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Basis of Λ

- **Monomial**

$$m_\lambda := \sum_{\alpha \text{ distinct permutations of } \lambda} x^\alpha$$

- **Elementary**

$$e_r := \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$$

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots = e_1^{m_1} e_2^{m_2} \cdots$$

- **Complete**

$$h_r := \sum_{|\lambda| = r} m_\lambda$$

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots = h_1^{m_1} h_2^{m_2}$$

$m_\lambda, e_\lambda, h_\lambda$ $\mathbb{Z}$-basis of Λ

- **Power sums**

$$p_r := \sum_{i \geq 1} x_i^r \qquad r > 0$$

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots$$
$$= p_1^{m_1} p_2^{m_2} \cdots$$

$\mathbb{Q}$ - basis of $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$

$$E(t) := \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t)$$

$$H(t) := \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

$$P(t) := \sum_{r \geq 1} p_r t^{r-1}$$

$$E(t) H(-t) = 1$$

$$(*) \qquad e_0 h_n - e_1 h_{n-1} + e_2 h_{n-2} + \cdots = 0$$

$e_0 = h_0 = 1$

$h_1 = e_1$

$h_2 = e_1 h_1 \cancel{+} - e_2$

$\vdots$

$e \leftrightarrow h$ determine each other

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

$$\lambda' = (\lambda'_1, \lambda'_2, \dots)$$

$\lambda = (4,3,1,1)$ $\qquad$ $\lambda' = (4,2,2,1)$

dual partition

$$\lambda'_i = \# \{ j \mid \lambda_j \geq i \}$$

$$m_i(\lambda) = \# \{ j \mid \lambda_j = i \}$$

$$= \lambda'_i - \lambda'_{i+1}$$

<u>Prop</u>

$$e_{\lambda'} = m_\lambda + \sum_{\mu > \lambda} a_{\lambda\mu} m_\mu$$

later in the reverse lexicographic order

<u>Pf</u>

$$e_{\lambda'} = e_1^{m'_1} e_2^{m'_2} \dots \qquad m'_i = m_i(\lambda')$$

$$= (x_1 + x_2 + \dots)^{m'_1} (x_1 x_2 + x_1 x_3 + \dots)^{m'_2} \dots$$

Largest term

$$x_1^{m_1'}\,(x_1x_2)^{m_2'}\,(x_1x_2x_3)^{m_3'}\cdots$$

$$= x_1^{m_1'+m_2'+\cdots}\;x_2^{m_2'+m_3'+\cdots}$$

$$= x_1^{\lambda_1}\,x_2^{\lambda_2}\cdots$$

$$m_i' = m_i(\lambda') = \lambda_i - \lambda_{i+1}$$

$$e_{\lambda'} = m_\lambda + \text{lin comb of } m_\mu \text{ with } \mu \text{ later than } \lambda. \quad \square$$

I.e. matrix relating $e_{\lambda'}$ with m_λ is upper triangular with 1's along diagonal.

$\Rightarrow$ e_λ are $\mathbb{Z}$-basis

$\Rightarrow$ $\Lambda = \mathbb{Z}[e_1, e_2 \ldots]$

Involution on Λ

$$\omega : e_\lambda \longmapsto h_\lambda$$

$$e_r \longmapsto h_r$$

$\omega(h_r) = e_r$

(apply ω to (*))

$\omega^2 = id_\Lambda$

h_λ $\mathbb{Z}$-basis & h_r are

alg indep / $\mathbb{Z}$

$$\Lambda = \mathbb{Z}[h_1, h_2, \dots]$$

$f_\lambda = \omega(m_\lambda)$ "forgotten" symmetric functions

$$P(t) = H'(t)/H(t)$$

$$P(-t) = E'(t)/E(t)$$

$$H(t)P(t) = H'(t)$$

$$\boxed{n h_n = \sum_{r=1}^{n} p_r h_{n-r}}$$

$h_1 = p_1$

$$h_2 = \frac{1}{2}(p_1^2 + p_2)$$

$$h_n \in \mathbb{Q}[p_1, \dots, p_n]$$

$$p_n \in \mathbb{Q}[h_1, \dots, h_n]$$

$$\mathbb{Q}[p_1, \dots, p_n] = \mathbb{Q}[h_1, \dots, h_n]$$

$n \to \infty$

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \dots]$$

p_i's are alg indep.

