1 Jabr Model

For the OPF model construction we the network as directed graph $(\mathbf{B}, \mathbf{L})$ where $\mathbf{B}$ is the set of Buses and and $\mathbf{L} \subset \mathbf{B} \times \mathbf{B}$ is the set of branches of the network and for each adjacent buses k, m both (k, m) and (m, k) are in $\mathbf{L}$. So the line l adjacent to k, m is modeled by two edges in the arc $\{(k, m), (m, k)\}$. L can be partitioned in $\mathbf{L}_0$ and L_1 with $|L_0| = |L_1|$ where every line l, adjacent to the buses k, m and with a transformer at k, is oriented so that $(k, m) \in L_0$ and $(m, k) \in \mathbf{L}_1$. We also consider a set $\mathcal{G}$ of generators, partitioned into (possibly empty) subsets $\mathcal{G}_k$ for every bus $k \in \mathbf{B}$. We consider the following convex Jabr relaxation of the OPF problem:

$$\inf_{\substack{P_g^G, Q_g^G, c_{km}, \\ s_{km}, S_{km}, P_{km}, Q_{km}}} \quad F(x) := \sum_{g \in \mathcal{G}} F_g(P_g^G) \tag{1}$$

Subject to: $\forall km \in \mathbf{L}$

$$\begin{aligned}
& c_{km}^2 + s_{mk}^2 \leq c_{kk} c_{mm} \quad \text{Jabr constraint} && (2)\\
& P_{km} = G_{kk} c_{kk} + G_{km} c_{km} + B_{km} s_{km} && (3)\\
& Q_{km} = -B_{kk} c_{kk} - B_{km} c_{km} + G_{km} s_{km} && (4)\\
& S_{km} = P_{km} + j Q_{km} && (5)\\
& \text{Power balance constraints: } \forall k \in \mathbf{B} && \\
& \sum_{km \in L} S_{km} + P_k^L + i Q_k^L = \sum_{g \in \mathcal{G}(k)} P_g^G + i \sum_{g \in \mathcal{G}(k)} Q_g^G && (6)\\
& \text{Power flow, Voltage, and Power generation limits:} && \\
& P_{km}^2 + Q_{km}^2 \leq U_{km} && (7)\\
& V_k^{\min^2} \leq c_{kk} \leq V_k^{\max^2} && (8)\\
& P_g^{\min} \leq P_g^G \leq P_g^{\max} && (9)\\
& c_{kk} \geq 0 && (10)\\
& V_k^{\max} V_m^{\max} \geq c_{km} \geq 0 && (11)\\
& -V_k^{\max} V_m^{\max} \leq s_{km} \leq V_k^{\max} V_m^{\max} && (12)\\
& c_{km} = c_{mk}, \; s_{km} = -s_{mk}. && (13)
\end{aligned}$$

This relaxation is in general not exact. We can recover exactness thanks to the following result:

Proposition 1. *Model* (1) *with the additional* loop constraint (14) *for every loop in a cycle basis of* $(\mathbf{B}, \mathbf{L})$ *is exact, we refer to this new model as the* Exact Jabr formulation

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subset [n] \\ |A| = 2k}} (-1)^k \prod_{h \in A} s_{k_h k_{h+1}} \prod_{h \in A^c} c_{k_h k_{h+1}} = \prod_{k=1}^{n} c_{k_i, k_i}. \tag{14}$$

In cite, auxiliary branches were added to the network, dividing each loop in smaller loops, to decrease the degree of the polynomials defining the loop constraint. Then Mc Cormick linearization was applied. The problem with this approach is that one exiliary branch is added for every branch in the loop. This result suggests the following approaches to either find a feasible solution or move along the space of feasible solutions.

- Since the loop constraint is multilinear in it's variables, we can consider linear relaxations called *Flower inequalities*, which generalize che classical Mc Cormick relaxations of products of variables.
- Such relaxation is exact on tree Networks (also known as radial networks). Our objective is, given a network $\mathcal{N} = (\mathbf{B}, \mathbf{L})$ which can also not be a tree, consider a radial subnetwork $\mathcal{N}' = (\mathbf{B}, \mathbf{L}')$, with $\mathbf{L}' \subset \mathbf{L}$ and consider the Jabr model on $\mathcal{N}'$. This solution is not necessarily feasible for the original problem $\mathcal{N}$, our objective is to iteratively recover a feasible solution for $\mathcal{N}$. Since the Jabr relaxation is exact on $\mathcal{N}'$ it follows that the constraints

  2 are respected, the constraints which are violated are the flow constraints on the leaves. We can try to recover feasibility my moving along the solution to the Jabr and Loop constraints.
- Given a feasible solution, find feasible directions.

2 Linearization of loop constraints

To find feasible relaxations of the loop constraint we follow cite. It must be noted that a major difference in our approach is that the OPF is not a multilinear optimization problem. So first we show that the same results in cite can be applied to the OPF.

Consider a set of multilinear constraints:

$$\sum_{I \in \mathcal{I}_j} c_I^j \prod_{v \in I_j} x_v \quad \forall I \leq b_j \quad \forall j \in \{1, 2, \dots, m\} \tag{15}$$

$$x_v \in [l_v, u_v] \quad \forall v \in V \tag{16}$$

Where V denotes the variables and $\mathcal{I}_j \in \mathcal{P}(V)$, for $j = 1, \dots, m$ are the variables of the monomials appearing in the j-th homogenous constraint. A straight forward linearization is to introduce a variable z_I for every subset I of variables appearing in the constraints. Thus we obtain the following equivalent problem.

$$\sum_{I \in \mathcal{I}_j} c_I^j z_I \leq b_j \quad \forall j \in \{1, 2, \ldots, m\} \tag{17}$$

$$z_I = \prod_{i \in I} x_i \quad \forall I \in \mathcal{E} \coloneqq \cup_{j=1}^m I_j \tag{18}$$

$$x_v \in [l_v, u_v] \quad \forall v \in V \tag{19}$$

By affine afformation we can assume the variables x_v to be in the form $c_v \in [0, 1]$. Note that such affine transformations need to be handled with care, we will cover this in subsection 2.1. Since contraint (17) is now linear, we are now interested in the linearization of the following set $Pr \coloneqq \{(x, z) \in [0, 1]^V \times [0, 1]^{\mathcal{E}} \mid z_I = \prod_{i \in I} \forall I \in \mathcal{E}\}$. If such constraints were the only ones, and if the cost was also multilinear, we would know that the solution would be on one of the vertices of the hypercube and the observation that follows would be trivially true. Since in the OPF the cost is not multilinear and there are other types of constraints we show that this is also a relaxation for Pr.

