Combinatorics and Geometry

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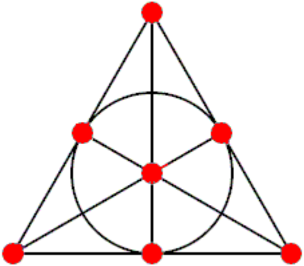
Projective plane

- A a set of points
- A collection of subsets of A called lines.

Axioms

- Two distinct points lie in a unique line.
- Two distinct lines meet in a unique point.
- There exist four points not all in a line

Properties

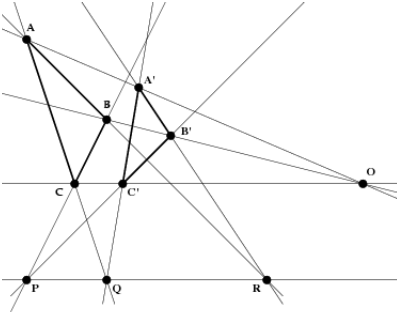
- No finiteness required.
- Fano (1892): projective plane consisting of seven points and seven lines.
- 

Fano plane

Finite projective planes

- Each line has $q + 1$ points for some q.
- The total number of points is $q^2 + q + 1$.
- In Fano's case $q = 2$.

Desargues

- 

Desargues theorem

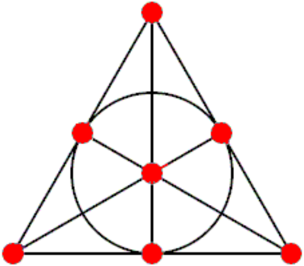
- Desargues theorem does not follows from the axioms.

Coordinates

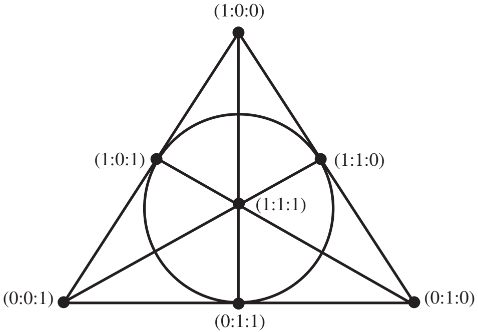
- Desargues holds if and only if we can put coordinates.

- Use Desargues theorem to define sum, multiplication by scalars, etc.

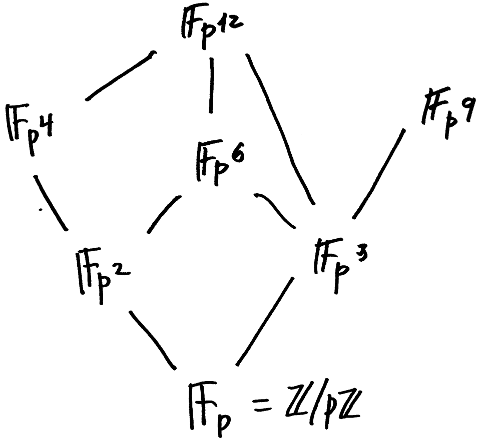
- For a finite plane we get a finite field $\mathbb{F}_q$ of $q = p^n$ elements, where p is a prime.



Fano plane



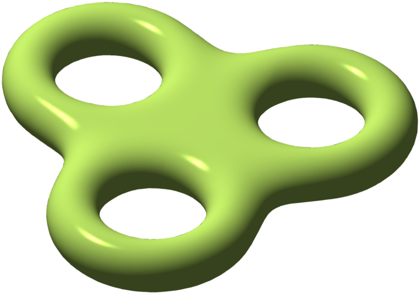
Fano plane coordinates


$\mathbb{F}_{p^{12}}$
$\mathbb{F}_{p^4}$
$\mathbb{F}_{p^6}$
$\mathbb{F}_{p^9}$
$\mathbb{F}_{p^2}$
$\mathbb{F}_{p^3}$
$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

Algebraic varieties

- X zero locus of polynomials $F_1, \ldots, F_m$ in variables $x_1, \ldots, x_n$.
- If the coefficients of F_i are integers we can consider $X(\mathbb{F}_q)$.
- What is the relation between $X(\mathbb{C})$ and $X(\mathbb{F}_q)$?

Example

- $$X : \quad y^2 = f(x)$$
  with $f \in \mathbb{Z}[x]$ square-free of degree 8.
- Algebraic curve of genus $g = 3$.
- 

  $X(\mathbb{C})$ genus 3 curve

Example

- By Weil for all n

$$\#X(\mathbb{F}_{p^n}) = p^n + 1 - \sum_{i=1}^{2g} \alpha_i^n, \qquad |\alpha_i| = p^{\frac{1}{2}}$$

- Hence with $q = p^n$

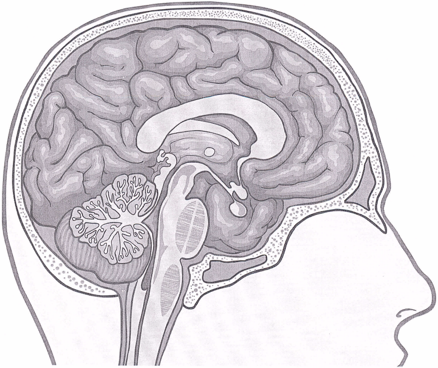
$$|\#X(\mathbb{F}_q) - q - 1| \leq 2g\sqrt{q}$$

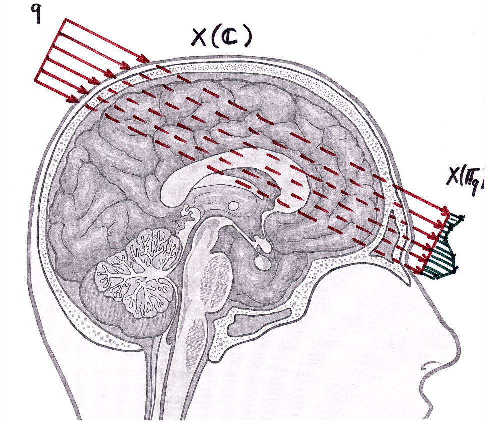
- Counting $\leftrightarrow$ Geometry.

