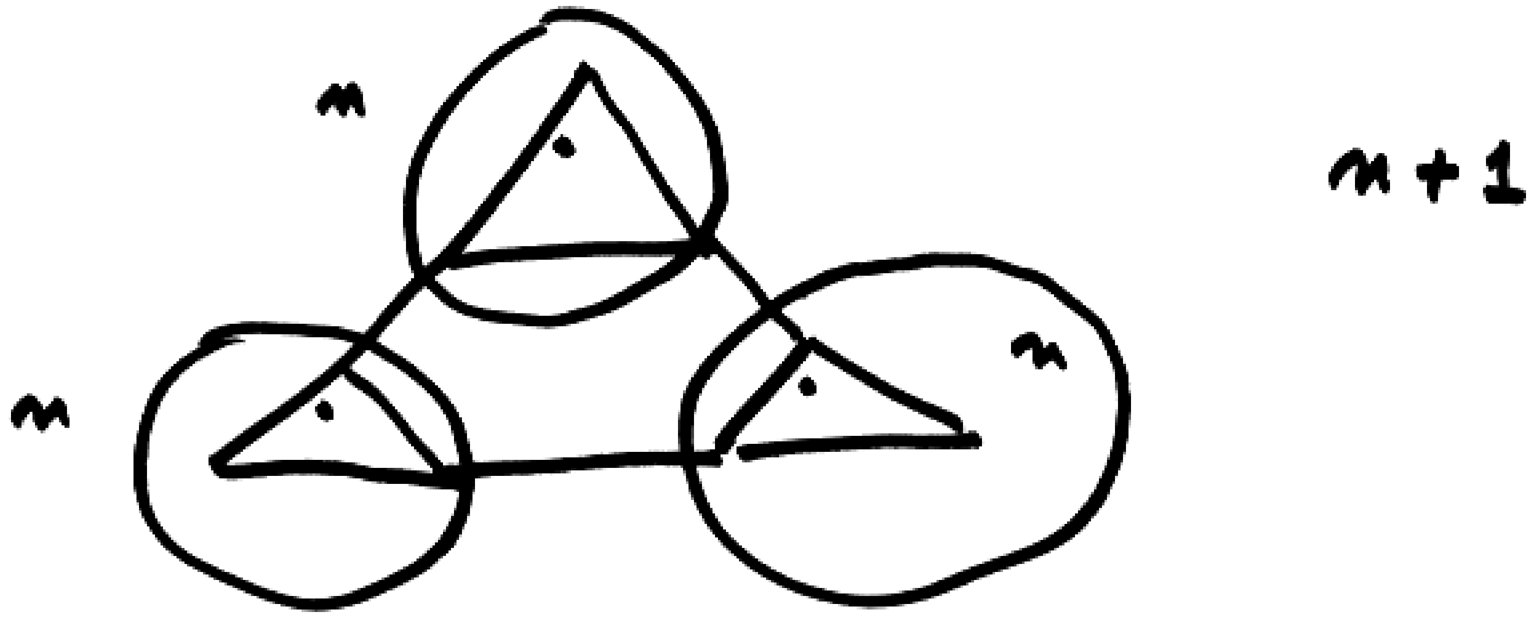
Feb 06, 2007



positions in case $n+1$

$= 3 \times$ positions in case n

$C_n :=$ # positions in case n

$$\begin{cases} C_{n+1} = 3\, C_n \\ C_1 = 3 \end{cases}$$

$3, 3^2, 3^3, \dots$

<u>Claim</u> $C_n = 3^n$

$C_{n+1} = 3^{n+1} = 3 \times C_n$

position $\longleftrightarrow$ $(p_1, p_2, \dots, p_n)$

$p_i = 0, 1, 2$
= peg # where disk i is

<u>$n=1$</u>

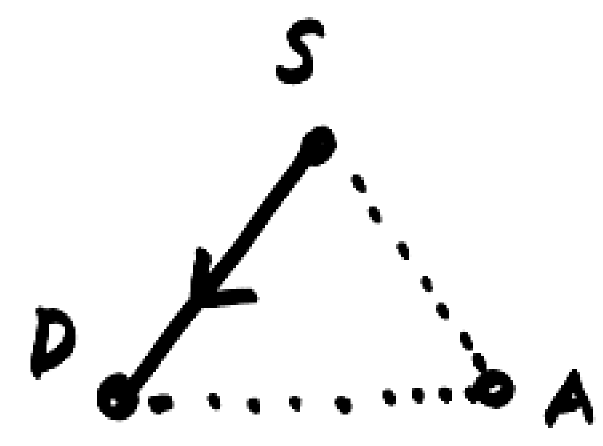
(0), (1), (2)

