Feb 27, 2007

Permutations σ, τ

$$\underset{2^{nd}}{\sigma} \cdot \underset{1^{st}}{\tau}$$

Eventually drop $\cdot$ altogether

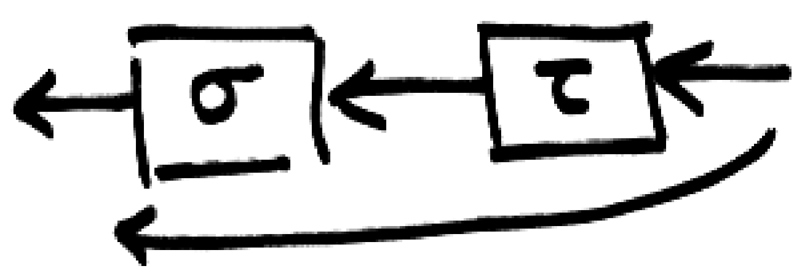
$$\sigma\tau$$

$$\sigma = (1\,2\,3)(4\,5)$$

$$\tau = (1\,4\,3)(2\,5\,6)$$

$\sigma, \tau \in S_6$

$$\sigma \cdot \tau$$



$$\sigma\tau = (3\,2\,4\,1\,5\,6)$$

$$2 \xleftarrow{\sigma} 1 \xleftarrow{\tau} 3$$

$$4 \xleftarrow{\sigma} 5 \xleftarrow{\tau} 2$$

$$1 \xleftarrow{\sigma} 3 \xleftarrow{\tau} 4$$

$$5 \xleftarrow{\sigma} 4 \xleftarrow{\tau} 1$$

$$6 \xleftarrow{\sigma} 6 \xleftarrow{\tau} 5$$

$$3 \xleftarrow{\sigma} 2 \xleftarrow{\tau} 6$$

Defn σ, τ commute if

$$\sigma\tau = \tau\sigma$$

Typically does not hold.

$\tau^2\sigma$

$$\leftarrow \boxed{\tau} \leftarrow \boxed{\tau} \leftarrow \boxed{\sigma} \leftarrow$$

$(1\,2\,3) \neq (1\,3\,2)$

$\sigma \in S_n$ the order of σ

is the smallest positive power

$\sigma^k = \underbrace{\sigma \cdot \cdots \cdot \sigma}_{k \text{ times}}$

which is the identity.

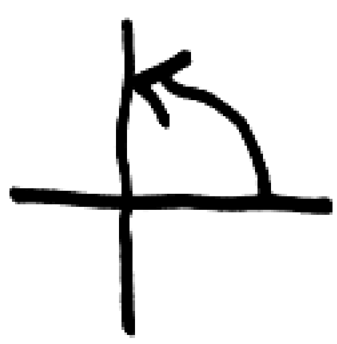
$\sigma = (4 \;\; 6)$ $\qquad\qquad$ $\sigma^2 = \sigma \cdot \sigma = 1$

$\sigma^{-1} = \sigma$

order 2

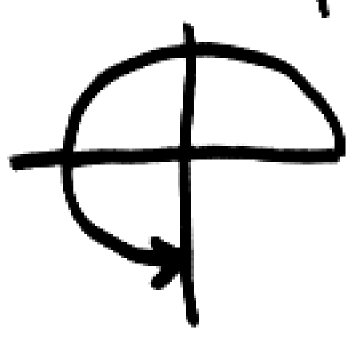
$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

