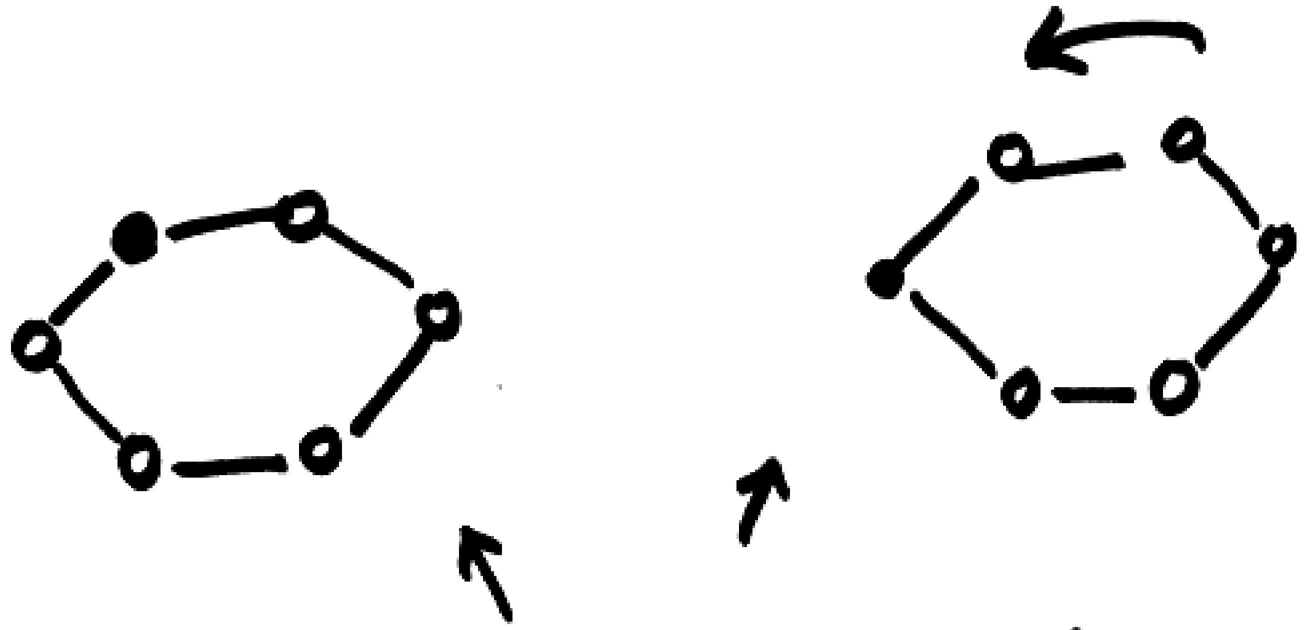
March 20, 2007

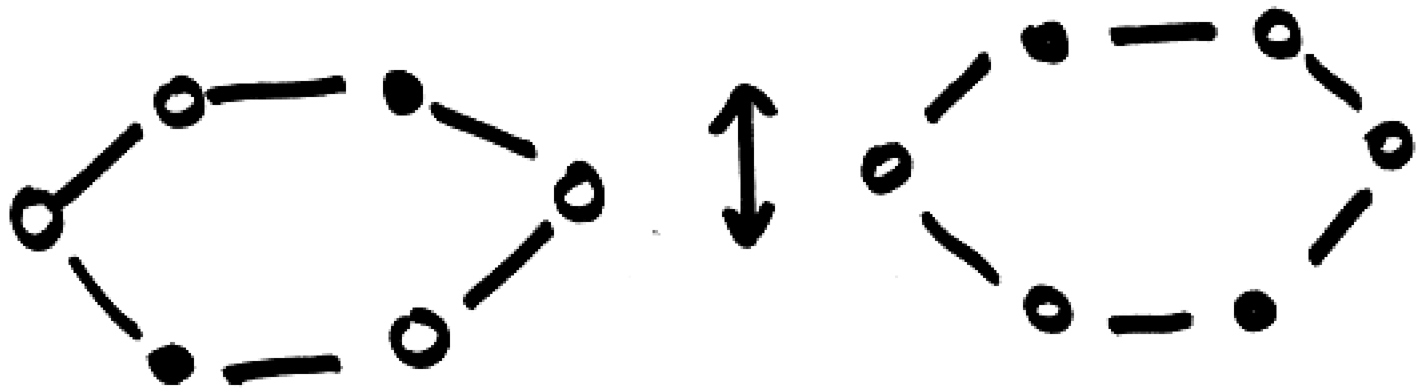
Count necklaces

$n = 6$ beads

$m = 2$ colors