<u>$n=2$</u>

$(0,0)$, $(1,0)$, $(2,0)$
$(0,1)$, $(1,1)$, $(2,1)$
$(0,2)$, $(1,2)$, $(2,2)$

<u>$n=3$</u> $(0,0,*)$ $(1,0,*)$, $(2,0,*)$
$\vdots$

Solving puzzle optimally what happens with last disk, n?



Disk n just moves from $S \longrightarrow D$?

$\underline{n=7}$ $\circlearrowleft 7 \Rightarrow$ disk # 1 $\circlearrowright$

PASCAL TRIANGLE

n $\binom{n}{1}$ $\binom{n}{2}$ $\binom{n}{3}$ $\binom{n}{k}$

0: 1

1: 1 1

2: 1 2 1

3: 1 3 3 1

4: 1 4 6 4 1

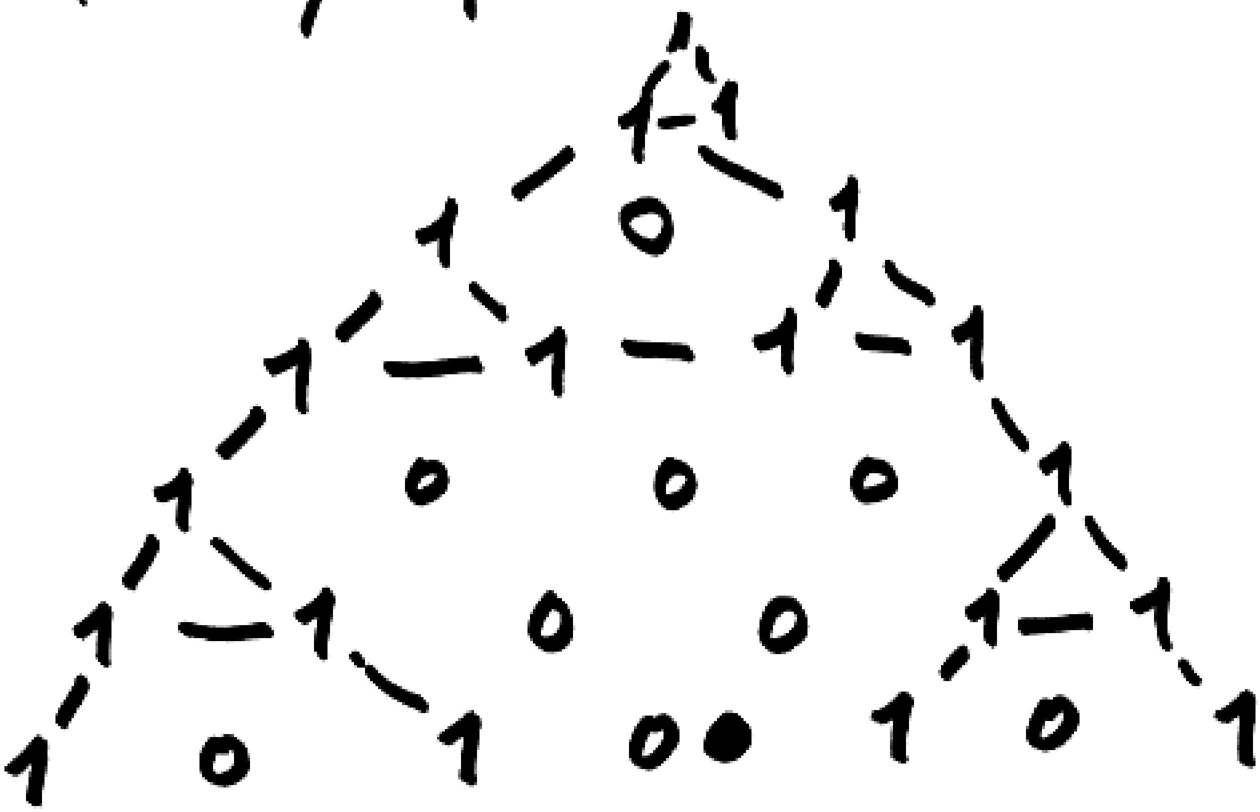
5: 1 5 10 10 5 1

6: 1 6 15 20 15 6 1

"n chose k"

