Wigner transform:

$$W(f,g)(x,y) = \iint e^{-2\pi i(xp+yq)} V(f,g)(p,q)\,dp\,dq$$
$$= \int e^{-2\pi i\, xp} f(x+\tfrac{1}{2}p)\overline{g(x-\tfrac{1}{2}p)}\,dx$$

f, g Schwartz functions in $\mathbb{R}$

We have the following summation fmla

$$\sum_{m,n\in\mathbb{Z}} (-1)^{mn} A(f,g)(m,n)\, e^{2\pi i(mx+ny)}$$

$$= \sum_{n\in\mathbb{Z}} e^{2\pi i n y} f(x+n) \cdot \sum_{m\in\mathbb{Z}} e^{-2\pi i m y} g(x+m)$$

where $A(f,g)(\mu,\nu) = \int_{\mathbb{R}} f(x+\tfrac{1}{2}\nu) g(x-\tfrac{1}{2}\nu) e^{-2\pi i \mu x}\,dx$

Pf We compute the $\mu,\nu \in \mathbb{Z}^2$ Fourier coeff of the right hand side

$$\int_0^1\int_0^1 \sum_{m,n} e^{2\pi i n y} f(x+n)\, e^{-2\pi i m y} g(x+m)\, e^{-2\pi i(\mu x+\nu y)}\,dx\,dy$$

make $n-m=\nu$

$$= \int_0^1 \sum_n e^{-2\pi i \mu x} f(x+n) g(x+n-\nu)\,dx$$

$$= \sum_n \int_n^{n+1} e^{-2\pi i \mu x} f(x) g(x-\nu)\,dx$$

$$= \int_{\mathbb{R}} e^{-2\pi i \mu x} f(x) g(x-\nu)\,dx$$

$$= (-1)^{\mu\nu} A(f,g)(\mu,\nu)$$ □

(2)

<u>Fact fmla in $\mathbb{R}^n$</u>

Q is a $n \times n$ real pos def matrix.

$z = x + iy \in \mathfrak{h}$

$a \in \mathbb{N}$

$\mu, \nu \in \mathbb{Q}^n$

Consider the product

$$e^{\frac{\pi}{2ay} {}^t u_2 Q^{-1} u_2} \sum_{n \in \mathbb{Z}^r + \mu} e^{-\pi i a {}^t n Q n \bar{z} + 2\pi i {}^t n (u_2 - a\nu)}$$

$$\cdot\, e^{\frac{\pi a}{2y} {}^t u_1 Q^{-1} u_1} \sum_{l \in \mathbb{Z}^r + a\mu} e^{\frac{\pi i}{a} {}^t l Q l z + 2\pi i {}^t l (u_1 + \nu)}$$

Now we make the substitution

$l = an + m$

$$= e^{\frac{\pi}{2ay} (a^2 {}^t u_1 Q^{-1} u_1 + {}^t u_2 Q^{-1} u_2)} \cdot \sum_{m \in \mathbb{Z}^r} e^{\frac{\pi i}{a} {}^t m Q m z + 2\pi i {}^t m (u_1 + \nu)}$$

$$\cdot \sum_{n \in \mathbb{Z}^r + \mu} e^{-2\pi {}^t n Q n a y + 2\pi i {}^t n (Q m z + a u_1 + u_2)}$$

Now we apply Poisson summation to the inner sum and we get $(D = \det Q)$

$$\frac{\sqrt{D}}{(2ay)^{r/2}} \cdot e^{\frac{\pi}{2ay}\left(a^2\, {}^t u_1 Q^{-1} u_1 + {}^t u_2 Q^{-1} u_2\right)} \cdot \sum_{m \in \mathbb{Z}^r} e^{\frac{\pi i}{a}\, {}^t m Q m z + 2\pi i\, {}^t m (u_1 + v)}$$

$$\cdot \sum_{n \in \mathbb{Z}^r} e^{-\frac{\pi}{2ay}\, {}^t(n - Qmz - au_1 - u_2)\, Q^{-1} (n - Qmz - au_1 - u_2)} \cdot e^{2\pi i\, {}^t n \mu}$$

$$= \frac{\sqrt{D}}{(2ay)^{r/2}} \cdot e^{-\frac{\pi}{y}\, {}^t u_1 Q^{-1} u_2} \cdot \sum_{m, n \in \mathbb{Z}^r} e^{2\pi i ({}^t m \cancel{\nu} + {}^t n \mu)}\, e^{\frac{\pi i}{a}\, {}^t m n}$$

$$\cdot\, e^{-\frac{\pi}{y}\left({}^t v_1 u_1 + {}^t v_2 u_2\right)}$$

$$\cdot\, e^{-\frac{\pi}{2ay}\left({}^t m Q m |z|^2 - {}^t m n (z + \bar z) + {}^t n Q^{-1} n\right)}$$

where $v_1 = {}^t m \bar z - {}^t n\, Q^{-1}$
$v_2 = ({}^t m z - {}^t n\, Q^{-1})/a$

