# Hypergeometric local systems over $\mathbb{Q}$ with Hodge vector $(1,1,1,1)$

Giulia Gugiatti* and Fernando Rodriguez Villegas†

**Abstract**

In this note we consider all irreducible rank-4 hypergeometric local systems $\mathbb{H}$ defined over $\mathbb{Q}$ that support a rational one-dimensional variation of Hodge structure of weight 3 and Hodge vector $(1,1,1,1)$. Up to a natural equivalence there are only 47 cases. The first 14 cases have maximally unipotent monodromy at one point and have been extensively studied in the literature. We show that all 47 local systems are associated to families of generically smooth threefolds and we analyze the geometry and arithmetic at their conifold point.

## Contents



## Introduction

In this note we consider all irreducible rank-4 hypergeometric local systems $\mathbb{H}$ defined over $\mathbb{Q}$ that support a rational one-dimensional variation of Hodge structure (VHS) of weight 3 and Hodge vector $h = (1,1,1,1)$.

A hypergeometric local system $\mathbb{H}$ is determined by standard hypergeometric parameters $\alpha, \beta$. We consider them up to isomorphism and pullback by the inversion map $\iota\colon t \mapsto 1/t$ (which exchanges $\alpha$ and $\beta$). Here $t$ is our choice of parameter in $\mathbb{P}^1$ normalized so that $\mathbb{H}$ is defined over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the local monodromy at $t = 1$ is a pseudo-reflection; following the Physics literature we call $t = 1$ the *conifold* of the family. Up to this equivalence there are only 47 cases. We list them in tables 1 and 2.

The first 14 cases of our list (table 1) have maximally unipotent monodromy (MUM) at one of the singularities (normalized to $t = 0$) and have been extensively studied in the literature[1].

*Dipartimento di Matematica “Tullio Levi-Civita”, Università di Padova, Via Trieste, 63, 35131 Padova.
Email address: giulia.gugiatti@math.unipd.it

†Math Section, ICTP, Leonardo Da Vinci Building, Strada Costiera 11, 34151 Trieste.
Email address: villegas@ictp.it

[1] Our presentation is necessarily limited and may not include all relevant citations.

They correspond to one-parameter families of Calabi-Yau threefolds and play an important role in mirror symmetry [CDlOGP91, BvS95, DM06, CDL$^+$16]. Moreover, their conifold points give rise to modular forms of weight 4 (see equation (1) below) and to certain supercongruences (first conjectured in [Vil03] and then established in [Kil06, McC12, Yui13, FM14, LTYZ21]) between truncations of the hypergeometric series at $t = 1$ and Fourier coefficients of the modular form[2]. We refer to the 14 cases as MUM cases. The remaining 33 cases (table 2) do not have a MUM point.

Our two main goals are:

1. To show that in fact all of the 47 cases are associated to families of generically smooth (but non-compact) threefolds.
2. To analyze the geometry and arithmetic at the conifold point of each local system.

For $\mathbb{H}$ a hypergeometric local system corresponding to the parameters $\alpha, \beta$, we call the hypergeometric local system $\widetilde{\mathbb{H}}$ corresponding to the parameters $\widetilde{\alpha}, \widetilde{\beta}$ with $\widetilde{\alpha}_i = \alpha_i + 1/2, \widetilde{\beta}_j = \beta_j + 1/2$ the *total twist* of $\mathbb{H}$. If $\mathbb{H}$ is irreducible, the operation of total twist yields an involution. Moreover, if $\mathbb{H}$ is defined over $\mathbb{Q}$, then so is $\widetilde{\mathbb{H}}$, and the rational VHS it supports has the same Hodge vector as that supported by $\mathbb{H}$.

The 47 cases can be classified as follows:

- The cases $1, 30, 37$ are equivalent to their total twists.
- The cases $2 - 14$ are the total twists of the cases $15 - 27$.
- The cases $28 - 38$, excluding cases 30 and 37, are the total twists of the cases $39 - 47$.

Thus our 47 cases give rise to 25 pairs $((\alpha, \beta), (\widetilde{\alpha}, \widetilde{\beta}))$.

**Goal 1.** To realize $\mathbb{H}$ geometrically as a family of threefolds we proceed as follows. First, we consider the canonical pair $(Z, \pi)$ of the gamma vector $\gamma^{\text{red}}$ corresponding to $\mathbb{H}$ (see §1.1 and §1.2). This is typically of high dimension. Then, we apply to this pair a range of dimension reduction techniques (see § 1.3) to arrive at a family of threefolds. Our techniques in § 1.3.2 extend the dimension reduction method of [BCM15] (consisting of splitting the gamma vector into a union of gamma vectors with relatively prime entries). We also realize the total twist operation geometrically in terms of double covers of canonical pairs and fibrations in odd-dimensional quadric bundles § 1.3.3. We work out a few cases explicitly (§1.4).

**Goal 2.** For each of the 47 cases $\mathbb{H}$ degenerates at $t = 1$. Local monodromy at $t = 1$ fixes a dimension 3 subspace with mixed Hodge structure (MHS) of weights 2 and 3 (see (27)). The weight 2 piece $H_1$ has rank one and Hodge vector $(0, 1, 0)$, while the weight 3 piece $H_2$ has rank 2 and Hodge vector $(1, 0, 0, 1)$.

Geometrically (§2.1), $H_1$ arises from the two rulings of the even-dimensional quadric determined by the Hessian $H$ of the polynomial defining the fiber $Z_1 = \pi^{-1}(1)$. This quadric is the exceptional divisor of the blowup of $Z_1$ at its unique ordinary double point. On the other hand, $H_2$ is related to the middle cohomology of the resolution of $Z_1$.

Correspondingly, at the arithmetic level (§2.2), the Euler factor $L_p$ of the associated $L$-series for good primes $p$ [§11-12][RRV22] factors as

$$L_p(T) = (1 - \sigma(p)pT)\left(1 - a_pT + p^3T^2\right) \tag{1}$$

[2]In two cases the works [BKSZ22, Kil23] further find special regular points, so called rank-two attractor points, giving rise to modular forms of weight four and two.

where $\sigma$ is a quadratic character and $a_p$ is conjecturally[3] the $p$-th coefficient of a Hecke eigenform $f$ of weight 4 and some level.

We show that $\sigma$ corresponds to the quadratic extension $K/\mathbb{Q}$ given by the field of definition of the two rulings of the quadric in the blow-up of $Z_1$. The field $K$ equals $\mathbb{Q}(\sqrt{\mathrm{disc}(H)})$, where $\mathrm{disc}(H)$ is the signed discriminant of the Hessian $H$. We give a formula (equation (31)) for $\mathrm{disc}(H)$ in terms of $\gamma$. (These facts about the rank 1 piece $H_1$ hold for general symplectic irreducible hypergeometric local systems $\mathbb{H}$ of any rank defined over $\mathbb{Q}$.) In the tables we list the discriminant $D$ of $K$ (which equals $\mathrm{disc}(H)$ up to squares).

We identify all modular forms $f$, excluding those corresponding to cases 38 and 47 (whose conductor 5400 is too large), by computing several $a_p$'s numerically and then searching in the database LMFDB [LMF24].

Finally, using the determination of $K$, we also identify a rank one local system $\epsilon_t$, corresponding to a quadratic extension of the form $\mathbb{Q}(\sqrt{Et})/\mathbb{Q}(t)$ for some integer $E$, with the property that $\widetilde{\mathbb{H}} \simeq \epsilon_t \otimes \mathbb{H}$. We give a formula for $E$, up to squares, in terms of $\gamma$ (equation (33)), and in the tables we also list $E$.

**Related and future work** Detailed statements and complete proofs of the results in this note, along with explicit equations for all 47 families of threefolds, will appear in a subsequent publication. The MUM cases of our list underlie one-parameter families of Calabi–Yau threefolds. The threefolds are mirrors to certain Calabi–Yau weighted complete intersections $D$, and the families are (expected to be) Landau–Ginzburg mirrors to certain Fano 4-fold weighted complete intersections $X$ with $D$ as anticanonical section. It would be interesting to understand whether all the 47 cases underlie families of Calabi–Yau threefolds, and/or play a role in mirror symmetry.

