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$p > 2$

$m = 2$

$$\Phi := \left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right) \qquad \Psi := \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)$$

B 2×2 symmetric non-degenerate form on L dim 2 space$/\mathbb{F}_p$

one of two: i) xy
ii) $x^2 - \epsilon y^2$, $\epsilon \notin \mathbb{F}_p^{*2}$

i) B represents 0
ii) does not represent 0
$({}^T u B u = 0)$

$$\Phi = \Psi - {}^T\Psi$$

$\rightsquigarrow$ Heisenberg group $H \ni (\zeta, u)$, $\zeta \in \mu_p$, $u \in L \oplus L^*$

$\eta : \mu_p \to \mathbb{C}^*$

Schrödinger repn.

$H \circlearrowleft V := \{ f : L \to \mathbb{C} \}$

~~so~~ $L^* \simeq L$ via B

use notation $h = (\zeta, \ell_1, \ell_2)$

$\ell_1, \ell_2 \in L$

H acts on V

- $\underline{(\zeta, 0, 0)}$ center

  $f \longmapsto \zeta f$, $\quad f \in V$ $\quad$ $\underline{\text{scalar}}$

- $\underline{(0, \ell_1, 0)}$

  $f \longmapsto f(\cdot + \ell_1)$ $\quad$ $\underline{\text{translation}}$

- $\underline{(0, 0, \ell_2)}$

  $f \longmapsto \phi(\cdot, \ell_2) f$ $\quad$ $\underline{\text{character}}$

$\underline{G := SL_2(\mathbb{F}_p)}$

$G \longrightarrow \mathrm{Aut}_0(H) := \{\text{Autom of } H \text{ preserve center}\}$

$\sigma \in G$

$$(\zeta, u)\sigma := (\overset{\nu_\sigma(u)}{\zeta}, u\sigma)$$

$\nu_\sigma(u)$ to be determined.

$\nu_\sigma(0) \equiv 1$

$\sigma \in SL_2(\mathbb{F}_p) \qquad \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$$U \ni (\ell_1, \ell_2) \longmapsto (\ell_1, \ell_2)\sigma = (\alpha \ell_1 + \gamma \ell_2, \; \beta \ell_1 + \delta \ell_2)$$

preserves Φ; i.e.

$${}^T\!\left(\begin{array}{c|c} \alpha I_2 & \beta I_2 \\ \hline \gamma I_2 & \delta I_2 \end{array}\right)\left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right)\left(\begin{array}{c|c} \alpha I_2 & \beta I_2 \\ \hline \gamma I_2 & \delta I_2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right)$$

Define:

$$\nu_\sigma(u) := \left(\frac{\psi(u\sigma, u\sigma)}{\psi(u, u)} \right)^{1/2} \qquad (p > 2)$$

$\underline{\sigma \in G}$

$$(\zeta_1, u_1)(\zeta_2, u_2) = (\zeta_1 \zeta_2 \psi(u_1, u_2), u_1 + u_2)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$(\zeta_1 \nu_\sigma(u_1), u_1\sigma) \cdot (\zeta_2 \nu_\sigma(u_2), u_2\sigma)$$

$$\stackrel{?}{=} (\zeta_1 \zeta_2 \psi(u_1, u_2) \nu_\sigma(u_1 + u_2), (u_1 + u_2)\sigma)$$

$$= \left(\xi_1 \xi_2 \frac{\nu_\sigma(u_1)\, \nu_\sigma(u_2)}{\psi(u_1\sigma, u_2\sigma)},\ u_1\sigma + u_2\sigma \right)$$

$$(u_1 + u_2)\sigma = u_1\sigma + u_2\sigma \ \checkmark$$

$$\nu_\sigma(u_1) \cdot \nu_\sigma(u_2) = \frac{\psi(u_1, u_2)}{\psi(u_1\sigma, u_2\sigma)}\, \nu_\sigma(u_1 + u_2)$$

<u>Check</u> our choice of ν_σ verifies this

$$\frac{\psi(u_1\sigma, u_1\sigma)}{\psi(u_1, u_1)} \cdot \frac{\psi(u_2\sigma, u_2\sigma)}{\psi(u_2, u_2)} = \left[\frac{\psi(u_1, u_2)}{\psi(u_1\sigma, u_2\sigma)} \right]^2 \cdot \frac{\psi(u_1^\sigma + u_2^\sigma,\ u_1^\sigma + u_2^\sigma)}{\psi(u_1\bullet + u_2\bullet,\ u_1\bullet + u_2\bullet)}$$

$$\psi(u_1 + u_2, u_1 + u_2) = \psi(u_1, u_1)\, \psi(u_2, u_2)\, \psi(u_1, u_2)\, \psi(u_2, u_1)$$

″

σ's . . .

$$1 = \left[\frac{\psi(u_1,u_2)}{\psi(u_1\sigma,u_2\sigma)}\right]^2 \frac{\psi(u_1\sigma,u_2\sigma)\cdot\psi(u_2\sigma,u_1\sigma)}{\psi(u_1,u_2)\,\psi(u_2,u_1)}$$

$$= \frac{\phi(u_1,u_2)}{\phi(u_1\sigma,u_2\sigma)} \quad \checkmark$$

$$G \longrightarrow \mathrm{Aut}_0(H)$$

$$\sigma \longmapsto \big((\zeta,u)\mapsto(\zeta\nu_\sigma(u),u\sigma)\big)$$

Schrödinger

$$\begin{tikzcd} H \arrow[r, "\rho"] \arrow[d, "\sigma"'] & GL(V) \\ H \arrow[ur, dotted, "\rho^\sigma"'] & \end{tikzcd}$$

ρ, ρ^σ have center acting the same way. i.e. (Stone von Neumann)

$\Rightarrow \rho \cong \rho^\sigma$

i.e. $\quad R(\sigma)^{-1}\rho(h)R(\sigma) = \rho(h^\sigma)$

Any two choices of R are off by an $\mathrm{Aut}(\rho)$ = scalars by Schur.

