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Source: *The American Mathematical Monthly*, Vol. 86, No. 5 (May, 1979), pp. 380-382
Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America
Stable URL: https://www.jstor.org/stable/2321099
Accessed: 31-07-2024 13:05 UTC

Research for this paper was partially supported by NRC grants A-8054 and A4748.

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CLASSROOM NOTES

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ON THE USE OF A DIFFERENTIABLE HOMOTOPY IN THE PROOF OF THE CAUCHY THEOREM

R. VÝBORNÝ

1. Introduction. The homotopy version of the Cauchy Theorem has been well presented on several occasions (see, e.g., [1], [2]). The aim of this paper is to present a particularly simple proof using a differentiable homotopy. We offer two proofs, one for continuously differentiable f and one for the general case.

2. Notation. An open connected set in $\mathbb{C}$ will be denoted by G, its boundary by ∂G. The letter f will stand for a function which possesses a derivative f' everywhere in G. We shall integrate over paths. *A path* ϕ will be a continuous map $\phi:[0,1]\to G$ (or $\mathbb{R}^2$); a continuously differentiable path will be called *a road*. A path ϕ is said to be *closed* if $\phi(0)=\phi(1)$. In the sequel S will denote the square $[0,1]\times[0,1]$ and σ the path whose geometric image is ∂S.

3. Differentiable homotopy. DEFINITION. Two paths ϕ and ψ are said to be homotopic if there exists a function $H:S\to G$ such that (i) $H(0,s)=\phi(s), H(1,s)=\psi(s)$ for every $s\in[0,1]$; (ii) H is continuous on S; and either (iii) $H(t,0)=a, H(t,1)=b$ for all $t\in[0,1]$; or (iv) $H(t,0)=H(t,1)$ for all $t\in[0,1]$.

The function H is called a homotopy.

REMARK. If ϕ and ψ are homotopic and (iii) holds, one speaks of a homotopy with fixed ends; if (iv) holds, one says that ϕ and ψ are homotopic as closed paths. Since no confusion can arise in this paper, we shall use the abbreviated term homotopy (or homotopic).

DEFINITION. Two paths ϕ and ψ are said to be smoothly homotopic if they are homotopic and the function H satisfies

(v) H_t, H_s, H_{ts} exist and are continuous on S. (On ∂S the derivatives are understood to be "one-sided.")

THEOREM 1. *If ϕ and ψ are two smoothly homotopic roads then*

$$\int_\phi f = \int_\psi f. \tag{1}$$

The first proof uses the additional assumption that f' is continuous in G. Consider

$$I = \int_\sigma f dH = \int_\sigma f(H(t,s))(H_t(t,s)dt + H_s(t,s)ds). \tag{2}$$

By Green's Theorem

$$I = \int\int_S \left(\frac{\partial f H_s}{\partial t} - \frac{\partial f H_t}{\partial s}\right) dt ds.$$

However,

$$\frac{\partial f H_s}{\partial t} = f' H_t H_s + f H_{st} = f' H_s H_t + f H_{ts} = \frac{\partial f H_t}{\partial s}.$$

Consequently, $I = 0$. Using either (iii) or (iv) to evaluate I directly, we have

$$I = \int_\phi f - \int_\psi f. \tag{3}$$

In the second proof, we also consider the integral (2). In view of (3), it is sufficient to show that $I = 0$. Assume, contrary to the desired result, that $I \neq 0$. Divide the square S into four equal squares S_1', S_2', S_3', S_4' as indicated in Figure 1, and denote their "boundary" paths by $\sigma_1', \sigma_2', \sigma_3', \sigma_4'$, respectively. Obviously,

$$I = \int_{\sigma_1'} f + \int_{\sigma_2'} f + \int_{\sigma_3'} f + \int_{\sigma_4'} f.$$

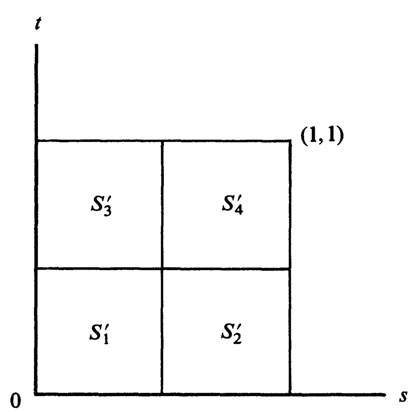

FIG. 1

It follows that there exists an integer $i, 1 \leqslant i \leqslant 4$ such that

$$\left|\int_{\sigma_i'} f\right| \geqslant \frac{|I|}{4}.$$

Denote $\sigma_1 = \sigma_i'$ and $S_1 = S_i'$. Now repeat this quartering process with S_1 and σ_1 and continue the process infinitely. We obtain a sequence of squares $S_1, S_2, \ldots$ and corresponding paths $\sigma_i, i = 1,2,\ldots$ such that

$$\left|\int_{\sigma_i} f\right| \geqslant \frac{|I|}{4^i}, i = 1,2,\ldots; \tag{4}$$

and $\bigcap_{i=1}^{\infty} S_i = \{(s_0, t_0)\}$. Let M be the common bound for $|H|, |H_t|, |H_s|$ and ϵ a positive number satisfying $8M^2\epsilon < |I|$. By the differentiability of f at $H(s_0,t_0) = z_0$ there exist a positive number δ and a function η such that the equation

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

with $|\eta(z)| < \epsilon$, holds for all $|z - z_0| < \delta$.

Let n be so large that $H(S_n) \subset \{z; |z - z_0| < \delta\}$; such an integer n certainly exists because H is continuous and the squares S_i shrink down to the point (s_0, t_0). Now the integral $\int_{\sigma_n} f(H_t dt + H_s ds)$ is the sum of three integrals

$$\int_{\sigma_n} f(z_0)\,dH, \quad \int_{\sigma_n} f'(z_0)(H - z_0)\,dH \quad \text{and} \quad \int_{\sigma_n} (H - z_0)\eta\,dH.$$

Direct evaluation shows that the first two integrals are zero. For the third one, we first observe that

$$|H(s,t) - H(s_0,t_0)| \leqslant M[|s - s_0| + |t - t_0|] \leqslant \frac{2M}{2^n}$$

and then we have the obvious estimate

$$\left|\int_{\sigma} \eta(H - z_0)\,dH\right| \leqslant \epsilon 2M \frac{1}{2^n} M \frac{4}{2^n} < \frac{|I|}{4^n}.$$

This contradicts (4). Hence $I = 0$ and the proof is complete.

4. Generalizations. There are several possible ways to generalize Theorem 1 to paths (not necessarily smooth). First, one can observe that the second proof remains valid if H_t and H_s exist with a possible exception of finitely many points, if they are bounded and if $H_t(\cdot,s)$ is piecewise continuous on $[0,1]$ for every $s \in [0,1]$ and $H_s(t,\cdot)$ is piecewise continuous on $[0,1]$ for every $t \in [0,1]$.

Using Theorem 1 and the homotopy H defined by $H(t,s) = t\psi(s) + (1-t)\phi(s)$ one can prove

THEOREM 2. *If ϕ and ψ are two roads such that the distance between ϕ and ψ is smaller than the distance of ϕ from ∂G and if either $\phi(0) = \psi(0)$, $\phi(1) = \psi(1)$ or ϕ and ψ are closed roads, then* (1) *holds.* (*Distance between ϕ and ψ is* $\sup\{|\phi(s) - \psi(s)|, s \in [0,1]\}$.)

The homotopy version of the Cauchy Theorem in full generality can be obtained from Theorem 2 using arguments similar to those presented in **[1]**.

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