Distribution of Betti numbers

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(joint w/ T. Hausel)

We consider (smooth) irreducible varieties X over $\mathbb{C}$.

Let

$$b_i(X) := \dim H^i(X, \mathbb{C})$$

For simplicity we assume

$$b_{2i+1}(X) = 0$$

Let

$$E(X; q) := \sum_{i=0}^{\dim X} b_{2i}(X) q^{d-i}$$

$$d := \dim X$$

What shape do these (even) Betti numbers have for large d?

i) Poincaré duality

$$b_i = b_{2d-i}$$

$d := \dim X$

(ii) Hard Lefschetz

$$b_0 \leq b_2 \leq \cdots \leq b_{2[d/2]}$$

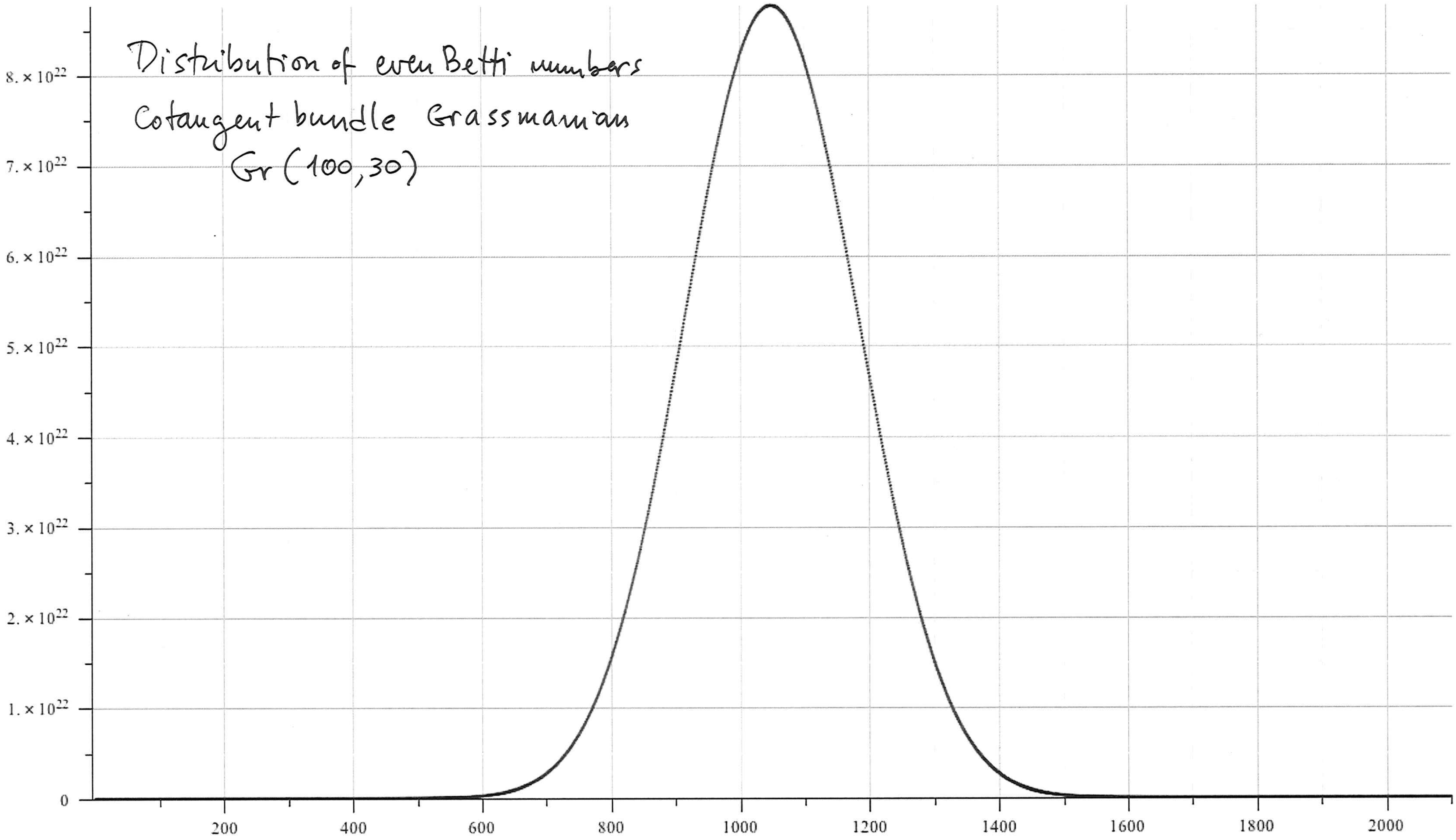
Hence the sequence

$b_0, b_2, \ldots, b_{2d}$

is unimodal and symmetric

If X is smooth and projective.

o) $X = T^*$ grassmannian


Distribution of even Betti numbers
Cotangent bundle Grassmannian
Gr (100,30)
8. × 10^22
7. × 10^22
6. × 10^22
5. × 10^22
4. × 10^22
3. × 10^22
2. × 10^22
1. × 10^22
0
200
400
600
800
1000
1200
1400
1600
1800
2000

It turns out that there are interesting varieties that satisfy a weak form of the Hard Lefschetz theorem.

This implies that:

There exists a k such that

$$b_0 \leq b_2 \leq \cdots \leq b_{2k}$$

<u>Examples</u>

- Toric hyper Kähler
- Nakajima quiver varieties (quiver with no edge loop)
- Moduli space of Higgs bundles

How far do the (even) Betti numbers grow?