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# 1 Geometric realisation

In this section we address the problem of the geometric realisation of the 47 local systems in tables 1 and 2 as families of threefolds. We start by spelling out our notion of gamma vector and by reviewing the construction of the canonical pair associated to a gamma vector with relatively prime entries.

## 1.1 Family parameters and gamma vectors

For $\mathbb{H}$ an irreducible hypergeometric local system of rank $m$ corresponding to parameters $\alpha = (\alpha_0, \ldots, \alpha_{m-1}), \beta = (\beta_0, \ldots, \beta_{m-1})$, consider the rational function $Q = q_\infty / q_0$, where:

$$q_\infty = \prod_{j=0}^{m-1} \left(x - e^{2\pi \mathrm{i} \alpha_j}\right) \qquad q_0 = \prod_{j=0}^{m-1} \left(x - e^{2\pi \mathrm{i} \beta_j}\right) \tag{2}$$

The function $Q$ is defined over the subfield $F \subset \mathbb{C}$ generated by the coefficients of $q_\infty$ and $q_0$. Following [RRV22, § 2], we call $Q$ the *family parameter*, and we say that $\mathbb{H}$ *is defined over* $F$.

[3] This is known for the MUM cases [Sch86, Vil03, Kil06, McC12, Yui13, FM14, LTYZ21].

If $\alpha_i, \beta_j$ are rational numbers, so that $F$ is a subfield of a cyclotomic field, $\mathbb{H}$ has a motivic interpretation [Kat90, §5.4]. In this note we restrict to cases where $F = \mathbb{Q}$. Then $\mathbb{H}$ supports a rational VHS whose Hodge vector can be computed combinatorially [CG11, Fed17, Vil19].

An equivalent way to write $Q$ is as follows

$$Q = \prod_{i=1}^{l} \left( x^{|\gamma_i|} - 1 \right)^{-\mathrm{sign}(\gamma_i)} \tag{3}$$

for certain non-zero integers $\gamma_i$ with zero sum. The order of the $\gamma_i$ is irrelevant for (3), and we may hence normalize $\gamma := (\gamma_1, \ldots, \gamma_l)$ by requiring it to be weakly-increasing.

We call a vector $\gamma$ of non-zero integer entries that sum to zero and cannot be all grouped in pairs of integers of opposite signs a *gamma vector*. For example, $(-3, 2, 1)$ is a gamma vector, and so is $(-3, -1, 2, 1, 1)$, but $(-2, -1, 1, 2)$ is not. (This is a minor extension of the notion of gamma vector in [RRV22].) We say that a gamma vector $\gamma$ is associated to $\mathbb{H}$ if it satisfies (3). Note that if $\gamma = (\gamma_1, \ldots, \gamma_l)$ satisfies (3) then so does $\gamma^{(n)} := (\gamma_1, \ldots, \gamma_l, -n, +n)$, $n \neq 0 \in \mathbb{N}$, but there is a unique gamma vector $\gamma^{\mathrm{red}}$ associated to $\mathbb{H}$ with no pair of entries summing to zero.

If $\gamma$ is a gamma vector for $\mathbb{H}$, the vector $\widetilde{\gamma}$ obtained by replacing each odd entry $\gamma_i$ of $\gamma$ with the pair $2\gamma_i, -\gamma_i$ is a gamma vector for its total twist $\widetilde{\mathbb{H}}$. Indeed, letting $Q$ and $\widetilde{Q}$ be the correspinding family parameters, one has

$$\widetilde{Q}(x) = Q(-x) = \prod_{\gamma_i \text{ even}} \left( x^{|\gamma_i|} - 1 \right)^{-\mathrm{sign}(\gamma_i)} \prod_{\gamma_i \text{ odd}} \left( \frac{x^{|2\gamma_i|} - 1}{x^{|\gamma_i|} - 1} \right)^{-\mathrm{sign}(\gamma_i)} \tag{4}$$

We call $\widetilde{\gamma}$ the total twist of $\gamma$. In general $\widetilde{\gamma} \neq \widetilde{\gamma}^{\mathrm{red}}$; for example, the total twist of $\gamma = (-6, -1, 2, 2, 3)$ is $\widetilde{\gamma} = (-6, -3, -2, 1, 2, 2, 6)$, and $\widetilde{\gamma}^{\mathrm{red}} = (-3, 1, 2)$.

In tables 1 and 2, third column, we list the vectors $\gamma^{\mathrm{red}}$ for all the 47 cases. For reasons of space we write $n^a$ if the entry $n$ appears $a$ times. Note that for all cases $\gamma^{\mathrm{red}}$ has relatively prime entries (this is not true in general).

## 1.2 Canonical pairs and toric models

From now on, $\mathbb{H}$ is an irreducible hypergeometric local system defined over $\mathbb{Q}$. Let $\gamma = (\gamma_1, \ldots, \gamma_l)$ be an associated gamma vector with $\gcd\{\gamma_1, \ldots, \gamma_l\} = 1$. Then a candidate geometric realisation for $\mathbb{H}$ is the canonical pair $(Z, \pi)$ associated to $\gamma$.

**Definition 1.** The *canonical pair* associated to $\gamma$ is the pair $(Z, \pi)$ given by

$$Z := \left( \sum_{i=1}^{l} z_i = 0 \right) \subset \mathbb{T}^{l-1} \quad \text{and} \quad \pi := \frac{z^\gamma}{\mathcal{M}_0} \colon Z \to \mathbb{C}^\times \tag{5}$$

where $\mathbb{T}^{l-1} \simeq (\mathbb{C}^\times)^{l-1}$ is a torus with homogeneous coordinates $z_j$, $z := (z_1, \ldots, z_l)$, $z^\gamma := \prod_{j=1}^{l} z_j^{\gamma_j}$, and $\mathcal{M}_0 := \prod_{j=1}^{l} {\gamma_j}^{\gamma_j}$ (this equals $M$ in [Vil19] up to a sign).

The dimension of $Z$ is $d := l - 2$, and we say that $(Z, \pi)$ has dimension $d$. It is simple to see that $Z_t := \pi^{-1}(t)$ is singular if and only if $t = 1$, and $Z_1$ has a unique ordinary double point. We expect (see [RRV22, §4]) the following identity of local systems on $U = \mathbb{C}^\times \setminus \{1\}$ to hold:

$$\mathbb{H} = \mathrm{gr}_k^W \, PR_c^k \pi_{U!} \mathbb{C} \tag{6}$$

where $k \coloneqq d-1$ is the dimension of $Z_t$, $\pi_U\colon \pi^{-1}(U) \to U$ is the restriction, and $\mathrm{gr}^W_k PR^k_c \pi_{U\,!}\mathbb{C}$ denotes the local system with stalk at $t$ the top graded piece of the primitive compactly supported cohomology $PH^k_c(Z_t, \mathbb{C})$. In this note we assume (6). The arithmetic counterpart of (6) has been studied in [BCM15], which shows that the hypergeometric series over $\mathbb{F}_q$ corresponding to $\gamma$ [BCM15, Def. 1.1, Thm. 1.3] equals the number of $\mathbb{F}_q$–rational points of a suitable completion of $Z_t$ (modulo certain corrections).

It can be convenient to consider different equations for $(Z, \pi)$, called toric models in [RRV22, §4]. Toric models arise as specialisation of certain multi-parametric Laurent polynomials first appearing in [GKZ94, Bat93, BB96, Sti98]. We briefly review the construction.

