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$H \subset GL(V)$

$H \curvearrowright V = \{ f : L \to \mathbb{C} \}$

$\sigma \in SL_2(\mathbb{F}_p) =: G$

$$R(\sigma)^{-1}\, h\, R(\sigma) = h^{\sigma}$$

1) $\sigma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$

$R(\sigma) := f(l) \mapsto f(al)$

2) $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$R(\sigma) := \mathcal{F}$

$$\mathcal{F} f(l) = \frac{1}{|L|^{1/2}} \sum_{l' \in L} f(l')\, b(l, l')$$

$B : L \times L \longrightarrow \mathbb{F}_p$ symmetric, non-deg.

$b = \theta \circ B$

3) $\sigma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $\quad x \in \mathbb{F}_p$

i) $u = (l_1, 0)$ $\quad u\sigma = (l_1, x l_1)$

$h = (1, u)$

$h^\sigma f(l) = ?$

$$h^\sigma = (1, l_1, x l_1) = (\psi(l_1, x l_1)^{-1}, 0, 0)(1, \underbrace{l_1, 0}_{(l_1,0)}) \cdot (1, \underbrace{0, x l_1}_{(0, x l_1)})$$

$$= (1, l_1, 0) \cdot (1, 0, x l_1) \cdot (\psi(l_1, x l_1), l_1, x l_1)$$

$$(h^\sigma f)(l) = \frac{b(l \overset{+l_1}{,} x l_1)}{b(l_1, x l_1)} f(l + l_1)$$

$$(hf)(l) = f(l + l_1)$$

$$\Phi = \left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right) \qquad {}^t(l_1, 0)\, \Phi \begin{pmatrix} 0 \\ l_2 \end{pmatrix} = {}^t l_1 B l_2$$

<u>Claim</u>

$$R(\sigma) := f(l) \longmapsto b(l, l)^{\frac{x}{2}} f(l) \qquad (p > 2)$$

$$R(\sigma)^{-1} h R(\sigma) = h^\sigma$$

<u>pf</u> apply lhs to function $f \in V$

$$(R(\sigma)^{-1} h)\, b(\ell,\ell)^{x/2} f(\ell)$$

$$= R(\sigma)^{-1}\, b(\ell+\ell_1, \ell+\ell_1)^{x/2} f(\ell+\ell_1)$$

$$= b(\ell,\ell)^{-x/2}\, b(\ell+\ell_1, \ell+\ell_1)^{x/2} f(\ell+\ell_1)$$

rhs $\qquad b(\ell+\ell_1, x\ell_1) / b(\ell_1, x\ell_1)$

$$\frac{b(\ell+\ell_1, \ell_1)^x}{b(\ell_1, \ell_1)^x} = b(\ell, \ell_1)^x$$

$\leftarrow 1/2\,?$

$$\Psi = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)$$

$$({}^T\ell_1, 0)\ \Psi \begin{pmatrix} 0 \\ x\ell_1 \end{pmatrix} = {}^T\ell_1\, B\, x\, \ell_1$$

$$b(\ell,\ell)^{-x/2}\, b(\ell,\ell)^{x/2}\, b(\ell_1,\ell)^{x/2}\, b(\ell,\ell_1)^{x/2}\, b(\ell_1,\ell_1)^{x/2}$$

$$= b(\ell_1,\ell)^{x}\, b(\ell_1,\ell_1)^{x/2}$$

$\vdots$

<u>summarize</u>

1) $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \xmapsto{R} \big( f(\ell) \longmapsto f(a\ell) \big)$

2) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xmapsto{R} \mathcal{F}$

3) $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xmapsto{R} \big( f(\ell) \longmapsto b(\ell,\ell)^{x/2} f(\ell) \big)$

<u>Bruhat decomposition</u>

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \alpha\gamma^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\gamma^{-1} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix}$$

<u>$\gamma \neq 0$</u> $\qquad \begin{pmatrix} 0 & -\gamma^{-1} \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\alpha\delta - \beta\gamma = 1 \Rightarrow \beta = (\alpha\delta - 1)\gamma^{-1}$$

<u>Define</u> : $R(\sigma) = R(\cdots)\, R(\cdots) \cdots$
with the previous choice.

$$R(\sigma_1)\, R(\sigma_2)\, R(\sigma_1\sigma_2)^{-1} = \pm 1$$

for $\sigma_i$ with $\gamma_i \neq 0$ $\qquad$ depends only
$\sigma = \sigma_1\sigma_2$ " $\gamma \neq 0$ $\qquad$ on $B$.

A) $B \sim xy$ $\qquad +1$

B) $B \sim x^2 - \varepsilon y^2$ $\qquad -1$ $\qquad \varepsilon \neq \square$ in $\mathbb{F}_p^\times$

Take $\pm R$ correspondingly.

$R: G \to Gl(V)$

A) principal series

B) cuspidal series

$R$ is in general not irreducible.

