Dec 30, 1991

## § Factorization Lemma

Let $\mathfrak{h}_g := \{ Z \in \mathbb{C}^{g\times g} : \operatorname{Im} Z > 0 \}$ be the Siegel upper half space of genus $g$ and let

$$Sp_{2g}(R) := \{ M \in R^{2g\times 2g} : {}^tM J M = J \},$$

for any commutative ring with identity $R$, where

$$J = \left( \begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array} \right).$$

As it is well known, $Sp_{2g}(\mathbb{R})$ acts on $\mathfrak{h}_g$ via

$$Z \longmapsto M \circ Z := (AZ+B)(CZ+D)^{-1}$$

where $M \in Sp_{2g}(\mathbb{R})$ is given by $M = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$ in block form.

Consider the standard theta series

$$\theta^{(g)}(u,Z) = \theta(u,Z) := \sum_{n\in\mathbb{Z}^g} e^{\pi i\, {}^tn Z n}\, e^{2\pi i\, {}^tn \cdot u}$$

vectors considered as column vectors

for $u \in \mathbb{C}^g$, $Z \in \mathfrak{h}_g$.

It is known that it transforms as a modular form under the action of the group

$$\Gamma^{(g)} = \Gamma_\theta := \left\{ M \in Sp_{2g}(\mathbb{Z}) : M = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \text{ diagonals of } {}^tAC \text{ and } {}^tBD \text{ are even} \right\}$$

of finite index in $\Gamma^{(g)} = \Gamma := Sp_{2g}(\mathbb{Z})$

Explicitly, for $M \in \Gamma_\theta$, $M = \left(\frac{A \mid B}{C \mid D}\right)$ we have

$$\theta\left( {}^t(CZ+D)^{-1} \cdot u,\ (AZ+B)(CZ+D)^{-1} \right)$$
$$= \zeta_M \cdot \det(CZ+D)^{1/2} \cdot e^{\pi i ( {}^t u \cdot (CZ+D)^{-1} C \cdot u)}\ \theta(u, Z)$$

where $\zeta_M^8 = 1$ and depends only on $M$ once we fix a branch for the square root for $CZ+D$.

We will drop the superindices $(g)$ if there is no risk of confusion.

We now specialize the above to the following.

Given $\tau_1, \tau_2 \in \mathfrak{h}_1$ consider

$$Z = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in \mathfrak{h}_2$$

and

$$M = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{array}\right) \in \Gamma_\theta^{(2)} \quad \checkmark$$

we find

$$AZ+B = \begin{pmatrix} \tau_1 & 0 \\ 0 & -1 \end{pmatrix} \qquad {}^t(CZ+D)^{-1} = \xi \begin{pmatrix} \tau_2 & -\tau_1 \\ 1 & 1 \end{pmatrix}$$

$$CZ+D = \begin{pmatrix} 1 & -1 \\ \tau_1 & \tau_2 \end{pmatrix} \qquad (CZ+D)^{-1} C = \xi \begin{pmatrix} \tau_2 & 1 \\ -\tau_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$
$$= \xi \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

So $$M \circ Z = \xi \cdot \begin{pmatrix} \tau_1 \tau_2 & \tau_1 \\ \tau_1 & -1 \end{pmatrix}$$

where, for convinience, we put $\xi = (\tau_1+\tau_2)^{-1}$.

Plugging all of this into the transformation formula for $\theta^{(2)}$ we get

$$\theta^{(2)}\left( u_1 \xi \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix} + u_2 \xi \begin{pmatrix} -\tau_1 \\ 1 \end{pmatrix}, \ \xi \cdot \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_1 & -1 \end{pmatrix} \right) =$$

$$= \zeta_M \cdot \xi^{-1/2} \cdot e^{\pi i \xi (u_1+u_2)^2} \, \theta^{(1)}(u_1,\tau_1)\, \theta^{(1)}(u_2,\tau_2)$$

(It is clear that $\theta^{(2)}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}\right)$ factors into $\theta^{(1)}(u_1,\tau_1)\,\theta^{(1)}(u_2,\tau_2)$ ).

