SOME REMARKS ON LEHMER'S CONJECTURE

Fernando Rodriguez Villegas

Department of Mathematics
University of Texas at Austin
Austin, TX 78712

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1. The *logarithmic Mahler measure* of a non-zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is defined as

$$m(P) = \int_0^1 \cdots \int_0^1 \log \left| P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}) \right| d\theta_1 \cdots d\theta_n \tag{1}$$

and its *Mahler measure* as $M(P) = e^{m(P)}$, the geometric mean of $|P|$ on the torus

$$T^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | \ |z_1| = \ldots = |z_n| = 1\}. \tag{2}$$

When $n = 1$ Jensen's formula gives the identity

$$M(P) = |a_0| \prod_{|\alpha_\nu|>1} |\alpha_\nu| \ , \tag{3}$$

where $P(x) = a_0 \prod_{\nu=1}^d (x - \alpha_\nu)$, from which we clearly obtain that $M(P) \geq 1$ if $P \in \mathbb{Z}[x]$. By a theorem of Kronecker if $M(P) = 1$ for $P \in \mathbb{Z}[x]$ then P is *cyclotomic*, i.e., P is monic and its roots are either 0 or roots of unity.

In the early 30's Lehmer [Le] famously asked whether the is an absolute lower bound for $M(P)$ when $P \in \mathbb{Z}[x]$ and $M(P) > 1$. The purpose of this note is to point out a simple reformulation of this question in terms of the logarithmic embedding of units of a number field and, given this setting, to propose a natural generalization.

2. We start with some general observations about $m(P)$. First of all, the fact that the integral in (1) is finite for all non-zero P does need a proof. Here is a sketch. Using Jensen's formula we find, as in (3), that

$$m(P) = m(a_0) + \frac{1}{(2\pi i)^n} \sum_{\nu=1}^d \int_{T^{n-1}} \log^+ |\alpha_\nu(y)| \frac{dy}{y}, \tag{4}$$

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where $y = (y_1, \cdots, y_{n-1})$, $dy/y = dy_1/y_1 \cdots dy_{n-1}/y_{n-1}$, $\log^+(x) = \max\{\log|x|, 0\}$, and $a_0(y), \alpha_v(y), d$ are the leading coefficient, roots and degree, respectively, of P viewed as a polynomial in x_n. The α_ν's are algebraic functions of $y \in \mathbb{C}^{n-1}$, continuous and piecewise smooth, except at those y's where $a_0(y)$ vanishes (where some will go off to infinity).

We can apply the above procedure to any variable x_n on the torus T^n. It is not hard to see that we may change coordinates in such a way that $a_0(y)$ is actually constant, completing the proof by induction on n.

This last remark can be expanded. Let Δ be the Newton polytope of P; i.e., the convex hull of the exponents $m \in \mathbb{Z}^n$ of monomials $x^m = x_1^{m_1} \cdots x_n^{m_n}$ such that if

$$P = \sum_{m \in \mathbb{Z}^n} c_m x^m \tag{5}$$

then $c_m \neq 0$.

We define a *face* τ of Δ as the non-empty intersection of Δ with a half-space in $\mathbb{R}^n$. Chose a parameterization $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ of the affine subspace of smallest dimension containing τ; k is the *dimension* of the face τ. Define

$$P_\tau = \sum_{m \in \mathbb{Z}^k} c_{\phi(m)} x^m \ , \tag{6}$$

a polynomial whose own Newton polytope is $\phi^{-1}(\tau)$. We call P_τ the *face polynomial* associated to the face τ. It depends on a choice of ϕ but note that by changing variables in the integral $m(P_\tau)$ is actually independent of that choice.

It is not hard to see that for any facet (co-dimension 1 face) $\tau \subset \Delta$ we can choose ϕ and system of coordinates in T^n so that, in the notation of (4), $P_\tau = a_0(y)$. By (4) and induction on n we conclude [Sm] that

$$m(P_\tau) \leq m(P), \qquad \text{for all faces} \quad \tau \subset \Delta \, . \tag{7}$$

In particular,

$$m(P) \geq 0, \qquad \text{for} \quad 0 \neq P \in \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \, . \tag{8}$$

Also, since clearly $m(PQ) = m(P) + m(Q)$, we have that

$$m(Q) \leq m(P), \qquad \text{if} \quad Q \mid P, \quad 0 \neq P, Q \in \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \, . \tag{9}$$

Though Lehmer's conjecture is about polynomials in one variable, polynomials in more variables are also relevant due to the following result [Bo]. For any $0 \neq P \in \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ and $0 \neq (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we have

$$\lim_{k \to \infty} m(Q_k) = m(P) \qquad \text{where} \quad Q_k(t) = P(t^{a_1 k}, \ldots, t^{a_n k}) \tag{10}$$

That is, there are one variable polynomials Q with $m(Q)$ as close to $m(P)$ as desired. (We should note that (10) is not an immediate consequence of general results about integration but requires a somewhat delicate analysis.)

