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$\dim V = n$

$$\dim \operatorname{Sym}^k V = \binom{n+k-1}{k}$$

$\operatorname{Sym}^k V$ "$=$" homog. polynomials of deg k in n variables

$$k=2 \qquad \binom{n+1}{2} = \frac{(n+1)n}{2}$$

$$\#\{ 1 \le i_1 \le i_2 \le \dots \le i_k \le n \}$$

$$k_j = \#\{ i_l = j \}$$

$$1 \le 1 \le 2 \le 3 \le 3 \le 4$$

$n = \cancel{4}\,5 \qquad k = 6$

1	2	3	4	5
2	1	2	1	0

$$k_1 + k_2 + \dots + k_n = k$$

$$k_j \geqslant 0$$

n bins

For each j put k_j particles in bin j.

empty bin

$$00|0|00|0|$$

To describe bins we need $n-1$ $|$

Total number of 0's & $|$'s $= k+n-1$

We're counting: $n+k-1$ things

k of which are 0

(or $n-1$ of which are 1)

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

$$\sum_{k\geq 0} \dim \operatorname{Sym}^k V \, T^k = \frac{1}{(1-T)^n}$$

$$\operatorname{Sym}^k(V \oplus W) \simeq \bigoplus_{j=0}^{k} \operatorname{Sym}^j V \otimes \operatorname{Sym}^{k-j} W$$

$n = \dim V$

$m = \dim W$

$$\frac{1}{(1-T)^{n+m}} = \frac{1}{(1-T)^n} \cdot \frac{1}{(1-T)^m}$$

Similarly for $\Lambda^k V$

$$(1+T)^n (1+T)^m = (1+T)^{n+m}$$

Categorification...

V, W representation of G

$\mathrm{Hom}(V, W)$

$\simeq V^* \otimes W$

$$V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$$
$$v^* \otimes w \longmapsto (v \longmapsto v^*(v) \cdot w)$$

<u>Pick bases</u> $v_1, \dots, v_n$; $v_1^*, \dots, v_n^*$; $w_1, \dots, w_m$

$$v_j^* \otimes w_i (v_k) = v_j^*(v_k) w_i$$
$$= \delta_{jk} w_i$$

In terms of matrices

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} \end{array}$$

$\{ v_j^* \otimes w_i \}$ basis of $V^* \otimes W$

We extend G-action from $\underset{\text{trivial repn}}{\dim W = 1}$ to arbitrary W.

($\mathrm{Hom}(V, W) \simeq V^*$)

<u>First Projection Fmla</u>

Repn V of G $\quad \rho : G \longrightarrow GL(V)$

$$V^G = \{ v \in V \mid gv = v \ \text{ all } g \in G \}$$

if non-zero it's a subrepn of V

$$V = V^G \oplus W$$

$$W^G = \{0\}_{\text{trivial}}$$

V^G is the "isotypical" component of V

$g \in G$

$\rho(g) : V \to V$

typically $\rho(g)$ is not G-linear

Average over G

$$\varphi := \frac{1}{|G|} \sum_{g \in G} \rho(g) \quad \in \operatorname{End}(V)$$

is G-linear

$$\rho(h)^{-1} \varphi \, \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(h^{-1} g h)$$

$$= \frac{1}{|G|} \sum_{g \in G} \rho(g)$$

$$= \varphi$$

<u>Prop</u> $\varphi : V \to V^G$ is a projection

<u>Pf</u> $v = \varphi(w)$

$$\rho(h)v = \frac{1}{|G|}\sum_{g\in G}\rho(hg)(w)$$

$$= \frac{1}{|G|}\sum_{g\in G}\rho(g)(w)$$

$$= \varphi(w) = v$$

$v \in V^G$

$$\varphi(v) = \frac{1}{|G|}\sum_{g\in G}\rho(g)(v)$$

$$= \frac{1}{|G|}\sum_{g\in G} v$$

$$= v$$

$\operatorname{Im}\varphi = V^G, \quad \varphi\circ\varphi = \varphi \quad \square$

Take trace of φ

$$\operatorname{tr}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(\rho(g))$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

$$= \dim V^G$$

$$\begin{tikzcd}
V = V^G \arrow[d, "id"] & \oplus & W \arrow[d, "\varphi"] \arrow[dr, "0"] & \\
V^G & \oplus & W &
\end{tikzcd}$$

In space of functions $G \to \mathbb{C}$

Define

$$(\alpha, \beta) := \frac{1}{|G|} \sum_{g \in G} \bar{\alpha}(g) \beta(g)$$

In this language

$$\boxed{\dim V^G = (1, \chi_V)}$$

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \bar{\alpha}(g^{-1}) \beta(g^{-1})$$

α, β characters

$$\alpha(g^{-1}) = \overline{\alpha}(g)$$

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta}(g)$$

$\underline{G = S_3}$ $\qquad V$ standard repn

$V \otimes V$

	1	3	2
	1	(12)	(123)
χ_V	2	0	-1
$\chi_{V \otimes V}$	4	0	1

$$(1, \chi_{V \otimes V}) = \frac{1}{6} (4 \times 1 \times 1 + 0 \times 1 \times 3 + 1 \times 1 \times 2) = \frac{6}{6} = 1$$

$(V \otimes V)^G$ is 1-diml.

$\mathrm{Hom}(V, W)^G$

$$\boxed{\begin{aligned} g v^*(g v) &= v^*(v) \\ g v^*(v) &= v^*(g^{-1} v) \end{aligned}}$$

$$\varphi \longleftrightarrow v^* \otimes w$$

$$g\varphi \longleftrightarrow g v^* \otimes g w$$

$$g\varphi(v) = (g v^*)(v) \cdot (g w)$$

$$= v^*(g^{-1} v) \cdot g w$$

$$g\varphi(v) = g\varphi(g^{-1} v)$$

scalar!

$$\varphi(g^{-1} v) = v^*(g^{-1} v)\, w$$

$$g\varphi(g^{-1} v) = v^*(g^{-1} v)\, g w$$

$$\boxed{g\varphi(v) = g\varphi(g^{-1}v)}$$

$g\varphi = \varphi$ all g ?

$\updownarrow$

$\varphi = g\varphi g^{-1}$

$\updownarrow$

φ is G-linear

$$\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$$

$\uparrow$ G-linear maps from V to W

E.g. V, W are irreducible

$$\dim \mathrm{Hom}(V,W)^G = \begin{cases} 1 & V \simeq W \\ 0 & \text{otherw} \end{cases}$$

What is the character of $\mathrm{Hom}(V,W)$?

$V^* \otimes W$

$$\boxed{\bar{\chi}_V \cdot \chi_W = \chi_{\mathrm{Hom}(V,W)}}$$

$$\dim \mathrm{Hom}(V,W)^G = \frac{1}{|G|} \sum_{g \in G} \bar{\chi}_V(g)\chi_W(g)$$

$$= (\chi_V, \chi_W)$$

I.e. V, W are repn of G we have

$$\dim \mathrm{Hom}_G(V,W) = (\chi_V, \chi_W)$$

Inner product $(\ ,\ )$ is completely intrinsic.

<u>Schur's lemma</u> V, W irred.

$$(\chi_V, \chi_W) = \begin{cases} 1 & V \cong W \\ 0 & \text{otherw.} \end{cases}$$

So χ_V's are <u>orthonormal</u>

V irred

<u>Cor</u> - Repn are uniquely determined by their characters

- V is irred iff $(\chi_V, \chi_V) = 1$
- $\#\{V \text{ irred}\} \leq \#$ conj classes
- $(\chi_V, \chi_{reg}) = \chi_V(1) = \dim V$

V irred