$0 \le k \le n$

Parity of the Pascal triangle



$$(1+x)^3 = 1 + 3x + 3x^2 + 1$$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$\vdots$$

$$\binom{n}{k} = \#\left\{\text{ways to choose } k \text{ things out of } n\right\}$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} =$$

$n = 4$

1, 2, 3, 4

1, 2	2, 3	3, 4
1, 3	2, 4	
1, 4		

$\binom{4}{2} = 6$

$\binom{n}{2} = \frac{1}{2} \; n \times (n-1)$

1st choice

2nd choice

account for the order

otherwise we are double counting.

$\underline{n=4}$

$\boxed{1, 2}$	$\boxed{2, 1}$	3, 1	4, 1
1, 3	2, 3	3, 2	4, 2
1, 4	2, 4	3, 4	4, 3

$$\binom{n}{2} = \frac{n \times (n-1)}{2}$$

$$\binom{4}{2} = \frac{4 \times 3}{2} = 6$$

$n = 5$ $\quad \binom{5}{2} = 10$

1, 2	2, 3	3, 4	4, 5
1, 3	2, 4	3, 5	
1, 4	2, 5		
1, 5			

$$\frac{n \times (n-1)}{2} = \frac{5 \times 4}{2} = 10$$

$$\binom{n}{3} = \frac{1}{6}\, \underset{\uparrow \atop \text{1st}}{n}\, \underset{\uparrow \atop \text{2nd}}{(n-1)}\, \underset{\uparrow \atop \text{3rd}}{(n-2)}$$

