-1}{2} \rfloor$ where d is *the minimum distance* of the code (the set containing all the m codewords).
- The minimum distance d is a simple measure of the goodness of a code.

Classical Sphere Packing Results

In coding theory, given a message length n and a minimum distance d, a fundamental problem is to construct the largest code with minimum distance d. This maximum size is denoted by $A_q(n,d)$ or $B_q(n,d)$ if only linear codes are considered.

Theorem 1 (Hamming Bound)

$B_q(n,d) \leq A_q(n,d) \leq \frac{q^n}{\sum_{i=o}^{t} \binom{n}{i}(q-1)^i}$
Where $t := \lfloor \frac{d-1}{2} \rfloor$.

Theorem 2 (Singleton Bound)

For $d \leq b$, $A_q(n,d) \leq q^{n-d+1}$. Further more if an $[n,k,d]$ linear code over $\mathbb{F}_q$ exists, then $k \leq n-d+1$.

Proof of Hamming bound

Let $\mathcal{C} \subset \mathbb{F}_q^n$ be a code. And let t be the maximum radius such that the union $\cup_{c\in\mathcal{C}} B_t(c)$ is disjoint. As in the euclidean case it can be seen that t must be half of the minimum distance, that is $t = \lfloor \frac{d-1}{2} \rfloor$. Then $| \cup_{c\in\mathcal{C}} B_t(c)| \leq |\mathbb{F}_q^n| = q^n$ and $| \cup_{c\in\mathcal{C}} B_t(c)| = |\mathcal{C}| b_t$. Where b_t is the size of the ball of radious t. To conclude observe that $b_t = \sum_{i=1}^t s_i$ where s_i is the size of the sphere of radious i. The latter equals the number of way we can choose exactly i non null coordinates in a vector of lenght n, thus $s_i = \binom{n}{i}(q-1)^i$. Thus $| \cup_{c\in\mathcal{C}} B_t(c)| = |\mathcal{C}| \sum_{i=1}^t \binom{n}{i}(q-1)^i \leq q^n$. By taking the maximum over $|\mathcal{C}|$ this concludes the proof.

We observe how knowing the sphere size was a central part of the proof.

Other metrics

- The Hamming metric is suitable when errors occur bit-wise with equal probability. For different error scenarios, alternative metrics must be considered.
- For each metric it is important to try to get analogous results to 1 and 2 bounds.
- Generally each metric is studied individually.
- A different approach is to try to get analogous results on a family of metrics.
- In this presentation we show our results on the family of projective metrics.

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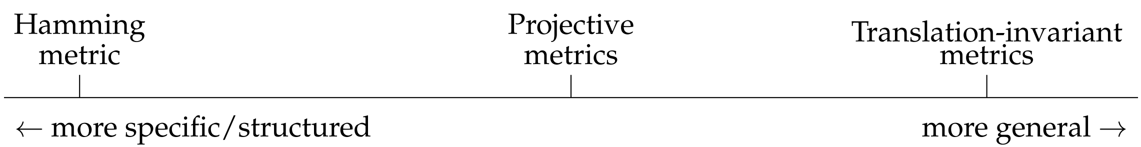

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The **projective weight function** $\mathrm{wt}_{\mathcal{F}}(\cdot) : V \to \mathbb{N}_{\geq 0}$ corresponding to $\mathcal{F}$ is

$$\mathrm{wt}_{\mathcal{F}}(x) := \min\{t \in \mathbb{N}_{\geq 0} \mid x \text{ is in the linear span of } t \text{ projective points } \langle f_i \rangle \in \mathcal{F}\}$$

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The **projective metric** $d_{\mathcal{F}}(\cdot, \cdot) : V \times V \to \mathbb{N}_{\geq 0}$ corresponding to $\mathcal{F}$ is

$$d_{\mathcal{F}}(x, y) := \mathrm{wt}_{\mathcal{F}}(y - x).$$

Hamming metric

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & \boxed{1} \end{pmatrix}$$

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$$\begin{pmatrix} 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & \boxed{1} \end{pmatrix} \quad \to \mathcal{F} = \{\text{spans of standard basis vectors}\}$$

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Rank metric

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Sum-Rank metric

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

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Cover metric (rows and columns)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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Phase-rotation metric

$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \boxed{1} & 0 \end{pmatrix}$$

