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χ be an irreducible character

$$\boxed{\varphi_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \bar{\chi}(g)\, g} \in \mathbb{C}[G]$$

is G-linear as an endom.
or equiv. $\varphi_\chi \in$ center of $\mathbb{C}[G]$

- $U \subseteq V$ subrepresentation

$$gU \subseteq U$$

$$\Rightarrow \quad \varphi_\chi U \subseteq U$$

- Schur's Lemma

U is irred., $\varphi_\chi : U \to U$

$$\varphi_\chi = \begin{cases} \mathrm{id}_U, & \chi_U = \chi \\ 0, & \chi_U \neq \chi \end{cases}$$

- V arbitrary repn

$$V = \bigoplus_U U^{a_U} \qquad U \text{ irred}, \quad a_U \in \mathbb{Z}_{\geq 0}$$

$\varphi_\chi : V \longrightarrow V$

$$\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & I_U & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

$\Rightarrow$ φ_χ is a projection

& $\operatorname{Im} \varphi_\chi = U^{a_U}$, $\chi_U = \chi$

True any decomposition of V in to irreducible $\operatorname{Im} \varphi_\chi$ is called the <u>χ-isotypical</u> component of V, denoted $V^\chi \subseteq V$ (canonically defined).

- Any map of repn $U \rightarrow V$ with $\chi_U = \chi$ has image in V^χ

Extreme silly case

$G = \{1\}$, $\chi = 1$.

V vector space dim n

$$V \simeq V^{\chi} = \cancel{k}\, \mathbb{C} v_1 \oplus \cdots \oplus \mathbb{C} v_n$$

a copy of the trivial repn

for any basis $v_1, \ldots, v_n$

In general

$$V \simeq \bigoplus_{\chi} V^{\chi}$$

<u>Example</u>

$G = \langle \sigma \rangle$ cyclic order n

$$V^{\chi} = \{ v \in V \mid \sigma v = \chi(\sigma) v \}$$

$\chi \in \mathrm{Hom}(G, \mathbb{C}^{\times})$

$$\varphi_{\chi} = \frac{1}{n} \sum_{k=0}^{n-1} \overline{\chi}(\sigma^k)\, \sigma^k$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \chi(\sigma)^{-k} \sigma^{k}$$

Concretely $\sigma = (12\cdots n)$ act on $(x_1, \ldots, x_n)$ permuting indices

$\zeta_n =$ n^{th}-primitive root of 1

$\chi_j(\sigma) = \zeta_n^{j}$ $\quad j \bmod n$

projection to the χ_j-component of V is

$$x_i \mapsto \frac{1}{n} \sum_{k=1}^{n} \zeta_n^{-jk} x_{k+i}$$

$j=0$ $\quad \langle \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \rangle = V^{\chi_0}$

$\vdots$

V^{χ_j}

$$V \simeq \bigoplus_{j=0}^{n-1} V^{\chi_j}$$ ↖ 1-diml.

Serre Topics in galois theory.

Solve equs in a group G.

E.g. $[x, y] = z$ for fixed $z \in G$

$$[x,y] = xyx^{-1}y^{-1}$$

say $z = 1$ $\quad xy = yx$

i.e. find or count all commuting pairs of elements in G.

Recall:

$$\delta(x) := \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

$$\delta = \frac{1}{|G|} \sum_{\chi} \chi(1)\, \chi$$

$$N(z) = \#\{ (x,y) \mid [x,y] = z \}$$

$$= \sum_{x, y \in G} \delta([x,y] z^{-1})$$

$$= \frac{1}{|G|} \sum_{\chi} \chi(1) \sum_{x,y} \chi([x,y]z^{-1})$$

How do we compute

$$\sum_{x,y} \chi([x,y]z^{-1}) \quad ?$$

(Ideas go back Frobenius)

Fix $y \in G$

$$\varphi = \frac{1}{|G|} \sum_{x} x y x^{-1}$$

- G- linear
- In a irred. repn $V \leftrightarrow \chi$

$$\varphi = \lambda \, id_V \qquad \text{by Schur}$$

- $tr(\varphi) = \lambda \, \chi(1)$

$$= \chi(y)$$

$$\Rightarrow \quad \lambda = \frac{\chi(y)}{\chi(1)}$$

- Take trace of $\varphi \cdot z$

$$\frac{\chi(y)}{\chi(1)}\,\chi(z) = \frac{1}{|G|}\sum_{x} \chi(xyx^{-1}z)$$

Take $z = y^{-1}$ and sum over y

$$\frac{1}{|G|^2}\sum_{x,y} \chi([x,y]) = \frac{1}{|G|}\sum_{y} \frac{\chi(y)\chi(y^{-1})}{\chi(1)}$$

$$= \frac{1}{|G|}\frac{1}{\chi(1)} \sum_{y} \chi(y)\overline{\chi(y)}$$

$$= \frac{1}{\chi(1)}$$

$$\frac{1}{|G|}\sum_{y\in G} \chi(y)\overline{\chi}(y) = (\chi, \chi)$$

$$= 1$$

I.e.

$$\frac{1}{|G|^2}\sum_{x,y} \chi([x,y]) = \frac{1}{\chi(1)}$$

Define

$$\varphi := \frac{1}{|G|^2} \sum_{x,y} [x,y]$$

- also G-linear

$$z\,[x,y]\,z^{-1} = [zxz^{-1}, zyz^{-1}]$$

- $\varphi = \lambda \, \mathrm{id}_V$ by Schur

take trace

$$\mathrm{tr}(\varphi) = \lambda\, \chi(1)$$

$$= \frac{1}{|G|^2} \sum_{x,y} \chi([x,y])$$

$$= \frac{1}{\chi(1)}$$

$$\Rightarrow \quad \lambda = \frac{1}{\chi(1)^2}$$

- take trace φz^{-1}

$$\frac{1}{\chi(1)^2}\chi(z^{-1}) = \frac{1}{|G|^2} \sum_{x,y} \chi([x,y]z^{-1})$$

$$N(z) = |G| \sum_{\chi} \chi(1) \frac{\chi(z^{-1})}{\chi(1)^2}$$

$$\boxed{N(z) = |G| \sum_{\chi} \frac{\chi(z^{-1})}{\chi(1)}}$$

If $z = 1$.

