Nov 8, 2007

$$1 + t + 3t^2 + 3t^3 + 3t^4 + t^5 + t^6$$

$$= \sum_{k=0}^{6} N(k)\, t^k$$

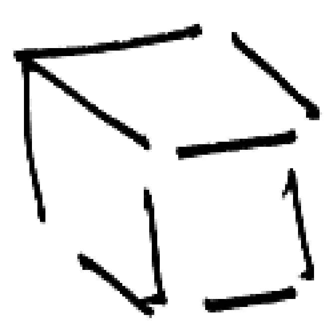
$N(k) = \#\{$ necklaces with k black beads $\}$

$G \subset S_n$ cycle indicator

$$Z_G(t_1, t_2, \ldots, t_n) = \frac{1}{|G|} \sum_{g \in G} t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$$

$m_i = \#$ of i-cycles in g acting on $\{1, 2, \ldots, n\}$

E.g.



$G =$ gp of rotations of cube

$X =$ faces of cube

Better: $t_i \leftrightarrow p_i = \sum_j x_j^{i}$

$Z_G(p_1, p_2, p_3, \ldots, p_n)$ symmetric function

Polyá's theorem

$G \circlearrowright X$ $\qquad \# X = n$

$C = \text{colors}$ $\qquad \# C = m$

$G \circlearrowright \{ \varphi : X \to C \}$ colorings

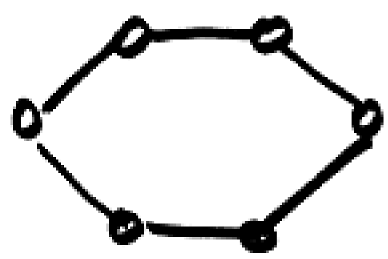
$$(g\varphi)(x) = \varphi(g^{-1}x)$$

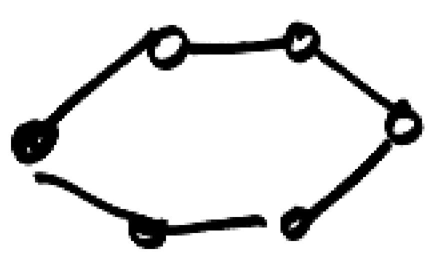
$$\sum_{\text{orbits } \varphi} \prod_{i=1}^{n} x_{\varphi(i)}$$

$x_1, \dots, x_m$

$x_j \leftrightarrow \text{color } j$

Eg: $m = 2$ $\qquad x_1, x_2$ (↑ white, ↑ black)

1  weight $= x_1^6$

t  $\quad " \quad = x_1^5 x_2$

$\vdots$

$t = x_2$
$1 = x_1$

A coloring φ gets a <u>weight</u>

$$w(\varphi) := \prod_{i=1}^{n} x_{\varphi(i)}$$

$$\boxed{\sum_{\varphi \bmod G} w(\varphi) = Z_G(p_1, p_2, \dots, p_n)\Big|_{\substack{x_i = 0 \\ i > m}}}$$

coeff of x^w on the lhs is
= # of orbits of colorings with
weight w

If $m=2$ $\qquad x_1^{w_1} x_2^{w_2}$

coeff of this on lhs

= # necklaces with
w_1 beads of color 1
w_2 " " " 2

What if $G = S_n$?

What is $Z_{S_n}(p_1, p_2, \ldots, p_n)$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} p_{\lambda(\sigma)}$$

$$= \frac{1}{n!} \sum_{\lambda} \#\{\sigma \in S_n \mid \lambda = \lambda(\sigma)\}\, p_\lambda$$

$$= \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda$$

$$= h_n$$

Polya's theorem $\Rightarrow$

Burnside

$$\#\,\text{orbits } G \circlearrowright \text{colorings} = Z_G(m, m, \dots, m)$$