Notice now that if

$$u_1 = t\tau_1 + s$$
$$u_2 = t\tau_2 - s$$

with $s, t \in \mathbb{C}$, then

$${}^t(CZ+D)^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}.$$

$$\xi \begin{pmatrix} \tau_2 & -\tau_1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \tau_1 & 1 \\ \tau_2 & -1 \end{pmatrix}\begin{pmatrix} t \\ s \end{pmatrix} = \xi \begin{pmatrix} 0 & \tau_1+\tau_2 \\ \tau_1+\tau_2 & 0 \end{pmatrix}\begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$$

So replacing in the equality above $u_1$ by $u_1 + t\tau_1 + s$ and $u_2$ by $u_2 + t\tau_2 - s$ we get

$$\theta^{(2)}\left(u_1 \xi \binom{\tau_2}{1} + u_2 \xi \binom{-\tau_1}{1} + \binom{s}{t}, \xi \begin{pmatrix}\tau_1\tau_2 & \tau_1 \\ \tau_1 & -1\end{pmatrix}\right) =$$

$$= \zeta_M \cdot \xi^{-1/2} \cdot e^{\pi i \xi (u_1+u_2+t/\xi)^2} \cdot \theta^{(1)}(u_1+t\tau_1+s,\tau_1)\,\theta^{(1)}(u_2+t\tau_2-s,\tau_2)$$

Finally we rearrange the exponential factor to get

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$$e^{-2\pi i \xi u_1 u_2}\, \theta^{(2)}\left(u_1 \xi \binom{\tau_2}{1} + u_2 \xi \binom{-\tau_1}{1} + \binom{s}{t}, \xi \begin{pmatrix}\tau_1\tau_2 & \tau_1 \\ \tau_1 & -1\end{pmatrix}\right) =$$

$$= \zeta_M\, \xi^{-1/2} \cdot e^{\pi i t^2/\xi} \cdot \left[ e^{\pi i \xi u_1^2} e^{2\pi i t u_1}\, \theta^{(1)}(u_1+t\tau_1+s,\tau_1)\right]$$

$$\cdot \left[ e^{\pi i \xi u_2^2} e^{2\pi i t u_2}\, \theta^{(1)}(u_2+t\tau_2-s,\tau_2)\right]$$

We want now to expand both sides in a Taylor $(u_1 u_2)^n$ series in $u_1, u_2$ and equate the coefficients to find the identities we are looking for.

Let us start with the left hand side. The terms involving $u_1$ or $u_2$ in the series defining

$$\theta^{(2)}\left(u_1 \xi \binom{\tau_2}{1} + u_2 \xi \binom{-\tau_1}{1} + \binom{s}{t}, \xi \begin{pmatrix}\tau_1\tau_2 & \tau_1 \\ \tau_1 & -1\end{pmatrix}\right)$$

are of the form

$$e^{2\pi i \xi (n_1 \tau_2 + n_2) u_1} \cdot e^{2\pi i \xi (-n_1 \tau_1 + n_2) u_2}$$

where $(n_1, n_2) \in \mathbb{Z}^2$.

We hence need an expansion for

$$e^{-2\pi i \xi u_1 u_2} \; e^{2\pi i \xi (n_1 \tau_2 + n_2) u_1} \; e^{2\pi i \xi (-n_1 \tau_1 + n_2) u_2}$$

using the expansion for the exponential we get

$$\sum_{n \geq 0} \frac{(-2\pi i \xi u_1 u_2)^n}{n!} \sum_{m_1 \geq 0} \frac{(2\pi i \xi (n_1 \tau_2 + n_2) u_1)^{m_1}}{m_1!} \cdot \sum_{m_2 \geq 0} \frac{(2\pi i \xi (-n_1 \tau_1 + n_2) u_2)^{m_2}}{m_2!}$$

Keeping only the terms $u_1^k u_2^j$ with $k = j$ (the diagonal) we get

$$\sum_{n \geq 0} \frac{(-2\pi i \xi u_1 u_2)^n}{n!} \cdot \sum_{m \geq 0} \frac{((2\pi i \xi)^2 (n_1 \tau_2 + n_2)(-n_1 \tau_1 + n_2))^m}{(m!)^2} \cdot (u_1 u_2)^m$$

$$= \sum_{n \geq 0} \left( \sum_{j=0}^{n} \frac{(-1)^{n-j} \left[ (n_1 \tau_2 + n_2)(-n_1 \tau_1 + n_2) 2\pi i \xi \right]^j}{(n-j)!\,(j!)^2} \right) (2\pi i \xi u_1 u_2)^n$$

$$= \sum_{n \geq 0} \left( \sum_{j=0}^{n} (-1)^j \binom{n}{j} \cdot \frac{\left[ (n_1 \tau_2 + n_2)(-n_1 \tau_1 + n_2) 2\pi i \xi \right]^j}{j!} \right) (-1)^n \frac{(2\pi i \xi u_1 u_2)^n}{n!}$$

