Nov 1, 2007

Symmetric functions

$$\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{S_n}$$

$$= \bigoplus_{k \geq} \Lambda_n^k$$

Λ_n^k deg k

$m \geq n$ $\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$

$x_j \longmapsto 0 \qquad j > n$

$x_j \longmapsto x_j, \quad 0 \leq j \leq n$

$m \geq n \geq k$

$\rightarrow$ ~~surj~~bijective: $\rho_{m,n}$

$\rightarrow$ surjective: $\rho_{m,n}$ if $m \geq n$

$$\Lambda^k = \varprojlim_m \Lambda_m^k$$

$\in$

$f = f_0, f_1, \dots$ $\qquad f_n \in \Lambda_n^k$

$$f_m(x_1, \dots, x_n, 0, 0 \dots, 0)$$
$$= f_n(x_1, \dots, x_n)$$

$$\rho^k_{m,n}(f_m) = f_n$$

Monomial symmetric fctns

$\lambda_1 \geq \lambda_2 \geq \dots$ partition

$|\lambda| = \lambda_1 + \lambda_2 + \dots = k$

$\ell(\lambda) =$ length $=$ # non-zero parts

$=$ # variables

$\ell(\lambda) \leq n$

Define

$$m_\lambda = \sum_\alpha x^\alpha$$

$\alpha =$ distinct permutations of $\lambda_1, \lambda_2 \dots$

$$x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 = m_\lambda$$
$$\in \Lambda^4_3$$

$\lambda = (2, 1, 1)$

$|\lambda| = 4$, $\ell(\lambda) = 3$

$$m_\lambda = x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4$$
$n = 4$
$$+ x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + x_3 x_2^2 x_4$$
$$+ x_1 x_2 x_3^2 + x_1 x_4 x_3^2 + x_2 x_4 x_3^2$$
$$+ x_1 x_2 x_4^2 + x_1 x_3 x_4^2 + x_2 x_3 x_4^2$$

For $n \geq k$

With $|\lambda| = k$ $\quad (\ell(\lambda) \leq k \leq n)$

m_λ's are a $\mathbb{Z}$ basis of Λ_n^k

$m \geq n \geq k$ $\quad \rho^k_{m,n}$ bijective

$\rightsquigarrow \quad m_\lambda \in \Lambda^k = \varprojlim \Lambda^k_n$

They are $\mathbb{Z}$-basis for Λ^k

Λ^k is a free $\mathbb{Z}$-module of rank $p(k) := \#$ partitions of k.

$$\Lambda := \bigoplus_{k \geq 0} \Lambda^k$$

$$\rho_n := \bigoplus_{k \geq 0} \rho^k_n : \Lambda \longrightarrow \Lambda_n$$

Λ is a ring <u>symmetric function</u>

(<u>not</u> the inverse limit of Λ_n in the category of rings

$$\prod(1+x_i) \notin \Lambda$$

it <u>is</u> the inverse limit of Λ_n in the category of graded rings)

$p(k)$ grows rapidly with k

$$p(200) = 3972999029388$$

$$p(k) \sim \frac{e^{\pi\sqrt{2/3 k}}}{4\sqrt{3}\, k}$$

<u>Hardy-Littlewood</u>

<u>Other bases of Λ</u>

Elementary functions

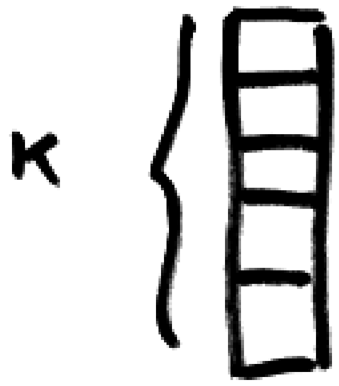
$$m_{(1^k)} = \sum_{i_1<\cdots<i_k} x_{i_1} x_{i_2} \cdots x_{i_k} =: e_k$$

$$\lambda = (\lambda_1, \lambda_2, \ldots) = 1^{m_1} 2^{m_2} \cdots$$

$$m_i := \#\{\lambda_j = i\}$$

<u>multiplicities</u>

$$1^k = (\underbrace{1, 1, \ldots,}_{k} 1)$$



$$E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i \geq 1} (1 + x_i t)$$

$$= 1 + (x_1 + x_2 + \cdots) t + (x_1 x_2 + x_1 x_3 + \cdots + x_2 x_3 + \cdots \ddots) t^2 + \cdots$$

