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§1 Brief summary of the conjectures of HLRV

$\mu = (\mu^1, \dots, \mu^k)$ $|\mu^i| = n$, $i = 1, \dots, k$
$\mu^i = \mu^i_1 \geq \mu^i_2 \geq \cdots$
curve genus g, k punctures

Character variety

$$\mathcal{M}_\mu := \{ [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_k = I_n \} // Gl_n$$

$x_i, y_i \in GL_n$, $z_i \in C_i$ conjugacy class in Gl_n

$[x, y] = x y x^{-1} y^{-1}$

$$C_i \sim \begin{pmatrix} \overbrace{\alpha_1^i \ddots \alpha_1^i}^{\mu_1^i} & & 0 \\ & \overbrace{\alpha_2^i \ddots \alpha_2^i}^{\mu_2^i} & \\ 0 & & \ddots \end{pmatrix}$$

$\alpha_1^i, \alpha_2^i, \dots$ distinct

purely topological

generic choice of eigenvalues

We wanted to compute $\# \mathcal{M}_\mu (\mathbb{F}_q)$. [To use Weil conjectures to get to the geometry

If non-empty $\mathcal{M}_\mu$ is an affine non-singular of pure dimension

$$d_\mu := (2g + k - 2) n^2 - \sum_{i,j} (\mu^i_j)^2 + 2$$

[Quiver picture

puncture ↕ leg

It is hard to deal with these varieties directly in general. For $n = 2$, $k = 4$ and $\mu^i = 1^2$, $i = 1, \dots 4$ we have

$$\tilde{D}_4 \quad \begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 1 & - & 2 & - & 1 \\ & & | & & \\ & & 1 & & \end{array}$$

$$\mathcal{M}_\mu \simeq S \setminus \{ \times \} =: S^0$$

where S is a smooth projective cubic surface.

This boils down to Fricke's relation between

$a_i := tr(A_i)$, $i = 1, 2, 3$ $a_4 := tr(A_1 A_2 A_3)$

(Hitchin slides)

$x_i := tr(A_j A_k)$, $\{i, j, k\} = \{1, 2, 3\}$

One equation

$$0 = x_1 x_2 x_3 + \text{quadr in } x_i \text{ w/coeffs in } a_i$$

For a cubic surface and generic finite field we have

$$\# S(\mathbb{F}_q) = q^2 + 7q + 1 \quad \text{polynomial count}$$

matching the Betti numbers 1 7 1 as S is polynomial count. Here generic means Frob$_q$ acts trivially on the 27 lines on S, which span H^2.

Hence:

$$\# M_\mu(\mathbb{F}_q) = q^2+7q+1 - (3(q-1)+3)$$
$$= q^2+4q+1$$

To express our result for general μ let

$$\Omega(x^1, \dots, x^k; q, t) := \sum_\lambda \frac{\tilde{H}_\lambda(x^1; q,t) \cdots \tilde{H}_\lambda(x^k; q,t)}{a_\lambda(q,t)},$$

where

$\tilde{H}_\lambda(x; q,t) =$ (modified) Macdonald polynomial

$$a_\lambda(q,t) := \langle \tilde{H}_\lambda, \tilde{H}_\lambda \rangle_* = \prod (q^{a+1} - t^l)(q^a - t^{l+1})$$

From now on we set $g=0$.

Define

$$\mathbb{H} := (q-1)(1-t) \operatorname{Log} \Omega$$
$$= \sum_{n \geq 1} \mathbb{H}_n$$

Log = plethystic logarithm

$$(1 - \text{monomial})^{-1} \longmapsto \text{monomial}$$
$$\cdot \longmapsto +$$

$\mathbb{H}_n(x^1, \dots, x^k)$ is a symmetric function in the separate set of variables $x^1, \dots, x^k$ with coefficients in $\mathbb{Q}(q,t)$ but in fact in $\mathbb{Q}[q,t]$ (Anton)

Define further:

$\mathbb{H}_\mu := \langle \mathbb{H}_n, h_\mu \rangle$ (picks up coefficient of m_μ)

$h_\mu := h_{\mu^1}(x^1) \cdots h_{\mu^k}(x^k)$

1

4 1

E.g. $\mathbb{H}_{(1^2)(1^2)(1^2)(1^2)} = q + t + 4,$

<u>Euler specialization</u>

<u>THM</u>

$$\# \mathcal{M}_\mu(\mathbb{F}_q) = q^{\frac{1}{2} d_\mu} \mathbb{H}_\mu\left(q, \frac{1}{q}\right)$$

<u>Cor</u> The E-polynomial of $\mathcal{M}_\mu$ equals the rhs.

<u>Pf</u> It is a consequence of a result of Katz about polynomial count varieties □

Conclude several geometric facts about $\mathcal{M}_\mu$: E is palindromic, Euler characteristic of $\mathcal{M}_\mu / GL_1^{2g}$ for $g > 1$, connectedness if non-empty, ...

