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$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F}_p \right\}$$

$$[\,(a,b,c),\ (x,y,z)\,] = [\,0, 0, ay - bx\,)$$

$$(a,b,c)\cdot(x,y,z) = (x+a, b+y, ay+z+c)$$

$$(x,y,z)(a,b,c)(x,y,z)^{-1} = (a, b, bx - ay + c)$$

EX. Compute character of

$$\mathrm{Ind}_{N^*}^{H}(\eta)$$

$$\eta : N^* \longrightarrow \mathbb{C}^\times$$
$$(\zeta, \ell) \longmapsto \eta(\zeta)$$

$$\eta : \mu_p \hookrightarrow \mathbb{C}^\times$$

$$\chi(g) = \begin{cases} \eta(\zeta)\cdot p^n & \text{if } u = 0 \\ 0 & u \neq 0 \end{cases}$$

$g = (\zeta, u)$

$\Rightarrow$ $(\zeta, 0)$ acts as scalar namely multiplication by $\eta(\zeta)$.

$$U = L \oplus L^*$$

$$\Phi = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

$e_1, \dots, e_n$ basis of L } symplectic basis
$e_1^*, \dots, e_n^*$ " " L^*

$$\Phi(e_i, e_j^*) = \delta_{ij}$$

<u>Note</u> Any nondegenerate alternating bilinear form can be obtained this way.

<u>Pf.</u> Φ <u>Alternating</u>.

$\Phi(u, u) = 0$ $u \in U$.

$\Rightarrow$ $\Phi(u, v) = -\Phi(v, u)$ <u>skew symmetric</u>

($\not\Leftarrow$ if $2 = 0$)
$\Leftarrow$ if $2 \neq 0$

Pick $e_1 \neq 0$

since Φ non-degenerate

There exists $u \in U$ s.t.

$$\Phi(e_1, u) = \alpha \neq 0$$

let $\quad e_1^* = u/\alpha$

$$\Phi(e_1, e_1^*) = 1$$

$$\Phi|_{\langle e_1, e_1^* \rangle} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Take orthog complement of this subspace ... by induction

$$\Phi \simeq \begin{pmatrix} \begin{array}{cc|} 0 & 1 \\ -1 & 0 \\ \hline \end{array} & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ 0 & & \ddots \end{pmatrix}$$

$$e_1, e_1^*, e_2, e_2^*, \ldots \qquad \square$$

$$N^* := \{ (\zeta, \ell^*), \ \zeta \in \mu_p, \ \ell^* \in L^* \}$$

abelian subgroup of H

- $1 \longrightarrow \mu_p \longrightarrow H \rightarrow U \rightarrow 0$

$$(\zeta, u)(\zeta', u') = (\zeta\zeta' \psi(u, u'), u + u')$$

$\psi : U \times U \to \mu_p$

$\Theta : (\mathbb{F}_p, +) \to \mu_p$ non-trivial additive character

$\psi = \Theta \circ \Psi$

$\Psi : U \times U \to \mathbb{F}_p$ bilinear form

$\Phi = \Psi - {}^t\Psi$

$[(\zeta, u), (\zeta', u')] = (\phi(u, u'), 0)$

$\phi = \Theta \circ \Phi$

$\phi(u, u') = \psi(u, u') / \psi(u', u)$

Fix once and for all $\eta : \mu_p \hookrightarrow \mathbb{C}^\times$

$$\eta : N^* \to \mathbb{C}^\times$$
$$(\zeta, \ell^*) \mapsto \zeta$$

$V = \mathrm{Ind}_{N^*}^{H}(\eta)$ <u>Schrödinger repn</u>

irred. and determined by the action of center (by choice of η)

Example $p = 2$, $n = 1$

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \psi = \theta \circ \Psi$$

or
$$\Psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(\zeta, u)^2 = (\zeta^2 \cdot \psi(u,u), 2u \cdot)$$
$$= (\psi(u,u), 0)$$

(ζ, u) of order 4 $\Leftrightarrow$ $\psi(u,u) = -1$

or $\Psi(u,u) = 1$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \Psi(u,u) = 1$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Q_8 $\quad x^2 + xy + y^2 = \Psi\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$

$\Rightarrow$ 3×2 elements of order 4

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad xy \quad$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

D_4 $\quad \Rightarrow$ 1×2 elements of order 4

$$V = \mathrm{Ind}_{N^*}^{H}(\eta)$$

$$:= \{ F : H \to \mathbb{C} \mid F(n^* h) = \eta(\zeta) F(h) \}$$
$$n^* \in N^*$$

H acts on the right.

