<u>Sep 25, 2007</u>

$z \in G$

$$N(z) := \# \underbrace{\{ (x_1, y_1, \ldots, x_g, y_g) \in G^{2g} \mid [x_1, y_1] \cdots [x_g, y_g] = z \}}_{}$$

$U(z) \quad ||$

We proved

$$N(z) = \sum_{\chi} \left(\frac{|G|}{\chi(1)} \right)^{2g-1} \chi(z)$$

$u \in Z(G)^{2g} \qquad u = (u_1, u_2, \ldots, u_{2g})$

$$(x_1, y_1, \ldots, x_g, y_g) \longmapsto (u_1 x_1, u_2 y_1, \ldots, u_{2g} y_g)$$

$$[x, y] = [ux, vy] \qquad u, v \in Z(G)$$

This is an action of $Z(G)^{2g}$ on $U(z)$.
It has no fixed points.

$$\Rightarrow \quad |Z(G)|^{2g} \mid \# U(z) = N(z)$$

for all z

Let θ be an irreducible char of G. Consider

$$T_\theta := \sum_{z \in G} \bar{\theta}(z) N(z)$$

$$= \sum_{z \in G} \bar{\theta}(z) \sum_{\chi} \left(\frac{|G|}{\chi(1)}\right)^{2g-1} \chi(z)$$

$$= \sum_{\chi} \left(\frac{|G|}{\chi(1)}\right)^{2g-1} \underbrace{\sum_{z \in G} \bar{\theta}(z) \chi(z)}_{= |G| \cdot \langle \theta, \chi \rangle}$$

$$= \underbrace{\left(\frac{|G|}{\theta(1)}\right)^{2g-1} \cdot |G|}_{\in \mathbb{Q}}$$

since $\langle \theta, \chi \rangle = \begin{cases} 1 & \theta = \chi \\ 0 & \theta \neq \chi \end{cases}$

On the other hand from the defn T_θ is an algebraic integer

$\Rightarrow T_\theta \in \mathbb{Z}$.

Note $|Z(G)|^{2g} \mid N(Z)$

$$\Rightarrow \quad |Z(G)|^{2g} \mid T_\Theta$$

$$\mathbb{Z} \ni \frac{T_\theta}{|Z(G)|^{2g}} = \left(\frac{|G|/|Z(G)|}{\Theta(1)}\right)^{2g-1} \frac{|G|}{Z(G)}$$

$$G_1 := G/Z(G)$$

$$\left(\frac{|G_1|}{\theta(1)}\right)^{2g-1} \cdot |G_1| \in \mathbb{Z}.$$

EX. Since true for all $g \in \mathbb{N}$

$$\Rightarrow \quad \frac{|G_1|}{\Theta(1)} \in \mathbb{Z}$$

or

$$\theta(1) \mid |G_1|$$

I.e. We proved that the dimension of an irred. repn of divides

$$[G : Z(G)] \mid |G|$$

$G = S_5$ character table

Review conj classes and their sizes in S_n

conj. class in S_n $\longleftrightarrow$ cycle decomp.

label by a partition of n.

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

$$n = |\lambda| = \lambda_1 + \lambda_2 + \cdots$$

<u>Notation</u> $\lambda \vdash n$

Notation by multiplicities

$$\lambda = 1^{m_1} 2^{m_2} \cdots$$

$$\underbrace{(\cdot)(\cdot)\cdots(\cdot)}_{m_1} \; \underbrace{(\cdot\cdot)\cdots\cdots(\cdot\cdot)}_{m_2} \cdots$$

$m_i = \#\{\lambda_j = i\}$

conj classes in S_n $\longleftrightarrow$ $\lambda \vdash n$

How many elements are there in the conjugacy class labeled by λ?