rotation

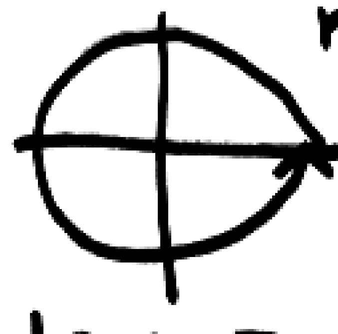


r

r^2



r^3



$r^4 = 1$

order $= 4$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

$$= (1\,4)(2\,5\,3)$$

$$\rho = (1\,4)\,(2\,5\,3)$$

$$\rho^2 = (1)(4)\,(2\,3\,5)$$

$$\rho^3 = (1\,4)\,(2)(3)(5) \qquad = \rho \cdot \rho^2$$

$$\rho^4 = (1)(4)\,(2\,5\,3) \qquad = \rho \cdot \rho^3$$

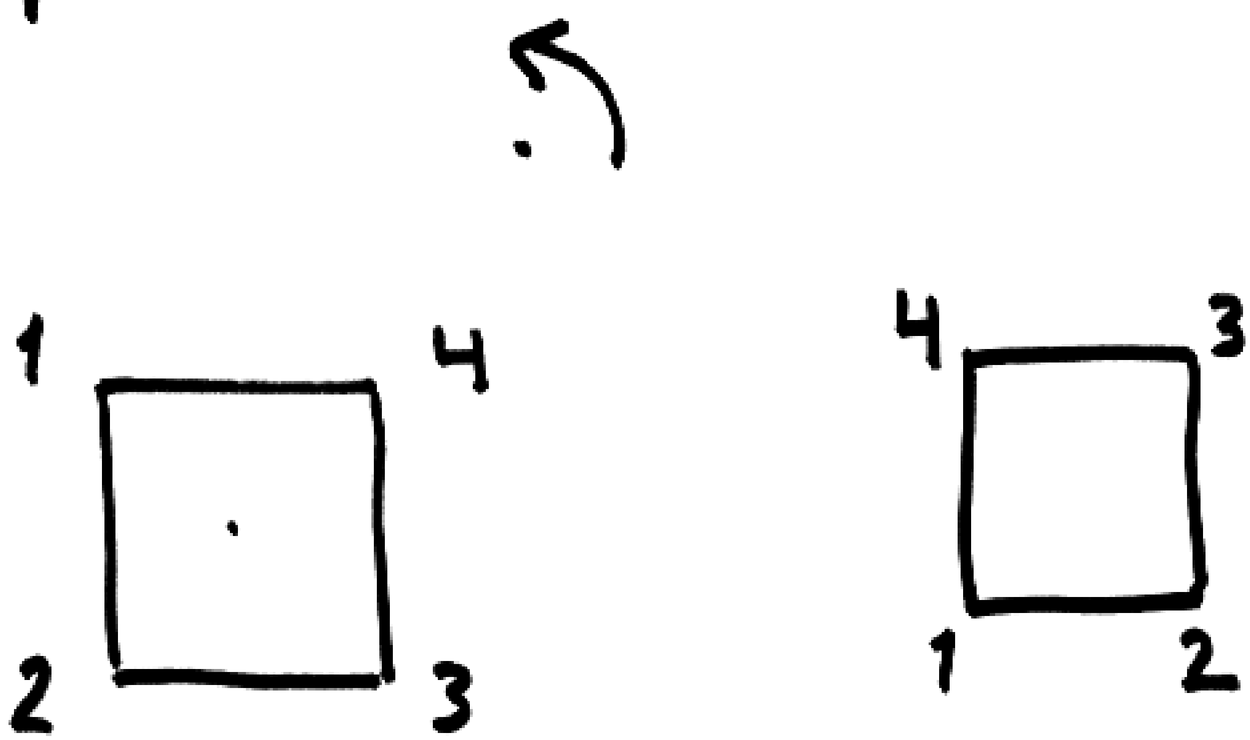
$$\rho^5 = (1\,4)\,(2\,3\,5) \qquad = \rho \cdot \rho^4$$

$$\rho^6 = (1)(4)\,(2)(3)(5) \qquad = \rho \cdot \rho^5$$

<u>order 6</u>.