Definition 1 (Standard form relaxation)**.** *Let the polyhedral PrR be defined by the linear constraints* (20a)-(20d)*.*

G: fixalignment

$$z_I \leq x_v \qquad \forall v \in I \in \mathcal{E} \tag{20a}$$

$$z_I + \sum_{v \in I} (1 - x_v) \geq 1 \qquad \forall I \in \mathcal{E} \tag{20b}$$

$$z_I \geq 0 \qquad \forall I \in \mathcal{E} \tag{20c}$$

$$x_v \in [0, 1] \qquad \forall v \in V \tag{20d}$$

The corresponding Standard Form Relaxation for problem with homogeonus cost and constraints if often very weak. As done in cite we augment 1 with *Flower Inequalities*, which are additional inequalities valid for Pr. Again the main difference with cite: McCormick stikes back, is that we cannot restict the hypercuber C to its vertices becase other point could also be optimal for the OPF problem. Se we check the additional flower inequalities are still valid for Pr.

Definition 2 (extendend flower inequalities.)**.** *Let $I \in \mathcal{E}$ and let $J_1, \ldots, J_k \in \mathcal{E} \cup \mathcal{S}$ be such that $I \subset \bigcup_{i=1}^k J_i$ and $I \cap J_i \neq \emptyset$ holds for $i = 1, 2, \ldots, k$. The* extended flower inequality *with center I and petals $J_1, \ldots, J_k$ is defined as*

$$z_I + \sum_{i=1}^k (1 - z_{J_i}) \geq 1 \tag{21}$$

The extended flower relaxation $FR \subset [0, 1]^{\mathcal{E} \cup \mathcal{S}}$ are the elements $x \in [0, 1]^{\mathcal{E} \cup \mathcal{S}}$ for which all the extended flower inequalities hold.

Proposition 2. *For all $x \in Pr$ and $I \in \mathcal{E}$, $J_1, \ldots, J_k \in \mathcal{E} \cup \mathcal{S}$ such that $I \subset \bigcup_{i=1}^k J_i$ and $I \cap J_i \neq \emptyset$ for $i = 1, 2, \ldots, k$. Then* extended flower inequality (21) *with center I and petals $J_1, \ldots, J_k$ holds for x. In particular $Pr \subset FR$.*

Proof. For $|I| = 1$ this is trivially true. For $|I| = n > 1$, wlog $I = \{1, \dots, n\}$. We want to see that for any $x \in C_{\cup_k J_k} := [0,1]^{\cup_{k=1}^K J_K}$ we have $a(x) = \prod_{i \in I} x_i + \sum_{k=1}^K (1 - \prod_{j \in J_k} x_j) - 1 \geq 0$. Consider the face $F = \{x \in [0,1]^{\cup_{k=1}^K J_K} \mid x_1 = 0\}$. Then $C_{\cup J_k} = C_{\cup J_k} \cap (F + \langle e_1 \rangle)$. Thus we only need to show that for every $x \in F$ the function $f_x(x_1) := a(x + x_1 e_1)$ is positive. This is an affine function, thus it sufficient to show that it is positive at $x_1 = 0$ and $x_1 = 1$. For $x_1 = 0$ this is trivially true. For $x_1 = 1$, we have $f_x(1) = \prod_{i \in I \setminus \{1\}} x_i + \sum_{k=1}^K (1 - \prod_{j \in J_k \setminus \{1\}} x_j) - 1 \geq 0$ by induction on $|I|$. □

By taking $J_i = \{x_i\}$ for all $x_i \in I$, since $z_{J_i} = x_i$, we have:

Corollary 2.1. *The* standard form relaxation *as in defition 1 is a relaxation of Pr. That is $Pr \subset PrR$.*

G: ok, to write better

G: Il resto del paper McCormick strikes back drovebbe valere anche qui, perchè parla della struttura dei rilassamenti, che non dipende da Pr.

2.1 Handling affine trasformations

In the beginning of the sections, we assumed that the variables $x_i \in [0,1]$ because affine transformation of homogenous constraints remain homogenous.But it must be noted that for each non linear affine trasformation, that is when the lower bound of the corresponding variable is not zero, the number of monomials increases. More precisly, given a monomial defined by $I \in \mathcal{E}$, let $I' \subset I$ be the subset of variables in I for which a nonlinear trasformation is applied. Then the monomial I is split into $2^{|I'|+1}$ new monomials. When the size of such I' is large this greatly increases the number of auxiliary variables z_J which must be introduced. Thus applying many non linear affine transformation can be very costly and complicates the handling of the constraints. For this reason, instead of applying non linear affine transformation, for each variables $v \in V$ such that $x_v \in [l_v, u_v]$ and $l_v * u_v \neq 0$, we split the problem in two new subproblems having $v_x \in [l_v, 0]$ and $v_x \in [0, u_v]$ respectively. This way linear transformations can be applied in the subproblems. This creates many subproblems, many of which are unfeasible for the OPF, we diminish the number of subproblems we need to solve thanks to some unfeasibility conditions. Then, instead of solving each subproblem in a random order, we rewrite the subproblems as a unique mixed integer programming problem.

Observation 3.

