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$$1 \to H \to S_4 \to S_3 \to 1$$

defining repn of S_4: D

$$\mathrm{Sym}^2 D$$

$$\left[\begin{aligned} (x_1 + x_2)(x_3 + x_4) &=: y_1 \\ (x_1 + x_3)\ (x_2 + x_4) &=: y_2 \\ (x_1 + x_4)\ (x_3 + x_2) &=: y_3 \end{aligned}\right.$$

defining repn of S_3 acting on the y's

Suppose $x_1, \ldots, x_4$ roots of a polynomial f with gal group S_4

F splitting field of f

$$\begin{tikzcd} L = K(y_1, y_2, y_3) \arrow[r, dash] \arrow[dr, dash, "S_3"'] & F = K(x_1, \ldots x_4) \arrow[d, dash, "S_4"] \\ & K \end{tikzcd}$$

Cardano

$$f = \prod_{i=1}^{4} (X - x_i)$$

$$g = \prod_{i=1}^{3} (X - y_i) \quad \text{cubic resolvent of } f. \quad \in K[X].$$

$$f = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = x^3 + b_2 x^2 + b_1 x + b_0$$

$$\begin{cases} b_2 = -a_2 \\ b_1 = a_1 a_3 - 4a_0 \\ b_0 = 4a_0 a_2 - a_1^2 - a_0 a_3^2 \end{cases}$$

E.g.

$$f = x^4 + x + 1$$

$$g = x^3 - 4x - 1$$

$$z_1 = x_1 x_2 + x_3 x_4$$
$$z_2 = x_1 x_3 + x_2 x_4$$
$$z_3 = x_1 x_4 + x_2 x_3$$

$$\begin{aligned} y_1 &= (x_1 + x_2)(x_3 + x_4) \\ &= x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 \\ &= z_2 + z_3 \end{aligned}$$

V standard repn of S_4

Let's compute: $\Lambda^k V$

	1	6	8	6	3	
	1	(12)	(123)	(1234)	$(12)(34)$	
U	1	1	1	1	1	
U'	1	-1	1	-1	1	
V	3	1	0	-1	-1	
V'	3	-1	0	1	-1	
W	2	0	-1	0	2	
$\Lambda^2 V$	3	-1	0	1	-1	$= V'$
$\Lambda^3 V$	1	-1	1	-1	1	$= U'$

$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left(\chi_V^2(g) - \chi_V(g^2) \right)$$

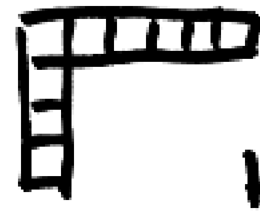
$\binom{3}{2} = 3 = \dim \Lambda^2 V$

$$\chi_{\Lambda^3 V}(g) =$$

$\Lambda^3 V = \det$

g acts like $\det \rho_V(g)$

$= \mathrm{sign}(g)$

We'll see $\Lambda^k V \longleftrightarrow$  Hook diagram

$$\chi_{\Lambda^3 V}(g) = \sum_{i_1 < i_2 < i_3} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3}$$

express in terms of $\sum_i \lambda_i^r = \chi(g^r)$

<u>Newton's fmlas.</u>

$$\prod_i (1 - \lambda_i T) = \exp\left(-\sum_{r \geq 1} N_r \frac{T^r}{r}\right)$$

$$N_r := \sum_i \lambda_i^r$$

$$N_1$$

$$\frac{1}{2!}(N_1^2 - N_2)$$

$$\frac{1}{3!}(N_1^3 - 3N_1 N_2 + 2N_3)$$

$$\vdots$$

$$\chi_{\Lambda^3 V}(g) = \frac{1}{6}\left(\chi_V(g)^3 - 3\chi_V(g)\chi_V(g^2) + 2\chi_V(g^3)\right)$$

V standard repn of S_4

k	0	1	2	3
$\Lambda^k V$	U	V	V'	U'

$\otimes U'$

all irreducible.

<u>Hint</u>. $\lambda_{i_1} \cdots \lambda_{i_k} = \dfrac{\lambda_1 \cdots \lambda_{n-1}}{\lambda_{j_1} \cdots \lambda_{j_{n-k}}}$

$= \det \cdot \bar{\lambda}_{j_1} \cdots \bar{\lambda}_{j_{n-k}}$

$$\varphi := \sum_{g \in G} \alpha(g)\, g \quad \in \mathbb{C}[G]$$

$\alpha : G \longrightarrow \mathbb{C}$

or think of it as an element of $\mathrm{End}(V)$ where V is a repn of G.

<u>Q</u>: When is φ G-linear?

<u>A</u>: α has to be a class function (i.e. constant on conjugacy classes)

<u>Pf</u>

$$\varphi(hv) = \sum_{g\in G} \alpha(g)\, g(hv)$$

$$= \sum_{g\in G} \alpha(hgh^{-1})\, hgh^{-1}(hv)$$

$$= h \cdot \sum_{g\in G} \alpha(hgh^{-1})\, g(v)$$

G-linear $\Leftrightarrow$ $= h\varphi(v)$

when does

$$\sum_{g\in G} \alpha(hgh^{-1})g(v) \overset{?}{=} \sum_{g\in G} \alpha(g)g(v)$$

α class function $\Longrightarrow$ $=$

Must be true for any repn. Take V = regular repn & take $v = 1 \in G$

must have $\alpha(hgh^{-1}) = \alpha(g)$ $\square$

What we did is compute center of $\mathbb{C}[G]$.

$$\varphi h = h\varphi \quad \text{for all } h\in G$$

center of $\mathbb{C}[G] = \{ \sum_g \alpha(g) g \mid \alpha$ is a class fcn $\}$

<u>Cor</u> # irred repn = # conj classes

<u>Pf</u> α class function

$(\alpha, \chi_U) = 0$ all irred U.

$$\varphi = \frac{1}{|G|} \sum_g \bar{\alpha}(g) g \in \mathrm{End}_G(V)$$

V irred. By Schur

$$\varphi = \lambda \, id_V$$

Take trace

$$\underset{\overset{\shortparallel}{0}}{(\alpha, \chi_U)} = tr(\varphi) = \lambda \cdot \dim U$$

$\Rightarrow \lambda = 0 \Rightarrow \varphi = 0$

$\Rightarrow \varphi = 0$ on <u>any</u> representation.

Take the regular repn $\Rightarrow \alpha = 0$ $\square$

Isotypical components

V repn , U irred repn

Consider

$$\varphi := \frac{\dim U}{|G|} \sum_{g \in G} \overline{\chi_U}(g)\, g$$

is in the center of the group ring

(i.e. acts G-linearly on any repn)

$$\varphi : V \longrightarrow V \qquad G\text{-linear}$$

Γ_V irred. Schur

$$\varphi = \lambda\, \cancel{\dim}\, i\, V$$

take trace

$$\lambda \dim V = \frac{\dim U}{|G|} \sum_{g \in G} \overline{\chi_U}(g)\, \chi_V(g)$$

$$= \dim U \, (\chi_U, \chi_V)$$

$$= \begin{cases} \dim U & U \simeq V \\ 0 & U \not\simeq V \end{cases}$$

$$\Rightarrow \quad \lambda = \begin{cases} 1 & U \simeq V \\ 0 & U \not\simeq V \end{cases}$$

V arbitrary

$$V = \bigoplus_{W \text{ irred}} W$$

<u>Claim</u> $\operatorname{Im} \varphi$ is U-isotypical component of V.