R is well defined up to scalars

$$G \xrightarrow{R} PGL(V) := GL(V)/\mathbb{C}^\times$$
$$\sigma \longmapsto R(\sigma)$$

<u>projective representation</u>

$$R(\sigma_1)R(\sigma_2) = R(\sigma_1\sigma_2)$$
up to scalars.

$$R(\sigma_1\sigma_2)\,c(\sigma_1,\sigma_2) = R(\sigma_1)R(\sigma_2)$$

<u>GOAL</u>: Choose R so $c \equiv 1$.

[$c(\sigma_1,\sigma_2) \in \mathbb{C}^\times$ $\quad c$, 2-cocycle.
want to split this cocycle]

Look at specific elements:

1) $\sigma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $\quad a \in \mathbb{F}_p^\times$

$$\nu_\sigma(u)^2 = \frac{\Psi(u\sigma, u\sigma)}{\Psi(u,u)} \qquad \Psi = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)$$

$u = (\ell_1, \ell_2)$

$u\sigma = (a\ell_1, a^{-1}\ell_2)$

$$\Psi(u\sigma, u\sigma) = B(a\ell_1, a^{-1}\ell_2)$$
$$= a.a^{-1} B(\ell_1, \ell_2)$$

$$= B(\ell_1, \ell_2)$$
$$= \Psi(u, u)$$

$$\Rightarrow \nu_\sigma(u) = 1$$

i) $u = (\ell_1, 0)$, $u\sigma = (a\ell_1, 0)$

$h = (1, \ell_1, 0)$, $h^\sigma = (1, a\ell_1, 0)$

$$h f(\ell) = f(\ell + \ell_1)$$
$$h^\sigma f(\ell) = f(\ell + a\ell_1)$$

ii) $u = (0, \ell_2)$, $u\sigma = (0, a^{-1}\ell_2)$

$h = (1, 0, \ell_2)$, $h^\sigma = (1, 0, a^{-1}\ell_2)$

$$h f(\ell) = \phi(\ell, \ell_2) f(\ell)$$
$$h^\sigma f(\ell) = \phi(\ell, a^{-1}\ell_2) f(\ell)$$

<u>want:</u> $R(\sigma) \in GL(V)$

$$R(\sigma)^{-1} h R(\sigma) = h^\sigma$$

<u>choose</u> $R(\sigma) f(\ell) = f(a\ell)$

Check it works!

(i) $\quad h = (1, \ell_1, 0), \quad h^\sigma = (1, a\ell_1, 0)$

$$
\begin{aligned}
&\left[R(\sigma)^{-1}\, h\, R(\sigma)\, f\right](\ell) \\
&= \left[R(\sigma)^{-1} h\right] f(a\ell) \\
&= R(\sigma)^{-1} f(a(\ell + \ell_1)) \\
&= R(\sigma)^{-1} f(a\ell + a\ell_1) \\
&= f(\ell + a\ell_1) \\
&= h^\sigma f(\ell) \ \checkmark
\end{aligned}
$$

(ii) $\quad h = (1, 0, \ell_2), \quad h^\sigma = (1, 0, a^{-1}\ell_2)$

generators for H
$(\sigma, 0, 0)$
$(1, \ell, 0)$
$(1, 0, \ell)$

$$
\begin{aligned}
&\left[R(\sigma)^{-1}\, h\, R(\sigma)\, f\right](\ell) \\
&= \left[R(\sigma)^{-1} h\right] f(a\ell) \\
&= R(\sigma)^{-1} \phi(\ell, \ell_2) f(a\ell) \\
&= \phi(a^{-1}\ell, \ell_2) f(\ell) \\
&= \phi(\ell, a^{-1}\ell_2) f(\ell) \\
&= h^\sigma f(\ell) \ \checkmark
\end{aligned}
$$

2) $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{F}_p)$

<u>Fourier transform on V</u>

$V = \{f : L \to \mathbb{C}\}$

$$\mathcal{F}(f)(\ell) := \hat{f}(\ell)$$

$$= \frac{1}{|L|^{1/2}} \sum_{\ell_1 \in L} f(\ell_1) b(\ell, -\ell_1)$$

$b = \theta \cdot B$ multiplicative version of B

(i) $u = (\ell_1, 0)$, $\nu_\sigma(u) = 1$

$$u\sigma = (\ell_1, 0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (0, \ell_1)$$

<u>Claim</u> $R(\sigma) = \mathcal{F}^{-1}$ works

<u>pf</u> $R(\sigma)^{-1} h R(\sigma) = h^\sigma$

$$\mathcal{F} h = h^\sigma \mathcal{F}$$

lhs applied to f

$$\frac{1}{|L|^{1/2}} \sum_{\ell' \in L} f(\ell' + \ell_1) b(\ell, -\ell')$$

$$= \frac{1}{|L|^{1/2}} \sum_{\ell' \in L} f(\ell') b(\ell, -\ell' + \ell_1)$$

$$= b(\ell, \ell_1) \frac{1}{|L|^{1/2}} \sum_{\ell' \in L} f(\ell')\, b(\ell_1, -\ell')$$

$$= b(\ell, \ell_1)\, \mathcal{F} f(\ell)$$

$$= [h^\sigma \mathcal{F}]\, f(\ell)$$

$$\Phi = \left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right)$$

$$h^\sigma = (0, \ell_1)$$

$$\Phi((\ell_1, 0), (0, \ell_2)) = B(\ell_1, \ell_2)$$