MODULAR MAHLER MEASURES I

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Introduction The *logarithmic Mahler measure* of a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is defined as

$$m(P) = \int_0^1 \cdots \int_0^1 \log \left| P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n}) \right| d\theta_1 \cdots d\theta_n$$

and its *Mahler measure* as $M(P) = e^{m(P)}$, the geometric mean of $|P|$ on the torus

$$T^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | \, |z_1| = \dots = |z_n| = 1\}.$$

If α is an algebraic number and $P \in \mathbb{Z}[x]$ with $P(\alpha) = 0$ has relatively prime coefficients and minimal degree then Jensen's formula of complex analysis gives

$$M(P) = \text{height of } \alpha.$$

For example, if $\alpha = n/m \in \mathbb{Q}^\star$ with m, n relatively prime integers then

$$M(mx - n) = \max\{|m|, |n|\}.$$

In the early 80's Smyth [Sm] discovered the following remarkable identity

$$m(1 + x + y) = L'(\chi, -1),$$

where $L(\chi, s)$ is the Dirichlet series associated to the quadratic character χ of conductor 3. A handful of similar formulas were known at the time, [Bo1], [Ra], in apparent isolation from the general (though mostly conjectural) world of special values of L-series. Recently Deninger [De] bridged the gap by showing how to interpret $m(P)$, when P does not vanish on T^n, as a Deligne period of a mixed motive. This combined with Beilinson's conjectures makes identities such as Smyth's somewhat less of a mystery, though certainly not less beautiful.

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Deninger's work also prompted Boyd [Bo1] to search numerically for other identities of the form

$$m(P) = c \cdot L'(0),$$

where $P \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, $c \in \mathbb{Q}^\star$, and L is an L-series somehow associated to P. Perhaps we should clarify right away what we mean by 'numerically' here. Once P and $L(s)$ are chosen, both quantities $m(P)$ and $L'(0)$ can be computed as complex numbers to any desired degree of precision (tough in general not without certain amount of ingenuity). The ratio of such approximations is of course a rational number, but what one looks for is a situation where the ratio is very close to a rational number c of relatively low height (e.g. $c = 1$).

Boyd found many such 'numerically verified' identities. For example, let

$$P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k, \qquad k \in \mathbb{C},$$

and let

$$B_k = \frac{L_k'(0)}{m(P_k)}, \qquad k \in \mathbb{N},$$

where $L_k(s)$ is the L-function of the elliptic curve E_k determined by $P_k = 0$ if $k \neq 4$ and $L(\chi, s-1)$ with χ the quadratic Dirichlet character of conductor 4 if $k = 4$. Then the numerical data suggests that

$$(?_1) \qquad B_k \in \mathbb{Q}^\star, \qquad k \in \mathbb{N}.$$

(Here though not strictly necessary we assume E_k is modular, an essentially academic assumption in light of the work of Wiles.)

What is known is the following (see n$^{\text{o}}$ 15). The case $k = 4$, for which $P_k = 0$ is of genus 0, is very similar to that of Smyth's and indeed it can be proved by direct manipulations of the integral defining $m(P_4)$ that $B_4 = 1/2$. For $k = 1$ (despite P_1 vanishing on the torus T^2) Deninger showed that the conjecture of Bloch–Beilinson implies that $B_1 \in \mathbb{Q}^\star$.

It turns out that a suitable modification of $(?_1)$ allows values of k with $k^2 \in \mathbb{N}$ and we find that for $k = 4\sqrt{2}$, where the corresponding elliptic curve has complex multiplication, we can prove that $B_k = 1$ and for $k = 3\sqrt{2}$, where E_k is isomorphic to the modular curve $X_0(24)$, a theorem of Beilinson's guarantees that $B_k \in \mathbb{Q}^\star$. Apart from a few other similar values of k the truth of $(?_1)$ remains, however, unproven.

Boyd produced many other examples of families of Laurent polynomials which appear to satisfy the analogue of $(?_1)$, finding, with uncanny intuition, the right set of conditions that these polynomials should meet.

The purpose of this expository paper is to give, on one hand, a gentle introduction to the conjecture of Bloch–Beilinson (chapter II) and on the other, to clarify the relation of Boyd's examples to this conjecture. As we will see in chapter III, Boyd's construction gives a new way of producing elements in $K_2(E)$ for certain elliptic curves E defined over $\mathbb{Q}$.

We have strived to present the ideas with a minimum of cohomological machinery in an attempt to make them as widely accessible as possible. We give however few complete proofs and refer the reader to [RV] (a more technical and in depth presentation) for what is lacking here.

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I. – Generalities

1. Mahler's measure

When $n = 1$ Jensen's formula gives the identity

$$M(P) = |a_0| \prod_{|\alpha_\nu|>1} |\alpha_\nu| \,,$$

where $P(x) = a_0 \prod_{\nu=1}^{d}(x - \alpha_\nu)$, and it is in fact in this guise that this quantity first seems to have arisen in an article by D. H. Lehmer in the early 1930's [Le]. Lehmer's primary interest was the discovery of new large primes and for this end he considered, after Pierce [Pi], the factors of the integers $\Delta_n = \prod_{\nu=1}^{d}(\alpha_\nu^n - 1)$ associated to a monic polynomial $P \in \mathbb{Z}[x]$. Here Pierce and Lehmer were following the old tradition of restricting the search for primes to numbers of a special kind; indeed, for $P = x - 2$ we have $\Delta_n = 2^n - 1$, the numbers considered by Mersenne.

The prime factors of Δ_n must satisfy certain congruences modulo n and are quite restricted. Factoring Δ_n is therefore much easier than factoring a random number of the same size. Pierce gives the following example.

$$\frac{\Delta_{61}}{\Delta_1} = 4459734401, \qquad P = x^3 + x + 1,$$

and we may test it is prime by means of 119 trial divisions as opposed to the 6655 we would need if we did not know its special form. In the same vein, Lehmer shows that

$$\frac{\Delta_{107}}{\Delta_1} = 323351425103273, \qquad P = x^3 - x - 1,$$

is prime. Incidentally, it is interesting to reflect on what was considered a large prime number in the 20's and 30's.

To maximize the savings we would need to choose polynomials such that Δ_n remains relatively small for large n. It is easy to check, as Lehmer does, that

Δ_n grows with n roughly like $M(P)^n$. Naturally, Lehmer then searched for monic polynomials $P \in \mathbb{Z}[x]$ with a small value of $M(P)$ (other than the minimum possible $M(P) = 1$ attained by products of cyclotomic polynomials, which give rise to uninteresting sequences Δ_n).

In what must have been a fantastic computational tour de force Lehmer found a polynomial of degree 10 with $M(P) = 1.17628\ldots$, and writes "All efforts to find a better equation of degree 12 and 14 have been unsuccessful." Remarkably, this value of $M(P)$ still stands today as the smallest known despite many further intensive searches. In fact, it is not even known if there is a minimum value for $M(P) > 1$. We refer the reader to [Bo1-3] for further details and bibliography.

Mahler introduced his measure in the early 60's [Ma] in order to prove inequalities between various quantities measuring the size of a polynomial and its factors. These inequalities are useful in the theory of transcendental numbers (see for example [Ma], [Ph], and [So]), where $M(P)$ enters as one of several possible height functions.

Mahler's measure also appears as the topological entropy of certain dynamical systems [LSW] and in connection with the spectra of certain discrete differential operators [Sa].

2. Special values of L-functions

The prototypical example of what we would like to discuss is given by Dirichlet's class number formula. To simplify the exposition we will only consider the case of a real quadratic field $F \subset \mathbb{R}$, where the group of units $\mathcal{O}_F^\star$ is of rank one. First, let us recall that, in general for any number field K, the zeta function of K, $\zeta_K(s)$, has an analytic continuation to all s (except for a simple pole at $s = 1$), satisfies a functional equation as $s \mapsto 1 - s$, and has a zero at $s = 0$ of order the rank of the finitely generated group $\mathcal{O}_K^\star$.

