# The $L$-function of the quintic

Fernando Rodriguez Villegas
*University of Texas at Austin*
villegas@math.utexas.edu

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## 1 Introduction

This is a report on work our group did at the workshop *Higher rank L-functions* in Benasque, July 2009. The goal was to compute the full $L$-function of the principal piece of the middle cohomology of the quintic

$$X_\psi : \qquad x_1^5 + \cdots + x_5 - 5\psi x_1 \cdots x_5 = 0,$$

for arbitrary $\psi \in \mathbb{Q}$.

Concretely, fix $\psi \in \mathbb{Q}$ with $\psi^5 \neq 1$. Then $X_\psi$ is a smooth projective Calabi-Yau threefold. Consider the abelian subgroup of automorphisms

$$A := \{(\zeta_1, \ldots, \zeta_5) \,|\, \zeta_i^5 = 1, \zeta_1 \cdots \zeta_5 = 1\},$$

acting by $x_i \mapsto \zeta_i x_i$ and let $V = V_\psi$ be the subspace of $H^3(X_\psi, \mathbb{C})$ fixed by $A$.

Our goal is to:

i) Compute the complete $L$-function $\Lambda(V, s)$ of $V$, i.e, compute all of its Euler factor including those for bad primes and at infinity.

ii) Check numerically the functional equation of $\Lambda(V, s)$ and determine the corresponding sign.

iii) Check, if possible, the modularity of $L(V, s)$.

## 2 The $L$-function

By the general recipe (described in Serre [**?**] for the total space $H^k(X)$ of a smooth projective variety $X$) the shape of the $L$-function is as follows

$$\Lambda(V, s) = N^{s/2} L_\infty(V, s) \prod_p L_p(V, p^{-s})^{-1},$$

where $N$ is the conductor, a positive integer, $L_\infty$ is a product of gamma factors and $L_p(V,T)$ is a polynomial, generically of degree equal to $\dim V$.

## 2.1 Gamma factors and numerical test of the functional equation

The gamma factors are determined by the Hodge numbers of $V$. It is known that $\dim V = 4$ and that in fact $h^{p,q}(V) = 1$ for $p = 0, 1, \ldots, 3$ and $p + q = 3$. This yields the following value for the Euler factor at $\infty$.

$$L_\infty(V, s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-1).$$

Let

$$L(V, s) := \prod_p L_p(V, p^{-s})^{-1} = \sum_{n\geq 1} \frac{a_n}{n^s}.$$

By using the Mellin transform we can write

$$\Lambda(V, s) = \int_0^\infty \varphi(t) t^s \, \frac{dt}{t},$$

where

$$\varphi(t) := \sum_{n\geq 1} a_n \, k\left(\frac{nt}{\sqrt{N}}\right), \qquad k(t) := \frac{1}{\pi\sqrt{t}} K_1\left(4\pi\sqrt{t}\right) \tag{1}$$

and $K_1$ is the usual $K$-Bessel function. The point is that $k(t)$ is the inverse Mellin transform of $L_\infty(V, s)$.

It is know that

$$K_1(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}, \qquad x \to \infty.$$

Since $V$ is pure of weight 3, being a subspace of $H^3(X_\psi)$ where $X_\psi$ is smooth and projective, we know that

$$a_n = O(n^{3/2+\epsilon}),$$

for any $\epsilon > 0$. Hence the definition (1) gives $\varphi$ as a sum of exponentially decaying terms. To compute it to a given accuracy we will need, as a rule of thumb, a number of terms in the series proportional to $\sqrt{N}$. The size of $N$ will therefore be crucial for the feasibility of the calculations.

Since $V$ is a piece of $H^3$ the expected functional equation is

$$\Lambda(4-s) = \epsilon\Lambda(s), \qquad \epsilon = \pm 1.$$

By taking the inverse Mellin transform this is equivalent to

$$\varphi(t^{-1}) = \epsilon t^4 \varphi(t).$$

Our numerical test will be to compute an approximation to the ratio

$$\varphi(t^{-1})/t^4\varphi(t)$$

for $t \approx 1$. The result should be close to $\epsilon = \pm 1$.