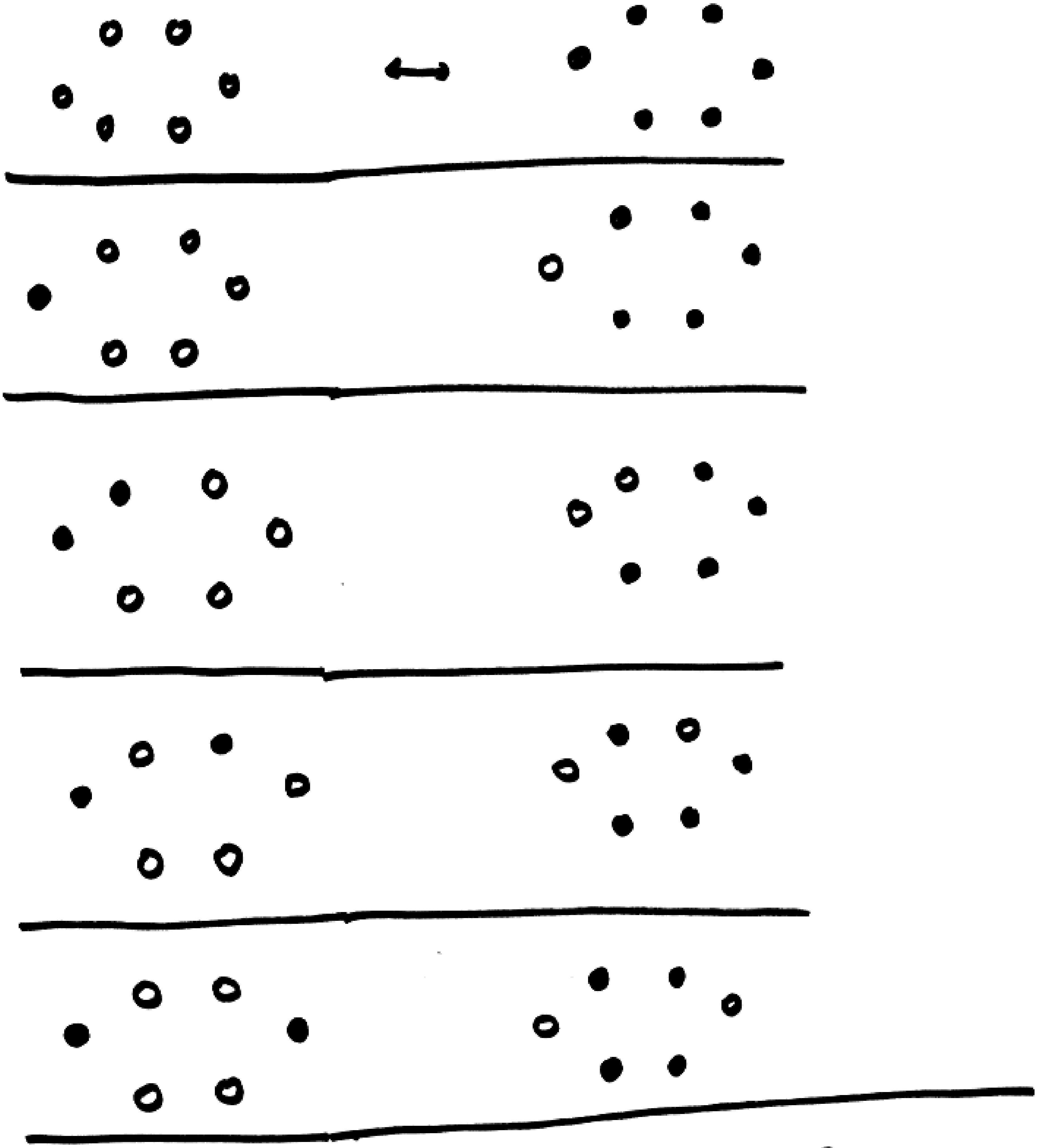


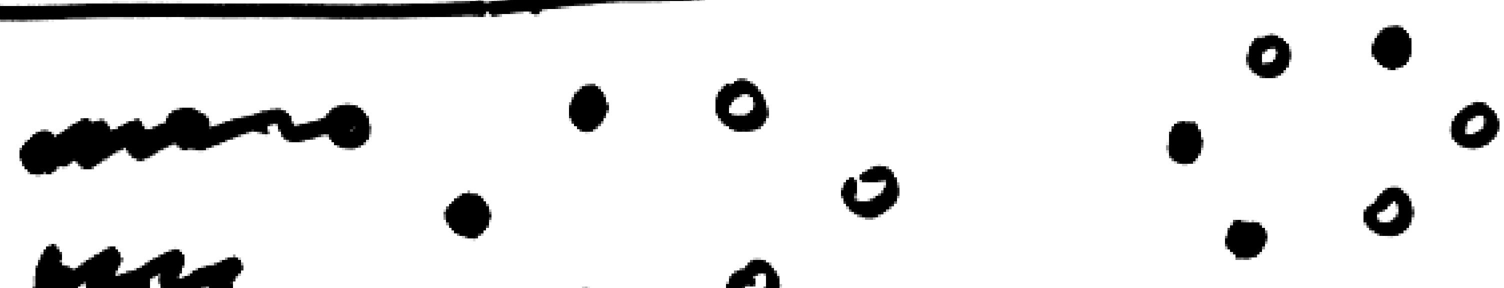
same necklace



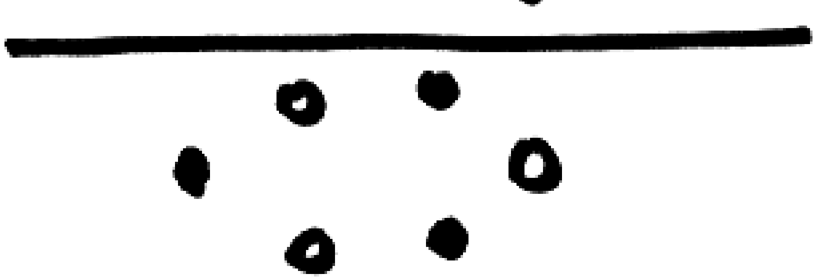
How many (different) necklaces can we get?

