March 8, 2007

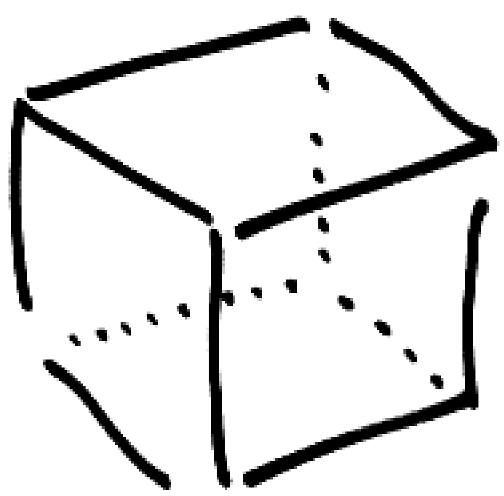
$S_n = \{\text{permutations of } n \text{ things}\}$

$|S_n| = n!$

↑

# of elements.

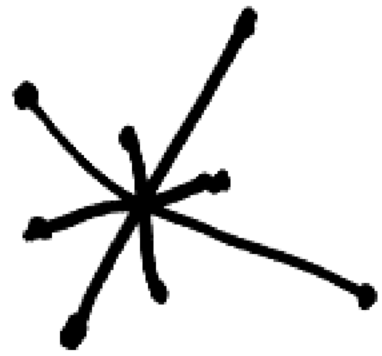
n	1	2	3	4	5	6	
$n!$	1	2	6	24	120	720	...



group
Rotations of cube
has 24 elements

Is it S_4 in disguise?

4 diagonals



S_n

transposition $\quad (i\,j)$

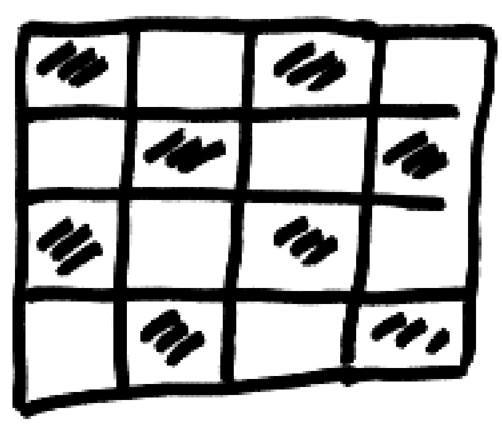
$$i \longleftrightarrow j$$

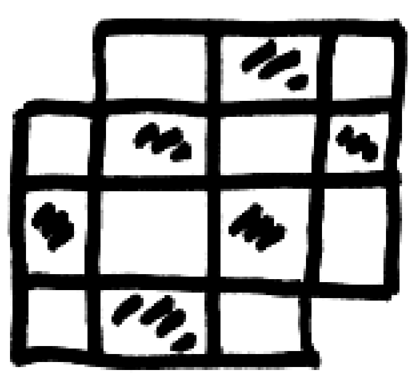
- All transposition generate S_n.

Any permutation can be obtained as a sequence of swaps (transpositions).

Even/odd permutations

Parity.





Dominos 

can you tile with dominos?

Each tile covers 1 □

1 ■

but there is a different number of these.

15-puzzle

1	2	3	4	
5	6	7	8	
9	10	11	12	?
13	$\boxed{15}$	$\boxed{14}$		

S. Lloyd

$\boxed{14 \mid 15}$

(14 15)

can't be done.

All permutations that can be achieved in this puzzle are even. But (14 15) is odd.

A transposition is odd.

product of odd → even permutation

odd · odd = even
even · even = even
odd · even = odd

$\oplus$	0	1
0	0	1
1	1	0

$\cdot$	+1	−1
+1	+1	−1
−1	−1	+1

$$(123) = \underset{-1\,\cdot\,-1}{(13)(12)} \quad \to \text{even} \; = +1$$

$$(12\cdots k) = (1k)\cdots(14)(13)(12)$$

$$(1234) = \underset{-1\,\cdot\,-1\,\cdot\,-1}{(14)(13)(12)} \quad \to \text{odd.} \; = -1$$

Any permutation is a ~~production~~ product of transpositions

$$\sigma = \tau_1 \cdot \tau_2 \cdot \cdots \cdot \tau_N$$

$$\mathrm{sgn}(\sigma) = \underbrace{(-1)\cdot(-1)\cdots(-1)}_{N}$$

$$\boxed{\mathrm{sgn}(\sigma) := (-1)^N}$$

$$(123) = (21)(23)$$

$$(123) = (21)(23)(12)(12)$$

Potentially the trouble is that writing $\sigma = \tau_1 \cdots \tau_N$ is not unique and hence $(-1)^N$ may depend on how we do it.

