Jan 30, 2007

Chinese rings

initial position $11\cdots1$

all rings are on

GRAY	BINARY	
$11\cdots1$	$10\cdots010$	n even
	$1\cdots0101$	n odd

$$\underbrace{1\oplus1\oplus\cdots\oplus1}_{n} = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

What number is this?

n odd

$$(10\cdots10101)_2 = ?$$

$$\cdots + 16 \quad 4 + 1$$

$$1+4+16+\cdots+$$

$$= 1+4+4^2+4^3+\cdots+4^k$$

geometric series

k	
0	1
1	$1+4=5$
2	$1+4+4^2=21$
3	$1+4+4^2+4^3=85$
	$\vdots$
	$\dfrac{4^{k+1}-1}{4-1}$

geometric series, $a^{\neq 1}$ number

$$S_k := 1+a+a^2+\dots+a^k = \frac{a^{k+1}-1}{a-1}$$

$$S_{k+1} = 1+a+\dots+a^{k+1}$$

$$= 1 + a(\underbrace{1+a+\dots+a^k}_{S_k})$$

$$S_{k+1} = 1 + aS_k$$

verify $\quad S_k \stackrel{?}{=} \dfrac{a^{k+1}-1}{a-1}$

$$(a-1)\,S_k \overset{?}{=} a^{k+1}-1$$ ③

$$\begin{aligned} a\,S_k - S_k &= a(1+a+\dots+a^k) \\ &\quad - (1+a+\dots+a^k) \\ &= a+a^2+\dots+a^{k+1} \\ &\quad - (1+a+\dots+a^k) \\ &= a^{k+1}-1 \checkmark \end{aligned}$$

Find # steps to solve the puzzle

<u>n odd</u>

$$\frac{1}{3}\left(2^{n+1}-1\right) = \frac{2}{3}\cdot 2^n - \frac{1}{3}$$

<u>n even</u>

position

$$\frac{2}{3}\left(2^n-1\right) = \frac{2}{3}\cdot 2^n - \frac{2}{3}$$

2^n = # of positions in puzzle

steps $\approx \frac{2}{3} \times$ positions

Ruler function

$\rho(m) =$ bit that changes in gray code $m-1 \mapsto m$

$=$ # bits that change in binary code $m-1 \mapsto m$

m	GRAY 4 3 2 1	BINARY	$\rho(m)$
0	0 0 0 0	0 0 0 0	
1	0 0 0 1	0 0 0 1	1
2	0 0 1 1	0 0 1 0	2
3	0 0 1 0	0 0 1 1	1