## 2.2 Hypergeometric trace and Euler factors

Let $S$ be the finite set of primes $p$ consisting of $p = 5$ and those satisfying $\psi^5 \equiv 1 \bmod p$ or $\psi \equiv \infty \bmod p$ (i.e., such that $p$ divides the denominator of $\psi$). Any prime $p$ outside $S$ is a good prime and the corresponding Euler factor has the form

$$L_p(T) = 1 + aT + bpT^2 + ap^3T^3 + p^6T^4, \qquad a, b \in \mathbb{Z}, \qquad p \notin S. \tag{2}$$

The coefficients $a$ and $b$ that determine the whole polynomial can be computed using the $p$-adic methods of Dwork. We will give the final expression obtained in [**?**] in terms of the hypergeometric trace of Katz, which we now define.

Let $\mathbb{Q}_{(p)}$ be the ring of rational numbers with denominator coprime to $p$. Fix $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ vectors in $\mathbb{Q}_{(p)}^r$ with $0 \leq \alpha_j, \beta_j < 1$ and $f$ a positive integer. For $m = 0, 1, \ldots, q-2$, with $q := p^f$, we define a $p$-adic analogue of the Pochammer symbol

$$(x)^*_{m,q} := \frac{\Gamma^*_q\left(x + \frac{m}{1-q}\right)}{\Gamma^*_q(x)}, \qquad x \in \mathbb{Q}_{(p)} \subseteq \mathbb{Z}_p. \tag{3}$$

where to simplify the notation we set

$$\Gamma^*_p(x) := \Gamma_p\left(\{x\}\right), \qquad \Gamma^*_q(x) := \prod_{\nu=0}^{f-1} \Gamma^*_p(p^\nu x), \qquad x \in \mathbb{Q}_{(p)} \subseteq \mathbb{Z}_p.$$

To alleviate the notation we will drop the dependence on $q$ when there is no risk of confusion.

For $x \in \mathbb{Q}_{(p)}$ and $m = 0, 1, \ldots, q-2$ we let

$$\eta_m(x) := \sum_{\nu=0}^{f-1} \left\{p^\nu \left(x + \frac{m}{1-q}\right)\right\} - \{p^\nu x\}$$

and extend the definition to $x = (x_1, \ldots, x_r) \in \mathbb{Q}_{(p)}^r$ by setting

$$\eta_m(x) := \sum_{j=1}^{r} \eta_m(x_j).$$

We assume now that for all $m = 0, 1, \ldots, q-2$ we have

$$\eta_m(\alpha) - \eta_m(\beta) \in \mathbb{Z} \tag{4}$$

and define

$$H_q\left( \begin{array}{c} \alpha \\ \beta \end{array} \middle| z \right) := \frac{1}{1-q} \sum_{m=0}^{q-2} (-p)^{\eta_m(\alpha)-\eta_m(\beta)} \, q^{\xi_m(\beta)} \prod_{j=1}^{r} \frac{(\alpha_j)^*_m}{(\beta_j)^*_m} \ \mathrm{Teich}(z)^m. \tag{5}$$

where

$$\xi_m(\beta) := \#\left\{j \,|\, \beta_j = 0\right\} - \#\left\{j \,|\, \beta_j + \frac{m}{1-q} = 0\right\}.$$

*Remark* 2.0.1 We have normalized the hypergeometric trace $H_q$ of Katz so that it resembles the classical hypergeometric series. We should point, however, that $\alpha$ gives the exponents of the local monodromy at $\infty$ and $\beta$ those at 0, whereas classically the exponents at 0 would be given as $1-\beta_j$ instead of $\beta_j$.

