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Representation ring

$R(G)$ free abelian group on isom. classes of repn of G. $/ ([V]+[W]-[V\oplus W])$

i.e. in $R[G]$

$$[V\oplus W] = [V] + [W]$$

Grothendieck gp.

$$\mathbb{N} \rightsquigarrow \mathbb{Z}$$

$$\mathbb{N}\times\mathbb{N} / \sim$$

$(m,n) \sim (m',n')$
$m+n' = m'+n$

An element of $R[G]$

$$\sum_U a_U [U] \qquad U \text{ irreducibles}, \quad a_U \in \mathbb{Z}.$$

this is called a "virtual" representation of G.

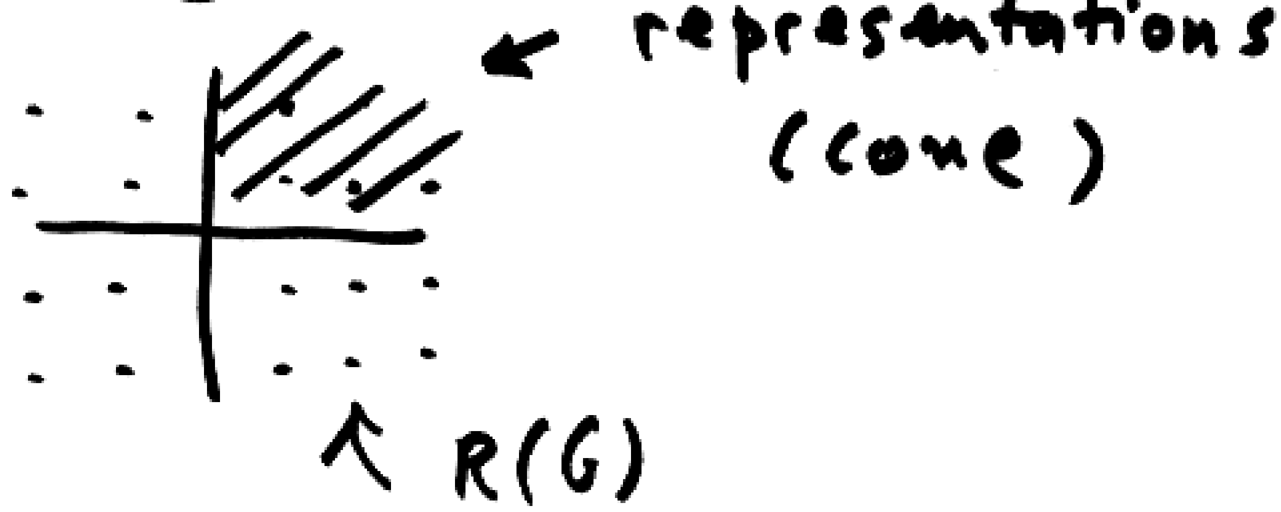
$$[V] - [W]$$

Product $\qquad [V] \cdot [W] := [V \otimes W]$

$$R(G) \longrightarrow \{\text{class functions}\}$$
$$[V] \longmapsto \chi_V \qquad \varphi : G \to \mathbb{C}$$

ring homomorphism, injective (χ determines the representation)

the image of $R[G]$ contains a basis of {class functions}. i.e. image is a lattice in {class fctns}.



Pairing in $R(G)$

$$\langle [V], [W] \rangle := \dim \operatorname{Hom}_G(V, W)$$

for V, W representations.

$$R(G) \overset{\imath}{\hookrightarrow} \{\text{class fctns}\}$$

isometry with $\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_g \overline{\varphi(g)}\psi(g)$

$$\imath(R(G) \otimes_{\mathbb{Z}} \mathbb{C}) = \{\text{Class fctns}\}$$

Multiplication in $\{\text{class functions}\}$?

pointwise product. i.e.

$$\varphi \cdot \psi(g) := \varphi(g) \cdot \psi(g)$$

Another product

$$\{\text{class functions on } G\} \hookrightarrow \mathbb{C}[G] = \{\text{functions on } G\}$$

$$\sum_g \varphi(g) g \cdot \sum_g \psi(g) g$$

$$= \sum_g \Big[\underbrace{\sum_h \varphi(h)\psi(h^{-1}g)}_{=\,\theta(g)} \Big] g$$

