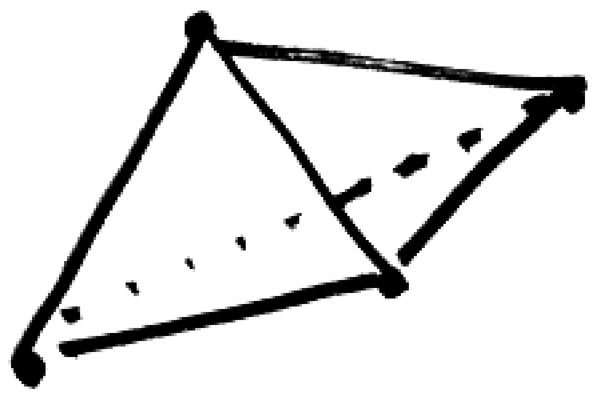
Oct 16, 2007

Tetrahedron

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}1\\-1\\-1\end{pmatrix} \quad \begin{pmatrix}-1\\-1\\1\end{pmatrix} \quad \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

$v_1 \quad v_2 \quad v_3 \quad v_4$



$$\sum_i v_i = 0$$

Action of S_4 (standard repn)

Permutes the v_i's.

Restrict to A_4 these are rotations of $\mathbb{R}^3$

$$R_{v_1} = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$e_1 \to e_2 \to e_3 \to e_1$

$v_3 \to v_2 \to v_4 \to v_3$ $\quad v_1$ $\quad$ order 3

(234)

R_E $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad v_1 \longleftrightarrow v_2$

$v_3 \longleftrightarrow v_4$

order 2

$(1\,2)(3\,4)$

$R_F = R_v$

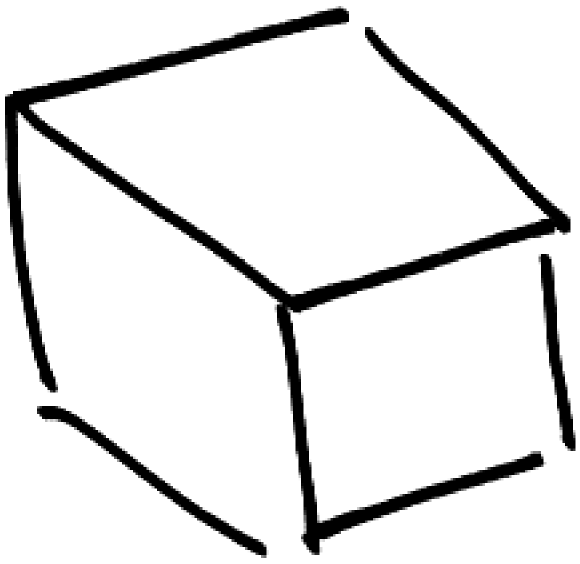
group of rotations of  $\cong A_4$ (permutations of vertices)

standard repn of A_4 real 3-dim'l

		order
4	$R_F = R_v$	3
$\frac{1}{2}$ 6	R_E	2

$$1 + 4 \times (3-1) + \tfrac{1}{2} 6 \times (2-1) = 12$$

<u>cube / octahedron</u>



$\begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix}$ vertices

		order
$3 = \frac{1}{2} 6$	R_F	4
$6 = \frac{1}{2} 12$	R_E	2
$4 = \frac{1}{2} 8$	R_V	3

$$1 + 4 \times (3-1) + 6 \times (2-1) + 3 \times (4-1) = 24$$

$$G \simeq S_4$$

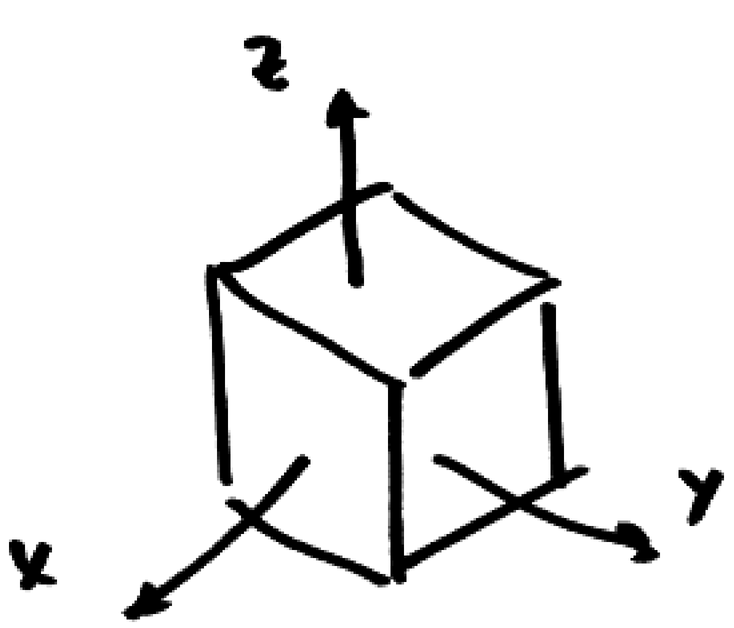
e.g. action on ~~d~~ 4 diagonals.