$E(t) \in \Lambda[\![t]\!]$

For $\lambda = (\lambda_1, \lambda_2, \ldots)$ define

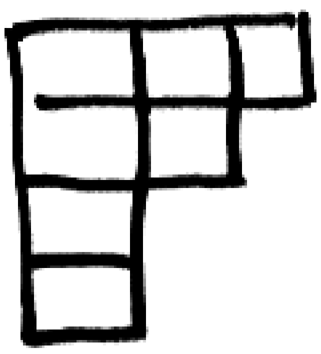
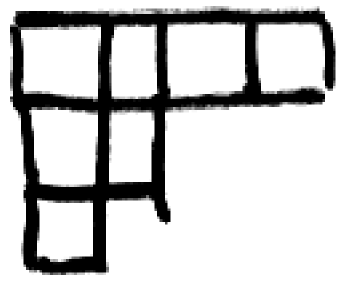
$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots \in \Lambda$$

<u>Prop</u> $\quad e_{\lambda'} = m_\lambda + \sum_{\mu} a_{\lambda,\mu} m_\mu$

$a_{\lambda,\mu} \in \mathbb{Z}$

$\mu > \lambda$ in reverse lexic. order

$|\mu| = |\lambda|$

$\lambda =$  $\lambda' =$ 

flip

$\rightsquigarrow \{e_{\lambda'}\}$ is a $\mathbb{Z}$-basis of $\Lambda^{|\lambda|}$

$= \{e_\lambda\}$

<u>Cor</u> $\Lambda = \mathbb{Z}[e_1, e_2, e_3, \dots]$

Fundamental theorem of symmetric functions.

<u>Complete symmetric functions</u>

$$h_r = \sum_{|\lambda| = r} m_\lambda, \quad h_0 = 1, \quad h_1 = e_1 = x_1 + x_2 + \cdots$$

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

$$h_2 = m_{(1,1)} + m_{(2)}$$
$$= x_1 x_2 + x_1 x_3 + \cdots + x_2 x_3 + \cdots$$
$$x_1^2 + x_2^2 + \cdots$$

$$e_2 = m_{(1,1)} = x_1 x_2 + x_1 x_3 + \dots + x_2 x_3 + \dots$$ 7

$$\frac{1}{1-x_1 t} = 1 + x_1 t + x_1^2 t^2 + \dots$$

$$\frac{1}{1-x_2 t} = 1 + x_2 t + x_2^2 t^2 + \dots$$

$$\vdots \qquad \vdots$$

$$E(t) = \prod_{i \geq 1} (1 + x_i t)$$

$$H(t) = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

$$\boxed{E(t) \cdot H(-t) = 1}$$

$$(*) \qquad \sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0, \qquad n > 0$$

$$\omega : \Lambda \longrightarrow \Lambda$$
$$e_r \longmapsto h_r$$

Apply ω to $(*)$

$$\sum_{r=0}^{n} (-1)^r h_r \, \omega(h_{n-r}) = 0$$

$$(-1)^n \sum_{r=0}^{n} (-1)^r h_{n-r}\, \omega(h_r) = 0$$

we must have

$$\omega(h_r) = e_r$$

Hence $\omega^2 = id_\Lambda$.

$$\Lambda = \mathbb{Z}[h_1, h_2, \dots]$$

i.e. h_i are alg. indep over $\mathbb{Z}$.

Define

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots$$

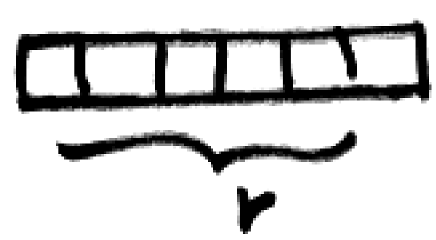
$$= h_1^{m_1} h_2^{m_2} \cdots$$

$$\lambda = (\lambda_1, \lambda_2, \dots) = 1^{m_1} 2^{m_2} \dots$$

$$\omega(h_\lambda) = e_\lambda$$
$$\omega(e_\lambda) = h_\lambda$$

- <u>Power sums</u>

$$p_r := \sum_{i\geq 1} x_i^r = m_{(r)}$$

$r>0$ $\qquad (r)$ 

dual to (1^r)

$$P(t) := \sum_{r\geq 1} p_r \, t^{r-1}$$

$$= \sum_{r\geq 1} \sum_{i\geq 1} x_i^r \, t^{r-1}$$

$$= \sum_{i\geq 1} \sum_{r\geq 1} x_i^r \, t^{r-1}$$

$$= \sum_{i\geq 1} \frac{x_i}{1 - x_i t}$$

$$= \frac{H'(t)}{H(t)}$$

$$\boxed{n h_n = \sum_{r=1}^{n} p_r \, h_{n-r}}$$

$$\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \dots]$$

p_i are alg. indep / $\mathbb{Q}$.

$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots$

$\mathbb{Q}$-basis for $\Lambda_{\mathbb{Q}}$ (not a $\mathbb{Z}$-basis)