~~Sep~~ Oct 9, 2007

$SU(2) \subseteq SL_2(\mathbb{C})$

${}^T g^{-1} = \bar{g}$

$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ $\quad |a|^2 + |b|^2 = 1$

$tr(g) = a + \bar{a} \in \mathbb{R}$

$G \hookrightarrow SU(2) \subseteq SL_2(\mathbb{C}) \subseteq Gl_2(\mathbb{C})$

gives a 2-dim'l of repn of G

G non-abelian this repn is not real.

$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$

$\mathbb{C} = \mathbb{R} + \mathbb{R}i$

$j^2 = -1$, $\quad \bar{a}j = ja$, $\quad a \in \mathbb{C}$

$(a+bj)(c+dj) = (ac - b\bar{d}) + (b\bar{c} + ad)j$

$\mathbb{H} \longrightarrow M_2(\mathbb{C})$ algebra homomorphism

$a+bj \longmapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$

$SU(2) \leftrightarrow \mathbb{H}_1^\times := \{ x \in \mathbb{H} \mid n(x) = 1 \}$

$n(a+bj) = (a+bj)(\bar{a} - j\bar{b})$
$= |a|^2 + |b|^2$

$H_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \subseteq \mathbb{H}_1^\times \leftrightarrow SU(2)$

<u>upshot</u> χ_V real valued $\not\Rightarrow$ V is real.

If χ_V real valued then ?

$\chi_{V^*}(g) = \overline{\chi_V(g)} = \chi_V(g)$

$\varphi : V \xrightarrow{\sim} V^*$ G - linear

Bilinear form fixed by G

$B(u,v) := \varphi(u)(v)$

non-degenerate b/c φ is an isom.

V irreducible.

$B(gu, gv) = \varphi(gu)(gv)$
$= g\varphi(u)(gv)$
$= \varphi(u)(g^{-1}gv) = \varphi(u)(v)$

By Schur's lemma φ and hence B is unique up to scalars.

$$B'(u,v) := B(v,u)$$

G-linear Bilinear form on V.

$$\Rightarrow B' = \varepsilon B$$

$$\varepsilon^2 = 1 \Rightarrow \varepsilon = \pm 1$$

I.e. $$\boxed{B(v,u) = \varepsilon B(u,v)}$$

Alternatively,

$$V^* \otimes V^* \simeq \mathrm{Sym}^2 V^* \oplus \Lambda^2 V^*$$

$$1 = \langle \chi_{V^*}, \chi_{V^*} \rangle = \frac{1}{|G|} \sum_g \chi_{V^*}(g)^2$$

$$= \langle \chi_{V^* \otimes V^*}, 1 \rangle$$

$$\Rightarrow \dim_{\mathbb{C}} (V^* \otimes V^*)^G = 1$$

We also have the Hermitian form H (unique up to scalars by Schur)

G-linear, Hermitian, pos. defn.

$$V \xrightarrow{B} V^* \xrightarrow{H} V$$

$v \in V$ there exists $\psi(v) \in V$ s.t.

$$B(v, w) = H(\psi(v), w) \quad \text{all } w \in V$$

$$\psi : V \to V$$

conjugate linear isom

$$\lambda B(v,w) = B(\lambda v, w) = H(\psi(\lambda v), w)$$

$$\lambda B(v,w) = \lambda H(\psi(v), w)$$
$$= H(\bar{\lambda}\psi(v), w)$$

$$\Rightarrow \psi(\lambda v) = \bar{\lambda}\psi(v)$$

$$B(gu, v) = B(u, g^{-1}(v))$$
$$= H(\psi(u), g^{-1}v)$$
$$= H(g\psi(u), v)$$

$$\psi(gu) = g\psi(u)$$

$$\psi^2 : V \longrightarrow V$$

$$\psi^2(gu) = g\psi(u)$$

$$\psi^2(\lambda u) = \psi(\psi(\lambda u))$$
$$= \psi(\bar{\lambda}\psi(u))$$
$$= \lambda\psi^2(u)$$

ψ^2 is $\mathbb{C}$-linear !!

