3.26pt

Hypergeometric Motives

Fernando Rodriguez Villegas

The Abdus Salam International Centre for Theoretical Physics

May 2015

Collaborators

H. Cohen, Ph. Candelas, S. Baig, F. Beukers, X. de la Ossa,
A. Mellit, P. Molin, D. Roberts, M. Vashenko, M. Watkins

L-functions

- $$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1}$$

L-functions

- $$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1}$$

- Conductor: N, a positive integer.

L-functions

- $$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1}$$
- Conductor: N, a positive integer.
- Euler factors: $L_p(T) \in \mathbb{Z}[T]$ polynomial generically of degree d.

L-functions

- $$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1}$$

- Conductor: N, a positive integer.

- Euler factors: $L_p(T) \in \mathbb{Z}[T]$ polynomial generically of degree d.

- Weight w

$$L_p(T) = \prod_{i=1}^{d} (1 - \xi_i T), \qquad |\xi_i| = p^{w/2}, \qquad p \nmid N.$$

L-functions

- $$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1}$$
- Conductor: N, a positive integer.
- Euler factors: $L_p(T) \in \mathbb{Z}[T]$ polynomial generically of degree d.
- Weight w
$$L_p(T) = \prod_{i=1}^{d} (1 - \xi_i T), \qquad |\xi_i| = p^{w/2}, \qquad p \nmid N.$$
- $L_\infty(s)$ product of gamma factors

L-functions

- $$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1}$$
- Conductor: N, a positive integer.
- Euler factors: $L_p(T) \in \mathbb{Z}[T]$ polynomial generically of degree d.
- Weight w
$$L_p(T) = \prod_{i=1}^{d} (1 - \xi_i T), \qquad |\xi_i| = p^{w/2}, \qquad p \nmid N.$$
- $L_\infty(s)$ product of gamma factors
- Expect functional equation
$$\Lambda(w + 1 - s) = \epsilon \Lambda(s), \qquad \epsilon = \pm 1$$

Computing Λ

- Inverse Mellin transform

$$L_\infty(s) = \int_0^\infty k_\infty(t) t^s \frac{dt}{t}$$

Computing Λ

- Inverse Mellin transform

$$L_\infty(s) = \int_0^\infty k_\infty(t) t^s \frac{dt}{t}$$

- $$\sum_{n \geq 1} \frac{a_n}{n^s} := \prod_p L_p(p^{-s})^{-1}.$$

  converges in some half-plane.

Computing Λ

- Inverse Mellin transform

$$L_\infty(s) = \int_0^\infty k_\infty(t) t^s \frac{dt}{t}$$

- $$\sum_{n \geq 1} \frac{a_n}{n^s} := \prod_p L_p(p^{-s})^{-1}.$$

  converges in some half-plane.

- $$\Lambda(s) = \int_0^\infty \phi(t) t^s \frac{dt}{t}, \qquad \phi(t) := \sum_{n \geq 1} a_n k_\infty\left(\frac{nt}{\sqrt{N}}\right).$$

Computing Λ

- Inverse Mellin transform

$$L_\infty(s) = \int_0^\infty k_\infty(t) t^s \frac{dt}{t}$$

- $$\sum_{n \geq 1} \frac{a_n}{n^s} := \prod_p L_p(p^{-s})^{-1}.$$

  converges in some half-plane.

- $$\Lambda(s) = \int_0^\infty \phi(t) t^s \frac{dt}{t}, \qquad \phi(t) := \sum_{n \geq 1} a_n k_\infty\left(\frac{nt}{\sqrt{N}}\right).$$

- Functional equation

$$\phi(t^{-1}) = \epsilon t^{w+1} \phi(t)$$

Computing Λ (cont'd)

- k_∞ is of rapid decay $\Longrightarrow$ can compute ϕ if we know

Computing Λ (cont'd)

- k_∞ is of rapid decay $\Longrightarrow$ can compute ϕ if we know
- $$k_\infty, \quad N, \quad \epsilon, \quad a_n, \quad n = 1, 2, \ldots, \approx \sqrt{N}$$

Computing Λ (cont'd)

- k_∞ is of rapid decay $\Longrightarrow$ can compute ϕ if we know
- $$k_\infty, \quad N, \quad \epsilon, \quad a_n, \quad n = 1, 2, \ldots, \approx \sqrt{N}$$
- Functional equation for $\phi \Longrightarrow$ can compute $\Lambda(s)$ (Riemann).