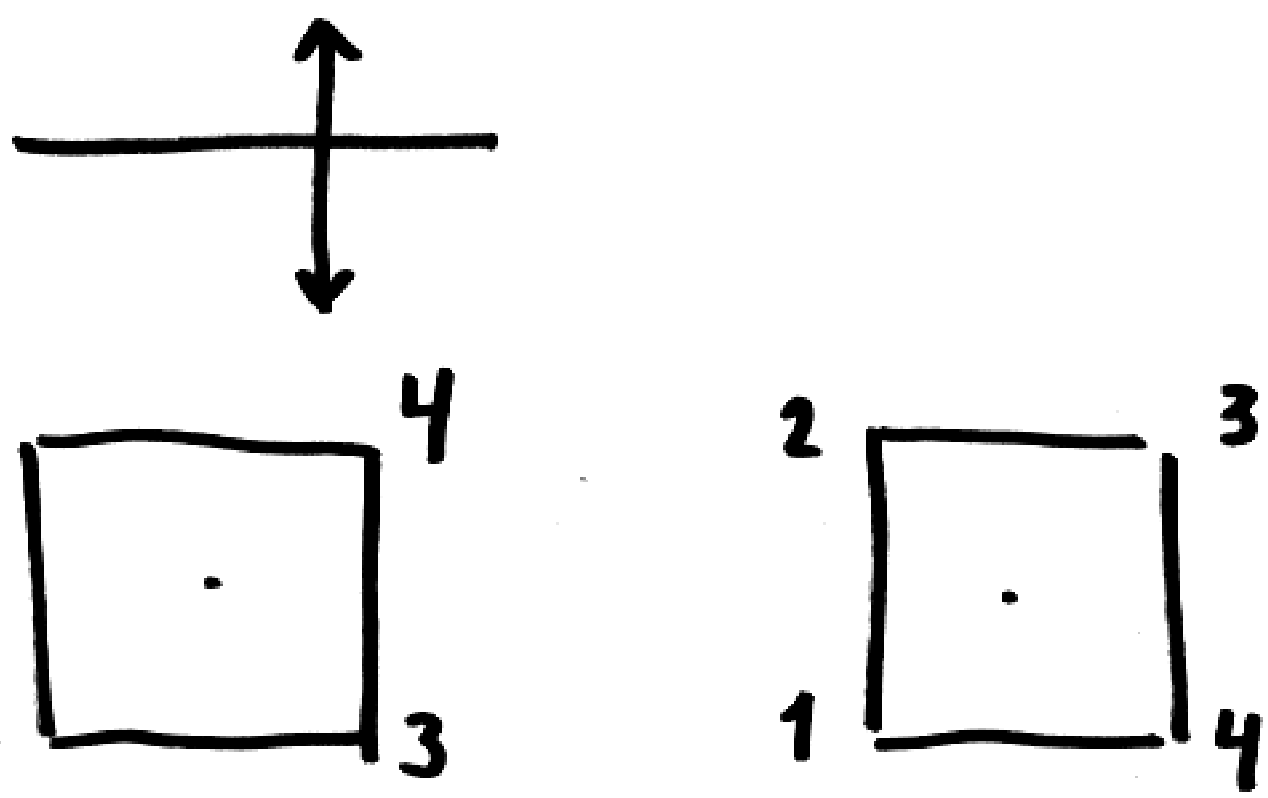
	1	2	3	4	5
ρ	4	5	2	1	3
ρ^2	1	3	5	4	2
ρ^3	4	2	3	1	5
ρ^4	1	5	2	4	3
ρ^5	4	3	5	1	2
ρ^6	1	2	3	4	5

r



$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

S



$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\boxed{srs = r^{-1}}$$

$r^{-1} =$ 

$= r^3$ 

$$r^4 = 1$$

$$r \cdot r^3 = 1$$

$$r^{-1}(r \cdot r^3) = r^{-1} \cdot 1 = r^{-1}$$

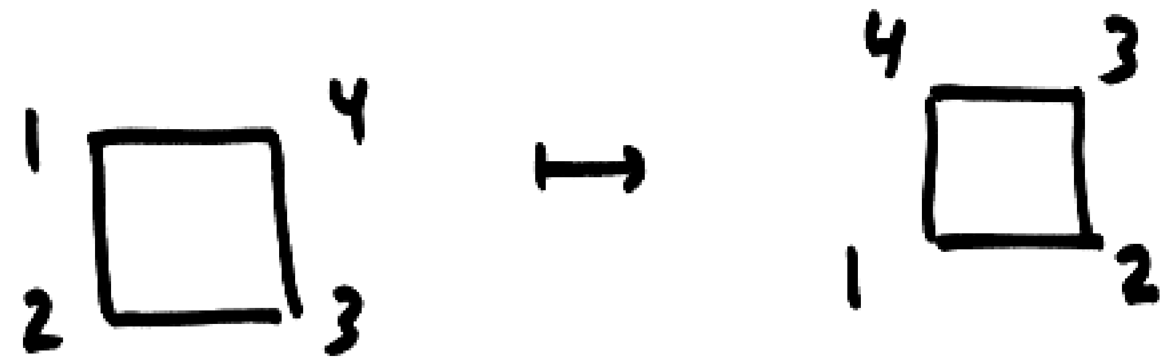
$$(r^{-1} \cdot r)\, r^3 = r^{-1}$$

$$1 \cdot r^3 = r^{-1}$$

$$r^3 = r^{-1}$$

March 1, 2007

r



$r = (1\,2\,3\,4)$

$r^{-1} = (1\,4\,3\,2)$
$= (4\,3\,2\,1)$

$\sigma = \underset{\substack{\uparrow \\ a}}{(1\,2)}\,\underset{\substack{\uparrow \\ b}}{(3\,4)}$

