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Representations of $SL_2(\mathbb{F}_p)$

Weil representation.

$$\{ f : \mathbb{R} \longrightarrow \mathbb{C},\ C^\infty \} \circlearrowright$$

Two operators

$$P = \frac{1}{2\pi i}\frac{d}{dx}, \quad Q = x$$

$$PQ(f) = \frac{1}{2\pi i}\frac{d}{dx}(x f(x))$$

$$= \frac{1}{2\pi i}\left(f(x) + x\frac{df}{dx} \right)$$

$$QP(f) = \frac{1}{2\pi i}\, x \frac{d}{dx} f$$

$$\boxed{PQ - QP = \frac{1}{2\pi i}}$$

Exponentiate this

Assume $e^{2\pi i t P}$, $e^{2\pi i t Q}$ t real

Baker-Hausdorff Lemma

A, B Hermitian operators, t real

$$e^{2\pi i t A} B e^{-2\pi i t A} = B + 2\pi i t \overset{AB-BA}{[A,B]} + \frac{(2\pi i t)^2}{2!} [A,[A,B]] + \dots$$

If $\underset{= \frac{1}{2\pi i}}{[A,B]}$ is a scalar

$$e^{2\pi i t A} B e^{-2\pi i t A} = B + t$$

$$e^{2\pi i t A} f(B) e^{-2\pi i t A} = f(B+t)$$

$$U_t := e^{2\pi i t P} \qquad V_t := e^{2\pi i t Q}$$

$$U_t V_s U_t^{-1} = e^{2\pi i s t} V_s$$

$$[U_t, V_s] = e^{2\pi i s t}$$

$$\boxed{U_t V_s = e^{2\pi i s t} V_s U_t}$$

(Weyl)

Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F}_p \right\}$$

$$(a, b, c) \cdot (x, y, z) = (x+a, y+b, bx + z + c)$$

$$[(a, b, c), (x, y, z)] = (0, 0, bx - ay)$$

center

$$(a, b) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (a, b) \cdot (y, -x) = ay - bx$$

$U = \mathbb{F}_p^2$ bilinear form on U
skew symmetric

non-degenerate

i.e. $bx - ay = 0$ for all x, y

$\Leftrightarrow$ $a = b = 0$

$$Z = \{ (0, 0, c) \}$$

$$H / Z \simeq U$$

Let $N = \{ (0, b, c) \}$

$|N| = p^2$ abelian

Note : $N \trianglelefteq H$

$$\eta : N \longrightarrow \mathbb{C}^\times \quad \text{homomorphism.}$$
$$(0, b, c) \longmapsto \zeta_p^c$$

ζ_p = primitive p^{th} - root of 1.

$V := \mathrm{Ind}_N^H (\eta)$

$\dim V = [H : N] = p.$

<u>Claim</u> V is irred.

<u>Pf</u> Using fmla for the character of an induced repn.

$$\underset{\in N}{\chi((a, b, c))} = \sum_{x \in \mathbb{F}_p} \eta((0, b, -bx + c))$$

$$= \sum_{x \in \mathbb{F}_p} \zeta_p^{-bx + c}$$

$$0 \to Z \to H \to U \to 0$$
$$Z \cong \mathbb{Z}/p\mathbb{Z}$$

central extension of U by Z.

conjugacy classes.

$$(a,b,c)\,(x,y,z)\,(a,b,c)^{-1} = (0,0,bx-ay)\,(x,y,z)$$

p classes of size 1 i.e. $z \in Z$.

and $(p^3-p)/p = p^2-1$ classes of size p.

total: p^2+p-1

$p^2 = |U|$ 1-dim'l repn of H.