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots$$

$$H = \sum h_n t^n$$

$$E = \sum e_n t^n$$

$$\omega : H \longleftrightarrow E \qquad P(t) = \sum_{r \geq 1} p_r t^{r-1}$$

$$\omega : P(t) \longmapsto P(-t)$$

$$\omega(p_r) = (-1)^{r-1} p_r$$

$$\omega(p_\lambda) = (-1)^{(\lambda_1 - 1) + (\lambda_2 - 1) + \cdots} p_\lambda$$

$$= (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$$

$$= \varepsilon_\lambda p_\lambda$$

$\varepsilon_\lambda := (-1)^{|\lambda| - \ell(\lambda)}$ = sgn of ~~the~~ a permutation of type λ.

$$z_\lambda := |\operatorname{Stab}_{S_n}(\sigma_\lambda)|$$

$\sigma_\lambda \in S_n$ of type λ.

$$= \prod_{i \geq 1} m_i!\, i^{m_i}$$

$$\underbrace{(\cdot)(\cdot)(\cdot)(\cdot)}_{m_1}\ \underbrace{(\cdot\,\cdot)\ (\cdot\,\cdot)\ \cdots (\cdot\,\cdot)}_{m_2}\ \cdots\cdots$$

$$m_1!\, 1^{m_1}\quad m_2!\, 2^{m_2}\quad m_3!\, 3^{m_3} \cdots$$

$$m_i!\ i^{m_i}$$

permuting the m_i i-cycles

cyclically permuting inside each i-cycle

$$H(t) = \prod_{i \geq 1} (1 - x_i t)^{-1} = \sum_{r \geq 0} h_r t^r$$

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \frac{H'(t)}{H(t)}$$

$$H(t) = \exp\left(\sum_{r \geq 1} p_r \frac{t^r}{r} \right)$$

$$= \prod_{r \geq 1} \exp(p_r t^r / r)$$

$$= \prod_{r \geq 1} \sum_{m_r \geq 0} p_r^{m_r} \frac{t^{r m_r}}{r^{m_r} m_r!}$$

$$=$$

$\sum_r r m_r = n$ partition of n

$\lambda = 1^{m_1} 2^{m_2} \cdots$

$$= \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda \quad t^{|\lambda|}$$

$$\boxed{h_r = \sum_{|\lambda| = r} z_\lambda^{-1} p_\lambda}$$

~~[illegible]~~ $\underline{\lambda\text{-ring}}$

Λ free λ-ring in one variable

$\Lambda \longrightarrow R$

$h_1 = p_1 = e_1 \longmapsto x$

$e_r \longmapsto \lambda^r(x)$ $\quad \Lambda^r$ wedge

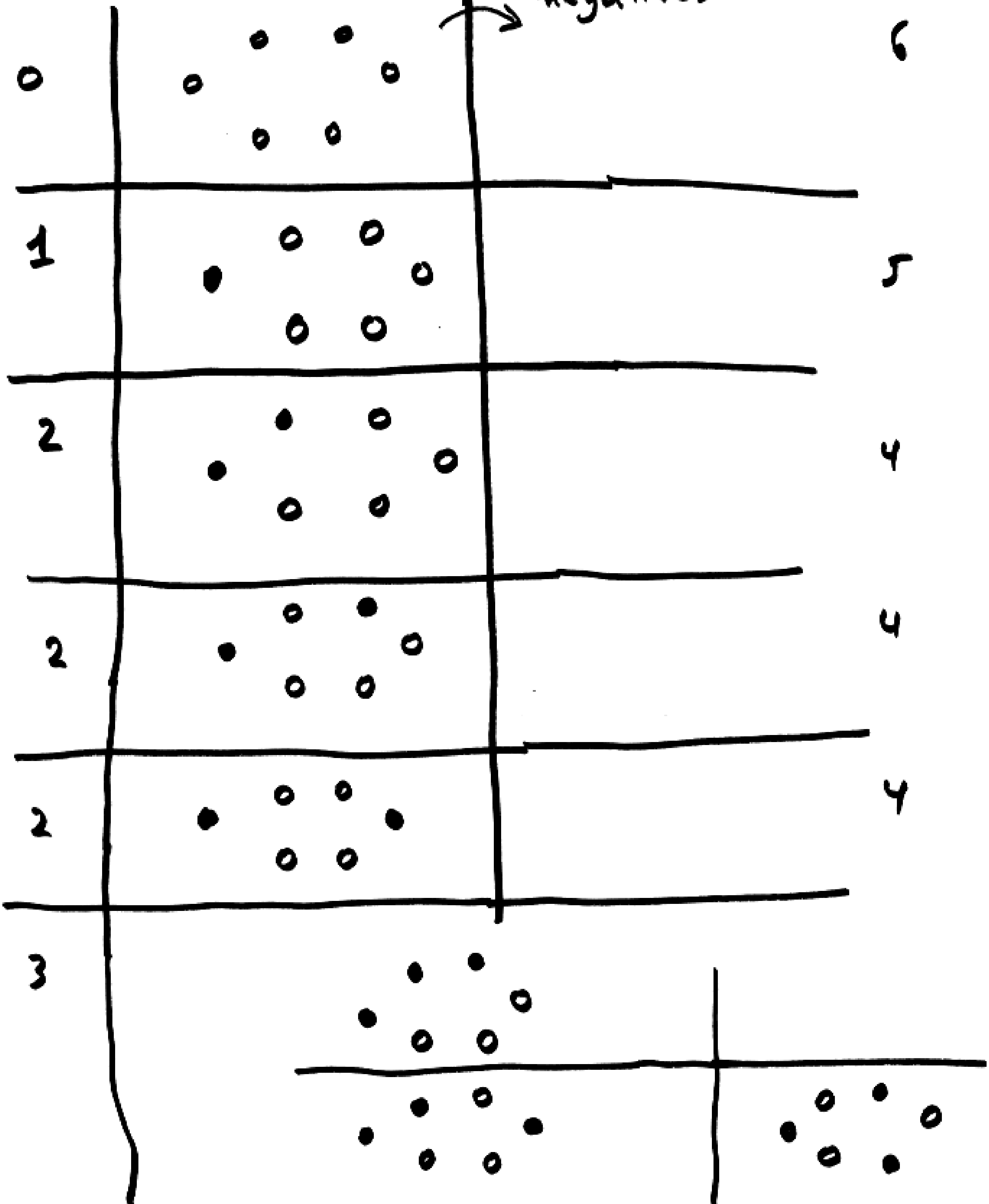
$h_r \longmapsto \sigma^r(x)$ $\quad \mathrm{Sym}^r$