<u>Conjecture</u>

- $\sum_{i,k} h_c^{iik}(\mathcal{M}_\mu) q^i t^k = (qt^2)^{\frac{1}{2} d_\mu} \mathbb{H}_\mu\left(\frac{1}{q}, t^2 q\right)$
- $h_c^{ijk}(\mathcal{M}_\mu) = 0$ if $i \neq j$

For our running example $d_\mu = 2$

$(qt^2)\left(\frac{1}{q} + qt^2 + 4\right) = (1 + 4q)t^2 + q^2 t^4$

this matches a calculation of the MHS of S^0 (Anton)

$\mathcal{M}_\mu$ diffeom. some moduli space of parab. Higgs bundles on a curve of genus g. On the Higgs side the complex structure on our curve matters.

The cohomology on the Higgs side is pure.

§2 Connection to Hilbertschemes

Similar fmlae to our Ω appear in work of Garsia-Haiman on $H_n :=$ Hilb$_n(\mathbb{C}^2)$. Concretely, applying Atiyah-Bott to H_n and $\mathcal{F} = \mathcal{O}_{Z_n} \otimes P$ where $Z_n = \sigma^{-1}(0)$, $\sigma : H_n \longrightarrow \mathbb{C}^{2n}/S_n$ zerofiber and $P =$ Procesi bundle on H_n.

The contribution to $\chi(\mathcal{F})$ from the fixed point I_μ, $|\mu| = n$ is

$$(-1)^{n-1}\frac{(1-q)(t-1)\,\Pi_\mu\,\phi_\mu\,\tilde{H}_\mu(x)\,q^{n(\mu')}t^{n(\mu)}}{a_\mu}$$

$$\phi_\mu(q,t) = \sum_{i,j} q^{j-1}t^{i-1}$$

$$\Pi_\mu(q,t) = \prod_{i,j}{}' (1-q^it^j), \qquad (i,j)\neq(0,0)$$

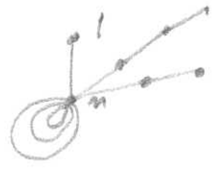
In partic. the (q,t)-Catalan polynomials correspond to $\mathcal{F} = \mathcal{O}_{Z_n} \otimes \mathcal{O}(m)$ or $m \geq 1$

$$C_n^{(m)}(q,t) = \sum_{|\mu|=n} (-1)^{n-1}\frac{(1-q)(t-1)\Pi_\mu\,\phi_\mu\,(q^{n(\mu')}t^{n(\mu)})^{m+1}}{a_\mu}$$

I claim that these appear naturally related to $\mathbb{H}_n$. More precisely, with $k = m+3$

$$(-1)^{n-1}C_n(q,t) = (-1)^{mn}\langle \mathbb{H}_n, P_n \otimes h_{(n-1,1)} \otimes \underbrace{S_{(1^n)} \otimes \cdots \otimes S_{(1^n)}}_{m+1}\rangle$$

[On quiverside: $h_{(n-1,1)}$ corresponds to a short leg



Pf sketch

i) Write $\Omega = \sum_\lambda A_\lambda \tilde{H}_\lambda(x,\cdot q,t)\,T^{|\lambda|}$

Then

$$\langle \operatorname{Log} \Omega, \sum_{n\geq 1} h_{(n-1,1)}(x) \rangle = \sum_{\lambda} A_\lambda \phi_\lambda(q,t) T^{|\lambda|} \Big/ \sum_{\lambda} A_\lambda T^{|\lambda|}$$

since

$$\tilde{H}_\lambda(1, y, 0, \cdots ; q, t) = 1 + \phi_\lambda(q,t) y + O(y^2)$$

and hence

$$\operatorname{Log} \Omega = \operatorname{Log}\Big(\sum_\lambda A_\lambda T^{|\lambda|} + \sum_\lambda A_\lambda \phi_\lambda(q,t) T^{|\lambda|} y + O(y^2)\Big)$$

the coefficient of y equals

$$\sum_\lambda A_\lambda \phi_\lambda(q,t) T^{|\lambda|} \Big/ \sum_\lambda A_\lambda T^{|\lambda|}$$

[Tamas fmla

Pairing with $h_{(n-1,1)}(x)$ is to pick the coefficient of $m_{(n-1,1)}$. For $x = (1, y, 0, \cdots)$ this is the coefficient of y.

ii) Consider the u-specialization on Λ

$$p_r(x) \mapsto 1 - u^r$$

denote it by $f \mapsto f[1-u]$ (a plethystic operation)

Also let $[f]$ = coefficient of degree n in $f[1-u]$ (a polynomial of degree $\leq n$ for $f \in \Lambda_n$)

We have

$$\langle f, p_n \rangle = f[1-u] / 1-u \Big|_{u=1}$$

$$[f] = (-1)^n \langle f, s_{(1^n)} \rangle$$

$$[\tilde{H}_\lambda(x;q,t)] = q^{n(\lambda')}t^{n(\lambda)}$$

There seems to be a dictionary between natural questions on the Hilbert scheme side and on the Higgs/character variety side.

