Feb 13, 2007

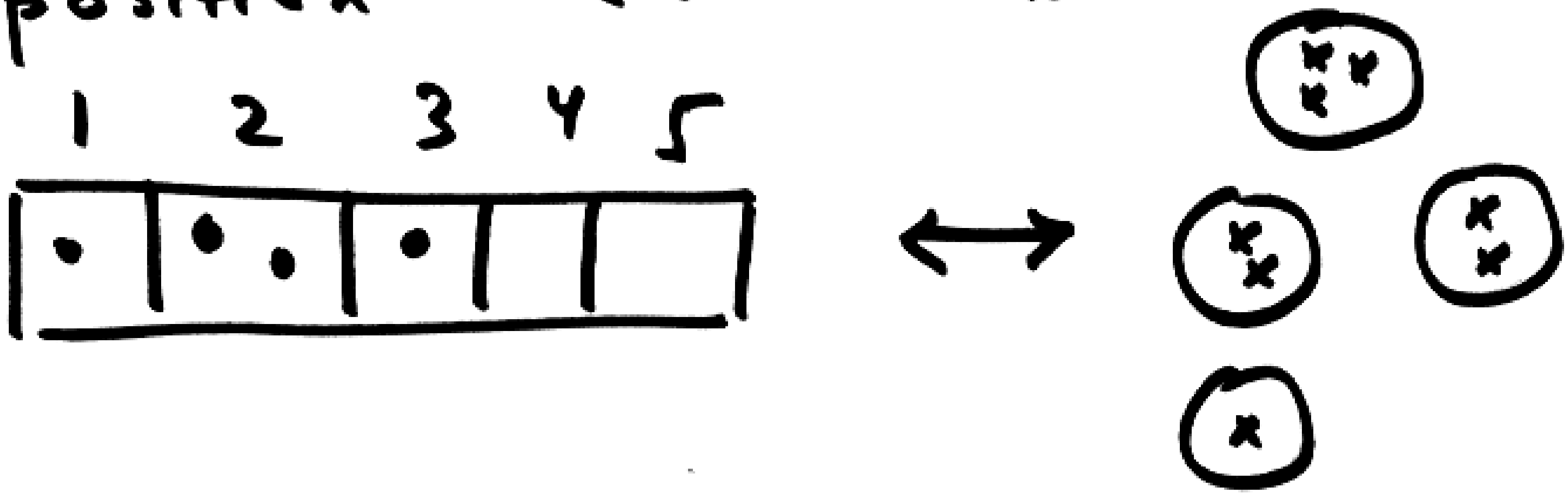
Nimble

Rule Move one penny to the left (any # of steps)

Nimble $\longleftrightarrow$ Nim

penny $\longleftrightarrow$ pile

position $\longleftrightarrow$ size



001	001	001	000
011	010	011	
100	011		

Write position of H's

$n_1, n_2, \ldots, n_k$

P position $\longleftrightarrow$ $n_1 \oplus \cdots \oplus n_k = 0$

$$\begin{array}{cc}
1 & 001 \\
3 & 011 \\
4 & 100 \\
\hline
6 & 110
\end{array}$$

$\to$ N position

Strategy: make it a P-position

Move

H T H H T T

$\downarrow$

H H H T T T

$$\begin{array}{c}
001 \\
010 \\
011 \\
\hline
000
\end{array}$$

$\to$ P-position

$$\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 0 \end{array}$$

$N \to P$

- identify leftmost column with sum of 1.
- In that column find a row with a 1.
- Change that row to get 0 sum

$$\to \cdots\cdots \boxed{1} \;\; \times \; \times \; \times \; \times \; \cdots \; \times$$

$$\uparrow$$

$$\rightsquigarrow$$

$$\cdots\cdots \boxed{0} \;\; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$\begin{array}{c|ccc} 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & \boxed{1} & 0 & 0 \\ \hline & 1 & 1 & 0 \end{array} \quad \longrightarrow \quad \begin{array}{c|ccc} 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ \hline & 0 & 0 & 0 \end{array}$$

NIM

$n_1, n_2, \ldots, n_K$ # of objects in each pile

P-position $\iff$ $n_1 \oplus \cdots \oplus n_K = 0$

- A move from $n_1 \oplus \cdots \oplus n_K = 0$ will mess this up
- Any N-position can be made into P.