Pick $d$ vectors in $\mathbb{Z}^l$ that, together with $e \coloneqq (1, \dots, 1)$, span $\ker(\gamma\colon \mathbb{Z}^l \to \mathbb{Z})$. Denote by $M$ the $d \times l$ integer matrix with the $d$ vectors as rows, and call $m_1, \dots, m_l$ its columns. Let $\mathbb{T}^d$ be $d$-dimensional torus with coordinates $x_1, \dots, x_d$. Let $u$ be a coordinate on $\mathbb{C}^\times$, let $u_1, \dots, u_l$ be coordinates on $\mathbb{C}^l$, and let $\iota\colon \mathbb{C}^\times \hookrightarrow \mathbb{C}^l$ be a map such that $\prod_{j=1}^l u_j^{\gamma_i} \circ \iota = \mathrm{id}$. Then, via the change of coordinates sending $z_j \mapsto \iota(u)_j x^{m_j}$, the pair $(Z, \pi)$ is the same as the pair formed by

$$\left(\sum_{j=1}^{l} \iota(u)_j x^{m_j} = 0\right) \subset \mathbb{T}^d \times \mathbb{C}^\times \quad \text{and} \quad \frac{\mathrm{pr}_2}{\mathcal{M}_0} = \frac{u}{\mathcal{M}_0} \tag{7}$$

where $x$ is the vector with entries $x_i$ and $\mathrm{pr}_2$ is the projection onto the second factor. We say that (7) is a *toric model* for $(Z, \pi)$. Concretely, one may choose a vector $\kappa \in \mathbb{Z}^l$ such that $\kappa \cdot \gamma = 1$ and set $u_j = u^{\kappa_j}$. We call $M$ and $\kappa$ an *$M$-matrix* and a *$\kappa$-vector* for $\gamma$.

**Example 2.** Consider case 28 in table 2. The vector $\gamma^{\mathrm{red}}$ has length $l = 10$, thus the pair $(Z, \pi)$ is a one-parameter family of 7-folds. We construct a toric model for $(Z, \pi)$.

We can choose $M$ as:

$$M = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

and $\kappa = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0)$. We obtain the hypersurface:

$$\left(1 + \frac{x_5 x_6 x_7}{x_1} + \frac{x_1}{x_2 x_3 x_4 x_8} + x_2 x_5 x_6 x_7 + x_3 + u x_4 + x_5 x_8 + x_6 + x_7 + \frac{1}{x_8} = 0\right) \subset \mathbb{T}^8 \times \mathbb{C}^\times$$

with the projection onto $u/\mathcal{M}_0$, where $\mathcal{M}_0 = -27/4$.

### 1.3 Dimension reduction

In general, the VHS supported by $\mathbb{H}$ has Hodge weight smaller than the dimension of the fibers of any associated canonical pair $(Z, \pi)$, thus it equals $\mathrm{gr}^W_k PR^k_c \pi_{U!}\mathbb{Q}$ only up to Tate twist. Then, it is natural to ask whether $\mathbb{H}$ also arises from the variation of cohomology of a morphism with fibers of dimension equal to the Hodge weight. For example, our 47 local systems all support VHS of weight 3 but most of them are associated to canonical pairs with odd-dimensional fibers of higher dimension. We will show that we may however find corresponding families of affine threefolds in all cases.

In this section we discuss a range of cases where out of a gamma vector of length $l$ one can build a pair $(Y, w)$ of dimension $d - 2j < d$, where $d = l - 2$.

If $\gamma$ is union of $s$ gamma vectors $\gamma^i$ with $\gcd\{\gamma^i_j\} = 1$, following [BCM15, §6] one may build a pair $(Y, w)$ of dimension $d - 2(s-1)$ out of the canonical pairs $(Z_i, \pi_i)$ of the $\gamma^i$. In § 1.3.1 we explain the relation between $(Y, w)$ and $(Z, \pi)$.

In § 1.3.2 we consider the more general case of a vector $\gamma$ with $\gcd\{\gamma_j\} = 1$ that is union of $s$ gamma vectors and a pair of the form $(-n, n)$. Out of $\gamma$ we construct a pair $(Y, w)$ of dimension $d - 2s$. By special cases of our construction, we are able to:

(1) Describe the various canonical pairs associated to $\mathbb{H}$ in terms of each other.

(2) Generalise the construction of canonical pair §1.2 to an arbitrary gamma vector.

(3) Generalise [BCM15, §6] to gamma vectors that are union of $s$ arbitrary gamma vectors.

We give details here for (2) and (3).

In § 1.3.3, we realize the total twist $\widetilde{\mathbb{H}}$ geometrically in terms of a double cover of the canonical pair of a gamma vector $\gamma$ for $\mathbb{H}$, and we relate the construction to the canonical pair of the twist $\widetilde{\gamma}$.

### 1.3.1 Union of gamma vectors with relatively prime entries

Assume that $\gamma = (\gamma_1, \ldots, \gamma_l)$ splits into $s \geq 2$ gamma vectors $\gamma^i$, $i = 0, \ldots, s-1$, with $\gcd\{\gamma^i_j\} = 1$. We write $\gamma = \gamma^0 \cup \cdots \cup \gamma^{s-1}$ and, for convenience, we order the entries of $\gamma$ according to the splitting. We denote by $l_i$ the length of $\gamma^i$ and let $d_i \coloneqq l_i - 2$.

**The pair $(Y, w)$.** The paper [BCM15, §6] associates to $\gamma$ a pair $(Y, w)$ of dimension $d - 2(s-1)$ as follows. For all $i$ let $(Z_i, \pi_i)$ be the canonical pair associated to $\gamma^i$. Then $(Y, w)$ is defined by:

$$Y \coloneqq Z_1 \times \cdots \times Z_s \subset \mathbb{T}^{l_0 - 1} \times \cdots \times \mathbb{T}^{l_{s-1} - 1} \quad \text{and} \quad w \coloneqq \pi_1 \cdot \cdots \cdot \pi_s \colon Y \to \mathbb{C}^\times \tag{8}$$

The hypergeometric sum over $\mathbb{F}_q$ corresponding to $\gamma$ is related to the number of $\mathbb{F}_q$-rational points of the varieties $Y_t$ ([BCM15, Thm. 6.1]).

**Remark 3.** There is a notion of convolution $\star$ of local systems [Kat90, Ch. 5]. It is a standard fact that, if a local system $\mathbb{V}$ is the convolution $\mathbb{V} = \mathbb{V}_0 \star \cdots \star \mathbb{V}_{s-1}$ of $s$ local systems $\mathbb{V}_i$ and $(Z_i, \pi_i)$ is a geometric realisation of $\mathbb{V}_i$, then $(Z_0 \times \cdots \times Z_s, \pi_i \cdot \cdots \cdot \pi_s)$ is a geometric realisation of $\mathbb{V}$.

In our case, the fact that $\gamma = \gamma^0 \cup \cdots \cup \gamma^{s-1}$ corresponds to $\mathbb{H}$ being the convolution $\mathbb{H} = \mathbb{H}_0 \star \cdots \star \mathbb{H}_{s-1}$, where $\mathbb{H}_i$ is the local system corresponding to $\gamma^i$. This explains the construction of $(Y, w)$ from a geometric perspective. More details on the VHS associated to $(Y, w)$ are given below.

**Connection with $(Z, \pi)$.** We pass to toric models for $(Z_i, \pi_i)$ to make the relation between $(Y, w)$ and $(Z, \pi)$ manifest. Note that the difference between the dimensions of the two pairs is $d - (l - 2s) = 2(s-1)$.

It is simple to see that $(Y, w)$ can be rewritten as:

$$Y = (F_0 = F_1 = \cdots = F_{s-1} = 0) \subset \mathbb{T}^{l-s-1} \times \mathbb{C}^\times \qquad w = \frac{u}{\mathcal{M}_0} \tag{9}$$

where[4]

$$F_0 = \sum_{j=1}^{l_0} \left( \frac{u}{\prod_{i=1}^{s-1} u_i} \right)^{\kappa_j^0} x_{\mathbf{0}}^{m_j^0}, \qquad F_i = \sum_{j=1}^{l_i} u_i^{\kappa_j^i} x_{\mathbf{i}}^{m_j^i} \quad i \geq 1$$

and $x_{\mathbf{i}j}$ $(i = 0, \dots, s-1, j = 1, \dots, d_i)$, $u_1, \dots, u_{s-1}$ are coordinates on $\mathbb{T}^{d_0+\cdots+d_{s-1}} \times (\mathbb{C}^\times)^{s-1} \simeq \mathbb{T}^{l-s-1}$, while $u$ is a coordinate on $\mathbb{C}^\times$.