Polya's theory of counting.





TOTAL = 13



We can give the number of necklaces with 6 beads and any # of colors with one formula

Group

$G :=$ Symmetries of the hexagon

Total symmetries = 12

$1, r, r^2, r^3, r^4, r^5$

$s_0, s_1, s_2, s_3, s_4, s_5$

Dihedral group D_6 of order 12

Think of pictures of necklaces

Two pictures are the same necklace if we can take one to the other by some $g \in G$.

How many pictures?

Each spot in the hexagon can be one of two possibilities

The total number $= \underbrace{2 \times 2 \times \cdots \times 2}_{6}$

$= 2^6$

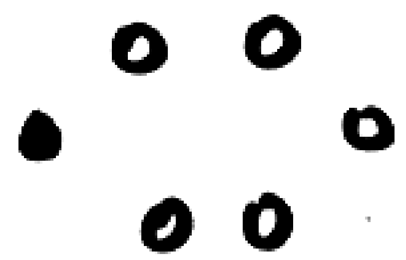
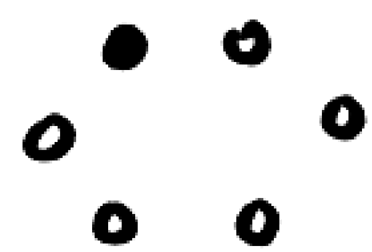
$= 64$

all pictures corresponding to same necklace

Say two pictures $x, y \in X$ are <u>equivalent</u> (i.e. represent same necklace) if

$$y = gx \quad , \qquad g \in G$$

E.g.

$x =$  $\qquad y =$ 

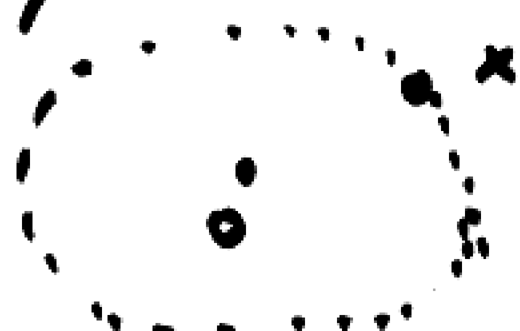
x equivalent to y because

$$y = r^{-1} x$$

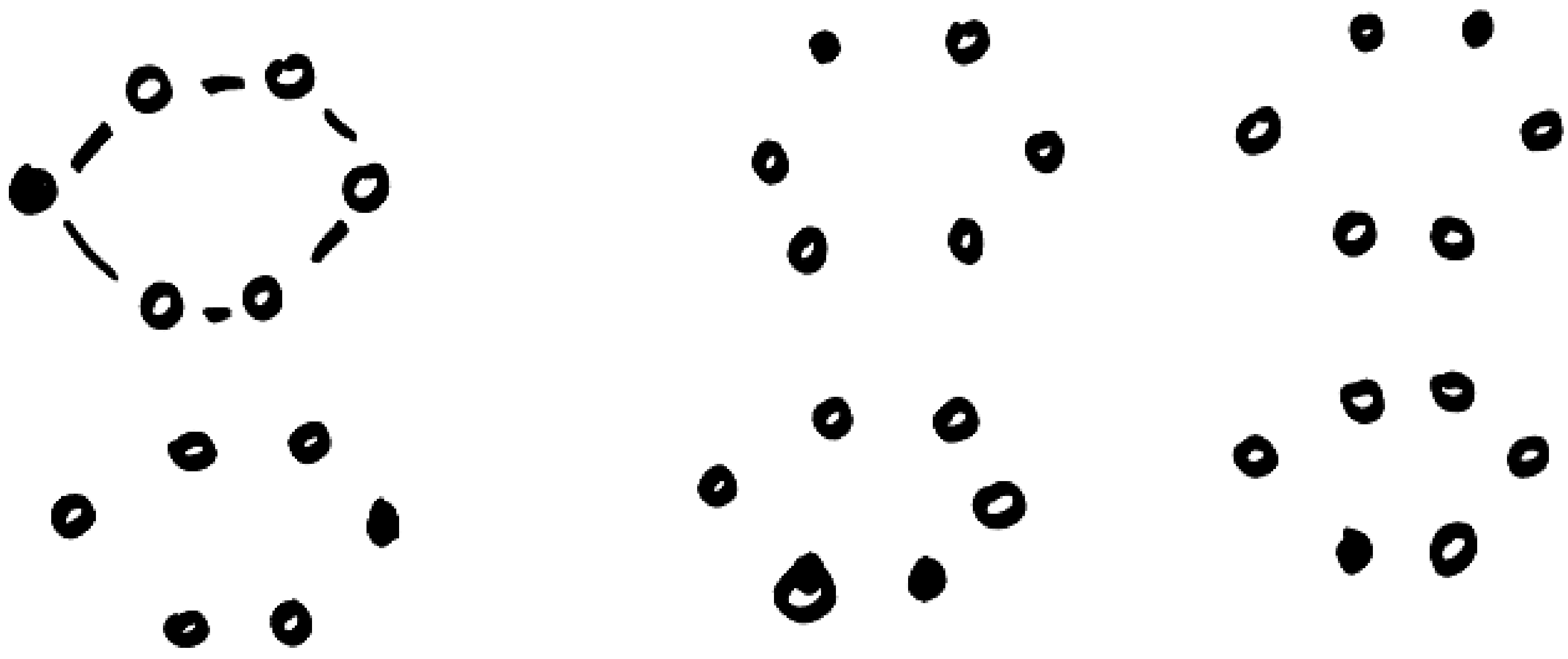
The set of all pictures equivalent to x is called the <u>orbit</u> of x