We can consider some examples using the following

<u>THM</u> (Katz)

Suppose X is polynomial count and has pure cohomology.

Then

$$\# X(\mathbb{F}_q) = E(X; q)$$

1) <u>Toric hyper Kähler</u>

Q = graph

$\mathcal{M}$ = Nakajima quiver variety

dimension vector = $(1, \dots, 1)$

$E(\mathcal{M}, q)$ = external activity polynomial

$= T_Q(1, q)$

$T_Q :=$ Tutte polynomial of Q

Let $E_n(q) := T_{K_n}(1, q)$, where

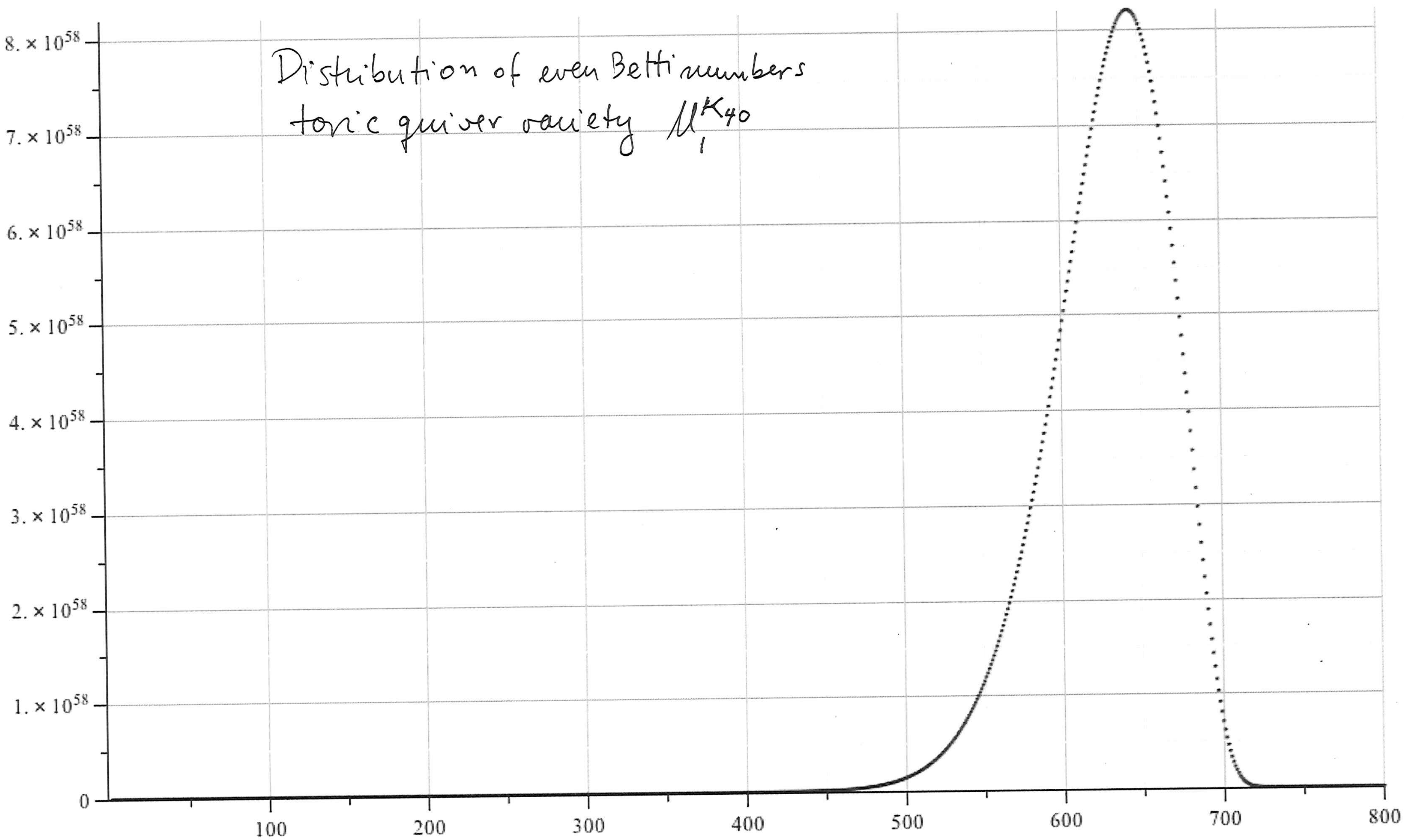
$K_n :=$ complete graph on n vertices.

Generating function:

$$\sum_{n \geq 1} E_n(q) \frac{T^n}{n!}$$

$$= (q-1) \log \sum_{n \geq 0} q^{\binom{n}{2}} \frac{(T/(q-1))^n}{n!}$$

n	E_n
1	1
2	1
3	$q+2$
4	q^3+3q^2+6q+6
5	$q^6+4q^5+10q^4+20q^3+\cdots$


Distribution of even Betti numbers
toric quiver variety $\mathcal{M}_1^{K_{40}}$
8. × 10^58
7. × 10^58
6. × 10^58
5. × 10^58
4. × 10^58
3. × 10^58
2. × 10^58
1. × 10^58
0
100
200
300
400
500
600
700
800

2) Hilbert scheme of points of $\mathbb{C} \times \mathbb{C}$

Ellingsrud - Stromme, Göttsche

$$b_{2i} = \# \left\{ \lambda \;\middle|\; \begin{array}{l} |\lambda| = n \\ \ell(\lambda) = i \end{array} \right\}$$