Analogy

$\#X(\mathbb{F}_q)$ like Radon transform data.

$X(\mathbb{C})$





q
$X(\mathbb{C})$
$X(\mathbb{H}_q)$

Weil conjectures

- What can we recover of $X(\mathbb{C})$ from the $\#X(\mathbb{F}_q)$ data?
- Suppose $X(\mathbb{C})$ is smooth, compact and $\#X(\mathbb{F}_q) = C(q)$ for a certain polynomial C.
- Let $b_j(X) := \dim H^j(X, \mathbb{C})$, the Betti numbers of X.

Weil conjectures

- Then $b_{2i+1}(X) = 0$ and
-
$$C(q) = \sum_{i=0}^{\dim X} b_{2i}(X)\, q^i$$

Examples

- $X = \mathbb{P}^1$, projective line

$$C(q) = q + 1$$

- $b_0 = b_2 = 1,\ b_1 = 0.$
- $X = \mathbb{P}^2$, projective plane

$$C(q) = q^2 + q + 1$$

- $b_0 = b_2 = b_4 = 1,\ b_1 = b_3 = 0.$

Consequences

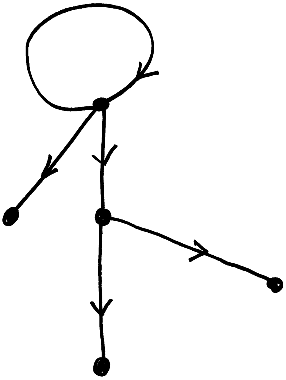
- Such counting polynomials $C(q)$ have non-negative integer coefficients.
- We may sometimes compute Betti numbers by counting.

Classical example

- $X = G(k,n)$ the Grassmanian of all dimension k subspaces in a fixed n dimensional space.
- $$\#G(k,n)(\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}$$
  q-binomial coefficient.
- E.g.
  $$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1.$$

Quivers

- A quiver Q is a directed graph



Q a quiver

Quivers representation

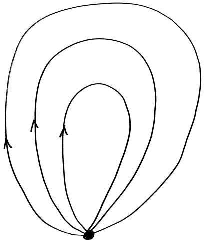
- A representation of Q is an assignment:

$\bullet \longmapsto$ Vector space

$\bullet \to \bullet \longmapsto$ Linear map

Representation of Q

Example



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- Representations up to isomorphism

$$\updownarrow$$

- Tuples $(A_1, \ldots, A_g)$ of $n \times n$ matrices up to simultaneous conjugation.

Classification

- For S_1 this is Jordan's problem.
- Classification of representations up to isomorphims in general? E.g. S_g for $g > 1$.
- Difficult linear algebra problems.

Kac

- Kac (early 80's): count over finite fields.
- Fix Q and a dimension vector α.
- The number of absolutely indecomposable representations up to isomorphism equals

$$A_\alpha(q)$$

a polynomial in q.

Kac conjecture

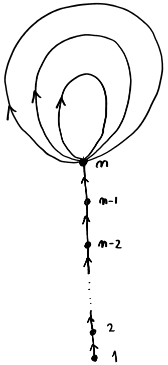
- $A_\alpha(q)$ has non-negative integer coefficients.
- Crawley-Boevey and van der Bergh proved it for α indivisible.
- With Hausel and Letellier we extended the proof to the general case.

Quiver variety

- CB-vB's argument:
- $$A_\alpha(q) = \sum_i \dim\left(H_c^{2i}(\mathcal{Q}_\alpha;\mathbb{C})\right) q^{i-d_\alpha}$$
- $\mathcal{Q}_\alpha$ is an associated smooth Nakajima quiver variety of dimension $2d_\alpha$.
- Indivisible α crucial for the existence of $\mathcal{Q}_\alpha$.

Quiver variety

- We consider an extended quiver $\tilde{Q}$ and dimension vector $\tilde{\alpha}$ adding legs to every vertex.
- E.g. for S_g and dimension $\alpha = n$



Extended quiver

- Get associated Nakajima quiver variety $\tilde{\mathcal{Q}}_{\tilde{\alpha}}$ of dimension $2d_{\tilde{\alpha}}$ ($\tilde{\alpha}$ is indivisible).

- The i-th leg gives an action of the symmetric group S_{α_i} on the cohomology of $\tilde{\mathcal{Q}}_{\tilde{\alpha}}$.

- We prove

$$A_\alpha(q) = \sum_i \dim \left(H_c^{2i}(\mathcal{Q}_{\tilde{\alpha}}; \mathbb{C})_\epsilon\right) q^{i-d_{\tilde{\alpha}}}$$

where ϵ is the sign character of $S_{\alpha_1} \times S_{\alpha_2} \cdots$