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Λ = symmetric functions

- $\langle , \rangle$ on Λ characterized by

pos definite

dual basis u_λ, v_λ

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(x) v_\lambda(y)$$

- $\omega : \Lambda \longrightarrow \Lambda$ is an isometry

- $\langle h_\lambda, m_\mu \rangle = \delta_{\mu\lambda}$

$$\langle p_\lambda, p_\lambda \rangle = \frac{1}{z_\lambda}$$

- Schur functions s_λ

$$s_\lambda = a_{\lambda+\delta} / a_\delta$$

quotient of two skew-symmetric

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ $\alpha_i \in \mathbb{Z}_{\geq 0}$

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

$$a_\alpha = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, \sigma(x^\alpha)$$

$$\sigma(x^\alpha) := x_1^{\alpha_{\sigma(1)}} x_2^{\alpha_{\sigma(2)}} \cdots \cdots x_n^{\alpha_{\sigma(n)}}$$

$$= \det\left(x_j^{\alpha_i} \right)$$

If $\alpha_i = \alpha_j$ for any $i \neq j$ then $a_\alpha = 0$

may assume $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$

$n = 5$

$$\delta = (n-1, n-2, \ldots, 1, 0)$$

$$\alpha = \lambda + \delta \qquad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots$$

λ is a partition of length $\ell(\lambda) \leq n$

$$a_\alpha = \det\left(x_i^{\lambda_j + n - j} \right)$$

In partic. if $\alpha = \delta$.

$$a_\delta = \det (x_i^{n-j})$$
$$= \text{Vandermonde}$$
$$= \prod_{i<j} (x_i - x_j)$$

$a_\alpha = a_{\lambda+\delta}$ are $\mathbb{Z}$-basis for skew-symmetric polynomials in $x_1, \dots x_n$

$\ell(\lambda) \leq n$ $\qquad \Lambda_n$

$a_\delta \mid a_{\lambda+\delta}$

Multiplying by a_δ : skew-symmetric $\updownarrow$ Λ_n

$$\boxed{s_\lambda := \frac{a_{\lambda+\delta}}{a_\delta}} \qquad \ell(\lambda) \leq n.$$

s_λ is a $\mathbb{Z}$-basis of Λ_n

Fix λ and increase n the s_λ's are compatible $n \to \infty$

$$s_\lambda \in \Lambda$$

s_λ a $\mathbb{Z}$-basis of Λ

<u>Claim</u> $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

<u>Pf</u> We need to check

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

We know

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda h_\lambda(x) m_\lambda(y)$$

$$a_\delta(x) a_\delta(y) \prod_{i,j} (1 - x_i y_j)^{-1} = a_\delta(x) \sum_{\sigma, \alpha} h_\alpha(x) \varepsilon(\sigma) y^{\alpha + \sigma(\delta)}$$

$$a_\delta(y) = \sum_\sigma \varepsilon(\sigma) y^{\sigma(\delta)}$$

with $e_r = 0$ if $r < 0$

$$E^{(k)}(t) := \prod_{\substack{i=1 \\ i \neq k}}^{n} (1 + x_i t) = \sum_{r=0}^{n-1} e_r^{(k)}(x) t^r$$

$$H(t) := \prod_{i=1}^{n} (1 - x_i t)^{-1} = \sum_{r=0}^{n} h_r(x) t^r$$

$$H(t) E^{(k)}(-t) = (1 - x_k t)^{-1}$$

coeff of t^{α_i}

$$\sum_{j=1}^{n} h_{\alpha_i - n + j} (-1)^{n-j} e^{(k)}_{n-j} = x_k^{\alpha_i}$$

$$\overset{H_\alpha}{\overset{\parallel}{(h_{\alpha_i - n + j})}} \overset{E_*^{(k)}}{\overset{\parallel}{\left((-1)^{n-j} e^{(k)}_{n-j}\right)}} = \left(x_k^{\alpha_i}\right)$$

$$\Rightarrow \det(H_\alpha) \det(E_*^{(k)}) = a_\alpha$$

in particular if $\alpha = \delta$

$$H_\delta = (h_{j-i}) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$h_0 = 1$

$$\sum_\lambda h_\lambda(x)\, m_\lambda(y)$$
$$= \sum_\alpha h_\alpha(x)\, y^\alpha$$