Partitions with given length

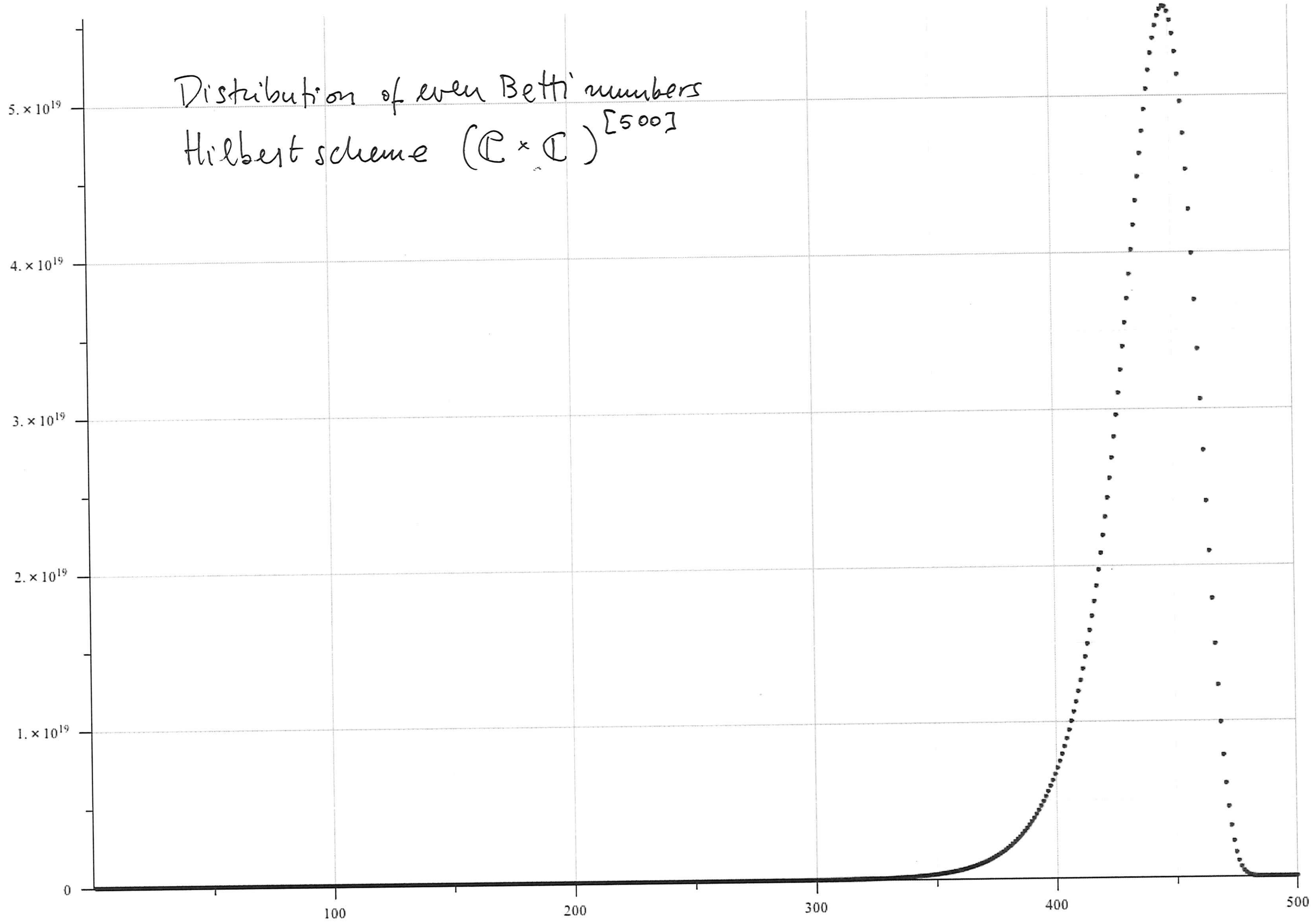
$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

$\lambda_\ell \neq 0$, $\ell = \ell(\lambda)$

Duality:

$\lambda' = (\lambda'_1, \lambda'_2, \ldots)$

$\lambda'_1 =$ largest part of λ'
$=$ length of λ


Distribution of even Betti numbers
Hilbert scheme $(\mathbb{C}\times\mathbb{C})^{[500]}$
5. × 10^19
4. × 10^19
3. × 10^19
2. × 10^19
1. × 10^19
0
100
200
300
400
500

3) Quiver varieties

A combinatorial formula of Hua gives the number of points over $\mathbb{F}_q$ via counting absolutely irreducible representations of the quiver.