$\sigma = ab$

disjoint cycles $\qquad ab = ba$

$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$

In general $\qquad \underset{\substack{\nearrow \\ 2^{nd}}}{a} \cdot \underset{\substack{\nwarrow \\ 1^{st}}}{b}$

$$(a \cdot b \cdot c)^{-1} = c^{-1} \cdot b^{-1} \cdot a^{-1}$$

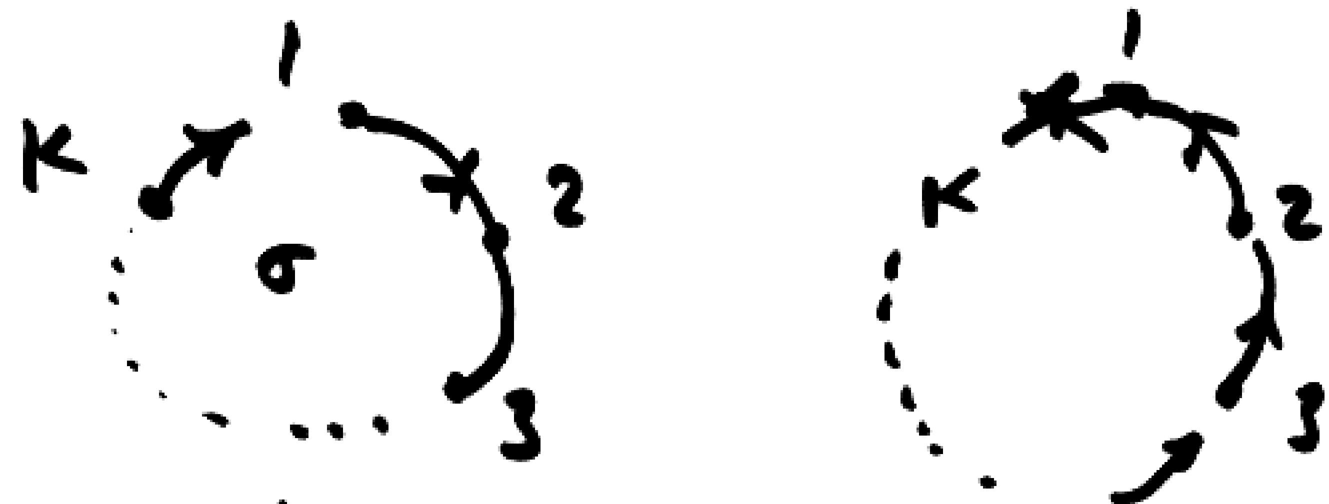
$$\sigma = (12)(34)$$

$$\sigma^{-1} = (34)^{-1}(12)^{-1}$$
$$= (34)(12)$$
$$= (12)(34)$$

$$\sigma = (123)(45)(687)$$

$$\sigma^{-1} = (687)^{-1}(45)^{-1}(123)^{-1}$$
$$= (786)(54)(321)$$

$$(123 \dots k)^{-1} = (k \dots 321)$$



$$(12345)^{-1} = (54321) = (32154)$$

Permutation Puzzle

15-puzzle, Rubik's cube, ...

Basic Moves: $U_1, U_2, \dots, U_N$
(include the inverses)

What are all the possible sequences of moves?

Look at all possible products of these permutations. They form a group G.

- How big is G?

- Differently: S_n permutations on n things, what is a set of basic permutations that will give all.

~~[illegible]~~ $\longrightarrow$ 12345

2 3 4 5 1

2 3 4 1 5

2 3 1 4 5

2 1 3 4 5

1 2 3 4 5

Bubbling algorithm for sorting

$$(1\,2\,3\,4\,5) = (15)(14)(13)(12)$$

<u>not</u> disjoint

Any cycle is a product of transpositions

$$(5\;7\;8) = (58)(57)$$

Any permutation is a product of cycles

$\longrightarrow$ every permutation is a product of 2-cycles (transpositions)

How many 2-cycles in S_4?