If now $z = \frac{b + \sqrt{-d}}{2a}$ is a CM point then with $a = 1$ in the identity

$|z|^2 = \frac{c}{a}$ $\qquad c = \frac{b^2 + d}{4a}$

$z + \bar z = \frac{b}{a}$

$y = \frac{\sqrt{d}}{2a}$

and $-\frac{\pi}{2y}\left({}^t m Q m |z|^2 - {}^t m n (z + \bar z) + {}^t n Q^{-1} n\right) = -\frac{\pi}{\sqrt{d}}\left(c\, {}^t m Q m - {}^t m n b + a\, {}^t n Q^{-1} n\right)$

(4)

We let $\tilde{Q} = \left(\begin{array}{c|c} 2cQ & -bI_r \\ \hline -bI_r & 2aQ^{-1} \end{array}\right)$ and assume level$(Q) \mid a$

then

$$\pi i \, {}^t(m,n)\, \tilde{Q} \binom{m}{n} \left(\frac{1}{2} + \frac{i}{2\sqrt{d}} \right) = \pi i \left(c\, {}^t m Q m - {}^t m n b + a\, {}^t n Q^{-1} n \right)$$

$$- \frac{\pi}{\sqrt{d}} \left(c\, {}^t m Q m - {}^t m n b + a\, {}^t n Q^{-1} n \right)$$

If d is odd then b is odd, and we get the crossed term $(-1)^{mn}$

The quadratic maps $c\, {}^t m Q m$ and $a\, {}^t n Q^{-1} n$ are linear mod 2 and are hence given by $m \mapsto 2\, {}^t\mu_0 m$ $\quad n \mapsto 2\, {}^t\nu_0 n$ for some $\mu_0, \nu_0 \in \mathbb{Q}^n$ $\quad 2\mu_0, 2\nu_0 \in \mathbb{Z}^n$.

So if d is odd, $u_1 = u_2 = 0$, $\mu = \mu_0$, $\nu = \nu_0$ the r.h.s of the identity gives the value

$$\frac{\sqrt{D}}{(\sqrt{d}/a)^{r/2}} \, \Theta_{\tilde{Q}} \left(\frac{1}{2} + \frac{i}{2\sqrt{d}} \right)$$

as a product of theta series of rank r

If the characteristic $\begin{bmatrix} \mu_0 \\ \nu_0 \end{bmatrix}$ is odd then we get 0.

Example $\dim Q = 2$

$\operatorname{disc} Q = -31$ $\quad$ $\operatorname{disc}(a,b,c) = -23$

$31 \mid a$ $\quad$ $[31, 15, 2] \sim [2, 1, 3]$ $\quad C_1$

$[93, 139, 52] \sim [1, -1, 6]$ $\quad C_2$

$[93, 77, 16] \sim [2, -1, 3]$ $\quad C_3$

$$\operatorname{disc} Q = -31 \qquad \begin{array}{ccc} Q_1 & Q_2 & Q_3 \\ \begin{bmatrix} 2 & 1 \\ 1 & 16 \end{bmatrix} & \begin{bmatrix} 10 & 3 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 10 & -3 \\ -3 & 4 \end{bmatrix} \end{array}$$

$$Q_1C_1 = \left[\begin{array}{cc|cc} 4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ \hline -1 & 0 & 8 & 4 \\ 0 & -1 & 4 & 8 \end{array}\right] \quad Q_1C_2 = \left[\begin{array}{cc|cc} 4 & 2 & -1 & 0 \\ 2 & 4 & 0 & 1 \\ \hline -1 & 0 & 8 & 4 \\ 0 & 1 & 4 & 8 \end{array}\right] \quad Q_1C_3 = \left[\begin{array}{cc|cc} 6 & -3 & 0 & -2 \\ -3 & 6 & 2 & 2 \\ \hline 0 & 2 & 6 & 3 \\ -2 & 2 & 3 & 6 \end{array}\right]$$

$$Q_2C_1 = \left[\begin{array}{cc|cc} 2 & 0 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ \hline 0 & -1 & 6 & 0 \\ -1 & 0 & 0 & 12 \end{array}\right] \quad Q_2C_2 = \left[\begin{array}{cc|cc} 2 & 0 & 0 & -1 \\ 0 & 4 & & \\ \hline 0 & 1 & 6 & 0 \\ -1 & 0 & 0 & 12 \end{array}\right] \quad Q_2C_3 = \left[\begin{array}{cc|cc} 4 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ \hline 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 6 \end{array}\right]$$

