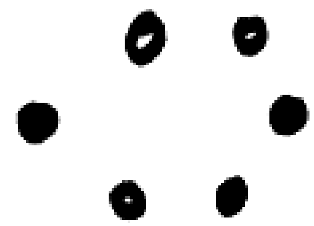
March 27, 2007

$G \circlearrowright X$

G: group (dihedral group order 12 symmetries of hexagon)

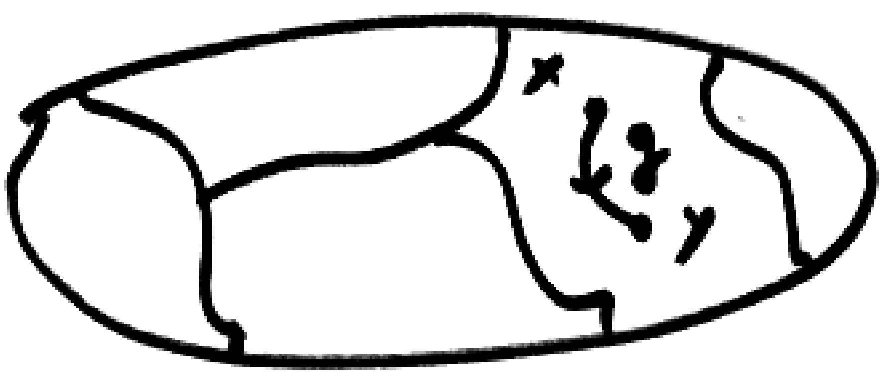
X: set ("pictures")

E.g.



$g \in G \qquad g \cdot x \in X$

- $G \cdot x := \{ y \in X \mid y = gx \}$



__orbit__ of x ("necklace")

- __Stabilizer__ of x

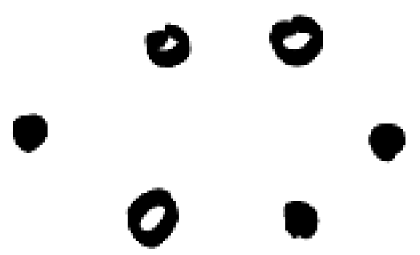
$\mathrm{Stab}_G(x) := \{ g \in G \mid gx = x \}$

subgroup of G

Fact. $\# G\cdot x \cdot |\mathrm{Stab}_G(x)| = |G|$

In particular, the size of an orbit is a factor of $|G|$.

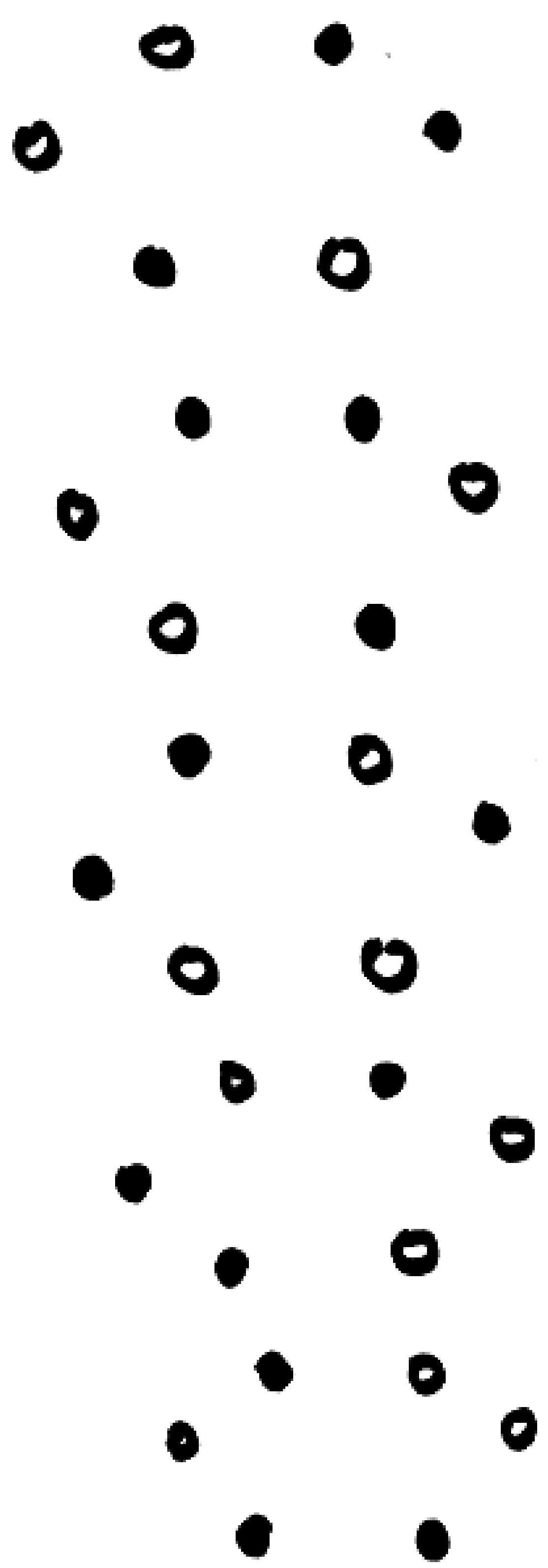
Eg.