$\rho(m) =$ highest power of 2 dividing m $+ 1$

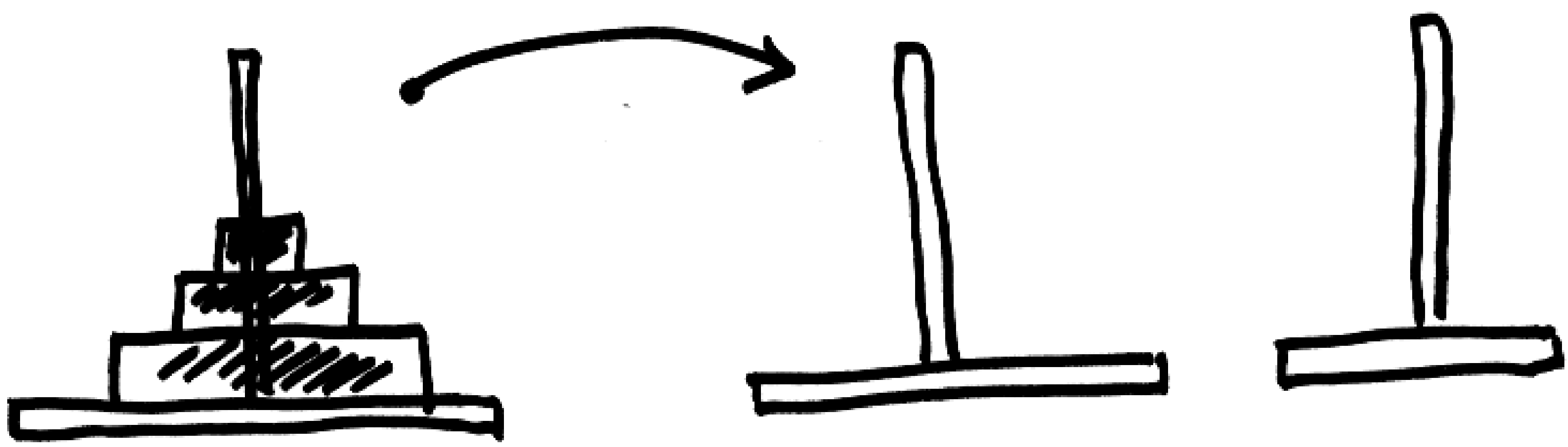
$12 = 2^2 \times 3$

2 highest power of 2

$\rho(12) = 3$

This gives a recipe to construct the Gray code (solve the lights puzzle or pick a binary lock)

Hanoi Towers



- one ring at a time
- with no:

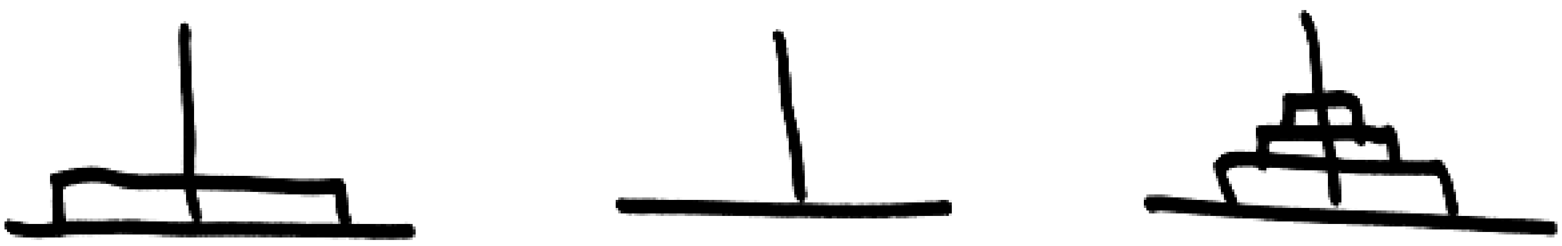
Unique optimal solution

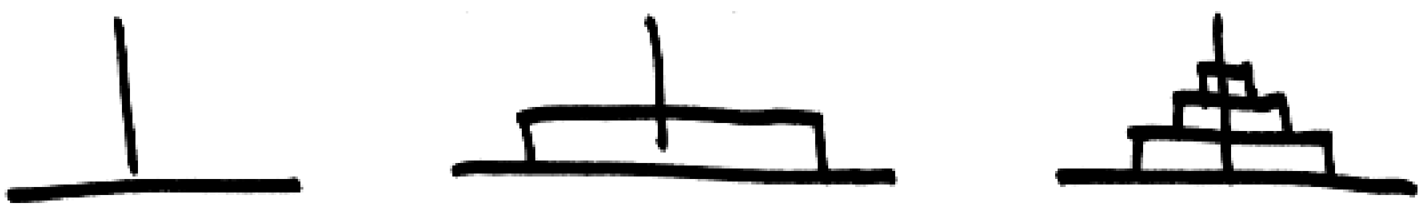
(optimal = least number of steps)

Suppose we know how to solve it if we had 3 rings.