$Q_3C_1 =$ " $\quad$ $Q_3C_2 =$ " $\quad$ $Q_3C_3 =$ "

We know take $z = \dfrac{b+\sqrt{-d}}{2a_1 a}$ corresp to the ideal

$\mathfrak{a}\mathfrak{a}_1$ in $\mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{-d})$ and such that

z/a corresp to $\mathfrak{a}_1\mathfrak{a}^2$, $\quad az$ to $\mathfrak{a}_1$, and $a_1 z/d$ to $\mathfrak{a}\mathfrak{D}$

Then $|z|^2 = \dfrac{b^2+d}{4a_1^2a^2} = \dfrac{c}{a_1 a}$ $\qquad$ $c = \dfrac{b^2+d}{4a_1 a}$

$z + \bar{z} = \dfrac{b}{a_1 a}$ $\qquad$ $y = \dfrac{\sqrt{d}}{2a_1 a}$

b odd

$b \mid d$ hence $b \mid c$

$a \mid c$

$2d$ prime to $a_1 a_2$

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Then

$$\frac{-\pi}{2ay}\left({}^t m Q m |z|^2 - {}^t m n (z+\bar{z}) + {}^t n Q^{-1} n \right)$$

$$= \frac{-\pi}{a\sqrt{d}} \underbrace{\left({}^t m Q m \frac{c}{} - {}^t m n \frac{b}{} + {}^t n Q^{-1} n \, a a_1 \right)}_{{}^t(m,n)\tilde{Q}\binom{m}{n}}$$

$$2\pi i \, {}^t(m,n)\tilde{Q}\binom{m}{n} \cdot \left(\frac{b+\sqrt{-d}}{2ad} \right) \equiv (\ldots) + {}^t m Q m \cdot c -$$

$p = 1523$

$\mathfrak{p}_3 = [3,1,127]$

$\mathfrak{p}_{19} = [19,15,23]$ generator $\quad 19, \frac{15+\sqrt{}}{2}$

$\mathfrak{p}_{19}^2 \sim \mathfrak{p}_3 \quad -53,1 \quad \mathfrak{p}_{19}^2\bar{\mathfrak{p}}_3 = \left(\frac{53+\sqrt{-p}}{2}\right)$

$\mathfrak{p}_{19}^3 \sim \mathfrak{p}_3^2 \quad 457,5 \quad \mathfrak{p}_{19}^3\bar{\mathfrak{p}}_3^2 = \left(\frac{-457+5\sqrt{-p}}{2}\right)$

$\bar{\mathfrak{p}}_{127} = [127,-1,3] \sim \mathfrak{p}_3 \quad \frac{-1+\sqrt{-p}}{2} \qquad \frac{53+\sqrt{-p}}{2} = 27 + \frac{-1+\sqrt{-p}}{2} \in \bar{\mathfrak{p}}_3$

$\bar{\mathfrak{p}}_{43} = [43,-5,9] \sim \mathfrak{p}_3^2 \quad \frac{-5+\sqrt{-p}}{2} \qquad = 38 + \frac{15+\sqrt{}}{2} \in \mathfrak{p}_{19}$

$\mathfrak{p}_{127} = [127,1,3] \sim \bar{\mathfrak{p}}_3 \quad \frac{1+\sqrt{-p}}{2}$

$\mathfrak{p}_{43} = [43,+5,9] \sim \bar{\mathfrak{p}}_3^2 \quad \frac{+5+\sqrt{-p}}{2} \quad \cancel{\mathfrak{p}_{127}\bar{\mathfrak{p}}_3 = \left(\frac{-1+\sqrt{-p}}{2}\right)}$

$\cancel{\mathfrak{p}_{43}\bar{\mathfrak{p}}_3^{-2} = \left(\frac{-5+\sqrt{-p}}{2}\right)}$

$\cancel{\mathfrak{p}_{19}^2\mathfrak{p}_{127}^{-1}}$

$$\begin{array}{r} 1523 \\ +\ 457 \\ \hline 1980 \end{array} \qquad 990 \qquad \begin{array}{r} 457 \\ 25 \\ \hline 432 \end{array} \qquad 9900 \quad 4500 \quad 450$$

$\mathfrak{p}_{127}\mathfrak{p}_3 = \left(\frac{1+\sqrt{-p}}{2}\right) \qquad \mathfrak{p}_{127}\mathfrak{p}_{19}^2\cdot(3) = \left(\frac{1+\sqrt{-p}}{2}\right)\left(\frac{53+\sqrt{-p}}{2}\right)$