Hamming metric

$$\begin{pmatrix} 0 & \colorbox{pink}{1} & 0 & 0 & \colorbox{pink}{1} & 0 & \colorbox{pink}{1} \end{pmatrix} \quad \rightarrow \mathcal{F} = \{\text{spans of standard basis vectors}\}$$

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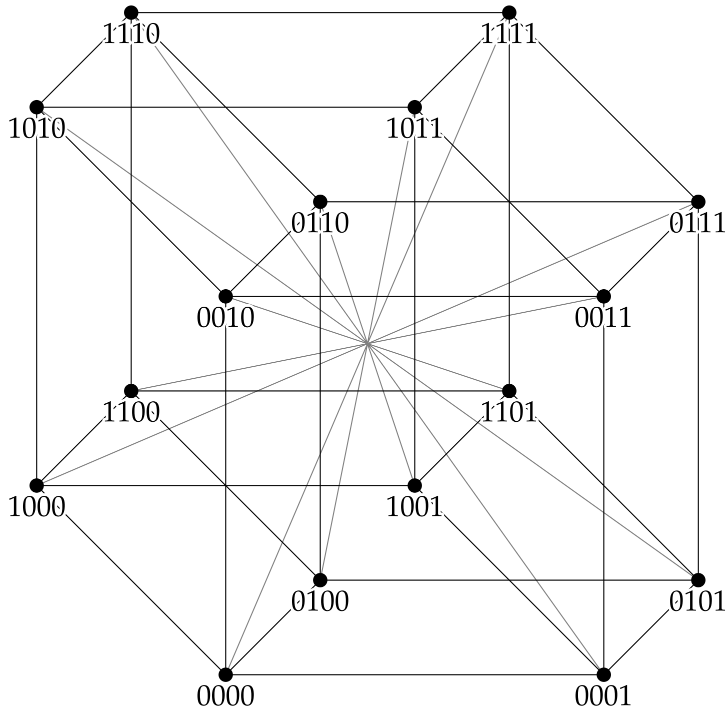
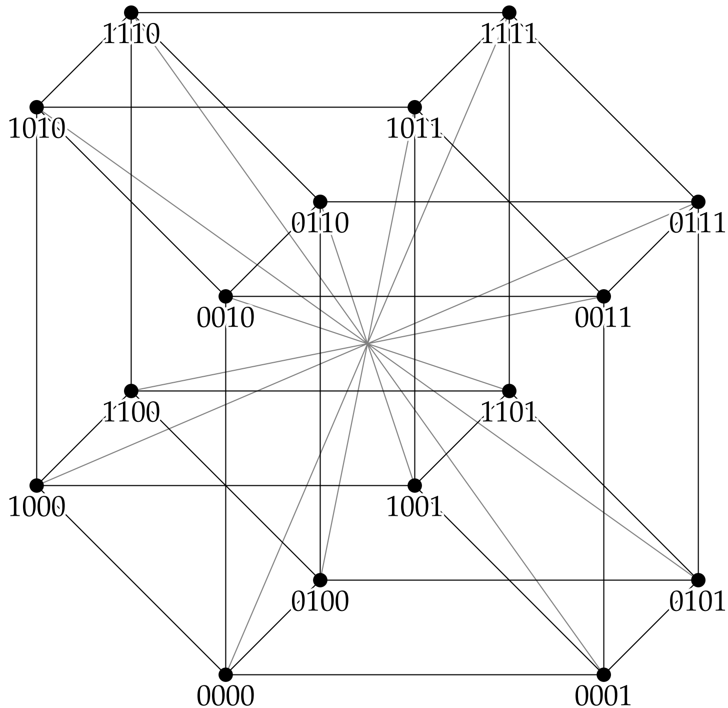
$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \colorbox{pink}{1} & \colorbox{pink}{1} & \colorbox{pink}{1} & \colorbox{pink}{1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \colorbox{pink}{1} & 0 \end{pmatrix} \quad \rightarrow \mathcal{F} = \{\text{spans of standard basis vectors or all-1}\}$$