G: add observation that they cannot be all positive or all negative. Can we also say something more? maybe not

Let $C = \{k_1, \dots, k_n\} \subset \mathbf{B}$ be a cycle. The variables s_h are in the form $s_h \in [-u_{s_h}, u_{s_h}]$ where $h = (k_i, k_{i+1})$ for all $i = 1, \dots, n$. We can then substitute s_h with $u_h s'_h = s_h$ where $s'_h \in [-1, 1]$. We then define the sign variables

$\sigma_h \in \{0,1\}$ for each $h \in C$, where $\sigma_h = 0$ if s_h is negative and 1 if it is positive. We can now rewrite the loop constraint as:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subset [n] \\ |A|=2k}} (-1)^k (\prod_{h \in A} (2\sigma_h - 1) u_h) z_A = z'_C \tag{22}$$

Where we substitute the monomial $\prod_{h \in A} s_{k_h k_{h+1}} \prod_{h \in A^c} c_{k_h k_{h+1}}$ with z_A and the product $\prod_{k=1}^{n} c_{k_i,k_i}$ with z'_C. For each even subset $A \subset [n]$ we introduce the binary variable $\lambda_A \in \{0,1\}$ which is 0 if $\prod_{h \in A}(2\sigma_h - 1)$ is -1 and $\lambda_A = 1$ otherwise. The loop constraint becomes:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subset [n] \\ |A|=2k}} (-1)^k (2\lambda_A - 1) U_A z_A = z'_C \tag{23}$$

The product $\lambda_A z_A$ can easily be linearized. To enforce the relation $2\lambda_A - 1 = \prod_{h \in A}(2\delta_h - 1)$, simply note that $\lambda_A = 0$ if and only if there is an odd number of δ_h equal to 0, that is, there exists $m_A \in \mathbb{Z}$ such that:

$$\lambda_A + 2m_A = \sum_{h \in A} \delta_h \tag{24}$$

G: In realtà questo non è specifico all'OPF, va bene per tutti i vincoli omogenei e si può fare al posto di fare le trasformazioni affini!

3 Other cuts

Let I a set of indices and x_v, $v \in I$, continuous variables such that $x_v \in [l_v, u_v] \subset [-1,1]$. Let also the set I be partitioned in two set such that $I = J \oplus K$ and

$$l_v > 0, \qquad u_v = 1, \qquad \forall v \in J, \tag{25a}$$
$$l_v = -u_v, \qquad u_v < 1, \qquad \forall v \in K. \tag{25b}$$

If we define $z_I \coloneqq \prod_{v \in I} x_v$ and $I' \coloneqq I \setminus \{v\}$, then the following lower and upper bounds trivially hold:

$$z_I \le x_v \prod_{v' \in I'} u_{v'}, \qquad \forall v \in J, \tag{26a}$$
$$z_I \le |x_v| \prod_{v' \in I'} u_{v'}, \qquad \forall v \in K, \tag{26b}$$
$$z_I \ge -|x_v| \prod_{v' \in I'} u_{v'}, \qquad \forall v \in K, \tag{26c}$$
$$z_I \ge x_v \prod_{v' \in I'} l_{v'}, \qquad \forall v \in J,\ I' = J' \oplus K' \,:\, K' = \emptyset, \tag{26d}$$
$$z_I \ge -x_v \prod_{v' \in I'} u_{v'}, \qquad \forall v \in J,\ I' = J' \oplus K' \,:\, K' \neq \emptyset. \tag{26e}$$

Lemma 4. *The following inequality holds:* ***Questo considera l'iperpiano tangente nel punto avente coordinate $u_v, u_v, \dots, u_v, \prod u_v$. Si riesce a fare lo stesso per altri punti del cuboide? Possibile che qualcuno lo ha già fatto?***

$$z_I + \sum_{v \in I} c_v(u_v - x_v) \geq \prod_{v \in I} u_v, \quad c_v := \prod_{v' \in I \setminus \{v\}} u_{v'}. \tag{27}$$

Proof. Because we are dealing with a multilinear inequality, it is sufficient to verify that it holds for every vertex of the multidimensional rectangular cuboid

$$\mathfrak{C} := \prod_{v \in I} [l_v, u_v] \subset [-1, 1]^{|I|}.$$

For such a vertex x, we have either $x_v = l_v$ or $x_v = u_v$ for every $v \in I$. Define $I_1 := \{v \in I \,|\, x_v = l_v\}$ and $I_2 := \{v \in I \,|\, x_v = u_v\}$, and J_1, J_2, K_1, K_2 analogously. By defining $k_1 := |K_1|$, we then have

$$\begin{aligned} z_I + \sum_{v \in I} c_v(u_v - x_v) &= \prod_{v \in I_1} l_v \prod_{v \in I_2} u_v + \sum_{v \in I_1} c_v(u_v - x_v) + \sum_{v \in I_2} c_v(u_v - x_v) = \\ &= \prod_{v \in I_1} l_v \prod_{v \in I_2} u_v + \sum_{v \in I_1} c_v(u_v - l_v) = \\ &= (-1)^{k_1} \prod_{v \in K_1 \cup I_2} u_v \prod_{v \in J_1} l_v + \sum_{v \in J_1} c_v(1 - l_v) + 2 \sum_{v \in K_1} c_v u_v = \\ &= A + B + C. \end{aligned}$$

Now, we rewrite each of the three terms. First,

$$\begin{aligned} A = (-1)^{k_1} \prod_{v \in K_1 \cup I_2} u_v \prod_{v \in J_1} l_v &= (-1)^{k_1} \prod_{v \in K_1 \cup I_2} u_v \prod_{v \in J_1} u_v \prod_{v \in J_1} l_v = \\ = (-1)^{k_1} \prod_{v \in I} u_v \prod_{v \in J_1} l_v, \end{aligned}$$

where the second equality holds because $u_v = 1, \forall v \in J_1$. Second,

$$\begin{aligned} B = \sum_{v \in J_1} c_v(1 - l_v) &= \sum_{v \in J_1} (\prod_{v' \in I \setminus \{v\}} u_{v'})(1 - l_v) = \sum_{v \in J_1} (\prod_{v' \in I} u_{v'})(1 - l_v) = \\ = (\prod_{v \in I} u_v) \sum_{v \in J_1} (1 - l_v), \end{aligned}$$

where the third equality holds because $u_v = 1 \, \forall v \in J_1$. Finally,

$$\begin{aligned} C = 2 \sum_{v \in K_1} c_v u_v &= 2 \sum_{v \in K_1} (\prod_{v' \in I \setminus \{v\}} u_{v'}) u_v = 2 \sum_{v \in K_1} (\prod_{v' \in I} u_{v'}) = \\ = 2k_1 (\prod_{v \in I} u_v). \end{aligned}$$