3. Let us go back to polynomials in one variable. If we want to find polynomials $P \in \mathbb{Z}[x]$ with positive but small $m(P)$, by (7) and (9) (and Gauss lemma) we may as well restrict ourselves to minimal polynomials of algebraic units.

Let F be a number field of degree n. Let I be the set of embeddings $\sigma : F \longrightarrow \mathbb{C}$ and V the real vector space of formal linear combinations

$$\sum_{\sigma \in I} \alpha_\sigma [\sigma], \qquad \alpha_\sigma \in \mathbb{R}\,. \tag{11}$$

We have the decomposition

$$V = V^+ \oplus V^-\,, \tag{12}$$

where $V^\pm$ is the subspace of V where complex conjugation acts like ± 1. We let $n_\pm = \dim_{\mathbb{R}} V^\pm$ (in terms of the standard notation $n_+ = r_1 + r_2$ and $n_= r_2$).

By Dirichlet's theorem the image of the unit group $\mathcal{O}_F^*$ by the log map

$$\begin{array}{cccc} l_1 : & \mathcal{O}_F^* & \longrightarrow & V \\ & \epsilon & \mapsto & \sum_{\sigma \in I} \log |\epsilon^\sigma| \, [\sigma] \end{array} \tag{13}$$

is a discrete subgroup $L_1 \subset V$ of rank $r = n^+ - 1$.

On V we define the L^1-norm

$$\left|\left| \sum_{\sigma \in I} \alpha_\sigma [\sigma] \right|\right|_1 := \sum_{\sigma \in I} |\alpha_\sigma| \tag{14}$$

and we let

$$\mu_{1,1}(F) := \min_{l \in L_1 \setminus \{0\}} ||l||_1 \tag{15}$$

(the reason for this indexing will become clear shortly).

For any unit $\epsilon \in \mathcal{O}_F^*$ we have $|\mathbb{N}_{F/\mathbb{Q}}(\epsilon)| = 1$ hence

$$\sum_{\sigma \in I} \log |\epsilon^\sigma| = 0\,. \tag{16}$$

Let $P \in \mathbb{Z}[x]$ be the (monic) minimal polynomial of ϵ and let n_ϵ be its degree. From (3) and (16) it follows that

$$||l_1(\epsilon)||_1 = \frac{2n}{n_\epsilon}\, m(P)\,. \tag{17}$$

This simple observation allows us to reformulate Lehmer's conjecture as follows.

Conjecture. *(Lehmer) There exists an absolute constant $\delta_1 > 0$ such that*

$$\mu_{1,1}(F) \geq \delta_1, \qquad \textit{for all number fields } F \textit{ with } r \geq 1. \tag{18}$$

4. Let V be a vector space over $\mathbb{R}$ of dimension n and $L \subset V$ a discrete subgroup of rank $r \geq 1$. A choice of basis $v_1, \dots, v_n$ for V determines L^1-norms on $\Lambda^k V$ for $k = 1, \dots, n$ by

$$\left|\left| \sum_{1 \leq j_1 < \cdots < j_k \leq n} a_{j_1, \dots, j_k} v_{j_1} \wedge \cdots \wedge v_{j_k} \right|\right|_1 := \sum_{1 \leq j_1 < \cdots < j_k \leq n} |a_{j_1, \dots, j_k}| \tag{19}$$

For each $1 \leq k \leq r$ we define (with respect to the chosen basis)

$$\mu_k(L) := \min ||l_1 \wedge \cdots \wedge l_k||_1 \tag{20}$$

where the minimum is taken over all $l_1, \dots, l_k \in L$ which are linearly independent over $\mathbb{R}$.

If A is the $n \times k$ integral matrix whose i-th column consists of the coordinates of l_i in the basis $v_1, \dots, v_n$ then, as it is easily seen,

$$||l_1 \wedge \cdots \wedge l_k||_1 = \sum_{A'} |\det A'| \tag{21}$$

where A' runs over all $k \times k$ minors of A.