As it happens $(-1)^N$ is always the same

$$(1\,2\,3) = \tau_1 \cdots \tau_N$$

necessarily N is even.

$\rightsquigarrow$ Even/odd permutations

$$\sigma = \underset{+1}{(1\,2\,3)} \cdot \underset{-1}{(4\,5)} \cdot \underset{-1}{(6\,7\,8\,9)}$$

$$= +1$$

$$= (1\,3)(1\,2)\,(4\,5)\,(6\,9)(6\,8)(6\,7)$$

($$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1\cdot5\cdot9 - 2\cdot6\cdot7 + \cdots$$)

even · even = even

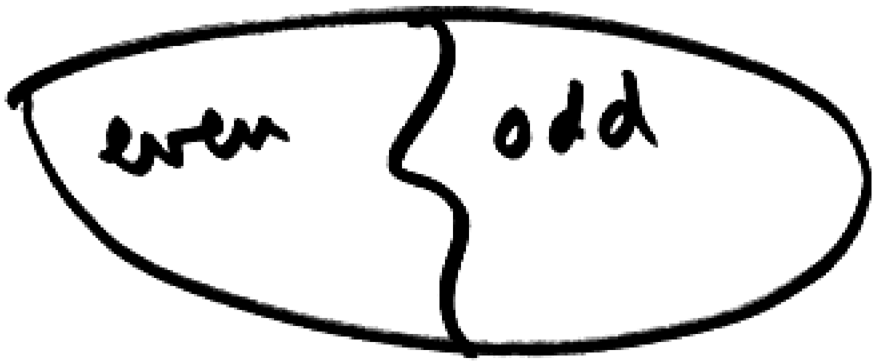
All even permutations forms a subgroup of S_n.

$\text{even}^{-1} = \text{even}$.

A_n = alternating group.

$|A_n| = \frac{1}{2} n!$ $\quad (n > 1)$

S_n



$$\underset{\text{even}}{\sigma} \longmapsto \underbrace{(12)\,\sigma}_{\text{odd}}$$

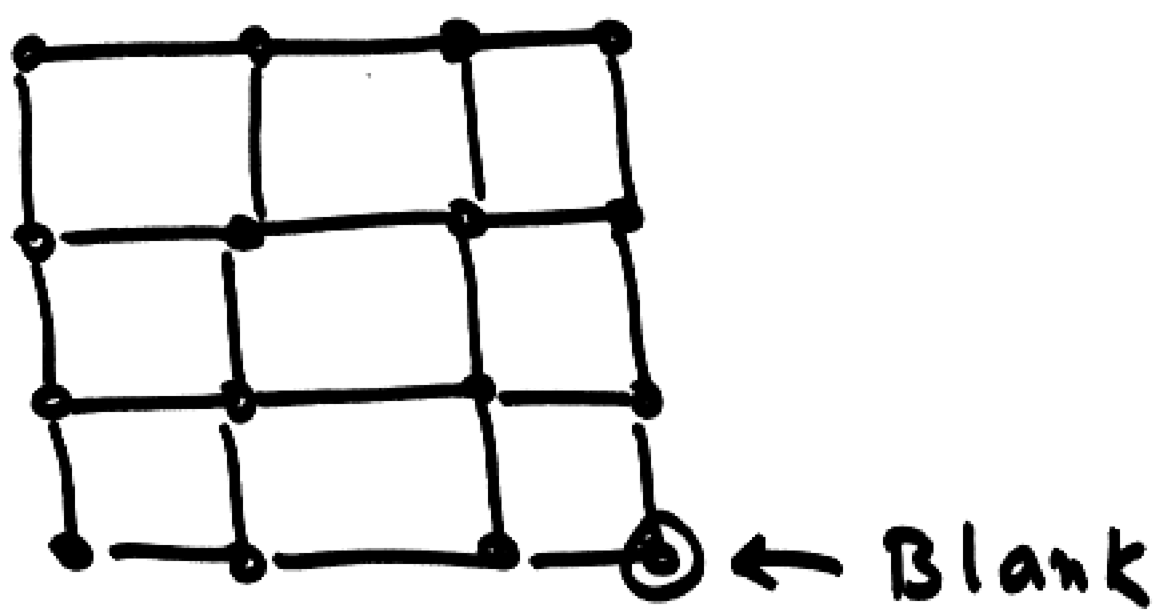
$$\underset{\text{even}}{(12)\,\eta} \longleftarrow \underset{\text{odd}}{\eta}$$

