Feb 20, 2007

15 - puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Goal: scrambled position to this

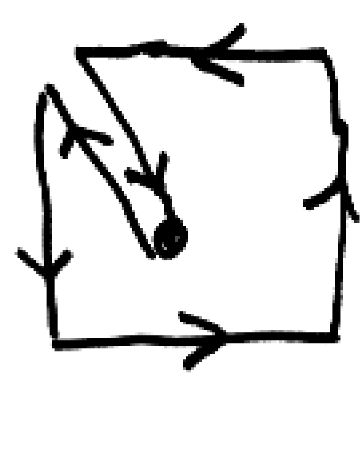
Move: Exchange a number w/blank (if neighbors)

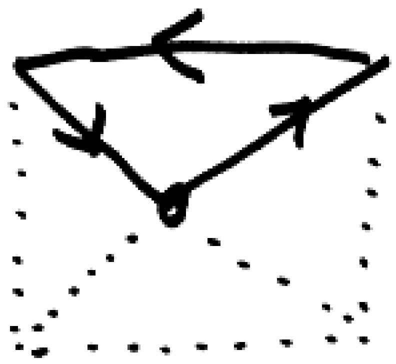
Moves permute the numbers.

$$\sigma := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & & 4 \\ & 0 & \\ 2 & & 3 \end{vmatrix} \begin{vmatrix} 0 & & 4 \\ & 1 & \\ 2 & & 3 \end{vmatrix} \begin{vmatrix} 2 & & 4 \\ & 1 & \\ 0 & & 3 \end{vmatrix} \begin{vmatrix} 2 & & 4 \\ & 1 & \\ 3 & & 0 \end{vmatrix} \begin{vmatrix} 2 & & 0 \\ & 1 & \\ 3 & & 4 \end{vmatrix}$$

$$\begin{vmatrix} 0 & & 2 \\ & 1 & \\ 3 & & 4 \end{vmatrix} \begin{vmatrix} 1 & & 2 \\ & 0 & \\ 3 & & 4 \end{vmatrix}$$





$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

transposition (swap two numbers)

Permutations can be "multiplied"

$$\underset{2^{nd}}{\tau} \cdot \underset{1^{st}}{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\leftarrow \boxed{\tau} \leftarrow \boxed{\sigma} \leftarrow$$

$$\sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

<u>Note</u> $\sigma \cdot \tau \neq \tau \cdot \sigma$

Not commutative

Every permutation has inverse

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

$\sigma^{-1} \sigma = 1$ ← permutation don't do anything

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\sigma \cdot \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1$$

$$\sigma \cdot 1 = \sigma$$

$$1 \cdot \sigma = \sigma$$

<u>All</u> permutations of n things $= S_n$

$$|S_n| = n!$$

15-puzzle : permutations of 15 #'s w/ blank in its final position. ?

$S = \{4, 10, 12\}$

0	1	2	3	4	5	6	7	8	9
0	0	0	0	1	1	1	1	0	0

10	11	12	13	14	15	16	17	18	19
2									

Feb 22, 2007

Permutations

- 1, identity, do-nothing
- $\sigma \rightsquigarrow$ inverse σ^{-1}

  $\sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = 1$
- $(\sigma \cdot \tau) \cdot \rho = \sigma \cdot (\tau \cdot \rho)$

  associative

  $= \sigma \cdot \tau \cdot \rho$

A **group** G (of permutations of n things) is a set of permutations

- $1 \in G$
- $\sigma \in G$, $\sigma^{-1} \in G$
- $\sigma, \tau \in G$, $\sigma \cdot \tau \in G$

For example

1) $G = S_n$ (all permutations)