We then have that the trace of the geometric Frobenius $\mathrm{Frob}_q$ on $V$ is given as a number in $\mathbb{Q}_p$ by

$$\mathrm{Tr}\left(\mathrm{Frob}_q\Big|_{V_\psi}\right) = H_q\left(\begin{array}{cccc|c} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \\ 0 & 0 & 0 & 0 & \end{array}\psi^{-5}\right), \qquad \psi \not\equiv 0 \bmod p. \tag{6}$$

If we abbreviate the right hand side by $H_q$ then we have

$$a = -H_p, \qquad\qquad b = \tfrac{1}{2p}(H_p^2 - H_{p^2}).$$

For a prime $p \neq 5$ such that $\psi^5 \equiv 1 \bmod p$ we may still compute the right hand side of (6). These are the traces of an operator with characteristic polynomial

$$L_p(T) = (1 - \left(\tfrac{5}{p}\right) pT)(1 - a_p T + p^3 T^2), \qquad p \neq 5, \quad \psi^5 \equiv 1 \bmod p, \tag{7}$$

where $a_p$ is the $p$-th coefficient of the Hecke eigenform of weight 4 and level 25 discovered by Schoen, which gives the trace of Frobenius acting on $H^3$ of a resolutions of singularities of the conifold $X_1$. Again, with the above notation

$$a_p + \left(\tfrac{5}{p}\right) p = -H_p, \qquad \left(\tfrac{5}{p}\right) a_p + p^2 = \tfrac{1}{2p}(H_p^2 - H_{p^2}).$$

For a prime $p \neq 5$ such that $\psi \equiv \infty \bmod p$ the right hand side of (6) gives the constant value 1 for all $f$. Hence the associated characteristic polynomials is simply

$$L_p(T) = 1 - T, \qquad p \neq 5, \quad \psi \equiv \infty \bmod p. \tag{8}$$

This seems to be the right answer.

If $\psi \equiv 0 \bmod p$ the formula (6) breaks down. However, the variety $X_0$ is the Fermat hypersurface

$$x_1^5 + \cdots x_5^5 = 0,$$

whose $L$-series was calculated by Weil in terms of Hecke characters. It is not difficult to work out the Hecke character corresponding to $V_0$. Let $K := \mathbb{Q}(\zeta_5)$, where $\zeta_5$ is a primitive fifth root of unity and let $\mathcal{F} = (1-\zeta_5)^2$. A prime $p \neq 5$ factors in the ring of integers $O_K$ of $K$ into primes

$$(p) = \prod_{i=1}^{s} \mathcal{P}_i,$$

where $s \mid 4$. The class number of $K$ is 1 and we can in fact choose generators $\alpha_i$ of $\mathcal{P}_i$ such that

$$\alpha_i \equiv 1 \bmod \mathcal{F}, \qquad i = 1, 2, \ldots, s. \tag{9}$$

Indeed, it is not hard to verify that a generator $\epsilon$ of $O_K^\star$ generates $(O_K/\mathcal{F})^\star$ and hence given any generator of $\mathcal{P}_i$ we can multiply it by an appropriate power of $\epsilon$ to obtain $\alpha_i$.

Let $\sigma$ be the generator of $\mathrm{Gal}(K/\mathbb{Q})$ that takes $\zeta_5$ to $\zeta_5^2$. We define the Hecke character $\phi$ by setting

$$\phi(\mathcal{P}_i) := \alpha_i^{1+2\sigma^2+3\sigma^3}.$$

It is a short calculation to verify that this is well defined independent of the choice of $\alpha_i$ satisfying (9). Then the Euler factor at $p$ is

$$L_p(T) := \prod_{i=1}^{s}(1 - \phi(\alpha_i)T^{4/s}).$$

For $p = 5$ we have

$$L_5(T) = 1.$$

The $L$-function of this Hecke character has functional equation of the form

$$\Lambda(s) := N^{s/2}L_\infty(s)L(\phi, s) = \Lambda(4 - s),$$

where $L(\phi, s) := \prod_p L_p(p^{-s})$. It is known that

$$N = \mathrm{disc}(K/\mathbb{Q}) \cdot \mathbb{N}_{K/\mathbb{Q}}(\mathcal{F}) = 5^3 \cdot 5^2 = 5^5.$$