1-1 correspondence between even and odd permutations

$\#$ even $= \#$ odd

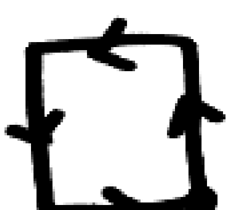
$\Rightarrow \#$ even $= \frac{1}{2}$ total.

4. The permutations we can get from moves in the 15-puzzle are <u>all</u> even.

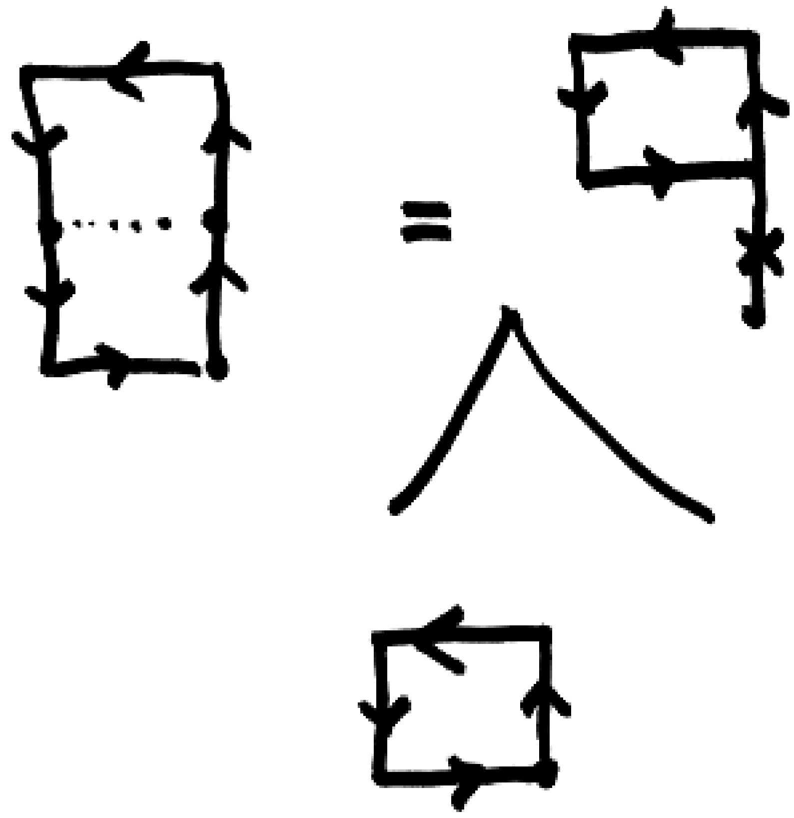


$$\begin{array}{cc|cc|cc|} 2 & 1 & 2 & 0 & 0 & 2 \\ 3 & 0 & 3 & 1 & 3 & 1 \end{array}$$

$$\begin{array}{cc|cc|} 3 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}$$

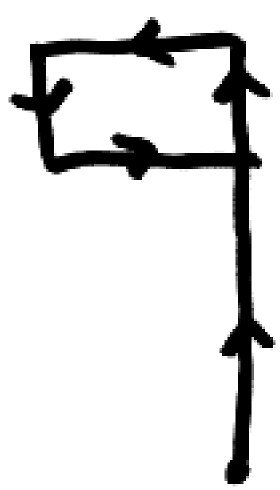
 $\rightsquigarrow (1\ 3\ 2)$

even



Any path can be decomposed as a sequence of little squares moves.

Each little square is 3-cyle



All moves are even.

RATE
YOUR
MIND
PAL

RATE
YOUR
MIND
PLA

you can solve this by swapping
LA and AA