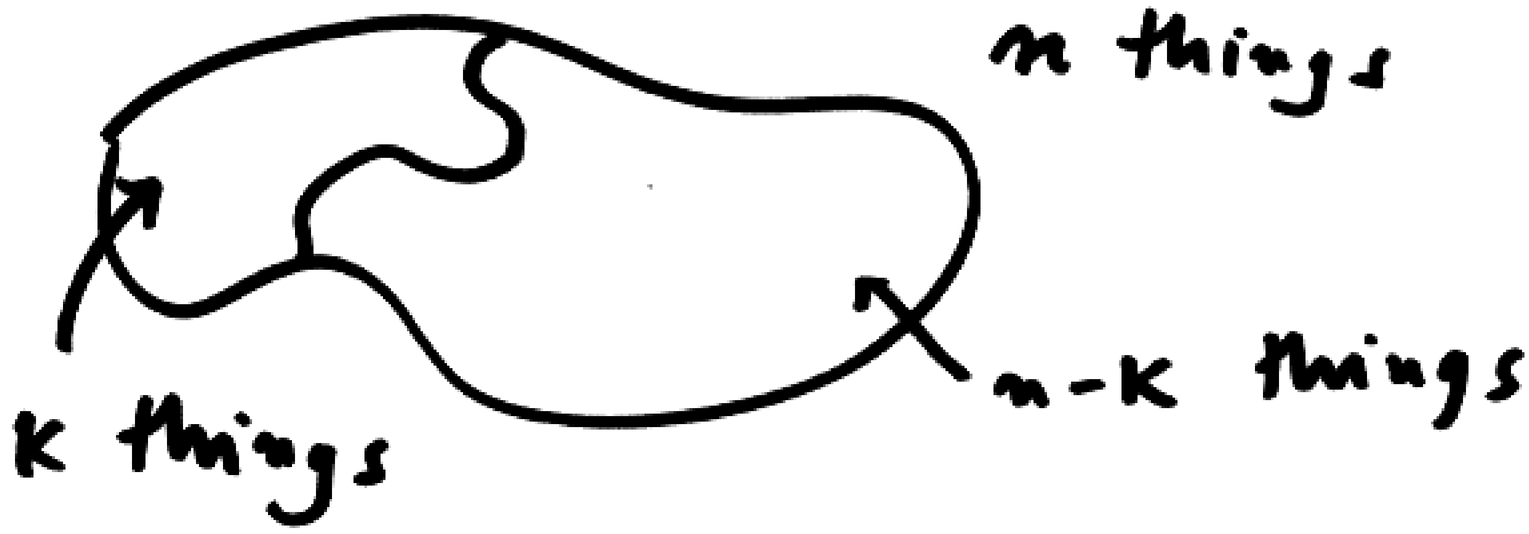
overcounting e.g. $\{1, 2, 3\}$

$$\text{6 times} \left\{ \begin{array}{l} 1, 2, 3 \\ 1, 3, 2 \\ 2, 1, 3 \\ 2, 3, 1 \\ 3, 1, 2 \\ 3, 2, 1 \end{array} \right.$$

$$\binom{n}{4} = \frac{1}{4!} n(n-1)(n-2)(n-3)$$

$$\binom{5}{3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

Symmetry in Pascal's triangle



Picking k $\longleftrightarrow$ Picking $n-k$

E.g. $n = 5$

Pick two things $\qquad \binom{5}{2} = 10$

$\updownarrow$

Pick 3 things $\qquad \binom{5}{3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= \frac{n(n-1)\cdots\cdots 2\,1}{k!\,(n-k)!}$$

$$\binom{7}{4} = \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1} = \frac{7\cdot 6\cdot 5\cdot 4\cdot \overbrace{3\cdot 2\cdot 1}}{4\cdot 3\cdot 2\cdot 1\cdot \underbrace{3\cdot 2\cdot 1}}$$