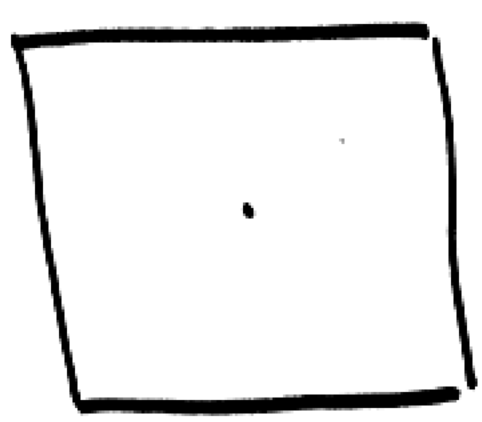
2) $G = \{1\}$

<u>trivial group</u>

$$1^{-1} = 1$$

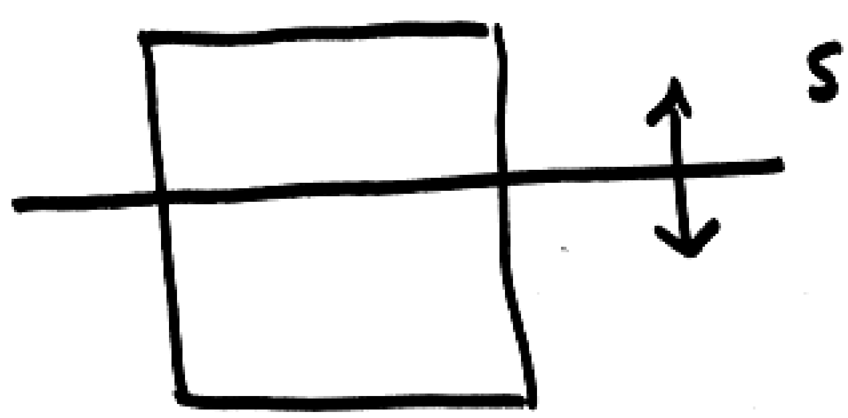
$$1 \cdot 1 = 1$$

<u>Symmetries of the square</u>

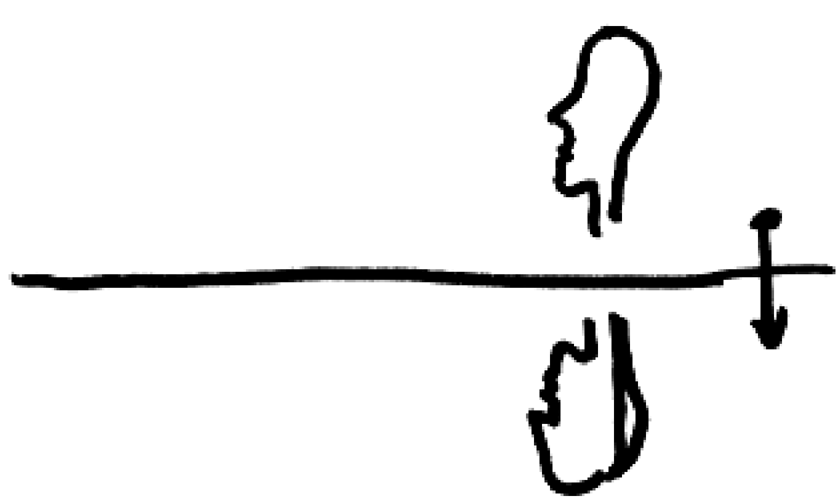


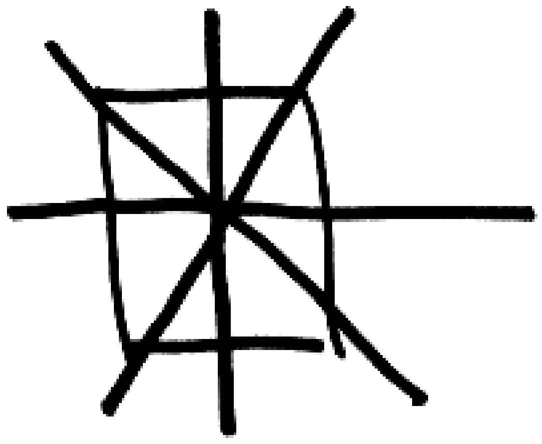
r

rotation
1/4 turn

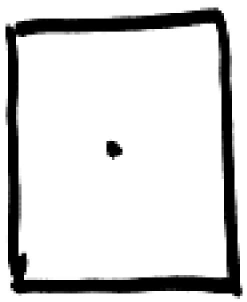


reflection