We then have

$$z_I + \sum_{v \in I} c_v(u_v - x_v) = A + B + C =$$
$$= (\prod_{v \in I} u_v)((-1)^{k_1} \prod_{v \in J_1} l_v + \sum_{v \in J_1} (1 - l_v) + 2k_1).$$

To conclude, we just need to prove that

$$(-1)^{k_1} \prod_{v \in J_1} l_v + \sum_{v \in J_1} (1 - l_v) + 2k_1 \geq 1,$$

but this is true because

$$(-1)^{k_1} \prod_{v \in J_1} l_v + \sum_{v \in J_1} (1 - l_v) + 2k_1 =$$
$$= ((-1)^{k_1} - 1) \prod_{v \in J_1} l_v + 2k_1 + \prod_{v \in J_1} l_v + \sum_{v \in J_1} (1 - l_v) =$$
$$\geq ((-1)^{k_1} - 1) \prod_{v \in J_1} l_v + 2k_1 + 1 \geq (-1)^{k_1} + 2k_1 \geq 1,$$

where the first inequality holds because $\prod_{v \in J_1} l_v + \sum_{v \in J_1} (1 - l_v) \geq 1$ because of the standard multilinear relaxation. cita, the second inequality holds because $0 < l_v < 1$ and so $0 \leq \prod_{v \in J_1} l_v < 1$, and the last inequality holds because $k_1 \geq 0$ integer. $\square$

Observation 5. *Note that, with the same notation of the proof above,* (27) *is tight for every vertex of* $\mathfrak{C}$ *for which* $|I_1| \leq 1$.

3.1 A new(?) class of contraints

We now consider when, given a vertex a of the cuboid $\mathfrak{C}$, we can define a separating hyperplane π for Pr, that is, when $\pi : \mathbb{R}^n \to \mathbb{R}$ such that either $\pi(x) \leq \prod_{v \in I} x_v$ for all x in $\mathfrak{C}$ for all x in the vertex or $\pi(x) \geq \prod_{v \in I} x_v$ for all x in $\mathfrak{C}$. Furthermore, we want a good cut, so we look for hyperplanes such that $\pi(y) = \prod_{v \in I} y_v$ for all vertices y in $\mathfrak{C}$, adjacent to a. In this section we denote the vertex opposite to a, a^{op}. Choosing a vertex, corresponds to picking for each interval $v \in I$, either the lower bound or the upper bound, so a^{op} is the vertex having as $a^{\text{op}}_v = l_v$ if $a_v = u_v$ and $a^{\text{op}}_v = u_v$ otherwise. Thus by defining $d_v := (a^{\text{op}}_v - a_v)e_v$, a vertex $x \in \mathfrak{C}$ corresponds to a unique $J \subset I$ such that

$$x = a + \sum_{v \in J} d_v$$

for this reason we denote x as x_J. First we observe that since in $\mathbb{R}^{n+1}$ fixed $n+1$ affinely independent points there exists a unique hyperplane containing all these points, then there exists a unique hyperplane such that $\pi(y) = \prod_{v \in I} y_v$ for all vertices y in $\mathfrak{C}$, adjacent to a and for $y = x$.

Observation 6. *The hyperplane* $\pi_a(x) := \prod_{v\in I} a_v + \sum_{v\in I} C_v a_v(x_v - a_v)$*, where* $C_v := \prod_{v'\in I\setminus\{v\}}$*, is the only hyperplane such that* $\pi(y) = \prod_{v\in I} y_v$ *for all vertices* y *in* $\mathfrak{C}$*, adjacent to* a*.*

We now look for conditions for when this uniquely defined hyperplane is also a separating hyperplane for Pr.

Proposition 7. *The hyperplane* π_a *is a separating hyperplane for* Pr *if and only if either for all* $J \subset I$*,* $J = \{j_1, \ldots, j_s\}$*,* $k = 1, \ldots, s-1$ *and* $J_k := \{j_1, \ldots, j_k\}$*, the following holds:*

$$\sum_{k=1}^{s-1} \left(\prod_{\substack{v\in I\\ v\neq j_k}} a_v - \prod_{\substack{v\in I\\ v\neq j_k}} x_v^{J_k} \right) (a_{j_s}^{op} - a_{j_s}) \geq 0 \tag{28}$$

or for all $J \subset I$*,* $J = \{j_1, \ldots, j_s\}$*:*

$$\sum_{k=1}^{s-1} \left(\prod_{\substack{v\in I\\ v\neq j_k}} a_v - \prod_{\substack{v\in I\\ v\neq j_k}} x_v^{J_k} \right) (a_{j_s}^{op} - a_{j_s}) \leq 0 \tag{29}$$

Proof. Let $p(x) := \prod_{v\in I} x_v$. Observe that for all $J \subset I$ with $|J| = 1$, we have $\pi(x) = p(x)$. Now fix $J \subset I$ and denote $J_h = \{j_1, \ldots, j_h\} \subset J$ for $h = 1, \ldots, s$. Then we have:

$$\pi(x_J) = \sum_{h=1}^{s-1} \pi(x_{J_{h+1}}) - \pi(x_{J_h}) + \pi(x_{J_1}) = \sum_{h=1}^{s-1} \left(\prod_{\substack{v\in I\\ v\neq j_h}} a_v \right) (a_{j_h}^{\mathrm{op}} - a_{j_h}) + p(x_{J_1}) \tag{30}$$