What 3-diml repn of S_4 is it?

(is irreducible...)

Is V' b/c $\det = +1$ and

det of V is sgn...

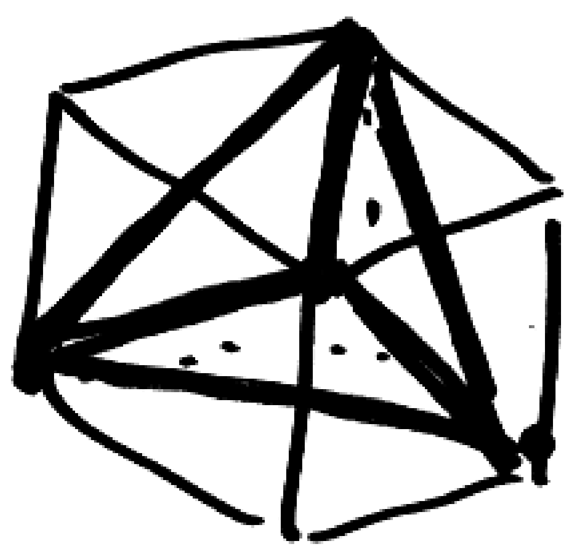


$$\begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}$$

trace = +1

(checks)

vertices whose coordinates multiply to +1. Their convex hull is a tetrahedron (the one we had before)



Stab of this tetrahedron $\cong A_4 \subseteq S_4$

rotations of cube $\cong$ signed permutation matrices with $\det = +1$

Icosahedron / Dodecahedron

$\frac{1}{2} 12 = 6$	R_V	5
$\frac{1}{2} 30 = 15$	R_E	2
$\frac{1}{2} 20 = 10$	R_F	3

$$1 + 6 \times (5-1) + 15 \times (2-1) + 10 \times (3-1) = 60$$

$$G \simeq A_5$$

<u>Recall</u>

$$SU(2) \simeq \mathbb{H}_1^\times$$

topologically this is 3- sphere.

$$\mathbb{H}^0 := \{ \mathrm{tr} = 0 \}$$ <u>pure quaternions</u>

$$\mathrm{tr}(x) = x + \bar{x} = 0 \quad \leftrightarrow \quad \bar{x} = -x$$

$$x \in \mathbb{H}^0, \qquad n(x) = -x^2$$

$$x^2 \in \mathbb{R}$$

In partic $x \in \mathbb{H}^0 \cap \mathbb{H}_1^\times$

$$x^2 = -1 \qquad S^2$$

$$1 \to \{\pm 1\} \to \underset{\mathbb{H}_1^\times}{\underset{\cong}{SU(2)}} \xrightarrow{r} SO(3) \to 1$$