E.g. $15, 13, 5$

$$5 \to 2$$

15, 13, 5

$\downarrow$

10

1

	8	4	2	1
15	1	1	1	1
13	1	1	0	1
5	0	1	0	1
	0	1	1	1

N - position

$\longrightarrow$

1	1	1	1	15
1	1	0	1	13
0	0	1	0	2
0	0	0	0	

P - position

2

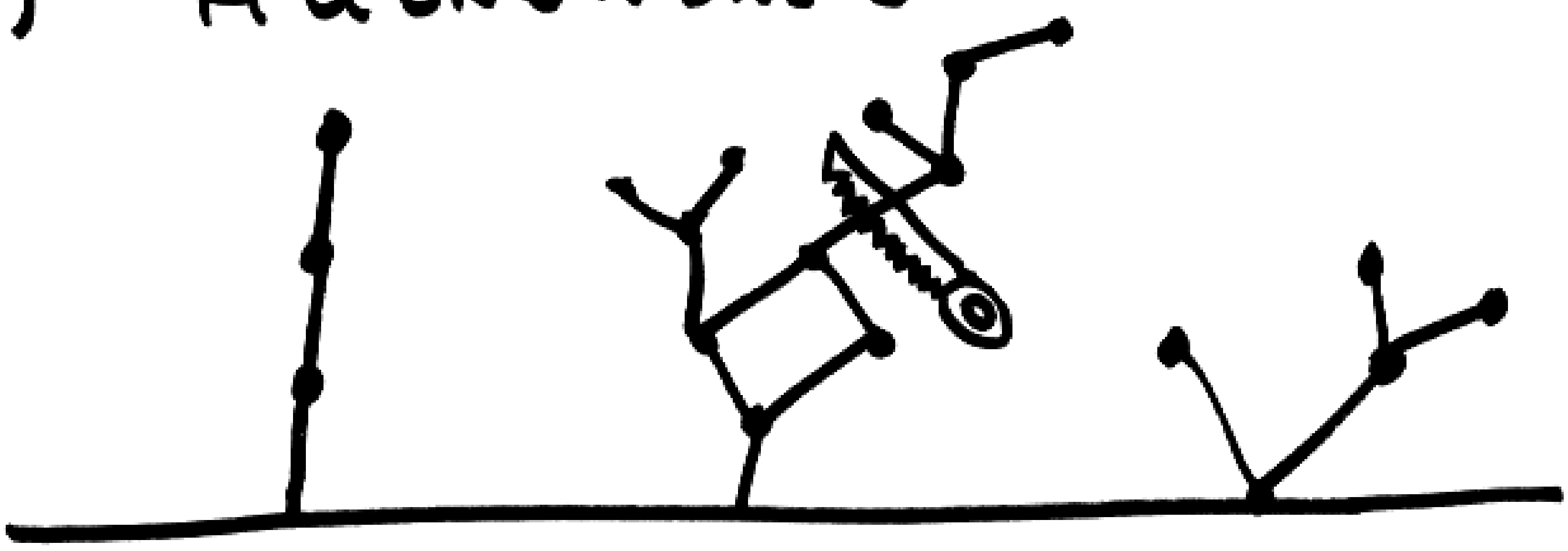
$$\begin{array}{c} 1111 \\ 1101 \\ 0101 \\ \hline 0111 \end{array} \longrightarrow \begin{array}{c|c} 1111 & 15 \\ 1010 & 10 \\ 0101 & 5 \\ \hline 0000 & \end{array}$$