And similarly:

$$p(x_J) = \sum_{h=1}^{s-1} p(x_{J_{h+1}}) - p(x_{J_h}) + p(x_{J_1}) = \sum_{h=1}^{s-1} \left(\prod_{\substack{v\in I\\ v\neq j_h}} a_v \right) (a_{j_h}^{\mathrm{op}} - a_{j_h}) + p(x_{J_1}) \tag{31}$$

From which the thesis follows. □

Since Inequality (30) needs to hold for all subsets of I, it is in general non-trivial to assess for which vertices $a \in \mathfrak{C}$, the hyperplane π_a serves as a separating hyperplane. However, vertices at which the product π attains its maximum (or minimum) value on the cuboid always define separating hyperplanes:

Proposition 8. *If*

$$\max_{x\in\mathfrak{C}} \prod_{v\in I} x_v = \prod_{v\in I} a_v \quad \text{or} \quad \min_{x\in\mathfrak{C}} \prod_{v\in I} x_v = \prod_{v\in I} a_v,$$

then π_a *is a separating hyperplane. In particular in the first case we have* $\pi_a(x) \leq p(x)$ *for all* x *in* $\mathfrak{C}$ *and in the second case* $\pi_a(x) \geq p(x)$*.*

Proof. Assume that $\prod_{v\in I} a_v$ attains the maximum. The argument for the minimum case is analogous.

By Proposition 7, it suffices to verify that for all $J \subset I$ and $j \in J$:

$$\left(\prod_{\substack{v\in I\\ v\neq j}} a_v - \prod_{\substack{v\in I\\ v\neq j}} x_v^J\right)(a_j^{\mathrm{op}} - a_j) \leq 0.$$

First, consider the case $\prod_{v\in I} a_v \geq 0$:
If $\prod_{\substack{v\in I\\ v\neq j}} a_v \geq 0$, then $a_j \geq 0$ and $a_j \geq a_j^{\mathrm{op}}$. Since

$$a_j \prod_{\substack{v\in I\\ v\neq j}} a_v \geq a_j \prod_{\substack{v\in I\\ v\neq j}} x_v^J,$$

it follows that

$$\prod_{\substack{v\in I\\ v\neq j}} a_v \geq \prod_{\substack{v\in I\\ v\neq j}} x_v^J.$$

Hence,

$$\left(\prod_{\substack{v\in I\\ v\neq j}} a_v - \prod_{\substack{v\in I\\ v\neq j}} x_v^J\right)(a_j^{\mathrm{op}} - a_j) \leq 0.$$

If $\prod_{\substack{v\in I\\ v\neq j}} a_v \leq 0$, then $a_j \leq 0$ and $a_j \leq a_j^{\mathrm{op}}$. Since

$$a_j \prod_{\substack{v\in I\\ v\neq j}} a_v \geq a_j \prod_{\substack{v\in I\\ v\neq j}} x_v^J,$$

we have

$$\prod_{\substack{v\in I\\ v\neq j}} a_v \geq \prod_{\substack{v\in I\\ v\neq j}} x_v^J,$$

and again

$$\left(\prod_{\substack{v\in I\\ v\neq j}} a_v - \prod_{\substack{v\in I\\ v\neq j}} x_v^J\right)(a_j^{\mathrm{op}} - a_j) \leq 0.$$

Now consider the scenario where $\prod_{v\in I} a_v < 0$. In this case all signs are reversed accordingly and the argument is the same.

In all cases, the required inequality is satisfied. Hence, by Proposition 7, π_a is a separating hyperplane. $\square$

We note that computing the cuts induced by the maximum and minimum values of the product $\prod_{v\in V} x_v$ over the given intervals can be done in $O(n)$ time. The procedure is as follows:

1. If all interval bounds are positive, then the maximum of the product is achieved by taking the upper bound for each coordinate (and the minimum is achieved by taking the lower bound for each coordinate).

2. Otherwise, for each variable v, choose as a_v the endpoint of its interval $\left[l_v,\, u_v\right]$ that has the largest absolute value, i.e.,

$$a_v \;:=\; \underset{x\in\{l_v,u_v\}}{\arg\max}\;\left|x\right|.$$

3. If the number of chosen endpoints a_v that are negative is even, then $\left(a_v\right)_{v\in V}$ gives the *unique maximum* of the product; otherwise, it gives the *unique minimum*.

4. If the configuration $\left(a_v\right)_{v\in V}$ yields the maximum (respectively, minimum), then to obtain the minimum (resp. maximum), simply switch one coordinate v^* of $\left(a_v\right)_{v\in V}$ whose ratio $\frac{a^{\mathrm{op}}_{v^*}}{a_{v^*}}$ is minimal. This change flips the sign of the product, thus moving from a maximum-attaining point to a minimum-attaining point (and vice versa).

Let $\mathfrak{C}^n \subset \mathbb{R}^n$ be an n-dimensional cuboid (box). Define an $(n-1)$-dimensional face of this cuboid by

$$\mathfrak{C}^{n-1} \;:=\; \prod_{i=1}^{n-1}[l_i, u_i]$$

where l_i is one of the boundary values in the ith coordinate.

1. **Restriction:** Any separating (supporting) hyperplane for $\mathfrak{C}^n$ is induces a separating hyperplane for $\mathfrak{C}^{n-1}$.

2. **Extension:** Conversely, any hyperplane that separates $\mathfrak{C}^{n-1}$ (i.e. supports it) can be extended to a hyperplane defined by (or "anchored at") some vertex of $\mathfrak{C}^n$.

Therefore, since the only hyperplanes π_a that could possibly remain separating hyperplanes after extension to the full cuboid $\mathfrak{C}^n$ must already separate the $(n-1)$-dimensional face $\mathfrak{C}^{n-1}$, we proceed by induction on n.