$$Gx = \{ y \mid y = gx \text{ for some } g \}$$

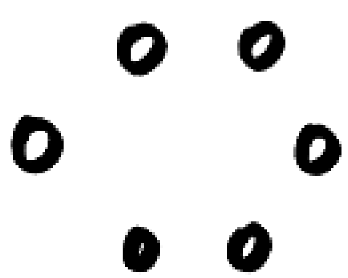
Say G is all rotations in the plane



E.g.

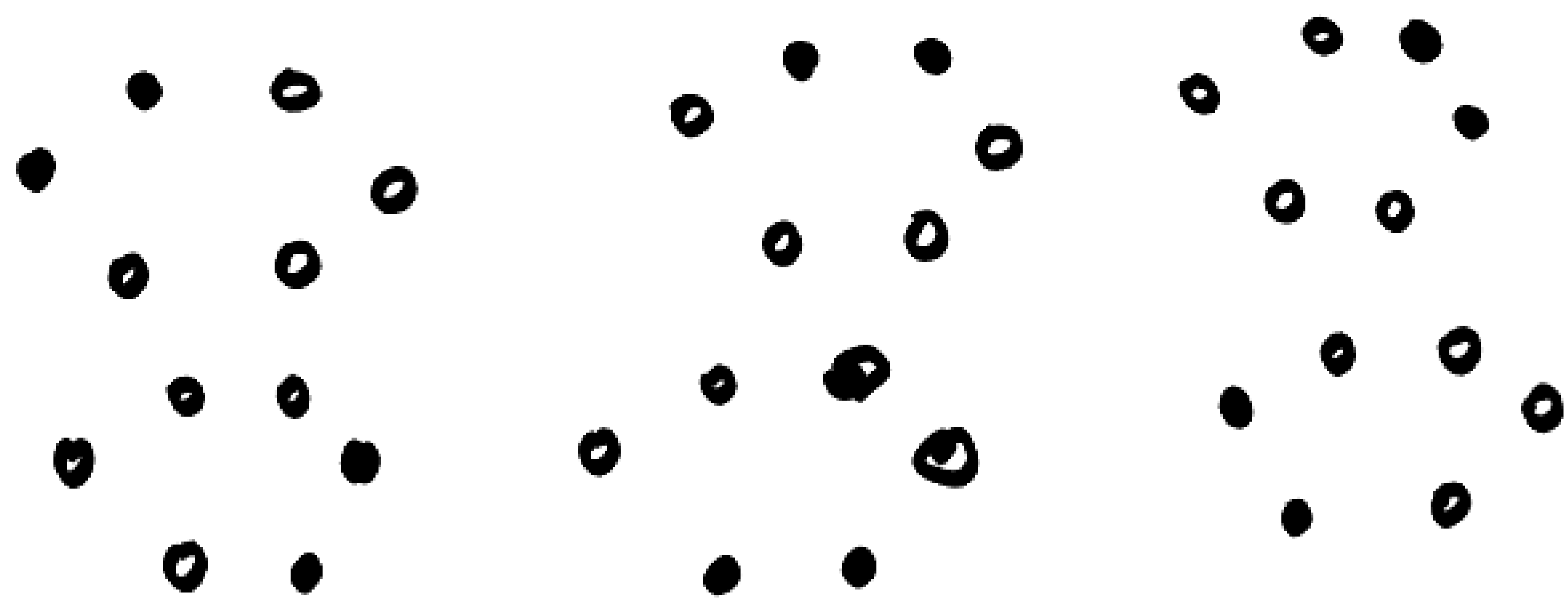
1) 

6 elements in this orbit.

2)  orbit has 1 element.