Dirichlet's class number formula implies the following

$$(*) \qquad \zeta_F'(0) \sim_{\mathbb{Q}^\star} \log(|\epsilon|), \qquad \epsilon \in \mathcal{O}_F^\star, \quad \epsilon \neq \pm 1$$

where for $a, b \in \mathbb{C}^\star$

$$a \sim_{\mathbb{Q}^\star} b \quad \Longleftrightarrow \quad a = rb \text{ for some } r \in \mathbb{Q}^\star.$$

The ingredients in this formula are:

(1) an arithmetic/geometric object $\mathcal{O}$ (the ring of integers $\mathcal{O}_F$ of the real quadratic field F),
(2) an associated finitely generated abelian group of rank one $K(\mathcal{O})$ (the group of units $\mathcal{O}_F^\star$),
(3) an associated L-function $L(\mathcal{O}, s)$ with a simple zero at $s = 0$ (the zeta function $\zeta_F(s)$), and
(4) a non-zero homomorphism $r : K(\mathcal{O}) \longrightarrow \mathbb{R}$ (the function $\log(|\cdot|)$).

With this setup (*) becomes

$$(**) \qquad L'(\mathcal{O},0) \sim_{\mathbb{Q}^\star} r(\alpha), \qquad \alpha \in K(\mathcal{O}) \setminus K(\mathcal{O})_{\mathrm{tor}},$$

where $K(C)_{\mathrm{tor}}$ is the torsion subgroup of $K(C)$. (Note that in general, r necessarily vanishes on torsion elements as the additive group $\mathbb{R}$ has no torsion.)

We expect other lists of objects (1)-(4) to also satisfy (**). First let us notice some compatibilities in (*). The zeta function ζ_F is defined by an Euler product

$$\zeta_F(s) = \prod_p F_p(s)^{-1}, \qquad \mathrm{Re}\,(s) > 1,$$

where

$$F_p(s) = \prod_{\mathcal{P}|p} (1 - \mathbb{N}\mathcal{P}^{-s})$$

is the Euler factor at p. For primes p unramified in $F/\mathbb{Q}$ we have

$$F_p(s)|_{s=1} = \frac{\#(\mathcal{O}_F/p\mathcal{O}_F)^\star}{p^2}.$$

Note that $s=1$ corresponds to $s=0$ under the functional equation and that we may interpret $(\mathcal{O}_F/p\mathcal{O}_F)^\star$ as $K(\mathcal{O} \bmod p)$, that is, as the group K of the reduction of $\mathcal{O}$ modulo p.

The other example we will discuss in this paper is that of an elliptic curve E defined over $\mathbb{Q}$. We will take a Néron model $\mathcal{E}$ of E over $\mathbb{Z}$ as $\mathcal{O}$, $L(E,s)$ as its corresponding L-function $L(\mathcal{O},s)$, and $K_2(\mathcal{E})$, a K-theory group to be described below, as $K(\mathcal{O})$. In this case (**) is then the conjecture of Bloch and Beilinson for elliptic curves.

II. – The conjecture of Bloch–Beilinson

In this chapter we give a short, low-brow introduction to the conjectures of Bloch and Beilinson for elliptic curves. We have benefited greatly (and borrowed freely) from a number of sources including [Be], [CCGLS], [De], [Gi], [MS], [Ram], [Sch] to which we refer the reader for further details.

3. K_2 **of fields** In order to describe $K_2(\mathcal{E})$ we will proceed in stages. First, we define $K_2(F)$ for a field F. Our main reference will be Milnor's book [Mi].

In its original definition $K_2(F)$ is, loosely speaking, the group of non-trivial relations satisfied by elementary matrices of arbitrary size with entries in F [Mi §5]. By a theorem of Matsumoto [Mi §11], however, it can also be described as the universal target of symbols on F. More precisely, a *symbol* on F is a bimultiplicative map

$$c : F^\star \times F^\star \longrightarrow A$$

to an abelian group A (here written multiplicatively) such that

$$c(x, 1-x) = 1, \qquad x \neq 0, 1.$$

By Matsumoto, there is a (universal) symbol

$$\begin{array}{ccc} F^\star \times F^\star & \longrightarrow & K_2(F) \\ (x,y) & \mapsto & \{x,y\} \end{array}$$

such that composition gives a bijection

$$\left\{ \begin{array}{c} \text{homomorphism} \\ K_2(F) \longrightarrow A. \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{symbol on } F \\ \text{with values in } A \end{array} \right\}.$$

In other words, any symbol c on F with values in A is of the form $c(x,y) = \phi(\{x,y\})$ for a unique homomorphism $\phi : K_2(F) \longrightarrow A$.

Examples. 1) If $F = \mathbb{Q}_p$ then the *Hilbert symbol*, with values in ± 1, is defined as

$$(a,b)_p = \begin{cases} 1 & \text{if } x^2 - ay^2 - bz^2 = 0 \text{ has a solution in } \mathbb{Q}_p \\ -1 & \text{otherwise} \end{cases}$$

2) Given a discrete valuation v on F with maximal ideal $\mathcal{M}$ and residue field k, Tate defined the *tame symbol* at v

$$(x,y)_v \equiv (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \mod \mathcal{M}.$$

Note that in particular $(x,y)_v = 1$ if $v(x) = v(y) = 0$.

We will let the corresponding map be

$$\lambda_v : K_2(F) \longrightarrow k^\star.$$

3) The analogue of 1) for $F = \mathbb{R}$ is the symbol

$$(x,y)_\infty = \begin{cases} -1 & \text{if } x, y < 0 \\ +1 & \text{otherwise.} \end{cases}$$

4) If F is a finite field then $K_2(F) = 1$; i.e. all symbols on F are trivial.

4. The regulator We will eventually only consider elliptic curves defined over $\mathbb{Q}$ but for now let C be an arbitrary compact Riemann surface and let $\mathbb{C}(C)$ be its field of rational functions. Let $f \in \mathbb{C}(C)^\star$ and let S be the set of its zeros and poles. Then, locally in $C \setminus S$, we can define a branch of $\log f$, which gives rise to a global, closed differential 1-form on $C \setminus S$

$$d \log f = \frac{df}{f} = d \log |f| + i\, d \arg f.$$

Given two non-zero functions $f, g \in \mathbb{C}(C)^\star$ we define

$$\eta(f,g) = \log|f|\, d\arg g - \log|g|\, d\arg f, \qquad f,g \in \mathbb{C}(C)^\star,$$

a real C^∞ differential 1-form on $C \setminus S$, where $S \subset C$ is the set of zeros and poles of f and g. The following properties of $\eta(f,g)$ are easy to check

$$\begin{aligned} \eta(g,f) &= -\eta(g,f) \qquad f,g \in \mathbb{C}(C)^\star \\ \eta(f_1 f_2, g) &= \eta(f_1,g) + \eta(f_2,g), \qquad f_1,f_2,g \in \mathbb{C}(C)^\star \end{aligned}$$

Also, η is closed since

$$0 = \operatorname{Im}\left(\frac{df}{f} \wedge \frac{dg}{g}\right) = d\eta.$$

Given a finite set $S \subset C$ consider the first homology group $H_1(C \setminus S, \mathbb{Z})$ and let $H^1(C \setminus S, \mathbb{R})$ be the real vector space of linear forms $\xi : H_1(C \setminus S, \mathbb{Z}) \longrightarrow \mathbb{R}$. Given two finite sets $S \subset S' \subset C$ there is a natural injection $H^1(C \setminus S, \mathbb{R}) \longrightarrow H^1(C \setminus S', \mathbb{R})$, which we will regard as an inclusion. For a closed loop γ in $C \setminus S$ we let $[\gamma]$ be its homology class.

Since $\eta(f,g)$ is closed, we may associate to it an element of $H^1(C \setminus S, \mathbb{R})$, where S is any finite set containing the zeros and poles of f and g, by integration. More precisely, for γ a closed path in $C \setminus S$ the map

$$\gamma \mapsto \int_\gamma \eta(f,g)$$

only depends on the homology class of γ in $H_1(C \setminus S, \mathbb{Z})$ and hence determines an element $\overline{r}(f,g) \in H^1(C \setminus S, \mathbb{R})$.