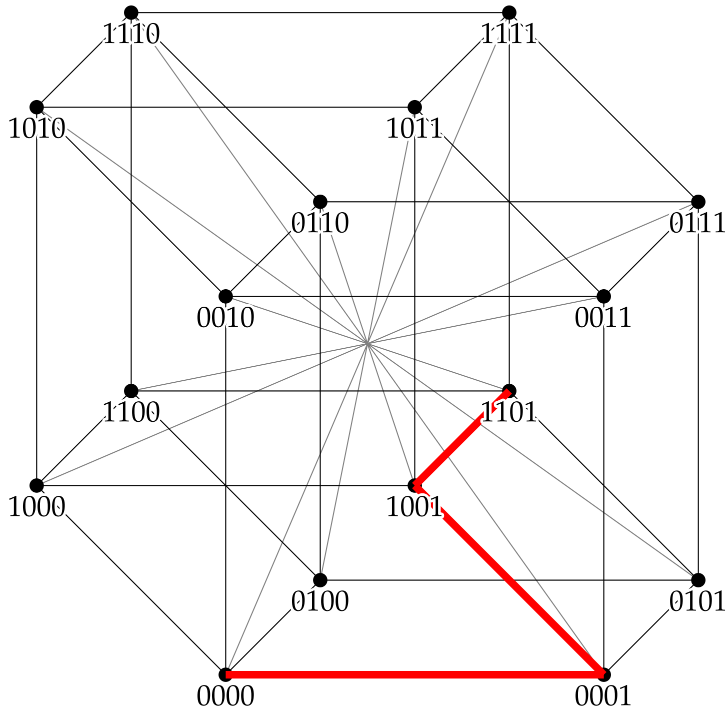
Vertices: vectors of $\mathbb{F}_2^4$



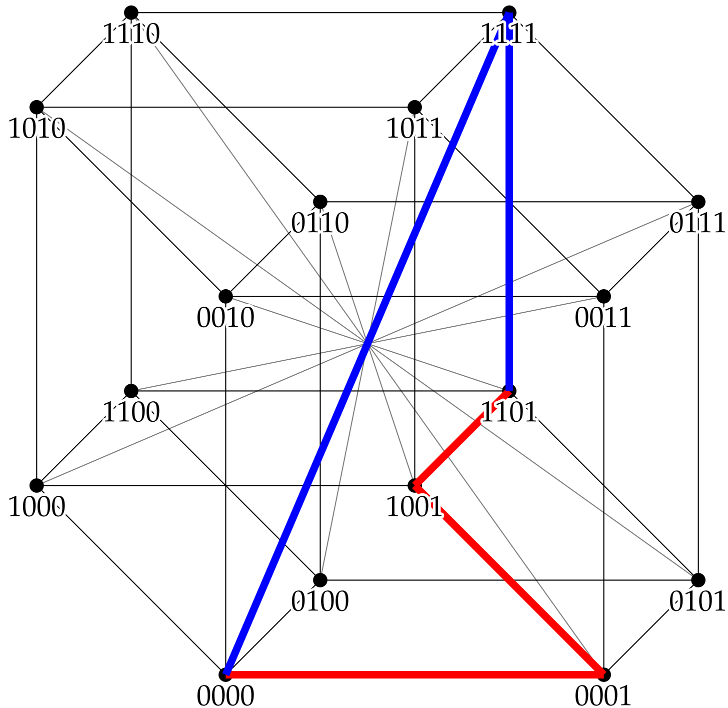
Distance from 0000 to 1101:



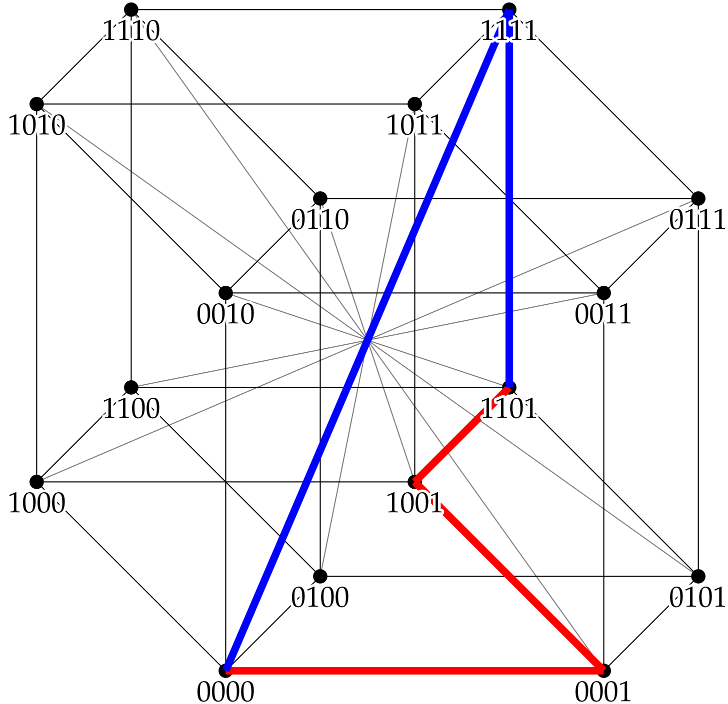
Distance from 0000 to 1101: red: 3,



Distance from 0000 to 1101: red: 3, blue: 2



Graph distance on Clebsch graph $=$ **Phase-rotation metric/distance** on $\mathbb{F}_2^4$