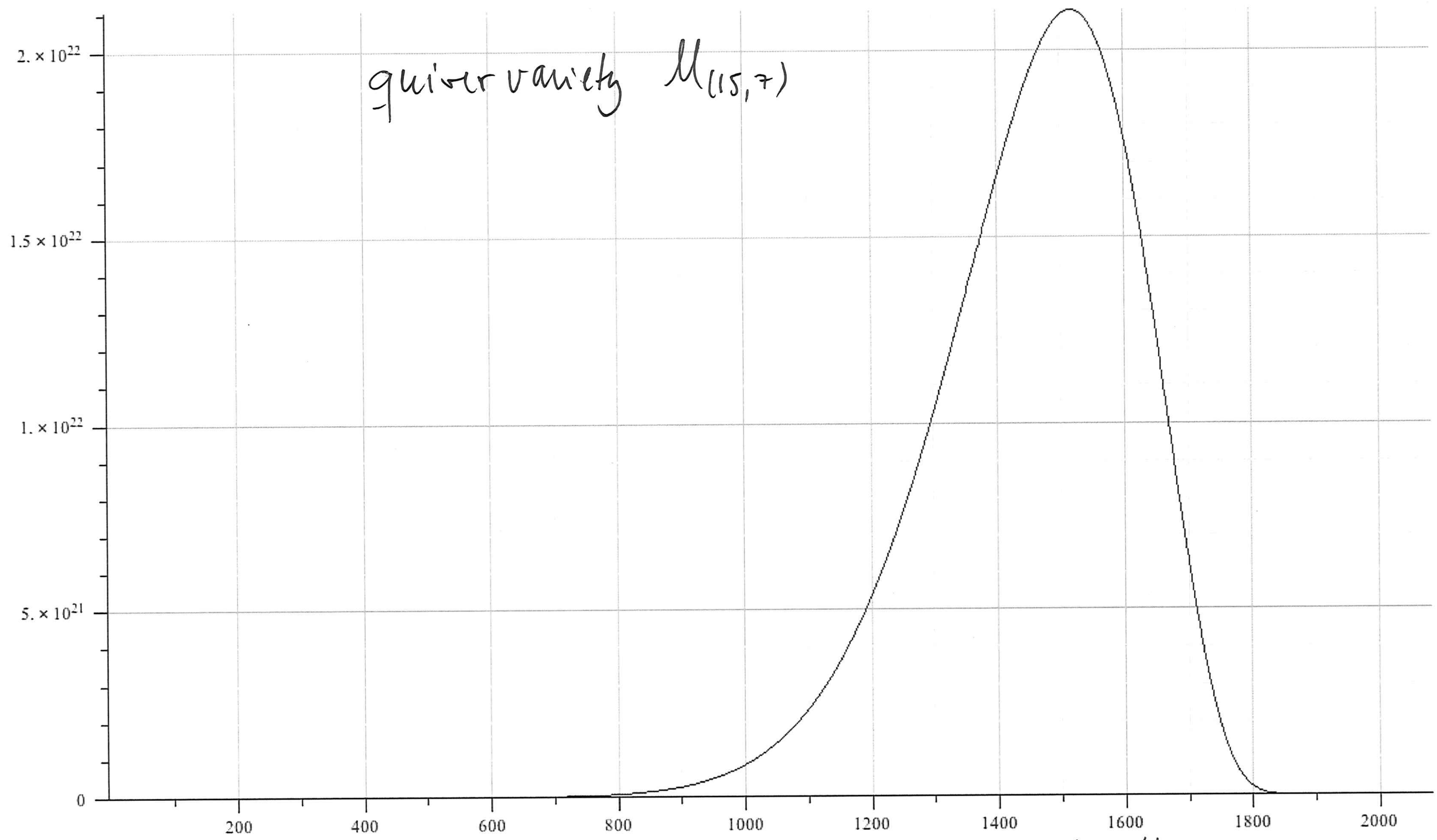
E.g.

10 loops

$$b_{3862} = 1$$

Max Betti number

$$b_{3036} \simeq 2.1 \times 10^{22}$$



Distibution of even Betti numbers quiver variety $\mathcal{M}_{(15,7)}$

Some results

0) Grassmanian

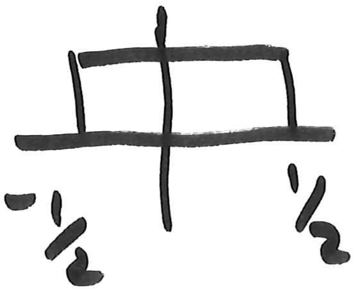
$\# Gr(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}$