we have

$$a_\delta(x) \sum_{\sigma,\alpha} h_\alpha(x)\, \varepsilon(\sigma)\, y^{\alpha+\sigma(\delta)}$$

$$\beta = \alpha + \sigma(\delta) \qquad \alpha = \beta - \sigma(\delta)$$

$$= a_\delta(x) \sum_{\sigma,\beta} h^{(x)}_{\beta-\sigma(\delta)}\, \varepsilon(\sigma)\, y^\beta$$

$$\| \quad \sum_\beta a^{(x)}_\beta y^\beta$$
$$= \sum_{\gamma_1>\gamma_2>\cdots} a_\gamma(x)\, a_\gamma(y)$$

detour

m_λ, e_λ, h_λ, p_λ, s_λ

Jacobi-Trudi identities

$$s_\lambda = \det\left(h_{\lambda_i - i + j} \right)$$

where $h_r = 0$ if $r < 0$

$$= \det\left(e_{\lambda'_i - i + j} \right)$$

so $\det H_\delta = 1$

$$\det(E_\delta^{(k)}) = a_\delta$$

$$\det(H_\alpha) = a_\alpha / a_\delta = s_\lambda$$

if $\alpha = \lambda + \delta$

$$\boxed{s_\lambda = \det\left(h_{\lambda_i - i + j} \right)}$$

same kind of thing

$$\boxed{s_\lambda = \det\left(e_{\lambda'_i - i + j} \right)}$$

Apply ω

$$\boxed{\omega(s_\lambda) = s_{\lambda'}}$$

$e_\lambda \leftrightarrow h_\lambda$

$p_\lambda \circlearrowleft (-1)^{\varepsilon(\lambda)}$

$s_\lambda \leftrightarrow s_{\lambda'}$

Character theory of S_n

$$\psi : S_n \longrightarrow \Lambda^n$$
$$\sigma \longmapsto p_{\lambda(\sigma)}$$

$\lambda(\sigma)$ = partition giving the cycle decomp. of σ

$$p_{\lambda(\sigma)} = p_1^{m_1} p_2^{m_2} \cdots$$
$$m_i := \# \{ i\text{-cycles in } \sigma \}$$

$$S_m \times S_n \hookrightarrow S_{m+n}$$

image is unique up to conjugation.

$$\psi(\sigma \times \tau) = \psi(\sigma)\,\psi(\tau)$$

f is a class function on S_n define

$$
\begin{aligned}
\mathrm{ch}(f) &:= \langle f, \psi \rangle_{S_n} \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma)\, \psi(\sigma) \\
&= \sum_{|\rho| = n} \frac{1}{z_\rho} f_\rho\, p_\rho
\end{aligned}
$$

$f_\rho := f(\sigma)$ σ is type ρ.

$R^n :=$ $\mathbb{Z}$-module spanned by the irred. characters of S_n

$(S_0 := \{1\}, \ R^0 = \mathbb{Z})$

$$R := \bigoplus_{n \geq 0} R^n$$

$$f \cdot g := \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} (f \times g)$$

$f \in R^m$
$g \in R^n$
$f \times g \in R^m \times R^n$

$$S_m \times S_n \hookrightarrow S_{m+n}$$

~~[illegible]~~

$f \times g$ a virtual character of

$S_m \times S_n$

~~can~~ consider $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} (f \times g)$

<u>Claim</u> We get a ring structure on R.

Define:

$$\langle f, g \rangle := \sum_{n \geq 0} \langle f_n, g_n \rangle_{S_n}$$

pos def bilinear form on R.

<u>MAIN THM</u> $ch: R \to \Lambda$

is a isometric isomorphism of rings.

<u>Proof</u> $ch(f \cdot g) = ch(f) \cdot ch(g)$

$f \in R^m,\ g \in S^n$

$$\langle \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} (f \times g), \psi \rangle_{S_{m+n}}$$

$$= \langle f \times g , \mathrm{Res}_{S_m \times S_n}^{S_{m+n}} (\psi) \rangle_{S_m \times S_n}$$

$$\psi(\rho) = p_{\lambda(\rho)}$$

$$\rho = \sigma \times \tau$$

$$\psi(\sigma \times \tau) = \psi(\sigma)\, \psi(\tau)$$

$$= \langle f, \psi \rangle_{S_m} \cdot \langle g, \psi \rangle_{S_n}$$

$$= \mathrm{ch}(f) \cdot \mathrm{ch}(g)$$