Oct 18, 2007

- $$\begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}$$

u, v, uv
basis

$$\mathrm{tr} = 2\cos\theta + 1$$

- $e = e_0 + e_1 i + e_2 j + e_3 k$

$\mathbb{H}^0 \circlearrowleft$ conjugation by e

$$e\, i\, e^{-1} = \begin{array}{l} (\ldots \overset{0}{''}\,)\; 1 \\ (\ldots\;)\; i \\ \quad\vdots \end{array}$$

first column of matrix r(e)

$$\begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & \cdots \\ \cdots & \\ & \end{pmatrix}$$

$$\mathrm{tr} = 4\cos^2\frac{\theta}{2} - 1$$

Finite subgps of $SU(2)$
double cover of $SO(3)$

$Q_8 := \{\pm 1, \pm i, \pm j, \pm k\} \subseteq SU(2)$

(recall this gives a 2-diml repn of Q_8 quaternion group which has real traces but is not real)

$1 \to \{\pm 1\} \to SU(2) \to SO(3) \to 1$

image of Q_8 is $D_2 \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$

Start with $D_n \subseteq SO(3)$ pull back to $SU(2)$ get generalized quaternions.

Extension is non-split

$1 \to \{\pm 1\} \to \langle i \rangle \to \mathbb{Z}/2 \to 0$

V standard repn of S_n

<u>THM</u> $\Lambda^k V$ is irreducible
$0 \leq k \leq n-1$

Pf $W = U \oplus V$ defining repn of S_n: $\sigma e_i = e_{\sigma(i)}$

$$\Lambda^k W = (\Lambda^k V \otimes \Lambda^0 U) \oplus (\Lambda^{k-1} V \otimes \Lambda^1 U)$$
$$= \Lambda^k V \oplus \Lambda^{k-1} V$$

will prove that

$$\langle \chi_k, \chi_k \rangle = 2$$

$$\chi_k := \chi_{\Lambda^k W}$$

$$\chi_k(\sigma) = \ ?$$

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \xrightarrow{\rho_n(\sigma)} e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_k)}$$

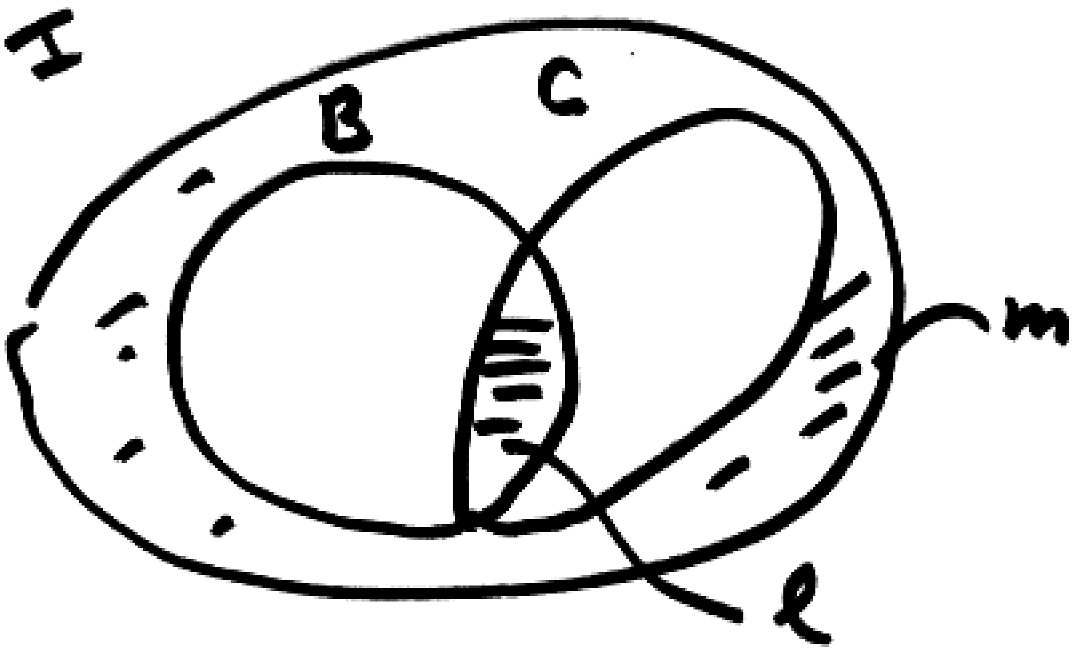
$$i_1 < i_2 < \cdots < i_k$$

$$\sigma : \underset{\substack{\cap \\ \{1,2,\dots,n\}}}{\overset{=: B}{\{i_1, \dots, i_k\}}} \longmapsto \{\sigma(i_1), \dots, \sigma(i_k)\}$$

only contribution to $\chi_k(\sigma)$ is when these sets are the same.

