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Remark Field of char p, not alg. closed, ring, ...

$$S_2 \circlearrowright \mathbb{Z}^2 = V \qquad e_1 \leftrightarrow e_2$$

$$V^{\pm} = \{\sigma v = \pm v\} \qquad \sigma = (12)$$

$$V^+ = \mathbb{Z}(1,1)$$
$$V^- = \mathbb{Z}(1,-1)$$
$$V^+ \oplus V^- \subsetneq V$$

$$a(1,1) + b(1,-1) = (a+b, a-b) = (c,d)$$

$$\begin{cases} c+d = 2a \\ c-d = 2b \end{cases}$$

Character

$$\rho : G \to GL(V)$$

$$\chi : G \to \mathbb{C}$$
$$g \mapsto Tr(\rho(g))$$

- χ depends only on the isom class of ρ.

- χ a <u>class function</u>

$$\chi(hgh^{-1}) = \chi(g)$$

$$\Uparrow$$

$$\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1}$$

<u>Properties</u>

1) χ class function

2) $\chi_{V \oplus W} = \chi_V + \chi_W$

3) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

$$\rho : G \to Gl(V)$$
$$\rho' : G \to Gl(W)$$

$$\rho \otimes \rho' : G \to Gl(V \otimes W)$$

$$\begin{pmatrix} (\cdot) & (\cdot) & \\ (\cdot) & (\cdot) & \\ & & \end{pmatrix}$$

$\rho'(g)$

$\rho(g)$

<u>Kronecker</u> <u>prodruct</u>

$$\rho(g) = \left(a_{ij} \right)$$

$$\rho'(g) = \left(a'_{ij} \right)$$

$$\rho \otimes \rho'(g) = \begin{pmatrix} a_{11}a'_{11} & & & & \\ & a_{11}a'_{22} & & & \\ & & \ddots & & \\ & & & a_{22}a'_{11} & \\ & & & & \ddots \end{pmatrix}$$

$$\text{trace} = a_{11}\left(a'_{11} + \cdots + a'_{mm} \right) + a_{22}\left(a'_{11} + \cdots + a'_{mm} \right) \\ \vdots \\ = \left(a_{11} + a_{22} + \cdots + a_{nn} \right)\left(a'_{11} + \cdots + a'_{mm} \right)$$

λ $\quad v \quad$ eigenvector for $\rho(g)$

λ' $\quad w \quad$ " $\quad$ " $\quad \rho'(g)$

$\lambda\lambda'$ $\quad v \otimes w \quad$ " $\quad$ " $\quad \rho \otimes \rho'(g)$

$$\sum_{\lambda, \lambda'} \lambda \lambda' = \sum_{\lambda} \lambda \cdot \sum_{\lambda'} \lambda'$$

$$\chi_{V^*} = \overline{\chi_V}$$

$$\rho^*(g) = {}^t\rho(g^{-1})$$

$$\chi_{V^*}(g) = tr(\rho^*(g))$$
$$= tr({}^t\rho(g^{-1}))$$
$$= tr(\rho(g^{-1}))$$

eigenvalues of $\rho(g^{-1}) = \rho(g)^{-1}$
$= (\text{eigenvalues of } \rho(g))^{-1}$
$= (\overline{\ \cdots\ })$

b/c g is finite order $\Rightarrow$ eigenv. are roots of unity.

4) V is a permutation repn.

$\chi_V(g) = \#$ fixed elts of g

Character Table

$G = S_3$

		1	(12)	(123)
trivial	U	1	1	1
sgn	U'	1	-1	1
std	V	2	0	-1

5) $\chi_V(1) = \dim V = \deg \rho$

$S_3 \circlearrowleft \{1,2,3\}$ defining repn

$\chi : 3 \quad 1 \quad 0$

$\chi_V + \chi_{triv} = \chi$

$\Rightarrow \chi_V : 2 \quad 0 \quad -1$

$V \otimes V$

$\chi : \quad 4 \quad 0 \quad 1$

$$\chi = \overset{4}{\chi_U} + \overset{0}{\chi_{U'}} + \overset{1}{\chi_V}$$

$$\Rightarrow \quad V \otimes V \cong U \oplus U' \oplus V$$

<u>Regular repn</u>

$$\chi_{reg}(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$$

For $G = S_3$

$$\chi_{reg} : \quad 6 \quad 0 \quad 0$$

$$= \chi_U + \chi_{U'} + 2\chi_V$$

$$\text{Reg} = R \cong U \oplus U' \oplus V \oplus V$$

<u>$Sym^2 V$</u>

$$V = \{ (a_1, a_2, a_3) \mid a_1 + a_2 + a_3 = 0 \}$$

Pick basis: $v_1 = (1, -1, 0)$

$v_2 = (0, 1, -1)$

$\tau = (123) \quad \sigma = (12)$

$\tau v_1 = (0,1,-1) = v_2$

$\tau v_2 = (-1,0,1) = -v_1 - v_2$

$\sigma v_1 =$
$(-1,1,0) = -v_1$

$\sigma v_2 = (1,0,-1) = v_1 + v_2$

$\tau : \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ $\quad \sigma = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$tr(\tau) = -1$ $\quad tr(\sigma) = 0$

Basis for $Sym^2 V$

$v_1 \cdot v_1 , \quad v_1 \cdot v_2 , \quad v_2 \cdot v_2$

(images $v_1 \otimes v_1$, $v_1 \otimes v_2$, $v_2 \otimes v_2$)

Think of $Sym^2 V$ as deg 2 homog. polynomials in v_1 , v_2.