Computing Λ (cont'd)

- k_∞ is of rapid decay $\Longrightarrow$ can compute ϕ if we know
- $$k_\infty, \quad N, \quad \epsilon, \quad a_n, \quad n = 1, 2, \ldots, \approx \sqrt{N}$$
- Functional equation for $\phi \Longrightarrow$ can compute $\Lambda(s)$ (Riemann).
- Typically calculation breaks up into:

  $$L_p(T), \quad p \notin S, \qquad L_p(T), \quad p \in S, \qquad L_\infty(s), \quad N, \quad \epsilon$$

  S finite set of primes.

Computing Λ (cont'd)

- In practice: only know some of the data.

Computing Λ (cont'd)

- In practice: only know some of the data.
- Test functional equation for ϕ: evaluating both sides at $t \approx 1$.

Computing Λ (cont'd)

- In practice: only know some of the data.
- Test functional equation for ϕ: evaluating both sides at $t \approx 1$.
- Use test to find missing data.

Computing Λ (cont'd)

- In practice: only know some of the data.
- Test functional equation for ϕ: evaluating both sides at $t \approx 1$.
- Use test to find missing data.
- Robust test.

Computing Λ (cont'd)

- In practice: only know some of the data.
- Test functional equation for ϕ: evaluating both sides at $t \approx 1$.
- Use test to find missing data.
- Robust test.
- Bootstrap.

Computing Λ (cont'd)

- In practice: only know some of the data.
- Test functional equation for ϕ: evaluating both sides at $t \approx 1$.
- Use test to find missing data.
- Robust test.
- Bootstrap.
- HGM implementation in MAGMA (M. Watkins)

Source of L-functions

- Automorphic Forms.

Source of L-functions

- Automorphic Forms.
- Cohomology of algebraic varieties.

Source of L-functions

- Automorphic Forms.
- Cohomology of algebraic varieties.
- Typically appear as piece of a bigger object

Source of L-functions

- Automorphic Forms.
- Cohomology of algebraic varieties.
- Typically appear as piece of a bigger object
- Subspace $V \subseteq H$

$$L_p(T) = \det(1 - \mathrm{Frob}_p|_V\, T), \qquad p \nmid N$$

Automorphic Forms

- Usual modular forms:

$$d = \dim V = 2, \qquad \dim H \approx cN$$

Automorphic Forms

- Usual modular forms:

$$d = \dim V = 2, \qquad \dim H \approx cN$$

- Siegel modular forms Sp_4: can compute for $N \approx 500$

Automorphic Forms

- Usual modular forms:

$$d = \dim V = 2, \qquad \dim H \approx cN$$

- Siegel modular forms Sp_4: can compute for $N \approx 500$
- SL_4 happy if can compute $L_2(T)$.

Algebraic Varieties

- Quintic threefold

$$X : F(x_1, \ldots, x_5) = 0$$

Algebraic Varieties

- Quintic threefold
$$X : F(x_1, \ldots, x_5) = 0$$
- $$H := H^3(X, \mathbb{Q}), \qquad \dim H = 204.$$

Algebraic Varieties

- Quintic threefold

$$X : F(x_1, \ldots, x_5) = 0$$

- $$H := H^3(X, \mathbb{Q}), \qquad \dim H = 204.$$

- Dwork pencil

$$X_\psi : x_1^5 + \cdots x_5^5 - 5\psi x_1 \cdots x_5 = 0$$

Algebraic Varieties

- Quintic threefold
$$X : F(x_1, \ldots, x_5) = 0$$
- $$H := H^3(X, \mathbb{Q}), \qquad \dim H = 204.$$
- Dwork pencil
$$X_\psi : x_1^5 + \cdots x_5^5 - 5\psi x_1 \cdots x_5 = 0$$
- $$A \subseteq \mathrm{Aut}(X_\psi), \qquad x_i \mapsto \zeta_i x_i, \qquad \zeta_1^5 = \cdots = \zeta_5^5 = \zeta_1 \cdots \zeta_5 = 1$$

Algebraic Varieties

- Quintic threefold
$$X : F(x_1, \ldots, x_5) = 0$$
- $$H := H^3(X, \mathbb{Q}), \qquad \dim H = 204.$$
- Dwork pencil
$$X_\psi : x_1^5 + \cdots x_5^5 - 5\psi x_1 \cdots x_5 = 0$$
- $$A \subseteq \mathrm{Aut}(X_\psi), \qquad x_i \mapsto \zeta_i x_i, \qquad \zeta_1^5 = \cdots = \zeta_5^5 = \zeta_1 \cdots \zeta_5 = 1$$
- $$V := H^A, \qquad \dim V = 4$$

Hodge numbers

- Refinement of rank d that determines $L_\infty(s)$.