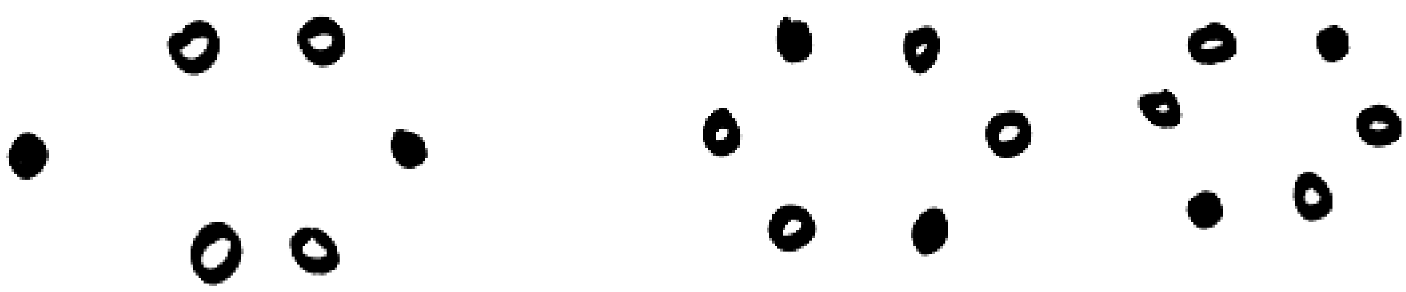
3) 

orbit has 2 elements.

4) 

6 elements.

5)



3 elements

<u>Fact</u> size of an orbit divides the order of G

Z A factor of $|G|$ <u>may</u> not be the size of an actual orbit

$x \in X$ <u>Stabilizer</u> of x in G

$$G_x := \{ g \in G \mid gx = x \}$$

is a subgroup of G.

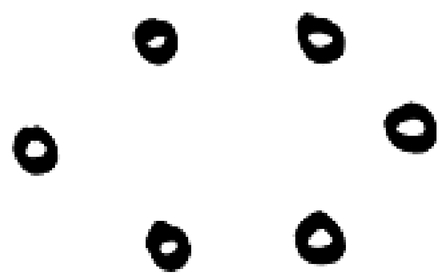
$$g_1, g_2 \in G_x$$

$$\Rightarrow \quad g_1 g_2 \in G_x$$

$$g_2 x = x$$

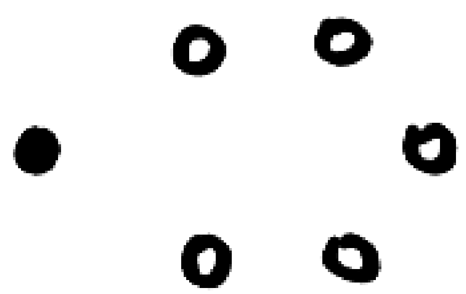
$$(g_1 g_2) x = g_1 (g_2 x) = g_1 x = x$$

Examples

1) 

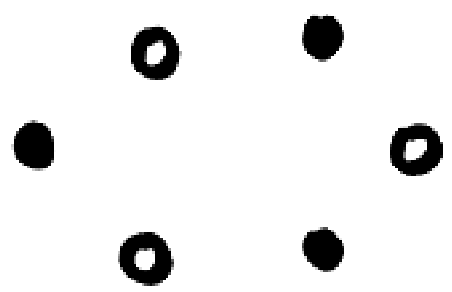
$G_x = G$

$|G_x| = 12$

2) 

$G_x = \{1, s_0\}$

$|G_x| = 2$

3) 

$G_x = \{1, r^2, r^{-2}, s_0, s_2, s_4\}$

$r^6 = 1 \quad r^4 \cdot r^2 = 1$

$r^4 = r^{-2}$

$|G_x| = 6$

$$\boxed{\underset{\text{orbit}}{\underset{\uparrow}{\#Gx}} \cdot \underset{\text{stabilizer}}{\underset{\uparrow}{|G_x|}} = |G|}$$

March 22, 2007

Count Necklaces

X = pictures

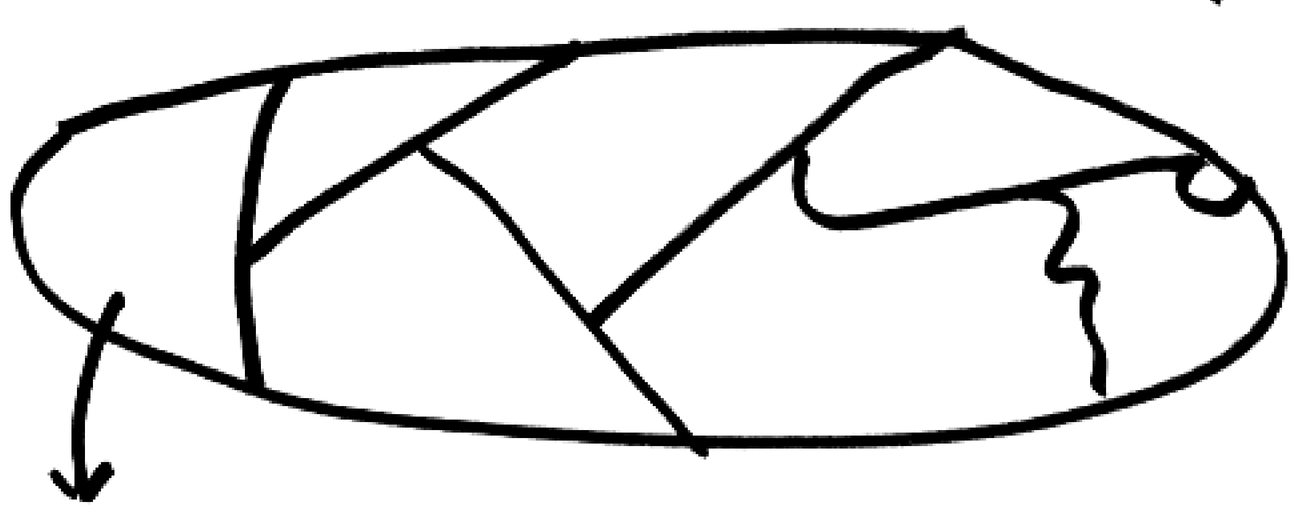
$n = 6$ # beads

$m = 2$ # colors

Two pictures of same necklace.

