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$R^n :=$ $\mathbb{Z}$-span of irred. characters of S_n

$R := \bigoplus_{n \geq 0} R^n$ $\qquad \left(\begin{array}{l} S_0 := \{1\} \\ R^0 = \mathbb{Z}. \end{array} \right.$

product on R

$$f \cdot g = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (f \times g)$$

$f \in R^n$, $g \in R^m$.

$$\psi : S_n \longrightarrow \Lambda^n$$
$$\sigma \longmapsto p_\sigma = p_1^{m_1} p_2^{m_2} \cdots$$

<u>THM</u> $\quad ch : R \to \Lambda \quad$ isomorphism isometry

- $f \in R^n \quad ch(f) := \langle f, \psi \rangle_{S_n}$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \psi(\sigma)$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) p_\sigma$$

$$= \sum_{|\rho|=n} \frac{1}{z_\rho} f(\rho)\, p_\rho$$

$$f = \sum f_n \qquad f_n \in R^n$$

$$g = \sum g_n \qquad g_n \in R^n$$

$$\langle f, g \rangle = \sum_{n \geq 0} \langle f_n, g_n \rangle_{S_n}$$

<u>Pf</u> $\quad ch(f \cdot g) = ch(f) \cdot ch(g)$

$$f \in R^n, \; g \in R^m$$

$$ch(f \cdot g) = \langle \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (f \times g), \psi \rangle_{S_{n+m}}$$

$$= \langle f \times g, \mathrm{Res}_{S_n \times S_m}^{S_{n+m}} \psi \rangle_{S_n \times S_m}$$

Frob. reciprocity

$$= ch(f)\, ch(g)$$

$$f, g \in R^n$$

$$\langle ch(f), ch(g) \rangle \quad \langle \sum_\rho \frac{1}{z_\rho} f(\rho) p_\rho ,$$

$\langle p_\rho, p_\rho \rangle = z_\rho$

$$\sum_\rho \frac{1}{z_\rho} g(\rho) p_\rho \rangle$$

$$= \sum_\rho \frac{1}{z_\rho} f(\rho) \cdot g(\rho)$$

$$= \langle f, g \rangle_{S_n}$$

$$z_\rho = |\mathrm{Stab}_{S_n}(\rho)|$$

conjugacy class of type ρ has cardinality

$$\frac{n!}{z_\rho}$$

Let η_n be the trivial character of S_n

$$ch(\eta_n) = \sum_\rho \frac{1}{z_\rho} p_\rho = h_n$$

$\eta_n \in R^n$

$\lambda = (\lambda_1, \lambda_2, \dots)$ $\quad |\lambda| = n$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$$

$$h_\lambda = ch(\eta_{\lambda_1} \eta_{\lambda_2} \cdots)$$

induce trivial character

$R^n \ni \underbrace{\eta_{\lambda_1} \eta_{\lambda_2} \cdots}_{\eta_\lambda}$ in $S_{\lambda_1} \times S_{\lambda_2} \times \cdots$

to $S_{\lambda_1 + \lambda_2 + \cdots} = S_n$

Let $\chi^\lambda \in R^n$ s.t.

$$ch(\chi^\lambda) = s_\lambda$$

concretely

$$\chi^\lambda := \det(\eta_{\lambda_i + j - i})$$ Jacobi Trudi

<u>claim</u> χ^λ is an irreducible character of S_n $n = |\lambda|$.

<u>Pf</u> we check

- $\langle \chi^\lambda, \chi^\lambda \rangle = 1$
- $\chi^\lambda(1) > 0$

$$\langle \chi^\lambda, \chi^\mu \rangle_{S_n} = \langle ch(\chi^\lambda), ch(\chi^\mu) \rangle$$

$|\lambda| = |\mu| = n$

$\chi^\lambda, \chi^\mu \in R^n$

$$= \langle s_\lambda, s_\mu \rangle$$

$$= \delta_{\lambda,\mu}$$

i.e. $\{ \chi^\lambda \}$ orthonormal basis (same cardinality as rank of R^n i.e. $p(n) = \#$ partitions of n)

$$s_\lambda = ch(\chi^\lambda)$$

$$= \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho$$

$\chi_\rho^\lambda :=$ value of χ^λ at a $\sigma \in S_n$ of cycle type ρ

$$\boxed{s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho}$$

i.e. character table of S_n gives the transition matrix (change of basis) between p_ρ and s_λ

want to show

$$\chi_{1^n}^\lambda > 0$$

$$\langle p_\rho, p_\mu \rangle = z_\rho \delta_{\rho,\mu}$$

$$\boxed{\chi_\rho^\lambda = \langle s_\lambda, p_\rho \rangle}$$

$$\chi^\lambda_{1^n} = \langle s_\lambda , p_{1^n} \rangle$$

$$p_{1^n} = p_1^n$$

Invert the relation

$$\boxed{p_\rho = \sum_\lambda \chi^\lambda_\rho \, s_\lambda}$$

$$p_{1^n} = p_1^n = \sum_\lambda \chi^\lambda_{1^n} \, s_\lambda$$

$$s_\lambda = a_{\lambda+\delta} / a_\delta$$

$$a_\delta \cdot \left(\sum_{i=1}^n x_i \right)^n = \sum_\lambda \chi^\lambda_{1^n} \, a_{\lambda+\delta}$$