$\mathfrak{p}_{43}\mathfrak{p}_3^2 = \left(\frac{+5+\sqrt{-p}}{2}\right) \qquad \mathfrak{p}_{43}\mathfrak{p}_{19}^3\cdot(3)^2 = \left(\frac{-457+5\sqrt{-p}}{2}\right)\left(\frac{5+\sqrt{-p}}{2}\right)$

$$\mathfrak{p}_{127}\mathfrak{p}_{19}^2 = \frac{+245 \mp 9\sqrt{-1523}}{2} \equiv \square \mod \sqrt{-p}$$

$$\mathfrak{p}_{43}\mathfrak{p}_{19}^3 = \frac{1}{9}\,\frac{-2285 + 5(-1523) + (25-457)\sqrt{-p}}{4}$$

$$= \frac{1}{9}\,\frac{-4950 - 216\sqrt{-p}}{2}$$

$$= -\frac{550 - 24\sqrt{-p}}{2} = -275 - 12\sqrt{-p} \equiv \square \mod \sqrt{-p}$$

$$\mathfrak{p}_{19}^7 = \frac{-38747 \pm 1167\sqrt{-p}}{2} \equiv \square \mod \sqrt{-p}$$

$\psi(\mathfrak{p}_{43})$

$\psi(\mathfrak{p}_{127})$

$$\boxed{\text{We get } \frac{L(\psi,1)}{\Omega_{A(p)}} = \left(2^7\cdot(5\cdot 7\cdot 71)\right)^2} \text{ as predicted.}$$

$\uparrow$ comes from $\prod_{\mathfrak{p}|p} c_{\mathfrak{p}}$ $\quad \left(\prod_{\lambda|3} c_\lambda = 1\right)$.

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$$Q_z(m,n) = |mz-n|^2/2y$$

Fact Fmla take

$$\mu = r/N \quad , \quad \nu = s/N$$

$$\sum_{m,n} e^{2\pi i (m\nu + n\mu)} \, e^{\frac{\pi i mn}{N}} \, e^{-\frac{\pi}{N} Q_z(m,n)}$$

$$= \sqrt{2yN} \quad \theta\begin{bmatrix} N\mu \\ \nu \end{bmatrix}\left(\frac{z}{N}\right) \theta\begin{bmatrix} \mu \\ -N\nu \end{bmatrix}(-N\bar{z})$$

$$f, g : \quad \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$$

$$\hat{g}(r) = \sum_{s \bmod N} g(s) \, e^{-\frac{2\pi i}{N} rs}$$

On rhs

$$\sqrt{2yN} \quad e^{-\frac{2\pi i rs}{N}} \theta\begin{bmatrix} 0 \\ s/N \end{bmatrix}\left(\frac{z}{N}\right) \theta\begin{bmatrix} r/N \\ 0 \end{bmatrix}(-N\bar{z})$$

pass $e^{-\frac{2\pi i rs}{N}}$ to lhs and add with factor $f(r)\,\hat{g}(s)$

On lhs

$$\sum_{m,n} \left[\sum_{r,s \bmod N} f(r)\hat{g}(s) \, e^{\frac{2\pi i rs}{N}} \, e^{\frac{2\pi i}{N}(ms+nr)} \right] e^{\frac{\pi i mn}{N}} \, e^{-\frac{\pi}{N} Q_z(m,n)}$$

On rhs

$$\sqrt{2yN} \quad \sum_{r \bmod N} f(r) \, \theta\begin{bmatrix} r/N \\ 0 \end{bmatrix}(-N\bar{z}) \, . \sum_{s \bmod N} \hat{g}(s) \theta\begin{bmatrix} 0 \\ s/N \end{bmatrix}\left(\frac{z}{N}\right)$$

$$\sum_{n\in\mathbb{Z}+r/N} \sum_{r \bmod N} f(r)\, e^{\pi i n^2(-N\bar{z})} = \sum_{n\in\mathbb{Z}} f(n)\, e^{\pi i n^2\left(\frac{-\bar{z}}{N}\right)}$$

$$\sum_{n\in\mathbb{Z}} \sum_{s \bmod N} \hat{g}(s)\, e^{\pi i n^2 \frac{z}{N}}\, e^{\frac{2\pi i n s}{N}} = N \sum_{n\in\mathbb{Z}} g(n)\, e^{\frac{\pi i n^2 z}{N}}$$

$$W(f,g)(m,n) := \sum_{r,s \bmod N} \bar{f}(r)\, \hat{g}(s)\, e^{\frac{2\pi i r s}{N}}\, e^{\frac{2\pi i}{N}(ms+nr)}$$