3

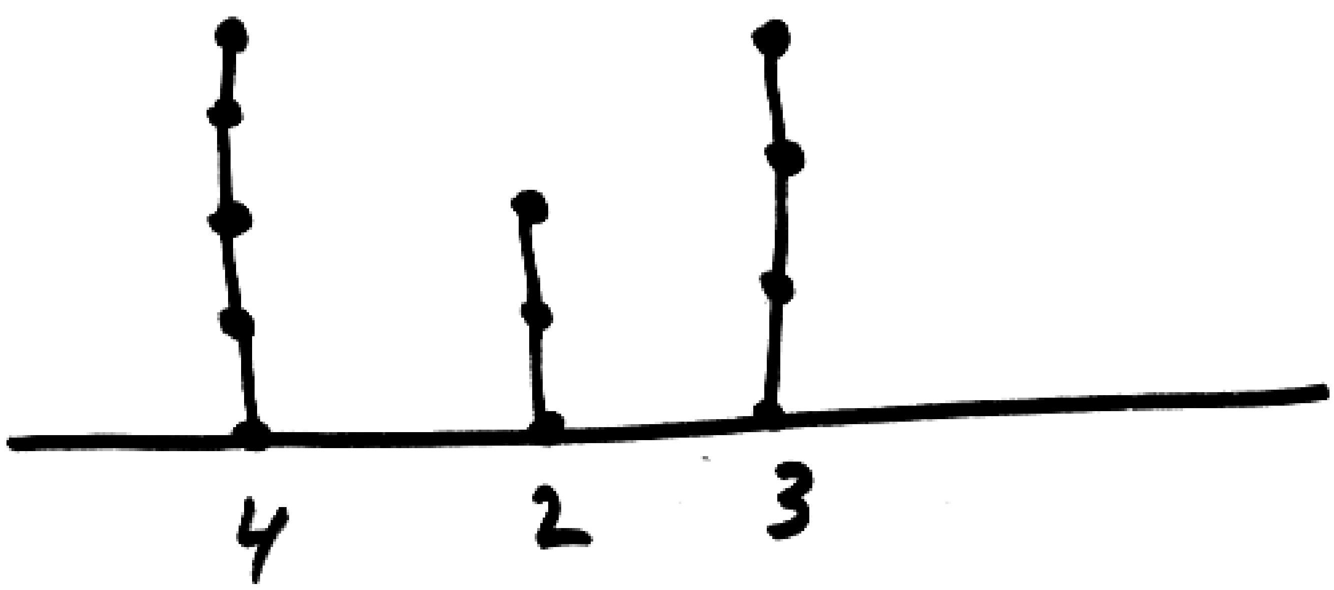
$$\begin{array}{c} 1111 \\ 1101 \\ 0101 \\ \hline 0111 \end{array} \longrightarrow \begin{array}{c|c} 1000 & 8 \\ 1101 & 13 \\ 0101 & 5 \\ \hline 0000 & \end{array}$$

Other examples of impartial games

3) Hackenbush



Move: Hack a piece off!