It commutes with the action of an abelian group $A \subseteq O(L)$, $\xi: L \to L$ preserving the form $B$.

$f \in V \mapsto f \circ \xi$
$f: L \to \mathbb{C}$

$G \circlearrowright V = \bigoplus_{\chi} V_\chi$ $\quad \chi$ characters of $A$

$A \circlearrowright G$

If actions commute, each $V_\chi$ is a representation of $G$

If $v \in V_\chi$ i.e. $av = \chi(a) v$

Then $a(gv) = g(av) = \chi(a)(gv)$

$gv \in V_\chi \qquad G \circlearrowright V_\chi$

All irred repn of G appear in $V_\chi$ for some $\chi$ (most $V_\chi$ are actually G-irred.)

<u>$G = SL_2(\mathbb{F}_p)$</u>

- $|G| = p(p^2-1) = |Gl_2(\mathbb{F}_p)| / (p-1)$

$$1 \to SL_2(\mathbb{F}_p) \to GL_2(\mathbb{F}_p) \xrightarrow{\det} \mathbb{F}_p^\times \to 1$$

$$|Gl_2(\mathbb{F}_p)| = (p^2-1)(p^2-p)$$

- $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_p^\times \right\}$ (split) <u>torus</u>
- $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_p \right\}$ <u>unipotent</u>
- $B = TN$ upper triangular matrices <u>Borel</u>
- $|B| = p(p-1)$ $[G:B] = p+1$
- $\mathbb{F}_{p^2} \overset{2}{|} \mathbb{F}_p$ $\mathbb{F}_p + \alpha\,\mathbb{F}_p$ $(p>2)$

$$\alpha^2 = \varepsilon \neq \square$$

Multiplication by $\mathbb{F}_{p^2}^\times$ gives map

$$\mathbb{F}_{p^2}^\times \longrightarrow Gl_2(\mathbb{F}_p)$$

( $\mathbb{F}_{p^2}$ is 2-dim'l vector space / $\mathbb{F}_p$ )

$\gamma \in \mathbb{F}_{p^2}^\times \qquad \gamma = a + \alpha b$

$$\gamma \longmapsto \begin{pmatrix} a & \varepsilon b \\ b & a \end{pmatrix}$$

$$\det = \text{norm}(\gamma) = \gamma \cdot \gamma'$$

$$\gamma' = a - \alpha b$$

$$= a^2 - \varepsilon b^2$$

~~man~~ quadr form 2-dim / $\mathbb{F}_p$
not representing zero.

~~Image of $\mathbb{F}_{p^2}^\times$ in $Gl_2(\mathbb{F}_p)$~~

$$C := \{ \gamma \in \mathbb{F}_{p^2}^\times \mid \text{norm}(\gamma) = 1 \}$$

$$C \hookrightarrow SL_2(\mathbb{F}_p)$$

(non-split) torus

$\mathbb{C}$

$$1 \to C \to \mathbb{F}_{p^2}^\times \xrightarrow{\text{norm}} \mathbb{F}_p^\times \to 1$$

$$\Rightarrow \quad |C| = \frac{p^2-1}{p-1} = p+1$$

- $N \triangleleft B$ in fact $N$ is the commutator subgroup of $B$.

$$1 \longrightarrow N \to B \to \mathbb{F}_p^\times \to 1$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \mapsto a$$

$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ normalizes $T$

$$w^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} w = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = {}^T\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$$\cdot N_G(T) = \langle T, w \rangle$$
$$= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}, a \in \mathbb{F}_p^\times \right\}$$

$N_G(T)/T \simeq S_2$ <u>Weyl group</u>

- Bruhat decomposition

$$G = \underset{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}{B} \,\dot\cup\, BwB$$

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \longmapsto \varphi(a)$$

$\varphi$ character of $\mathbb{F}_p^\times$

representation of $B$. Induce to $G$

We get $(p+1)$-dim'l rep'n of $G$.

$$\mathrm{Ind}_B^G(\varphi) = V_\varphi$$

Can compute the character.

Find that $\mathrm{Ind}_B^G(\varphi)$ is irred.

if $\varphi^2 \neq 1$.

For $\varphi^2 = 1$ $\mathrm{Ind}_B^G(\varphi)$ decomposes into two irreducibles.

.....

$V_\varphi \simeq V_{\varphi^{-1}}$

<u>principal series</u>

$\varphi : \mathbb{F}_p^\times \longrightarrow \mathbb{C}^\times$

| | | | dim | # |
|---|---|---|---|---|
| $\underline{\varphi^2 \neq 1}$ | $V_\varphi$ | dim $p+1$ | | $\frac{p-3}{2}$ |
| $\underline{\varphi = \text{quadr} = \tau}$ | $V_\tau^+, V_\tau^-$ | | $\frac{p+1}{2}$ | 2 |
| $\underline{\varphi = 1}$ | trivial | | 1 | 1 |
| | + Steinberg | | $p$ | 1 |