Recall the definition of the Laguerre polynomials

$$L_n^{\alpha}(x) = \sum_{j=0}^{n} \frac{(-x)^j}{j!\,(n-j)!} \frac{\Gamma(n+\alpha+1)}{\Gamma(j+\alpha+1)}$$

so that

$$L_n(x) = L_n^0(x) = \sum_{j=0}^{n} \frac{(-x)^j}{j!} \binom{n}{j}$$

So in terms of $L_n$ we get

$$= \sum_{n\geq 0} L_n\left(2\pi i \xi (n_1\tau_2+m_2)(-n_1\tau_1+m_2)\right)\cdot(-1)^n \frac{(2\pi i \xi u_1 u_2)^n}{n!}$$

Let now turn to the right hand side; both of its factors are similar so it will suffice to look at one, say the one involving $u_1$. Again the terms involving $u_1$ are

$$e^{\pi i \xi u_1^2}\, e^{2\pi i t u_1}\, e^{2\pi i n_1 u_1}$$

$$= e^{\pi i \xi u_1^2}\, e^{2\pi i (t+n_1) u_1}$$

To expand this we use the identity

$$e^{2xy-y^2} = \sum_{n\geq 0} H_n(x) \frac{y^n}{n!}$$

where

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \, n!}{k!\,(n-2k)!} (2x)^{n-2k}$$

are the Hermite polynomials.

We need

$$y = \sqrt{-\pi i \xi}\, u_1$$
$$x = \sqrt{-\pi i/\xi}\,.(t+n_1)$$

then

$$e^{\pi i \xi u_1^2}\, e^{2\pi i (t+n_1) u_1} = \sum_{n \geq 0} H_n\left(\sqrt{-\pi i/\xi}\,.(t+n_1)\right) \frac{\left(\sqrt{-\pi i \xi}\, u_1\right)^n}{n!}$$

Putting the other factor and writing only the diagonal terms $(u_1 u_2)^n$ we get

$$\sum_{n \geq 0} H_n\left(\sqrt{\tfrac{-\pi i}{\xi}}\,.(t+n_1)\right).\ H_n\left(\sqrt{\tfrac{-\pi i}{\xi}}\,(t+n_2)\right) \frac{(-1)^n}{2^n\, n!} \frac{(2\pi i \xi u_1 u_2)^n}{n!}$$

Performing the sum over $(n_1, n_2) \in \mathbb{Z}^2$ and equating coefficients of $(u_1 u_2)^n$ we get

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} e^{2\pi i(s n_1 + t n_2)} L_m\big(2\pi i \xi (n_1\tau_2+n_2)(-n_1\tau_1+n_2)\big) \cdot e^{\pi i \xi(\tau_1\tau_2 n_1^2 + 2\tau_1 n_1 n_2 - n_2^2)}$$

$$= \frac{J_M \cdot \xi^{-1/2}}{2^m m!} \sum_{m_1\in\mathbb{Z}} e^{2\pi i s m_1} \cdot H_m\left(\sqrt{\tfrac{-\pi i}{\xi}}\cdot(t+m_1)\right) e^{\pi i (t+n_1)^2\tau_1} \cdot \sum_{m_2\in\mathbb{Z}} e^{-2\pi i s m_2} H_m\left(\sqrt{\tfrac{-\pi i}{\xi}}(t+m_2)\right) e^{\pi i (t+n_2)^2\tau_2}$$

We will now simplify these fmlas somewhat according to wether $n$ is odd or even.

$\xi = \frac{1}{2iy}$

$\xi^{-1} = 2iy$

$n$ even

From the definiton of $H_{2n}$ we see that

$$H_{2m}(x) = h_{2m}((2x)^2)$$

where

$$h_{2n}(x) = \sum_{k=0}^{m} (-1)^k \frac{(2n)!}{k!\,(2n-2k)!} x^{m-k}$$

So we can rewrite the first factor above as

$$\sum_{n_1 \in \mathbb{Z}} e^{2\pi i s n_1} h_{2m}\left(-\frac{4\pi i}{\xi}\cdot(t+n_1)^2\right) e^{\pi i (t+n_1)^2 \tau_1}$$

and similarly the second one.

