Oct 11, 2007

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) =: \text{Schur indicator}$$

V irreducible repn

$$= \langle Sym^2 V, 1 \rangle - \langle \Lambda^2 V, 1 \rangle$$

If V cplx (χ_V is not real valued) then $V \not\cong V^*$ as repn so both multiplicities are zero.

If χ_V real valued only of these numbers equals 1 (the other 0)

$\varepsilon = +$ $\langle Sym^2 V, 1 \rangle = 1$

$\varepsilon = -1$ $\langle \Lambda^2 V, 1 \rangle = 1$

<u>Cor</u> If $|G|$ is odd. then every repn of G is cplx (except the trivial repn)

Pf $\frac{1}{|G|} \sum_{g \in G} \chi_V(g)$ = Schur indicator ②

since $g \mapsto g^2$ is a bijection

$$\langle \chi_V, 1 \rangle = \begin{cases} 1 & V \cong 1 \\ 0 & V \not\cong 1 \end{cases}$$

Prop.

~~Thm~~ All χ are real valued iff every element of G is conjugate to its inverse.

Pf χ real valued $\iff$ $\chi(g) = \chi(g^{-1})$ for all $g \in G$.

All χ real valued $\iff$ $\chi(g) = \chi(g^{-1})$
for all $g \in G$
all χ

$\iff$ $f(g) = f(g^{-1})$
all $g \in G$
f class function

Take $f = \delta_C$ C = conjugacy class

$$\delta_C(g) = \begin{cases} 1 & g \in C \\ 0 & g \notin C \end{cases}$$

If all χ real valued

$\Rightarrow \quad \delta_C(g) = \delta_C(g^{-1})$

if $g \in C \Rightarrow g^{-1} \in C$ $\square$

E.g. $G = S_n$ satisfies the hypotesis

· schur indicator of V = standard repn of S_4

$$\frac{1}{24}(3 + 6\times 3 + 8\times 0 + 6\times(-1) + 3\times 3)$$
$$= +1$$

$$S_4 \hookrightarrow \cong O(3) \subseteq Gl_3(\mathbb{R})$$

preserving usual metric

same for A_4 standard is real

$$A_4 \hookrightarrow O(3) \qquad \varepsilon = \frac{1 \pm \sqrt{5}}{2}$$

· A_5 Y, Z 3-dim'l repn

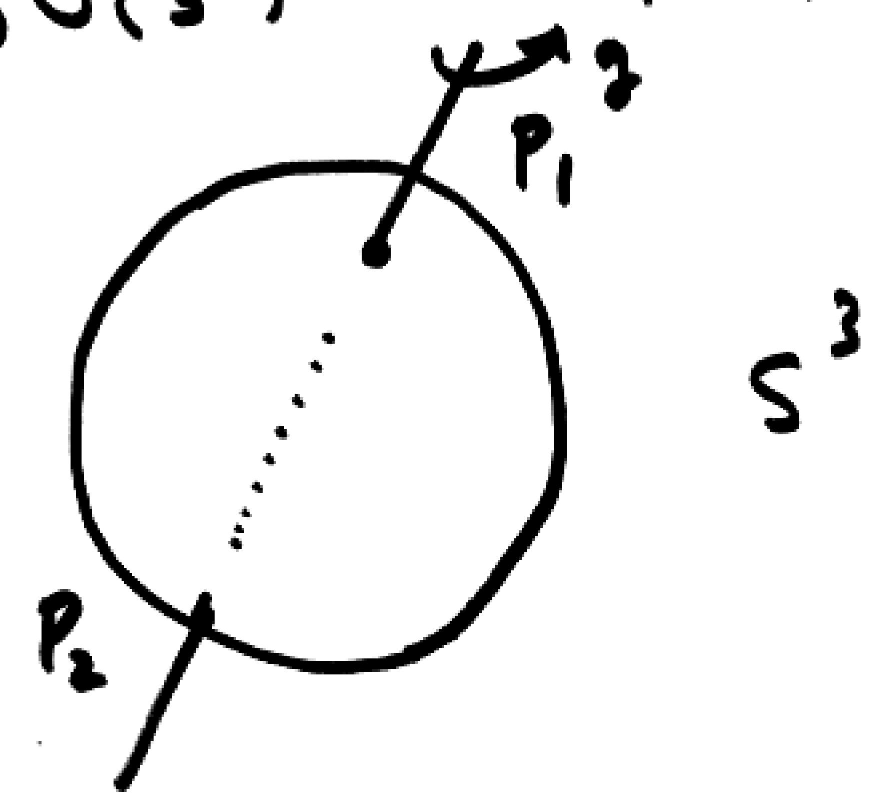
$$\text{schur} = \frac{1}{60}(3 + 20\times 0 + 15\times 3 + 12\varepsilon + 12\varepsilon')$$
$$= 1$$