Limiting distribution $n \to \infty$

for $\begin{bmatrix} n+k \\ k \end{bmatrix}$

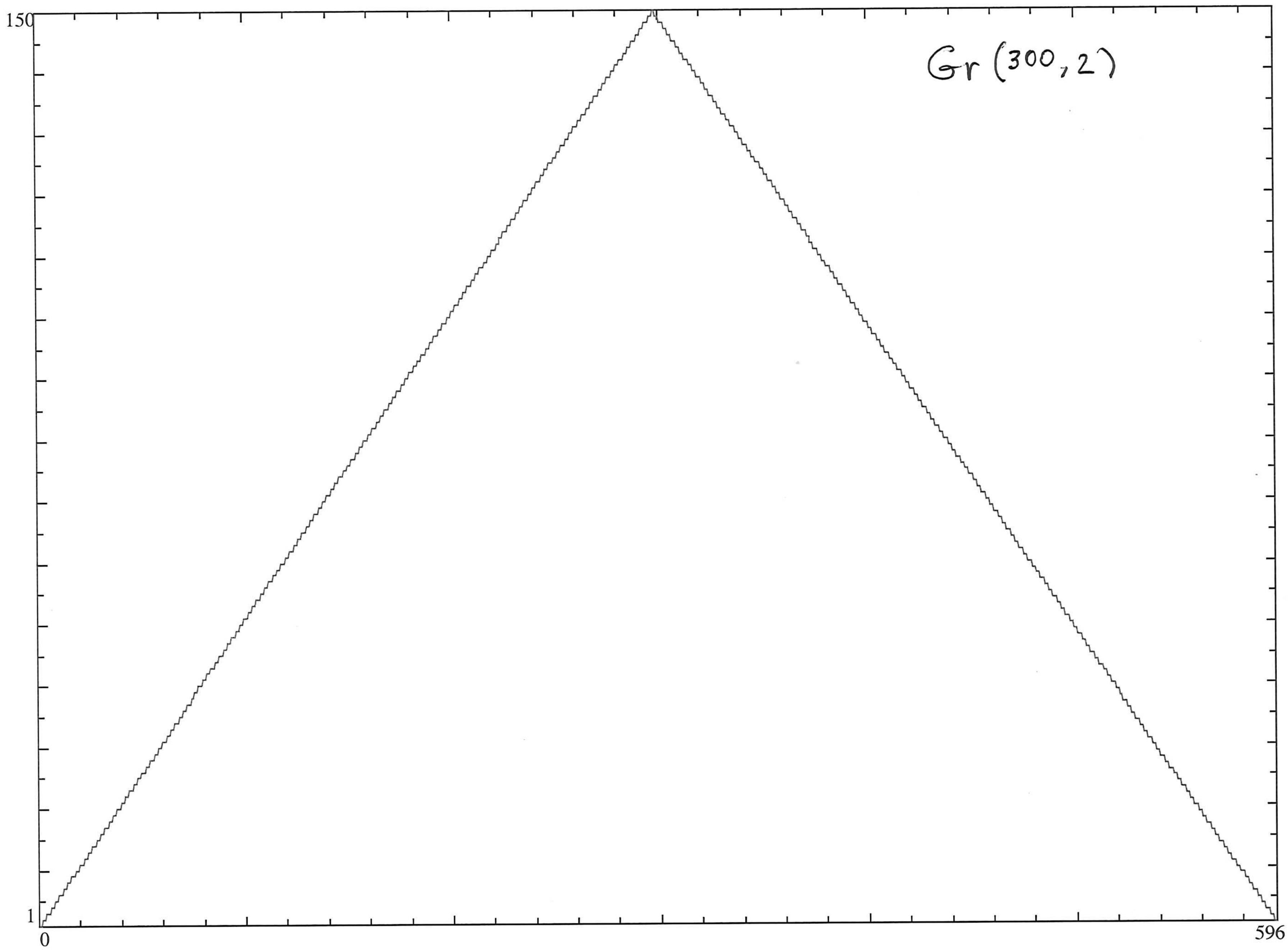
$=$ k^{th}- iterated convolution

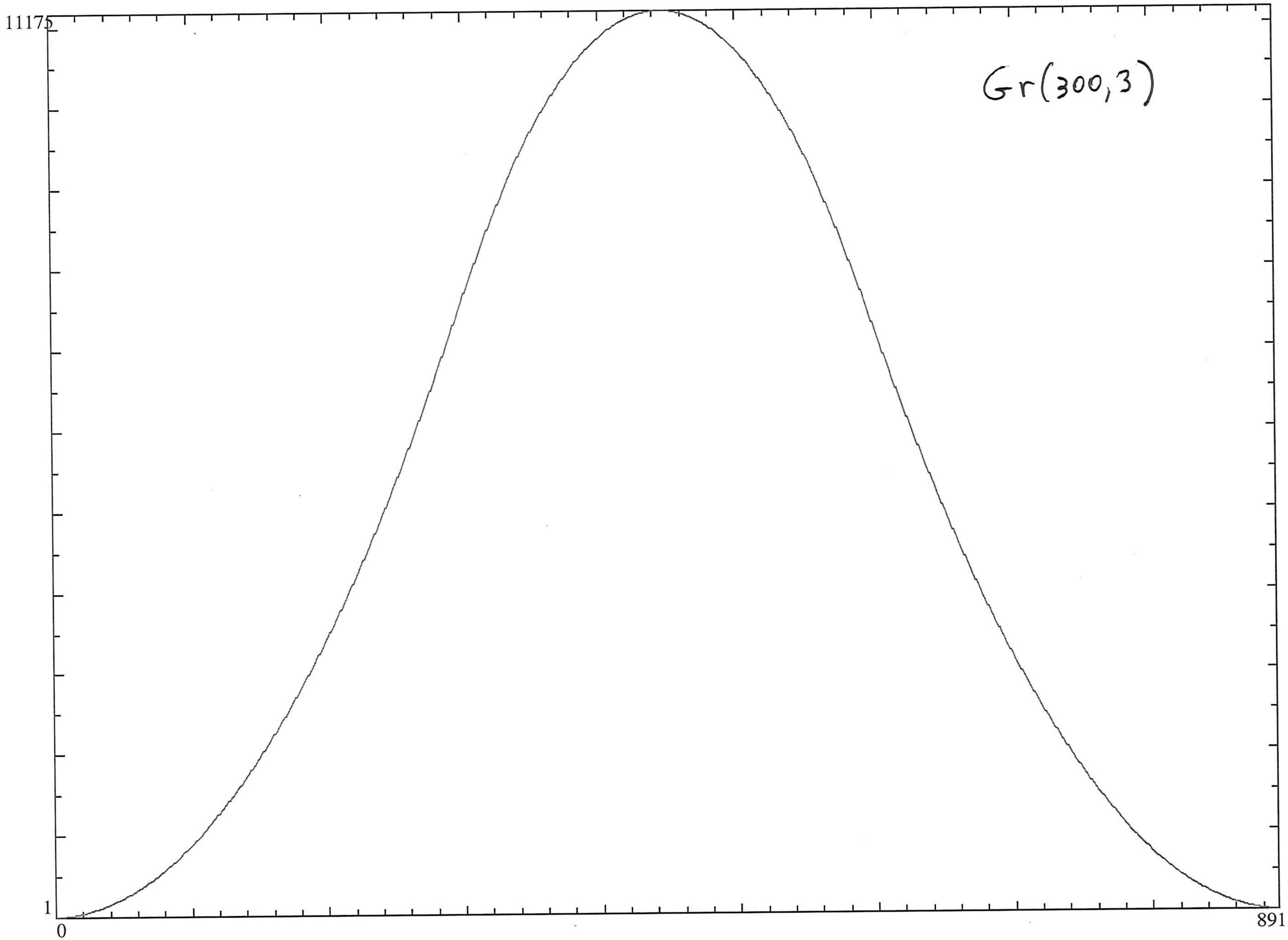