We can verify that $L_\infty(s)$ is our previously computed factor $(2\pi)^{-2s}\Gamma(s)\Gamma(s-1)$ directly. Indeed, the infinite type of $\phi$ is $\mu := 1 + 2\sigma^2 + 3\sigma^3$ and $1, \sigma^2$ and $\sigma, \sigma^3$ correspond to pairs of complex conjugate embeddings of $K$. For the first pair we have $(1, \sigma^2) + (0, \sigma^2)$ in $\mu$. This contributes a factor of $\Gamma_\mathbb{C}(s-1)$. For the second pair we have $(0, 3\sigma^3)$ in $\mu$, which contributes a factor of $\Gamma_\mathbb{C}$.

## 2.3 The conductor

The conductor $N$ is defined as a product over primes $\prod_p p^{f_p}$, where $f_p = 0$ for all but finitely many primes. The exponent $f_p$ itself is a sum of two terms: $r_p := \dim V - \dim V^I$, where $I$ is the inertia group at $p$, and a wild contribution $\delta_p$. Since

$$L_p(T) := \det\left(I - \mathrm{Frob}_q\big|_{V^I}\right)$$

we see that $r_p = \dim V - \deg L_p$.

## 2.4 Modularity

We would like to test whether $L(V, s)$ is modular. A natural choice of automorphic $L$-function to compare $L(V, s)$ with is the *spinor L-function* of a Siegel modular form. For $g = 2$ this $L$-function has an Euler factor for good primes $p$ of the form

$$L_p(T) = 1 - \lambda_p T + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})T^2 - \lambda_p p^{2k-3}T^3 + p^{4k-6}T^4,$$

where $k$ is the weight of the Siegel modular form and $\lambda_p$ and $\lambda_{p^2}$ are the eigenvalues of the Hecke operators $T_p$ and $T_{p^2}$.

For a Siegel modular $F$ form of level 1 Andrianov showed that the $L$-function

$$Z_F(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-1)\prod_p L_p(p^{-s})^{-1},$$

has a meromorphic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation

$$Z_F(k+1-s) = (-1)^k Z_F(s).$$

So if we expect $Z_F(s)$ to equal $L(V,s)$ we need the weight $k$ to equal 3. However, one needs to be aware that if $F$ is in the Maass space $L_p(T)$ is not pure of weight 3. Indeed in that case the $Z_F$ factors as $\zeta(s-k+1)\zeta(s-k+2)L(f,s)$ for an eigenform of weight $2k-2$ on $\Gamma_0(N)$. It is a conjecture of Arthur that if $F$ is not in the Maass space then there is an associated motive of $V$ rank 4 and pure weight 3 such that $L(V,s) = Z_F(s)$. Hence it is natural to expect that our $V$ is such a motive for some $F$.

Our situation is similar to the case of elliptic curves since a Siegel modular form $F$ of $g = 2$ and weight 3 determines a holomorphic differential in the corresponding Siegel threefold

$$F(z)\, dz_{1,1} \wedge dz_{1,2} \wedge dz_{2,2}, \qquad z = (z_{i,j}) \in \mathcal{H}_2.$$

Other than quadratic twists of $Z_F(s)$ for $F$ a Siegel eigenform of level 1 I could not find in the literature a description of the Euler factors for primes dividing the conductor.

# 3 Other analogous families

There turn out to be fourteen families $X_\psi$ of Calabi-Yau threefolds analogue to the quintic; i.e., hypergeometric and with $\beta = (0,0,0,0)$, (maximally unipotent monodromy at $\psi = \infty$). The values of $\alpha$ are given in the following table together with the level $N_1$ of the weight 4 modular form and the conductor $D$ of the Dirichlet character associated with the singularity at $\psi = 1$.