A point $w \in C$ defines a valuation v on $\mathbb{C}(C)$. To simplify the notation we will let $(\cdot,\cdot)_w$ (and λ_w) denote the tame symbol given by v. The point w also determines a linear form $\operatorname{Res}_w : H^1(C \setminus \{w\}, \mathbb{R}) \longrightarrow \mathbb{R}$, the residue map, as follows. Let $\phi : U \longrightarrow \mathbb{C}$ be a local chart in C with $w \in U$ and $\phi(w) = 0$ and let γ be the preimage by ϕ of a small circle in $\mathbb{C}$ centered at zero and oriented counterclockwise. Then

$$\begin{array}{rccc} \operatorname{Res}_w : & H^1(C \setminus \{w\}, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \xi & \mapsto & \frac{1}{2\pi}\,\xi([\gamma]) \end{array}.$$

Since the construction is local, Res_w actually gives a linear form $H^1(C \setminus S, \mathbb{R}) \longrightarrow \mathbb{R}$ for any finite set S, which is identically zero if w is not in S).

The differential form $\eta(f,g)$ form is related to the tame symbol as follows.

Lemma. *Let $w \in C$, $f,g \in \mathbb{C}(C)^\star$, and $S \subset C$ a finite set containing the zeros and poles of f and g. Then*

$$\operatorname{Res}_w \eta(f,g) = \log|(f,g)_w|.$$

Proof sketch. Both sides are bimultiplicative and skew-symmetric hence it is enough to check the cases: $(v(f), v(g)) = (0,0), (0,1), (1,1)$. For example, in the second case after taking coordinates we must verify that

$$\log|f(0)| = \frac{1}{2\pi i}\int_C \log|f|\,\frac{dz}{z} - \frac{1}{2\pi}\int_C \log|z|\,d\arg f,$$

where C is a circle around the origin with small radius oriented counterclockwise. By letting the radius of C go to zero we see that the second integral vanishes. The first integral equals the left hand side by Jensen's formula. $\square$

Remark Applying Stokes theorem, the lemma gives Weil's reciprocity law

$$\sum_{w\in C}(f,g)_w = 0$$

(compare [GH, p. 242]).

Now comes the crucial point.

Corollary. *Let $f \in \mathbb{C}(C)$ with $f \neq 0, 1$ and $S \subset C$ the finite set of zeros and poles of f and $1-f$. Then*

$$\eta(f, 1-f) = 0 \quad \text{in } H^1(C \setminus S, \mathbb{R}).$$

Proof. First assume that $C = \mathbb{P}^1$ and $f = z$ with $\mathbb{C}(C) = \mathbb{C}(z)$. Then the result follows from the lemma and the fact that $(\cdot,\cdot)_w$ is a symbol.

In the general case the claim follows by using the map $f : C\setminus S \longrightarrow \mathbb{P}^1\setminus\{0,1,\infty\}$ to reduce to the previous case. $\square$

Given a finite set $S \subset C$ we define

$$K_{2,S}(C) = \bigcap_{w\notin S} \ker\lambda_w \subset K_2(\mathbb{C}(C)).$$

(This notation is not standard.) By the preceding lemma and corollary, $(f,g) \mapsto \overline{r}(f,g)$ extends linearly to a map

$$\overline{r}:\ K_{2,S}(C) \longrightarrow H^1(C \setminus S, \mathbb{R}).$$

We may summarize neatly what we have proved in the following.

Proposition. *The diagram*

$$K_{2,S}(C)@ > \overline{r} >> H^1(C \setminus S, \mathbb{R})$$

$$@V\lambda_w VV@V \operatorname{Res}_w VV$$

$$\mathbb{C}^\star @ > \log|\cdot| >> \mathbb{R}$$

is commutative for every $w \in S$.

Note that

$$\bigcup_{S \subset C} K_{2,S}(C) = K_2(\mathbb{C}(C))$$

since $(f,g)_w = 1$ for all but finitely many $w \in C$. Hence, we could take direct limits over all $S \subset C$ and obtain a true symbol on $\mathbb{C}(C)$ (i.e. a map defined on all of $K_2(C)$), but we will not need this fact.

Finally, for E an elliptic curve defined over $\mathbb{R}$, let $c_0 \in H_1(E,\mathbb{Z})$ be the class of the cycle determined by the connected component of its real points (with some orientation) and define

$$\begin{array}{rccc} r: & K_{2,S}(C) & \longrightarrow & \mathbb{R} \\ & \alpha & \mapsto & \frac{1}{2\pi}\,\overline{r}(\alpha)(c_0). \end{array}$$

5. An example To illustrate the previous discussion, let us consider $C = \mathbb{P}^1, f = P(z)$ and $g = z$, where $P \in \mathbb{C}[z]$ is a polynomial and z is a generator of $\mathbb{C}(C)$. Let S be the set of zeros P and $S' = S \cup \{0, \infty\}$. Then, $\{P, z\}$ is an element of $K_{2,S'}(C)$ and $\overline{r}(P,z)$ an element of $H^1(C \setminus S', \mathbb{R})$. Note that $\{P,z\} \in K_{2,S}(C)$ if and only if P is monic and $P(0) = 1$.

On the other hand, let T be the circle $|z| = 1$ in C oriented counterclockwise. If P does not vanish on T then $[T]$ gives a class in $H_1(C \setminus S', \mathbb{Z})$ and we have

$$m(P) = \tfrac{1}{2\pi}\,\overline{r}(P,z)([T]),$$

since $\log|z|$ vanishes identically on T.

Let us now make a preliminary connection with values of L-series. Consider the following family of polynomials

$$P_k(x) = x^2 - kx + 1,$$

depending on a parameter $k \in \mathbb{C}$. If k is real and $|k| > 2$ then the roots of P_k, $S_k = \{\epsilon_k, \epsilon_k^{-1}\}$ with say $|\epsilon_k| > 1$, are real, $\{P_k, z\} \in K_{2,S_k}(C)$, and $\overline{r}(P_k, z) \in H^1(C \setminus S_k, \mathbb{R})$. In this range of k we also have that P_k does not vanish on the unit circle T. Hence

$$\log|\epsilon_k| = m(P_k) = \tfrac{1}{2\pi}\,\overline{r}(P_k,z)([T]), \qquad k \in \mathbb{R}, \quad |k| > 2.$$

If k is an integer with $|k| > 2$ then $F_k = \mathbb{Q}(\epsilon_k)$ is a real quadratic field and by Dirichlet's class number formula (see n$^{\text{o}}$ 2)

$$\zeta'_{F_k}(0) \sim_{\mathbb{Q}^\star} m(P_k), \qquad k \in \mathbb{Z}, \quad |k| > 2.$$

As we shall see, Boyd's numerical examples are, in a sense, a generalization of this fact to two variable polynomials P_k determining elliptic curves.

6. $K_2(E)$ **and** $K_2(\mathcal{E})$ Let now E be an elliptic curve defined over $\mathbb{Q}$ and $\mathcal{E}$ a Néron model over $\mathbb{Z}$. Let us fix an algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$. As in n° 4 we define for every finite set $S \subset E(\overline{\mathbb{Q}})$

$$K_{2,S}(E) = \bigcap_{w \notin S} \ker \lambda_w \subset K_2(\mathbb{Q}(E)),$$

where $\mathbb{Q}(E)$ is the field of rational functions on E and w runs through all points in $E(\overline{\mathbb{Q}}) \setminus S$. (Again, this notation is not standard.) Note that the inclusion $\mathbb{Q}(E) \hookrightarrow \mathbb{C}(E)$ yields a homomorphism $K_2(E) \longrightarrow K_2(E/\mathbb{C})$ and $K_{2,S}(E)$ is just the pull-back of $K_{2,S}(E/\mathbb{C})$ by this map since non-algebraic points of $E/\mathbb{C}$ induce the trivial valuation on $\mathbb{Q}(E)$. (Here $E/\mathbb{C}$ denotes E viewed as a Riemann surface.) By composition the regulator of n° 4 now gives a map $\overline{r} : K_{2,S}(E) \longrightarrow H^1(E \setminus S, \mathbb{R})$ and correspondingly $r : K_{2,S}(E) \longrightarrow \mathbb{R}$.