$x =$ 

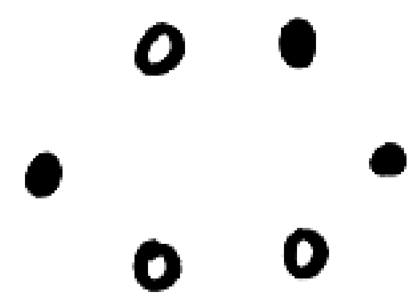
what is the size of Gx ?

~~use~~ x

r, r^2, r^3, r^4, r^5



$s \updownarrow$

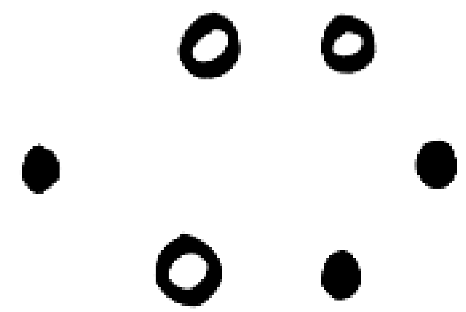


⋮

$\# Gx \geq 7$

$\Longrightarrow \# Gx = 12$

$\mathrm{Stab}_G(x) = \{1\}$

$x =$ 

r, r^2, r^3, r^4, r^5 do not fix x

$s_0, s_1, s_2, s_3, s_4, s_5$ "

• $\Rightarrow \quad \#Gx = 12$

How many orbits are there?

<u>Burnside's Lemma</u>

$$\boxed{\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} F(g)}$$

$F(g) = \#\{x \in X \mid gx = x\}$

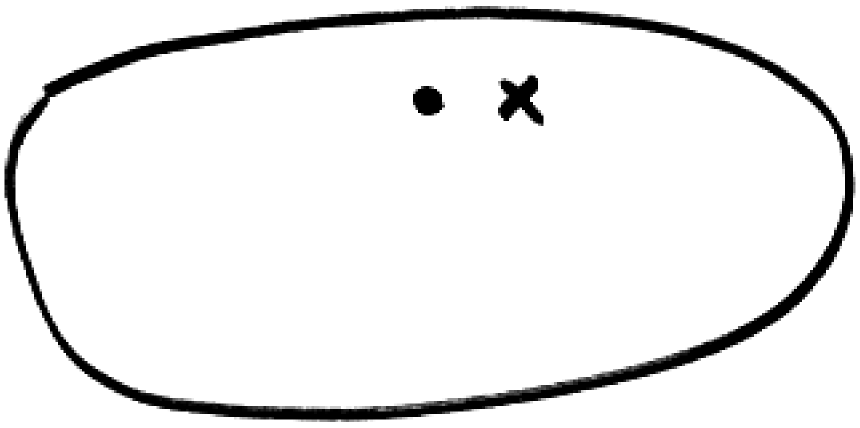
<u>Proof</u>

$$\sum_{g \in G} F(g)$$

$x \in X$

How many times does it get counted in this sum?

X



It will be counted in $F(g)$ if $\boxed{gx = x}$

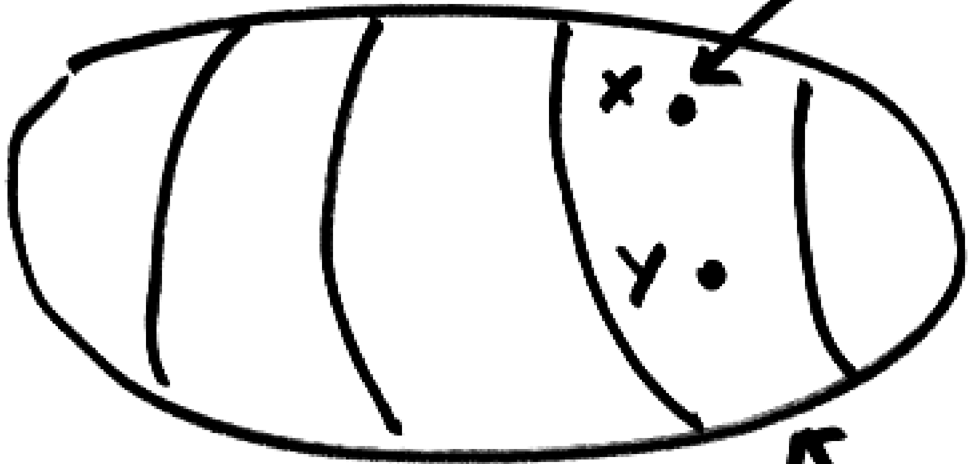
Total contribution of x to

$$\sum_{g \in G} F(g)$$

is $\{ g \in G \mid gx = x \} = \mathrm{Stab}_G(x)$

Each $x \in X$ contributes $|\mathrm{Stab}_G(x)|$ to the sum.