$$\chi_k(\sigma) = \sum_{\substack{\#B=k \\ \sigma B = B}} \operatorname{sgn}(\sigma|_B) \qquad B \subseteq \overset{\overset{I}{\shortparallel}}{\{1,2,\dots,n\}}$$

$$\langle \chi_k, \chi_k \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\substack{\sigma B = B \\ \sigma C = C \\ \#B = \#C = k}} \operatorname{sgn}(\sigma|_B)\operatorname{sgn}(\sigma|_C$$



Fix B, C with these properties

Let σ run over $\sigma \in S_n$ s.t.

$$\sigma B = B$$
$$\sigma C = C$$

This σ ~~[illegible]~~ gives 4 permutations

$$\sigma|_{I \setminus (B \cup C)} = h$$

$$\sigma|_{B \setminus B \cap C} = b$$

$$\sigma|_{C \setminus B \cap C} = c$$

$$\sigma|_{B \cap C} = d$$

$$\begin{array}{l} \sigma B = B \\ \sigma C = C \end{array} \quad \longleftrightarrow \quad (h, b, c, d)$$

Let $\ell = \# B \cap C$

$$m = \# \; I \setminus (B \cup C)$$

$$= n - 2k + \ell$$

$k > \ell$
(ie: $B \neq C$)

$$m = n - 2k + \ell$$

$$\sum_{h \in S_m} \sum_{b, c \in S_{k-\ell}} \sum_{d \in S_\ell}$$

$$\operatorname{sgn}(\sigma|_B) = \operatorname{sgn}(b) \cdot \operatorname{sgn}(d)$$

$$\operatorname{sgn}(\sigma|_C) = \operatorname{sgn}(c) \cdot \operatorname{sgn}(d)$$

$$(*) = \sum_{h \in S_m} \sum_{b, c \in S_{k-\ell}} \sum_{d \in S_\ell} \operatorname{sgn} b \operatorname{sgn} c \underset{1}{\cancel{\operatorname{sgn}^2 d}}$$

$$= m! \, \ell! \left(\sum_{b \in S_{k-\ell}} \operatorname{sgn} b \right) \left(\sum_{c \in S_{k-\ell}} \operatorname{sgn} c \right)$$

⑥

$$\frac{1}{r!}\sum_{b\in S_r} \operatorname{sgn} b = \langle \operatorname{sgn}, 1\rangle$$

$$= \begin{cases} 1 & r=1 \\ 0 & r>1 \end{cases}$$

i.e.

$(*) = 0$ unless $l = k-1$

in that case its value is

$(k-1)!$

$m!$

$m = n - 2k + l$
$= n - 2k + k - 1$
$= n - k - 1$

$n - k + 1$

How many?

$\binom{n}{k-1}$ $(n-k+1)(n-k)$

Total contribution:

$$\frac{1}{m!}(n-k-1)!\,(k-1)!\binom{n}{k-1}(n-k+1)(n-k)$$

$$= 1$$

Contributions from $B = C$

$$\sum_{h \in S_m} \sum_{d \in S_k} 1 = m!\,k!$$

$= k!\,(n-k)!$

$m = n-k$

How many sets? $\binom{n}{k}$

Total contribution

$$\frac{1}{n!}(n-k)!\,k! \cdot \binom{n}{k} = 1$$

$\langle \chi_k, \chi_k \rangle = 2$ $\square$

* * *

Induced Representations

V representation G $\quad W \subseteq V$ fixed by H

$H \leq G$ subgroup

If $V = \bigoplus_{\sigma \in G/H} \sigma W$ left cosets space

$\dim V = [G:H] \dim W$

then say V is <u>induced</u> from W and write $V = \mathrm{Ind}_H^G(W)$

Example

G acting on G/H by left multiplication.