Rather than defining the group $K_2(E)$ itself, which would take us to far afield, we will appeal to the following property it satisfies: there is an injection

$$K_2(E) \otimes \mathbb{Q} \hookrightarrow K_{2,\emptyset} \otimes \mathbb{Q}.$$

In other words, $K_2(E)$ may be regarded, up to torsion, as the subgroup of $K_2(\mathbb{Q}(E))$ where all tame symbols λ_w for $w \in E(\overline{\mathbb{Q}})$ vanish.

Similarly, $K_2(\mathcal{E})$ may be viewed, up to torsion, as the subgroup of $K_2(E)$ (hence also of $K_{2,\emptyset}(E)$) of elements satisfying certain conditions at the primes of split-multiplicative reduction of E. Note that since the regulator r vanishes on torsion elements the value $r(\alpha)$ for $\alpha \in K_2(\mathcal{E})$ makes sense.

7. The conjecture of Bloch–Beilinson

Conjecture. *(Bloch–Beilinson) Let E be a modular elliptic curve defined over $\mathbb{Q}$ and $\mathcal{E}$ a Néron model of E. Then*

$$L'(E,0) \sim_{\mathbb{Q}^\star} r(\alpha), \qquad \alpha \in K(\mathcal{E}) \setminus K(\mathcal{E})_{\mathrm{tor}}. \tag{$?_2$}$$

(The assumption of modularity, an essentially academic one in light of the work of Wiles, is not really necessary as we could just as well formulate the conjecture in terms of $L(E,2)$ by means of the functional equation; as usual, the version at $s=0$ is cleaner.)

As in the case of Dirichlet's class number formula there are internal compatibilities in the conjecture. The L-function of E is given by

$$L(E,s) = \prod_p F_p(s)^{-1}, \qquad \mathrm{Re}\,(s) > 3/2,$$

where for primes p of good reduction

$$F_p(s) = (1-\alpha_p p^{-s})(1-\beta_p p^{-s}), \qquad \alpha_p\beta_p = p, \qquad \alpha_p+\beta_p = a_p = p+1-\#E(\mathbb{Z}/p\mathbb{Z}).$$

As a consequence of a theorem of Bass and Tate [Ta] the kernel of all tame symbols in $K_2(F)$, where F is the function field of the reduction of E modulo p, is finite of order $(1-\alpha_p p)(1-\beta_p p)$. We may formulate this result as follows

$$F_p(s)|_{s=2} = \frac{\#K_{2,S}(E \bmod p)}{p^3};$$

note that $s=0$ corresponds to $s=2$ under the functional equation satisfied by $L(E,s)$. (Compare this discussion with that of n$^{\text{o}}$ 2.)

Finally, note that the primes p of split-multiplicative reduction for E, those appearing in the conditions defining $K_2(\mathcal{E})$, are precisely those for which $F_p(0)=0$.

III. – Boyd's examples

In this chapter, the paper [CCGLS], which was referred to us by Boyd, was of critical importance as it triggered the main ideas.

8. Construction Given a Laurent polynomial $P \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, with x, y indeterminates, we let $\Delta(P)$ be its Newton polygon; i.e. $\Delta(P) \subset \mathbb{R}^2$ is the convex hull of the finitely many points $(m,n) \in \mathbb{Z}^2$ for which the coefficient of $x^m y^n$ in P is non-zero. Notice that $\Delta(P)$ is by construction a convex polygon with vertices in $\mathbb{Z}^2$; we will call such a set a *convex lattice polygon* or *clp* for short. By *lattice points* we will mean points in $\mathbb{Z}^2$. Throughout the rest of the paper we will denote by $\mathbb{T} \subset \mathbb{A}^2$ the special affine open set $\{(x,y) \in \mathbb{C}^2 | xy \neq 0\}$.

Given a clp Δ, it is known [Da] that for a generic Laurent polynomial $P \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with $\Delta(P) = \Delta$ the equation $P=0$ defines an affine curve in $\mathbb{T}$ of genus equal to the number of interior lattice points of Δ. For example, a generic polynomial $P \in \mathbb{C}[x,y]$ of degree $d>0$ has Newton polygon the triangle of vertices $(0,0), (d,0), (0,d)$ and $P=0$ has genus $\frac{1}{2}(d-1)(d-2)$, as is well known. Similarly, a hyperelliptic equation $y^2 = f(x)$ with $f \in \mathbb{C}[x]$ of degree $d>0$ has generically genus $[\frac{1}{2}(d-1)]$, the number of lattice points inside the triangle of vertices $(0,0), (0,2), (d,0)$.

We will use τ to denote a side of a clp Δ and write $\tau < \Delta$. We will always parameterize a side clockwise around Δ and in such a way that $\tau(0), \tau(1), \cdots$ are the consecutive lattice points in τ. To every side we associate a one-variable polynomial

$$P_\tau(t) = \sum_{k=0}^{\infty} c_{\tau(k)} t^k \ \in \mathbb{C}[t], \qquad \tau < \Delta,$$

where

$$P = \sum_{(m,n)\in\mathbb{Z}^2} c_{(m,n)} x^m y^n \ \in \mathbb{C}[x^{\pm1}, y^{\pm1}],$$

(both sums are, naturally, finite). Note that since any side τ has at least two lattice points (namely, its endpoints) all roots of P_τ are non-zero.

Let $P \in \mathbb{C}[x^{\pm1}, y^{\pm1}]$ be irreducible. The equation $P(x,y) = 0$ defines an affine curve C' in $\mathbb{T}$. Let C be the normalization of the projective closure of C'; C is a compact Riemann surface with function field isomorphic to the fraction field of $\mathbb{C}[x^{\pm1}, y^{\pm1}]/(P)$. We view x, y as rational functions on C and let $S \subset C$ be the set of zeros and poles of x and y. The following result is basically contained in [CCGLS§].

Proposition. *Let $P(x,y), x, y, C$ and S be as above, $\Delta = \Delta(P)$ the Newton polygon of P, and*

$$\mathcal{T} = \{\ t \in \mathbb{C}^\star \mid \prod_{\tau<\Delta} P_\tau(t) = 0\ \}.$$

Then

$$(x,y)_w \in \mathcal{T} \qquad \textit{for all } w \in S$$

and the map

$$\begin{aligned} S &\longrightarrow \ \mathcal{T} \\ w &\mapsto (x,y)_w \end{aligned}$$

is surjective.

The proof uses Puiseux expansions to associate valuations to the sides of Δ, cf. [Le] and [CCGLS].

It will be convenient to define one more concept. Let $P \in \mathbb{C}[x^{\pm1}, y^{\pm1}]$ and let $\Delta = \Delta(P)$ be its Newton polygon. We will say that P is *tempered* if the set $\mathcal{T}$ of roots of $\prod_{\tau<\Delta} P_\tau$ consists of roots of unity only. (If the P_τ's are monic and have integer coefficients then this is equivalent to requiring that $m(P_\tau) = 0$ for all $\tau < \Delta$.)

Corollary. *With the notation of the proposition we have*

$$\{x,y\}^N \in K_{2,\emptyset}(C) \quad \textit{for some } N \in \mathbb{N} \qquad \Leftrightarrow \qquad P \textit{ is tempered.}$$

We now turn to Boyd's construction. The corollary gives us a way to produce elements in $K_{2,\emptyset}(E)$, where E is an elliptic curve defined over $\mathbb{Q}$. Indeed, what we need is a tempered Laurent polynomial $P \in \mathbb{Q}[x^{\pm1}, y^{\pm1}]$ such that the affine curve $P = 0$ in $\mathbb{T}$ is birational to E. The simplest way to do this is to consider a clp with only one interior lattice point and assign numbers to the lattice points on its boundary so that the resulting polynomial is tempered. This leaves the coefficient corresponding to the unique interior point free and hence, in fact, we obtain a *family* of tempered polynomials P_k, depending on a parameter k, all with the same Newton polygon. For all but finitely many choices of $k \in \mathbb{Q}$ the equation $P_k(x, y) = 0$ will yield a curve $C_k/\mathbb{Q}$ of genus 1 together with an element $\{x, y\}^N \in K_{2,\emptyset}(C_k)$ (for some $N \in \mathbb{N}$ independent of k). (We say "curves of genus 1" instead of "elliptic curves" as the curves do not necessarily have rational points.) We describe these families of curves in more detail in the next n°.