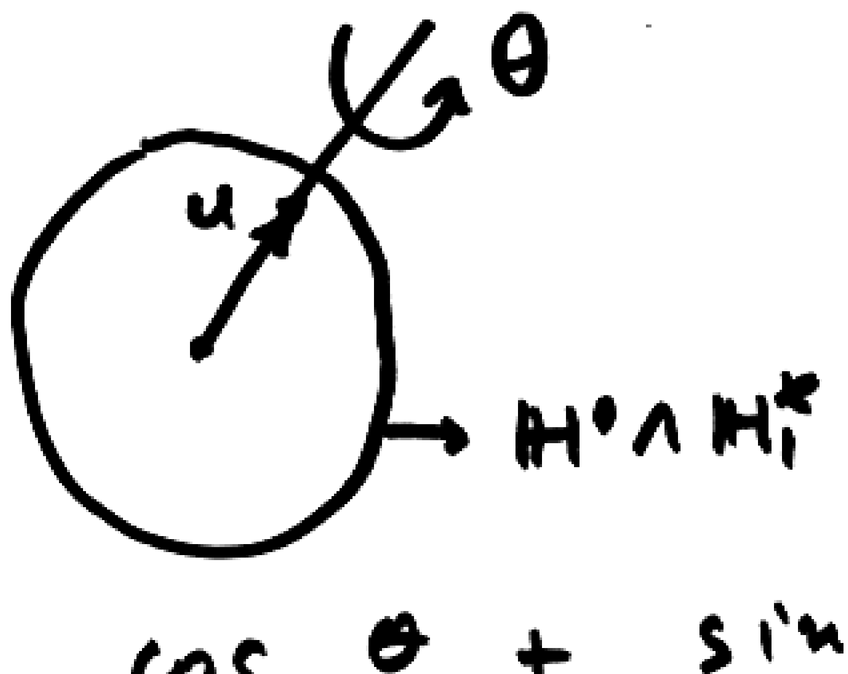
$x \in \mathbb{H}_1^\times$

$y \in \mathbb{H}^0$, $\quad x y x^{-1} \in \mathbb{H}^0$

& $\quad n(x y x^{-1}) = n(y)$

So conjugating by x gives a isometry of $\mathbb{R}^3$ euclidean

$u \in S^2$ rotation with axis u and angle θ



$\subseteq \mathbb{H}^0 \simeq \mathbb{R}^3$

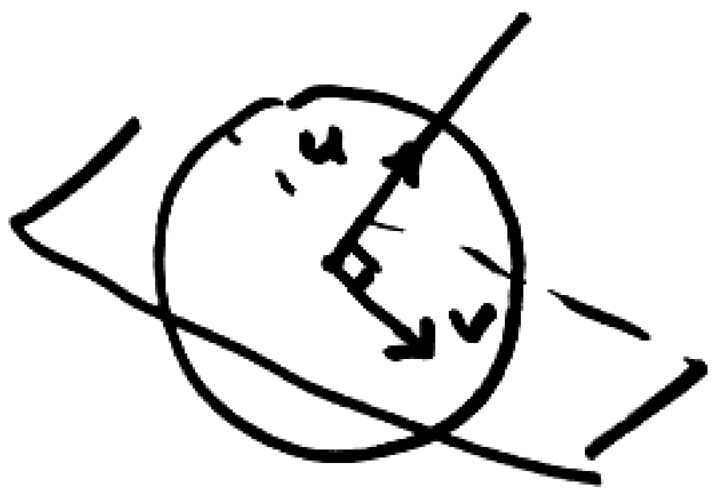
$$e := \underset{\alpha}{\underset{\parallel}{\cos\frac{\theta}{2}}} + \underset{\beta}{\underset{\parallel}{\sin\frac{\theta}{2}}} \cdot u \qquad \alpha^2 + \beta^2 = 1$$

$$\bar{e} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\, u$$

$$1 = e \cdot \bar{e}\ , \qquad e^{-1} = \bar{e}$$

$$\begin{aligned} e u e^{-1} &= (\alpha + u\beta)\, u\, (\alpha - \beta u) \\ &= (\alpha u - \beta)(\alpha - \beta u) \\ &= \alpha^2 u + \alpha\beta - \alpha\beta + \beta^2 u \\ &= (\alpha^2 + \beta^2) u = u \end{aligned}$$

$$u \perp v \in \mathbb{H}^0 \cap \mathbb{H}_1^\times$$



$$\begin{aligned} \langle u, v \rangle &= -\frac{1}{2}\left((u+v)^2 - u^2 - v^2 \right) \\ &= -\frac{1}{2}(uv + vu) \end{aligned}$$

$$u \perp v \iff \boxed{uv = -vu}$$

$$\begin{aligned} e v e^{-1} &= (\alpha + u\beta)\, v\, (\alpha - u\beta) \\ &= (\alpha v + uv\beta)(\alpha - u\beta) \\ &= \alpha^2 v - \alpha\beta\, vu + \alpha\beta\, uv - uvu\beta^2 \end{aligned}$$