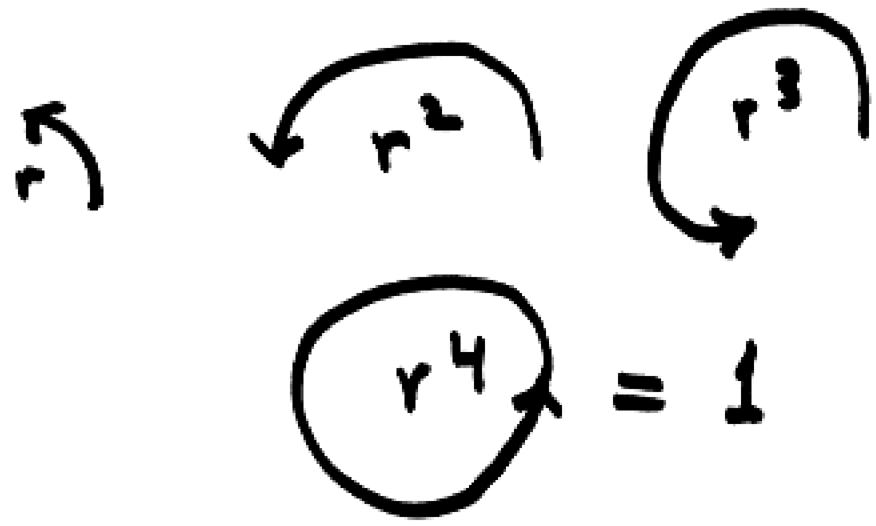




4 reflections



4 rotations



$r^2 = r \cdot r$

$r^3 = r \cdot r \cdot r$

$r^4 = r \cdot r \cdot r \cdot r$

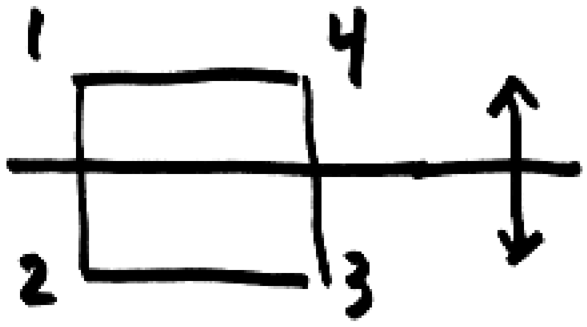
These are all the symmetires of the square.

Total of 8 symmeties.

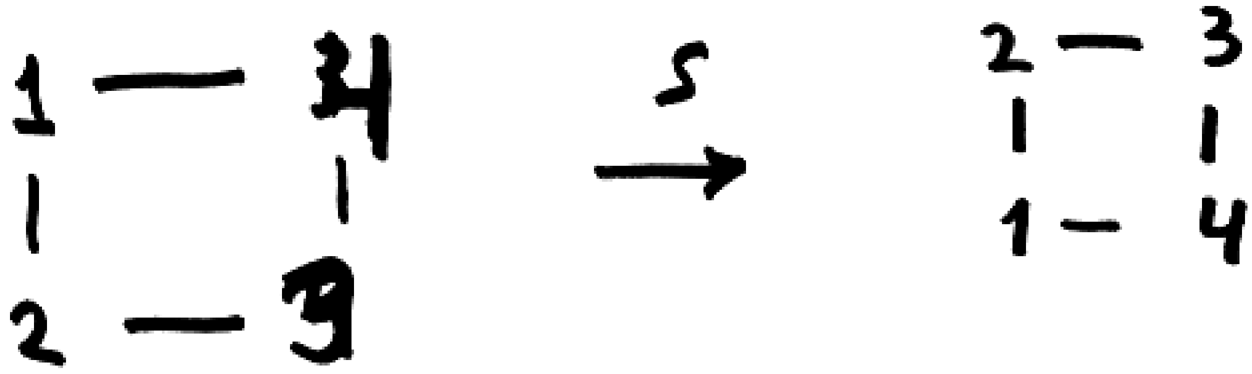
These symmetries form a group.