of

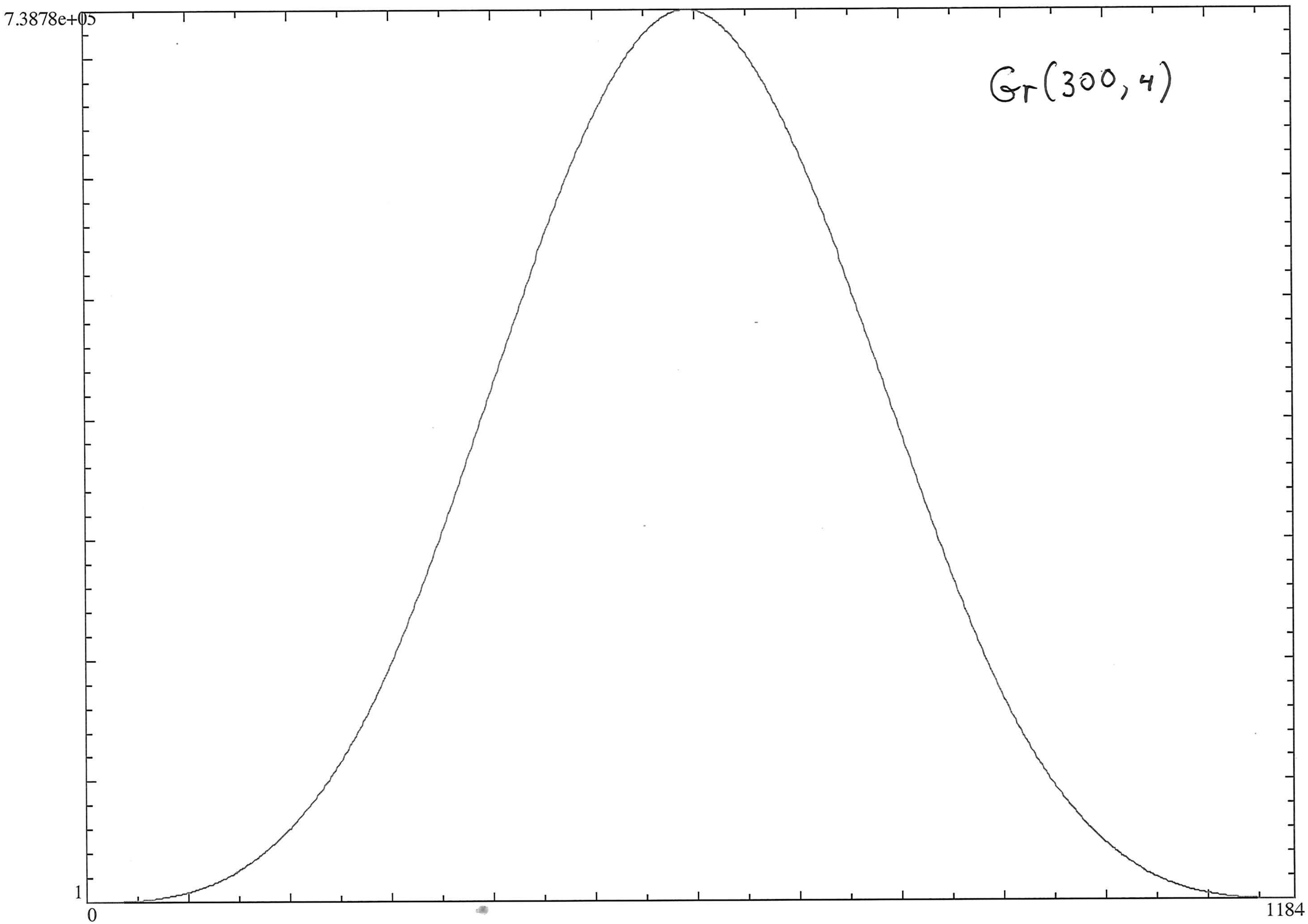


up to scaling.

Box spline: piecewise polynomial approximation to gaussian.


Gr (300, 2)
150
1
0
596


11175
1
0
891
Gr(300,3)