$\#X = 2^6 = 64$

X



all pictures of same necklace

G = group of symmetries of the regular hexagon

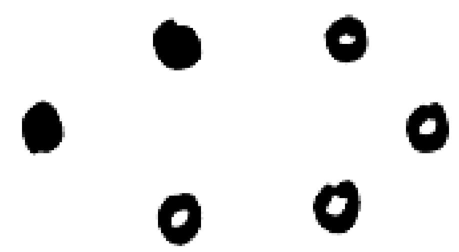
$= D_6$ (dihedral group of order 12)

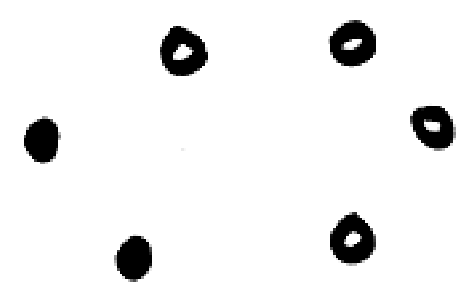
Group G acts on the set X

$x \in X$, $g \in G$

$g \cdot x \in X$

E.g. if $g = r$

$x =$ 

$g \cdot x =$ 

The <u>orbit</u> of $x \in X$ under G

$Gx = \{ y \mid gx , g \in G \}$

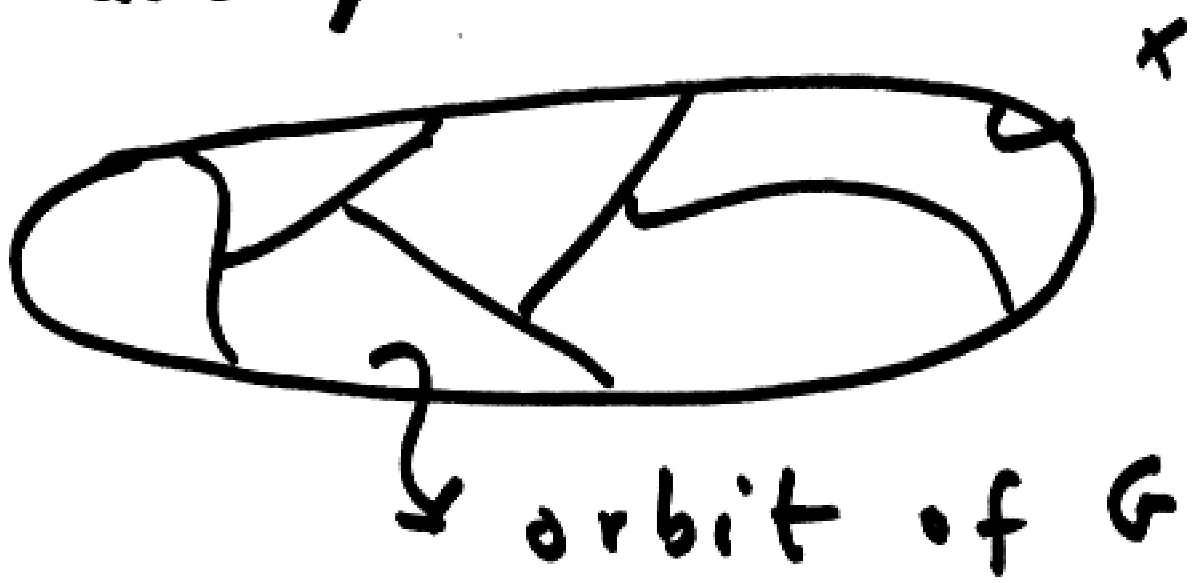
In terms of necklaces two pictures $x, y \in X$ are pictures of the same necklace

$$y = gx$$

For some $g \in G$.

I.e. y is in the orbit of x

or x and y are in the same orbit.



Alternatively: $x, y \in X$

$$x \sim y$$

if $y = gx$ for some $g \in G$.

Defines <u>equivalence relation</u> in X

- $x \sim x$

$$x = 1 \cdot x$$

- $x \sim y \Rightarrow y \sim x$

$y = gx$ for some $g \in X$

$x = h * y \qquad h\,?$

$x = g^{-1} \cdot y$

- $x \sim y,\ y \sim z \Rightarrow x \sim z$

$y = gx, \quad z = hy$

$$\Rightarrow \quad z = h(gx) = (h \cdot g)x$$

Equivalence classes $\longleftrightarrow$ orbits

$$Gx = \{ y \mid y \sim x \}$$

Typically orbits have different sizes complicates counting them.