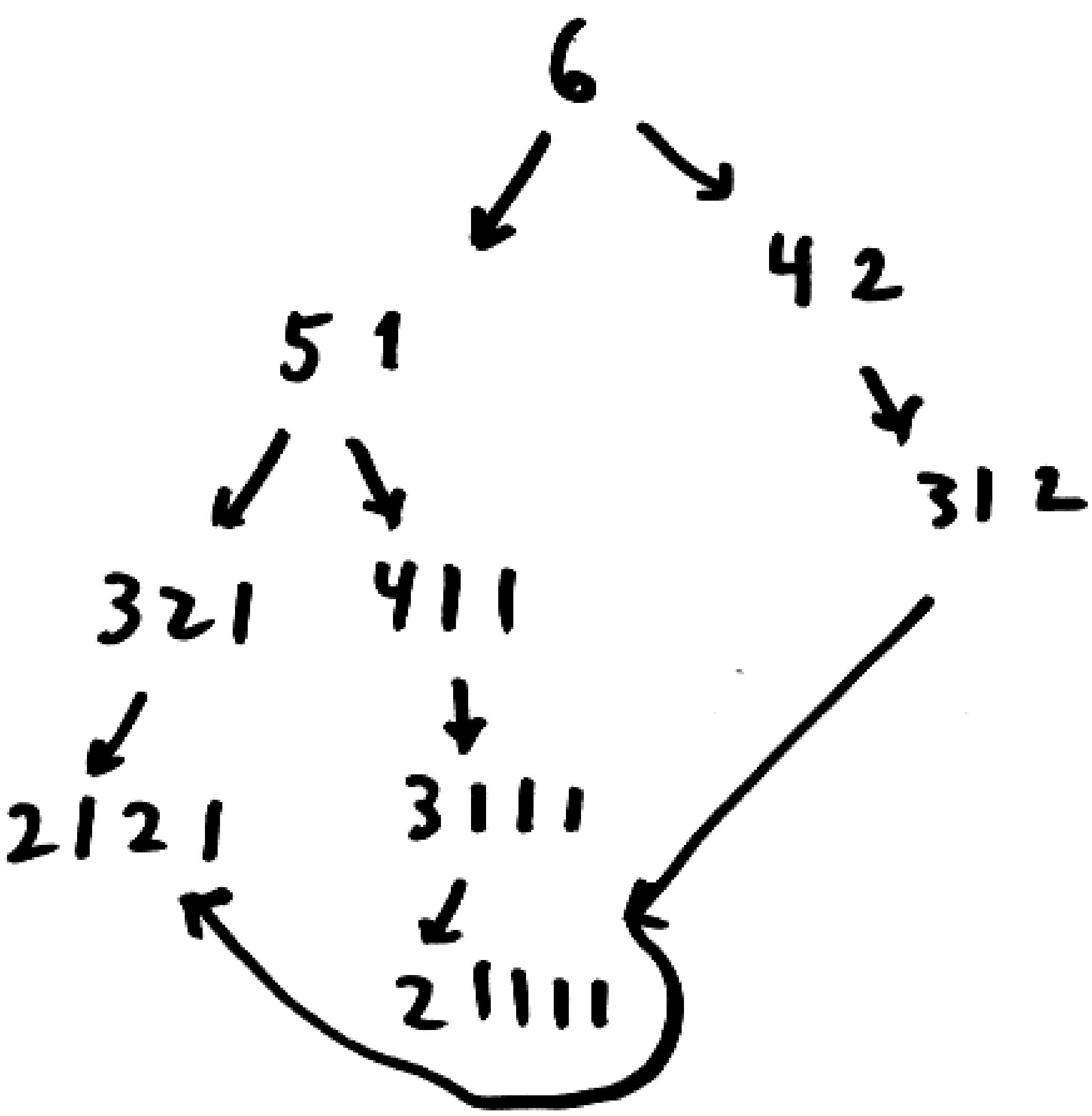
Each bamboo shoot $\longleftrightarrow$ pile

segment $\longleftrightarrow$ penny

4) <u>Grundy's game</u>

<u>Start</u> pile w/ n things

<u>Rule</u> Pick a pile and divide it into two unequal piles



<u>sum of games</u>

Γ_1, Γ_2 impartial games

$$\Gamma = \Gamma_1 \oplus \Gamma_2$$

A <u>move</u> in Γ is either a move Γ_1 or a move in Γ_2

Similarly $\Gamma_1, \dots, \Gamma_k$

$$\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k$$

E.g. Nim with k-piles is the sum $\oplus$ of k 1-pile games.