On the other hand, a toric model for $(Z, \pi)$ is given by:

$$(F_0 + w_1 F_1 + \cdots + w_{s-1} F_{s-1} = 0) \subset \mathbb{T}^d \times \mathbb{C}^\times \qquad \text{with} \quad \frac{\mathrm{pr}_2}{\mathcal{M}_0} \tag{10}$$

where $x_{\mathbf{i}j}$, $u_1, \dots, u_{s-1}$, $w_1, \dots, w_{s-1}$ are coordinates on $\mathbb{T}^{l-s-1} \times (\mathbb{C}^\times)^{s-1} \simeq \mathbb{T}^d$.

**Remark 4.** If a polytope $P$ has width one and projects onto the $(s-1)$-dimensional simplex, any Laurent polynomial $F$ with support $P$ can be written as $F = F_0 + w_1 F_1 + \cdots + w_{s-1} F_s$. This is known as *Cayley trick*. It is simple to see that $\gamma = \gamma^0 \cup \cdots \cup \gamma^{s-1}$ if and only if the Newton polytope $P$ of the polynomial $F(\cdot, \mathcal{M}_0 t)$ defining $Z_t$ (see (7)) satisfies these properties. In (10), in addition, we have chosen the vectors $m_j$ so that each $F_i$ defines a toric model for $(Z_i, \pi_i)$.

One finds an isomorphism of VHS on $U = \mathbb{C}^\times \setminus \{1\}$:

$$\mathrm{gr}_k^W R_c^k \pi_{U!} \mathbb{Q}\,(s-1) = \mathrm{gr}_{k-2(s-1)}^W R_c^{k-2(s-1)} w_{U!} \mathbb{Q} \tag{11}$$

where $w_U \colon w^{-1}(U) \to U$ is the restriction. (The local system $\mathbb{H}$ is (11) tensored with $\mathbb{C}$.)

Indeed, one can consider the partial compactification $Z_t \subset \widehat{Z}_t \subset \mathbb{T}^{l-s-1} \times \mathbb{P}^{s-1}$, where $\mathbb{P}^{s-1} \supset \mathbb{T}^{s-1}$ has local coordinates $w_1, \dots, w_{s-1}$, and the first projection $\phi_t \colon Z_t \to \mathbb{T} = \mathbb{T}^{l-s-1}$. The fibre of $\phi_t$ over a point in $Y_t \subset \mathbb{T}$ is the projective space $\mathbb{P}^{s-1}$, while the fiber over a point in $\mathbb{T} \setminus Y_t$ is a hyperplane in $\mathbb{P}^{s-1}$. (Note that if $s = 2$ $\phi_t \colon Z_t \to \mathbb{T}$ is the blow-up along $Y_t$.) It follows that:

$$\mathrm{gr}_k^W H_c^k(\widehat{Z}_t, \mathbb{Q}) = \mathrm{gr}_k^W \left( H_c^{k-2(s-1)}(Y_t, \mathbb{Q})(-(s-1)) \right)$$

On the other hand, one can show that the cohomology of $B_t = \widehat{Z}_t \setminus Z_t$ is controlled by those of the spaces $Z_i$ and does not contribute to $\mathrm{gr}_k^W H_c^k(\widehat{Z}_t, \mathbb{Q})$.

### 1.3.2 Generalisations of §1.3.1

We generalise § 1.3.1 to a gamma vector $\gamma = (\gamma_1, \dots, \gamma_l)$ with $\gcd\{\gamma_j\} = 1$ and of the form $\gamma = \gamma^0 \cup \cdots \cup \gamma^{s-1} \cup (-n, n)$, where $s \geq 1$, $\gcd\{\gamma_j^i\} = D_i \geq 1$, and $n \neq 0 \in \mathbb{N}$. We assume for simplicity that $D_0, \dots, D_{s-1}, n$ are pairwise coprime.

For all $i$, $\frac{\gamma^i}{D_i} := (\frac{\gamma_1^i}{D_i}, \dots, \frac{\gamma_{l_i}^i}{D_i})$ is a gamma vector with relatively prime entries. Let $M_i$ and $\kappa_i$ a $M$-matrix and a $\kappa$-vector for $\frac{\gamma^i}{D_i}$, pick $a_0, \dots, a_{s-1}, b$ such that $\sum_{i=0}^{s-1} a_i D_i + bn = 1$, and set:

$$F_0 = \sum_{j=1}^{l_0} \left( \frac{u^{a_0} u_s^n}{\prod_{m=1}^{s-1} u_m^{D_m}} \right)^{k_j^0} x_{\mathbf{0}}^{m_j^0} \quad F_i = \sum_{j=1}^{l_i} (u^{a_i} u_i^{D_0})^{k_j^i} x_{\mathbf{i}}^{m_j^i} \quad F_s = u_s^{D_0} + u^b$$

[4] We are replacing the $u$-coordinate $u_0$ of a toric model for $(Z_0, \pi_0)$ with the ratio $\frac{u}{\prod_{i=1}^{s-1} u_i}$. Of course this may be done for any of the $u_i$.

where $i = 1, \dots, s-1$. Then, a toric model for the canonical pair $(Z, \pi)$ of $\gamma$ is:

$$\left( F_0 + \sum_{i=1}^{s-1} w_i F_i + w_s F_s = 0 \right) \subset \mathbb{T}^d \times \mathbb{C}^\times \quad \text{with} \quad \frac{\mathrm{pr}_2}{\mathcal{M}_0} \tag{12}$$

We define a pair $(Y, w)$ by:

$$Y = (F_0 = F_1 = \cdots = F_s = 0) \subset \mathbb{T}^{d-s} \times \mathbb{C}^\times \quad \text{and} \quad w = \frac{u}{\mathcal{M}_0} \tag{13}$$

Letting $\widehat{Z}_t \subset \mathbb{T}^{d-s} \times \mathbb{P}^s$ be the partial compactification of $Z_t$ in the coordinates $w_i$, one can now show that:

$$\mathrm{gr}^W_k H^k_c(\widehat{Z}_t, \mathbb{Q}) = \mathrm{gr}^W_k \left( H^{k-2s}_c(Y_t, \mathbb{Q})(-s) \right) \quad \text{and} \quad \mathrm{gr}^W_k H^k_c(Z_t, \mathbb{Q}) \hookrightarrow \mathrm{gr}^W_k H^k_c(\widehat{Z}_t, \mathbb{Q}) \tag{14}$$

**Application (2): non-relatively prime entries.** Let $\gamma = (\gamma_1, \dots, \gamma_l)$ be a gamma vector with $\gcd\{\gamma_j\} = D > 1$.

As a special case of §1.3.2 ($s = 1, D_0 = D, n = 1, a_0 = 0, b = 1$), we find that the canonical pair $(Z^{(1)}, \pi^{(1)})$ associated to the vector $(\gamma_1, \dots, \gamma_l, -1, 1)$ is the hypersurface

$$\left( \sum_{j=1}^{l} u_1^{k_j} x^{m_j} + w_1(u_1^D + u) = 0 \right) \subset \mathbb{T}^{d+2} \times \mathbb{C}^\times \quad \text{with} \quad -\frac{\mathrm{pr}_2}{\mathcal{M}_0}$$

Then, to $\gamma$ we can associate the $d$-dimensional pair $(Y, w)$:

$$Y = \left( \sum_{j=1}^{l} u_1^{k_j} x^{m_j} = 0 \right) \subset \mathbb{T}^d \times \mathbb{C}^\times \qquad \text{and} \qquad w = \frac{u_1^D}{\mathcal{M}_0} \tag{15}$$

This generalises the construction of canonical pair § 1.2.