Stabilizer

$$\mathrm{Stab}_G(x) := \{ g \in G \mid gx = x \}$$

$$\mathrm{Stab}_G(x) \subseteq G$$

Is a subgroup

- $g_1, g_2 \in \mathrm{Stab}_G(x)$

$$g_1 x = x$$
$$g_2 x = x$$

$$(g_1 g_2) x = g_1 (g_2 x) = g_1 x = x$$

$$\Rightarrow g_1 g_2 \in \mathrm{Stab}_G(x)$$

- $g \in \mathrm{Stab}_G(x)$

$$gx = x$$

$$g^{-1}(gx) = g^{-1}x$$

$$x = (g^{-1} \cdot g) x = g^{-1} x \quad \Rightarrow g^{-1} \in \mathrm{Stab}_G(x)$$

$$\boxed{\#\, Gx \cdot |\mathrm{Stab}_G(x)| = |G|}$$

In particular, the size of an orbit always divides the order of the group.

2 If d divides $|G|$ there may not be an orbit of size d.

Burnside's Lemma

If $g \in G$ let

$$F(g) := \#\,\{x \in X \mid gx = x\}.$$

$$\boxed{\#\,\text{orbits} = \frac{1}{|G|} \sum_{g \in G} F(g)}$$

"average of # of fixed points"

