Number Theory Seminar UT
Junior Geometry UT Oct 5 1998

10.166 Reflexive loops

Sept 10, 1998

A reflexive loop is the closed polygonal path joining successively a sequence of lattice points such that:

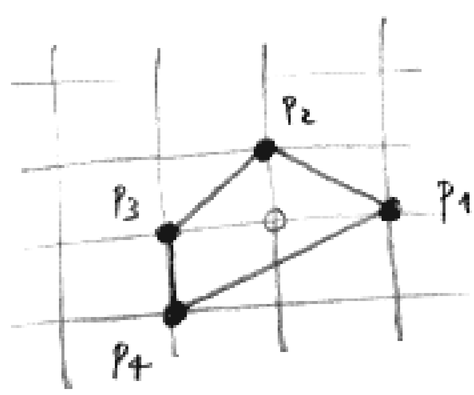
$p_i \in \mathbb{Z}^2$, $i = 1, 2, \dots, n$

(*) p_i, p_{i+1} is a basis of $\mathbb{Z}^2$ for $i = 1, 2, \dots, n$

(we take indices modulo n so that $p_{n+1} = p_1$).

For the moment we will also assume the loop is convex and circles always counterclockwise. Let us call it positive.

Example



$\mathcal{L}$

An equivalent formulation of (*) is:

(*') the triangle $0, p_i, p_{i+1}$ has area $1/2$

equivalently, by Pick's theorem, it contains no other lattice point.

Given a reflexive loop we may form its dual $\mathcal{L}^*$ by taking the polygonal path joining the tangent vectors. More precisely:

Let $p_i' = p_{i+1} - p_i \qquad i = 1, 2, \dots, n$ then intercalating between the p_i''s any lattice point on the segments joining p_i' and p_{i+1}' we obtain a new reflexive loop $\mathcal{L}^*$.

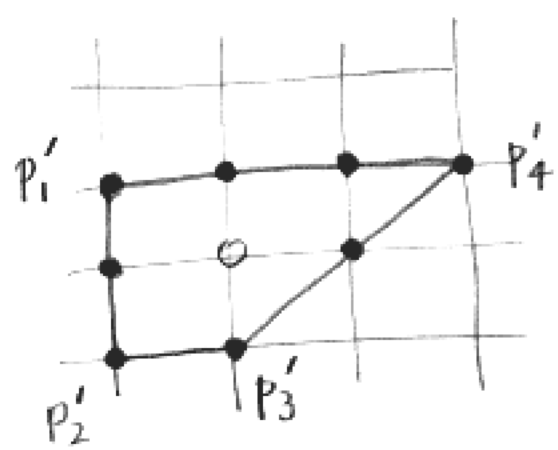
A reflexive loop determines a closed path in $\mathbb{R}^2 \setminus \{0\}$ and has, therefore, a well defined <u>winding number</u> $w(\mathcal{L})$.

<u>Theorem</u>

(1) $\mathcal{L}^*$ is a reflexive loop, which is positive if $\mathcal{L}$ is.

(2) $(\mathcal{L}^*)^*$ is the reflection of $\mathcal{L}$ through the origin

(3) $$\boxed{\# \partial \mathcal{L} + \# \partial \mathcal{L}^* = 12\, w(\mathcal{L})}$$



$\mathcal{L}^*$

Notice that duality gives

vertices $\leftrightarrow$ sides
sides $\leftrightarrow$ vertices

When $\mathcal{L}$ is a positive reflexive loop of winding number 1 we may associate to it a toric surface and in fact non-singular.

Noether's fmla (a special case of the Hirzebruch-Riemann-Roch theorem) gives the identity

$$12(1+p_a) = K^2 + c_2$$

and in our case these have the values

$p_a = 0 \qquad K^2 = \# \partial \mathcal{L} \qquad c_2 = \# \partial \mathcal{L}^*$

Q: What happens in the case $w(\mathcal{L}) > 1$ with this interpretation?