Hodge numbers

- Refinement of rank d that determines $L_\infty(s)$.
- $$h^{p,q} \in \mathbb{Z}_{\geq 0}, \qquad\qquad p + q = w$$

Hodge numbers

- Refinement of rank d that determines $L_\infty(s)$.
- $$h^{p,q} \in \mathbb{Z}_{\geq 0}, \qquad p + q = w$$
- $$h^{p,q} = h^{q,p}, \qquad \sum_{p,q} h^{p,q} = d$$

Hodge numbers

- Refinement of rank d that determines $L_\infty(s)$.
- $$h^{p,q} \in \mathbb{Z}_{\geq 0}, \qquad p+q=w$$
- $$h^{p,q} = h^{q,p}, \qquad \sum_{p,q} h^{p,q} = d$$
- Hodge vector (up to Tate twists)

$$\mathbf{h} := (h^{w,0}, h^{w-1,1}, \ldots, h^{0,w})$$

Hodge numbers

- Refinement of rank d that determines $L_\infty(s)$.
- $$h^{p,q} \in \mathbb{Z}_{\geq 0}, \qquad p+q=w$$
- $$h^{p,q} = h^{q,p}, \qquad \sum_{p,q} h^{p,q} = d$$
- Hodge vector (up to Tate twists)
$$\mathbf{h} := (h^{w,0}, h^{w-1,1}, \ldots, h^{0,w})$$
- $$h^{p,p} = h^{p,p}_{+} + h^{p,p}_{-}$$

Hypergeometric Motives

- $q_0, q_\infty \in \mathbb{Z}[T]$, coprime, same degree d, roots are roots of unity.

Hypergeometric Motives

- $q_0, q_\infty \in \mathbb{Z}[T]$, coprime, same degree d, roots are roots of unity.
- Get associated family of motives $\mathcal{H}(t)$ with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Hypergeometric Motives

- $q_0, q_\infty \in \mathbb{Z}[T]$, coprime, same degree d, roots are roots of unity.
- Get associated family of motives $\mathcal{H}(t)$ with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- $\mathcal{H}(t)$ has rank d and a computable weight w in terms of q_0, q_∞.

Hypergeometric Motives

- $q_0, q_\infty \in \mathbb{Z}[T]$, coprime, same degree d, roots are roots of unity.
- Get associated family of motives $\mathcal{H}(t)$ with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- $\mathcal{H}(t)$ has rank d and a computable weight w in terms of q_0, q_∞.
- Can also compute the Hodge numbers: $\Longrightarrow L_\infty(s)$

Hypergeometric Motives

- $q_0, q_\infty \in \mathbb{Z}[T]$, coprime, same degree d, roots are roots of unity.
- Get associated family of motives $\mathcal{H}(t)$ with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- $\mathcal{H}(t)$ has rank d and a computable weight w in terms of q_0, q_∞.
- Can also compute the Hodge numbers: $\Longrightarrow L_\infty(s)$
- For fixed $t \in \mathbb{Q}$: formula for $L_p(T)$ for $p \notin S$ (Katz's hypergeometric trace)

Examples

- Belyi polynomials $c := a + b$

$$\mathbb{Q}[x]/(B(a,b;t)), \qquad B(a,b;t) := x^a(1-x)^b - \frac{a^a b^b}{c^c}t$$

Examples

- Belyi polynomials $c := a + b$

$$\mathbb{Q}[x]/(B(a,b;t)), \qquad B(a,b;t) := x^a(1-x)^b - \frac{a^a b^b}{c^c}t$$

- $$\frac{q_\infty}{q_0} = \frac{T^c - 1}{(T^a - 1)(T^b - 1)}$$

Examples

- Belyi polynomials $c := a + b$

$$\mathbb{Q}[x]/(B(a,b;t)), \qquad B(a,b;t) := x^a(1-x)^b - \frac{a^a b^b}{c^c} t$$