$$\sum_{g \in G} F(g) = \#\text{orbits} \cdot |G|$$



contributes $|\mathrm{Stab}_G(x)|$

all y in the orbit of x contribute the <u>same</u> amount

namely $|\mathrm{Stab}_G(y)| = \dfrac{|G|}{\# Gy}$

$$Gy = Gx$$

Total contribution of orbit is

$$|\mathrm{Stab}_G(x)| \cdot \# Gx = |G|$$

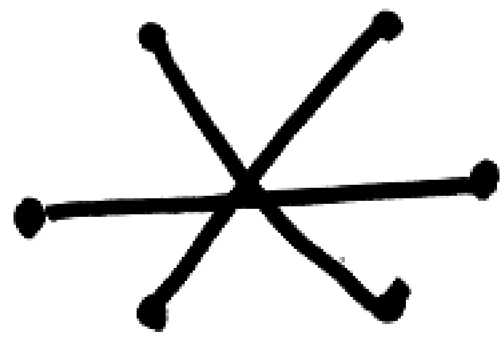
Total sum $\quad |G| \cdot \#\text{orbits}$

$$= \sum_{g \in G} F(g) \qquad \square$$

Cycle indicator

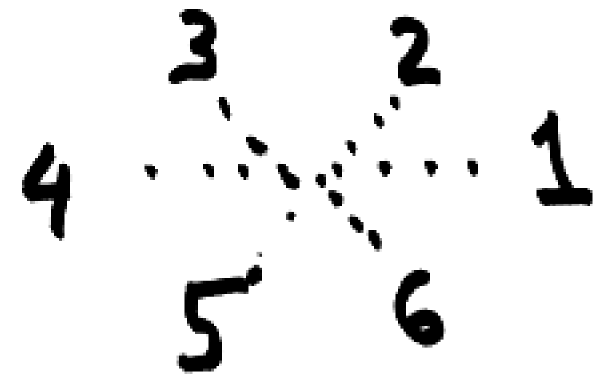
How do we compute $F(g)$?

Say $g = r^3$ what is $F(g)$?



If we have m colors then

$$F(r^3) = m \cdot m \cdot m = m^3$$



, $r^3 = \underbrace{(14)(25)(36)}_{3 \text{ cycles}}$

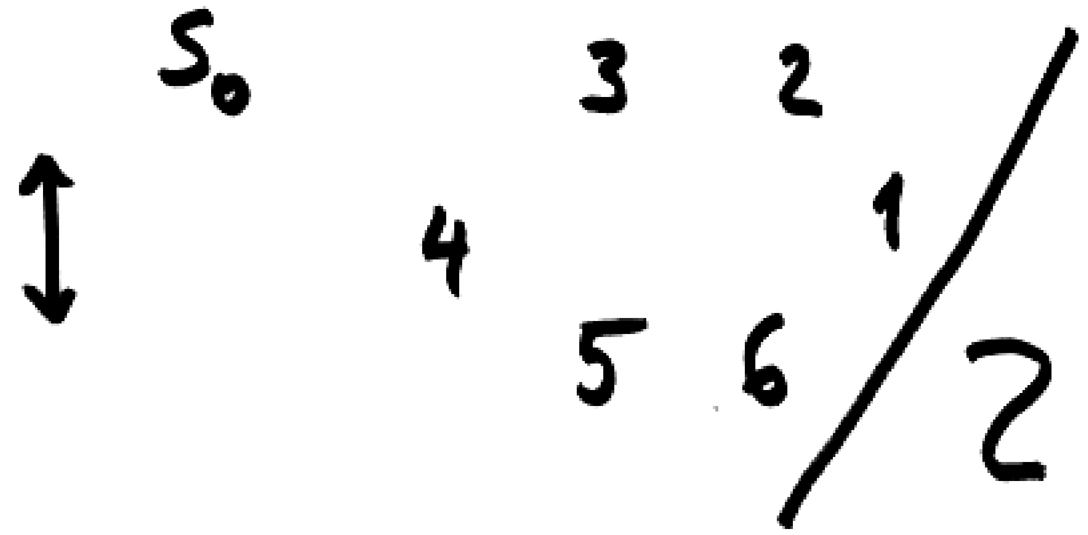
In general $F(g) = m^{\ell(g)}$

$\ell(g) :=$ # cycles in g

<u>length of g</u>

$$F(g) = m^{\ell(g)}$$

s_0



$s_0 = (35)(26)$

$\ell(g)$ includes the 1-cycles.

$$\ell(s_0) = 4$$

$$F(s_0) = m^4$$

<u>cycle indicator</u>

$$Z_G(x_1, x_2, \ldots) = \frac{1}{|G|} \sum_{g \in G} x_1^{k_1(g)} \cdot x_2^{k_2(g)} \cdots$$

$K_i(g) :=$ # cycles in g of length i

For $g = s_0$ $\qquad s_0 = (35)(26)$

$K_1(s_0) = 2$

$K_2(s_0) = 2$

$K_3(s_0) = 0$

$\vdots$

Contribution to Z_G : $\qquad x_1^2 x_2^2$

Formal way to keep track of the cycle decomposition of all $g \in G$.