D_4 (<u>dihedral</u> group)

Each symmetry gives a permutations of the vertices



s



$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

The eight symmetries of the square result in eight permutations of the vertices.

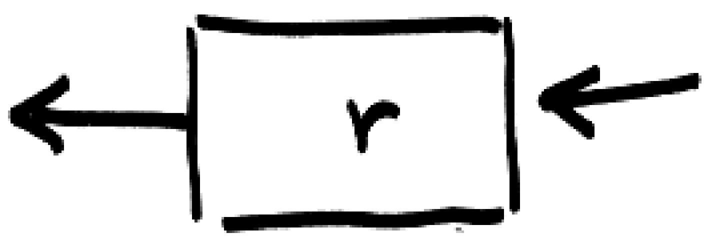
Total number of possible permutations is 24.

I.e. NOT every permutation is obtained as a symmetry of the square.

The 8 permutation form a <u>group</u>.

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

<u>cycle notation</u>

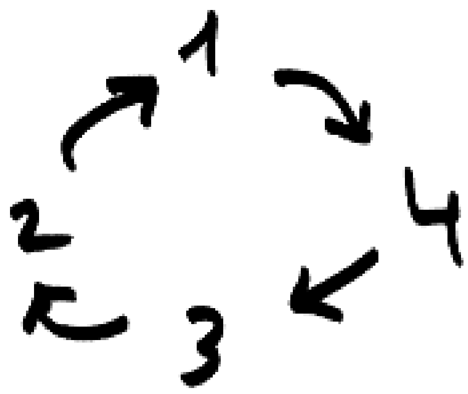


chase the numbers:

$$1 \longmapsto 4 \longmapsto 3 \longmapsto 2 \longmapsto 1 \longmapsto 4 \longmapsto 3 \ldots$$

Write r as a bunch of cycles

$$r = (1\ 4\ 3\ 2)$$



$$r = (2\ 1\ 4\ 3)$$

r is a <u>4 - cycle</u>.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$$

In cycle notation:

$$\sigma = \underbrace{(1\ 5\ 2)}_{\text{3-cycle}}\ \underbrace{(3\ 4)}_{\text{2-cycle}}$$

Note that the cycles are <u>disjoint</u>

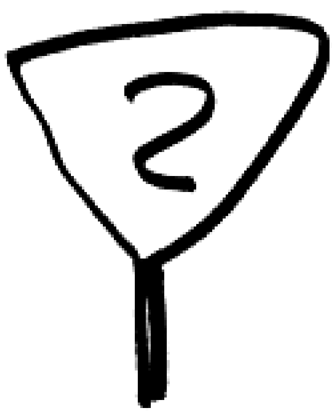
$$\sigma = (3\,4)(1\,5\,2)$$
$$= (4\,3)(5\,2\,1)$$

$$\sigma = (3\,4)\cdot(1\,5\,2) = (1\,5\,2)\cdot(3\,4)$$

cycles commute
disjoint

disjoint
a, b are cycles

$$\boxed{a \cdot b = b \cdot a}$$



NON disjoint cycles
may not commute.

$a = (1\,2)$
$b = (2\,3)$

$$a \cdot b = (1\,2)\cdot(2\,3) = (1\,2\,3)$$
$$b \cdot a = (2\,3)(1\,2) = (1\,3\,2)$$

$$(1\,2\,3) \neq (1\,3\,2)$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$

cycle notation

$(4\ 2)\ (3\ 1)\quad (5)$

Typically not write 1-cycles

$\sigma = (4 2)(3 1)$

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in S_5
identity: $(1)(2)(3)(4)(5)$

$= (\,)$