We should emphasize that though there is not a single instance of an elliptic curve $E/\mathbb{Q}$ for which we know that $K_2(\mathcal{E}) \otimes \mathbb{Q}$ is one-dimensional (or even finite-dimensional!) it is actually quite hard to construct elements in this group. Most previous examples relied on the original method of Bloch involving functions with divisors supported on torsion points of E [BG].

9. Tempered families Let us say that two clp's are *equivalent* if they differ by a translation and/or a change of basis of $\mathbb{Z}^2$. Correspondingly, let us say that two Laurent polynomials in $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ are *equivalent* if they differ by a factor $x^m y^n$ with $m, n \in \mathbb{Z}$ and/or an invertible change of variables $x \mapsto x^a y^b, y \mapsto x^c y^d$ of $\mathbb{T}$ with $a, b, c, d \in \mathbb{Z}$. Clearly, equivalent polynomials have equivalent Newton polygons and define curves in $\mathbb{T}$ which are birational over their field of definition. We may hence restrict ourselves to polynomials up to equivalence.

Our first task is to describe up to equivalence all clp with only one interior lattice point. We will call such a clp *reflexive* (after [Ba]). (In higher dimensions the notion of reflexive polyhedron requires more than just having only one interior lattice point, see [Ba].) It is known [Sc] that there are only finitely many. (More generally, for fixed $m, n \in \mathbb{N}$ there are only finitely many non-equivalent convex lattice polytopes in $\mathbb{R}^n$ with m interior lattice points [He].) In fact, there are only 16 classes and it is not hard to write down representatives (see fig. 1).

Tempered polynomials with relatively prime integer coefficients and fixed Newton polygon have finitely many possibilities for their P_τ's. If the Newton polygon is reflexive then it is easy to check that they should be among the following (up to changes of the form $\pm P_\tau(\pm t)$)

TABLE 1

FIGURE 1. Equivalence classes of reflexive clp's

0	1	2	3	4
1	0	−2	0	1
1	0	−1	0	0
1	0	−1	0	1
1	0	0	0	−1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	0
1	0	1	0	1
1	0	2	0	1
1	1	−1	−1	0
1	1	0	−1	−1
1	1	0	0	0
1	1	0	1	1

0	1	2	3	4
1	1	1	0	0
1	1	1	1	0
1	1	1	1	1
1	1	2	1	1
1	2	0	−2	−1
1	2	1	0	0
1	2	2	1	0
1	2	2	2	1
1	2	3	2	1
1	3	3	1	0
1	3	4	3	1
1	4	6	4	1

(to clarify the notation: the first line in the second column corresponds to $P_\tau(t) = t^2 + t + 1$).

Examples (a) Consider the Hesse family of elliptic curves

$$x^3 + y^3 + z^3 - kxyz.$$

Upon setting $z = 1$ we obtain a tempered family whose Newton polygon is the triangle of vertices $(0,0), (3,0), (0,3)$ (after translation by $(1,1)$).

(b) The family

$$x + \frac{1}{x} + y + \frac{1}{y} - k$$

is tempered with Newton polygon the square of vertices $(0,1), (0,-1), (-1,0), (1,0)$.

(c) Upon setting $z = 1$ in

$$(x-y)(x-z)(y-z) - kxyz$$

we obtain a tempered family with Newton polygon the hexagon of vertices $(1,0), (2,0), (2,1), (1,2), (0,2), (0,1)$ (after translation by $(1,1)$).

We fix the following notation: given a family of polynomials P_k as above and a value of $k \in \mathbb{Q}$ such that the affine curve $P_k = 0$ is non-singular in $\mathbb{T}$ we let $C_k/\mathbb{Q}$ be the smooth projective curve of genus 1 that it determines (after projective closure and normalization) and let E_k be its Jacobian, an elliptic curve defined over $\mathbb{Q}$,

Remark By analogy we may say that $P \in \mathbb{C}[x, 1/x]$ is tempered if the coefficients of its largest and smallest power of x are roots of unity. Then, for example, a polynomial $P \in \mathbb{Z}[x]$ is tempered if and only if its roots are units in $\overline{\mathbb{Q}}$. Any reflexive clp in one dimension is equivalent to the segment $[-1,1]$ and hence, up to equivalence, the only tempered family of polynomials in one variable with a reflexive Newton polygon is that of n° 5.

10. What fits the facts We may now give the general form of question $(?_1)$ of the introduction. Its formulation is due to Boyd and was based largely on the analysis of numerical data.

Conjecture. *With the above notation*

$$(?_3) \qquad B_k = \frac{L'(E_k,0)}{m(P_k)} \in \mathbb{Q}^\star \qquad \textit{for all sufficiently large } k \in \mathbb{Z}.$$

(The requirement that k be sufficiently large guarantees both that $P_k = 0$ is non-singular in $\mathbb{T}$ and that P_k does not vanish on T^2, see n° 11 below.)

In order to relate this question to the Bloch–Beilinson conjecture we need on more result. Let $P \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ be a tempered Laurent polynomial, irreducible over $\overline{\mathbb{Q}}$ and not vanishing on the torus T^2. As in n° 8, let $C/\mathbb{Q}$ be the normalization of the projective closure of the curve in $\mathbb{T}$ determined by $P = 0$ and regard x, y as functions in $\mathbb{Q}(C)$.

Theorem. *With the above notation*

$$m(P) = \tfrac{1}{2\pi}\, \overline{r}(x,y)([\gamma])$$

for some non-trivial cycle $\gamma \in H_1(C,\mathbb{Z})$ fixed by complex conjugation.

The homology class $[\gamma]$ is related to the class of the torus T in $H_2(\mathbb{P}^2 \setminus C, \mathbb{Z})$ by the tube homomorphism (see [Gr]). The identity is a consequence of the dimension 2 analogue of the lemma of n° 4 (see [RV] for more details).

Note that if C is an elliptic curve then $[\gamma]$ is a non-zero integer multiple of c_0 (the class of the connected component of the real points of C, see n° 7). Therefore, $m(P)$ is up to an non-zero integer factor precisely the quantity associated to $\{x, y\}$ appearing in the Bloch–Beilinson conjecture!

Hence all that remains to check is that some multiple of $\{x, y\}$ actually lies in $K_2(\mathcal{E}_k)$, where $\mathcal{E}_k$ is a Néron model of E_k. This we have not been able to confirm, but given the numerical evidence it seems safe to suggest that

$$(?_4) \qquad k \in \mathbb{Z} \qquad \Rightarrow \qquad \{x,y\}^M \in K_2(\mathcal{E}_k), \quad \text{for some } M \in \mathbb{N}.$$

Granting $(?_4)$ then, $(?_3)$ would follow from the Bloch–Beilinson conjecture.

IV. – Modular Mahler Measures

11. Expansions Let P_k be one of the tempered families of n° 9. By translating if necessary, we may assume that the unique lattice point of $\Delta(P_k)$ is the origin. We then have

$$P_k(x,y) = k - P(x,y),$$

for a Laurent polynomial $P \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with no constant term. We will study $m(P_k)$ as a function of the complex parameter k.