For us counting orbits with m colors

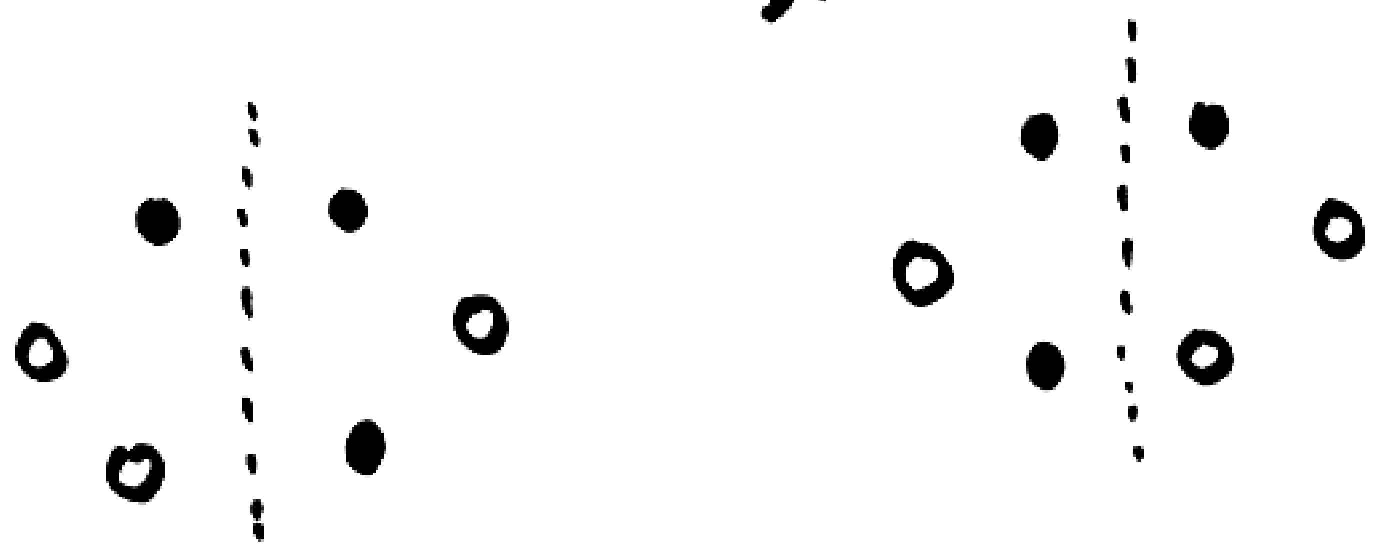
$Z_G(m, m, m, \dots)$

If $x_i = m$ then

$$x_1^{k_1(g)} x_2^{k_2(g)} \cdots = m^{k_1(g)+k_2(g)+\cdots}$$
$$= m^{\ell(g)}$$

$$Z_G(m, m, m, \ldots) = \frac{1}{|G|} \sum_{g \in G} m^{\ell(g)}$$

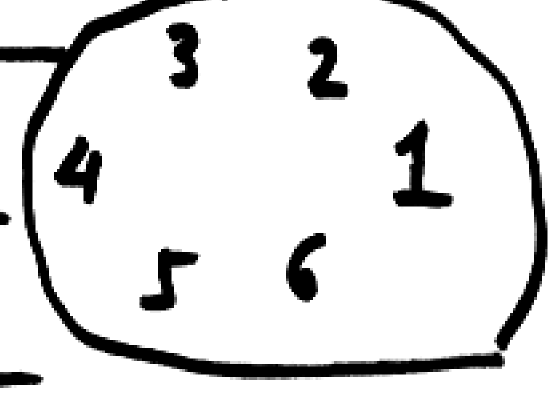
$= \#$ orbits.



<u>Cycle indicator</u> for rotations of hexagons (no flips allowed)

$G = \{1, r, r^2, r^3, r^4, r^5\}$

		k_1	k_2	k_3	k_4	k_5	k_6
1	1	6					
r	(123456)	0	0	0	0	0	1
r^2	(135)(246)			2			
r^3	(14)(25)(36)		3				
r^4	([illegible])			2			
r^5	(654321)						1



(642)(531)