$\chi^\lambda_{1^n}$ = coeff of $x^{\lambda+\delta}$ in the lhs.

$$a^{\lambda+\delta} = \sum_\sigma \varepsilon(\sigma) \, x^{\sigma(\lambda+\delta)}$$

$$= x^{\lambda+\delta} + \cdots$$

$$\chi^{\lambda}_{1^n} = \left[a_\delta \cdot \left(\sum_{i=1}^{m} x_i \right)^n \right]_{\lambda+\delta}$$

<u>**A=B**</u>

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x + x^{-1})^{2n} = (x + x^{-1})^n (x + x^{-1})^n$$

constant coeff.

$$\text{lhs} = \sum_{k=0}^{2n} \binom{2n}{k} x^k x^{-(2n-k)}$$

$$k = 2n - k$$
$$k = n$$

$$\text{const. term} = \binom{2n}{n}$$

$$\sum_k \binom{n}{k} x^k x^{-(n-k)} \cdot \sum_j \binom{n}{j} x^j x^{-(n-j)}$$

$$x^{2k-n} \qquad x^{2j-n}$$

$$2k - n = -(2j - n)$$
$$k = j$$

const term: $\sum_k \binom{n}{k}^2$

$$\chi^{\lambda}_{1^n} = \Big[\underbrace{\prod_{i<j}(x_i - x_j)}_{= a_\delta} \Big(\sum_{i=1}^{n} x_i \Big)^n \Big]_{\lambda+\delta}$$

$$\Big(\sum_{i=1}^{n} x_i \Big)^n = \sum_{\substack{k_1, \dots, k_n \geq 0 \\ k_1 + \cdots + k_n = n}} \frac{n!}{k_1! \cdots k_n!} x_1^{k_1} \cdots x_n^{k_n}$$

$$k = (k_1, k_2, \dots, k_n)$$

$$a_\delta = \sum_{\sigma} \varepsilon(\sigma) \, x^{\sigma(\delta)}$$

$$x^{\sigma(\delta)+k}$$

$$\sigma(\delta) + k = \lambda + \delta = \mu$$

$$k = \mu - \sigma(\delta)$$

$$\delta = (n-1, n-2, \dots, 1, 0)$$

$$k_i = \mu_i - n + \sigma(i)$$

$$\chi^{\lambda}_{1^n} = \sum_{\sigma} \frac{n! \quad \varepsilon(\sigma)}{\prod_{i=1}^{n} (\mu_i - n + \sigma(i))!}$$

$$= n! \det\left(\frac{1}{(\mu_i - n + j)!}\right)$$

$\mu! := \mu_1! \mu_2! \cdots \mu_n!$

$$= \frac{n!}{\mu!} \det\left(\mu_i(\mu_i - 1) \cdots (\mu_i - n + j + 1)\right)$$

$$\frac{\mu_i!}{(\mu_i - n + j)!} = \mu_i(\mu_i - 1) \cdots (\mu_i - m + j + 1)$$

highest deg in μ_i is $n-1$

$$= \frac{n!}{\mu!} \det\left(\mu_i^{\,j-1}\right)$$

use rows/column operations to kill off smaller deg terms in each entry

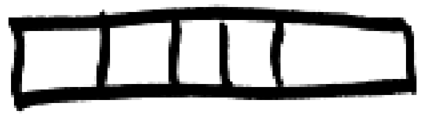
$$\boxed{\chi^\lambda_{1^n} = \frac{n!}{\mu!} \prod_{i<j} (\mu_i - \mu_j) \qquad \mu_i = \lambda_i + n - i}$$

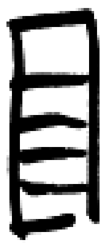
$\mu_1 > \mu_2 > \cdots$

$\Rightarrow \chi^\lambda_{1^n} > 0$!!

- $\chi^{(n)} = \eta_n$ trivial char of S_n

  $\chi^{(1^n)} = \varepsilon_n$ sign repn.

   $\longleftrightarrow$ trivial

   $\longleftrightarrow$ sgn

- $\chi^{\lambda'} = \varepsilon \chi^{\lambda}$

  $\chi^{\lambda'}_{\rho} = \langle s_{\lambda'}, p_{\rho} \rangle$

  apply ω

  $= \langle s_{\lambda}, \varepsilon_{\rho} p_{\rho} \rangle$

  $= \varepsilon_{\rho} \cdot \chi^{\lambda}_{\rho}$

i.e. $\omega \circlearrowleft \Lambda^n$ corresponds to multiplication by sgn $= \varepsilon$

- $G \subset S_n$

  $Z_G(p_1, p_2, \dots) = ch(\chi_G)$

  $\chi_G = \mathrm{Ind}_G^{S_n}(1)$