Given π_a^{n-1} for vertex $a \in \mathfrak{C}^{n_1}$, consider the extended vertex $a' = (a, a_n) \in \mathfrak{C}^n$, then we have:

$$\pi^n_{a'}(x') = \pi^{n-1}_a(x)a_n + C_n^{a'}(x_n - a_n)$$

Assume that $p(x) \leq \pi_a^{n-1}$ for all $x \in \mathfrak{C}^{n-1}$, and $a_n \geq 0$ (otherwise the argument is analogous), then for all $x' \in \mathfrak{C}^{n_1} \times \{a_n\}$:

$$p(x') = a_n p(x) \leq a_n \pi_a^{n-1}(x) = \pi^n_{a'}.$$

We now try do understand when $\pi_{a'}^{n-1}$ also separates the remaining vertices on $\mathfrak{C}^{n-1} \times \{b_n\}$, where $b_n = a_n^{op}$. That is when for all $x \in \mathfrak{C}^{n-1}$ we have:

$$b_n \prod_{i=1}^{n} \leq \pi_{a'}^{n} = a_n \pi^{n-1}(x) + \prod_{i=1}^{n-1} a_i(b_n - a_n)$$

expanding the definition of π^{n-1}:

$$b_n \prod_{i=1}^{n} x_i \leq a_n(\sum_{i=1}^{n-1} C_i^a(x_i - a_i) + p(a)) + \prod_{i=1}^{n-1} a_i(b_n - a_n)$$

which is equivalent to:

$$b_n(p(x) - p(a)) \leq a_n(\sum_{i=1}^{n-1} C_i^a(x_i - a_i)) \tag{32}$$

We now define the sets:

$$C_a^+ := \{x \in \mathfrak{C}^{n-1} \mid p(x) - p(a) > 0\} \tag{33}$$
$$C_a^- := \{x \in \mathfrak{C}^{n-1} \mid p(x) - p(a) < 0\} \tag{34}$$
$$C_a^0 := \{x \in \mathfrak{C}^{n-1} \mid p(x) - p(a) = 0\} \tag{35}$$

Then, equation (32), is equivalent to the following three equations holding:

G: this is not a big help because we still need to check a lot of points rip

$$b_n \leq a_n \inf_{x \in C_a^+} \frac{\pi^{n-1}(x) - p(a)}{p(x) - p(a)} (\geq 0) \tag{36}$$
$$b_n \geq a_n \sup_{x \in C_a^-} \frac{\pi^{n-1}(x) - p(a)}{p(x) - p(a)} \tag{37}$$
$$0 \leq \inf_{x \in C_a^0} \pi^{n-1}(x) - p(a) (\geq 0) \tag{38}$$

We notice that equation (38) is always satisfied. Unfortunately, this still means that to know if the extension of π_a^{n-1} is a separating hyperplane for $\mathfrak{C}^n$ we need to check 2^{n-1} points. However some extensions can be easily excluded, since for an adjacent vertex x of a, we have $\pi^{n_1}(x) = p(x)$, thus if $x \in C_a^+$ we have

$$b_n \leq a_n \inf_{x \in C_a^+} \frac{\pi^{n-1}(x) - p(a)}{p(x) - p(a)} \leq a_n * 1$$

otherwise if $x \in C_a^-$ we have

$$b_, \geq \sup_{x \in C_a^-} \frac{\pi^{n-1}(x) - p(a)}{p(x) - p(a)} \geq a_n * 1$$

In particular, if both $C_a^+ \cap \delta(a)$ and $C_a^- \cap \delta(a)$ (where $\delta(a)$ are the neighbouring verticeso of a) are not empty, that is, if a si neither a local maximum, nor a local minimum, then the extension of π_a^{n-1} is a separating hyperplane for $\mathfrak{C}^n$ if and only if $a_n = b_n$ (thus, most often it isn't). Thus we have:

Proposition 9. *If $l_v \neq u_v$ for all v, and a is a vertex of $\mathfrak{C}$ defining a separating hyperplane π_a, then a is either a local minimum, or a local maximum.*

We look for sufficient and necessary conditions for when if $\pi_{\bar{a}}$ is a separating hyperplane, if $a \in \mathfrak{C}$ is and adjacent vertex of a, then also π_a is a separating hyperplane.

Proposition 10. *If a is an adjacent vertex of $\bar{a}$ and $\pi_{\bar{a}}$ is a separating hyperplane for Pr, such that $\pi_{\bar{a}}(x) \leq p(x)$ for all x in $\mathfrak{C}$, let $d_j := \bar{a}_j^{op} - \bar{a}_j \geq 0$ for all j. Then π_a is also a separating hyperplane for Pr such that $\pi_a(x) \leq p(x)$ if and only if*

$$C_v^a d_v \leq C_v^{\bar{a}} d_v \text{ for all } v \text{ in } J \setminus \bar{v} \tag{39}$$

Conversely the proposition also works by changing the sign of all inequalities.

Proof. Let $\bar{v} \in [n]$ be such that $\bar{a} - d_{\bar{v}} = a$.First we observe that $p(\bar{a}) = p(a) - C_{\bar{v}}^a d_{\bar{v}}$, by definition of $C_{\bar{v}}^a$. Let $x^J := \bar{a} + \sum_{j \in J} d_j$.Then if $\bar{v}$ is contained in J, we have:

$$\pi_a(x^J) = p(a) + \sum_{j \in J \setminus \bar{v}} C_v^a d_v \tag{40}$$

$$= p(\bar{a}) + C_{\bar{v}}^a d_{\bar{v}} + \sum_{j \in J \setminus \bar{v}} C_v^a d_v \tag{41}$$

$$\leq p(\bar{a}) + C_{\bar{v}}^{\bar{a}} d_{\bar{v}} + \sum_{j \in J \setminus \bar{v}} C_v^{\bar{a}} d_v = \pi_{\bar{a}}(x^J) \leq p(x^J)(41) \tag{42}$$