- $$\frac{q_\infty}{q_0} = \frac{T^c - 1}{(T^a - 1)(T^b - 1)}$$

- Legendre family of elliptic curves: $H^1(E_t)$

$$E_t : \quad y^2 = x(x-1)(x-t)$$

Examples

- Belyi polynomials $c := a + b$

$$\mathbb{Q}[x]/(B(a,b;t)), \qquad B(a,b;t) := x^a(1-x)^b - \frac{a^a b^b}{c^c} t$$

- $$\frac{q_\infty}{q_0} = \frac{T^c - 1}{(T^a - 1)(T^b - 1)}$$

- Legendre family of elliptic curves: $H^1(E_t)$

$$E_t : \quad y^2 = x(x-1)(x-t)$$

- $$\frac{q_\infty}{q_0} = \frac{(T+1)^2}{(T-1)^2}$$

Examples

- Belyi polynomials $c := a + b$

$$\mathbb{Q}[x]/(B(a,b;t)), \qquad B(a,b;t) := x^a(1-x)^b - \frac{a^a b^b}{c^c} t$$

- $$\frac{q_\infty}{q_0} = \frac{T^c - 1}{(T^a - 1)(T^b - 1)}$$

- Legendre family of elliptic curves: $H^1(E_t)$

$$E_t : \quad y^2 = x(x-1)(x-t)$$

- $$\frac{q_\infty}{q_0} = \frac{(T+1)^2}{(T-1)^2}$$

- Dwork pencil piece: V

$$\frac{q_\infty}{q_0} = \frac{T^5 - 1}{(T-1)^5}$$

Hodge vectors

- By Griffiths transversality $\mathbf{h}$ is a symmetric composition of d.

Hodge vectors

- By Griffiths transversality $\mathbf{h}$ is a symmetric composition of d.
- Total number: $2^{\lfloor d/2 \rfloor}$

Hodge vectors

- By Griffiths transversality $\mathbf{h}$ is a symmetric composition of d.
- Total number: $2^{\lfloor d/2 \rfloor}$
- $$\begin{array}{llll} (2), & (1,1) & & \\ (3), & (1,1,1) & & \\ (4), & (2,2), & (1,2,1), & (1,1,1,1) \\ (5), & (2,1,2), & (1,3,1), & (1,1,1,1,1) \end{array}$$

Rank 24

- Rank $d = 24$. Number of possible Hodge vectors: 4096.

Rank 24

- Rank $d = 24$. Number of possible Hodge vectors: 4096.
- Total number of family of motives: $464,247,183$

Rank 24

- Rank $d = 24$. Number of possible Hodge vectors: 4096.
- Total number of family of motives: $464,247,183$
-

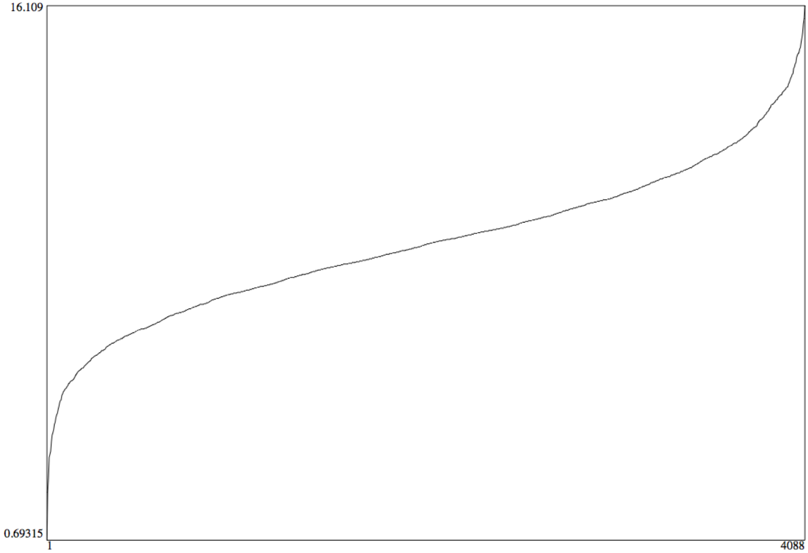
h	#
[9, 1, 1, 2, 1, 1, 9]	0
[7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 7]	0
[1, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6, 1]	0
[4, 1, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 4]	0
[5, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 5]	0
[6, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 6]	0
[4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 4]	0
[4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 4]	0

