Group Representations

G finite group

acts on something.

$G \longrightarrow Aut(X)$ homomorphism

X = object, $Aut(X)$ = group of symmetries.

1) X = finite set

$Aut(X) = S(X)$ permutations of X.

$n = \# X$

$S(X) \simeq S_n = S(\{1, 2, \dots, n\})$

↑ isomorphism

label the elements of X

$X = \{x_1, x_2, \dots, x_n\}$, $X \simeq \{1, \dots n\}$

$$\boxed{\simeq \; \neq \; =}$$

$$\rho : G \longrightarrow S(X)$$

<u>permutation representation</u>

2) X = vector space over a field K

$Aut(X) = GL(V)$ $\qquad$ $\dim V < \infty$

If we pick a basis for V

$v_1, \ldots, v_n$ $\qquad$ $n = \dim V$

$$GL(V) \simeq Gl_n(K)$$

$n \times n$ invertible matrices$/K$

$$\rho : G \longrightarrow Gl(V)$$

<u>linear representation</u>

G = finite, $\quad V$ finite dim

ρ gives V the structure of a

$K[G]$ - module

$$K[G] = \left\{ \sum_{g \in G} a_g g \;\middle|\; a_g \in K \right\}$$

$$g_1 \cdot g_2 = g_1 \cdot g_2$$

Shorthand:

$$\rho(g)(v) = g \cdot v$$

The $\dim V$ is called the <u>degree</u> of the representation.

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1(g)"'] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "\varphi"] & V_2 \end{tikzcd}$$

map between representations

φ is a $K[G]$-module map.

$$\varphi(g \cdot v) = g \cdot \varphi(v), \quad g \in G$$

Isomorphism of representations.

Say V_1 and V_2 are isomorphic (equivalent) representations.

Two different choices of basis give rise to isom. repns.

From now on $K = \mathbb{C}$.

Examples

1) **deg 1** $\rho : \mathrm{Hom}(G, Gl_1(K))$

$$Gl_1(K) \cong K^\times$$

$$\rho : G \to \mathbb{C}^\times$$

$\rho(g)$ root of unity

in particular $|\rho(g)| = 1$.

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathbb{C}^\times \\ & G/[G,G] =: A \arrow[u, dashed] \end{tikzcd}$$

abelian

$A \to \mathbb{C}^\times$ [Finite Fourier Analysis

non-canonically isom to A

G has $[G:[G,G]]$ 1-dim'l repn.

Trivial repn $\rho : G \to \mathbb{C}^\times$, $g \mapsto 1$

$$S_n \longrightarrow \{\pm 1\} \subseteq \mathbb{C}^\times$$
$$\sigma \longmapsto \operatorname{sgn}(\sigma)$$

<u>alternating</u> ~~of~~ repn of S_n

2) <u>Regular repn</u>

$G \longrightarrow S(G)$ by left multiplication

3) In general $G \to S(X)$

X finite set $n = \# X$

$$g \cdot \left(\sum_{x \in X} a_x e_x \right)$$

$K^n \ni e_x$ standard basis

$a_x \in \mathbb{C}$

$$:= \sum_{x \in X} a_x e_{g \cdot x}$$

$$G \to S(X) \to Gl(K^n)$$

$$\sum_{x \in X} a_x \, e_{gx} = \sum_{x \in X} a_{g^{-1}x} \, e_x$$

Alternatively, think of the functions $x \mapsto a_x$. I.e.

$$V = \mathbb{C}^X \qquad a : X \to \mathbb{C}$$

$$(g.a)(x) = a(g^{-1}x)$$

$$e_x \leftrightarrow \delta_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

<u>$G = S_3$</u> trivial, alternating

G has a permutation repn

it acts on $\{1, 2, 3\}$

$$e_1, \; e_2, \; e_3$$

$\tau = (123)$ $\qquad \tau e_1 = e_{\tau 1} = e_2$

$\sigma = (12)$ $\qquad \tau e_2 = e_3$

$\qquad \tau e_3 = e_1$

$$\rho(\tau) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\sigma \quad e_1 \leftrightarrow e_2$

$$\rho(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

permutation matrices

$v = (a_1, a_2, a_3)$

$\tau v = (a_3, a_1, a_2)$

$\tau(a_1 e_1 + a_2 e_2 + a_3 e_3) = a_1 e_2 + a_2 e_3 + a_3 e_1$

$W' := \langle e_1 + e_2 + e_3 \rangle \subseteq V$

is stable by G

$$W = \left\{ \sum_{i=1}^{3} a_i e_i \;\middle|\; a_1 + a_2 + a_3 = 0 \right\}$$

$V = W \oplus W'$

V is reducible repn.