Where the first inequality follows direclty from the hypothesis. Similarly the same is done when $\bar{v} \notin J$:

$$\pi_a(x^J) = p(a) - C_{\bar{v}}^a d_{\bar{v}} + \sum_{j \in J} C_v d_v \tag{43}$$

$$= p(\bar{a}) + \sum_{j \in J} C_v^a d_v \tag{44}$$

$$\leq p(\bar{a}) + C_{\bar{v}}^{\bar{a}} d_{\bar{v}} + \sum_{j \in J} C_v^{\bar{a}} d_v = \pi_{\bar{a}}(x^J) \leq p(x^J) \tag{45}$$

Thus π_a is a separating hyperplane. Now we show that if the conditions doesn't hold than π_a is not a separating hyperplane. Consider $v' \in I \setminus \bar{v}$ such that $C_v^a d_v \leq C_v^{\bar{a}} d_v > 0$, then we have:

$$\pi_a(a - d_{\bar{v}} + d_{v'}) = \pi_{\bar{a}}(\bar{a}) + C_{v'}^a d_{v'} > p(\bar{a}) + C_{v'}^{\bar{a}} d_{v'} = \tag{46}$$

$$= p(\bar{a} + d_{v'}) = p(a - d_{\bar{v}} + d_{v'}) \tag{47}$$

□

Observation 11. *If $\bar{a}_{\bar{v}} \neq 0$, Equation (39) is equivalent to $\frac{a_{\bar{v}}}{\bar{a}_{\bar{v}}} C_v^{\bar{a}} = C_v^a \leq C_v^{\bar{a}}$ for all v in $J \setminus \bar{v}$.*

Proof. Since $C^a_{\bar v} = \prod_{\substack{j\in I\\ j\neq \bar v}} a_j = \prod_{\substack{j\in I\\ j\neq \bar v}} \bar a_j = C^{\bar a}_{\bar v}$, and

$$C^a_h = \frac{a_{\bar v}}{\bar a_{\bar v}} \bar a_{\bar v} \prod_{\substack{j\in I\\ j\neq \bar v}} a_j = \frac{a_{\bar v}}{\bar a_{\bar v}} \bar a_{\bar v} \prod_{\substack{j\in I\\ j\neq \bar v}} \bar a_j = \frac{a_{\bar v}}{\bar a_{\bar v}} C^{\bar a}_h.$$

□

We observe that it could still be the case that, even if the assumption does not hold, $\pi_{\bar a}$ could still be a separating hyperplane for Pr, but it may separate Pr from above instead of from below. In fact, this occurs whenever there are two adjacent vertices where one attains the maximum over the product and the other the minimum, as they both define supporting hyperplanes. We note that Condition (39) implies that if $1 - \frac{a_{\bar v}}{\bar a_{\bar v}} \leq 0$, then $p(\bar a + d_{\bar v}) \leq p(\bar a)$. Conversely, if $1 - \frac{a_{\bar v}}{\bar a_{\bar v}} \geq 0$, then $p(\bar a + d_{\bar v}) \geq p(\bar a)$. It is natural to examine how this condition behaves when $\bar a$ attains the maximum over the product, as it defines a supporting hyperplane. In this case, it is true that $p(\bar a + d_j) \leq p(\bar a)$ for all $j \in I \setminus \{\bar v\}$. However, in general, we do not have $1 - \frac{a_{\bar v}}{\bar a_{\bar v}} \leq 0$. If $\bar a_{\bar v} \geq 0$, this is equivalent to $d_{\bar v} \geq 0$, which would imply $p(\bar a + d_{\bar v}) \geq p(\bar a)$. Thus, the adjacent point still attains a maximum. On the other hand, if $\bar a_{\bar v} \leq 0$, then Condition (39) implies that $p(\bar a + d_{\bar v}) \geq p(\bar a)$. Therefore, Proposition 10 indicates that the only adjacent vertices to a maximizing vertex which also support Pr from below are themselves maximizing vertices.

VW: scrivo delle cose qui sotto che magari possono tornare utili

$$\begin{aligned} f_a(x) &= \prod_{v\in I} x_v + \sum_{v\in I} c_v(a_v - x_v) - \prod_{v\in I} a_v, \\ g_a(x) &= \prod_{v\in I} x_v - \sum_{v\in I} c_v x_v \end{aligned}$$

we are looking for $a \in \mathfrak{C}$ such that either

$$\begin{aligned} f_a(x) &\leq 0 && \forall x \in \mathfrak{C}, \text{ or} \\ f_a(x) &\geq 0 && \forall x \in \mathfrak{C}, \end{aligned}$$

that is

$$\begin{aligned} g_a(x) &\leq g_a(a), && \forall x \in \mathfrak{C}, \text{ or} \\ g_a(x) &\geq g_a(a), && \forall x \in \mathfrak{C}. \end{aligned}$$

So we are looking for a such that the function g_a attains its maximum or minimum at a. Note that $g_a(a) = (1 - |I|) \prod_{v\in I} a_v$.

VW: quindi secondo me, col fatto che $I \leq 2$, se $|a_v| = u_v$ dovrebbe essere facile far vedere che quello è sempre il massimo o il minimo, dipende da quanti segni meno ci sono

4 Bounds on Loop constraints violation

We want to confront different possible relaxations in order to pick the one which minimizes the violation of the loop constraint. The violation of the loop constraint comes from the fact that a solution x can violate the following equality:

$$z_I = \prod_{v \in I} x_v$$

We are thus interested in the following quantity,

$$\epsilon_I \coloneqq \sup_{(z_I, x) \in PrR} |z_I - \prod_{v \in I} x_v|$$

To give a bound on the violation of the loop constraint. We give a lower bound on ϵ_I:

Proposition 12. *Let I be a set of variables. Then*

$$\epsilon_I \geq \sup_{(z_i, x) \in \mathcal{C}(Pr)} |z_I - \prod_{v \in I} x_v| = U_I \left(\frac{1}{|I|}\right)^{\frac{1}{|I|-1}} \left(1 - \frac{1}{|I|}\right),$$

where $U_I \coloneqq \prod_{v \in I} \max(|u_v|, |l_v|)$.