$$t_i = m$$

Specialization: $p_i \longmapsto m$

$$p_i = \sum_{j \geq 1} x_j^i$$

$$x_j \longmapsto \begin{cases} 1 & j \leq m \\ 0 & j > m \end{cases}$$

$$e_i \longmapsto \ ?$$

$$h_i \longmapsto \ ?$$

$$H(t) = \prod (1 - x_i t)^{-1} \mapsto (1-t)^{-m}$$

$$E(t) = \prod (1 + x_i t) \mapsto (1+t)^{m}$$

$$\begin{cases} h_i = \binom{-m}{i}(-1)^i = \binom{m+i-1}{i} \\ e_i = \binom{m}{i} \end{cases}$$

orbits of colorings $S_n \circlearrowright$ $= \binom{m+n-1}{n}$

$= \dim \mathrm{Sym}^m V$

$\dim V = n$

$|000|00|0|0$

Inner product on Λ

Casimir element.

Build inner product by declaring h_λ & m_λ to be dual basis

$$\langle h_\lambda , m_\mu \rangle = \delta_{\lambda,\mu}$$

$$\sum_\lambda h_\lambda(x) \cdot m_\lambda(y)$$

$x_1, x_2, \ldots$
$y_1, y_2, \ldots$

Claim $\prod_{i,j} (1 - x_i y_j)^{-1}$

$$H(t) = \prod_{i \geq 1} (1 - x_i t)^{-1} = \sum_{n \geq 0} h_n t^n$$ (7)

$$= \sum_{\lambda} z_\lambda^{-1} p_\lambda(x) \, t^{|\lambda|}$$

Now consider the set of variables

$$x_i y_j$$

$$H(t) = \prod_{i,j \geq 1} (1 - x_i y_j t)^{-1}$$

$$= \sum_{\lambda} z_\lambda^{-1} p_\lambda(\{x_i y_j\}) \, t^{|\lambda|}$$

$$p_r(\{x_i y_j\}) = \sum_{i,j \geq 1} (x_i y_j)^r$$

$$= \sum_i x_i^r \sum_j y_j^r$$

$$= p_r(x) \, p_r(y)$$

$$\prod_{i,j\geq 1}(1-x_iy_j)^{-1} = \sum_{\lambda} z_\lambda^{-1}\, p_\lambda(x)\, p_\lambda(y)$$

$$\Rightarrow \langle p_\lambda, p_\mu \rangle = \frac{1}{z_\lambda}\, \delta_{\lambda,\mu}$$

$$\prod_{i,j\geq 1}(1-x_i y_j)^{-1} = \prod_{j\geq 1}\prod_{i\geq 1}(1-x_iy_j)^{-1}$$

$$= \prod_{j\geq 1} H(y_j)$$

$$= \prod_{j\geq 1}\sum_{i\geq 0} h_i(x)\, y_j^{i}$$

$$= \sum_{i_1 i_2 \cdots} h_{i_1} h_{i_2}^{(x)} \cdots y_1^{i_1} y_2^{i_2} \cdots$$

$$= \sum_{\lambda} h_\lambda(x)\, m_\lambda(y)$$

$$= \sum_{\lambda} \sum_{\alpha} h_\alpha\, y^\alpha$$

distinct permutations of λ

$$= \sum_{\lambda} h_\lambda \underbrace{\sum_{\alpha} y^\alpha}_{m_\lambda(y)}$$

$h_\alpha = h_\lambda$ all α

u_λ, v_λ basis of Λ

dualbasis $\leftrightarrow$ $\prod (1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(x) v_\lambda(y)$

$$\langle p_\lambda, p_\mu \rangle = \frac{1}{z_\lambda} \delta_{\lambda,\mu}$$

$\Rightarrow$ $\langle \, , \, \rangle$ is positive definite !!

$$\omega : \Lambda \to \Lambda$$

$$\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$$

$$\langle \omega p_\lambda, \omega p_\mu \rangle = \langle p_\lambda, p_\mu \rangle \varepsilon_\lambda \varepsilon_\mu$$

$$= \frac{1}{z_\lambda} \delta_{\lambda,\mu} = \langle p_\lambda, p_\mu \rangle$$

ω is an isometry