$$Z = \frac{1}{6}\left(x_1^6 + x_6 + x_3^2 + x_2^3 + x_3^2 + x_6 \right)$$

$$\boxed{Z = \frac{1}{6}\left(x_1^6 + 2x_6 + 2x_3^2 + x_2^3 \right)}$$

$m = 2$

$$\frac{1}{6}\left(2^6 + 2 \cdot 2 + 2 \cdot 2^2 + 2^3 \right)$$

$$= \frac{1}{6}(64 + 4 + 8 + 8) = \frac{84}{6} = 14$$

3 2 1

4 5

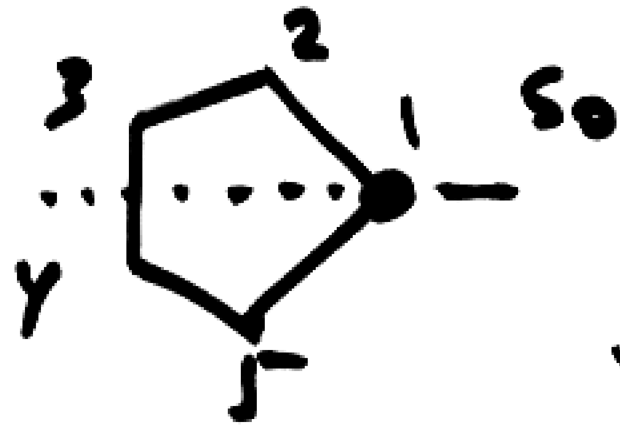
$r = (12345)$

1	1	x_1^5
r	(12345)	x_5
r^2	(13524)	x_5
r^3	$(14\ldots)$	x_5
r^4	(54321)	x_5

$$Z = \frac{1}{5}\left(x_1^5 + 4 x_5 \right)$$

March 29, 2007

flips



$$\overbrace{(1)\ (25)(34)}^{l=3}$$

cycle index $\qquad x_1^1 \cdot x_2^2$

color counting $\qquad x_i = m$

$$m \cdot m^2 = m^3$$

$$\frac{1}{10}\left(m^5 + 4m + 5m^3\right)$$

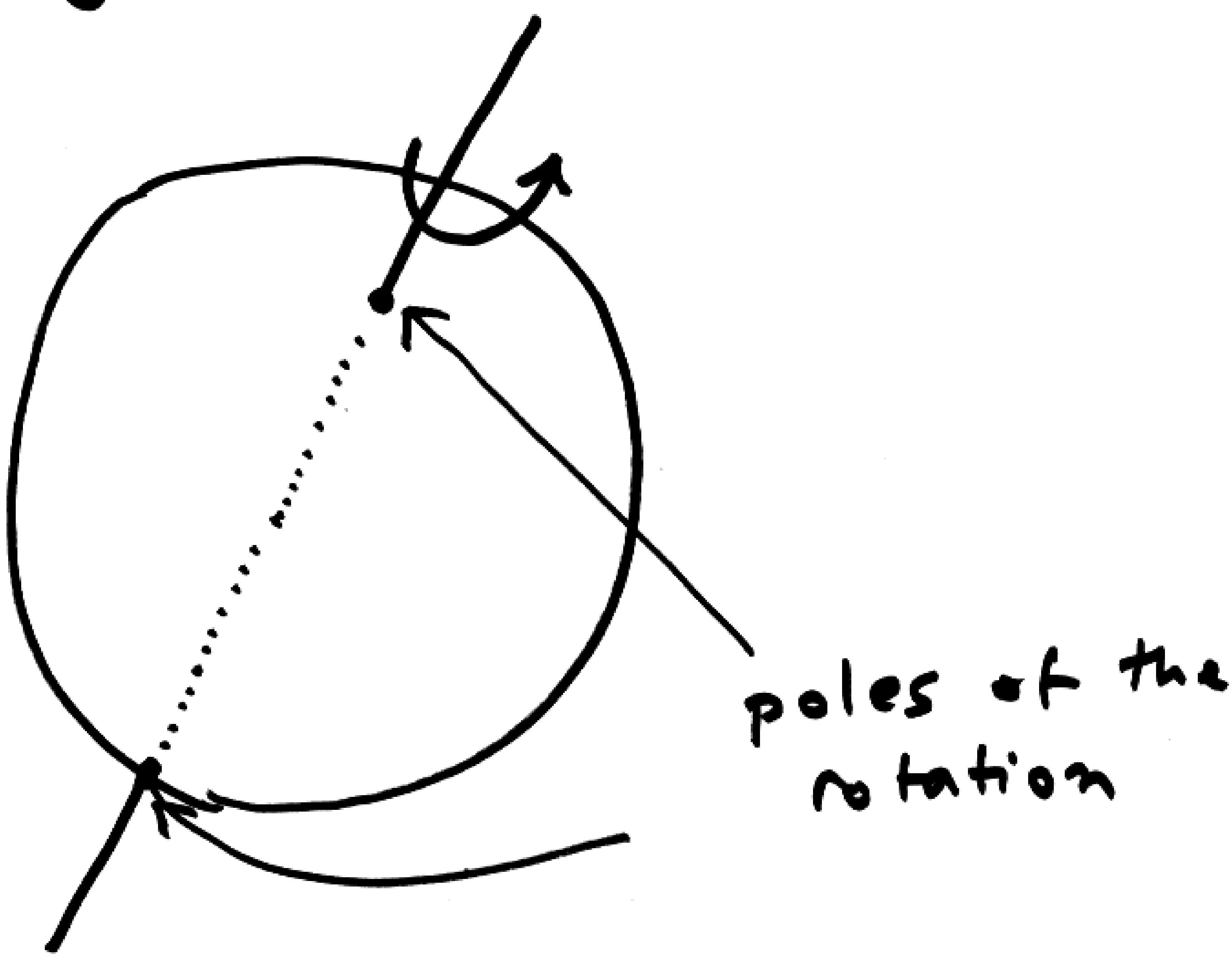
$m=3,$ $\qquad 3^5 + 4 \times 3 + 5 \times 3^3$

$$= \frac{1}{10}(243 + 12 + 135)$$

$$= \frac{390}{10} = 39$$

Finite group of rotations in $\mathbb{R}^3$

G



Let $\mathcal{P} = \{$ poles of rotations of $G \}$

Finite number of poles in $\mathcal{P}$.