<u>Example</u> Γ_1 = substraction game $S = \{1, 2\}$

Γ_2 = substraction $S = \{1, 3\}$

Position $\Gamma = \Gamma_1 \oplus \Gamma_2$

two piles $\quad n_1 \to \Gamma_1$

$\qquad\qquad\quad n_2 \to \Gamma_2$

Γ_1	Γ_2
5	6

Pick Γ_1: $\quad 5 \to 4$, $\quad 5 \to 3$

Pick Γ_2: $\quad 6 \to 5$, $\quad 6 \to 3$

$$
\begin{tikzcd}
 & (5,6) \arrow[dl] \arrow[d] \arrow[r] \arrow[dr] & (5,5) \\
(3,6) & (4,6) & (5,3)
\end{tikzcd}
$$

Feb 15, 2007

Impartial games

Γ_1, Γ_2

$\Gamma = \Gamma_1 \oplus \Gamma_2$

A <u>move</u> in Γ is a move in either Γ_1 or Γ_2

E.g. Nim with k-piles

$$\underbrace{\Gamma \oplus \cdots \oplus \Gamma}_{k\text{-times}}$$

of Γ = Nim w/ one pile

Labeling P/N positions in Γ_1 and Γ_2 is not enough to find the label in $\Gamma_1 \oplus \Gamma_2$

Γ_1 = substraction game $\quad S = \{1, 2\}$

Γ_2 = " $\quad S = \{1, 3\}$

$$\Gamma = \Gamma_1 \oplus \Gamma_2$$

$$n_1, \; n_2$$

$\boxed{\Gamma_1}$ $\quad S = \{1, 2\}$

⋮	
5	N
4	N
3	P
2	N
1	N
0	P

n P-position $\Updownarrow$ $3 \mid n$

$\boxed{\Gamma_2}$ $\quad S = \{1, 3\}$

⋮	⋮
5	N
4	P
3	N
2	P
1	N
0	P

n P-position $\Updownarrow$ $2 \mid n$

$\Gamma = \Gamma_1 \oplus \Gamma_2$

Γ_2 \ Γ_1	0	1	2	3	4	5	6	7
6	⋮							
5	N							
4	P	N	N	P				
3	N	P	N	N				
2	P	N	N	P	N			
1	N	P	N	N	P			
0	P	N	N	P	N	N	....	

1st column

$$\begin{tikzcd} P \arrow[r] & P \\ & P \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r] & ? \\ & N \arrow[u] \end{tikzcd}$$

1st row

$$\begin{tikzcd} N \arrow[r] & N \\ & P \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] & N \\ & N \arrow[u] \end{tikzcd}$$

We need something more elaborate than just N/p labels for each individual game.

We'll define a numerical value to a position in an impartial game. G (position) = 0, 1, 2, 3, ...

P - position $\longleftrightarrow$ G value $= 0$

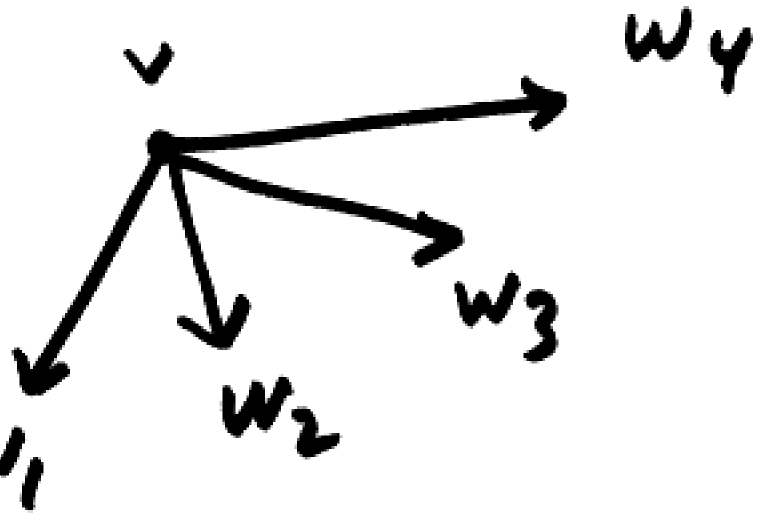
N - position $\longleftrightarrow$ G - value > 0