$$N(1) = |G| \sum_{\chi} 1$$

$$= |G| \; \#\{\chi\text{'s}\}$$

$$\#\{xy = yx\} = |G| \; \#\text{ conj. classes}$$

E.g. G abelian $\underline{z = 1}$ ✓

$\#$ conj classes $= |G|$

and $\#\{xy = yx\} = |G|^2$

$\underline{z \neq 1}$ $\quad N(z) = 0 \quad\quad 1 = z$

$$\Rightarrow 0 = |G| \sum_{\chi} \frac{\chi(z^{-1})}{\chi(1)}$$

$$= |G| \sum_{\chi} \chi(z^{-1})$$

$$= 0$$ ✓

<u>G abelian</u>

$$\frac{1}{|G|}\sum_{\chi} \chi(z) = \delta(z)$$

$$\frac{1}{|G|}\sum_{g\in G} \chi(g) = \delta(\chi)$$

$$\underset{g\in G}{\sum} \chi(g) \;\; \| \;\; (\chi, 1)$$

Extend to

$$N_g(z) := \#\{[x_1, y_1]\cdots[x_g, y_g] = z\}$$

Need

$$\frac{1}{|G|^g}\sum_{\substack{x_1\ldots x_g\\ y_1\cdots y_g}} \chi([x_1,y_1]\cdots[x_g,y_g]z^{-1})$$

By induction we see this equals

$$\frac{\chi(z^{-1})}{\chi(1)^{2g}}$$

$$N_g(z) = \sum_{\chi}\left(\frac{|G|}{\chi(1)}\right)^{2g-1}\chi(z^{-1})$$

<u>Frobenius mass formula</u>

$N_g(z)$ is a class function so certainly has an expression of this form, the miracle is that the coefficients are so simple.

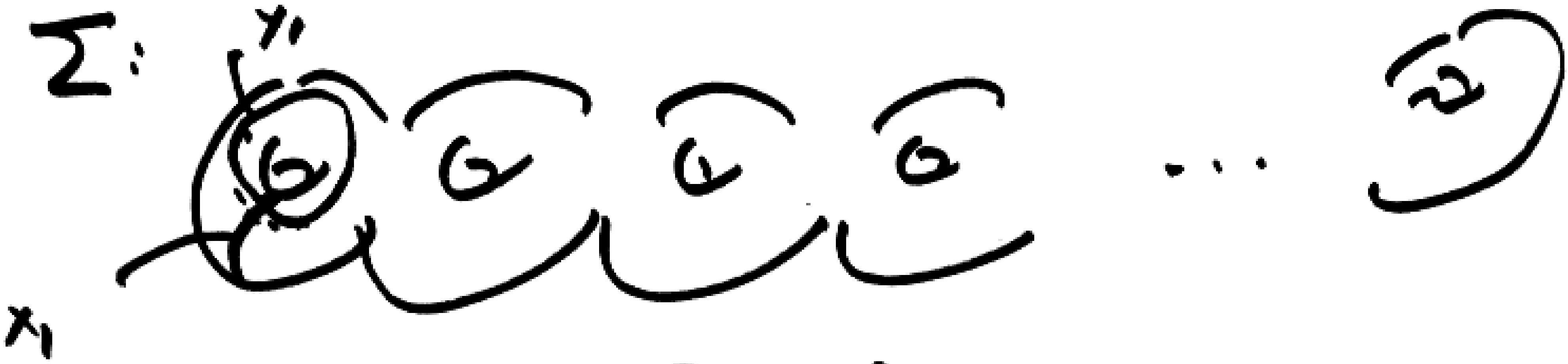
$\underline{z=1}$

$$N_g(1) = |G| \sum_{\chi} \left(\frac{|G|}{\chi(1)} \right)^{2g-2}$$

(*) $\quad [x_1, y_1] \cdots [x_g, y_g] = 1$

Riemann surface of genus g

Σ:



$$\Gamma_g := \pi_1(\Sigma, *)$$

$$= \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$$

$\mathrm{Hom}(\Gamma_g, G) \leftrightarrow$ solution to (*)

What happens as $g \to \infty$?

$$\frac{N_g(z)}{|G|^{2g}} = \frac{1}{|G|} \sum_{\chi} \frac{1}{\chi(1)^{2g-1}} \chi(z^{-1})$$

$$\xrightarrow[g \to \infty]{} \sum_{\chi(1)=1} \chi(z^{-1})$$

$$\frac{N_g(z)}{|G|^{2g}} \to \frac{1}{|G|} \sum_{\chi \in \mathrm{Hom}(G/G', \mathbb{C}^\times)} \chi(z^{-1})$$

$$= \frac{1}{|G|} \begin{cases} |G|/|G'| & z \in G' \\ 0 & z \notin G' \end{cases}$$

$$= \begin{cases} \frac{1}{|G'|} & z \in G' \\ 0 & z \notin G' \end{cases}$$

For large g

$$[x_1, y_1] \cdots [x_g, y_g] = z$$

has $\approx \frac{1}{|G'|}$ solutions $\times |G|^{2g}$ if $z \in G'$