If $z \mapsto z+2$ then $f(n) \mapsto f(n)\, e^{\frac{-2\pi i n^2}{N}}$

$g(n) \mapsto g(n)\, e^{\frac{2\pi i n^2}{N}}$

on lhs $(m,n) \mapsto (m, n+2m)$

$$W(f,g)(m,n) \longmapsto \cancel{W(f,g)(m,n)}\; e^{\frac{-2\pi i m^2}{N}} \cdot W\left(f, g(s) e^{\frac{2\pi i 2ms}{N}}\right)$$

$$\sum_{r,s} f(r)\, \hat{g}(s)\, e^{\frac{2\pi i r(s+2m)}{N}}\, e^{\frac{2\pi i}{N}(ms+nr)}$$

$$= \sum_{r,s} f(r)\, \hat{g}(s-2m)\, e^{\frac{2\pi i r s}{N}}\, e^{\frac{2\pi i}{N}(ms+nr)} \cdot e^{\frac{2\pi i}{N}(-2m^2)}$$

We want to find all pairs of fctns f & g s.t. $W(f,g)$ depends on $\phi(m,n)$ mod N where ϕ is a homog quadr polynomial, in order to get the most general factorization fmla which relates to Eisenstein series

$$f(r) \doteq e^{\frac{2\pi i}{N} u r^2} \qquad \hat{g}(s) = e^{\frac{2\pi i}{N} v s^2}$$

$$\sum_{r,s} e^{\frac{2\pi i}{N} u r^2} e^{\frac{2\pi i}{N} v s^2} e^{\frac{2\pi i}{N} rs} e^{\frac{2\pi i}{N}(ms+nr)}$$

$$= \sum_s \sum_r e^{\frac{2\pi i}{N} u (r + 2^{-1}u^{-1}(s+n))^2} e^{-\frac{2\pi i}{N} u^{-1} 4^{-1}(s+n)^2}$$

$$e^{\frac{2\pi i v s^2}{N}} e^{\frac{2\pi i}{N} ms}$$

$$= \left(\sum_r e^{\frac{2\pi i}{N} u r^2}\right) \cdot \sum_s e^{\frac{2\pi i}{N}(v s^2 - 4^{-1}u^{-1}s^2 - u^{-1}2^{-1}sn - 4^{-1}u^{-1}n^2 + ms)}$$

Assume $v - 4^{-1}u^{-1}$ is invertible

$$= (\quad) \cdot \sum_s e^{\frac{2\pi i}{N}(v - 4^{-1}u^{-1})(s + 2^{-1}(v-4^{-1}u^{-1})^{-1}(m - u^{-1}2^{-1}n))^2}$$

$$\cdot e^{-\frac{2\pi i}{N} 4^{-1}(v-4^{-1}u^{-1})^{-1}(m - u^{-1}2^{-1}n)^2} \cdot e^{-\frac{2\pi i}{N} 4^{-1}u^{-1}n^2}$$

$$= \left(\sum_r e^{\frac{2\pi i}{N} u r^2}\right)\left(\sum_s e^{\frac{2\pi i}{N}(v - 4^{-1}u^{-1})s^2}\right)$$

$$\cdot\, e^{-\frac{2\pi i}{N}\left((4v-u^{-1})^{-1}(m - u^{-1}2^{-1}n)^2 + 4^{-1}u^{-1}n^2\right)}$$

$$(4v-u^{-1})^{-1}(m^2 - u^{-1}mn + 4^{-1}u^{-1\,2} n^2) + 4^{-1}u^{-1}n^2$$

$$(4v-u^{-1})^{-1}\left[m^2 - u^{-1}mn + 4^{-1}\cancel{u^{-2}}n^2 + vu^{-1}n^2 - 4^{-1}\cancel{u^{-2}}n^2\right]$$

$$(4v-u^{-1})^{-1}\left[m^2 - u^{-1}mn + u^{-1}v n^2\right]$$

$$(4uv-1)^{-1}\left[u m^2 - mn + v n^2\right]$$

N odd

Hence $\quad f(r) = e^{\frac{2\pi i}{N} ur^2} \qquad \hat{g}(s) = e^{\frac{2\pi i}{N} vs^2}$

$$\boxed{\begin{array}{l} W(f,g)(m,n) = \left(\frac{-D}{N}\right)\cdot N \cdot e^{\frac{2\pi i}{N}(um^2 - mn + vn^2)D^{\nu}} \\ \\ D = 4uv-1 \text{ invertible mod } N \end{array}}$$

χ_1, χ_2 characters mod N

$$\sum_{r,s} \chi_1(r)\,\bar{\chi}_2(s)\, e^{\frac{2\pi i rs}{N}}\, e^{\frac{2\pi i}{N}(ms+nr)}$$