Also get $A_5 \hookrightarrow O(3)$

Finite groups of rotations in $\mathbb{R}^3$

$G \subseteq SO(3)$ $\quad |G| = n$

$1 \neq g \in G$



$g \longrightarrow \{P_1, P_2\}$ poles

Let $\mathcal{P} = \{$ set of poles of $G\}$

- G acts on $\mathcal{P}$

If P pole of h then

gP pole ghg^{-1}

$G \curvearrowright \mathcal{P}$

Let $N = \#$ of orbits of this action

$P_1, P_2, \dots, P_N$ representatives of these orbits.

$$G_i := \mathrm{Stab}_G(P_i)$$

$$m_i := \# |G_i| \qquad \mid n$$

$$2(|G| - 1) = \sum_{g \neq 1} \#\{P \in \mathcal{P} \mid gP = P\}$$

$$= \sum_{i=1}^{N} (|G_i| - 1) \cdot \frac{|G|}{|G_i|}$$

~~orbits~~

$$= \sum_{P \in \mathcal{P}} (|\mathrm{Stab}_G(P)| - 1)$$

$$1 - \frac{1}{n} = \frac{1}{2} \sum_{i=1}^{N} \left(1 - \frac{1}{m_i}\right)$$

assume $n > 1$

$m_i > 1$ by defn of $\mathcal{P}$

$$1 - \frac{1}{m_i} \geq 1/2$$

$$1 > 1 - \frac{1}{n} \geq \frac{N}{4}$$

$$\Rightarrow \quad N < 4$$

i.e. there are at most 3 orbits.

$\underline{N=1}$ $\times$

$\underline{N=2}$

$$1-\frac{1}{n}=\frac{1}{2}\left(1-\frac{1}{n_1}\right)+\frac{1}{2}\left(1-\frac{1}{n_2}\right)$$

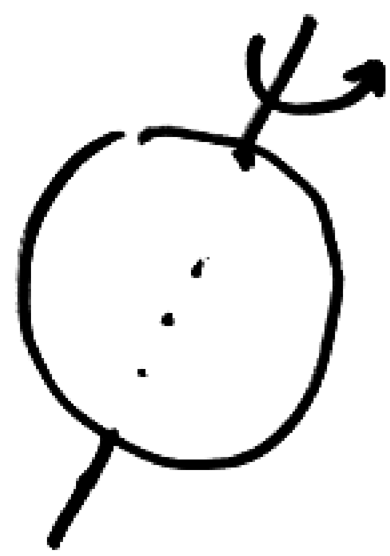
$$\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}$$