7.3878e+05
Gr(300, 4)
1
0
1184

1) Complete graph K_n

Limiting distribution

= Airy distribution up to scaling

Determined by its moments

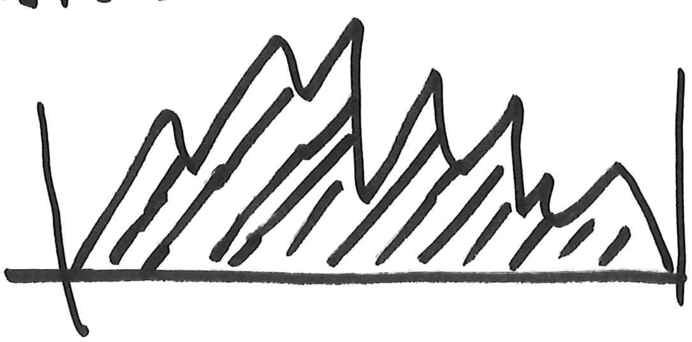
$M_0 = 1$

$M_1 = \frac{\sqrt{2\pi}}{4}$

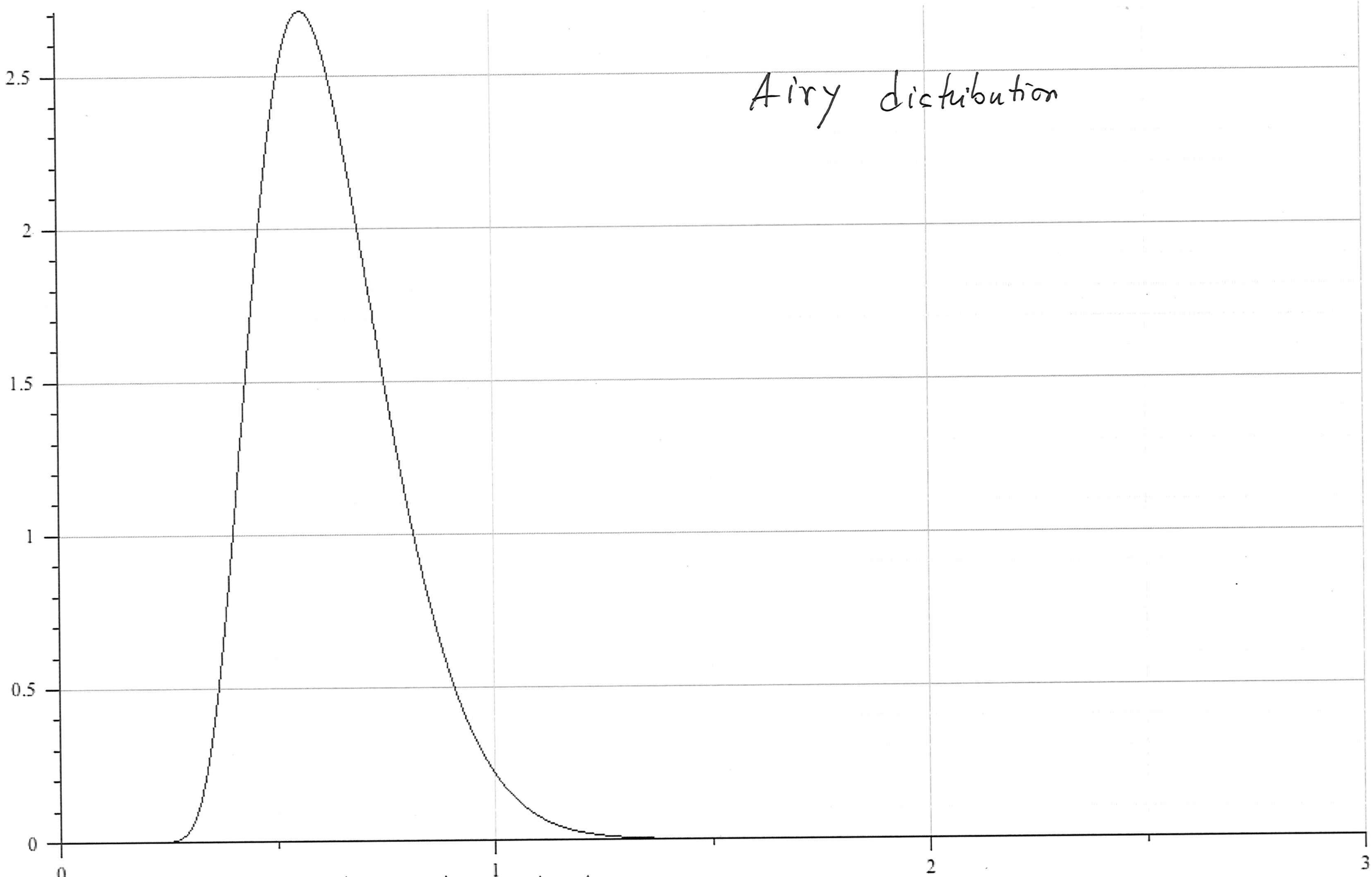
$M_2 = 5/12$

$\vdots$

Originally: Area in Brownian excursion



Many combinatorial problems.