| $\alpha$ | $N_1$ | $D$ |
|---|---|---|
| $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | 8 | 1 |
| $(\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4})$ | 9 | 24 |
| $(\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4})$ | 16 | 8 |
| $(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$ | 25 | 5 |
| $(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3})$ | 27 | 1 |
| $(\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4})$ | 32 | 1 |
| $(\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3})$ | 36 | 12 |
| $(\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6})$ | 72 | 1 |
| $(\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6})$ | 108 | 12 |
| $(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8})$ | 128 | 8 |
| $(\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6})$ | 144 | 8 |
| $(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$ | 200 | 1 |
| $(\frac{1}{6},\frac{5}{6},\frac{1}{6},\frac{5}{6})$ | 216 | 1 |
| $(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12})$ | 864 | 1 |

For all of these cases we can write an explicit model for $X_\psi$ (for the first thirteen cases as complete intersections in weigted projective spaces [**?**]; the fourteenth case is described in []) . The family carries a period satisfiying the corresponding hypergeometric differential equation with parameters $\alpha,\beta$. We again obtain a motive $V_\psi$ of rank 4 and pure weight 3 for good primes $p$ coming from a piece of the middle cohomology of $X_\psi$. The trace of $\mathrm{Frob}_q$ on $V_\psi$ is given by

$$\mathrm{Tr}\left(\mathrm{Frob}_q\big|_{V_\psi}\right) = H_q\left(\begin{array}{c}\alpha\\ \beta\end{array}\middle|\,\psi^{-m}\right), \qquad \psi \not\equiv 0 \bmod p$$

for some positive integer $m$. (For the quintic case $\alpha = (\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$ and $m = 5$.)

We posit that in fact for each choice of $\alpha$ from the above list there is a rank 4 motive $\mathcal{H}_t = \mathcal{H}_t(\alpha,\beta)$, of pure weight 3 for good primes, such that for $t \in \mathbb{Q}$ we have

$$\mathrm{Tr}\left(\mathrm{Frob}_q\big|_{\mathcal{H}_t}\right) = H_q\left(\begin{array}{c}\alpha\\ \beta\end{array}\middle|\,t\right), \qquad p \notin S,$$

where $S$ is the finite set of primes dividing $t, t-1, t^{-1}, \mathrm{denom}(\alpha)$ or $\mathrm{denom}(\beta)$.

Let us take, for example, $\alpha = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $t = -1$. Here are the first few values of $H_p$ and $H_{p^2}$.

| $p$ | $H_p$ | $H_{p^2}$ |
|---|---|---|
| 3 | 0 | −12 |
| 5 | −4 | 276 |
| 7 | 0 | −476 |
| 11 | 0 | −4972 |
| 13 | −84 | −1420 |
| 17 | 36 | 7620 |
| 19 | 0 | −21964 |
| 23 | 0 | 24932 |
| 29 | 140 | −62412 |

From the traces we can compute the coefficients of the Euler factor

$$L_p(T) = 1 + aT + pbT^2 + ap^3 + p^6T^4, \qquad a = -H_p, \quad b = \frac{1}{2p}(H_p^2 - H_{p^2}).$$

| $p$ | $a$ | $b$ |
|---|---|---|
| 3 | 0 | 2 |
| 5 | 4 | −26 |
| 7 | 0 | 34 |
| 11 | 0 | 226 |
| 13 | 84 | 326 |
| 17 | −36 | −186 |
| 19 | 0 | 578 |
| 23 | 0 | −542 |
| 29 | −140 | 1414 |

If we compare these with the corresponding table 7.6 of van Geemen and van Straaten we see that except for the signs of $a$ (and the value of $b$ for $p = 11$) they agree with those in the column for the Siegel modular form $F_7$. (Note that their polynomial is normalized as $T^4 - a_pT^3 + a_{p^2}T^2 - a_pp^3T + p^6$.)