$$2=\frac{n}{n_1}+\frac{n}{n_2}$$

$$\Rightarrow \quad n=n_1=n_2$$

$$\Rightarrow \quad G \subseteq SO(2) \simeq S^1$$

$$\Rightarrow \quad G \text{ cyclic}$$



$\underline{N=3}$

$$1+\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}$$

$n_1 \leq n_2 \leq n_3$

lhs > 1 $\Rightarrow$ $n_1 = 2$

$$\frac{1}{2} + \frac{2}{n} = \frac{1}{n_2} + \frac{1}{n_3}$$

lhs $> 1/2$

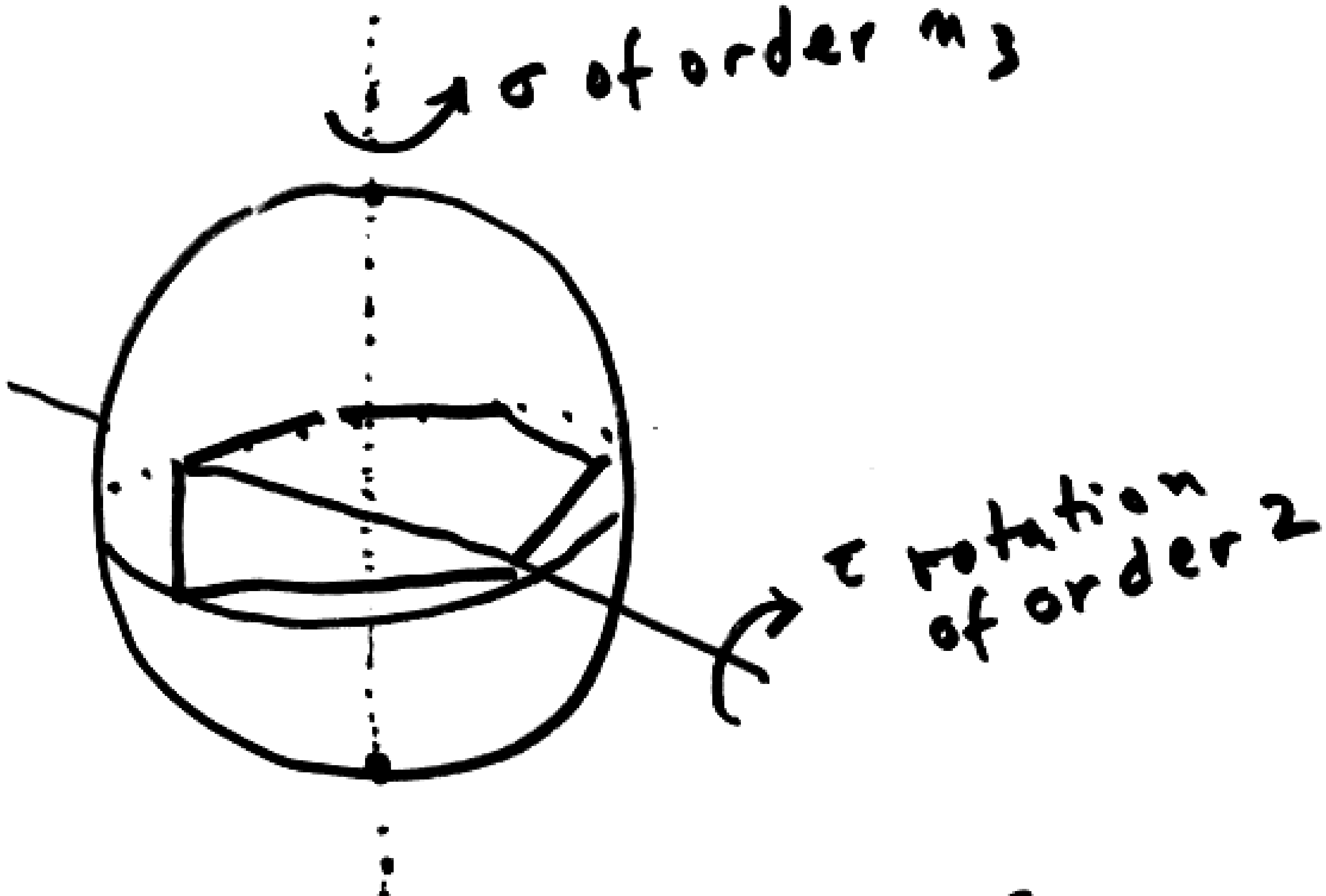
if $n_2 = 2$

get ∞ solutions $(2, 2, n_3)$

$n = 2n_3$

Is there such G?