The pair fits into the diagram:

$$\begin{tikzcd} Z = Y \arrow[d, "\pi"] \arrow[dr, "w"] & \\ \mathbb{C}^\times \arrow[r, "q_D"] & \mathbb{C}^\times \end{tikzcd} \tag{16}$$

where $(Z, \pi)$ is the canonical pair of $\frac{\gamma}{D}$, and $q_D$ is the $D$:1 cover of $\mathbb{C}^\times$ induced by the map $-u \mapsto u_1^d$. In particular,

$$R^k_c w_{U!}\mathbb{Q} = q_{D\star} R^k_c \pi_{U!}\mathbb{Q} \tag{17}$$

Let $\mathbb{H}_D$ be the local system corresponding to $\gamma$, and $\mathbb{H}$ that corresponding to $\frac{\gamma}{D}$. By (14) and studying $\widehat{Z}^{(1)}_t \setminus Z^{(1)}_t$, it is not difficult to see that:

$$0 \to \mathrm{gr}^W_{k+2} R^{k+2}_c \pi^{(1)}_{U!}\mathbb{C} \to R^k_c w_{U!}\mathbb{Q} \to \mathrm{gr}_k \underline{G}^D \to 0 \tag{18}$$

where $\underline{G}$ is the constant sheaf with stalk $H^{d+1}_c(\mathbb{T}^d, \mathbb{Q})(1) \simeq \mathbb{Q}^d$. It follows that

$$\mathbb{H}_D = q_{D\star}\mathbb{H} \tag{19}$$

**Application (3): union of arbitrary gamma vectors.** Let $\gamma = \gamma^0 \cup \cdots \cup \gamma^{s-1}$ be a gamma vector with $\gcd\{\gamma_j\} = 1$.

As a special case of §1.3.2 (remove $(-n, n)$ and the corresponding contributions, i.e. set $u_s = 1$, and ignore $w_s$, $F_s$), we find that $Z$ is isomorphic to the hypersurface

$$\left(F_0 + \sum_{i=1}^{s-1} w_i F_i = 0\right) \subset \mathbb{T}^d \times \mathbb{C}^\times$$

and the variety $Y \subset \mathbb{T}^{l-s-1} \times \mathbb{C}^\times$ is defined by the equations $F_0 = F_1 = \cdots = F_{s-1} = 0$.

It is easy to see that:

$$(Y, w) = (Y_0 \times \cdots \times Y_{s-1}, \prod_{i=0}^{s-1} w_i) \tag{20}$$

where $(Y_i, w_i)$ is the pair associated to $\gamma^i$ by (15). This generalises (8).

Let $(Z_i, \pi_i)$ be the canonical pair corresponding to $\gamma^i / \gcd\{\gamma^i_j\}$. By (14) and studying $\widehat{Z}_t \setminus Z_t$, one finds

$$\mathbb{H} = \mathrm{gr}^W_k\, R^k_c \pi_{U!}\mathbb{Q}(s-1) \hookrightarrow \mathrm{gr}^W_{k-2(s-1)}\, R^{k-2(s-1)}_c w_{U!}\mathbb{Q} \tag{21}$$

Unlike in §1.3.1, the cokernel of the morphism (21) is not always trivial, and is determined by the middle cohomology of the covers of $Z_i$ defined by $F_i = 0$.

### 1.3.3 The geometry of total twists

Let $\mathbb{H}$ and $\widetilde{\mathbb{H}}$ be a hypergeometric local system and its total twist. Note that $\widetilde{\mathbb{H}}$ fits into the split exact sequence of local systems on $U = \mathbb{C}^\times \setminus \{1\}$:

$$0 \to \mathbb{H} \to p_{U\star} p_U^\star \mathbb{H} \to \widetilde{\mathbb{H}} \to 0 \tag{22}$$

where $p\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the double cover $t \mapsto s^2$, and $p_U : p^{-1}(U) \to U$ is the restriction. Indeed by the projection formula

$$p_{U\star} p_U^\star \mathbb{H} \simeq \mathbb{H} \otimes p_{U\star}\mathbb{C}$$

and letting $\mathbb{V}$ be the quotient of $p_{U\star}\mathbb{C}$ by $\mathbb{C}$ (22) is the tensor by $\mathbb{H}$ of

$$0 \to \mathbb{C} \to p_{U\star}\mathbb{C} \to \mathbb{V} \to 0$$

**Double covers.** Assume that $\mathbb{H}$ is irreducible and defined over $\mathbb{Q}$, let $\gamma$ be an associated gamma vector and $(Z, \pi)$ be the canonical pair of $\gamma$ (in fact, any geometric realisation of $\mathbb{H}$ satisfying (6)).

We define a pair $(Z', \pi')$ via the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "P"] \arrow[d, "\Pi"] \arrow[dr, "\pi'"] & Z \arrow[d, "\pi"] \\ \mathbb{C}^\times \arrow[r, "p"] & \mathbb{C}^\times \end{tikzcd} \tag{23}$$

where $p$ is the double cover above. Equivalently, $Z'$ is the double cover of $Z$ determined by $\mathcal{M}_0 \cdot s^2 = u$ and $\pi'\colon Z' \to \mathbb{C}^\times$ is the projection onto $t$.

We find:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}_k\, R^k\pi_{U!}\mathbb{C}_Z \arrow[r] & \mathrm{gr}_k\, R^k\pi'_{U!}\mathbb{C}_Z \arrow[r] & \mathrm{gr}_k\, R^k\pi_{U!}\mathbb{C}_Z \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{H} \arrow[r] \arrow[u] & p_{U\star}p_U^\star\mathbb{H} \arrow[r] \arrow[u] & \widetilde{\mathbb{H}} \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{24}$$

where $\pi'_U \colon (\pi')^{-1}(U) \to U$ is the restriction and the vertical arrows are inclusions for $k \geq 0$, isomorphisms if $k > 0$. Indeed, by base change, the definition of $\pi'$, and the projection formula, we have that

$$R^k\pi'_!\mathbb{C}_{Z'} = R^k\pi_!\mathbb{C}_Z \otimes p_\star\mathbb{C}$$

which, combined with (22), implies (24).

**Relation to the canonical pairs of $\widetilde{\mathbb{H}}$.** Let $(\widetilde{Z}, \widetilde{\pi})$ be the canonical pair of the total twist $\widetilde{\gamma}$ of $\gamma$. The dimension of $(\widetilde{Z}, \widetilde{\pi})$ is $\widetilde{d} = l_0 + d$, where $l_0 = 2m$ is the number of odd entries of $\gamma$. Moreover, $\widetilde{\mathcal{M}}_0 = 4^{L_o}\mathcal{M}_0$, where $L_o$ is the sum of the odd entries of $\gamma$.

Let $x_1, \dots, x_d, y_1, \dots, y_{l_o}$ be coordinates on $\mathbb{T}^d \times \mathbb{T}^{l_o} \simeq \mathbb{T}^{\widetilde{d}}$. Let $\widetilde{u}$ be a coordinate on $\mathbb{C}^\times$ and set $u := 4^{L_e}\widetilde{u}$ where $L_e$ is the sum of the even entries of $\gamma$. Let $M = [m_1, \dots, m_l]$ and $\kappa = (\kappa_1, \dots, \kappa_l)$ be a $M$-matrix and a $\kappa$-vector for $\gamma$. Then, by the definition of $\widetilde{\gamma}$, a toric model for $(\widetilde{Z}, \widetilde{\pi})$ is

$$\left( \sum_{\gamma_j \text{ odd}} u^{\kappa_j}(2y_j - y_j^2)x^{m_j} + \sum_{\gamma_j \text{ even}} u^{\kappa_j}x^{m_j} = 0 \right) \subset \mathbb{T}^{\widetilde{d}} \times \mathbb{C}^\times \quad \text{with} \quad \frac{\mathrm{pr}_2}{\widetilde{\mathcal{M}}_0} = \frac{\widetilde{u}}{\widetilde{\mathcal{M}}_0} \tag{25}$$

For $t \neq 1$ let $\widetilde{Z}_t = \widetilde{\pi}^{-1}(t)$ be the fibre . The partial compactification $\widetilde{Z}_t \subset X_t \subset \mathbb{T}^d \times \mathbb{P}^{l_o}$, where $\mathbb{P}^{l_o}$ has local coordinates $y_j$, is smooth. By (25), the projection $\psi_t \colon X_t \to \mathbb{T}^d$ is a quadric bundle with $(l_0 - 1)$-dimensional fibres.

**Remark 5.** If a polytope $P$ has width two and projects onto the 2-dilation of the $r$-dimensional simplex, then any Laurent polynomial with support $P$ defines a (non-proper) quadric bundle with $(r-1)$-dimensional fibers. By definition of $\widetilde{\gamma}$, the polytope $\widetilde{P}$ associated to the fibre $\widetilde{Z}_t$ satisfies these properties (with $r = l_0$).