G: mhhh se l_v è più grande di zero l'errore è più piccolo di così.

Proof. We start with the case where $l_v = 0$. It can be easily (?) shown that the point which achieves the supremum is of the type $(z_I, tu_{v_1}, \ldots, tu_{v_k}, \ldots)$, where $k = |I|$, with $z_I = t \prod_{v \in I} u_v$. Thus, we calculate:

$$\sup_{t \in [0,1]} t \prod_{v \in I} u_v - \prod_{v \in I} tu_v = \sup_{t \in [0,1]} \prod_{v \in I} u_v (t - t^k).$$

The supremum is attained at $t = \left(\frac{1}{|I|}\right)^{\frac{1}{|I|-1}}$, and thus the error is

$$\prod_{v \in I} u_v \left(\frac{1}{|I|}\right)^{\frac{1}{|I|}} \left(1 - \frac{1}{|I|}\right)$$

.

In general, for l_v possibly negative, we observe that the relaxation introduced in Section ref is not convex. However, if restricted to each quadrant, it is convex, and we can apply the same argument. The error is then larger than the maximum error of the convexification of the graph of the monomial over each quadrant, and thus the thesis follows. ☐

We now consider how the error on the monomial approximation influences the error on the loop constraint:

$$\epsilon_C \coloneqq |\prod_{v\in V} c_{vv} - \sum_{k=0}^{\lfloor\frac{|C|}{2}\rfloor} \sum_{\substack{A\subset\mathcal{E}(C)\\|A|=2k}} (-1)^k \prod_{e\in A} c_e \prod_{e\in A^c} s_e| \tag{51}$$

$$\leq |z_C^v + \epsilon_C^v + \sum_{k=0}^{\lfloor\frac{|C|}{2}\rfloor} \sum_{\substack{A\subset\mathcal{E}(C)\\|A|=2k}} z_A + \epsilon_A| \tag{52}$$

$$\leq |\prod_{v\in V} \epsilon_C^v| + \sum_{k=0}^{\lfloor\frac{|C|}{2}\rfloor} \sum_{\substack{A\subset\mathcal{E}(C)\\|A|=2k}} |\epsilon_A| \tag{53}$$

$$\leq \bar{\epsilon}_C + \sum_{k=0}^{\lfloor\frac{|C|}{2}\rfloor} \sum_{\substack{A\subset\mathcal{E}(C)\\|A|=2k}} \bar{\epsilon}_C \cong 2^{|C|-1}\bar{\epsilon}_C \tag{54}$$

As done in cite, cycles can be decomposed into cycles of length 3 or 4 (resulting in McCormick relaxations of binomial), or larger cycles (resulting in generalized monomial relxations). We determine the optimal length subcycle length to minimize the loop constraint violation. To do this, given the constraints induced by the subcycles of C we relate these to the loop constraint over C to confront the error. First, given C, decomposed in the cycles $C_0, \dots, C_k$, let ϵ_{C_i} be the loop constraint violation over the subcycle C_i, and let $\epsilon^s_{C_i}$ be the constraint violation respect to the constraint $\prod_{v\in C_i} \sin(\sum_{e\in\mathcal{E}(C_i)} \theta_e) = 0$. Lastly, we denote by $LHS^c_{C_i}$ (resp. $RHS^s_{C_i}$) the left hand side (right hand side) of the constraint $\prod_{v\in C_i} c_{vv} \cos(\sum_{e\in\mathcal{E}(C_i)} \theta_e) = \prod_{v\in C_i} c_{vv}$. Consider:

$$\prod_{v\in C} c_{vv} \cos(\sum_{e\in\mathcal{E}(C)} \theta_e) = \prod_{v\in C} c_{vv}$$

By expanding the cosine, and substituting $\cos(\theta_e)|V_{e_0}||V_{e_1}|$ and $\sin(\theta_e)|V_{e_0}||V_{e_1}|$, by c_e and s_e respectively, we obtain the loop constraint. Alternatively, observe that:

$$\cos(\sum_{e\in\mathcal{E}(C)} \theta_e) = \cos(\sum_{i=0}^{k} \sum_{e\in C_i} \theta_e) = \cos(\sum_{i=0}^{k} \theta_{C_i})$$

Where $\theta_{C_i} \coloneqq \sum_{e\in C_i} \theta_e$. Thus, by expanding the last sum respect to the angles θ_{C_i} and multiplying by $\prod_{i=0}^{k} \prod_{v\in C_i} c_{vv}$, we obtain the loop constraint expressed respect in function of the LHS of the loop constraints of the subcycles:

$$\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor} (-1)^h \sum_{\substack{A\subset[k]\\|A|=2h}} \prod_{i\in A} LHS^c_{C_i} \prod_{i\in A^c} LHS^s_{C_i} = \prod_{i=0}^{k} \prod v \in C_i c_{vv}$$

Observe that $LHS^c_{C_i} = RHS^C_{C_i} + \epsilon^s_{C_i}$ and $LHS^s_{C_i} = \epsilon^s_{C_i}$, and that dividing by $\prod_{i=0}^k c_{v_i,v_i}$ we obtain the expression of the loop constraint over C. Thus:

$$\frac{1}{\prod_{i=0}^k c_{v_i,v_i}} \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^h \sum_{\substack{A \subset [k] \\ |A|=2h}} \prod_{i \in A} (RHS^c_{C_i} + \epsilon_{C_i}) \prod_{i \in A^c} \epsilon^s_{C_i}) = RHS_C$$

This sum can be divided in two sums, one corresponding to the left hand side of the loop constraint over C, and the other corresponding to the loop constraint violation, depending on ϵ^c and ϵ^s. By taking the absolute values, we obtain:

$$\epsilon_C \leq 2^{|k|-1} \bar{\epsilon}^A_C$$

Where $\bar{\epsilon}^A_C$ is the average of the products in the sum. Then, since $k = |C|/c$, where $c = |C_i|$ the optimal length of the subcycles is $c = 42 * y$ with y to be determined. MHHHHH va scritto meglio