Schur's lemma $\Rightarrow$ $\psi^2 = \lambda\, id_V$

($\lambda \neq 0$ ψ^2 is an isom).

$$H(\psi(u), v) = B(u,v) = \varepsilon B(v,u)$$

$$\varepsilon B(v,u) = \varepsilon\, H(\psi(v), u) = \varepsilon\, \overline{H(u, \psi(v))}$$

Do it again

$$H(\psi^2(u), v) = \varepsilon\, \overline{H(\psi(u), \psi(v))}$$
$$= \varepsilon^2 H(u, \psi^2(v))$$
$$= H(u, \psi^2(v))$$

$\psi^2(u) = \lambda u$

$$\bar{\lambda} H(u,v) = \lambda H(u,v)$$

$\Rightarrow \quad \lambda \in \mathbb{R}.$

Take $v = \psi(u)$

$$H(\psi(u), \psi(u)) = \varepsilon \overline{H(u, \psi^2(u))}$$
$$= \varepsilon \bar{\lambda} H(u,u)$$

H is positive defn so both

$u \neq 0$ $H(\psi(u), \psi(u)) > 0$

$H(u,u) > 0$

$\Rightarrow \quad \varepsilon \lambda > 0$

i.e. $\operatorname{sgn}(\lambda) = \varepsilon$

scale ψ by $1/\sqrt{|\lambda|}$ so that

$$\boxed{\psi^2 = \varepsilon}$$

If $\boxed{\varepsilon = +1}$ we can split

V as vector spaces over $\mathbb{R}$

$$V = V_+ \oplus V_- \qquad \psi^2 = 1$$

according to eigenspaces of ψ.

Since ψ commutes with the action of G hence $V_\pm$ are G-stable

$$\psi(iu) = -i\psi(u)$$

$$iV_+ = V_-$$

$$iV_- = V_+$$

$$V = V_+ \otimes \mathbb{C}$$

i.e. V is real !

If $\boxed{\varepsilon = -1}$ then

$$\psi^2 = -id_V$$

$$\psi(iu) = -i\psi(u)$$

i.e. V is an $\mathbb{H}$-module

and V is <u>not</u> real.

If $V = V_+ \otimes \mathbb{C}$, V_+ a real representation of G then V_+ carries a G-bilinear, symmetric form (take any pos defn one ~~of them~~ and average).

<u>Trichotomy</u>

- cplx (χ not real valued)
- real
- quaternionic

$\left.\begin{array}{l}\text{real}\\ \text{quaternionic}\end{array}\right\}$ (χ real valued)

~~χ_V real valued~~ real vs quaternionic depend on ε.

$\underline{\varepsilon=-1}$ $\quad 1 = \langle \chi_{\Lambda^2 V^*}, 1 \rangle = \dim (\Lambda^2 V^*)^G$

$\underline{\varepsilon=+1}$ $\quad 1 = \langle \chi_{Sym^2 V^*}, 1 \rangle = \dim (Sym^2 V^*)^G$

$$\langle \chi_{\Lambda^2 V^*}, 1 \rangle = \frac{1}{|G|} \sum_{g \in G} \tfrac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right)$$

χ realvalued

$$\Rightarrow \frac{1}{|G|} \sum \chi_V(g)^2 = \langle \chi_V, \chi_V \rangle = 1$$

$$= \frac{1}{2} - \frac{1}{2}\frac{1}{|G|}\sum_{g \in G} \chi_V(g^2)$$

$\frac{1}{|G|}\sum_{g \in G} \chi_V(g^2)$ $\quad$ __Schur indicator__

$$= \begin{cases} 1 & V \text{ real} \quad \varepsilon = +1 \\ -1 & \text{quaternionic} \quad \varepsilon = -1 \\ 0 & \text{cp} \mathbb{C}\text{x} \end{cases}$$