- Move top 3 disks

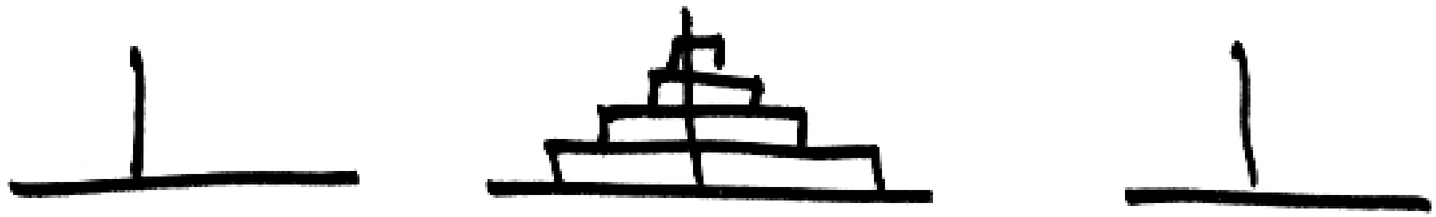
S $\longmapsto$ A





Move last disk to destination

- Move top 3 disks to destination



hanoi (n, S, D, A) =

- hanoi$(n-1, S, A, D)$
- move disk n from S to D
- hanoi$(n-1, A, D, S)$

Recursive procedure

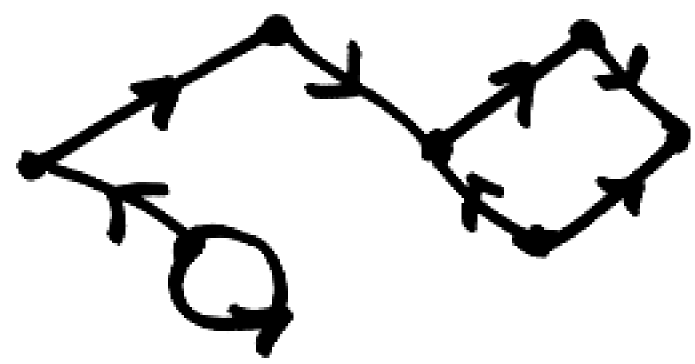
n	h_n	
1	1	2 − 1
2	3	4 − 1
3	7	8 − 1
4	15	16 − 1
5	31	32 − 1

Recursion: $h_n = 2h_{n-1} + 1$

In fact: $h_n = 2^n - 1$

What is the graph of the puzzle?