Let $\mathcal{K} \subset \mathbb{C}$ be the image of T^2 under $(x,y) \mapsto P(x,y)$. Since T^2 is compact and the map continuous we see that $\mathcal{K}$ is compact. Clearly

$$P_k \text{ vanishes on } T^2 \quad \Leftrightarrow \quad k \in \mathcal{K}.$$

In what follows it will be convenient to use $\lambda = 1/k$ as the parameter rather k. It is not hard to see that $\mathcal{K}$ cannot reduce to the origin. Define $R > 0$ as

$$\frac{1}{R} = \max_{(x,y)\in T^2} |P(x,y)|.$$

Then

$$1 - \lambda\, P(x,y) \ \text{ does not vanish on } \ T^2 \ \text{ for } |\lambda| < R.$$

We define

$$\widetilde{m}(\lambda) = -\log\lambda - \sum_{n=1}^{\infty} \frac{a_n}{n}\lambda^n, \qquad |\lambda| < R, \quad \lambda \notin (-\infty, 0],$$

where $\log\lambda$ is the usual branch of logarithm and

$$\begin{aligned} a_n &= \frac{1}{(2\pi i)^2}\int_{T^2} P(x,y)^n \,\frac{dx}{x}\frac{dy}{y}, \qquad n = 0,1,2,\ldots \\ &= [P(x,y)^n]_0, \end{aligned}$$

where for a Laurent polynomial $Q \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ we let $[Q]_0$ denote its the constant term. Note that $a_0 = 1$ and $a_1 = 0$.

Clearly $\widetilde{m}$ is holomorphic and

$$m(P_k) = \mathrm{Re}\,(\widetilde{m}(\lambda)), \qquad k = \frac{1}{\lambda}, \quad |\lambda| < R.$$

Let

$$\begin{aligned} u_0(\lambda) &= \frac{1}{(2\pi i)^2}\int_{T^2} \frac{1}{1-\lambda P(x,y)}\frac{dx}{x}\frac{dy}{y} \\ &= \sum_{n=0}^{\infty} a_n\lambda^n, \qquad |\lambda| < R. \end{aligned}$$

Then

$$\widetilde{m}(\lambda) = -\log\lambda - \int_0^\lambda (u_0(t)-1)\frac{dt}{t}.$$

12. Computing the coefficients Here is a way of computing the coefficients a_n in the expansion of $\widetilde{m}$ and u_0 which is often useful in practice. Write

$$P(x,y) = \sum_{j=1}^N c_j\, x^{r_j} y^{s_j}, \qquad c_j \neq 0, \qquad j = 1, \dots, N$$

and let

$$M = \begin{pmatrix} r_1 & r_2 & \dots & r_N \\ s_1 & s_2 & \dots & s_N \end{pmatrix} \in \mathbb{Z}^{2\times N}.$$

Then by the multinomial theorem

$$a_n = \sum_{\substack{w=(w_1,\dots,w_N)\in\mathbb{Z}_{\geq 0}^N \\ w_1+\cdots+w_N=n \\ Mw=0}} \frac{n!}{w_1!\cdot\cdots\cdot w_N!}\cdot c_1^{w_1}\cdot\cdots\cdot c_N^{w_N}.$$

In particular, a_n only depends on the kernel $\ker M$ of M. Note that in the situation of n$^{\rm o}$ 9 (the only one we will consider) M always has rank 2 and hence $\ker M$ has rank $N-2$, where N is the number of non-zero monomials in P.

Example. The simplest case of the formula for a_n is when P has exactly 3 non-zero monomials (the minimum possible) and hence $\ker M$ has rank 1. In this case the Newton polygon of the family is a reflexive triangle, of which there are 5 up to equivalence (see fig. 1). The corresponding functions u_0 are hypergeometric. We have the following three possibilities (unlisted coefficients a_n are zero):

TABLE 2

generator of $\ker M$	u_0	coefficients
$(1,1,1)$	$F(\frac13,\frac23;1;\ (-1)^\nu 3^3\lambda^3)$	$a_{3n} = (-1)^{\nu n}\frac{(3n)!}{n!^3}$
$(1,1,2)$	$F(\frac14,\frac34;1;\ (-1)^\nu 4^4\lambda^4)$	$a_{4n} = (-1)^{\nu n}\frac{(4n)!}{n!^2\,(2n)!}$
$(1,2,3)$	$F(\frac16,\frac56;1;\ (-1)^\nu 2^4 3^3\lambda^6)$	$a_{6n} = (-1)^{\nu n}\frac{(6n)!}{n!\,(2n)!\,(3n)!}$,

where F is the standard hypergeometric function and $\nu = 0$ or 1.

Theses cases arise as follows (it is a general fact about reflexive simplices in any dimension [Ba]). If $w = (w_1, w_2, w_3)$ with $w_j \in \mathbb{Z}_{\geq 0}$ generates $\ker M$ let $d = w_1 + w_2 + w_3$ and $d_j = d/w_j$ then

$$\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = 1, \qquad d_j \in \mathbb{Z}_{\geq 0}, \quad j = 1, 2, 3.$$

The solutions to this equation in positive integers are, respectively

$$(3,3,3), \qquad (2,4,4), \qquad (2,3,6).$$

(The degenerate case $(2,2,\infty)$ could be thought to correspond to the polynomials $x+1/x-k$ of n° 5.)

Let us mention that examples with other Newton polygons may also yield coefficients a_n given as products of factorials (and hence u_0 hypergeometric). For example, for n° 9 (b) and (c) $a_n = 0$ for n odd and

$$a_{2n} = \binom{2n}{n}^2, \qquad \sum_{j=0}^{n} (-1)^j \binom{2n}{j}^3 = (-1)^n \frac{(3n)!}{(n!)^3},$$

respectively (the last equality is an identity due to Dixon).

13. Modular expansions In the above examples u_0 turned out be a hypergeometric function, which satisfies a linear second order differential equation. In fact, this is true in all cases; u_0 satisfies a differential equation with only regular singularities, the Picard–Fuchs differential equation, of the form

$$(\mathbf{PF}) \qquad A\frac{d^2u_0}{d\lambda^2} + B\frac{du_0}{d\lambda} + Cu_0 = 0,$$

where A, B, C are polynomials in λ. The reason is that the integral defining $u_0(\lambda)$ implies (see [Gr]) that it is a period of a holomorphic differential on the curve defined by $1 - \lambda P(x,y) = 0$. In our case is not hard to see that there must be a second solution u_1 of the form

$$u_1(\lambda) = u_0(\lambda)\log\lambda + v_1(\lambda), \qquad |\lambda| < R,$$

with v_1 holomorphic and $v_1(0) = 0$.

We define

$$\tau = \frac{1}{2\pi i}\frac{u_1}{u_0}, \qquad q = e^{2\pi i\tau} = \lambda + \cdots .$$

A loop around $\lambda = 0$ takes τ to $\tau + n$ for some $n \in \mathbb{Z}$ and hence fixes q. It follows that we may locally invert q and write $\lambda(\tau) = q + \cdots$ as a power series in q. We let

$$c(\tau) = u_0(\lambda(\tau)) = 1 + \cdots$$
$$e(\tau) = c\frac{q\,d\lambda/dq}{\lambda} = 1 + \sum_{n=1}^{\infty} e_n q^n.$$

The functions λ, c, e behave like (meromorphic) modular forms for the monodromy group of (**PF**) of weights $0, 1, 3$ respectively. Often we may relate them to usual modular forms (we give some examples below).

Finally, we obtain the following expression for $\widetilde{m}$ as a function of τ. Note that $\lambda = 0$ corresponds to $\tau = i\infty$. The change of variables $t = \lambda(\tau)$ yields the following.