$$h_1 F(h) := F(h h_1)$$

$$N^* \ni n^* = (\zeta', e^*)$$
$$h = (\zeta^\circ, u)$$
$$n^* h = (\zeta \zeta' \psi(e^*, u), e^* + u)$$

$$f(e) := F((1, e))$$

Recover F from f

$$F((\zeta, u)) = F((\zeta, 0)(1, u)) = \eta(\zeta) F(1,$$

if $n^* = (\zeta^{-1}, 0)$ then

$$n^* (\zeta, u) = (1, u)$$
$$(\zeta, u) = (\zeta, 0)(1, u)$$

$$u = \ell^* + u_1$$

$$(1, u) = \underbrace{(\psi(\ell^*, u_1)^{-1}, \ell^*)}_{\in N^*}(1, u_1)$$

$$F((1, u)) = \psi(\ell^*, u_1)^{-1} F((1, u_1))$$

We only need to know F on $(1, \ell)$ since every $u \in U$ is congruent mod L^* to some unique $\ell \in L$

$$U = L \oplus L^*$$

$$F \rightleftarrows f$$

$$V \simeq \{f : L \to \mathbb{C}\}$$

How does H act here?

$(\zeta, 0)$

$$F(h(\zeta,0)) = F((\zeta,0)h) = \zeta F(h)$$

$(1, \ell^*)$

$$F((1,\ell)(1,\ell^*)) = F((\psi(\ell,\ell^*), \ell+\ell^*))$$

$$(1,\ell^*)(1,\ell) = (\psi(\ell^*,\ell), \ell^*+\ell)$$

$$\left(\frac{\psi(\ell,\ell^*)}{\psi(\ell^*,\ell)}, \ell^*\right)(1,\ell) = (\psi(\ell,\ell^*), \ell+\ell^*)$$

$$(\phi(\ell,\ell^*), \ell^*) \in N^*$$

$$F((\psi(\ell,\ell^*), \ell+\ell^*)) = \phi(\ell,\ell^*)\, F((1,\ell)) = \phi(\ell,\ell^*) \cdot f(\ell)$$

$$f \longmapsto \phi(\cdot,\ell^*) \cdot f$$

$$L \longrightarrow \mathbb{C}^\times$$
$$\ell \longmapsto \phi(\ell,\ell^*)$$

multiplication by a character of L

$\underline{(1, l_1)}$

$$F\left((1, l)(1, l_1)\right) = F\left((\psi(l, l_1), l + l_1)\right)$$

$\underline{\text{Choose}}$ $\Psi = \left(\begin{array}{c|c} 0 & I_n \\ \hline 0 & 0 \end{array}\right)$

$$\Psi(l, l_1) = 0$$

$$\psi(l, l_1) = 1$$

$$F\left((1, l)(1, l_1)\right) = F\left((1, l + l_1)\right)$$

$$f(l) \longmapsto f(l + l_1)$$

i.e. translation by L

From now on $n = 2$, $\dim U = 4$.

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{F}_p)$$

$$u = (l, l^*) \in U$$

$$u\sigma := (l, l^*)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$= (\alpha l + \gamma l^*, \beta l + \delta l^*)$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$SL_2(\mathbb{F}_p) \hookrightarrow Sp(U)$$

(i.e. $\Phi(u\sigma, v\sigma) = \Phi(u,v)$

$$\Phi = \left(\begin{array}{c|c} 0 & I_2 \\ \hline -I_2 & 0 \end{array}\right)$$

)

$$SL_2(\mathbb{F}_p) \hookrightarrow Sp(U)$$

want to extend this to $\mathrm{Aut}(H)$.

Define

$$(\zeta, u)\sigma := (\zeta \nu_\sigma(u), u\sigma)$$

with $\nu_\sigma(u)$ to be determined...

$$(\zeta_1, u_1)\cdot(\zeta_2, u_2) = (\zeta_1 \zeta_2 \psi(u_1, u_2), u_1 + u_2)$$

$$\nu_\sigma(u_1)\cdot \nu_\sigma(u_2) = \nu_\sigma(u_1 + u_2)\, \psi(u_1, u_2)$$

<u>claim</u> $\nu_\sigma(u) = \left(\dfrac{\psi(u\sigma, u\sigma)}{\psi(u,u)}\right)^{1/2}$

<u>$p > 2$</u> works.

<u>Idea</u>

$$\begin{tikzcd} \rho : H \arrow[r] \arrow[d, "\sigma"'] & GL(V) \\ H \arrow[ur, "\rho^\sigma := \rho \circ \sigma"'] & \end{tikzcd}$$

By Stone von Neumann $\rho \sim \rho^\sigma$

$$R(\sigma)^{-1} \rho(h) R(\sigma) = \rho(h^\sigma)$$

R well defined up to scalar by Schur.