yes $G = D_n$



rhs $\leq \frac{2}{n_2}$

$\frac{1}{2} < \frac{2}{n_2}$

$\Rightarrow \quad n_2 < 4$

if $\quad n_2 > 2 \quad$ then $n_2 = 3$

$$1 + \frac{2}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{n_3}$$

$$\frac{1}{6} + \frac{2}{n} = \frac{1}{n_3}$$

lhs $> 1/6 \quad \Rightarrow \quad n_3 < 6$

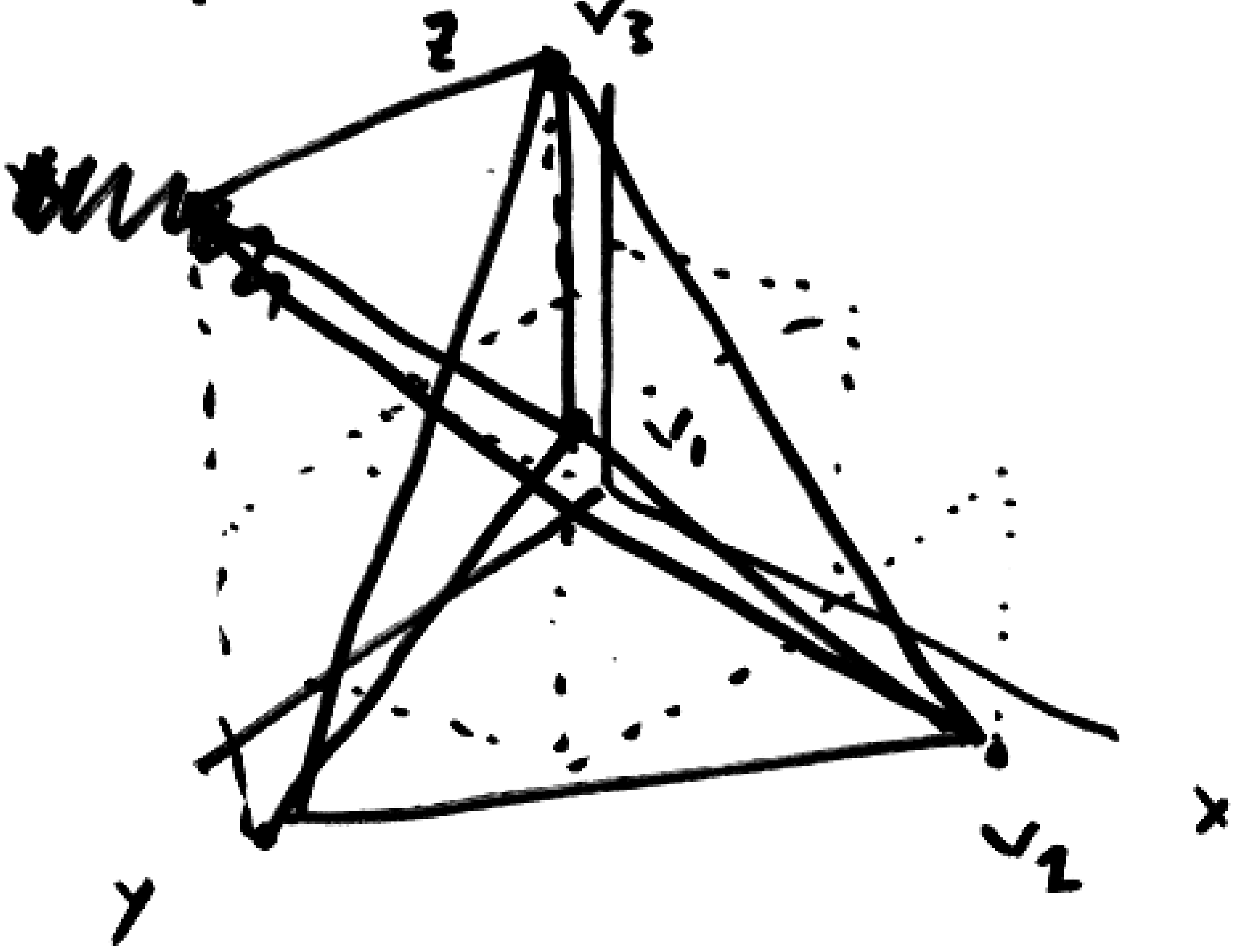
$n_2 = 3 \leq n_6 < 5$

$\Rightarrow \quad n_3 = 3, 4, 5$

(n_1, n_2, n_3)	n	G	name
$(2, 2, m)$	$2m$	dihedral	D_m
$(2, 3, 3)$	12	tetrahedral	A_4
$(2, 3, 4)$	24	cube/octahedron	S_4
$(2, 3, 5)$	60	icos/dodecahedron	A_5

Tetrahedron

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$



$$\langle v_i, v_j \rangle = -1 \qquad i \neq j$$

$$\langle v_i, v_i \rangle = 3$$

$$\begin{aligned} \mathrm{dist}(v_i, v_j)^2 &= \langle v_i, v_i \rangle + \langle v_j, v_j \rangle - 2\langle v_i, v_j \rangle \\ &= 8 \end{aligned}$$

center of mass $= 0$

Rotation about v_1: $v_2 \to v_4 \to v_3 \to v_2$

$e_1 \mapsto e_2 \to e_3 \to e_1$ $\Rightarrow$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = R_{v_1} = R_{F_1}$$

F_1 = face opposite v_1

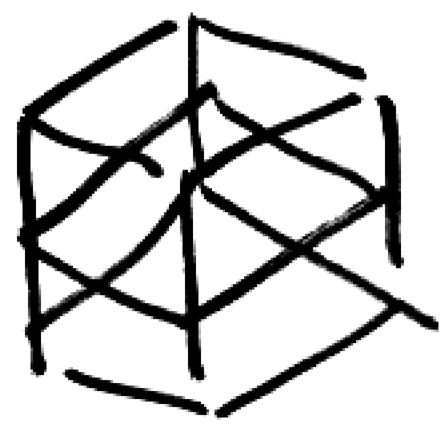
Rotations about midpoint of an edge

$v_1 \leftrightarrow v_2$
$v_3 \leftrightarrow v_4$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = R_E$$

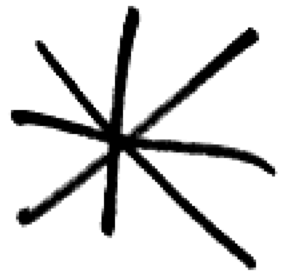
Standard repn of A_4.

<u>cube</u> vertices $\begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix}$



		order
3	R_F	4
6	R_E	2
4	R_V	3

$1 + 3\times 3 + 6 \times 1 + 4 \times 2$
$= 24$

$G \simeq S_4$  4 diagonals

ico/dodecahedron $\longleftrightarrow$ A_5