$\tau(v_1 \cdot v_1) = \tau v_1 \cdot \tau v_1 = v_2 \cdot v_2$

$\tau(v_1 \cdot v_2) = \tau v_1 \cdot \tau v_2 = v_2 \cdot (-v_1 - v_2)$

$\tau(v_2 \cdot v_2) = \tau v_2 \cdot \tau v_2 = (-v_1 - v_2) \cdot (-v_1 - v_2)$

$$\tau(v_1 \cdot v_1) = v_2 \cdot v_2$$

$$\tau(v_1 \cdot v_2) = -v_1 \cdot v_2 - v_2 \cdot v_2$$

$$\tau(v_2 \cdot v_2) = +v_1 \cdot v_1 + 2 v_1 \cdot v_2 + v_2 \cdot v_2$$

$v_1 \cdot v_1, \; v_1 \cdot v_2, \; v_2 \cdot v_2$

$$\tau : \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\sigma : \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\chi_{Sym^2 V} : \quad 3 \quad 1 \quad 0$$

$$Sym^2 V = U \oplus V$$

In fact $v_1 \cdot v_1 + v_1 \cdot v_2 + v_2 \cdot v_2$ is fixed by S_3

eigenvalues λ_i

$v_1, \dots, v_n$ eigenvectors

of g acting V dim n.

in $\mathrm{Sym}^k V$

$$g(v_{i_1} \cdots v_{i_k}) = (g v_{i_1}) \cdots (g v_{i_k})$$
$$= \underbrace{\lambda_{i_1} \cdots \lambda_{i_k}}_{\text{eigenvalue}} v_{i_1} \cdots v_{i_k}$$

$$\mathrm{tr}(g|_{\mathrm{Sym}^k V}) = \sum_{i_1 \leq \dots \leq i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

For $\underline{k=2}$

$$\sum_{i_1 \leq i_2} \lambda_{i_1} \lambda_{i_2}$$
$$= \frac{1}{2} \left[\left(\sum_i \lambda_i \right)^2 + \sum_i \lambda_i^2 \right]$$
$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2} \left[\chi_V(g)^2 + \chi_V(g^2) \right]$$

$V = \mathrm{std}$

$$\chi_{Sym^2 V}((12)) = \frac{1}{2}\left[\,0^2 + 2\,\right] = 1$$

$$\chi_{Sym^2 V}((123)) = \frac{1}{2}\left[\,(-1)^2 + (-1)\right] = 0$$

$\underline{\Lambda^2 V}$

$$V \otimes V \simeq Sym^2 V \oplus \Lambda^2 V$$

$$\hookrightarrow \chi_V^2$$

$$\chi_{\Lambda^2 V}(g) = \chi_V^2(g) - \frac{1}{2}\left[\chi_V(g)^2 + \chi_V(g^2)\right]$$

$$= \frac{1}{2}\left[\chi_V(g)^2 - \chi_V(g^2)\right]$$

$v_1, \dots, v_n$ eigenvectors of g on V

$v_{i_1} \wedge \dots \wedge v_{i_k}$ basis of for $\Lambda^k V$ $i_1 < \dots < i_k$

$\lambda_{i_1} \cdots \lambda_{i_k}$ eigenvalue

$$\chi_{\Lambda^k V}(g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

$= (-1)^{k}$ k^{th} coeff of char polynomial of $\rho(g)$

$$x^n + a_1 x^{n-1} + \cdots + a_n$$

$V =$ std of S_3

$\underline{\Lambda^2 V}$

$$\chi_{\Lambda^2 V} : \quad 1 \qquad 0 \qquad -1$$

$$\Lambda^2 V = U'$$

$$V \otimes V = Sym^2 V \oplus \Lambda^2 V$$
$$= U \oplus V + U'$$

$\dim V = n$

$\dim \Lambda^k V = \binom{n}{k}$ $\qquad v_{i_1} \wedge \cdots \wedge v_{i_k}$, $\quad i_1 < \cdots < i_k$

$$\dim \mathrm{Sym}^k V = \binom{n+k-1}{k}$$

$$v_{i_1} \cdots\cdots v_{i_k}$$

$$i_1 \leq \cdots\cdots \leq i_k$$

$$\sum_{k \geq 0} \dim \Lambda^k V \; T^k = \sum_{k=0}^{n} \binom{n}{k} T^k$$

$$= (1+T)^n$$

$$\sum_{k \geq 0} \dim \mathrm{Sym}^k V \, T^k = \sum_{k \geq 0} \binom{n+k-1}{k} T^k$$

$$= \frac{1}{(1-T)^n}$$