It is not difficult to see that the discriminant of $\psi_t$ is the fibre $Z_t = \pi^{-1}(t)$. Moreover, the double cover $P_t \colon Z'_t \to Z_t$ parametrizing degenerate quadrics is the map $P(\cdot, t)$ in (23). Indeed, by (25), one finds that $P_t$ is determined by the square root of the product

$$\delta_t = \prod_{\gamma_j \text{ odd}} u^{k_j}x^{m_j}$$

hence, since $\kappa \cdot \gamma = 1, \sum \gamma_j m_j = 0$ imply $\sum_{\gamma_j \text{ odd}} k_j = 1, \sum_{\gamma_j \text{ odd}} m_j = 0$ mod 2, by the square root of $u$.

One can show that $\mathrm{gr}_k^W H^c(\widetilde{Z}_t, \mathbb{C}) = \mathrm{gr}_k^W H^c(X_t, \mathbb{C})$. By this fact and the natural generalisation of [CGR22, Theorem 3] to quadric bundles with odd-dimensional fibres, one finds the short exact sequence:

$$0 \to \mathrm{gr}_k^W R^k\pi_{U!}\mathbb{C}_Z \to \mathrm{gr}_k^W R^k\pi'_{U!}\mathbb{C}_{Z'} \to \mathrm{gr}_k^W R^k\widetilde{\pi}_{U!}\mathbb{C}_{\widetilde{Z}} \to 0 \tag{26}$$

This is the first row of (24).

### 1.4 One-parameter families of threefolds

By the dimension reduction techniques outlined in §1.3, one can find a corresponding family of threefolds for all 47 cases of our list. Below we discuss a few examples and strategies.

**Case 28.** We continue with example 2. The vector $\gamma^{\mathrm{red}}$ has length 10 and splits into the union $\gamma^{\mathrm{red}} = \gamma^0 \cup \gamma^1 \cup \gamma^2$, where

$$\gamma^0 := (-4,-1,2,3) \quad \gamma^1 := (-1,-1,2) \quad \gamma^2 := (-1,-1,2)$$

thus we can consider the family of threefolds $(Y, w)$ in (8). Choosing

$$M_0 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \quad M_1 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$$

and $\kappa^0 = (0,-1,0,0)$, $\kappa^1 = \kappa^2 = (0,-1,0)$, by (9) $(Y, w)$ can be written as

$$Y = \left( x_1 + \frac{1}{u} x_2^2 x_5 x_6 + x_1^2 x_2 + 1 = x_3^2 + \frac{1}{x_5} + x_3 = x_4^2 + \frac{1}{x_6} + x_4 = 0 \right) \subset \mathbb{T}^6 \times \mathbb{C}^\times$$

where the $x_j$ are the coordinates on $\mathbb{T}^6$ and $u$ that on $\mathbb{C}^\times$, and $w$ is the projection onto $-4u/27$. Note that the fibers $Y_t = w^{-1}(t)$ may be written as hypersurfaces in $\mathbb{T}^4$.

One may also consider the splitting $\gamma^{\mathrm{red}} = (-4,2,2) \cup (-1,-1,-1,3) \cup (-1,-1,2)$ and proceed as in §1.3.2(3), or find a family of threefolds for case 39 (its total twist), to which the methods in §1.3.1 and §1.3.2(3) also apply, and use §1.3.3.

**Case 30.** The vector $\gamma^{\mathrm{red}}$ has length 12 and splits into the union $\gamma^{\mathrm{red}} = \gamma^0 \cup \gamma^1 \cup \gamma^2 \cap \gamma^3$, where

$$\gamma^0 := (-6,3,3) \quad \gamma^1 = \gamma^2 = \gamma^3 := (-1,-1,2)$$

thus we can obtain a family of threefolds $(Y, w)$ by §1.3.2(3). Choosing

$$M_0 = [1,0,2] \quad M_1 = M_2 = M_3 = [2,0,1]$$

$\kappa^0 = (0,1,0), \kappa^i = (0,-1,0), i = 1,2,3$, and $\kappa = (0,0,0,1,0,1,0,0,0,0,0,0)$, $Y$ can be written as:

$$\left( x_1 + \frac{1}{u_1 u_2 u_3} + x_1^2 = u x_2^2 + \frac{1}{u_1^3} + u x_2 = x_3^2 + \frac{1}{u_2^3} + x_3 = x_4^2 + \frac{1}{u_3^3} + x_4 = 0 \right) \subset \mathbb{T}^7 \times \mathbb{C}^\times$$

with $x_i, u_j$ the coordinates on $\mathbb{T}^7$ and $u$ that on $\mathbb{C}^\times$, and $w$ is the projection onto $u$.

Alternatively, one could write $\gamma^{\mathrm{red}} = (-6,-1,2,2,3) \cup (-1,-1,-1,3) \cup (-1,-1,2)$, and use that $(-6,-1,2,2,3)$ is the total twist of $(-3,1,2)$ combined with §1.3.3 and remark 3. Note that case 30 coincides with its total twist.

**Case 41.** The vector $\gamma^{\mathrm{red}}$ has length 14. One can check that $\gamma^{\mathrm{red}}$ does not split into the union of $s = 5$ gamma vectors, thus we cannot arrive at a family of threefold by the methods in §1.3.2(3). However, one may observe that $\gamma^{\mathrm{red}} = \gamma^0 \cup \gamma^1 \cup \gamma^2 \cap \gamma^3$, where

$$\gamma^0 := (-3,-3,6) \quad \gamma^1 := (-3,-2,-2,1,6) \quad \gamma^2 := (-2,1,1) \quad \gamma^3 := (-2,1,1)$$

and $\gamma^0$ is the total twist of $(-2,-1,3)$. Thus, by §1.3.2(2), §1.3.3, and remark 3, we can find a family of threefolds $(Y, w)$.

Concretely, $(Y, w) = (Y_0 \times Z_1' \times Z_2 \times Z_3, w_0 \times \pi_1' \times \pi_2 \times \pi_3)$ where $(Y_0, w_0)$ and $(Z_1', \pi_1')$ can be written as:

$$Y_0 = \left(x^2 + \frac{1}{u} + x = 0\right) \subset \mathbb{T}^1 \times \mathbb{C}^\times \quad w_0 = u^3/64$$
$$Z_1' = \left(1 + \frac{y^3}{s^2} + y = 0\right) \subset \mathbb{T}^1 \times \mathbb{C}^\times \quad \pi_1' = -4s^2/27$$

while $(Z_i, \pi_i)$, $i = 2, 3$ are both isomorphic to:

$$Z = \left(z + v + z^2 = 0\right) \subset \mathbb{T}^1 \times \mathbb{C}^\times \quad \pi = 4v$$

Alternatively, one can find a family of threefolds for case 31 (its total twist), to which the methods in §1.3.2(3) apply, and use §1.3.3.

# 2 The point $t = 1$

Let $\mathbb{H}$ be any of the 47 local systems in tables 1and 2, and let $\mathcal{H}(t)$ be the corresponding family of motives. In this section we analyze the geometry and the arithmetic of the motive $\mathcal{H}(1)$ at the point $t = 1$. Our analysis naturally generalises to any symplectic irreducible hypergeometric local system $\mathbb{H}$ defined over $\mathbb{Q}$. We thank S. Bloch for his help on the middle cohomology of the fibre $Z_1$ and on the action of Galois on the two rulings of an even-dimensional quadric.

## 2.1 Geometric perspective

The local monodromy of $\mathbb{H}$ at $t = 1$, being a pseudo-reflection of determinant 1, has Jordan form

$$h_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I_2.$$

We see that the invariant subspace $V \subseteq \mathbb{C}^4$ of $h_1$ is of dimension 3. The monodromy filtration on $V$ is of the form

$$0 \subseteq W_2 \subseteq W_3 = V$$

with $\dim W_2 = 1$ and $\dim W_3/W_2 = 2$ (see [Kul98, II §2]). Hence we have a short exact sequence of MHS:

$$0 \to \operatorname{gr}_2^W V = W_2 \to V \to \operatorname{gr}_3^W V = W_3/W_2 \to 0 \tag{27}$$

We interpret the above sequence geometrically as follows. The canonical pair $(Z, \pi)$ corresponding to a vector $\gamma$ associated to $\mathbb{H}$ has even dimension $d := 2n$. Let $F$ be a Laurent polynomial as in (7), so that $Z_1 = (F(\cdot, \mathcal{M}_0) = 0) \subset \mathbb{T}^d$.