θ <u>convolution</u> of φ & ψ

With this product

$\{$ class functions $\}$ = center of $\mathbb{C}[G]$

Character table of A_5

	1	20	15	12	12
	1	(123)	(12)(34)	(12345)	(21345)
U	1	1	1	1	1
V	4	1	0	-1	-1
W	5	-1	1	0	0
Y	3	0	-1	ε	ε'
Z	3	0	-1	ε'	ε

$1\,2\,3 \longmapsto i\,j\,k$

achieve this within A_5 by fixing sgn with 4 & 5 if needed.

$$\langle \chi_{\Lambda^2 V}, \chi_{\Lambda^2 V} \rangle_{A_5} = \frac{1}{60} (\dots) = 2$$

$$\Lambda^2 V \simeq Y \oplus Z$$

$$60-(1^2+4^2+5^2) = 18 = (\dim Y)^2 + (\dim Z)^2$$

$$\dim Y + \dim Z = 6$$

$$\rightarrow \dim Y = \dim Z = 3 \qquad \text{Ex.}$$

$$\begin{array}{c|ccccc} Y & 3 & a_2 & a_3 & a_4 & a_5 \\ Z & 3 & b_2 & b_3 & b_4 & b_5 \end{array}$$

$$a_2 + b_2 = 0$$
$$a_3 + b_3 = -2$$
$$a_4 + b_4 = 1$$
$$a_5 + b_5 = 1$$

orthog of char

$$0 = \langle \chi_Y, 1 \rangle = \langle \chi_Y, \chi_V \rangle = \langle \chi_Y, \chi_W \rangle$$

$$\rightarrow \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ +1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\langle \chi_Y, \chi_Y \rangle = 1$$

6

$\rightarrow \quad t^2 + t - 1 = 0$

$\rightarrow \quad t = \dfrac{-1 \pm \sqrt{5}}{2}$

$\varepsilon = \dfrac{1+\sqrt{5}}{2}$

$t = \dfrac{-1+\sqrt{5}}{2}$

$\rightarrow t + 1 = \varepsilon$

$-t = \varepsilon'$

$$\begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ t+1 \\ -t \end{pmatrix}$$

Note: χ_Y and χ_Z are galois conjugates

When is a repn defined over $\mathbb{R}$?

G acting on a vectorspace V_0 over $\mathbb{R}$.

$$V_0 \rightsquigarrow V = V_0 \underset{\mathbb{R}}{\otimes} \mathbb{C}$$

$\Rightarrow \quad \chi_V$ has real values.

Converse is not true

Example

$SU(2) \subset SL_2$

Preserves Hermitian form in $\mathbb{C}^2$ & $\det = 1$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad {}^T g^{-1} = \bar{g}$$

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad {}^T g^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$g \in SU(2) \Leftrightarrow \begin{array}{l} \bar{a} = d \\ -\bar{b} = c \end{array}$$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad a\bar{a} + b\bar{b} = 1$$

Note: $\operatorname{tr}(g) = a + \bar{a} \in \mathbb{R}$.

Suppose $G \hookrightarrow SU(2)$ G finite.

We get a 2-dim'l repn of G with real character. This repn cannot be real. (if G is non-abelian) b/c if it was it would fix a real pos defn. quadr form. (and $\det = +1$)

$G \hookrightarrow SO(2) \simeq S^1$ circle group.

th. $\to$ G is abelian!

Let $\mathbb{H}$ Hamilton quaternions

$$\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

$$k = ij = -ji$$

$$\mathbb{H} = \underset{\mathbb{R}+\mathbb{R}i}{\underset{\parallel}{\mathbb{C}}} \oplus \mathbb{C}j$$

$$\{a + bj \in \mathbb{H} \mid a, b \in \mathbb{C}$$

$$ja = \bar{a}j \qquad j^2 = -1$$

$$\mathbb{H} \longrightarrow M_2(\mathbb{C})$$

$$a + bj \longmapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$SU(2) \leftrightarrow \mathbb{H}_1^\times = \{x \in \mathbb{H} \mid x\bar{x} = 1\}$$

$$\{\pm 1, \pm i, \pm j, \pm k\} \hookrightarrow \mathbb{H}_1^\times$$