Furhermore, they notice that $L_p(T)$ seems to be the Euler factor of the $L$-function associated to $f_2 \otimes f_3$ where $f_2$ is a CM eigenform of weight 2, level 32 and trivial character and $f_3$ an eigenform of weight 3, level 32 and character $\left(\frac{-4}{\cdot}\right)$. With this information we can extend their calculation and check the agreement of this $L$-function with ours. Here is a table of the $p$-coefficients of $f_2$ and $f_3$ from Stein's database.

| $p$ | $f_2$ | $f_3$ |
|---|---|---|
| 2 | 0 | 0 |
| 3 | 0 | $4i$ |
| 5 | $-2$ | 2 |
| 7 | 0 | $-8i$ |
| 11 | 0 | $4i$ |
| 13 | 6 | $-14$ |
| 17 | 2 | 18 |
| 19 | 0 | $12i$ |
| 23 | 0 | $40i$ |
| 29 | $-10$ | $-14$ |
| 31 | 0 | $32i$ |
| 37 | $-2$ | $-30$ |

We see, for example, that the product of the $p$ coefficients of $f_2$ and $f_3$ matches the values of $H_p$.

## 3.1 Special cases

Inspired by this example we experimented with the motive $\mathcal{H}_{\pm 1}$ where $\alpha = (\frac{1}{2}, \ldots, \frac{1}{2})$ of length $r$. Let us denote the motive by $W_r^{\pm}$. Our previous example associated to $f_2 \otimes f_3$ is then $W_4^-$. Here is a table of the values of $H_p$.

$W_r^+$

| $p \backslash r$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 3 | 0 | $-1$ | 0 | $-1$ | 0 | $-1$ |
| 5 | 0 | 1 | $-6$ | 3 | 20 | $-59$ |
| 7 | 0 | $-1$ | 0 | 31 | 0 | 95 |
| 11 | 0 | $-1$ | 0 | $-33$ | 0 | $-481$ |
| 13 | 0 | 1 | 10 | 35 | $-300$ | 933 |
| 17 | 0 | 1 | $-30$ | 67 | $-60$ | $-59$ |
| 19 | 0 | $-1$ | 0 | 63 | 0 | 3519 |

$W_r^-$

| $p\backslash r$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 3 | −1 | 0 | −1 | 0 | −1 | 0 |
| 5 | −1 | −2 | 5 | −4 | −21 | 58 |
| 7 | 1 | 0 | −7 | 0 | −79 | 0 |
| 11 | −1 | 0 | −25 | 0 | 79 | 0 |
| 13 | −1 | 6 | 13 | −84 | −101 | −1102 |
| 17 | −1 | 2 | 19 | 36 | −699 | 614 |
| 19 | −1 | 0 | 15 | 0 | −161 | 0 |

There is a clear pattern that emerges, $W_r^{\pm}$ has $H_p = 0$ unless $p \equiv 1 \bmod 4$ when $\pm 1 = (-1)^{r-1}$.

It would seem that the $L$-function of $W_5^+$ equals $L(f_2 \otimes f_4, s)$, where $f_2$ is our previous CM form of weight 2 and $f_4$ is a modular form of weight 4 and level 32 with eigenvalues

| $p$ | $f_4$ |
|---|---|
| 2 | 0 |
| 3 | 8 |
| 5 | −10 |
| 7 | 16 |
| 11 | −40 |
| 13 | −50 |
| 17 | −30 |
| 19 | 40 |
| 23 | 48 |
| 29 | −34 |
| 31 | 320 |
| 37 | 310 |

## 3.2 Stirling and Dirichlet

It turns out that the discriminant $D$ giving the Dirichlet character of the linear factor of $L_p(T)$ corresponding to $\psi = 1$ can be given directly in terms of $\alpha$ and $\beta$ as follows.

The hypergeometric series $F\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| t\right)$ has a power series around $t = 0$ of the form

$$\sum_{n\geq 0} u_n \left(\frac{t}{K}\right)^n,$$

where

$$u_n := \prod_{\nu\geq 1} (\nu n)!^{\gamma_\nu}, \qquad K := \prod_{\nu \geq 1} \nu^{\nu\gamma_\nu}, \tag{10}$$

for certain integers $\gamma_\nu$, which are zero for almost all $\nu$. The relation between the parameters $\alpha, \beta$ and $\gamma$ is the following

$$\prod_{\nu\geq 1}(1-T^\nu)^{\gamma_\nu} = \frac{q_\infty(T)}{q_0(T)}, \qquad q_\infty(T) := \prod_j (1-e^{2\pi i\alpha_j}T), \qquad q_0(T) := \prod_j (1-e^{2\pi i\beta_j}T). \quad (11)$$

(We are assuming that $q_0$ and $q_\infty$ have coefficients in $\mathbb{Z}$.)