In the case of Nim k-piles

$n_1, \dots, n_k$

$$G = n_1 \oplus \cdots \oplus n_k$$

<u>Grundy function</u>



$$G(v) = \operatorname{mex}\{G(w_1), G(w_2), G(w_3), G(w_4)\}$$

$$G(v) := \operatorname{mex}\{\, G(w) \mid v \mapsto w \,\}$$

mex = minimum excludant

$$S \subseteq \{0, 1, 2, 3, \dots\}$$

mex(S) := smallest number which is NOT in S

$$\text{mex}\{0, 1, 4, 6, 9\} = 2$$

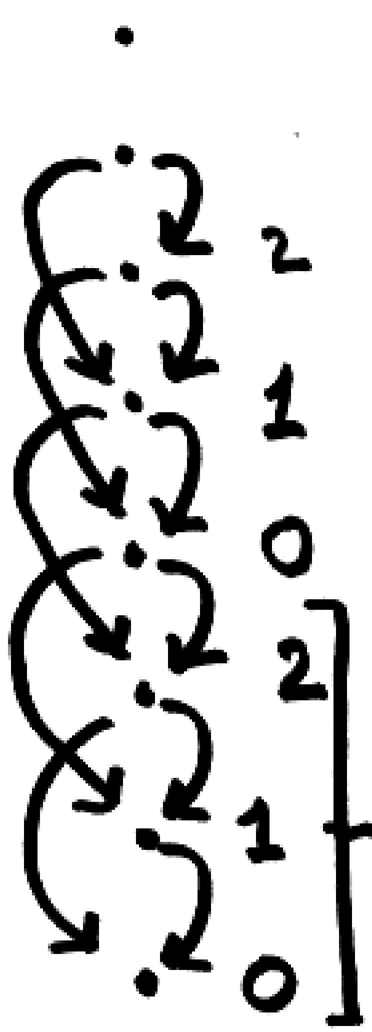
$$\text{mex}\{\phi\} = 0$$

$$\boxed{\text{mex}\{S\} = 0 \iff 0 \text{ is not in } S}$$

$$\text{mex}\{0, *, *, * \dots\} > 0$$

Γ_1

$\{1,2\}$



$\text{mex}\{0, 1\} = 2$

$\text{mex}\{0\} = 1$

$$G(v) = 0 \iff \text{P-position}$$

$$\Updownarrow$$

$$\text{mex}\{G(w) \mid v \mapsto w\} = 0$$

$$\Updownarrow$$

$$0 \neq G(w) \quad | v \mapsto w$$

$$G(v) > 0 \quad \longleftrightarrow \quad N\text{-position}$$

$$\Updownarrow$$

$$\operatorname{mex}\{ G(w) \mid v \mapsto w \} > 0$$

$$\Updownarrow$$

at least on child $v \mapsto w$
has $G(w) = 0$

(Sprague - Grundy)

THEOREM

$\Gamma_1, \Gamma_2, \dots, \Gamma_k$, $\Gamma := \Gamma_1 \oplus \cdots \oplus \Gamma_k$

$$G_\Gamma = G_{\Gamma_1} \oplus \cdots \oplus G_{\Gamma_k}$$

E.g. Γ_1 substraction $S = \{1, 2\}$
Γ_2 " $S = \{1, 3\}$

$G_{\Gamma_1}(n) = \{ 0, 1, 2, 0, 1, 2, \dots$

$G_{\Gamma_2}(n) = 0, 1, 0, 1, 0, 1, \dots$

5	1	0						
4	0	1						
3	1	0	3	1				
2	0	1	2	0	1			
1	1	0	3	1	0	3	1	
0	0	1	2	0	1	2	0	1
	0	1	2	3	4	5	6	7

8

$$\begin{array}{r} 1\,0 \\ 0\,1 \\ \hline 1\,1 \end{array}$$

$\Gamma_1 \quad \Gamma_2$

$(5, 6)$

$G_{\Gamma_2}(6) = 0$

$G_{\Gamma_1}(5) = 2$

$G_\Gamma(5,6) = 2 \oplus 0 = 2$

$\rightarrow$ N-position

winning move $2 \longrightarrow 0$

$5 \longmapsto 3$