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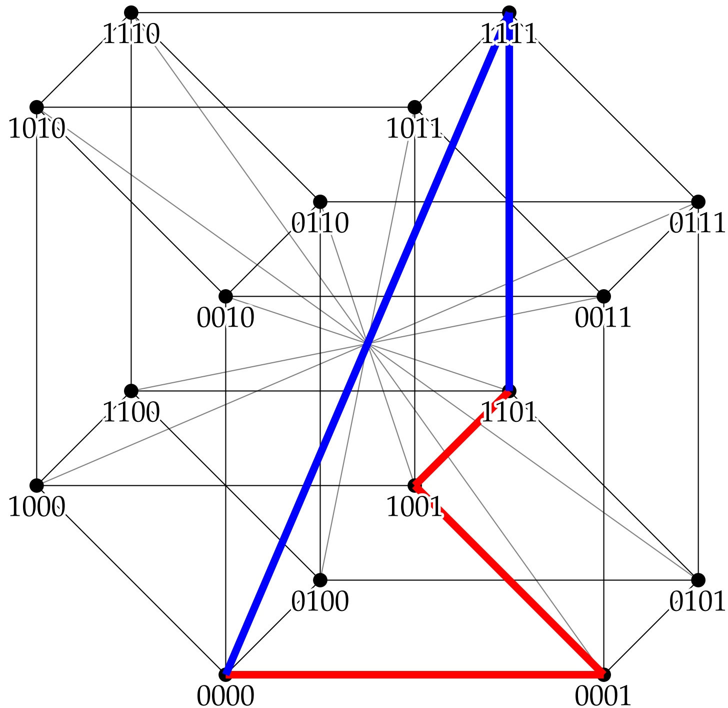
An edge is a Hamming error or the all-bits-flip error



Equivalent notions - 2
Subspace Arrangments

For a set $I \subset \mathcal{F}$ let $F_I := \langle f \mid f \in I \rangle$. For $t \in \mathbb{N}$ consider the subspace arrangement $\mathcal{A} := \{F_I \mid dim(F_I) = t\}$. Then we have $B_t(0) = \cup_{F_I \in \mathcal{A}} F_I$. By exclusion/inclusion we have

$$|B_t(0)| = \sum_{J \subset \mathcal{A}} (-1)^{|J|+1} | \cap_{F \in J} F| \tag{1}$$

An other version is obtained by considering the lattice: $\mathcal{L}_{\mathcal{A}}\{\cap_{F \in J} F | J \subset \mathcal{A}\}$ ordered by reverse inclusion.

$$|\mathbb{F}_q^n \setminus B_t(0)| = \sum_{x \in \mathcal{L}_{\mathcal{A}}} \mu(\mathbb{F}_q^n, x) card(x) \tag{2}$$

Where μ is the Möbius function of $\mathcal{L}_{\mathcal{A}}$.

Equivalent notions - 2

Subspace Arrangments

- For some subspace arrangements the Möbius function of some subspace arrangements is known.
- It may be useful to try to understand whether any of these can be induced by a projective metric.
- This also points in the direction of trying to study the homology group of the lattice associated to a sphere of a projective metric.
- Ideas on how this might work are very welcome! :)

A very important connection to Classical Coding theory is given by the following.

Definition 0.1

(Parent functions and Parent codes of $\mathcal{F}$) The **parent functions** *of $\mathcal{F}$ are $\mathbb{F}_q$-linear functions $\varphi : \mathbb{F}_q^{\mathbf{N}} \to V$ such that $\langle \varphi(e_i) \rangle = F_{\sigma(i)} \in \mathcal{F}$ for some $\sigma(i) \in S_{\mathbf{N}}$. The* **parent codes** *of $\mathcal{F} \subset \mathsf{Gr}_1(V)$ are the elements in the class $\bar{\mathcal{C}} := [\mathrm{Ker}(\varphi)]$ where $[\mathrm{Ker}(\varphi)]$ is the equivalence class of $\mathrm{Ker}(\varphi)$ and φ is a parent function of $\mathcal{F}$.*