A <u>graph</u> : collection of vertices & edges

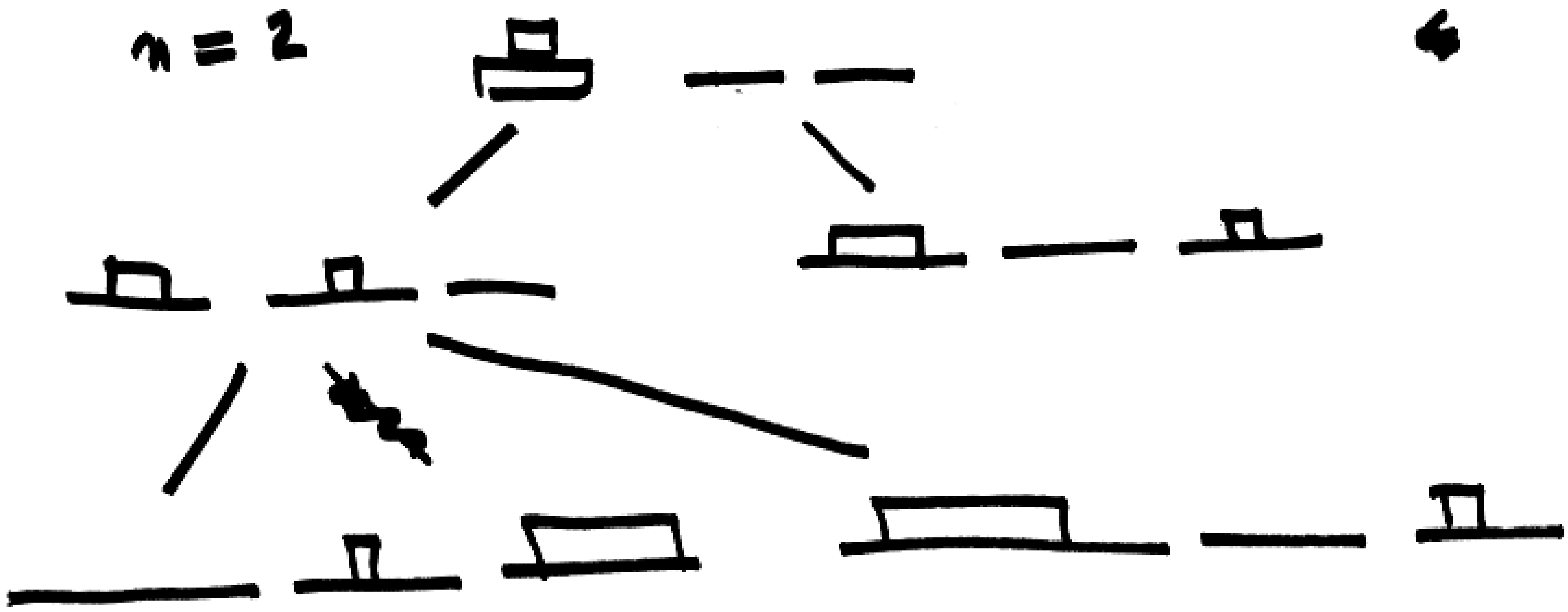


positions $\longleftrightarrow$ vertices

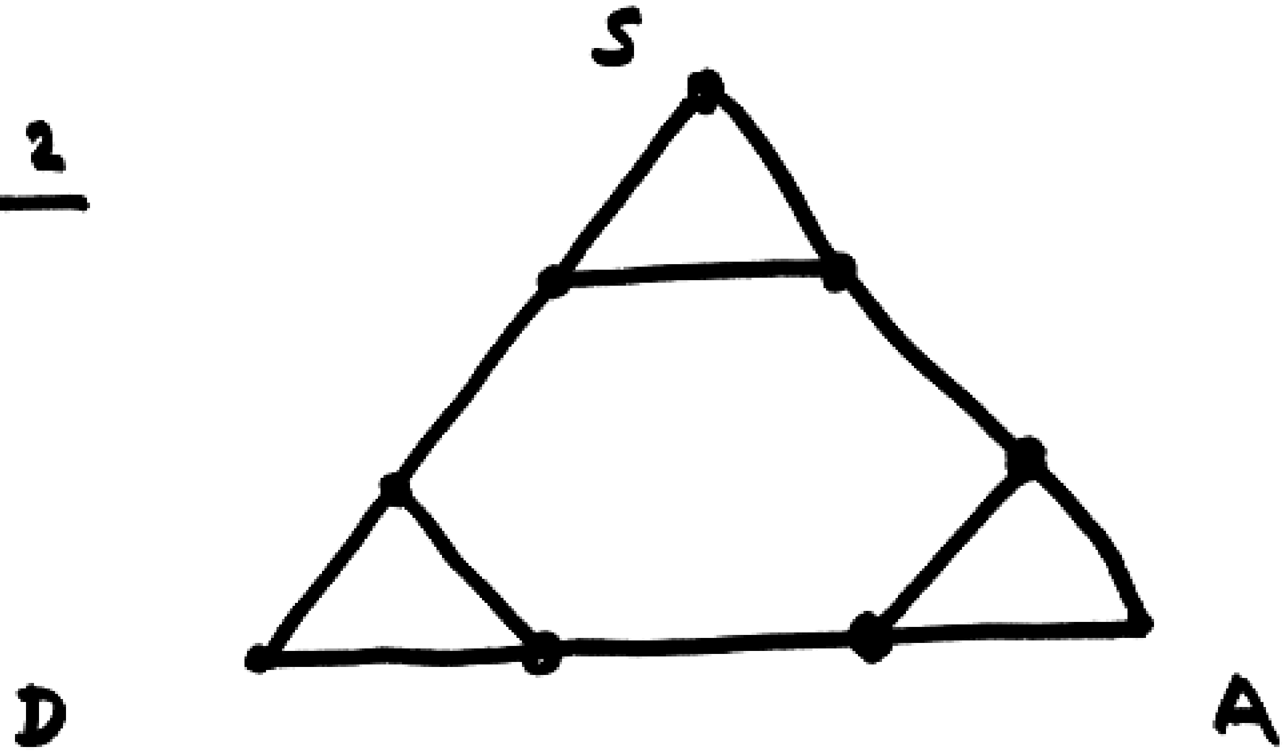
moves $\longleftrightarrow$ edges

Chinese rings graph