The exceptional divisor of the blow-up $\phi\colon \tilde{Z}_1 \to Z_1$ at the unique ordinary double point $p$ is the smooth $(2n-2)$-dimensional quadric $Q$ defined by:

$$\left(xHx^T = 0\right) \subset \mathbb{P}^{2n-1} \tag{28}$$

where $H$ is the Hessian of $F(\cdot, \mathcal{M}_0)$ at $p$, and $x = (x_1, \ldots, x_{2n})$ is a vector of coordinates on $\mathbb{P}^{2n-1}$.

Given a smooth $(2n-2)$-dimensional quadric $Q$ in $\mathbb{P}^{2n-1}$, it is a standard fact that the middle cohomology $H^{2n-2}(Q)$ is generated by the two rulings of $Q$, that is, the two families $\mathcal{L}_1$ and $\mathcal{L}_2$ of codimension $n$ linear subspaces of $\mathbb{P}^{2n-1}$ contained in $Q$.

Writing $\delta$ and $\eta$ for the difference and the sum of the two rulings and letting $i\colon Q \hookrightarrow \widetilde{Z}_1$ be the inclusion, one has that $i_\star\delta = 0$, while $i^\star H^{2n-2}(\widetilde{Z}_1)$ is generated by $\eta$. Then, the long exact sequence of the pair:

$$\cdots \to H^{2n-2}(\widetilde{Z}_1) \to H^{2n-2}(Q) \to H^{2n-2}(\widetilde{Z}_1, Q) \to H^{2n-1}(\widetilde{Z}_1) \to H^{2n-1}(Q) \to \cdots$$

yields, via the isomorphism $H^{2n-1}(Z_1) = H^{2n-1}(\widetilde{Z}_1, Q)$, the exact sequence of MHS:

$$0 \to \mathbb{Q}(-(n-1)) \to H^{2n-1}(Z_1, \mathbb{Q}) \to H^{2n-1}(\widetilde{Z}_1, \mathbb{Q}) \to 0 \tag{29}$$

which matches with (27) up to a Tate twist by $n-2$.

## 2.2 Arithmetic perspective

From the arithmetic point of view the exact sequence (27) corresponds to the following factorization of the Euler factor of the $L-series$ of $\mathcal{H}(1)$ [§11-12][RRV22] for good primes $p$

$$L_p(T) = (1 - \sigma(p)pT)\left(1 - a_pT + p^3T^2\right), \tag{30}$$

where $\sigma$ is a quadratic character and $a_p$ is (conjecturally) the $p$-th coefficient of a modular form $f$ of weight 4. This is equation (1) in the Introduction. Note that $\sigma(p)p$ has weight two.

The character $\sigma$ corresponds to the quadratic extension $K/\mathbb{Q}$ determined by the square root of

$$\sqrt{-(-1)^{\frac{d}{2}}\prod_{i=1}^{l}\gamma_i} \tag{31}$$

where $\gamma = (\gamma_1, \ldots, \gamma_l)$ is a gamma vector associated to $\mathbb{H}$ and $d = l-2$. (Note that two different gamma vectors associated to $\mathbb{H}$ determine by (31) the same quadratic field.) We interpret this in terms of the difference $\delta$ of the two rulings of $Q$.

Given any irreducible hypergeometric local system $\mathbb{H}$ over $\mathbb{Q}$ and any canonical pair $(Z, \pi)$ associated to it, it is not hard to show that the Hessian $H$ at $p$ of $F(\cdot, \mathcal{M}_0)$ has determinant $\det(H) = -\prod_{i=1}^{l}\gamma_i$ up to a square factor.

For a symmetric $m \times m$ matrix $A$ over a field $F$ of characteristic different from 2 (defining a quadratic form), we define the *signed discriminant* of $A$ as

$$\operatorname{disc}(A) \coloneqq (-1)^{\frac{m(m-1)}{2}}\det(A)$$

This is standard in Witt's theory [Lam05, II §2] and the choice of sign is also that in Picard–Lefschetz theory [Voi03, §3.2.1].

Let now $d = 2n$ be even. Given in general a smooth $(2n-2)$-dimensional quadric $(xAx^T = 0) \subset \mathbb{P}^{2n-1}$ over $F$, the field of definition $K$ of the two rulings $\mathcal{L}_1$ and $\mathcal{L}_2$ of the quadric is given by

$$K \coloneqq F\left(\sqrt{\operatorname{disc}(A)}\right) = F\left(\sqrt{(-1)^n\det(A)}\right)$$

where the equality holds since $n = (2n)(2n-1)/2 \bmod 2$. Hence $K$ is also the field of definition of the difference of the rulings $\delta$. The non-trivial element of $\operatorname{Gal}(K/F)$ acts by $\delta \mapsto -\delta$.

In our case $F = \mathbb{Q}$, and the quadratic extension corresponding to $\sigma$ (31) is $K = \mathbb{Q}(\sqrt{\operatorname{disc}(H)})/\mathbb{Q}$, the field of definition of the difference of the rulings $\delta$ on $Q \subset \widetilde{Z}_1$. It is not hard to verify that

$$\operatorname{sign}(\operatorname{disc}(H)) = (-1)^{\frac{r(r-1)}{2}}$$

where $r$ equals $\#\{\gamma_i < 0\} - \#\{\gamma_i > 0\}$ or, equivalently, the number of $\alpha_i$ or $\beta_i$ which are zero modulo $\mathbb{Z}$. In tables 1 and 2, column 4, we list the discriminants $D$ of the 47 quadratic fields.

We have identified the modular forms $f$, excluding those corresponding to cases 38 and 47, by computing several $a_p$'s numerically and then searching in the database LMFDB. They are listed in the 5th column of the tables (using the LMFDB label).

Comparing the finite hypergeometric sums associated to the motives $\mathcal{H}(t)$ and the twisted motives $\widetilde{\mathcal{H}}(t)$ (see §[RRV22, Eq. (10.1)]), we deduce that for all $t$ in $\mathbb{P}^1$:

$$\widetilde{\mathcal{H}}(t) \simeq \epsilon_t \otimes \mathcal{H}(t) \tag{32}$$

where $\epsilon_t$ is a character depending only on the hypergeometric parameters. Comparing the quadratic fields $K$ corresponding to $\mathcal{H}(1)$ and $\widetilde{\mathcal{H}}(1)$, we deduce that $\epsilon_t$ is the character corresponding to the quadratic field determined by the square root:

$$\sqrt{(-1)^{\frac{l_o}{2}}\mathcal{M}_0 \cdot t} \tag{33}$$

where, for a gamma vector $\gamma$ for $\mathcal{H}(t)$, $\mathcal{M}_0$ is as in Definition 1 and $l_o$ is the number of odd entries of $\gamma$. (As above, different choices of $\gamma$ determine by (33) the same quadratic field.) Note that $\epsilon_t$ is the same for $\mathbb{H}$ and $\widetilde{\mathbb{H}}$. In the last column of the tables we list for each case the discriminant $E$ of the quadratic field for $t = 1$, equal to $(-1)^{l_o}\mathcal{M}_0$ up to squares.