$rs = t \qquad s = r^{\nu}t$

$$\sum_{r,t} \chi_1(r)\,\chi_2(r)\,\bar{\chi}_2(t)\, e^{\frac{2\pi i t}{N}}\, e^{\frac{2\pi i}{N}(mr^{\nu}t + nr)}$$

$$\cancel{= \sum_t \sum_r \chi_1\chi_2(r)\, e^{\frac{2\pi i}{N} r(mt^{\nu}+n)}\, \bar{\chi}_2(t)\, e^{\frac{2\pi i t}{N}}}$$

$$\cancel{= \sum_t \bar{\chi}_1\chi_2(mt^{\nu}+n)\, \chi_2(t)\, e^{\frac{2\pi i t}{N}}\; G(\chi_1\bar{\chi}_2)}$$

$$\cancel{= \sum_t \bar{\chi}_1(mt^{\nu}+n)\, \chi_2(m+nt)\, e^{\frac{2\pi i t}{N}}\; G(\chi_1\chi_2)}$$

$$= \sum_r \chi_1\chi_2(r) \sum_t \bar{\chi}_2(t)\, e^{\frac{2\pi i}{N} t(1+mr^{\nu})}\, e^{\frac{2\pi i}{N} nr}$$

$$= \sum_r \chi_1\chi_2(r)\, \chi_2(1+mr^{\nu}) \cdot e^{\frac{2\pi i}{N} nr} \cdot G(\bar{\chi}_2)$$

$$= \sum_r \chi_1(r)\, \chi_2(r+m)\, e^{\frac{2\pi i nr}{N}} \cdot G(\bar{\chi}_2)$$

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(Talking to Nick) Finite Wigner transform

$$\sum_{x,y} \chi_1(x)\chi_2(y)\, e_N(xy)\, e_N(ax+by)$$

$$= \sum_u e_N(u) \sum_{xy=u} \chi_1(x)\chi_2(y)\, e_N(ax+by)$$

$a,b \neq 0$

$$= \bar{\chi}_1(a)\bar{\chi}_2(b) \sum_u e_N(u/ab) \underbrace{\sum_{xy=u} \chi_1(x)\chi_2(y)\, e_N(x+y)}_{\text{Kloosterman sum}}$$

see $\mathcal{H}\left(\begin{smallmatrix}\alpha\text{'s} \\ \beta\text{'s}\end{smallmatrix}\right)$ in his book; Fourier transf of Kloosterman

Mellin transf is easy product of Gauss sums.

$$\frac{\sqrt{D}}{(2ay)^{r/2}} \cdot e^{-\frac{\pi}{y}\, {}^t u_1 u_2} \cdot \sum_{m,n\in\mathbb{Z}^r} e^{2\pi i ({}^t m \mu + {}^t n \nu)} \cdot e^{\frac{\pi i}{a}\, {}^t m n}.$$

$$\cdot\, e^{-\frac{\pi}{y}\left(({}^t m A \bar{z} - {}^t n\, {}^t A^{-1}) u_1 + ({}^t m A z - {}^t n\, {}^t A^{-1}) \frac{u_2}{a}\right)}$$

$$\cdot\, e^{-\frac{\pi}{2ay}\left({}^t m Q m |z|^2 - {}^t m n (z+\bar{z}) + {}^t n Q^{-1} n\right)}$$

Note that

$$({}^t m A \bar{z} - {}^t n\, {}^t A^{-1}) \cdot ({}^t A m z - A^{-1} n)$$

$$= {}^t m Q m |z|^2 - {}^t m n (z+\bar{z}) + {}^t n Q^{-1} n$$

If $z = \dfrac{b_1 + \sqrt{-d}}{2a_1}$ then $|z|^2 = \dfrac{b_1^2 + d}{4a_1^2} = \dfrac{c_1}{a_1}$

$$z + \bar{z} = \frac{b_1}{a_1}$$

$$y = \frac{\sqrt{d}}{2a_1}$$

hence the quadr term gives

$$e^{-\frac{\pi}{a\sqrt{d}}\left(c_1\, {}^t m Q m - {}^t m n\, b_1 + a_1\, {}^t n Q^{-1} n\right)}$$

$${}^t u = \left({}^t m \left(\frac{b_1 + \sqrt{-d_1}}{2}\right) - {}^t n\, a_1 Q^{-1}\right)$$

$${}^t \bar{u} Q u = a_1\left(c_1\, {}^t m Q m - {}^t m n\, b_1 + a_1\, {}^t n Q^{-1} n\right)$$

$$u = \begin{pmatrix} \frac{b_1+\sqrt{-d_1}}{2} & 0 \\ 0 & -a_1 Q^{-1} \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

Fix Q pos def of rank r & disc $= D$

given (a, b, c) - binary pos def discr $= -d$ odd

consider

$$Q_C = \begin{pmatrix} 2cQ & -bI_r \\ -bI_r & 2aQ^{-1} \end{pmatrix}$$

$C =$ class of (a, b, c) modulo $\Gamma_0(N)$

$N =$ level of Q

We assume aQ^{-1} is integral i.e. $N \mid a$

The isometry class of Q_C only depends on C.