Rank 24

h	#
[6, 2, 1, 1, 1, 2, 1, 1, 1, 2, 6]	2
[8, 1, 1, 1, 2, 1, 1, 1, 8]	4
[1, 22, 1]	4
[8, 1, 1, 4, 1, 1, 8]	6
[6, 1, 2, 1, 1, 2, 1, 1, 2, 1, 6]	8
[6, 1, 3, 1, 2, 1, 3, 1, 6]	8
[10, 1, 2, 1, 10]	10
⋮	⋮
[1, 3, 4, 4, 4, 4, 3, 1]	6082776
[2, 5, 5, 5, 5, 2]	6850823
[1, 3, 8, 8, 3, 1]	6868016
[1, 5, 6, 6, 5, 1]	7637828
[1, 2, 4, 5, 5, 4, 2, 1]	7982874
[2, 4, 6, 6, 4, 2]	9504072
[1, 4, 7, 7, 4, 1]	9905208

Densities

Graph of logarithmic densities, rank $d = 24$



Useful features

- $\mathcal{H}(t)$: family over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Useful features

- $\mathcal{H}(t)$: family over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

- Study of $L_p(T)$ for tame primes p from behaviour at cusps.

Useful features

- $\mathcal{H}(t)$: family over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- Study of $L_p(T)$ for tame primes p from behaviour at cusps.
- Cusps $0, \infty$ with full Jordan blocks $\Longrightarrow$ Hecke characters.

Useful features

- $\mathcal{H}(t)$: family over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- Study of $L_p(T)$ for tame primes p from behaviour at cusps.
- Cusps $0, \infty$ with full Jordan blocks $\Longrightarrow$ Hecke characters.
- Families $\mathcal{H}(t)$ share many properties.

Useful features

- $\mathcal{H}(t)$: family over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- Study of $L_p(T)$ for tame primes p from behaviour at cusps.
- Cusps $0, \infty$ with full Jordan blocks $\Longrightarrow$ Hecke characters.
- Families $\mathcal{H}(t)$ share many properties.
- Study simple cases to uncover them.

Hypergeometric series

- Hypergeometric series $|t| < 1$

$$u(t) = {}_dF_{d-1}\begin{bmatrix}\alpha_1 & \dots & \alpha_d \\ \beta_1 & \dots & \beta_{d-1}\end{bmatrix} \mid t \Bigg] := \sum_{n\geq 0} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{t^n}{n!},$$

Hypergeometric series

- Hypergeometric series $|t| < 1$

$$u(t) = {}_dF_{d-1}\left[\begin{matrix}\alpha_1 & \dots & \alpha_d \\ \beta_1 & \dots & \beta_{d-1}\end{matrix} \mid t\right] := \sum_{n\geq 0} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{t^n}{n!},$$

- $$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$$

  is the Pochhammer symbol.

Hypergeometric series

- Hypergeometric series $|t| < 1$

$$u(t) = {}_dF_{d-1}\begin{bmatrix}\alpha_1 & \dots & \alpha_d \\ \beta_1 & \dots & \beta_{d-1}\end{bmatrix} \mid t\Bigg] := \sum_{n\geq 0} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{t^n}{n!},$$

-

$$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$$

  is the Pochhammer symbol.

- Satisfies linear differential equation of order d with regular singularities at $t = 0, 1, \infty$.

Hypergeometric series

- Hypergeometric series $|t| < 1$

$$u(t) = {}_dF_{d-1}\begin{bmatrix}\alpha_1 & \dots & \alpha_d & \\ \beta_1 & \dots & \beta_{d-1} & \mid t\end{bmatrix} := \sum_{n\geq 0} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{t^n}{n!},$$

- $$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$$

  is the Pochhammer symbol.

- Satisfies linear differential equation of order d with regular singularities at $t = 0, 1, \infty$.

- Gives rise to a monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \mathrm{GL}(V)$$

Hypergeometric series

- Hypergeometric series $|t| < 1$

$$u(t) = {}_dF_{d-1}\begin{bmatrix} \alpha_1 & \dots & \alpha_d \\ \beta_1 & \dots & \beta_{d-1} \end{bmatrix} \mid t \Bigg] := \sum_{n \geq 0} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{t^n}{n!},$$

- $$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$$

  is the Pochhammer symbol.