Returning to the number field situation of the previous section we define the invariants

$$\mu_{1,k}(F) := \mu_k(L_1) \tag{22}$$

where, as before, L_1 is the image of the units of F under the log map.

A general version of Lehmer's conjecture would then be

Conjecture. *For each $k \in \mathbb{N}$ there exists an absolute constant $\delta_k > 0$ such that*

$$\mu_{1,k}(F) \geq \delta_k, \qquad \textit{for all number fields } F \textit{ with } r \geq k \tag{23}$$

A straightforward calculation shows that the top invariant $\mu_{1,r}(F)$, with $r = n^+ - 1$ the rank of the unit group $\mathcal{O}_F^*$, equals the regulator of F. It is known [Zi], [Fr], [Sk] that the regulator of number fields is universally bounded below and hence the above conjecture is true for $k = r$.

In summary, we have seen (18) that Lehmer's conjecture can be phrased in terms of the L^1-norm of units under the log map. The above conjecture is an attempt to quantify, in what seems to be the most natural way, the question of what is the general shape of L_1, the discrete group of units under the log map.

5. We may carry these ideas a little further still. Borel [Bo] proved, generalizing Dirichlet's result for units, that for each $j > 1$ there is a regulator map reg_j

$$\begin{array}{rccc} l_j : & K_{2j-1}(F) & \longrightarrow & V \\ & \xi & \mapsto & \sum_{\sigma\in I} \mathrm{reg}_j(\xi^\sigma)\,[\sigma] \end{array} \tag{24}$$

whose image is a discrete subgroup L_j of $V^{\pm}$, with $\pm = (-1)^{j-1}$, of rank $n^{\pm}$ and covolume related to the value of the zeta function ζ_F of F at $s = j$. Here $K_{2j-1}(F)$ are the K groups defined by Quillen.

We now define

$$\mu_{j,k}(F) := \mu_k(L_j), \qquad \text{for } 1 \le k \le n_{\pm} \tag{25}$$

and we may ask: what is the nature of these invariants, how do they depend on the field F? does the analogue of Lehmer's conjecture hold?

Apart from their formal analogy with Lehmer's question, answers to such questions can be quite useful in practice as we now illustrate.

6. For general j, very little is known about the groups $K_{2j-1}(F)$ or the map reg_j. For $j = 2$, however, things can be made quite explicit (and of course $j = 1$ corresponds to the case of units). Indeed, up to torsion, $K_3(F)$ is isomorphic to the *Bloch group* $\mathcal{B}(F)$, defined by generators and relations as follows.

For any field F define

$$\mathcal{A}(F) := \left\{ \sum_i n_i[z_i] \in \mathbb{Z}[F] \mid \sum_i n_i(z_i \wedge (1 - z_i)) = 0 \right\} \tag{26}$$

where the corresponding term in the sum is omitted if $z_i = 0, 1$ and

$$\mathcal{C}(F) := \left\{ [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1 - xy] + \left[\frac{1-y}{1-xy}\right] \mid x, y \in F,\ xy \ne 1 \right\} . \tag{15}$$

It is not hard to check that $\mathcal{C}(F) \subset \mathcal{A}(F)$. Finally, let

$$\mathcal{B}(F) := \mathcal{A}(F)/\mathcal{C}(F) . \tag{27}$$

We recall the definition of the *Bloch–Wigner dilogarithm*. Starting with the usual dilogarithm

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad |z| < 1 \tag{28}$$

one defines

$$D(z) = \mathrm{Im}\,(\mathrm{Li}_2(z)) + \arg(1 - z) \log |z| \tag{29}$$

and checks that it extends to a real analytic function on $\mathbb{C} \setminus \{0, 1\}$, continuous on $\mathbb{C}$. See [Za] for an account of its many wonderful properties. It is obvious that

$$D(\bar{z}) = -D(z)\,. \tag{30}$$

The 5-term relation satisfied by D guarantees that, extended by linearity to $\mathcal{A}(F)$, it induces a well defined function on $\mathcal{B}(\mathbb{C})$ (still denoted by D).