What is $\sum_C \Theta_{Q_C}$?

or perhaps $\sum_{Q, C} \Theta_{Q_C}$?

Fact Fmla

without using decomp $A^tA = Q$

$$e^{\frac{\pi}{2ay}\,{}^tu_2Q^{-1}u_2} \sum_{n\in\mathbb{Z}^r+\mu} e^{-\pi i a\,{}^tnQn\bar{z} + 2\pi i\,{}^tn(u_2-a\nu)}$$

$$e^{\frac{\pi a}{2y}\,{}^tu_1Q^{-1}u_1} \sum_{\ell\in\mathbb{Z}^r+\mu \to a?} e^{\frac{\pi i}{a}\,{}^t\ell Q\ell z + 2\pi i\,{}^t\ell(u_1+\nu)}$$

$\ell = an+m$

$$e^{\frac{\pi}{2ay}(a^2\,{}^tu_1Q^{-1}u_1 + {}^tu_2Q^{-1}u_2)} \sum_{m\in\mathbb{Z}^r} e^{\frac{\pi i}{a}\,{}^tmQmz + 2\pi i\,{}^tm(u_1+\nu)}$$

$$\sum_{n\in\mathbb{Z}^r+\mu} e^{-2\pi\,{}^tnQnay + 2\pi i\,{}^tn(Qmz + au_1+u_2)}$$

Poisson summation in inner sum

$$\frac{\sqrt{D}}{(2ay)^{r/2}} e^{\frac{\pi}{2ay}(a^2\,{}^tu_1Q^{-1}u_1 + {}^tu_2Q^{-1}u_2)} \sum_{m\in\mathbb{Z}^r} e^{\frac{\pi i}{a}\,{}^tmQmz + 2\pi i\,{}^tm(u_1+\nu)}$$

$$\sum_{n\in\mathbb{Z}^r} e^{-\frac{\pi}{2ay}\,{}^t(n-Qmz-au_1-u_2)Q^{-1}(n-Qmz-au_1-u_2)} \; ? \; e^{2\pi i\,{}^tn\mu}$$

$${}^t\ell Q\ell = a^2\,{}^tnQn + a\,{}^tnQm + a\,{}^tmQn + {}^tmQm$$

$\cancel{{}^t\ell(u_1+\nu) = {}^t\ell u_1 + {}^t\ell\nu}$

$${}^t\ell(u_1+\nu) = a\,{}^tnu_1 + a\,{}^tn\nu + {}^tmu_1 + {}^tm\nu$$

$$\boxed{-\frac{\pi}{2ay}\left({}^t n\, Q^{-1} n - {}^t m n (z+\bar z) + {}^t m Q m\, |z|^2 \right) + \frac{\pi i}{a}\, {}^t m n}$$

$$-(z+\bar z) = -2z + (z-\bar z)$$
$$= -2z + 2iy$$

$$-\frac{\pi}{2ay}\left(-{}^t m n\, 2z\right) = -\frac{\pi}{2ay}\left(-(z+\bar z) - 2iy\right){}^t m n$$
$$= -\frac{\pi}{2ay}\left(-(z+\bar z)\,{}^t m n\right) + \frac{\pi i}{a}\,{}^t m n$$

$$-\frac{\pi}{2ay}\left(z^2\, {}^t m Q m\right) + \frac{\pi i}{a}\,{}^t m Q m\, z = \frac{-\pi}{2ay}\left(z^2 - 2iyz\right){}^t m Q m$$
$$= -\frac{\pi}{2ay}\left(z - 2iy\right) z\, {}^t m Q m$$
$$= -\frac{\pi}{2ay}\, z \bar z\, {}^t m Q m$$

$$\frac{\sqrt{D}}{(2ay)^{r/2}} \cdot e^{-\frac{\pi}{y}\, {}^t u_1 Q^{-1} u_2} \sum_{m,n \in \mathbb{Z}^r} e^{2\pi i({}^t m\mu + {}^t n\nu)} \cdot e^{\frac{\pi i}{a}\, {}^t m n}$$

$$\cdot\, e^{-\frac{\pi}{y}\left(\underbrace{({}^t m \bar{z} - {}^t n Q^{-1})}_{{}^t u} u_1 + \underbrace{({}^t m z - {}^t n Q^{-1})}_{{}^t v} u_2 \right)}$$

$$\cdot\, e^{-\frac{\pi}{2ay}\left({}^t m Q m |z|^2 - {}^t m n (z+\bar{z}) + {}^t n Q^{-1} n \right)}$$

$u = {}^t(u_1, \dots, u_n)$
$v = {}^t(v_1, \dots, v_n)$

$u_1 = {}^t(x_1, \dots, x_r) = \cancel{?}\, X$
$u_2 = {}^t(y_1, \dots, y_r) = \cancel{?}\, Y$