V basis e_σ $\quad \sigma \in G/H$

$G \ni g$: $\quad g e_\sigma = e_{g\sigma}$

$$W = \langle e_1 \rangle \qquad V = \bigoplus_{\sigma \in G/H} \langle \sigma e_1 \rangle$$

$$= \bigoplus_{\sigma \in G/H} \langle e_\sigma \rangle$$

I.e. we have

$$\mathrm{Ind}_H^G(\text{trivial repn of } H)$$

$G = S_4 \qquad H =$ 2-Sylow subgroup

$3 = [G : H]$

What is $\mathrm{Ind}_H^G(1_H)$

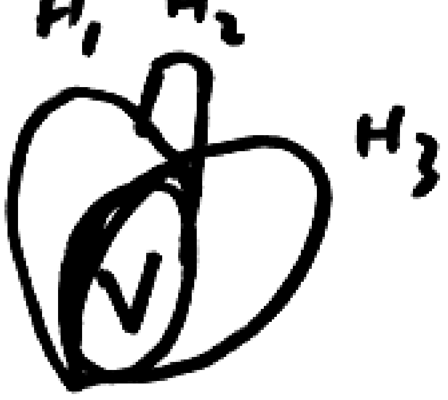
$H = \{ 1, (12)(34), (13)(24), (14),(23)$
$(12), (34), (1324), (1423) \}$

$V = \{ 1, (12)(34), (13)(24), (14)(23) \}$

$V \triangleleft S_4$

There are 3 - sylow subgps H_1, H_2, H_3

$V = H_i \cap H_j \quad i \neq j$



$g \in S_4$

$g\sigma = \sigma \qquad \sigma \in G/H$

Pick set of representatives for cosets $G/H \quad \{ g_\sigma \}$

$g \cdot g_\sigma = g_\sigma h \qquad$ some $h \in H$

$g = g_\sigma h g_\sigma^{-1}$

$g \in g_\sigma H g_\sigma^{-1}$

$$\begin{array}{cccccc} g \to & 1 & (12) & (123) & (1234) & (12)(34) \in V \\ & 3 & 1 & 0 & 1 & 3 \quad = \bigcap_{i=1}^{3} H_i \\ & \uparrow & & & & \\ & [G:H] & & & & \end{array}$$

$$\#\{\sigma \mid g_\sigma H g_\sigma^{-1} \ni g\}$$

Want to construct $\mathrm{Ind}_H^G(W)$

$\{g_\sigma\}$ choice of representatives of G/H

$$V := \bigoplus_{\sigma \in G/H} W_\sigma \qquad W_\sigma := W$$

$$v = \sum_\sigma \underbrace{g_\sigma \cdot w_\sigma}_{\in W_\sigma} \qquad w_\sigma \in W$$

$$g \in G \qquad \boxed{g g_\sigma = g_\tau h} \qquad \begin{array}{l} h \in H \\ \tau \in G/H \end{array}$$

$$gv := \sum_\sigma g(g_\sigma w_\sigma)$$

$$\begin{aligned} g(g_\sigma w_\sigma) &:= (g g_\sigma) w_\sigma \\ &= (g_\tau h) w_\sigma \\ &= g_\tau (h w_\sigma) \end{aligned}$$

$$g_\sigma w_\sigma \overset{g}{\longmapsto} g_\tau (h w_\sigma)$$

Define

$$g(g_\sigma w_\sigma) = g_\tau (h w_\sigma)$$

Need check this g makes V a representation of G which is W induced from H.

Fancy version

- W is $\mathbb{C}[H]$-module

$\mathbb{C}[G]$ is a free right $\mathbb{C}[H]$-module

$$V := \mathbb{C}[G] \underset{\mathbb{C}[H]}{\otimes} W$$

$$V = \mathrm{Ind}_H^G(W)$$

- $V = \left\{ f : G \longrightarrow W \,/\, f(hx) = h f(x) \quad h \in H \right\}$

Define action of G on V

$$(gf)(x) = f(xg)$$

$$V = \mathrm{Ind}_H^G (W)$$