Airy distribution

2) Hilbert scheme

Erdős - Lehner

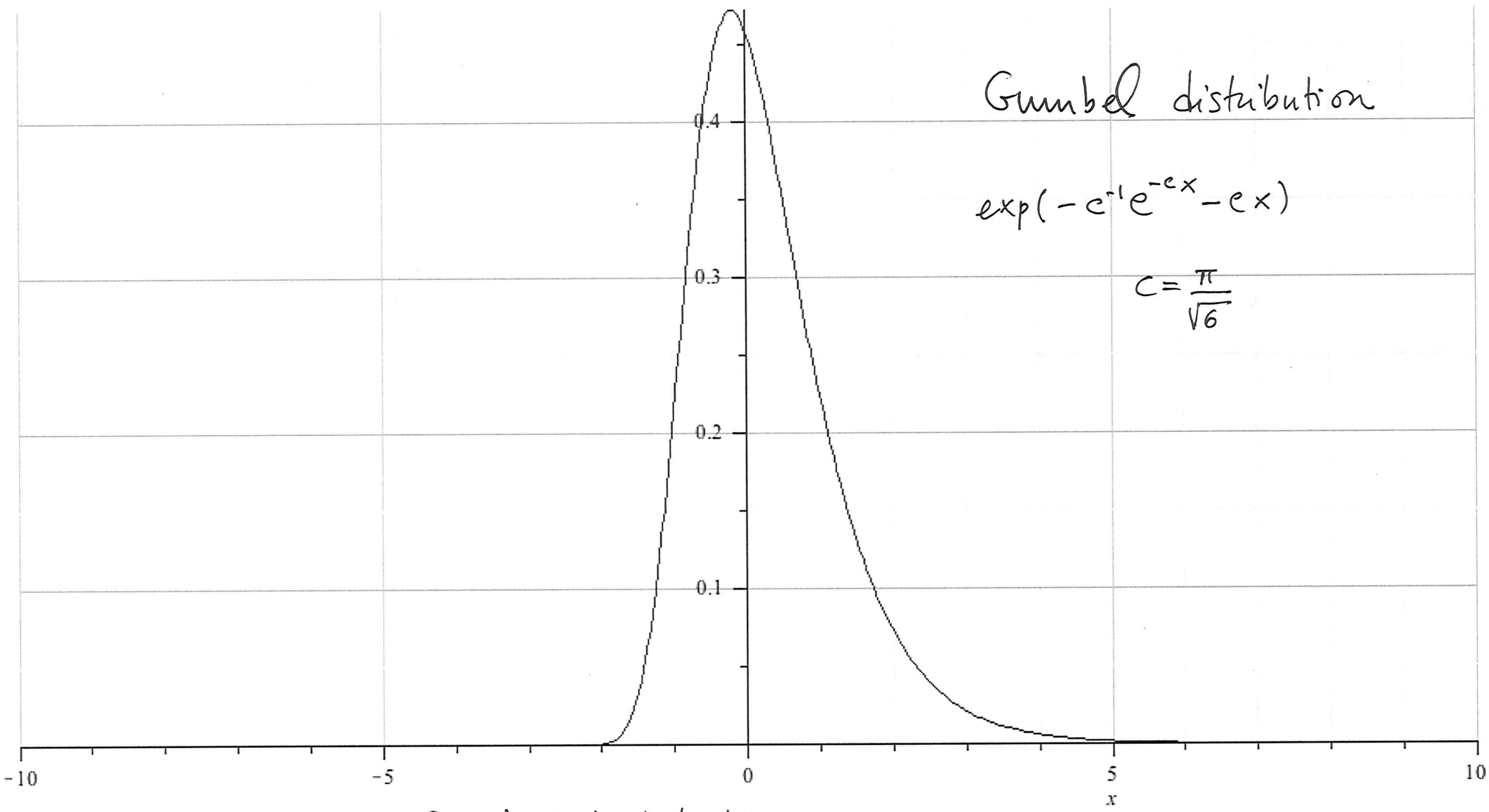
Limiting distribution is after scaling an instance of the Gumbel distribution, with density function

$$\exp(-c^{-1}e^{-cx} - cx)$$

$$c = \pi/\sqrt{6}$$

Extreme value distribution

$$\max(x_1, x_2, \ldots, x_n)$$



Gumbel distribution

3) Quiver varieties

Here we also expect to have the Airy distribution, up to scaling, in the limit as the dimension vector goes to ∞ generically.

Some Heuristics

(after Flajolet et al)
For the complete graphs K_n.
Write generating function as

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\phi(x,\eta,T)}\, dx =: F(\eta,T)$$

where

$$q = e^{-t} = 1+\eta$$

and

$$\phi(x,\eta,T) = -\tfrac{1}{2} t^{-1} x^2 + e^{ix + \frac{1}{2} t} \cdot (T/\eta)$$

$$\eta \log F(\eta,T) = C_0(T) + C_1(T)\,\eta + \cdots$$

$$C_k(T) := \sum_{n \geq 0} C_{n,n+k-1} \frac{T^n}{n!}$$

$C_{n,n+k-1}$ = # connected graphs on n vertices $n+k-1$ edges

$$\phi(x,\eta,T) = \left(\tfrac{1}{2}x^2 + e^{ix}T\right)\eta^{-1} + O(1)$$

Change variables to saddle point parameter:

$$ix = w(T)$$

$$T e^{w} = w$$

Then

$$\begin{cases} C_0 = w - \tfrac{1}{2}w^2 \\ C_1 = -\tfrac{1}{2}\log(1-w) + \tfrac{1}{2}w + \tfrac{1}{4}w^2 \\ C_k = \dfrac{E_{k-1}(w)}{(1-w)^{3(k-1)}}, \quad k > 1 \end{cases}$$

$E_k(1) = c_k$ <u>Wright constants</u>

$$\sum_{k\geq 1} C_k T^k = \log \sum_{m\geq 0} \frac{(1/6)_m (5/6)_m}{m!} \left(\tfrac{3}{2}T\right)^m$$

k	1	2	3	4	...
C_k	5/24	5/16	1105/1152	565/128	

$$C_k \sim C_k (1-w)^{-3(k-1)}, \quad k>1$$

We want to study asymptotics when $w \to 1$. get <u>coalesaing</u> <u>saddle points</u>.

$$\phi(x,\eta,T) \sim \eta^{-1}\left(-\tfrac{1}{2}w^2 + w + \tfrac{1}{2}(1-w)y^2 - \tfrac{i}{6}y^3 + O(y^4)\right) + O(1)$$

$x = y - iw$

$$\frac{1}{\sqrt{2\pi\eta_*}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\eta_*^{-1}\left(y_*^2 - \frac{i}{3}y_*^3\right)} dy_*$$

We expect this to be the generic behaviour for large dimension vectors and any quiver, or dimension vector $(1, 1, \dots, 1)$ and

any generic sequence of graphs with increasing number of vertices.

I.e. Nakajima quiver varieties

or toric hyperKähler

~~or~~