$Q = (a_{jk})$

$${}^t u Q v = ({}^t m \bar{z} - {}^t n Q^{-1})\, Q\, (m z - Q^{-1} n)$$
$$= {}^t m Q m |z|^2 - {}^t n m (z + \bar{z}) + {}^t n Q^{-1} n$$

$$\frac{\partial^2}{\partial x_j y_k}({}^t u X + {}^t v Y) = u_j v_k$$

$$\sum_{j,k} a_{jk} \frac{\partial^2}{\partial x_j y_k}({}^t u X + {}^t v Y) = \sum_{j,k} a_{jk} u_j v_k = {}^t u Q v$$

$A\, {}^t A = Q$

${}^t A Q^{-1} A = I_r$

${}^t u' = {}^t u A$
${}^t v' = {}^t v A$

make $u_1 = A X'$
$u_2 = A Y'$

then get $e^{-\frac{\pi}{y}\, {}^t X' Y'}\; e^{{}^t u X' + {}^t v Y'}$

with ${}^t u' v' = * Q_z$

Any symmetric monomial on the r.h.s is then non negative

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$${}^t n Q^{-1} n - {}^t n m Z - {}^t n \cancel{Q^{-1}} a u_1 - {}^t n \cancel{Q^{-1} u_2}$$

$$- \cancel{{}^t m n Z} + \cancel{{}^t m Q m Z^2} + \cancel{{}^t m u_1 a Z} + {}^t m \cancel{u_2} Z$$

$$- a \cancel{{}^t u_1 Q^{-1}} n + a \cancel{{}^t u_1} m Z + a^2 \cancel{{}^t u_1} Q^{-1} u_1 + a \cancel{{}^t u_1} Q^{-1} u_2$$

$$- {}^t u_2 \cancel{Q^{-1}} n + {}^t u_2 \cancel{m} Z + \cancel{{}^t u_2 Q^{-1}} a u_1 + {}^t u_2 Q^{-1} u_2$$

u_1

$$-\frac{\pi}{2ay}\left(- {}^t n Q^{-1} a + {}^t m a Z - a {}^t n Q^{-1} + a {}^t m Z + a^2 \cancel{{}^t u_1} Q^{-1} + 2a {}^t u_2 Q^{-1}\right) u_1$$

$$+\frac{\pi}{2ay}\left(\cancel{a^2 {}^t u_1 Q^{-1} u_1}\right)$$

$$+ 2\pi i \, {}^t m u_1$$

$Z - \bar{Z} = 2iy$

$$-\frac{\pi}{y} {}^t m Z + 2\pi i \, {}^t m = \pi \, {}^t m\left(-\frac{Z}{y} + 2i\right) = \pi \, {}^t m \frac{(-Z + 2iy)}{y} = \pi \, {}^t m \frac{(-\bar{Z})}{y} = -\frac{\pi}{y} {}^t m \bar{Z}$$

$$= \boxed{-\frac{\pi}{y}\left(- {}^t n Q^{-1} + {}^t m \bar{Z}\right) u_1 + -\frac{\pi}{y} {}^t u_2 Q^{-1} u_1}$$

u_2

$$-\frac{\pi}{2ay}\left(- {}^t m Q^{-1} + {}^t m Z - {}^t n Q^{-1} + {}^t m Z + {}^t \cancel{u_2 Q^{-1}}\right) u_2$$

$$+\frac{\pi}{2ay}\left({}^t \cancel{u_2 Q^{-1}} u_2\right)$$

$$= \boxed{-\frac{\pi}{ay}\left({}^t m Z - {}^t n Q^{-1}\right)}$$

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$$-\frac{\pi}{2ay}\left({}^t n Q^{-1} n - {}^t n m Z - {}^t m n Z + {}^t m Q m Z^2\right)$$

$$+ \frac{\pi i}{a} {}^t m Q m Z$$

$$Q = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \qquad 7Q^{-1} = \begin{bmatrix} 6 & -3 & -2 \\ -3 & 5 & 1 \\ -2 & 1 & 3 \end{bmatrix} \simeq \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

$$z = \frac{3+\sqrt{-19}}{2 \times 7}$$

$$y = \frac{\sqrt{19}}{2 \times 7}$$

$$|z|^2 = \frac{1}{7}$$

$$z + \bar{z} = \frac{3}{7}$$

$${}^t m Q m |z|^2 - {}^t m n (z+\bar{z}) + {}^t n Q^{-1} n$$

$$= \frac{1}{7}\left({}^t m Q m - 3\,{}^t m n + 7\,{}^t n Q^{-1} n \right)$$

mod 2 gives μ and ν

$$\mu = [0, 0, \tfrac{1}{2}]$$

$$\nu = [0, 1/2, \tfrac{1}{2}]$$