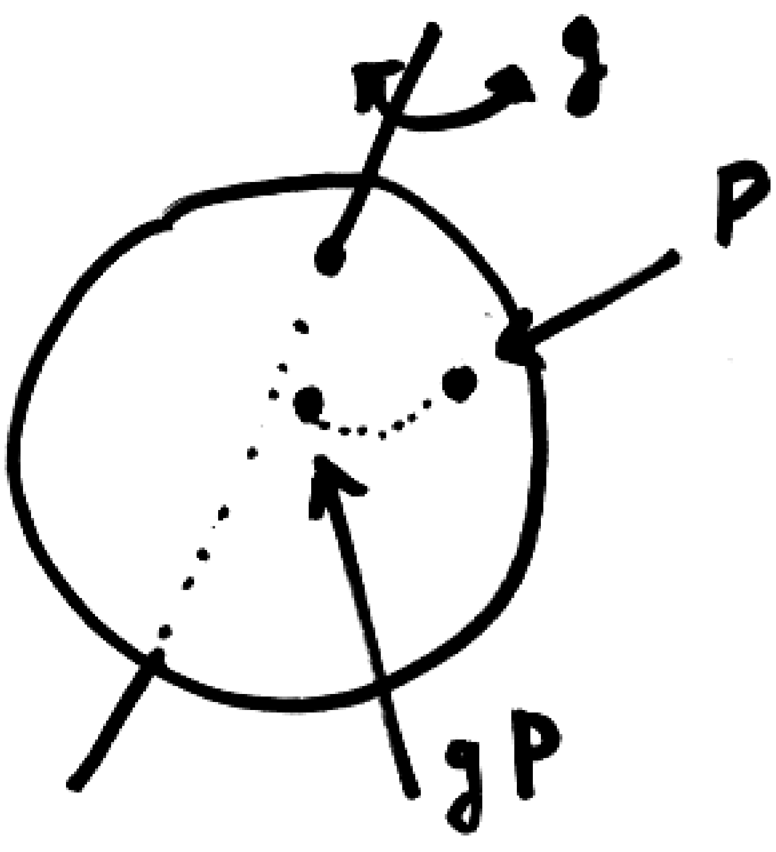
G acts on $\mathcal{P}$

. If $P \in \mathcal{P}$, $g \in G$

then $g \cdot P$

$\boxed{hP = P}$ for some $h \in G$
$h \neq 1$.

To see that gP is a pole
I need to find a rotation
in G that fixes gP.