Hamilltonian mechanics

Think of the polygonal path as a sequence of pairs of vectors ... (p, q) in $\mathbb{Z}^2$ with det $= 1$;

where p is the position and q the momentum. (i.e a direction) $p \wedge q = 1$

$p =$ some $p_j \in \mathcal{L}$
$q =$ some $q_k \in \mathcal{L}^*$
weave them right

We move from one pair to the other by either one of the following:

$$A: \begin{cases} p \longmapsto p+q \\ q \longmapsto q \end{cases} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

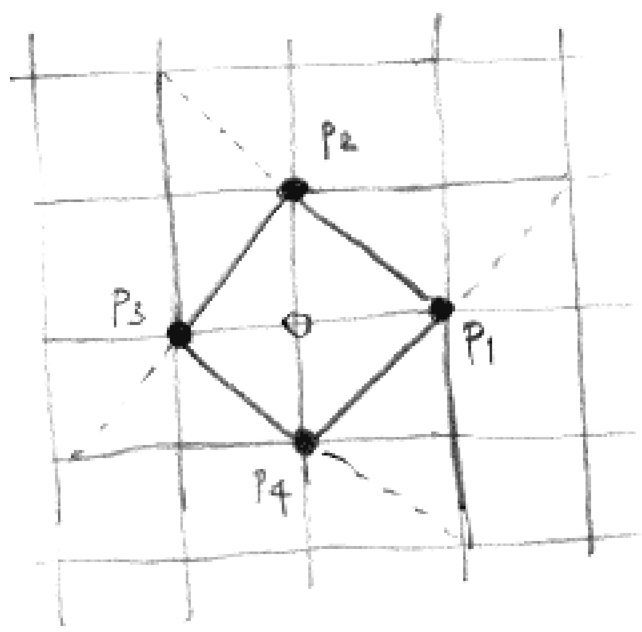
$$B: \begin{cases} p \longmapsto p \\ q \longmapsto q-p \end{cases} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

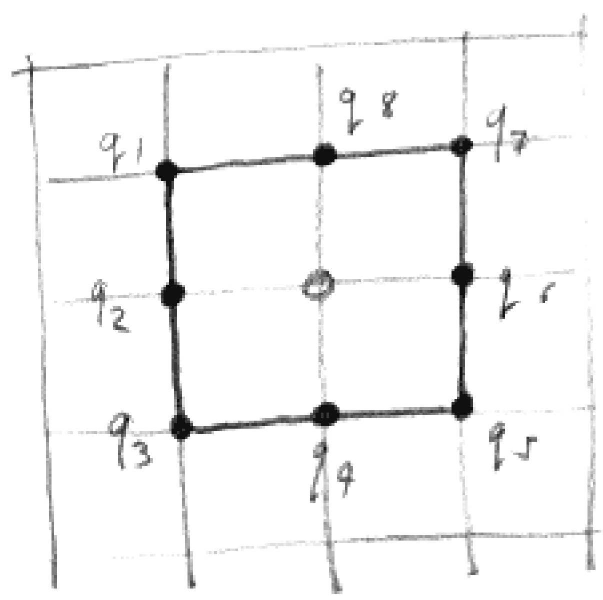
So we may describe a reflexive loop (modulo $SL_2(\mathbb{Z})$) by a sequence

$$A^{a_1} B^{b_1} \cdots A^{a_v} B^{b_v} = id$$

(v = # vertices of $\mathcal{L}$ or $\mathcal{L}^*$)

Example





$(p_1, q_1)(p_2, q_1)(p_2, q_2)(p_2, q_3)(P_3, q_3)(p_3, q_4, (p_3, q_5) \cdots$

$$A B^2 A B^2 A B^2 A B^2 = id$$

In fact we have a polygonal loop in $Sl_2(\mathbb{Z}) \cdot \tilde{\mathcal{L}}$

Consider

$$\tilde{A} : \quad t \longmapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad\qquad t \in [0,1]$$

$$\tilde{B} : \quad t \longmapsto \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

