Sep 4, 2007

Subrepn $W \subseteq V \circlearrowleft G$ **Quotient** $V/W \circlearrowleft G$

$GW \subseteq W$

Repn is irred if it has no ~~nontrivial~~ proper subrepn.

E.g. any 1-dim'l repn is irred.

Direct sums $V \oplus W$

Tensor **product** $g \cdot (v \otimes w) = gv \otimes gw$

Appendix B Fulton-Harris.

$\mathrm{Sym}^n V$, $\Lambda^n V$

quotients of $\underbrace{V \otimes \cdots \otimes V}_{n}$

$V \longrightarrow W$ G-linear

Ker, Im

$G = S_3$ 1-dim'l repn

- trivial
- sgn (or alternating)

defining repn

S_3 acts on $\{1, 2, 3\}$

$\rightsquigarrow$ linear repn. $V = \langle e_1, e_2, e_3 \rangle$

$$g e_i = e_{gi}, \quad g \in S_3$$

subrepn $\mathbb{C}(1, 1, 1) = W'$

$$W = \{ (a_1, a_2, a_3) \mid a_1 + a_2 + a_3 = 0 \}$$

$$V \simeq W \oplus W'$$

W 2-dim'l repn. **standard repn**

$\tau = (1\,2\,3)$ $\sigma = (1\,2)$

Consider W as $\langle \tau \rangle$-module repn

$$\rho : S_3 \longrightarrow GL(W)$$

~~$\rho(g)$~~

In general $\rho(g)$ is diagonalizable

$\rho(g)^N = 1$, $g^N = 1$, G finite

minimal polynomial of $\rho(g) \mid x^N - 1$

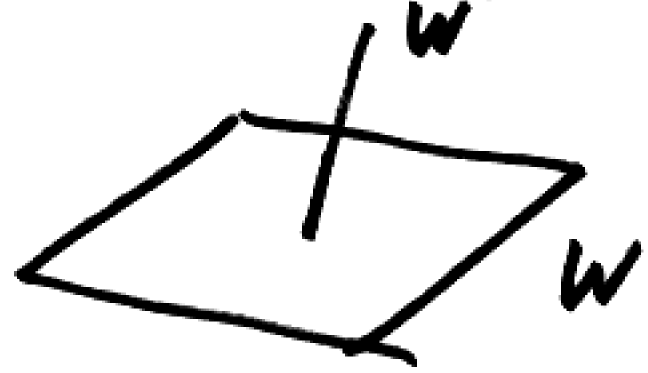
Char $= 0 \to$ distinct roots

$\to$ $\rho(g)$ is diagonalizable

$\begin{pmatrix} * & 1 & & \\ & * & 1 & \\ 0 & & * & 1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ finite order in char $p > 0$

Eigenspaces of τ acting on W

W: $\quad a_1 + a_2 + a_3 = 0$



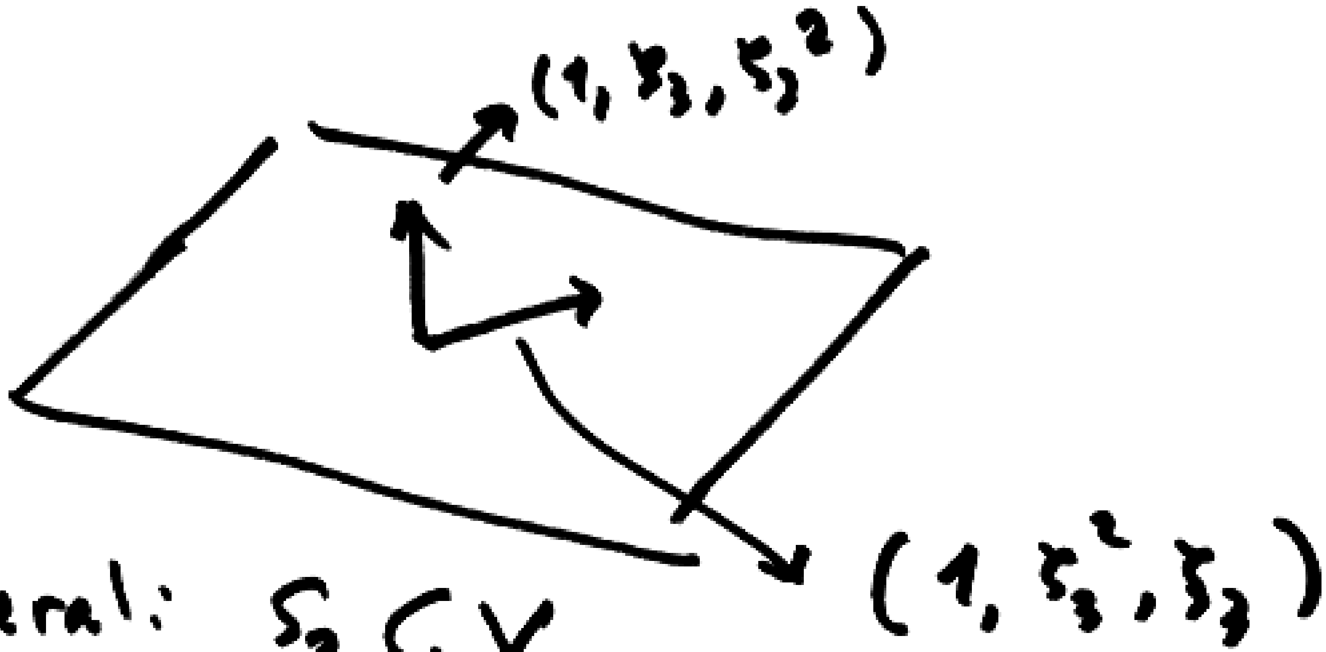
$\mathbb{C}(1, \zeta_3, \zeta_3^2) = W_{\zeta_3}$