- Satisfies linear differential equation of order d with regular singularities at $t = 0, 1, \infty$.

- Gives rise to a monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \mathrm{GL}(V)$$

- $V :=$ space of local solutions of the DE at $z = t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Hypergeometric series (cont'd)

- This local system is rigid.

Hypergeometric series (cont'd)

- This local system is rigid.
- The local monodromies h_0, h_1, h_∞ satisfy

$$h_0 h_1 h_\infty = \mathrm{id}_V$$

Hypergeometric series (cont'd)

- This local system is rigid.
- The local monodromies h_0, h_1, h_∞ satisfy

$$h_0 h_1 h_\infty = \mathrm{id}_V$$

- h_1 is a pseudo-reflection

Hypergeometric series (cont'd)

- This local system is rigid.
- The local monodromies h_0, h_1, h_∞ satisfy

$$h_0 h_1 h_\infty = \mathrm{id}_V$$

- h_1 is a pseudo-reflection
- Characteristic polynomials

$$q_\infty(T) := \prod_{j=1}^{d} (T - e^{2\pi i \alpha_j}), \qquad q_0(T) := \prod_{j=1}^{d} (T - e^{2\pi i (1-\beta_j)})$$

and h_∞, h_0 have full Jordan blocks.

Hypergeometric series (cont'd)

- This local system is rigid.
- The local monodromies h_0, h_1, h_∞ satisfy

$$h_0 h_1 h_\infty = \mathrm{id}_V$$

- h_1 is a pseudo-reflection
- Characteristic polynomials

$$q_\infty(T) := \prod_{j=1}^{d} (T - e^{2\pi i \alpha_j}), \qquad q_0(T) := \prod_{j=1}^{d} (T - e^{2\pi i (1-\beta_j)})$$

  and h_∞, h_0 have full Jordan blocks.

- ρ is irreducible if q_∞ and q_0 are coprime.

Hypergeometric trace

- $$H_p(\alpha, \beta \mid t) := \frac{1}{1-p} \sum_{\varrho} (-p)^{\eta_f(\varrho)} \frac{(\alpha_1)_{\infty,\varrho} \cdots (\alpha_r)_{\infty,\varrho}}{(\beta_1)_{0,\varrho} \cdots (\beta_s)_{0,\varrho}} \omega(t)^{(p-1)\varrho},$$

Hypergeometric trace

- $$H_p(\alpha, \beta \mid t) := \frac{1}{1-p} \sum_{\varrho} (-p)^{\eta_f(\varrho)} \frac{(\alpha_1)_{\infty,\varrho} \cdots (\alpha_r)_{\infty,\varrho}}{(\beta_1)_{0,\varrho} \cdots (\beta_s)_{0,\varrho}} \omega(t)^{(p-1)\varrho},$$

- For $x \in \mathbb{Q}/\mathbb{Z}$ with denominator coprime to p

$$(x)_{\nu,\varrho} := \frac{\Gamma_p(\{x-\varrho\}_\nu)}{\Gamma_p(\{x\}_\nu)}, \qquad \nu = 0, \infty, \quad \varrho \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z},$$

Hypergeometric trace

- $$H_p(\alpha, \beta \mid t) := \frac{1}{1-p} \sum_{\varrho} (-p)^{\eta_f(\varrho)} \frac{(\alpha_1)_{\infty,\varrho} \cdots (\alpha_r)_{\infty,\varrho}}{(\beta_1)_{0,\varrho} \cdots (\beta_s)_{0,\varrho}} \omega(t)^{(p-1)\varrho},$$

- For $x \in \mathbb{Q}/\mathbb{Z}$ with denominator coprime to p

$$(x)_{\nu,\varrho} := \frac{\Gamma_p(\{x-\varrho\}_\nu)}{\Gamma_p(\{x\}_\nu)}, \qquad \nu = 0, \infty, \quad \varrho \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z},$$

- Γ_p: Morita's p-adic gamma function

Hypergeometric trace

- $$H_p(\alpha, \beta \mid t) := \frac{1}{1-p} \sum_{\varrho} (-p)^{\eta_f(\varrho)} \frac{(\alpha_1)_{\infty,\varrho} \cdots (\alpha_r)_{\infty,\varrho}}{(\beta_1)_{0,\varrho} \cdots (\beta_s)_{0,\varrho}} \omega(t)^{(p-1)\varrho},$$