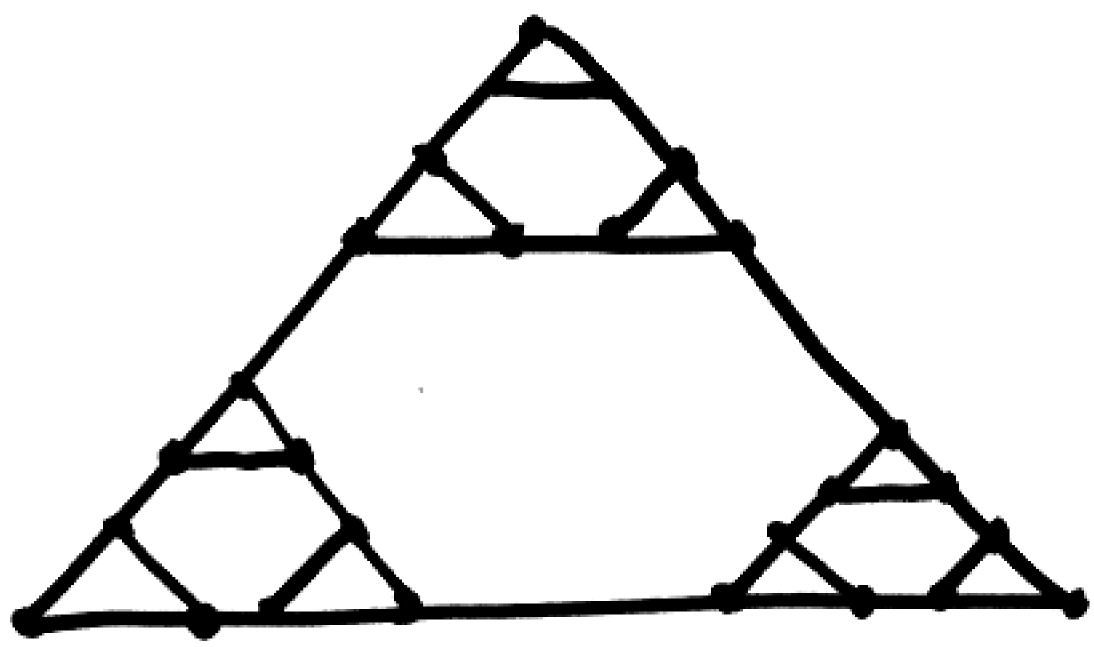
$n = 2$



n = 2



n = 3



Feb 1, 2007

Recursion

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

$= \#$ of permutations of n things

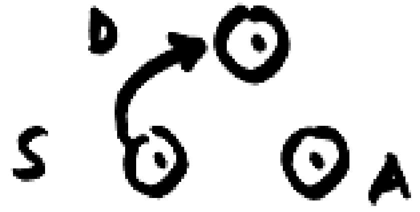
$n = 3$ $\qquad 3! = 3 \cdot 2 \cdot 1 = 6$