For example, on H_n we $\overset{\text{may}}{}$ take $\mathcal{F} = \mathcal{O}_{Z_n} \otimes B^{\otimes r-1} \otimes \mathcal{O}(s-1)$ on the other side this corresponds to pairing with

$$p_{(n)} \otimes h_{(n-1,1)}^{\otimes r} \otimes s_{(1^n)}^{\otimes s}$$

In the Euler specialization this counts the number of solutions in $GL_n(\mathbb{F}_q)$ to

$$z = w_1 \cdots w_r u_1 \cdots u_s$$

where $z \sim \begin{pmatrix} \alpha & & 0 \\ & \alpha^q & \\ 0 & & \ddots \; \alpha^{q^{n-1}} \end{pmatrix} / \overline{\mathbb{F}_q}$

w_i = reflection

u_j = unipotent

This is a q-analogue of the standard Hurwitz numbers. Lewis-Reiner-Stanton

Garsia-Haiman prove that $C_m^{(m)}(q,t)$ have non-negative coefficients. This is not $\overset{\text{a priori}}{}$ implied by our conjectures. On the Hilbert side it follows from the vanishing of the higher cohomology gps for the sheaves in question. For us it seems the repn of $S_n \times \cdots \times S_n$ on the legs is, up to a twist by sign, a permutation representation. This would imply traces of elements are non-negative.

⑦

§3 Extended qt-Catalan

It seems natural to define

$$C_\lambda(q,t) := \langle H|_n ,\ P_{(n)} \otimes h_{(n-1,1)} \otimes S_{\lambda^1} \otimes \cdots \otimes S_{\lambda^{k-2}} \rangle$$

Cauchy's fmla for Macdonald polyn

For $k=2$ gives

$$C_\phi(q,t) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$$

$\lambda := (\lambda^1, \ldots, \lambda^{k-2})$

GH prove that for $k=3$

$$C_\lambda(q,t) = \begin{cases} 1 & \text{if } \lambda = (1^n) \\ 0 & \text{otherw.} \end{cases}$$

Since $\langle \tilde H_\mu, S_\lambda \rangle = \tilde K_{\lambda,\mu}(q,t)$ the Kostka polynomials we have

$$C_\lambda = (-1)^{n-1} \sum_\mu (q-1)(1-t) \frac{\Pi_\mu \Phi_\mu}{a_\mu} \prod_{i=1}^{k-2} \tilde K_{\lambda^i \mu}(q,t)$$

Define C_λ^ν by

$$\prod_{i=1}^{k-2} \tilde K_{\lambda^i \mu} = \sum_\nu C_\lambda^\nu \tilde K_{\nu\mu}$$

for all μ. Then

$$C_\lambda(q,t) = C_\lambda^{(1^n)}(q,t)$$

These C_λ^ν for $k=4$ are structure constants for a commutative algebra on Λ_n. It is obtained by dualizing the coproduct

[New structure?

$$\Delta \tilde H_\lambda := \tilde H_\lambda \otimes \tilde H_\lambda$$

wrt the usual Hall inner product.

Then

$$\Delta S_\nu = \sum_{\rho,\eta} c^{\nu}_{\rho,\eta} \, S_\rho \otimes S_\nu$$

If we dualize wrt $\langle \cdot, \cdot \rangle_*$ in the sense

$$\langle \tilde{H}_\mu, f \rangle_* \langle \tilde{H}_\mu, g \rangle_* = \langle \tilde{H}_\mu, f \# g \rangle$$

then the $\tilde{H}_\mu / a_\mu$ are idempotents. Let $\tau : \Lambda \to \mathbb{C}$ be the trace function

$$\tau : \tilde{H}_\mu \mapsto 1$$

then

$$\tau(f \# g) = \sum_\lambda \frac{1}{a_\lambda} \langle \tilde{H}_\lambda, f \# g \rangle_*$$
$$= \langle \sum_\lambda \frac{\tilde{H}_\lambda \otimes \tilde{H}_\lambda}{a_\lambda}, f \otimes g \rangle_*$$

More generally,

$$\tau(f_1 \# \cdots \# f_k) = \langle \Omega, f_1 \otimes \cdots \otimes f_k \rangle_*$$

We have a Frobenius algebra structure on Λ_n hence a TQFT. The value assigned to a closed Riemann surface of genus g is

$$\tau(\omega^g) = \sum_\lambda a_\lambda^{g-1}$$

where ω is the handle operator

$$\omega := \sum_\lambda \tilde{H}_\lambda$$

This value equals a specialization of a general form of the Ω series, conjecturally related to the Higgs moduli space.