$$= \frac{7!}{4!\,3!}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

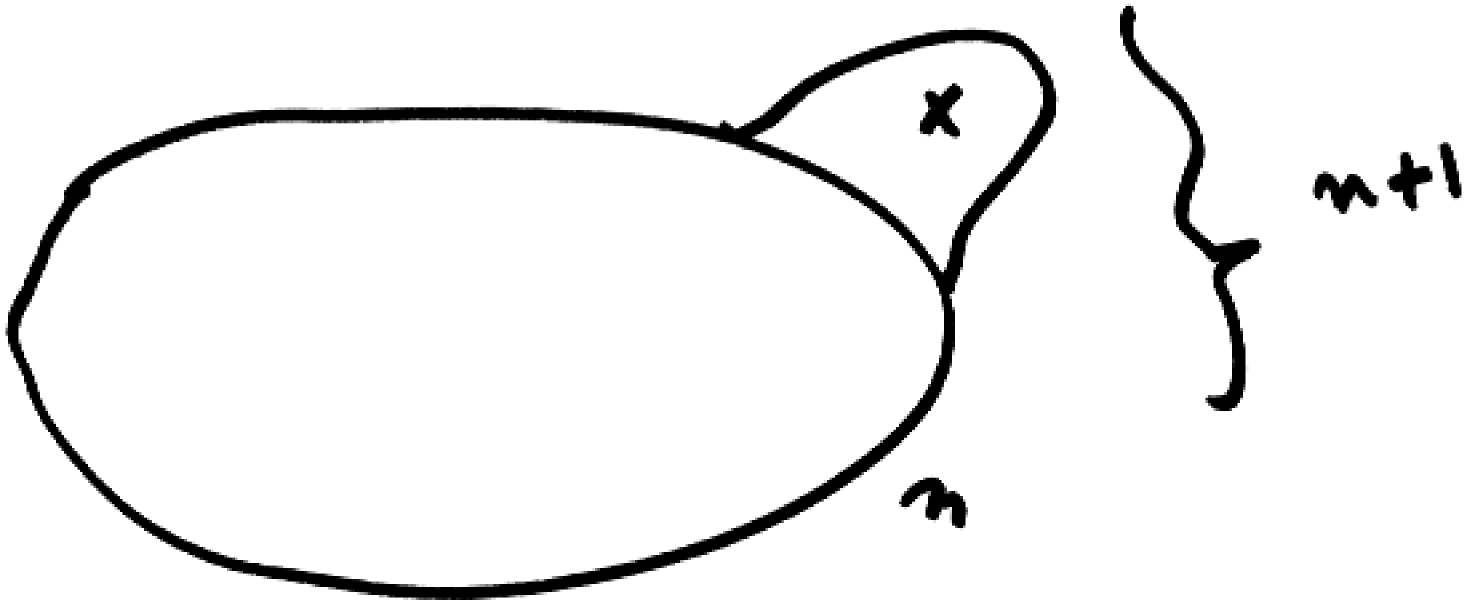
$$k \longleftrightarrow n-k$$

$$k \longmapsto n-k$$

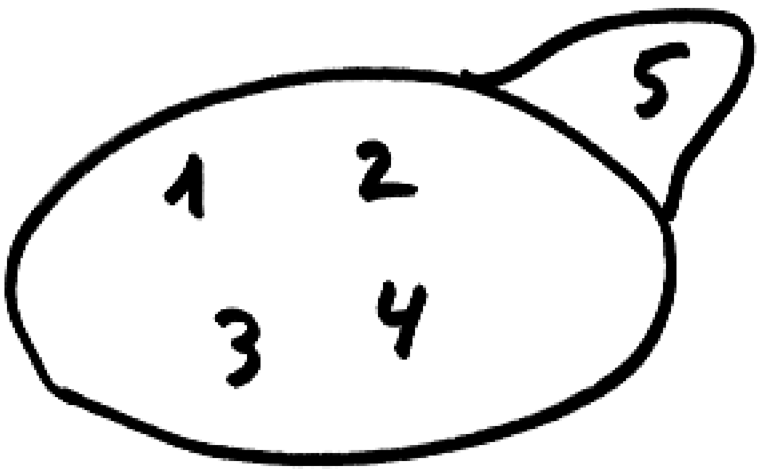
$$n-k \longmapsto n-(n-k) = k$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\boxed{\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}}$$



n=4



Excluding 5

1, 2 2, 3 3, 4
1, 3 2, 4
1, 4

$\binom{4}{2}$

Including 5

5, 1
5, 2
5, 3
5, 4

$\binom{4}{1}$

$$\binom{5}{2} = \binom{4}{2} + \binom{4}{1}$$

15 - puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

S. LLoyd

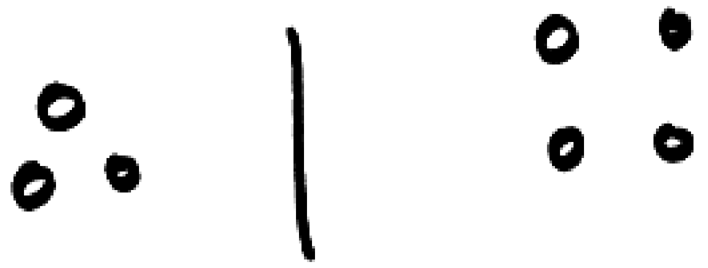
1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

cannot be done!

Feb 8, 2007

NIM

Two piles



good move → 

Strategy: achieve same number in both piles.

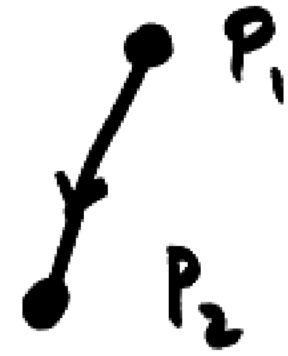
How does this extend to more piles?

Ch. Bouton 1901

Theory applies to impartial games.