Given $v \in \mathcal{V}$ the parent code $\mathcal{C}$ describes all the possible linear combinations of $\mathcal{F}$ that equal to v. That is we have: $v = \sum_{f \in \mathcal{F}} a_f f = \sum_{f \in \mathcal{F}} b_f f$ that is $v = \varphi(a) = \varphi(b)$ if and only if $a - b \in \mathcal{C}$. A series of important proprieties depends on this code, for example we have that: If $x \in \mathbb{F}_q^{\mathbb{N}}$ satisfies $\mathrm{wt}_H(x) \leq \frac{d_H(\mathcal{C})}{2}$, then $\mathrm{wt}_H(x) = \mathrm{wt}_{\mathcal{F}}(\varphi(x))$.

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  Flats spanned by points in $\mathcal{F}$; $\quad \mathrm{wt}_{\mathcal{F}}(v)$ is smallest rank of such a flat that contains v.

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  Certain code $\mathcal{C}$ (depends on $\mathcal{F}$); $\mathrm{wt}_{\mathcal{F}}(v)$ is Hamming weight of the coset $(v + \mathcal{C})$.

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- Please let me know if you know a (partial) answer in any of these contexts! :)

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Theorem 3 (General Singleton-type bound)

Let $\mathcal{C} \subseteq V$ be a subset and let $d = \min\{d_{\mathcal{F}}(x, y) \mid x \neq y \in \mathcal{C})\}$. Then

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Coincides with Singleton bounds for specific projective metrics!

Characterization of projective metrics

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Where two codes are equivalent if there exists a linear Hamming isometry sending one onto the other. The following result tells us that a projective metric is univocally determined by it's parent code.

Theorem 4

Let $\bar{P}r_{\mathbf{N}}(V)$ be the set containing the equivalence classes of projective metrics on V of size $\mathbf{N}$ and $\bar{G}r_{\mathbf{N}-N}(\mathbb{F}_q^{\mathbf{N}})$ be the set containing the equivalence classes of subspaces of $\mathbb{F}_q^{\mathbf{N}}$ of dimension $\mathbf{N}-N$. Then there exists a bijection:

$$\Psi : \bar{P}r_{\mathbf{N}}(V) \to \bar{G}r_{\mathbf{N}-N}(\mathbb{F}_q^{\mathbf{N}})$$
$$\bar{w}_{\mathcal{F}} \mapsto \bar{\mathcal{C}}_{\mathcal{F}}$$

Where $\bar{\mathcal{C}}_{\mathcal{F}}$ is the parent code of $\mathcal{F}$.

Characterization of projective isometries

What can we do?

Definition 0.2

An $\mathbb{F}$-isometry is a linear isomorphism $L : V \to V$ such that $L(\mathcal{F}) = \mathcal{F}$. The set of $\mathbb{F}$-isometries, with the operation of composition, forms a group denoted as $isom_{\mathbb{F}}(V)$.

Theorem 5

Let $stab_H(\mathcal{C})$ be the stabilizer of the parent code $\mathcal{C}$ respect to the Hamming isometries, then
$isom_{\mathcal{F}}(V) \cong stab_H(\mathcal{C})$

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Let $\mathcal{F} = \{\text{all 1-dim subspaces of } V\}$. Then $\mathrm{wt}_{\mathcal{F}}(x) = 1$ for all $x \neq 0$. This is the **discrete weight** $\mathrm{wt}_{\mathrm{Dis}}$.

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- $\mathrm{wt}_{\mathrm{Row}} \cup \mathrm{wt}_{\mathrm{Column}} = \mathrm{wt}_{\mathrm{Cover}}$

Current research

- Algorithms for calculating $\mathrm{wt}_{\mathcal{F}}(v)$ for $v \in V$
- Are there general methods for obtaining sphere sizes $|\{v \in V \mid \mathrm{wt}_{\mathcal{F}}(v) = t\}|$ for $t \in \mathbb{N}$?
- Is there a natural way do generilize other concepts of coding theory? Dual Codes? Perfect Codes? ecc...
- Approach?: using poset lattice of projective metrics, where $\mathrm{wt}_{\mathcal{F}} \preccurlyeq \mathrm{wt}_{\mathcal{G}}$ iff $\mathcal{F} \subseteq \mathcal{G}$