(actually $[H,H] = Z$ so these are all the 1-dim'l repn of H).

$$|H| = p^3 = p^2 + p^2(p-1)$$

p^2 1-dim'l

$p-1$ p-dim'l

$$= \zeta_p^c \cdot \begin{cases} p & b=0 \\ 0 & \text{otherw.} \end{cases}$$

So $(0,0,c)$ acts as a scalar ζ_p^c on V.

$$\langle \chi, \chi' \rangle = \frac{1}{p^3} \sum_{c \in \mathbb{F}_p} \bar{\zeta}_p^{c} \cdot p \cdot \zeta_p'^{c} \cdot p$$

$$= \frac{1}{p} \sum_c (\zeta_p^{-1} \zeta_p')^c$$

$$= \begin{cases} 1 & \text{if } \zeta_p = \zeta_p' \\ 0 & \end{cases}$$

I.e. the $p-1$ representations obtained by choosing different ζ_p's are all distinct & irreducible $\square$

Hence we have all irred. repn of H.

<u>Remark</u> These repn are uniquely characterized by how the center acts.

(Finite field version of Stone-von Neuman

General Case

U vector space $/\mathbb{F}_p$ of dim $2n$

$$1 \longrightarrow \mu_p \longrightarrow H \longrightarrow U \longrightarrow 0$$

$\mu_p = \{ p^{th} \text{ roots of } 1 \} \subseteq \mathbb{C}^\times$

$(\zeta, u) \in H$

$(\overset{\shortparallel}{c} \quad \overset{\shortparallel}{(a,b)})$

$$(\zeta, u) \cdot (\zeta', u') = (\zeta\zeta' \psi(u,u'), u+u')$$

$\psi : U \times U \longrightarrow \mu_p$

bilinear pairing with

$$\psi(u, u') \cdot \psi(u', u)^{-1} =: \phi(u, u')$$

non-degenerate perfect pairing

$\phi(u, u') = 1$ for all u'

$\Leftrightarrow u = 1$

$$\phi(u', u) = \phi(u, u')^{-1}$$

a**lternating**

$$\phi(u, u)^2 = 1$$

$$[(\zeta, u), (\zeta', u')] = (\phi(u, u'), 0)$$

p^{2n} 1-diml repn.

Analogue of N:

Consider $L \subseteq U$ subspace

$$N = \{(\zeta, \ell) \mid \ell \in L\}$$

$$[(\zeta, \ell), (\zeta', u')] = (\phi(\ell, u'), 0)$$

N abelian?

$$[(\zeta, \ell), (\zeta', \ell')] = (1, 0)$$

i.e. $\phi(\ell, \ell') = 1$ $\ell, \ell' \in L$

i.e. $\phi|_L \equiv 1$

$$\eta : \mu_p \hookrightarrow \mathbb{C}^\times$$

$$V_\eta := \mathrm{Ind}_{N^*}^{H}(\eta)$$

As before we get $p-1$ irred repn of H of dim p^n

$$\underbrace{p^{2n}}_{\text{1-dim'l}} + (p-1)(p^n)^2 = p^{2n+1} = |H|$$

$$\chi_\eta((\zeta,u)) = \begin{cases} \eta(\zeta)\, p^n, & u=0 \\ 0 & \text{otherw.} \end{cases}$$

Again $(\zeta, 0)$ center acts the scalar $\eta(\zeta)$.

$$[U_t, V_s] = \eta\left(e^{\frac{2\pi i}{p} st}\right)$$

<u>$n=1$</u>

$$(1, (s,0)) = V_s$$
$$(1, (0,t)) = U_t$$

Write additively

$$\phi = e^{\frac{2\pi i}{p}\Phi}$$

$$\Phi : U \times U \longrightarrow \mathbb{F}_p$$

$$\Phi(u, u') = -\Phi(u', u)$$

$(p > 2)$ skew-symmetric can find a symplectic basis, s.t. Φ looks like

$$\begin{array}{cc} L & L^* \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)} \end{array}$$

L = <u>Lagrangian subspace</u> of U

$$N^* := \{ (\zeta, \ell^*) \mid \ell^* \in L^* \}$$

$$[H : N^*] = p^n$$

$$\eta : N^* \longrightarrow \mathbb{C}^\times$$
$$(\zeta, \ell^*) \longmapsto \eta(\zeta)$$