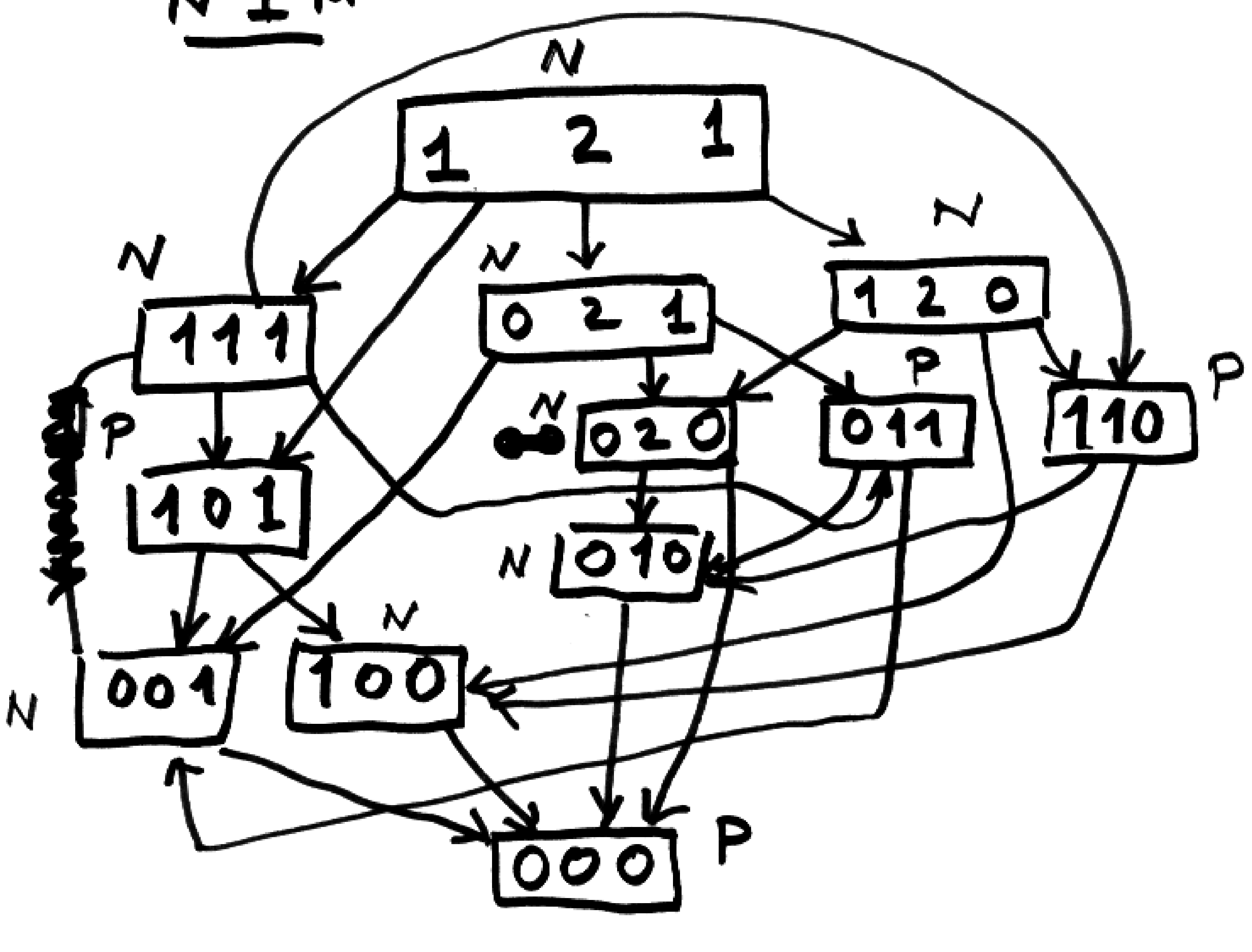
graph : vertices $\leftrightarrow$ position ②

edge $\leftrightarrow$ move



$P_1 \rightsquigarrow P_2$

<u>NIM</u>



Strategy

- Reach P
- opponent moves to N

Impartial game

- Two players, alternate
- same moves
- No chance
- complete information
- no ties / endgame