<u>$n$ odd</u>

Now

$$H_{2n+1}(x) = h_{2n+1}\left((2x)^2\right)\cdot 2x$$

where

$$h_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k\,(2n+1)!}{k!\,(2n-2k+1)!}\cdot x^{n-k}$$

So the first factor becomes

$$2\sqrt{\frac{-\pi i}{\xi}}\cdot\sum_{n_1 \in \mathbb{Z}} e^{2\pi i s n_1} h_{2n+1}\left(-\frac{4\pi i}{\xi}(t+n_1)^2\right)\cdot(t+n_1)\cdot e^{\pi i (t+n_1)^2 \tau_1}$$

and similarly the second one.

We specialize further by taking $t=\frac{1}{2}$ and $s=0$ or $\frac{1}{2}$.

In the case $s=0$ only the terms with even $n$ survive and if $s=\frac{1}{2}$ the odd ones. We get the following then.

$t = 1/2,\ s = 0$

even terms

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} (-1)^{n_2} L_{2m}(2\pi i \xi (n_1\tau_2 + n_2)(-n_1\tau_1 + n_2)) \cdot e^{\pi i \xi (\tau_1\tau_2 n_1^2 + 2\tau_1 n_1 n_2 - n_2^2)}$$

$$= \frac{s_M \cdot \xi^{-1/2}}{2^{2(m-1)}(2m)!} \cdot \sum_{n_1\geq 0} h_{2m}\left(-\frac{\pi i}{\xi} \cdot (2n_1+1)^2\right) \cdot e^{\pi i (n_1+1/2)^2 \tau_1} \cdot \sum_{n_2 \geq 0} h_{2m}\left(-\frac{\pi i}{\xi}(2n_2+1)^2\right) e^{\pi i (n_2 + 1/2)^2 \tau_2}$$

$t = 1/2,\ s = 1/2$

odd terms

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} (-1)^{n_1+n_2} L_{2m+1}(2\pi i \xi (n_1\tau_2 + n_2)(-n_1\tau_1 + n_2))\, e^{\pi i \xi (\tau_1\tau_2 n_1^2 + 2\tau_1 n_1 n_2 - n_2^2)}$$

$$= \frac{s_M \cdot (-\pi i) \cdot \xi^{-3/2}}{2^{2m-1} \cdot (2m+1)!} \sum_{n_1 \geq 0} (-1)^{n_1} h_{2m+1}\left(-\frac{\pi i}{\xi}(2n_1+1)^2\right) \cdot (2n_1+1)\, e^{\pi i (n_1+\frac{1}{2})^2 \tau_1} \sum_{n_2\geq 0} (-1)^{n_2} h_{2m+1}\left(-\frac{\pi i}{\xi}(2n_2+1)^2\right) \cdot (2n_2+1) \cdot e^{\pi i (n_2+\frac{1}{2})^2 \tau_2}$$

## Example

Take $d=7$ $\qquad \tau_1 = \frac{-1+\sqrt{-7}}{2} \qquad \tau_2 = \frac{1+\sqrt{-7}}{2}$

$t = 1/2 \qquad s=0 \qquad$ so odd terms are zero.

$\tau_1\tau_2 = -2$

$\xi = (\tau_1+\tau_2)^{-1} = 1/\sqrt{-7}$

$(n_1\tau_2+n_2)(-n_1\tau_1+n_2) = 2n_1^2 + n_1n_2 + n_2^2$

$\xi(\tau_1\tau_2 n_1^2 + 2\tau_1 n_1 n_2 - n_2^2) = -(2n_1^2+n_1n_2+n_2^2)/\sqrt{-7} + n_1n_2$

So $(-1)^{n_2} e^{\pi i \xi(\tau_1\tau_2 n_1^2 + 2\tau_1 n_1 n_2 - n_2^2)} = (-1)^{n_2+n_1n_2} e^{-\pi(2n_1^2+n_1n_2+n_2^2)/\sqrt{7}}$

$= e^{2\pi i(2n_1^2+n_1n_2+n_2^2)\left(\frac{-7+\sqrt{-7}}{2\times 7}\right)}$

So the l.h.s. gives

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} L_{2m}\left(\frac{2\pi}{\sqrt{7}}(2n_1^2+n_1n_2+n_2^2)\right) \cdot e^{2\pi i(2n_1^2+n_1n_2+n_2^2)\cdot\left(\frac{-7+\sqrt{-7}}{2\times 7}\right)}$$

$$= \frac{\sqrt{7}}{\pi} \cdot L(\psi^{4n+1}, 2n+1)$$

(according to calculation with Zagier)