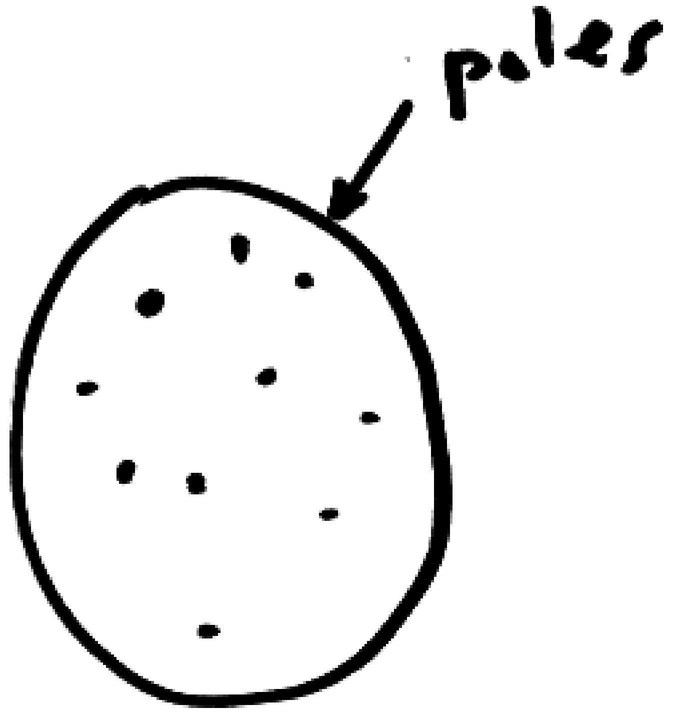
I.e. $r \in G$ s.t. $r(gP) = gP$

$$g^{-1}(gP) = P$$

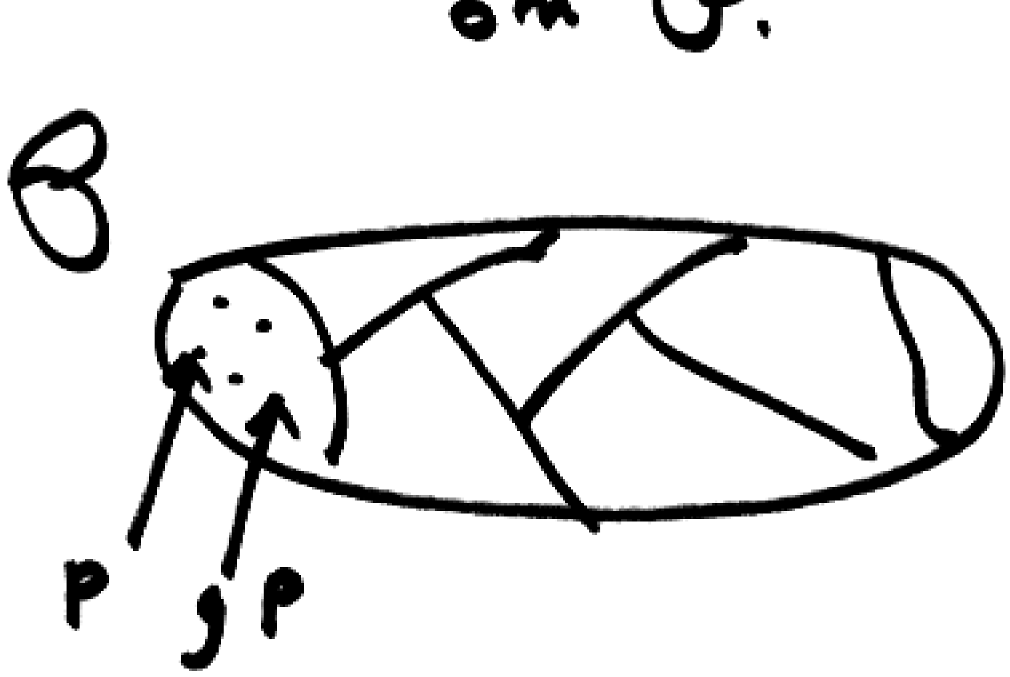
$$hg^{-1}(gP) = hP = P$$

$$\underbrace{ghg^{-1}}_{r}(gP) = gP$$

Found $r \in G \rightsquigarrow gP$ is a pole



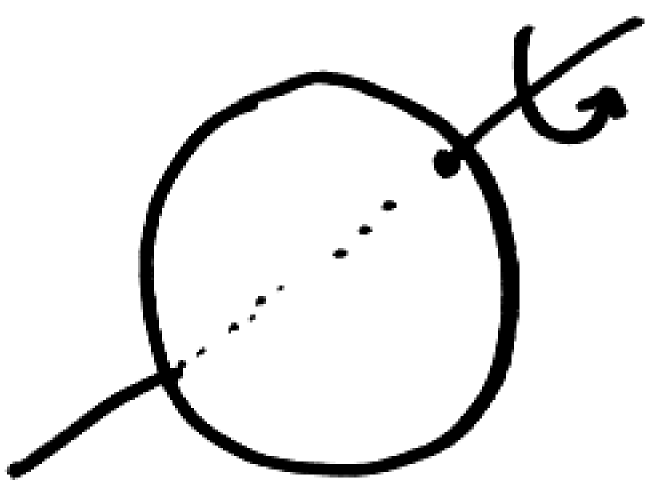
$N :=$ # orbits of G acting on $\mathcal{P}$.



$$N = \frac{1}{|G|} \sum_{g \in G} F(g)$$

by Burnside.

$$F(g) = \begin{cases} \#\wp & g = 1 \\ 2 & g \neq 1 \end{cases}$$



$$N = \frac{1}{|G|}\left(\#\wp + 2\left(\overbrace{|G| - 1}^{\text{\# of non-identity rotations}}\right)\right)$$

$$|G|N = \#\wp + 2|G| - 2$$

$$\boxed{|G|(N-2) = \#\wp - 2}$$

Assume $|G|>1$, I.e.
there is some non-identity
rotation $g \in G$. It fixes
two point. Hence $\# \mathcal{P} \geq 2$.

rhs $\geq 0 \Rightarrow N \geq 2$.

If $N=2$ then $\# \mathcal{P}=2$

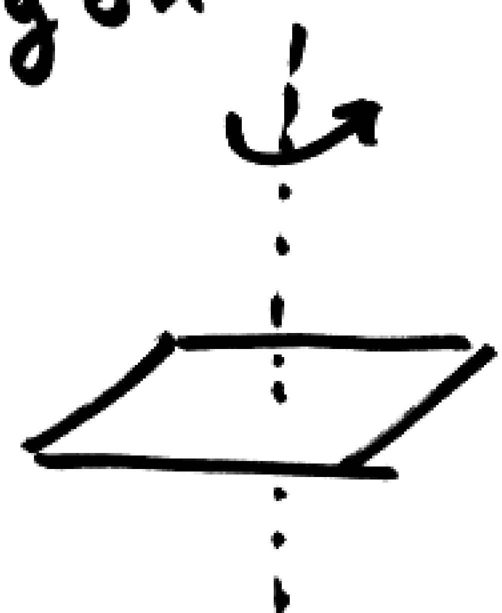
I.e. we have only have two poles
all rotations share same axis