Player unable to move loses.
(Normal play)
(opposite " wins)
misére play

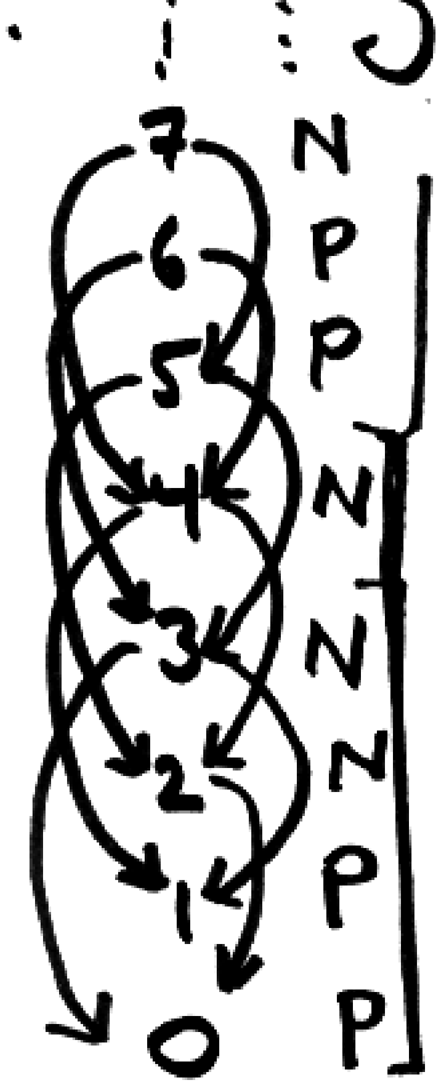
Substraction games

one pile

you take s objects from the pile where

$s \in S$

E.g. $S = \{2, 3\}$



positions labels repeat in the

P P N N N

pattern

E.g. if pile has 22 things

P	P	N	N	N
20	21	22	23	24

take 2 to reach a P position

NIM addition

(Nimbers)

$n \oplus m$

write n, m in binary

E.g. $n = 3$, $m = 5$

$$\begin{array}{lccc} m & 1 & 0 & 1 \\ n & 0 & 1 & 1 \\ \hline & 1 & 1 & 0 \end{array} \leftarrow \quad n \oplus m = 6$$

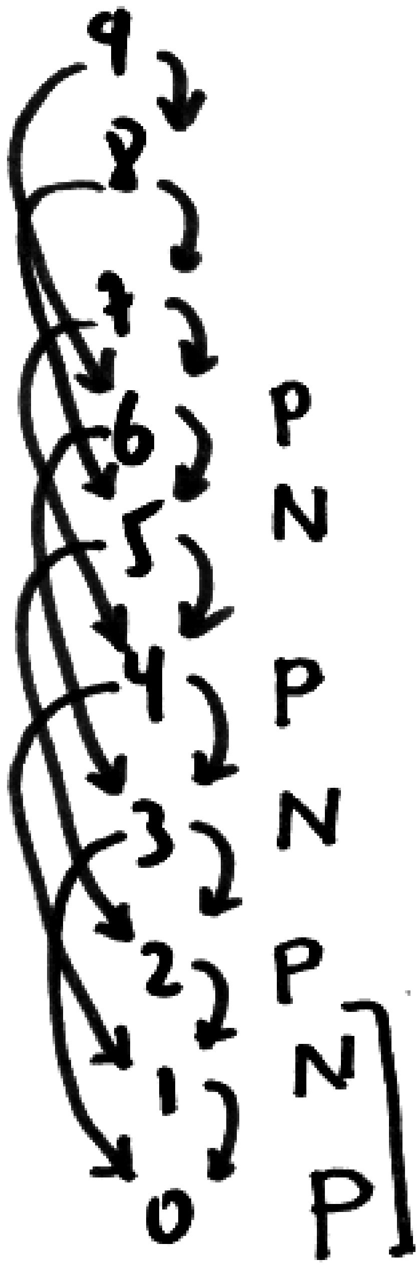
$$3 \oplus 5 = 6$$

$$n = m \longleftrightarrow n \oplus m = 0$$

$$\begin{array}{cccc} 1 & 1 & 0 & 1 \\ a & b & c & d \\ \hline 0 & 0 & 0 & 0 \end{array} \longleftrightarrow \begin{array}{l} d = 1 \\ c = 0 \\ b = 1 \\ a = 1 \end{array}$$

$(n_1, \ldots, n_k)$ P-position $\Leftrightarrow n_1 \oplus \cdots \oplus n_k = 0$

E.g. $S = \{1, 3\}$



PN
pattern repeats

What are the P positions in NIM?

two piles : (n, m)

$n = m \longleftrightarrow$ P position

<u>More piles</u>