(12) (13) (14)

(23) (24)

(34)

6 transpositions

In S_n? $\binom{n}{2}$

$(i\,j) = (j\,i)$

$|S_n| = n!$

$\sigma \in S_4$

$\sigma, \sigma \cdot \sigma, \sigma \cdot \sigma \cdot \sigma, \sigma \cdot \sigma \cdot \sigma \cdot \sigma, \ldots$

$\sigma, \sigma^2, \sigma^3, \sigma^4, \ldots$

$\sigma = (1234)$

$1, (1234), (1234)^2 = (13)(24)$

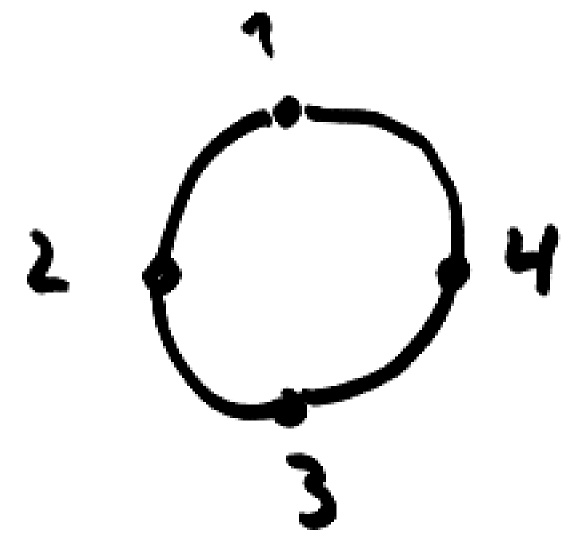
$$(1234)^3 = (1432)$$

$$(1234)^4 = 1$$

$$(1234)^5 = (1234)\cdot(1234)^4$$
$$= (1234)$$

$1,\ \sigma,\ \sigma^2,\ \sigma^3.$

total of 4 possible moves.



No one $\sigma \in S_n$ is not sufficient.
(if $n > 2$)

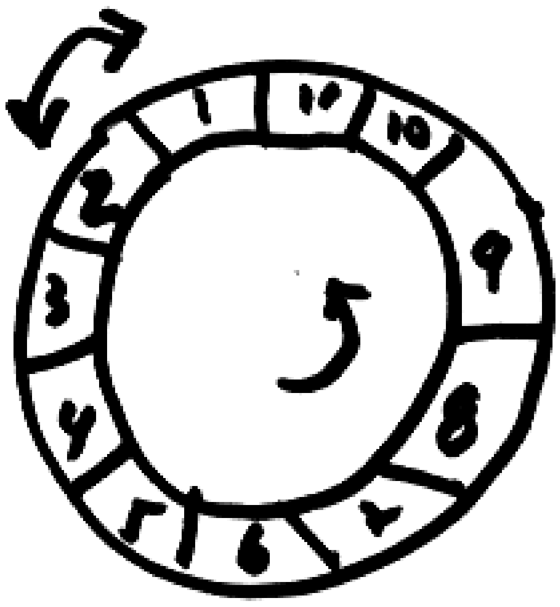
order of σ = least k > 0 such that

$$\sigma^k = \underbrace{\sigma \cdots\cdots \sigma}_{k} = 1.$$

Already two permutations σ, τ can generate very large groups.

E.g. σ, τ can generate all of S_n

S_4 ? $\sigma = (12)$, $\tau = (1234)$



- all swaps (transpositions) can be obtained from σ, τ

$$\tau \sigma \tau^{-1} = (1234)(12)(1234)^{-1}$$
$$= (1234)(12)(4321)$$
$$= (1)(23)(4)$$
$$= (23)$$

$$\tau^2 \sigma \tau^{-2} = (34)$$
$$\tau^3 \sigma \tau^{-3} = (41)$$

We have: $(12), (23), (34), (41)$ 8

(13) ?

(24) ?

$$(12)(23) = (123)$$

$$(23)(12) = (132)$$

$$(23)(12)(23)^{-1} = (23)(12)(23)$$
$$= (13)$$

$$(24) = (14)(12)(14)$$

All transpositions arise from σ and τ. $\rightsquigarrow$ All permutations arise from σ and τ.

For any n

$$\sigma = (12) \qquad \tau = (12 \cdots n)$$

they generate all of S_n.