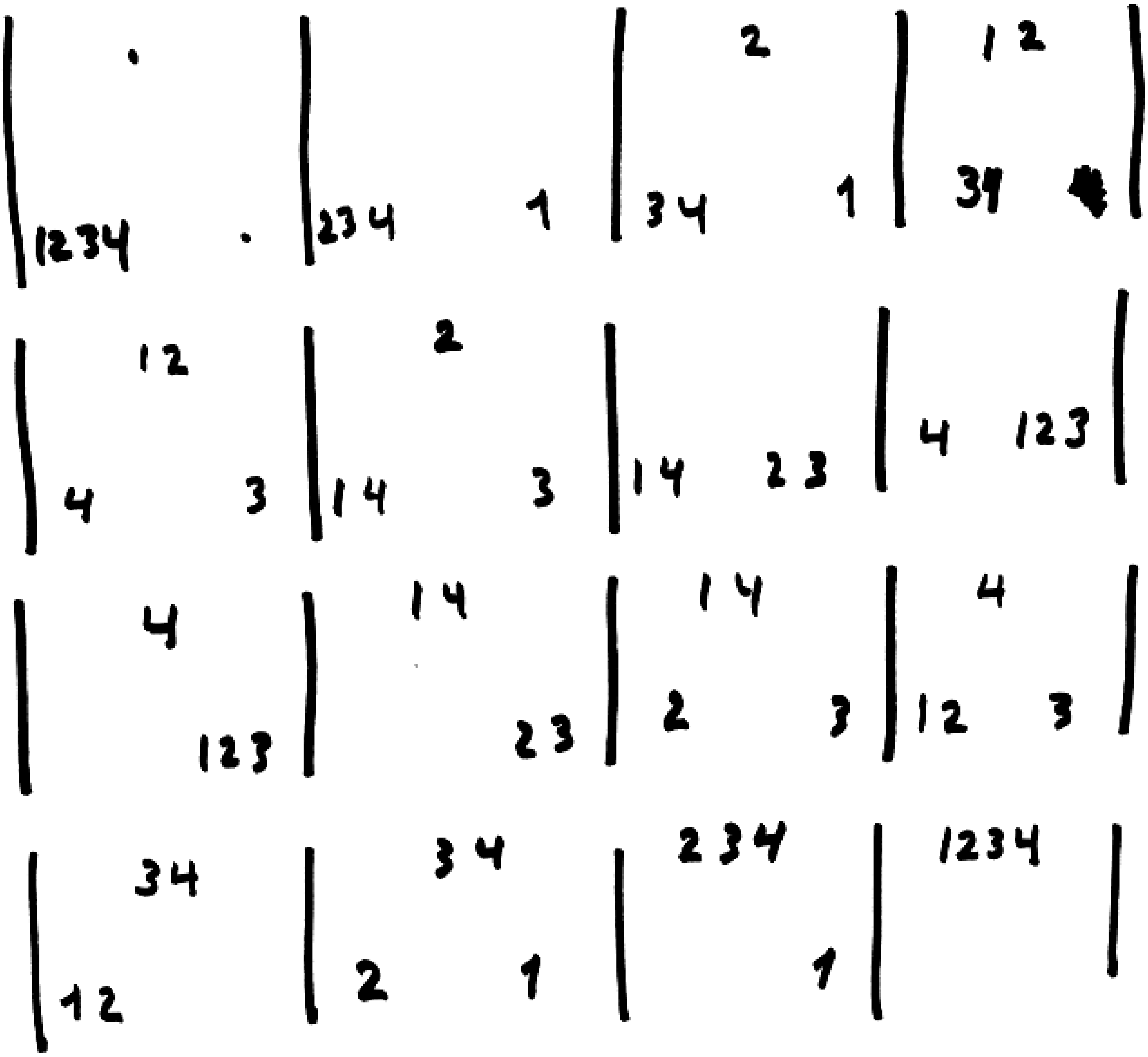
1 2 3
2 1 3
3 2 1
1 3 2
2 3 1
3 1 2

Recursively.

$$\text{factorial}(n) = \begin{cases} n * \text{factorial}(n-1) & n > 1 \\ 1 & n = 1 \end{cases}$$

n = 4



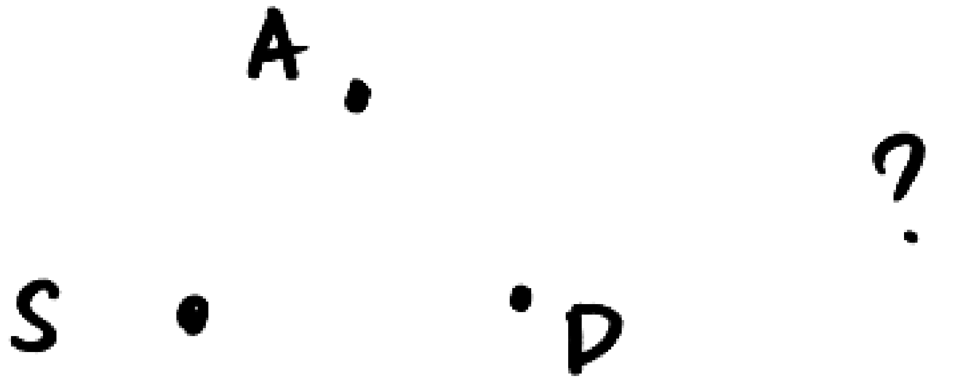


sequential

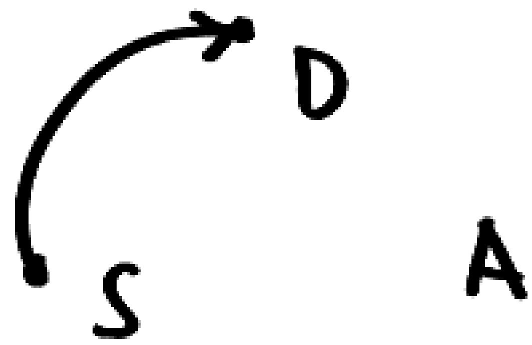
- disk 1 ↻ every other move
- in other moves: move disk other than 1.