On the r.h.s. we get

$$\frac{\zeta_M \cdot \sqrt[4]{-7}\, e^{\frac{\pi i}{4}}}{2^{2(n-1)}(2m)!} \cdot \Big( \underbrace{\sum_{n_1 \geq 0} h_{2m}\left(\pi\sqrt{7}\,(2n_1+1)^2\right) e^{\pi i (n_1+\frac{1}{2})^2 \left(\frac{-1+\sqrt{-7}}{2}\right)}}_{\ell_{2m}} \Big)^2$$

so let

$$L_{2m} := L(\psi^{4n+1}, 2n+1)$$

then

$$L_{2m} = \frac{\zeta_M \cdot \pi \sqrt[4]{-7}\, e^{\frac{\pi i}{4}}}{2^{2(n-1)}(2m)!\sqrt{7}} \cdot \ell_{2m}^2$$

Putting all the normalizing factors we get

$$\frac{1}{2}\,\frac{L_{2m}\,(2m)!}{\Omega^{4n+1}\left(\frac{\sqrt{7}}{2\pi}\right)^{2m}} = \zeta\,\frac{\pi\sqrt[4]{7}\,\ell_{2m}^2}{\sqrt{7}\,2^{2m-1}\,\Omega^{4n+1}\left(\frac{\sqrt{7}}{2\pi}\right)^{2m}} \qquad ; \ \zeta^8 = 1$$

$$= \frac{\zeta\, 2\sqrt[4]{7}\,\ell_{2n}^2}{\Omega^{4n+1}\left(\frac{\sqrt{7}}{\pi}\right)^{2n+1}}$$

$$z = \frac{-1+\sqrt{-7}}{2}$$

$$y = \mathrm{Im}(z) = \frac{\sqrt{7}}{2}$$

$$4\pi y = 2\sqrt{7}\,\pi$$

$$\boxed{b_{2n+1} = \left(\frac{\zeta^{1/2}\sqrt{2}\,\sqrt[8]{7}\,\ell_{2n}}{\Omega^{2n+1/2}\left(\frac{\sqrt{7}}{\pi}\right)^{n+1/2}}\right)^2}$$

(we're off by a $\frac{1}{2}$ !?)

$$\frac{2\sqrt[4]{7}\,\pi^{2n+1}}{\Omega^{4n+1}\sqrt{7}^{\,2n+1}} = \frac{2\sqrt[4]{7}\,\pi^{2n+1}}{\Omega^{4n+1}\,2\,\frac{\pi^{4n+1}}{(\sqrt[4]{7})^{4n+1}}\cdot\sqrt{7}}$$

Jan 4, 1991

The factors we are expanding in the rhs of the main fmla

$$e^{\pi i \xi u_1^2} \sum_{n_1 \in \mathbb{Z}} e^{2\pi i (t+n_1) u_1} e^{2\pi i n_1 s} e^{\pi i (t+n_1)^2 \tau_1}$$

and

$$e^{\pi i \xi u_2^2} \sum_{n_2 \in \mathbb{Z}} e^{2\pi i (t+n_2) u_2} e^{-2\pi i n_2 s} e^{\pi i (t+n_2)^2 \tau_2}$$

The ~~mth~~ coefficient of $u_1^m$ (in the first factor) is

$$\sum_{n_1 \in \mathbb{Z}} H_m\left(\sqrt{\frac{-\pi i}{\xi}}\,(t+n_1)\right) e^{2\pi i n_1 s} e^{\pi i (t+n_1)^2 \tau_1} \cdot \frac{(\sqrt{-\pi i \xi}\,)^m}{m!}$$

As in Jordan p. 415 we replace $u_1$ by $\frac{u_1}{\Omega_0}$

we get

<u>n even</u>

$$\left(\sum_{n_1 \in \mathbb{Z}} h_{2n}\left(-\frac{4\pi i}{\xi}(t+n_1)^2\right) e^{2\pi i s n_1} e^{\pi i (t+n_1)^2 \tau_1}\right) \cdot \frac{(-\pi i \xi)^n}{\Omega_0^{2n}(2n)!}$$

<u>n odd</u>

$$\left(\sum_{n_1 \in \mathbb{Z}} (t+n_1)\, h_{2n+1}\left(-\frac{4\pi i}{\xi}(t+n_1)^2\right) e^{2\pi i s n_1} e^{\pi i (t+n_1)^2 \tau_1}\right) \cdot \frac{2 \cdot (-\pi i \xi)^n}{\xi \cdot \Omega_0^{2n+1}(2n+1)!}$$