$$uvu = v$$

$$eve^{-1} = (\alpha^2 - \beta^2)v + 2\alpha\beta\, uv$$

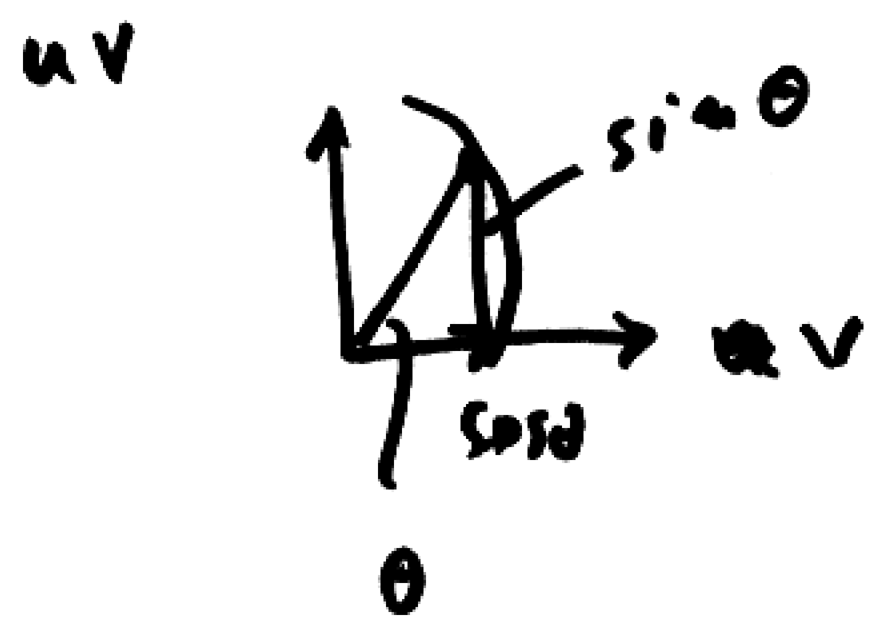
<u>Check</u>: u, v, uv orthonormal basis of $\mathbb{H}^0$

In this basis $r(e)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\alpha^2 - \beta^2 = \cos\theta \qquad 2\alpha\beta = \sin\theta$$

Indeed $r(e)$ is a rotation about u of angle θ



$$e = e_0 + i e_1 + j e_2 + k e_3$$

$$r(e) \in SO(3)$$

$$r(e) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & \cdots \\ \cdots & e_0^2 + e_2^2 - e_1^2 - e_3^2 & \cdots \\ \cdots & \cdots & e_0^2 + e_3^2 - e_1^2 - e_3^2 \end{pmatrix}$$

Euler parameters

$$e\, i\, e^{-1} = (e_0 + i e_1 + e_2 j + e_3 k)\, i\, (e_0 - i e_1 - j e_2 - k e_3)$$

$$= (e_0 + i e_1 + e_2 j + e_3 k)(i e_0 + e_1 - k e_2 + j e_3)$$

$$= e_0 e_1 - e_2 e_3 + e_2 e_3 - e_0 e_1 + \cdots$$

$$\mathrm{tr}(r(e)) = 3 e_0^2 - e_1^2 - e_2^2 - e_3^2$$

$$= 4 e_0^2 - 1$$

$$e = \cos \theta/2 + \sin \theta/2 \; u$$

$e_0 = \cos \theta/2$

$$\boxed{tr(r(e)) = 4\cos^2 \theta/2 - 1}$$

$\theta = 2\pi k/n$ $\quad gcd(k,n) = 1$

n	θ	$tr(r(e))$
1	0	3
2	π	-1
3	$\pm 2\pi/3$	0
4	$\pm \pi/2$	1
5	$\pm 2\pi/5$	$\varepsilon = (1+\sqrt{5})/2$
5	$\pm 4\pi/5$	$\varepsilon' = (1-\sqrt{5})/2$
6	$\pm \pi/3$	2

$e = \frac{1}{2}(1 + i + j + k)$

$\bar{e} = e^{-1}$

$e_0 = 1/2$

$tr(e) = 4e_0^2 - 1$
$= 0$

$$r(e) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$r(i) = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$$

1, e generate

$$\{\pm 1, \pm i, \pm j, \pm k, \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 1$$

maps to A_4 not isom to S_4

(e.g. non-trivial center)

Arithmetic with quaternions.

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

natural

$$\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$$

"naive" integral quaternions

<u>Hurwitz.</u>

$$\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k + \mathbb{Z}\tfrac{1}{2}(1+i+j+k)$$

Hurwitz quaternions

its units are binary tetrahedron gp.