Hypergeometric Motives

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- Motive of rank $d = 2$
$$V = H^1(E, \mathbb{Q})$$

L-function

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- $$L_\infty(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-1)$$

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- Hodge vector
$$\mathbf{h} := (h^{w,0}, h^{w-1,1}, \ldots, h^{0,w})$$

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- Artin representation rank d

$$\mathbf{h} = (d)$$

Motives in practice

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- How are these distributed according to Hodge vectors $\mathbf{h}$ and conductor N?

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- Gauss hypergeometric series

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- $V :=$ space of local solutions of the DE at $z = t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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- It has rank $d = 2$ and a certain computable weight w.

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- By Griffits transversality $\mathbf{h}$ is a symmetric composition of d.

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- Here h means height, M is a sort of discriminant of $\mathcal{H}(t)$ computable from q_0, q_∞.

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$\mathbf{h}$	#
[9, 1, 1, 2, 1, 1, 9]	0
[7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 7]	0
[1, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6, 1]	0
[4, 1, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 4]	0
[5, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 5]	0
[6, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 6]	0
[4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 4]	0
[4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 4]	0

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$\mathbf{h}$	#
[6, 2, 1, 1, 1, 2, 1, 1, 1, 2, 6]	2
[8, 1, 1, 1, 2, 1, 1, 1, 8]	4
[1, 22, 1]	4
[8, 1, 1, 4, 1, 1, 8]	6
[6, 1, 2, 1, 1, 2, 1, 1, 2, 1, 6]	8
[6, 1, 3, 1, 2, 1, 3, 1, 6]	8
[10, 1, 2, 1, 10]	10
$\vdots$	$\vdots$
[1, 3, 4, 4, 4, 4, 3, 1]	6082776
[2, 5, 5, 5, 5, 2]	6850823
[1, 3, 8, 8, 3, 1]	6868016
[1, 5, 6, 6, 5, 1]	7637828
[1, 2, 4, 5, 5, 4, 2, 1]	7982874
[2, 4, 6, 6, 4, 2]	9504072
[1, 4, 7, 7, 4, 1]	9905208

Densities

Graph of logarithmic densities, rank $d = 24$