$p_r \longmapsto \psi^r(x)$ $\quad$ Adams operations

Polya theory of counting

Necklaces 6 beads
2 colors

negatives



colorings $\circlearrowleft D_6$

2^6

"necklace" = orbit

$G \circlearrowright X$

C = set of colors

$$\text{colorings} := \{\varphi : X \to C\} = C^X$$

$$g\varphi(x) := \varphi(g^{-1}x)$$

<u>Burnside's Lemma</u>

$$\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g)$$

$$\mathrm{Fix}(g) = \#\{\varphi : X \to C \;\; \varphi|_{\substack{\text{orbit} \\ \text{of } g \text{ in } X}} = \text{constant}\} = \#C^{\#g\text{-orbits}}$$

g-orbits

$$G \subset S_n \simeq S(X)$$

$\# X = n$, $\# C = m$

g cycle type

λ partition of n $\leftarrow$ $\underbrace{(\cdot)(\cdot)\cdots(\cdot)}_{m_1}\ \underbrace{(\cdot\cdot)\cdots(\cdot\cdot)}_{m_2}\ \cdots$

$\#$ g orbits = length of λ

$$\mathrm{Fix}(g) = m^{\ell(\lambda)}$$

Cycle index indicator

$G \circlearrowright X$ $\quad \# X = n$

$$Z_G(t_1, t_2, \dots, t_n) = \frac{1}{|G|} \sum_{g \in G} t_1^{m_1} t_2^{m_2} \cdots$$

$m_i :=$ ~~# the~~ i-cycles in the g-orbits

$t = t_1 = t_2 \cdots$ $\quad t^{m_1 + m_2 + \cdots} = t^{\ell(g)}$

$G = D_6$ acting on 

1	$(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)$
r, r^{-1}	$(\cdot\cdot\cdot\cdot\cdot\cdot)$
r^2, r^{-2}	$(\cdot\cdot\cdot)(\cdot\cdot\cdot)$
r^3	$(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)$
s_1, s_3, s_5	$(\cdot)(\cdot)(\cdot\cdot)(\cdot\cdot)$
s_2, s_4, s_6	$(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)$

$$Z_{D_6} = \frac{1}{12}\Big(t_1^6 + 2t_6 + 2\, t_3^2 + t_2^3 + 3\, t_1^2 t_2^2 + 3\, t_2^3 \Big)$$

$$= \frac{1}{12}\left(t_1^6 + 2t_6 + 2t_3^2 + 4t_2^3 + 3t_1^2 t_2^2 \right)$$

$$\#\text{necklaces} = Z_{D_6}(m, m, \dots, m)$$

$$= \text{polynomial in } m$$

$$= \frac{1}{12}\left(m^6 + 3m^4 + 12m^3 + 8m^2 \right)$$

$$m = 2 \longrightarrow 13$$