## Tables

In the two tables we list the 14 MUM cases and the 33 remaining cases. We tabulate the parameters $\alpha, \beta$, the vector $\gamma^{\text{red}}$, the discriminant $D$ of the quadratic field associated to $\sigma$, the modular form $f$ as labeled in LMFDB, and the discriminant $E$ of the quadratic field associated to $\epsilon_1$.

| $n^\circ$ | $\alpha$ | $\gamma^{\text{red}}$ | $D$ | $f$ | $E$ |
|---|---|---|---|---|---|
| 1 | $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ | $(-2^4,1^8)$ | 1 | 8.4.a.a | 1 |
| 2 | $\left(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3}\right)$ | $(-3,-2^2,1^7)$ | 12 | 36.4.a.a | $-3$ |
| 3 | $\left(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right)$ | $(-4,-2,1^6)$ | 8 | 16.4.a.a | $-4$ |
| 4 | $\left(\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{5}{6}\right)$ | $(-6,-2,1^5,3)$ | 1 | 72.4.a.b | $-3$ |
| 5 | $\left(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$ | $(-3^2,1^6)$ | 1 | 27.4.a.a | 1 |
| 6 | $\left(\frac{1}{4},\frac{1}{3},\frac{2}{3},\frac{3}{4}\right)$ | $(-4,-3,1^5,2)$ | 24 | 9.4.a.a | 12 |
| 7 | $\left(\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}\right)$ | $(-6,1^4,2)$ | 12 | 108.4.a.a | 1 |
| 8 | $\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)$ | $(-4^2,1^4,2^2)$ | 1 | 32.4.a.a | 1 |
| 9 | $\left(\frac{1}{6},\frac{1}{4},\frac{3}{4},\frac{5}{6}\right)$ | $(-6,-4,1^3,2^2,3)$ | 8 | 144.4.a.f | 12 |
| 10 | $\left(\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}\right)$ | $(-6^2,1^2,2^2,3^2)$ | 1 | 216.4.a.c | 1 |
| 11 | $\left(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\right)$ | $(-5,1^5)$ | 5 | 25.4.a.b | 5 |
| 12 | $\left(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\right)$ | $(-8,1^4,4)$ | 8 | 128.4.a.b | 1 |
| 13 | $\left(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}\right)$ | $(-10,1^3,2,5)$ | 1 | 200.4.a.f | 5 |
| 14 | $\left(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}\right)$ | $(-12,-2,1^4,4,6)$ | 1 | 864.4.a.a | 1 |

**Table 1:** The 14 MUM cases. We omit $\beta = (0,0,0,0)$.

| $n^\circ$ | $\alpha$ | $\gamma^{\mathrm{red}}$ | $D$ | $f$ | $E$ |
|---|---|---|---|---|---|
| 15 ($\widetilde{2}$) | $(0,0,\frac{1}{6},\frac{5}{6}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-6,-1^7,2^5,3)$ | $-4$ | 12.4.a.a | $-3$ |
| 16 ($\widetilde{3}$) | $(0,0,\frac{1}{4},\frac{3}{4}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-4,-1^6,2^5)$ | $-8$ | 8.4.a.a | $-4$ |
| 17 ($\widetilde{4}$) | $(0,0,\frac{1}{3},\frac{2}{3}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-3,-1^5,2^4)$ | $-3$ | 24.4.a.a | $-3$ |
| 18 ($\widetilde{5}$) | $(\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-6^2,-1^6,2^6,3^2)$ | 1 | 27.4.a.a | 1 |
| 19 ($\widetilde{6}$) | $(\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-6,-4,-1^5,2^6,3)$ | 8 | 144.4.a.d | 12 |
| 20 ($\widetilde{7}$) | $(\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-6,-1^4,2^5)$ | 12 | 108.4.a.a | 1 |
| 21 ($\widetilde{8}$) | $(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-4^2,-1^4,2^6)$ | 1 | 32.4.a.a | 1 |
| 22 ($\widetilde{9}$) | $(\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-4,-3,-1^3,2^5)$ | 24 | 72.4.a.a | 12 |
| 23 ($\widetilde{10}$) | $(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-3^2,-1^2,2^4)$ | 1 | 216.4.a.c | 1 |
| 24 ($\widetilde{11}$) | $(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-10,-1^5,2^5,5)$ | 1 | 25.4.a.a | 5 |
| 25 ($\widetilde{12}$) | $(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-8,-1^4,2^4,4)$ | 8 | 128.4.a.b | 1 |
| 26 ($\widetilde{13}$) | $(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-5,-1^3,2^4)$ | 5 | 200.4.a.e | 5 |
| 27 ($\widetilde{14}$) | $(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}),\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $(-12,-1^4,2^3,4,6)$ | 1 | 864.4.a.a | 1 |
| 28 | $(0,0,\frac{1}{4},\frac{3}{4}),\ (\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3})$ | $(-4,-1^5,2^3,3)$ | $-24$ | 18.4.a.a | 12 |
| 29 | $(0,0,\frac{1}{4},\frac{3}{4}),\ (\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3})$ | $(-4,-1^4,2,3^2)$ | $-8$ | 54.4.a.c | $-4$ |
| 30 | $(0,0,\frac{1}{6},\frac{5}{6}),\ (\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3})$ | $(-6,-1^6,2^3,3^2)$ | $-3$ | 6.4.a.a | 1 |
| 31 | $(0,0,\frac{1}{6},\frac{5}{6}),\ (\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3})$ | $(-6,-1^5,2,3^3)$ | $-4$ | 54.4.a.d | $-3$ |
| 32 | $(0,0,\frac{1}{6},\frac{5}{6}),\ (\frac{1}{4},\frac{1}{3},\frac{2}{3},\frac{3}{4})$ | $(-6,-1^4,3^2,4)$ | $-24$ | 12.4.a.a | $-4$ |
| 33 | $(0,0,\frac{1}{6},\frac{5}{6}),\ (\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4})$ | $(-6,-1^3,-2,3,4^2)$ | $-4$ | 96.4.a. | $-3$ |
| 34 | $(0,0,\frac{1}{6},\frac{5}{6}),\ (\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$ | $(-6,-1^4,2,3,5)$ | $-20$ | 150.4.a.h | $-15$ |
| 35 | $(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}),\ (\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4})$ | $(-3^2,-2^2,1^2,4,4)$ | 1 | 864.4.a.a | 1 |
| 36 | $(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}),\ (\frac{1}{6},\frac{1}{4},\frac{3}{4},\frac{5}{6})$ | $(-3^3,-2^2,1^3,4,6)$ | 8 | 432.4.a.k | 12 |
| 37 | $(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}),\ (\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6})$ | $(-3^4,-2^2,1^4,6^2)$ | 1 | 72.4.a.b | 1 |
| 38 | $(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}),\ (\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$ | $(-5,-3^2,-2,1^3,10)$ | 1 | 5400.4.?.? | 5 |
| 39 ($\widetilde{28}$) | $(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4}),\ (0,0,\frac{1}{6},\frac{5}{6})$ | $(-4,-3,-2^2,1^5,6)$ | $-8$ | 48.4.a.c | 12 |
| 40 ($\widetilde{29}$) | $(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4}),\ (\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6})$ | $(-4,-3^2,-2^3,1^4,6^2)$ | 8 | 432.4.a.j | $-4$ |
| 41 ($\widetilde{31}$) | $(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3}),\ (\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6})$ | $(-2^4,-3^3,1^5,6^2)$ | 12 | 54.4.a.a | $-3$ |
| 42 ($\widetilde{32}$) | $(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3}),\ (\frac{1}{6},\frac{1}{4},\frac{3}{4},\frac{5}{6})$ | $(-2^4,-3^2,1^4,4,6)$ | 24 | 48.4.a.a | $-4$ |
| 43 ($\widetilde{33}$) | $(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3}),\ (\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4})$ | $(-3,-2^4,1^3,4^2)$ | 12 | 288.4.a.f | $-3$ |
| 44 ($\widetilde{34}$) | $(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3}),\ (\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$ | $(-5,-3,-2^3,1^4,10)$ | 12 | 450.4.a.o | $-15$ |
| 45 ($\widetilde{35}$) | $(\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}),\ (\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4})$ | $(-6^2,-1^2,3^2,4^2)$ | 1 | 864.4.a.a | 1 |
| 46 ($\widetilde{36}$) | $(\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}),\ (\frac{1}{4},\frac{1}{3},\frac{2}{3},\frac{3}{4})$ | $(-6^2,-1^3,2,3^3,4)$ | 24 | 216.4.a.a | 12 |
| 47 ($\widetilde{38}$) | $(\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}),\ (\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$ | $(-6^2,-1^3,2^2,3^2,5)$ | 5 | 5400.4.?.? | 5 |

**Table 2:** The 33 cases without a MUM point. We write $n$ ($\widetilde{m}$) to indicate that the case $n$ is the total twist of a previous case $m$.

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