By Stirling, as $n \to \infty$

$$u_n \sim \frac{\sqrt{\delta}}{(2\pi n)^{d/2}}\, K^n,$$

where

$$\delta := \prod_{\nu\geq 1} \nu^{\gamma_\nu}, \qquad d := -\sum_{\nu\geq 1}\gamma_\nu.$$

Numerically, it seems that $D$ is the discriminant of the quadratic extension of $\mathbb{Q}$ given by adjoining a square root of $(-1)^d\delta$.

# 4 Hypergeometric motives

## 4.1 Hodge numbers

We expect the above situation to be true in greater generality. Let $\gamma$ be a non-zero sequence of integers $\gamma = (\gamma_\nu)$ for $\nu \geq 1$, only finitely many of which are non-zero, and satisfying

$$\sum_{\nu\geq 1}\gamma_\nu \nu = 0, \qquad (12)$$

a condition we will call *regularity*. We associate to $\gamma$ a family of motives $V_t$ with $t \in \mathbb{P}^1$ defined over $\mathbb{Q}$. Our goal is to describe the $L$-function of $V_t$ completely.

To this end, define

$$\mathcal{L}^+(x) := \sum_{\nu\geq 1}\gamma_\nu\left(\tfrac{1}{2} - \{\nu x\}\right), \qquad x \in \mathbb{R},$$

where $\{\cdot\}$ denotes the ordinary fractional part of a real number. It is easy to check that $\mathcal{L}^+$ is periodic of period 1, locally constant, right continuous and satisfies

$$\mathcal{L}^+(-x) = -\mathcal{L}^-(x), \qquad (13)$$

where $\mathcal{L}^-(x) := \lim_{y\to x^-}\mathcal{L}^+(y)$. Also, $\mathcal{L}^+$ has only finitely many discontinuities. If we let $l(x) := \mathcal{L}^+(x) - \mathcal{L}^-(x)$ then $l$ takes integer values and is zero away from these discontinuities. In other words, the functions $\mathcal{L}^\pm$ have only jump discontinuities and the jumps are integral. We have $\sum_{x\in[0,1)} l(x) = 0$ and by (13) the symmetry $l(-x) = l(x)$.

Define the *weight* of $\gamma$ by

$$w := \max_{x\in[0,1)} \mathcal{L}^+(x) - \min_{x\in[0,1)} \mathcal{L}^+(x) - 1.$$

Clearly, $w$ is an integer and is in fact non-negative since $\mathcal{L}$ is not identically zero (we have assumed $\gamma$ is not zero). Note that $\max_{x\in[0,1)} \mathcal{L}^\pm(x) = -\min_{x\in[0,1)} \mathcal{L}^\pm(x) = (w+1)/2$ and hence $\mathcal{L}^\pm(x) + (w+1)/2$ takes values in $\mathbb{Z}_{\geq 0}$.

Define the *Hodge polynomial* of $\gamma$ by

$$h(T) := \sum_{l(x)>0} T^{\mathcal{L}^-(x)+(w+1)/2}[l(x)] \in \mathbb{Z}[T],$$

where $[l] := 1 + T + \cdots + T^{l-1}$ and the sum is over the finitely many $x \in [0,1)$ with $l(x) > 0$.

**Lemma 4.1.** *The Hodge polynomial is reciprocal of degree $w$ and has non-negative integer coefficients.*

$$h(T^{-1}) = T^{-w}h(T).$$

*Proof.* It is clear from the definition that the coefficients of $h(T)$ are non-negative integers. Let $x$ be the left endpoint of an interval in $(0,1)$ where $\mathcal{L}^+(x)$ achieves its maximum. Then $l(x) > 0$ and the corresponding term in the sum defining $h(T)$ has degree $\mathcal{L}^-(x)+(w+1)/2+l(x)-1 = w$. Hence the degree of $h$ is $w$.