Theorem. *With the above notation we have, locally around $\tau = i\infty$,*

$$\widetilde{m}(\lambda(\tau)) = -2\pi i\tau - \sum_{n=1}^{\infty} \frac{e_n}{n} q^n, \qquad q = e^{2\pi i\tau}.$$

14. Example 1. Let us consider the Hesse family of elliptic curves (see n° 9 (a)). The corresponding Newton polygon is a reflexive triangle corresponding to the first row of table 2. For this example it will be more convenient to formulate things in terms of the parameter $\mu = \lambda^3$ instead of λ. (A simple change of variables shows that $m(x^3 + y^3 + 1 - kxy)$ only depends on k^3.) We find

$$\begin{aligned} u_0(\mu) &= \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} \mu^n \\ &= F(\tfrac{1}{3}, \tfrac{2}{3}; 1; 3^3\mu), \end{aligned}$$

which satisfies the differential equation

$$\mu(27\mu - 1)\frac{d^2 u_0}{d\mu^2} + (54\mu - 1)\frac{du_0}{d\mu} + 6u_0 = 0;$$

a second solution around $\mu = 0$ is

$$u_1(\mu) = u_0(\mu)\log\mu + 15\mu + 513/2\mu^2 + 5018\mu^3 + \cdots.$$

In this case μ, c, e are modular of level 3:

$$\begin{aligned} c &= 1 + 6\sum_{n=1}^{\infty}\sum_{d|n} \chi(d)\, q^n, \\ e &= 1 - 9\sum_{n=1}^{\infty}\sum_{d|n} \chi(d) d^2\, q^n, \\ \mu &= \tfrac{1}{27}(1 - \frac{e}{c^3}) = q - 15q^2 + 171q^3 - 1679q^4 + \cdots, \end{aligned}$$

where $\chi = \left(\frac{d}{3}\right)$. In fact, μ is a Hauptmodul for the subgroup of $SL_2(\mathbb{Z})$ with fundamental domain $\mathcal{F}$ formed by the geodesic triangle in the upper-half plane of vertices $i\infty, 0, (1 + i/\sqrt{3})/2$ and its reflection along the imaginary axis. The value of $k^3 = 1/\mu$ at these vertices is respectively $\infty, 27, 0$.

The the theorem of n° 11 yields

$$\widetilde{m}(\mu) = \tfrac{1}{3}[\, -2\pi i\tau + 9\sum_{n=1}^{\infty}\sum_{d|n} \chi(d) d^2\, \frac{q^n}{n}\,],$$

and provides an analytic continuation for $\widetilde{m}$ to a cut μ-plane. The compact region $\mathcal{K}$ in this case consists of an hypocycloid with vertices at the roots of $k^3 - 27$. For every

$k \in \mathbb{C}\backslash\mathcal{K}$ we have $\operatorname{Re}[\widetilde{m}(1/k^3)] = m(x^3+y^3+1-kxy)$ since both sides are harmonic and agree on a neighborhood of $k=\infty$. In the interior of $\mathcal{K}$, however, the functions do not agree. In fact, at $k=0$, as we will see shortly, $\widetilde{m}=0$, whereas by the formula of Smyth quoted in the introduction, $m(x^3+y^3+1) = m(x+y+1) = L'(\chi,-1) \neq 0$. On the other hand, the functions do agree, by continuity, on the boundary of $\mathcal{K}$. In particular, if we let τ approach the cusp 0 within the fundamental domain (so that k approaches 3) we find that

$$\begin{aligned} m(x^3+y^3+1-3xy) &= 3\lim_{y\to 0}\left[\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)\, d^2\,\frac{e^{-2\pi n y}}{n}\right] \\ &= 3[\zeta(s)L(\chi,s-2)]\big|_{s=1} \\ &= 3L'(\chi,-1). \end{aligned}$$

(The second equality follows for example by shifting the line of integration in

$$\int_{\operatorname{Re} w=\eta_0} t^{-w}\Gamma(w)\zeta(s+w)L(\chi,s+w-2)\,dw\ , \qquad \eta_0 \gg 0.)$$

The above identity can also be proven directly as was pointed out to us by Boyd. Over $\mathbb{Q}(w)$, with w a primitive cubic root of unity,

$$x^3+y^3+1-3xy = (x+y+1)(x+w^2y+w)(x+wy+w^2)$$

and each term on the right hand side has measure $L'(\chi,-1)$ by Smyth's result.

Since e is an Eisenstein series we find, after some calculation, that

$$\begin{aligned} m(P_k) &= \tfrac{1}{3}\operatorname{Re}\Big[\,-2\pi i\tau + 9\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^2\,\frac{e^{2\pi i n\tau}}{n}\,\Big] \\ &= \operatorname{Re}\Big[\,\frac{3^3\sqrt{3}y}{4\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \chi(n)\frac{1}{(m3\tau+n)^2(m3\bar{\tau}+n)}\,\Big]\,, \end{aligned} \tag{16}$$

where $\tau = x+iy$ and, as before, $k^3 = 1/\mu(\tau)$ with $\tau \in \mathcal{F}$. (This shows that here that $(?_4)$ actually reduces to a case of Bloch's original conjecture involving the regulator of functions with divisors supported on torsion points.)

The above calculation also allows us to prove $(?_4)$ for $k=-6$. Indeed, in this case $P_k = 0$ determines an elliptic curve $E/\mathbb{Q}$ with complex multiplication by the ring of integers $\mathcal{O}_K$ of $K=\mathbb{Q}(w)$ (w a primitive 3rd root of unity) and $\tau = (1+\sqrt{-3})/2$; E has conductor 27 and minimal model $y^2+y=x^3$. (It is not hard to see that

$$y^2 = x^3 - 27k^2x^2 + 216k(k^3-27)x - 432(k^3-27)^2$$

is a Weierstrass equation for the curve $P_k=0$ over $\mathbb{Q}(k)$.)

By Deuring the L-function of E is $L(\overline{\psi},s)$, where ψ is a Hecke character of K; we find that $\psi((\alpha))=\alpha$ for $\alpha\in\mathcal{O}_K$ with $\alpha\equiv 1 \bmod 3\mathcal{O}_K$. Therefore, $2L(E,2)$ is visibly given by the last sum in (16) with $\tau=(1+\sqrt{-3})/2$ and we obtain

$$L'(E,0)=\tfrac{27}{4\pi^2}L(E,2)=\tfrac{1}{3}m(P_{-6}).$$

Finally, for $k=0$ we find $\tau=(1+i/\sqrt{-3})/2$ and by (15)

$$\begin{aligned}\widetilde{m}(P_0)&=c\sum_{\alpha\in\mathcal{O}_K}\left(\frac{\alpha}{\sqrt{-3}}\right)\frac{1}{\alpha^2\overline{\alpha}}\,, \qquad c>0\\ &=0.\end{aligned}$$

(To see that the sum vanishes change α to $w\alpha$.)

15. Example 2 Let us consider the family $P_k=x+1/x+y+1/y-k$ of the introduction (also n$^{\mathrm{o}}$ 9 (b)); the situation is very similar to the that of the previous example and we will hence be brief. We let $\mu=\lambda^2$ then

$$\begin{aligned}u_0(\mu)&=\sum_{n=0}^{\infty}\binom{2n}{n}^2\mu^n\\ &=F(\tfrac12,\tfrac12;1;2^4\mu),\end{aligned}$$

which satisfies the differential equation

$$\mu(16\mu-1)\frac{d^2u_0}{d\mu^2}+(32\mu-1)\frac{du_0}{d\mu}+4u_0=0;$$

a second solution at $\mu=0$ is

$$u_1(\mu)=u_0(\mu)\log\mu+8\mu+84\mu^2+2960/3\mu^3+\cdots.$$

In this case μ,c,e are modular of level 4:

$$\begin{aligned}c&=1+4\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)\,q^n,\\ e&=1-4\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^2\,q^n,\\ \mu&=\frac{\phi}{c^2}=q-8q^2+44q^3-192q^4+\cdots,\end{aligned}$$

where $\chi(n)=\left(\frac{n}{4}\right)$ and

$$\phi=\sum_{\substack{n\geq1\\ n\,\mathrm{odd}}}\sum_{d|n}d\,q^n=q+4q^3+6q^5+\cdots.$$

In fact, μ is a Hauptmodul for the subgroup of $SL_2(\mathbb{Z})$ with fundamental domain $\mathcal{F}$ formed by the geodesic triangle in the upper-half plane of vertices $i\infty, 0, 1/2$ and its reflection along the imaginary axis. The value of $k^2 = 1/\mu$ at these vertices is respectively $\infty, 16, 0$.

Although not a fact we will need, let us remark that by an identity of Jacobi (see e.g. [WW])

$$\theta^2 = F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 16\mu), \qquad \text{where} \qquad \theta = \sum_{n\in\mathbb{Z}} q^{n^2},$$

and therefore $c = \theta^2$.