Let us normalize things so that

$(p_1, q_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We have then a polygonal path $\tilde{\mathcal{L}}$ in $Sl_2(\mathbb{Z})$

described by

$$\tilde{A}^{a_1} \tilde{B}^{b_1} \cdots \tilde{A}^{a_v} \tilde{B}^{b_v}$$

If we project $\tilde{\mathcal{L}}$ to the first row we get $\mathcal{L}$

if we project to the second $\mathcal{L}^*$

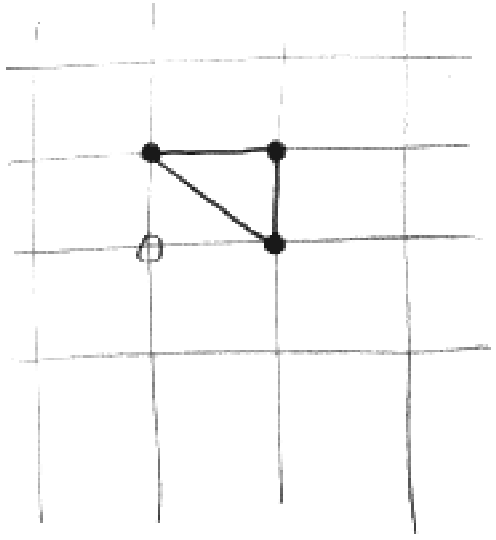
? This proves parts (1) & (2) of the theorem and also shows the winding numbers of $\mathcal{L}, \mathcal{L}^*$ are the same and equal the winding number of $\tilde{\mathcal{L}}$.

$\tilde{A}, \tilde{B}$ generate $\widetilde{SL_2}(\mathbb{Z})$ (preimage of $SL_2(\mathbb{Z})$ in universal cover of $SL_2(\mathbb{R})$) since A, B generate $SL_2(\mathbb{Z})$.

$$1 \longrightarrow \mathbb{Z} \rightarrow \widetilde{SL_2}(\mathbb{R}) \longrightarrow Sl_2(\mathbb{R}) \rightarrow 1$$

is $\pi_1(SL_2(\mathbb{R}))$

homotopy class of paths from the identity in $Sl_2(\mathbb{R})$



Has winding number 0.

So we obtain the relation

$$\tilde{A}\tilde{B}\tilde{A}\tilde{B}^{-1}\tilde{A}^{-1}\tilde{B}^{-1} = id$$

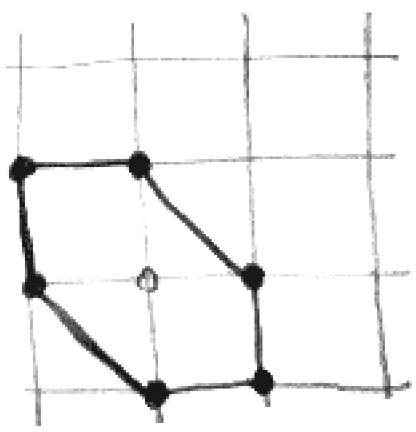
in $\widetilde{SL_2}(\mathbb{Z})$

Hence the map

$$\begin{aligned} \Phi : \widetilde{SL_2}(\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \tilde{A} &\longmapsto 1 \\ \tilde{B} &\longmapsto 1 \end{aligned}$$

is a well defined homomorphism.

On the other hand



$\mathcal{L} \rightarrow (AB)^6 = id$

has winding number 1 and $\Phi(\mathcal{L}) = 12$

Hence

$$\sum_{j=1}^{v} a_j + \sum_{j=1}^{v} b_j = 12 \cdot w$$

Cor: There are finitely many positive reflexive loops of (up to $SL_2(\mathbb{Z})$) fixed winding number