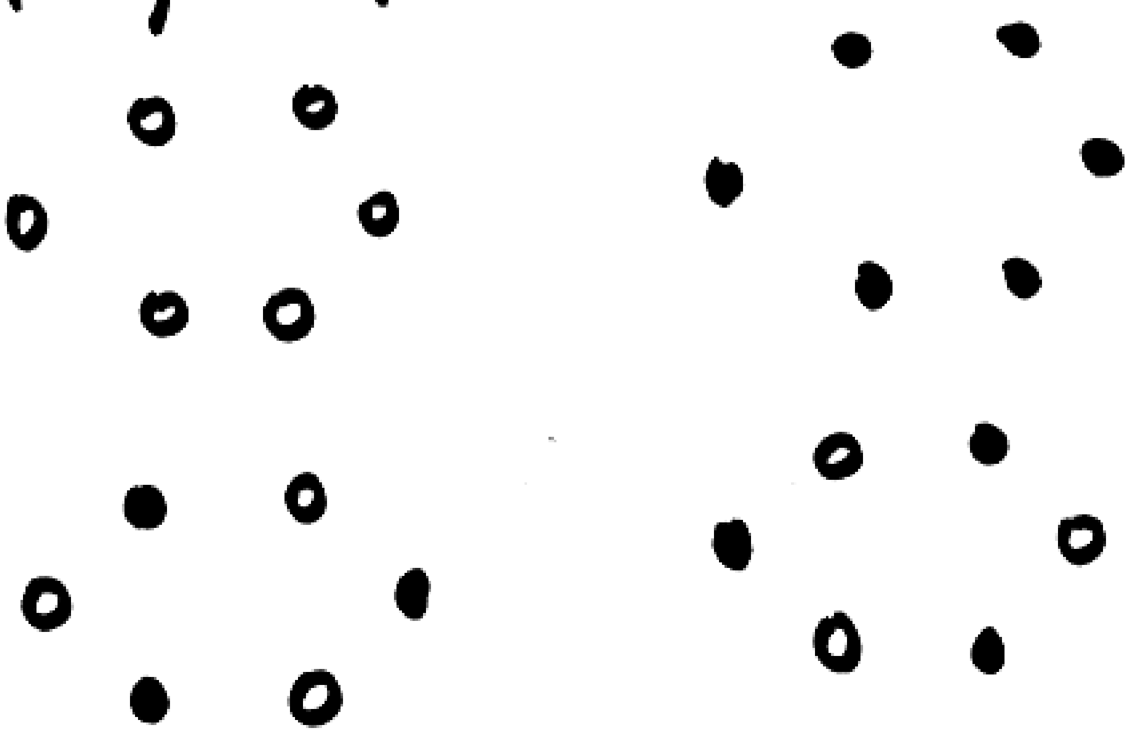
$G = D_6$

$F(r) = 2 = 2^1$

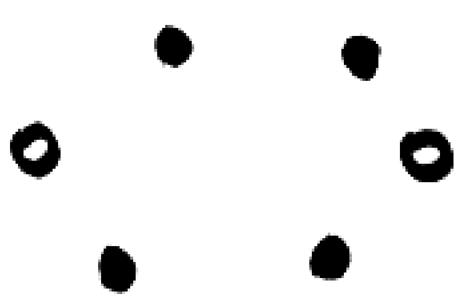
If $rx = x$ then $r^2 x = x$

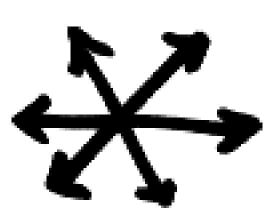
$r^3 x = x \ldots$

$F(r^2) = 4 = 2^2$



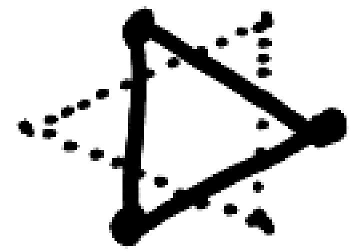
$F(r^3) = 8 = 2^3$ ~~previous ones~~



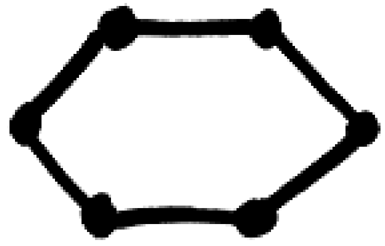


B

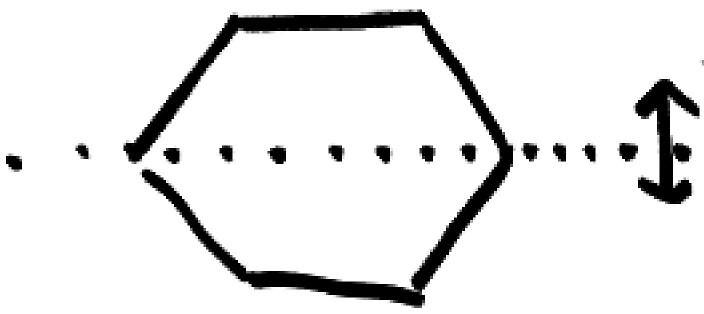
$F(r^4) = 2^2$

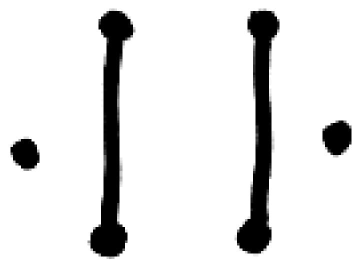


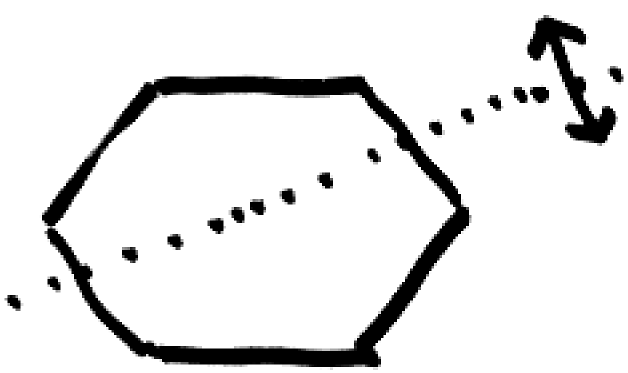
$F(r^5) = 2^1$

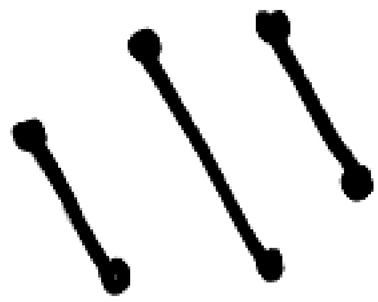


$F(s) = 2^4$









$F(s_1) = 2^3$

<u>Burnside</u>

$$\# \text{ necklaces} = \frac{1}{12}\Big(\underset{\substack{\uparrow \\ 1}}{2^6} + \underset{\substack{\uparrow \\ r}}{2^1} + \underset{\substack{\uparrow \\ r^2}}{2^2} + \underset{\substack{\uparrow \\ r^3}}{2^3} + \cdots$$

$$\underset{r^4}{\underset{\uparrow}{2^2}} + \underset{r^5}{\underset{\uparrow}{2}} + \underset{s_0,\, s_2,\, s_4}{2^4+2^4+2^4} + \underset{s_1,\, s_3,\, s_5}{2^3+2^3+2^3})$$

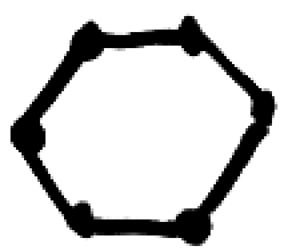
$$= \frac{1}{12}\left(2^6 + 2 \times 2^1 + 2 \times 2^2 + 2^3 + 3 \times 2^4 + 3 \times 2^3 \right)$$

$$\begin{array}{r} 64 \\ 20 \\ 48 \\ \underline{24} \\ 156 \end{array} = 12 \times 13$$

What if we had m colors?

$F(r) = m$

r



$$F(r^2) = m^2$$

$\vdots$

$$\#\text{necklaces} = \frac{1}{12}\Big(\underset{1}{\underset{\uparrow}{m^6}} + \underset{r,\, r^{-1}}{\underset{\uparrow}{2 \times m}} + \underset{r^2,\, r^{-2}}{\underset{\uparrow}{2 \times m^2}} + \underset{r^3}{\underset{\uparrow}{m^3}}$$

$$+ \underset{s_0,\, s_2,\, s_4}{\underset{\uparrow}{3 \times m^4}} + \underset{s_1,\, s_3,\, s_5}{\underset{\uparrow}{3 \times m^3}} \Big)$$

$$\boxed{\#\text{necklaces} = \frac{1}{12}\left(m^6 + 3m^4 + 4m^3 + 2m^2 + 2m \right)}$$

$\underline{m = 1}$ $\rightsquigarrow$ $\#\text{necklaces} = 1$

__Note__: In particular for any $m = 1, 2, \ldots$

we must have

$$m^6 + 3m^4 + 4m^3 + 2m^2 + 2m$$

is divisible by 12.

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}(n^2 - n)$$