As in example 1, e is an Eisenstein series (and hence we are also in the original situation of Bloch involving functions supported on torsion points.) The region $\mathcal{K}$ now consists of the interval $[-4, 4]$ and hence by continuity we find that

$$\begin{aligned} m(x + 1/x + y + 1/y - k) &= \tfrac{1}{2}\mathrm{Re}\,[\, -2\pi i\tau + 4\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^2\,\frac{q^n}{n}\,] \\ &= \mathrm{Re}\,[\,\frac{16y}{\pi^2}\sum_{m,n\in\mathbb{Z}}{}^{'}\chi(n)\frac{1}{(m4\tau+n)^2(m4\overline{\tau}+n)}\,] \end{aligned}$$

for *all* $k \in \mathbb{C}$ where $k^2 = 1/\mu(\tau)$ and $\tau \in \mathcal{F}$. (This identity was previously proved for $k = 1$ by Deninger [De].)

We also find that by taking limits as before that $m(x + 1/x + y + 1/y - 4) = 2L'(\chi, -1)$, which again may be established directly, and that we can prove the identities

$$\begin{aligned} m(x + 1/x + y + 1/y - 4\sqrt{2}) &= L'(A, 0), &\qquad A : y^2 = x^3 - 44x + 112, \\ m(x + 1/x + y + 1/y - \frac{4}{\sqrt{2}}) &= L'(B, 0), &\qquad B : y^2 = x^3 + 4x, \end{aligned}$$

since A, B are CM elliptic curves.

Finally, for $k = 3\sqrt{2}$ we obtain the modular curve $X_0(24)$ and then a theorem of Beilinson applies and we can prove $(?_1)$ in this case (see [MS] and [RV] for more details).

16. Example 3. Finally, we consider an example involving the regulator of functions not supported on torsion points, i.e. where e is not an Eisenstein series. Let $P_k(x, y) = x^2/y - y/x - 1/xy - k$ corresponding to the third row of table 2. With $\mu = \lambda^6$ we have

$$\begin{aligned} u_0(\mu) &= \sum_{n=0}^{\infty}\frac{(6n)!}{(3n)!(2n)!n!}\mu^n \\ &= F(\tfrac{1}{6}, \tfrac{5}{6}; 1; 2^4 3^3\mu), \end{aligned}$$

which satisfies the differential equation

$$\mu(432\mu - 1)\frac{d^2u_0}{d\mu^2} + (864\mu - 1)\frac{du_0}{d\mu} + 60u_0 = 0;$$

a second solution at $\mu = 0$ is

$$u_1(\mu) = u_0(\mu)\log\mu + 312\mu + 77652\mu^2 + 23485136\mu^3 \cdots .$$

We find that we can express μ, c, e in terms of the standard modular forms

$$E_4 = 1 + 240\sum_{n=1}^{\infty}\sum_{d|n} d^3\, q^n, \qquad E_6 = 1 - 504\sum_{n=1}^{\infty}\sum_{d|n} d^5\, q^n$$

as follows

$$\begin{aligned} c &= E_4^{1/4} = 1 + 60q - 4860q^2 + 660480q^3 - 105063420q^4 + \cdots \\ e &= \tfrac{1}{2}(E_4^{3/4} + \frac{E_6}{E_4^{3/4}}) = 1 - 252q + 53244q^2 - 11278368q^3 + 2431713276q^4 \cdots \\ \mu &= \tfrac{1}{864}(1 - \frac{E_6}{E_4^{3/2}}) = q - 312q^2 + 87084q^3 - 23067968q^4 + \cdots . \end{aligned}$$

It is not hard to see that these q-series are indeed in $\mathbb{Z}[[q]]$ as suggested by their first few terms. Notice that in contrast with example 1, the functions c, e, μ now have singularities (at $\tau = (1+\sqrt{-3})/2$ where E_4 vanishes).

Let $\mathcal{F}$ be the domain in the upper-half plane formed the geodesic triangle of vertices $i\infty, 0, (1+\sqrt{-3})/2$ and its reflection along the imaginary axis. The function μ gives a conformal mapping between $\mathcal{F}$ and some domain in the μ-plane. The values of $1/\mu$ at $i\infty, 0, (1+\sqrt{-3})/2$ are respectively $\infty, 432, 0$. As before the theorem of n$^{\text{o}}$ 12 provides an analytic continuation of $\widetilde{m}$ to a cut μ-plane.

As a numerical example, consider $\tau = i \in \mathcal{F}$ at which μ takes the value 864. Then

$$m(x^3 - y^2 - 1 - \sqrt[6]{864}\, xy) = \tfrac{1}{6}[\, 2\pi - \sum_{n=1}^{\infty}\frac{e_n}{n}e^{-2\pi n}\,] = 1.1133053133\cdots ,$$

where as before $e = 1+\sum_{n=1}^{\infty} e_n q^n$. This number seems to equal $\frac{1}{3}L'(E,0)$, where E is the elliptic curve with CM by $\mathbb{Z}[i]$ of minimal model $y^2 = x^3 - 27x$ and conductor 576. Despite the fact that E is a curve with CM we see no way of actually proving this apparent relation between the numbers in this case.

Although e is not an Eisenstein series, by analogy (and inspired by the "mirror symmetry" of physics, which actually informed much of this chapter) we may still consider the numbers r_n such that

$$e_n = \sum_{d|n} r_d\, d^2, \qquad n = 1, 2, 3, \ldots .$$

These identities define the numbers uniquely as one may see using the Möbius inversion formula

$$c_n = \frac{1}{n^2}\sum_{d|n}\mu(n/d)\, e_d,$$

where (here only) μ is the standard Möbius function. A priori we can only assert that the r_n's are rational. Remarkably, it appears that they are actually integers for all n. Here is a short table.

n	e_n	r_n
1	−252	−252
2	53244	13374
3	−11278368	−1253124
4	2431713276	151978752
5	−531387193752	−21255487740
6	117213742465056	3255937602498
7	−26029619407786176	−531216722607876
8	5809495541986361340	90773367805541376
9	−1301648449222030804956	−16069733941012586748
10	292541140545091693518744	2925411405456230806590

The power of d in the definition of r_n (namely 2) is somewhat natural since e is formally of weight 3; also, the analogue of the r_n's defined with a higher power of d are not integers. Notice that, whether $r_n\mathbb{Z}$ for all $n \in \mathbb{N}$ may be formulated purely in terms of classical forms, independently of everything else we have done.

In order to obtain the expansions of this chapter it was not really necessary for the family of polynomials to be tempered. We may ask: (how) is the tempered condition reflected in the expansions? The examples we computed seem to suggest the following

$$\begin{array}{c} P_k \textit{ is tempered} \\ \Updownarrow \\ \textit{there exists an } N \in \mathbb{N} \textit{ such} \\ \textit{that } r_n N^n \in \mathbb{Z} \textit{ for all } n \in \mathbb{N}\ . \end{array} \tag{?$_5$}$$

(Here r_n are the analogue of the numbers defined above.)

16. Final remarks and speculations

(1) The conjectures of Beilinson are more general than those outlined here; as with Dirichlet's class number formula, if the rank in question is $r > 1$ one compares an L-value with the determinant of an $r \times r$ matrix constructed using the regulator map.

(2) Recent work of Bloch and Kato make the conjectures more precise by attributing a meaning to the ratio of the L-value and the regulator (again in analogy to what happens with Dirichlet's class number formula).

(3) Given the formal similarities between the behavior of the coefficients e_n in the tempered case and those appearing in the expansions of "mirror

symmetry", it is tempting to ask whether there is a deeper connection. For example, do the r_n's have a natural interpretation as the number of objects of some sort? Or perhaps they are involved in a theta-lift of the kind considered by Borcherds?

(4) Much of chapters III and IV can be extended to higher dimensions (see [RV]). A particularly nice case is that of families of K3-surfaces, for example, those determined by $x^4 + y^4 + z^4 + w^4 - \lambda xyzw = 0$.

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