$\tau(1, \zeta_3, \zeta_3^2) = (\zeta_3, \zeta_3^2, 1)$

$= \zeta_3(1, \zeta_3, \zeta_3^2)$

$$\mathbb{C}(1, \zeta_3^2, \zeta_3) = W_{\zeta_3^2}$$

$$\tau(1, \zeta_3^2, \zeta_3) = \zeta_3^2(1, \zeta_3^2, \zeta_3)$$



In general: $S_3 \circlearrowright V$

$$\tau\sigma = \sigma\tau^{-1}$$

$$\tau v = \lambda v \qquad \lambda \in \mathbb{C}^\times$$

$$\tau\sigma v = \sigma\tau^{-1} v = \lambda^{-1}\sigma v$$

In W $\quad \sigma : W_{\zeta_3} \longleftrightarrow W_{\zeta_3^2}$

$$\sigma(1, \zeta_3, \zeta_3) = (\zeta_3, 1, \zeta_3^2)$$

$$= \zeta_3(1, \zeta_3^2, \zeta_3)$$

In basis $(1, \zeta_3, \zeta_3^2)$
$(1, \zeta_3^2, \zeta_3)$

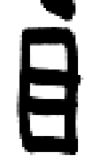
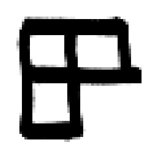
$$\rho(\tau) = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}$$

$$\rho(\sigma) = \begin{pmatrix} 0 & \zeta_3^2 \\ \zeta_3 & 0 \end{pmatrix}$$

No line of W is fixed by S_3
(i.e. both τ & σ)

$\Rightarrow$ W is irreducible.

<u>Summary</u>: Irred. repn of S_3

- trivial  $3 = 1+1+1$
- sgn  $3 = 3$
- standard  $3 = 2+1$

<u>Claim</u> Any repn of S_3 is a direct sum of copies of these.

<u>pf</u> Decompose V into $\rho(\tau)$

eigenspaces

$$V \simeq V_1 \oplus V_{\zeta_3} \oplus V_{\zeta_3^2}$$

$$\sigma \circlearrowright V_1 \simeq \underset{\substack{\wedge \\ \oplus \text{trivial}}}{V_1^+} \oplus \underset{\substack{\wedge \\ \oplus \text{sgn}}}{V_1^-}$$

$$V_{\zeta_3} \overset{\sigma}{\longleftrightarrow} V_{\zeta_3^2}$$

$$v_1, \ldots, v_m \qquad \zeta_3^{-1}\sigma(v_1), \ldots, \zeta_3^{-1}\sigma(v_m)$$

$$\langle v_i, \underbrace{\zeta_3^{-1}\sigma(v_i)}_{\substack{\| \\ w_i}} \rangle \simeq W$$

$$\tau : \begin{cases} v_i \longmapsto \zeta_3 v_i \\ w_i \longmapsto \zeta_3^2 w_i \end{cases}$$

$$\sigma : \begin{cases} v_i \longmapsto \zeta_3 w_i \\ w_i \longmapsto \zeta_3^2 v_i \end{cases}$$

$$V_{\zeta_3} \oplus V_{\zeta_3^2} \simeq \underbrace{W \oplus \cdots \oplus W}_{m}$$