- For $x \in \mathbb{Q}/\mathbb{Z}$ with denominator coprime to p

$$(x)_{\nu,\varrho} := \frac{\Gamma_p(\{x-\varrho\}_\nu)}{\Gamma_p(\{x\}_\nu)}, \qquad \nu = 0, \infty, \quad \varrho \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z},$$

- Γ_p: Morita's p-adic gamma function
- $$\{x\}_\infty := \{x\}, \qquad \{x\}_0 := 1 - \{-x\},$$

Chebyshev example

- Interlacing roots

$$q_\infty = \Phi_{30}, \qquad q_0 = \Phi_1\Phi_2\Phi_3\Phi_5,$$

Chebyshev example

- Interlacing roots

$$q_\infty = \Phi_{30}, \qquad\qquad q_0 = \Phi_1\Phi_2\Phi_3\Phi_5,$$

-

$$\begin{aligned} \alpha &= 1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30 \\ \beta &= 1, 1/2, 1/3, 2/3, 1/5, 2/5, 3/5, 4/5 \end{aligned}$$

Chebyshev example

- Interlacing roots

$$q_\infty = \Phi_{30}, \qquad q_0 = \Phi_1\Phi_2\Phi_3\Phi_5,$$

- $$\begin{aligned} \alpha &= 1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30 \\ \beta &= 1, 1/2, 1/3, 2/3, 1/5, 2/5, 3/5, 4/5 \end{aligned}$$

- $$\frac{q_\infty}{q_0} = \frac{(T^{30}-1)(T-1)}{(T^{15}-1)(T^{10}-1)(T^6-1)}.$$

Chebyshev example

- Interlacing roots

$$q_\infty = \Phi_{30}, \qquad q_0 = \Phi_1\Phi_2\Phi_3\Phi_5,$$

-

$$\begin{aligned}\alpha &= 1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30\\ \beta &= 1, 1/2, 1/3, 2/3, 1/5, 2/5, 3/5, 4/5\end{aligned}$$

-

$$\frac{q_\infty}{q_0} = \frac{(T^{30}-1)(T-1)}{(T^{15}-1)(T^{10}-1)(T^6-1)}.$$

-

$$u(t) := \sum_{n\geq 0} \frac{(30n)!n!}{(15n)!(10n)!(6n)!}\left(\frac{t}{M}\right)^n, \qquad M := \frac{30^{30}}{15^{15}\cdot 10^{10}\cdot 6^6}.$$

Chebyshev example

- Interlacing roots
$$q_\infty = \Phi_{30}, \qquad\qquad q_0 = \Phi_1\Phi_2\Phi_3\Phi_5,$$
- $$\begin{aligned} \alpha &= 1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30 \\ \beta &= 1, 1/2, 1/3, 2/3, 1/5, 2/5, 3/5, 4/5 \end{aligned}$$
- $$\frac{q_\infty}{q_0} = \frac{(T^{30}-1)(T-1)}{(T^{15}-1)(T^{10}-1)(T^6-1)}.$$
- $$u(t) := \sum_{n\geq 0} \frac{(30n)!n!}{(15n)!(10n)!(6n)!}\left(\frac{t}{M}\right)^n, \qquad M := \frac{30^{30}}{15^{15}\cdot 10^{10}\cdot 6^6}.$$
- $$\frac{(30n)!n!}{(15n)!(10n)!(6n)!} = 1, 77636318760, 53837289804317953893960, \cdots$$
are integral for every n.

Chebyshev example (cont'd)

- Monodromy group is finite.

Chebyshev example (cont'd)

- Monodromy group is finite.
- Series $u(t)$: Taylor expansion of an algebraic function of t.

Chebyshev example (cont'd)

- Monodromy group is finite.
- Series $u(t)$: Taylor expansion of an algebraic function of t.
- Degree over $\bar{\mathbb{Q}}(t)$: $483,840$.

Chebyshev example (cont'd)

- Monodromy group is finite.
- Series $u(t)$: Taylor expansion of an algebraic function of t.
- Degree over $\bar{\mathbb{Q}}(t)$: $483, 840$.
- $\mathcal{H}(t)$: Artin with Gal $\leq W(E_8)$ (generically $=$).