For $j = 2$ (24) can be formulated as follows

$$\begin{array}{rccc} l_2: & \mathcal{B}(F) & \longrightarrow & V \\ & \xi & \mapsto & \sum_{\sigma \in I} D(\xi^\sigma)\,[\sigma] \end{array} \tag{31}$$

((30) makes it clear that the image L_2 lies in V^-) whose image L_2 is a discrete subgroup of rank n^-.

An a priori lower bound for $||l_2(\xi)||_1$ even for the simplest case where L_2 is of rank 1 (namely, for a field with only one complex embedding) would be quite useful. For example, in [BRV1] we find that an identity between the Mahler measure of certain two-variable polynomials is equivalent to the following

$$D(7[\alpha] + [\alpha^2] - 3[\alpha^3] + [-\alpha^4]) = 0, \qquad \alpha = (-3 + \sqrt{-7})/4\,. \tag{32}$$

This was proved by Zagier by showing that it is a consequence of series of 5-term relations. Such calculations, however, can be quite hard and at present there is no known algorithm that is guaranteed to produce the desired result. Clearly if we knew a reasonable lower bound for the possible non-zero values of $|D(\xi)|$ for $\xi \in \mathcal{B}(\mathbb{Q}(\sqrt{-7}))$ a simple numerical verification would be enough to prove (32).

Similarly, many identities [BRV2] between the Mahler measure of certain two-variable polynomials and $\zeta_F(2)$ for a corresponding number field F, which by Borel's theorem are known up to an unspecified rational number, could be proved by a numerical check. For example, we can show that

$$m(x^2 - 2xy - 2x + 1 - y + y^2) = s\frac{1728^{3/2}}{2^6\pi^7}\zeta_F(2)\,, \tag{33}$$

with $s \in \mathbb{Q}^*$, where F is the splitting field $x^4 - 2x^3 - 2x + 1$, of discriminant -1728. However, though numerically s appears to be equal to 1 we cannot prove this at the moment. Again, a reasonable lower bound on $|D(\xi)|$ for non-torsion elements $\xi \in \mathcal{B}(F)$ would allow us to conclude that $s = 1$ by checking it numerically to high enough precision.

There is also some evidence that $\mu_{2,1}(F)$ might be universally bounded below, at least for fields with one complex embedding. Indeed, for a such a field one can construct a hyperbolic three dimensional manifold M by taking the quotient of hyperbolic space by a torsion-free subgroup of the group of units of norm 1 in a quaternion algebra over F ramified at all its real places. Its associated Bloch group element $\xi(M)$, obtained from a triangulation of M into ideal tetrahedra, satisfies

$D(\xi(M)) = \mathrm{vol}(M)$. On the other hand, the volume of hyperbolic 3-manifolds is known to be universally bounded below. The question becomes then, that of obtaining an upper bound for the index in $\mathcal{B}(F)$ of the subgroup generated by all such $\xi(M)$. This index is likely to be rather small; in fact, if we accept a precise form of Lichtembaum's conjecture, it should be essentially the order of $K_2(\mathcal{O}_F)$, an analogue of a class group. Unfortunately, there is no known upper bound for $|K_2(\mathcal{O}_F)|$ in terms of, say, the degree and discriminant of F.

Finally, to a hyperbolic 3-manifold M with one cusp one may associate [CCGLS] a two variable polynomial $A(x, y) \in \mathbb{Z}[x, y]$, called the *A-polynomial* of M. It zero locus parameterizes deformations of the complete hyperbolic structure of M.

It is known that

$$m(A_\tau) = 0 \tag{34}$$

for every face polynomial of A and that A is *reciprocal*, i.e. $A(1/x, 1/y) = x^a y^b A(x, y)$ for some $a, b \in \mathbb{Z}$. It is interesting that these two properties, which have a topological and K-theoretic origin, are, for A irreducible, precisely the known necessary conditions for a polynomial in $\mathbb{Z}[x, y]$ to have to have small Mahler measure (the first, an analogue of being the minimal polynomial of an algebraic unit, because of (7); the second because $m(P)$ is known to be universally bounded below for P non-reciprocal [Sm]).

Though the whole picture is still not completely clear yet one can prove [BRV2] for many M's identities of the form

$$2\pi m(A) = ||D(\xi(M))||_1 \tag{35}$$

where $\xi(M)$ is the Bloch group element associated to M. This suggests a direct link between Lehmer's conjecture and the size of the invariants $\mu_{2,1}$.

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Department of Mathematics University of Texas at Austin

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E-mail address: villegas@math.utexas.edu