Dual Repn

V , V^* dual vector space

Bilinear non-degenerate pairing

$$\langle\, , \rangle : \quad V^* \times V \longrightarrow \mathbb{C}$$
$$v^* , v \longmapsto v^*(v)$$

Want to define action on V^* such that this pairing is preserved

I. e.

$$\langle g \cdot v^*, g \cdot v \rangle = \langle v^*, v \rangle$$

$$\rho : G \longrightarrow GL(V)$$

want to define

$$\rho^* : G \rightarrow GL(V^*)$$

$$\langle \rho^*(g) v^*, \rho(g) v \rangle = \langle v^*, v \rangle$$

$$\varphi : V \longrightarrow V$$

$$^t\varphi : V^* \longrightarrow V^*$$

$$^t\varphi(v^*)(v) = v^*(\varphi(v))$$

$$\langle {}^t\varphi(v^*), v \rangle = \langle v^*, \varphi(v) \rangle$$

want

$$\langle v^*, \underbrace{{}^t\rho^*(g)\,\rho(g)}_{= \mathrm{id}_V} v \rangle = \langle v^*, v \rangle$$

$$^t\rho^*(g)\,\rho(g) = \mathrm{id}_V$$

$$^t\rho^*(g) = \rho(g^{-1})$$

$$\boxed{\rho^*(g) := {}^t\rho(g^{-1})}$$

Choosing basis

$$\langle v^*, v \rangle = {}^t v^* \cdot v$$

$$\langle {}^tA v^*, v \rangle =$$

$$= {}^t({}^tAv^*)v$$
$$= {}^tv^*Av = \langle v^*, Av\rangle$$

$$g\cdot({}^tv^*)\cdot gv = {}^tv^*\cdot v$$
$$\parallel$$
$${}^t({}^tg^{-1}v^*)\cdot gv = {}^tv^*\cdot v \checkmark$$

Complete Reducibility

$0 \subsetneq W \subsetneq V$ subrepn has a complement

pf Start with arbitrary pos defn Hermitian form on V

$$H_0(u,v) := \sum_{j=1}^{n} \bar{u}_j v_j$$

$u = (u_1, \ldots, u_n)$
$v = (v_1, \ldots, v_n)$

$$H_0(\lambda u, v) = \bar{\lambda}\, H_0(u,v)$$
$$H(v,u) = \overline{H(u,v)}$$

Want H Hermitian form
G-stable i.e.

$$H(gu, gv) = H(u,v)$$

b/c then orthog. complement to W, say, $W^\perp$ is the complement we want.

$$V = W \oplus W^\perp$$

$$v \in W^\perp$$

$$w \quad H(gv, w) = H(v, \underbrace{g^{-1}w}_{\in W}) = 0$$

all $w \in W$

$$\Rightarrow gv \in W^\perp$$

To find H we <u>average</u> over G

$$H(u,v) := \sum_{g \in G} H_0(gu, gv)$$

Schur's Lemma

V, W irred. repn. of G

$\varphi : V \to W$ G-linear

1) Either

φ is zero or φ is an isom.

2) If $V = W$ then $\varphi = \lambda \, id_V$

for some $\varphi \in \mathbb{C}$.

Pf 1) $\ker \varphi \subseteq V$ subrepn

Hence either $\ker\varphi = 0$ or $\ker\varphi = V$

2) Pick eigenvalue λ

$\ker(\varphi - \lambda \, id_V) \neq 0$

subrepn of G $\qquad \varphi(v) = \lambda v$

$\varphi(gv) = g\,\varphi(v) = \lambda g v$

V irred $\Rightarrow$ $\varphi = \lambda \, id_V$ $\quad \square$

Cor G abelian, V irred
$\Rightarrow \dim V = 1$

Pf. $g \in G$, $\rho(g) : V \to V$

$$\rho(g)(g' v) = \rho(gg')(v)$$
$$= \rho(g'g)(v)$$
$$= g' \cdot \rho(g)(v)$$

$\rho(g)$ is a scalar
Every subspace is fixed by G
Hence V irred $\Rightarrow \dim V = 1$ $\square$

Character

$$\rho : G \to Gl(V)$$

assoc. its **character** $\rho(g)$

$$g \mapsto \chi_\rho(g) := \mathrm{Tr}(g|_V)$$

well defined for the class of ρ

$$\mathrm{Tr}(\varphi^{-1}\rho(g)\varphi) = \mathrm{Tr}(\rho(g))$$

χ is a function on G (values in $\mathbb{C}$).