E.g. $\lambda = (2,2,1)$, $|\lambda| = 5$

$(\cdot)(\cdot\cdot)(\cdot\cdot)$

$G \circlearrowright X$, $\# Gx = [G, \mathrm{Stab}_G x]$

$X = \coprod$ orbits

$S_n \ni$

$\sigma = \underbrace{(\cdot)(\cdot)\cdots(\cdot)}_{m_1} \underbrace{(\cdot\cdot)\cdots(\cdot\cdot)}_{m_2} \cdots$

$\tau\sigma\tau^{-1} = \sigma$

$\mathrm{Stab}_{S_n}(\sigma) \subseteq S_{m_1} \times S_{2m_2} \times \cdots$

$(1\,2\cdots k) = (2\,3\cdots k\,1) = \cdots$

$$\mathrm{Stab}_{S_n}(\sigma) \simeq S_{m_1} \times \left[(\mathbb{Z}/2\mathbb{Z})^{m_2} \rtimes S_{m_2} \right]$$

$$\cdots \times \left[(\mathbb{Z}/d\mathbb{Z})^{m_d} \rtimes S_{m_d} \right] \cdots$$

wreath product of $\mathbb{Z}/d\mathbb{Z}$ by S_{m_d}

$$\mathbb{Z}/d\mathbb{Z} \wr S_{m_d}$$

$$|\mathrm{Stab}_{S_n}(\sigma)| = \prod_{d \geq 1} d^{m_d} \cdot m_d! =: z_\lambda$$

$$\text{size of conj class} = \frac{n!}{z_\lambda}$$

#'s	1	10	20	30	24	15	20
	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
1	1	1	1	1	1	1	1
sgn	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1
W	5	1	-1	-1	0	1	1
W'	5	-1	-1	1	0	1	-1
$\Lambda^2 V$	6	0	0	0	1	-2	0

$p(n) := \#\{\lambda \vdash n\}$

Partitions of 5

$(1,1,1,1,1)$ 1 $p(5) = 7$

$(2,1,1,1)$ 10

$(3,1,1)$ $5!/3^1 \cdot 1! \cdot 1^2 \cdot 2! = 20$

$(4,1)$ $5!/4^1 \cdot 1! \cdot 1^1 \cdot 1! = 30$

(5) $5!/5^1 \cdot 1! = 24$

$(2,2,1)$ $5!/2^2 \cdot 2! = 15$

$(3,2)$ $5!/3 \cdot 2 = 20$

$$\langle \chi_V, \chi_V \rangle = (4^2 + 2^2 \times 10 + 1^2 \times 20 + (-1)^2 \times 24 + (-1)^2 \times 20)/120$$

$$= 1$$

$\Rightarrow$ V is irred.

$\Lambda^2 V$ is irred. $\dim \binom{4}{2} = 6$

$120 - (1^2 + 1^2 + 4^2 + 4^2 + 6^2) = 50$

Two irred repn of dim 5

$\Lambda^3 V$ $\dim = 4$ $\Lambda^3 V \simeq ?$

$\Lambda^4 V$ $\dim = 1$ $\Lambda^4 V \simeq \mathrm{sgn}$

Note $\Lambda^2 V \otimes \mathrm{sgn} \simeq \Lambda^2 V$

Look at $\mathrm{Sym}^2 V$ $\dim = \binom{4+2-1}{2} = 10$

$\mathrm{Sym}^2 V$	10	4	1	0	0	2	1

$$(\chi_{\mathrm{Sym}^2 V}, 1) = \frac{1}{120}(10 + 4 \times 10 + \overset{1\times}{20} + 2 \times 15 + 1 \times 20) = 1$$

$$(\chi_{Sym^2 V}, sgn) = \frac{1}{120}\left(10 - 4\times 10 + 1\times 20 + 2\times 15 - 1\times 20\right.$$

$$= 0$$

$$(\chi_{Sym^2 V}, V) = 1$$

$$\chi_{Sym^2 V} - \chi_U - \chi_V = 5 \quad 1 \quad -1 \quad -1 \quad 0 \quad 1 \quad 1$$

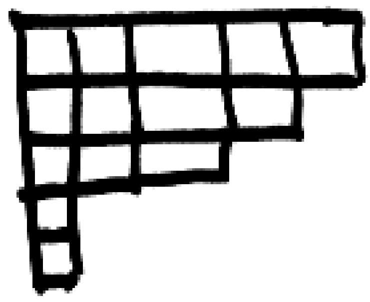
is irred. W

In general the representations of S_n can be labeled by partitions of n, in a natural way.

conj class S_n $\longleftrightarrow$ partitions of n $\longleftrightarrow$ irred. repn of S_n

<u>Ferrers diagram</u>

$\lambda = (\lambda_1, \lambda_2, \dots)$



λ_1
λ_2
λ_3
λ_4
λ_5
$\vdots$