Remarks

- disk that moves is given by ruler function.
- odd ↻ even ↺



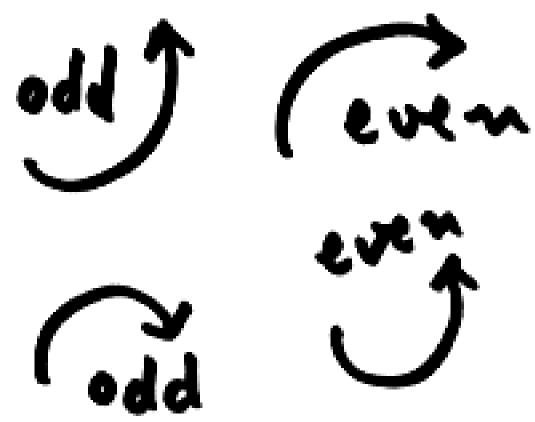
move in opposite direction.



n disks

<u>n even</u>

<u>n odd</u>



<u>PUZZLE</u> ⇝ graph

position ⟷ vertex

move ⟷ edge

Hanoi towers

position = arrangement of disks in a legal way.

To give a position is enough to say what disks are where.

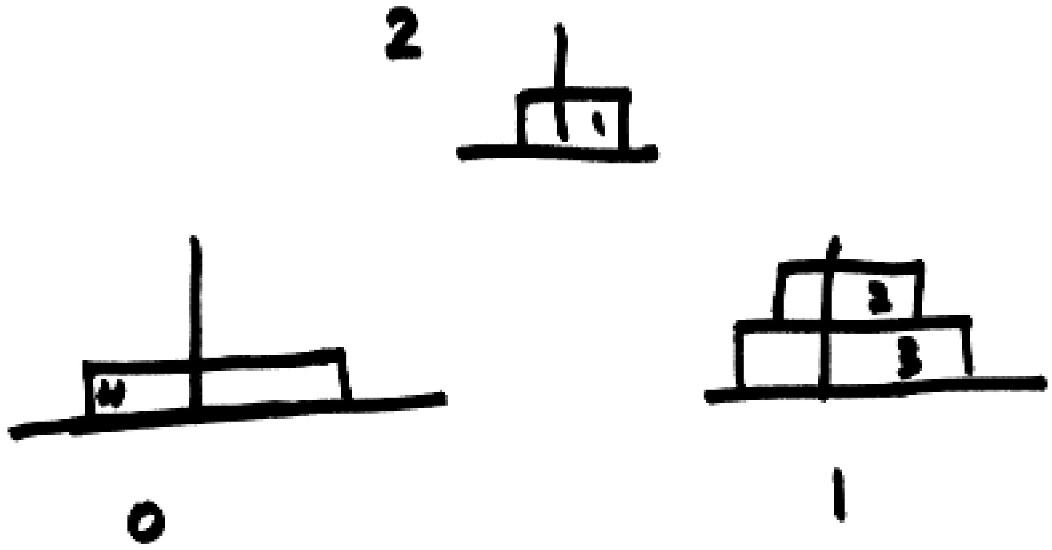
encode a position

$$(p_1, p_2, \dots, p_n)$$

ternary bits $\qquad p_i = 0, 1, 2$

$p_i :=$ peg # where disk i is.

