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$H \leq G$ $\qquad$ W repn of H

$V := \mathrm{Ind}_H^G(W)$ $\qquad$ $\dim V = [G:H]\cdot \dim W$
$= 3 \cdot 1 = 3$

$H = \langle (12) \rangle \leq S_3$

W sgn repn of H

$(12) \longmapsto -1$

$1 \longmapsto +1$

Pick set of repn's for G/H

g_σ: $\quad 1, \ (123), \ (132)$

$g \in S_3$

$$\boxed{g \cdot g_\sigma = g_\tau \cdot h_{\sigma,\tau}}$$

$\sigma = 1$

g	$g \cdot 1 = g_\tau \cdot h$
1	$1 \cdot 1 = 1 \cdot 1$
(12)	$(12)\cdot 1 = 1 \cdot (12)$
(13)	$(13)\cdot 1 = (123)\cdot(12)$
(23)	$(23)\cdot 1 = (132)\cdot(12)$
(123)	$(123)\cdot 1 = (123)\cdot 1$
(132)	$(132)\cdot 1 = (132)\cdot 1$

$\sigma = (123)$

$$(123)\cdot(12) = (13)$$

$$(132)\cdot(12) = (23)$$

g	$g\cdot(123) = g_\tau \cdot h$
1	$(123) = (123)\cdot 1$
(12)	$(23) = (132)\cdot(12)$
(13)	$(12) = 1\cdot(12)$
(23)	$(13) = (123)(12)$
(123)	$(132) = (132)\cdot 1$
(132)	$1 = 1\cdot 1$

$$(12)(123) = (23)$$
$$(12)(132) = (13)$$

$$(13)(123) = (12)$$
$$(13)(132) = (23)$$

$$(23)(123) = (13)$$
$$(23)(132) = (12)$$

$\sigma = (132)$

g	$g \cdot (132) = g_\tau \cdot h$
1	$(132) \cdot 1$
(12)	$(123)(12)$
(13)	$(132)(12)$
(23)	$1 \cdot (12)$
(123)	$1 \cdot 1$
(132)	$(123) \cdot 1$

$1 \quad (123) \quad (132)$

$$1 \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad (12) \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$(13) \mapsto \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (23) \longmapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

standard $\oplus$ sgn $=$ defining $\otimes$ sgn

defining $=$ standard $\oplus$ trivial

"mixture" of the permutation repn on G/H and W.

<u>Alternative defn.</u>

- $V := \{ f : G \to W \mid f(hx) = h f(x),\ h \in H \}$

Define left action of G on V

$$(gf)(x) = f(xg)$$

<u>Claim</u> $V \cong \mathrm{Ind}_H^G(W)$

- <u>Universal property</u>

U repn of G

$\varphi : W \to U$ H-linear

extends to a unique G-linear map

$\tilde{\varphi} : V \to U$

$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[d] & U \\ V \arrow[ur, dotted, "\tilde{\varphi}"'] & \end{tikzcd}$$

$\mathrm{Res}_H^G(U)$ Restriction of U from G to H.

$$\mathrm{Hom}_H(W, \mathrm{Res}_H^G U) = \mathrm{Hom}_G(\mathrm{Ind}_H^G W, U)$$

<u>Cor</u> (Frobenius Reciprocity)

$$(\chi_W, \chi_{\mathrm{Res}_H^G U})_H = (\chi_{\mathrm{Ind}_H^G W}, \chi_U)_G$$

<u>Example</u> $S_3 \hookrightarrow S_4$ natural embedding.

W defining repn of S_3.

$V := \mathrm{Ind}_{S_3}^{S_4} W$ $\dim V = 12$

g_σ : $1, \kappa, \kappa^2, \kappa^3$ $\kappa = (1234)$

Coset $g \in S_4$ is determined by $g(4)$.

$$(12) \cdot 1 = 1 \cdot (12)$$
$$(12) \cdot \kappa = \kappa^2 \cdot (132)$$
$$(12) \cdot \kappa^2 = \kappa \cdot (123)$$
$$(12) \cdot \kappa^3 = \kappa^3 \cdot (23)$$

$$k^{-2}(12)k = (13)(24)(12)(1234)$$
$$= (132)$$

$$k^{-1}(12)k^{2} = (1432)(12)(13)(24)$$
$$= (123)$$

$$k^{-3}(12)k^{3} = (1234)(12)(1432)$$
$$= (23)$$

$$(12) = \begin{array}{cccc} 1 & k & k^2 & k^3 \\ \left(\begin{array}{c|c|c|c} \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix} & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix} & 0 & 0 \\ \hline 0 & 0 & 0 & \begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix} \end{array}\right) \end{array}$$

<u>Character of $\mathrm{Ind}_H^G(W)$</u>

$$\chi(g) = \sum_{\substack{g\sigma=\sigma \\ \sigma \in G/H}} \chi_W(g_\sigma^{-1}\, g\, g_\sigma)$$

$$\left[\begin{array}{l} g\, g_\sigma = g_\tau \cdot h \\ \tau = \sigma \\ g_\sigma^{-1} g\, g_\sigma = h \end{array}\right.$$

<u>Note</u> terms in sum are indeed indep. of choice of representatives.

<u>Ex</u>. C conjugacy class in G

$$H \cap C = D_1 \,\dot\cup\, \cdots \,\dot\cup\, D_r$$

H-conjugacy classes

$$\chi_{\mathrm{Ind}_H^G W}(C) = \frac{|G|}{|H|} \cdot \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W(D_i)$$

In particular if W is trivial

$$\chi_{\mathrm{Ind}_H^G W}(C) = [G:H] \cdot \frac{|H \cap C|}{|C|}$$

<u>Symmetric Fctns</u>

$x_1, \ldots, x_n$ indeterminates.

$$\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$$

symmetric polynomials

graded ring $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$

$\Lambda_n^k = \{h \text{ symmetric deg } k\} \cup \{0\}$

$m \geq n \quad \rho_{m,n}$

$$\Lambda_m \longrightarrow \Lambda_n$$

setting $x_j = 0 \quad j > n$

preserves degree

$$\Lambda_{n+1}^k \longrightarrow \Lambda_n^k$$

$$f \in \Lambda^k = \varprojlim_n \Lambda_n^k$$

$f_0, f_1, f_2 \ldots$

$f_n \in \Lambda_n^k$ s.t.

$\rho_{m,n}(f_m) = f_n \qquad m \geq n$

$$\rho_n^k : \Lambda^k \longrightarrow \Lambda_n^k$$
$$f \longmapsto f_n$$

is an isom. for $n \geq k$.

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$

$\lambda_1 \geq \lambda_2 \geq \ldots$

$|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$ <u>size</u>

$\lambda_k > 0 \qquad \ell(\lambda) = k$ <u>length</u>

<u>Monomial symmetric fctns</u>

$$m_\lambda := \sum_\alpha x^\alpha$$

α all distinct permutations of $\lambda_1, \lambda_2, \ldots$

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

$$\deg m_\lambda = |\lambda|$$

$$m_\lambda \in \Lambda_{\ell(\lambda)}^{|\lambda|}$$

<u>**Ex.**</u> $\lambda = (2, 1, 1)$ $\qquad |\lambda| = 4$, $\ell(\lambda) = 3$

$$m_\lambda = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

$$m_\lambda \in \Lambda_3^4$$