We have

$$h(T^{-1}) = \sum_{l(x)>0} T^{-\mathcal{L}^-(x)-(w+1)/2}T^{1-l(x)}[l(x)] = \sum_{l(x)>0} T^{\mathcal{L}^-(-x)+l(-x)-(w+1)/2}T^{1-l(x)}[l(x)]$$

by (13) and the right hand side simplifies to give $T^{-w}h(T)$ finishing the proof. ◻

We refine the Hodge polynomial by definining for every $m \in \mathbb{Z}_{\geq 0}$

$$h_0^{(m)}(T) := \sum_{l(x)=m} T^{\mathcal{L}^-(x)+(w+1)/2} \in \mathbb{Z}[T]$$

so that $h(T) = \sum_{m\geq 0} h^{(m)}(T)[m]$. As in the proof of Lemma 4.1 we find

$$h_0^{(m)}(T^{-1}) = \sum_{l(x)=m} T^{-\mathcal{L}^-(x)-(w+1)/2} = \sum_{l(x)=m} T^{\mathcal{L}^+(-x)-(w+1)/2} = \sum_{l(x)=m} T^{\mathcal{L}^-(x)+m-(w+1)/2}$$

so that

$$h_0^{(m)}(T^{-1}) = T^{-w+m-1}h_0^{(m)}(T).$$

An alternative way to compute the Hodge polynomial of $\gamma$ is as follows.

**Proposition 4.2.** *We have*

$$h(T) = \sum_{l(x)<0} = T^{\mathcal{L}^+(x)+(w+1)/2}[-l(x)]. \tag{14}$$

*Proof.* Slightly deform $\mathcal{L}^+$ to a continuous function $L$ as follows. Replace a jump of $\mathcal{L}^+$ with $l(x) > 0$ by an increasing function in a small interval $(x-\epsilon, x+\epsilon)$ for some $\epsilon > 0$ going from $\mathcal{L}^-(x)$ to $\mathcal{L}^+(x)$. Similarly replace a jump with $l(x) < 0$ by a decreasing function going from $\mathcal{L}^+(x)$ to $\mathcal{L}^-(x)$.

Then

$$h(T) = \sum T^{L(x)+(w+1)/2},$$

where the sum is over $x \in [0,1)$ such that $L(x) \in \mathbb{Z}$ and $L(x) < L(x')$ for $x < x'$. Call such an $x$ a *point of increase*.

On the other hand the right hand side of (14) is a similar sum over $x \in [0,1)$ such that $L(x) \in \mathbb{Z}$ and $L(x) > L(x')$ for $x' < x$. Call such an $x$ a *point of decrease*. By periodicity, for a given $y \in \mathbb{Z}$ the number of points of increase with $x \in [0,1)$ with $L(x) = y$ is the same as those of decrease. This proves our claim. $\square$

We define a new refinement of the Hodge polynomial. For $m \in \mathbb{Z}_{\geq 0}$ let

$$h_\infty^{(m)}(T) := \sum_{l(x)=-m} T^{\mathcal{L}^+(x)+(w+1)/2} \in \mathbb{Z}[T].$$

Then

$$h(T) = \sum_{m\geq 0} h_\infty^{(m)}(T)[m].$$

Note that if we replace $\gamma$ by $-\gamma$ then $\mathcal{L}$ also changes sign and $h_0^{(m)}$ turns into $h_\infty^{(m)}$. Hence

$$h_\infty^{(m)}(T^{-1}) = T^{-w+m-1} h_\infty^{(m)}(T).$$

## 4.2 Tame primes

We will say that a prime $p$ is *tame* if it does not divide the denominators of $\alpha$ or $\beta$ and one of $v_p(t), v_p(t^{-1})$ or $v_p(t-1)$ is positive. We would like to describe the Euler factor $L_p(T)$ and power $f_p$ of $p$ in the conductor of the $L$-function associated to the motive $H_t(\alpha,\beta)$.