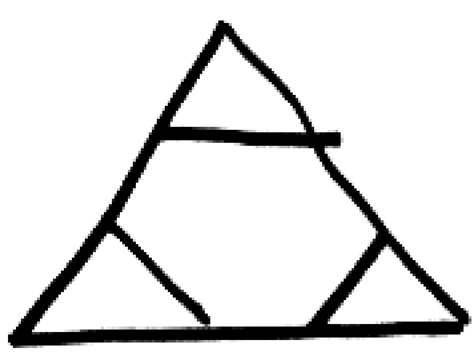
$n=4$



$\updownarrow$

$(2, 1, 1, 0)$

graph of puzzle



$\binom{n}{k} =$

$(1+x)^2 = 1 + 2x + x^2$ 1 2 1

$(1+x)^3 = 1 + 3x + 3x^2 + x^3$ 1 3 3 1

Pascal's triangle

$k \rightarrow$

$\downarrow n$

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
```

$\binom{5}{2} = 10$

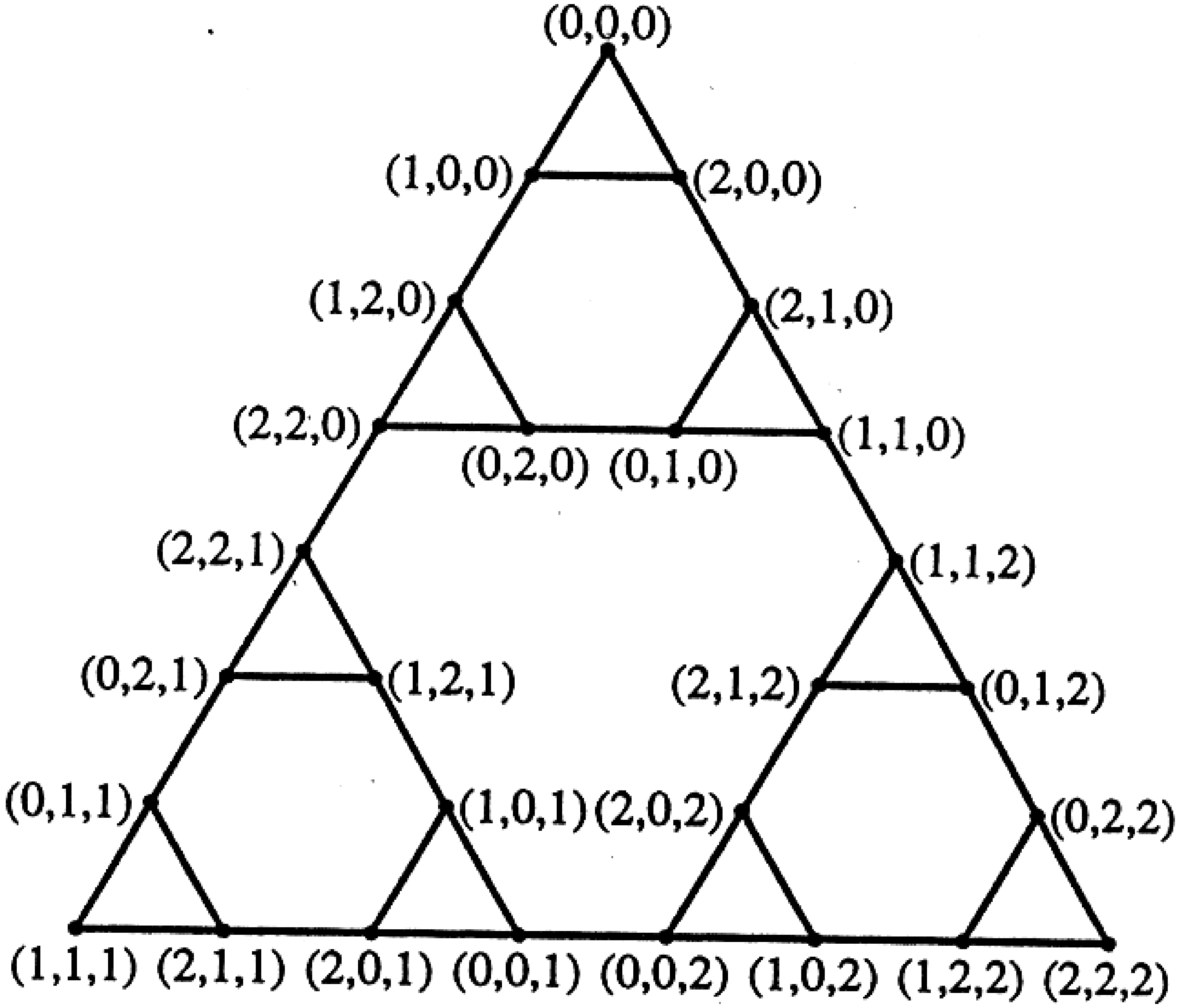
$\binom{n}{k}$ = n chose k

= # of ways to choose k things out of n

Pasal triangle w/parity

1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1


LA TOUR D'HANOÏ
VÉRITABLE CASSE-TÊTE ANNAMITE
JEU
RAPPORTÉ DU TONKIN
PAR LE PROFESSEUR N. CLAUS (DE SIAM)
MANDARIN
DU COLLÈGE LI-SOU-STIAN
BREVETÉ S.G.D.G.
A.U



Graph of Hanoi Towers puzzle with three disks.