$\rightarrow \quad 1, r, r^2, \ldots, r^{n-1}$

some n

I.e. ~~sym~~ rotations fixing
a n-gon

E.g. $\underline{n=4}$



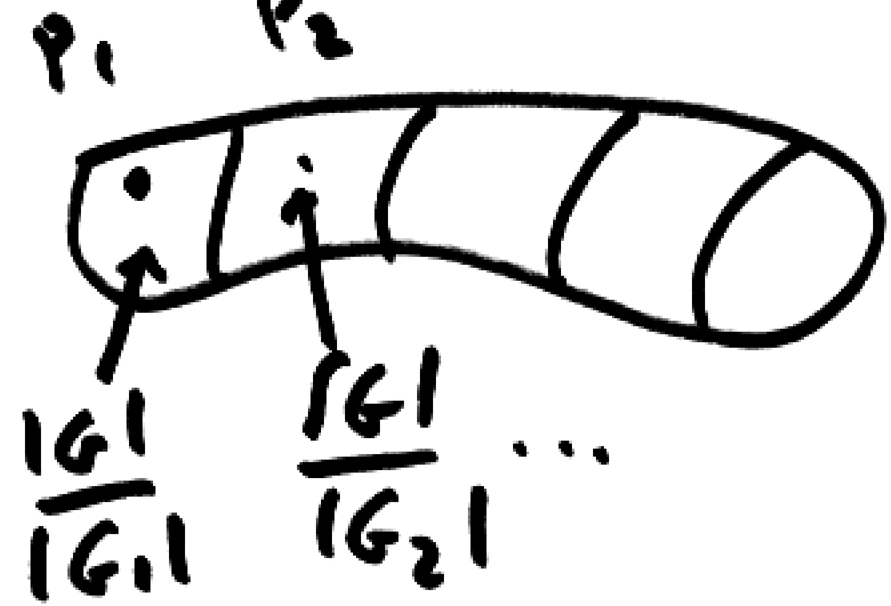
Say $N \geq 3$. $\# \mathcal{P} > 2$

$P_1, P_2, P_3, \ldots, P_N$ are poles one per orbit.

$$G_i := \mathrm{Stab}_G\, P_i$$

$$\# G P_i \cdot |G_i| = |G|$$

$$\# G P_i = \frac{|G|}{|G_i|}$$



8

$$\# \mathcal{P} = \frac{|G|}{|G_1|} + \frac{|G|}{|G_2|} + \dots + \frac{|G|}{|G_N|}$$

(II) $$\boxed{\frac{\# \mathcal{P}}{|G|} = \frac{1}{|G_1|} + \frac{1}{|G_2|} + \dots + \frac{1}{|G_N|}}$$

$$N = \frac{\# \mathcal{P}}{|G|} + 2\left(1 - \frac{1}{|G|}\right)$$

(from before).

(I) $$N = \underbrace{1 + 1 + \dots\dots + 1}_{N \text{ times}}$$

(I) − (II)

$$N - \frac{\# \mathcal{P}}{|G|} = \left(1 - \frac{1}{|G_1|}\right) + \left(1 - \frac{1}{|G_2|}\right) + \dots + \left(1 - \frac{1}{|G_N|}\right)$$

$$= 2\left(1 - \frac{1}{|G|}\right)$$

Finally:

$$\boxed{\sum_{i=1}^{N}\left(1-\frac{1}{|G_i|}\right) = 2\left(1-\frac{1}{|G|}\right)}$$

$(|G|>1 \rightarrow |G| \geqslant 2)$

rhs < 2

$G_i = \mathrm{Stab}_G\, P_i$

$|G_i| \geqslant 2$ since P_i is the pole of <u>some</u> non-trivial rotation $g_i \in G_i$.

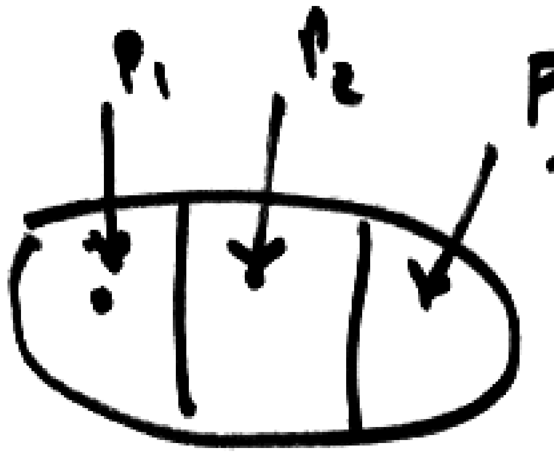
$$1 - \frac{1}{|G_i|} \geqslant \frac{1}{2}$$

$\Rightarrow$ $N < 4$

we know $N \geqslant 2$

$N = 2$ dealt with already

$\Rightarrow N = 3$



$G_1, \; G_2, \; G_3$

$n_i = |G_i|$

$n_1 \leq n_2 \leq n_3$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|G|}$$

$n_i \geqslant 2 \;, \quad |G| \geqslant 2$

We can't have $n_i \geqslant 3$
Otherwise

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

but lhs ≤ 1 rhs > 1

$\Rightarrow \boxed{n_1 = 2}$

$$\frac{1}{2} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|G|}$$

$$\frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} + \frac{2}{|G|}$$

If $n_2 = 2$ n_3 could be any thing. $n_3 = n$

$$\frac{1}{2} + \frac{1}{n} = \frac{1}{2} + \frac{2}{|G|}$$

$$\Rightarrow \quad |G| = 2n$$

$\rightsquigarrow$ $G = D_n$ dihedral group

G	n_1	n_2	n_3	
D_n	2	2	n	Dihedral n-gon
A_4	2	3	3	Tetrahedron
S_4	2	3	4	Cube/octahedron
A_5	2	3	5	Icosahedron/ Dodecahedron