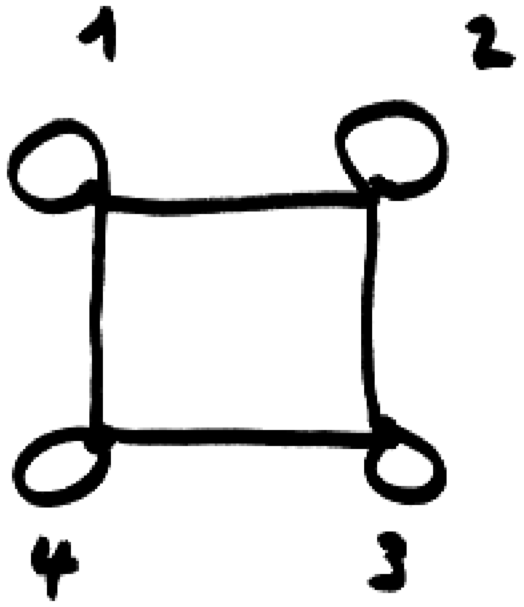
April 12, 2007



$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

(Adjacency matrix of the graph)

ij entry of $A = 1$ if $i — j$

0 otherwise

Initial state : $s_I = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$

Move : $t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}$

$s_F = s_I + At$ final state

goal $\quad S_F = 0$

$$S_I + At = 0$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 + t_4 \\ t_1 + t_2 + t_3 \\ t_2 + t_3 + t_4 \\ t_1 + t_3 + t_4 \end{pmatrix}$$

$$\boxed{S_I = -At}$$

$$\begin{cases} t_1 + t_2 + t_4 = -s_1 \\ t_1 + t_2 + t_3 = -s_2 \\ t_2 + t_3 + t_4 = -s_3 \\ t_1 + t_3 + t_4 = -s_4 \end{cases}$$

Linear system of equations

solving puzzle $\longleftrightarrow$ solving system of equations.

$n \times n$ matrix A

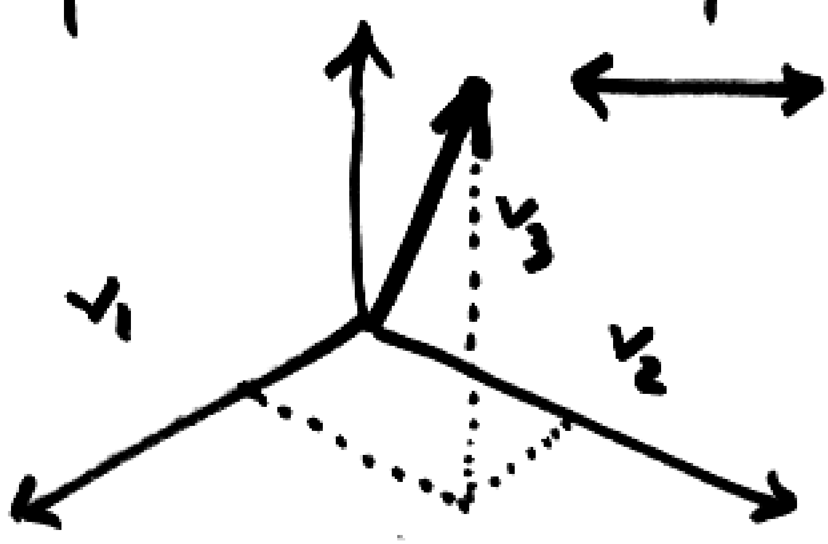
$A = (a_{ij})$ $\qquad a_{ij}$ numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & & a_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & & \cdots & a_{nn} \end{pmatrix}$$

<u>vectors</u>: $\quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

<u>$n=3$</u> $\quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\rightsquigarrow$ point in space



$\longleftrightarrow \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$V = \{ \text{all } n\text{-dim'l vectors} \}$

$A \quad n \times n$ matrix

$v \in V \quad$ vector

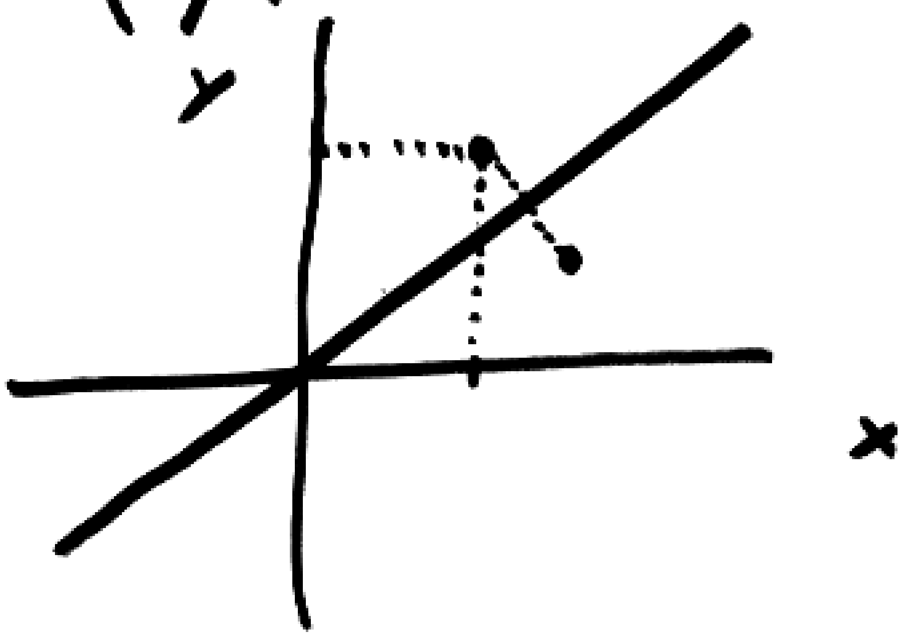
$\rightsquigarrow \quad A \cdot v \in V$

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\underline{n=2}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Effect of multiplication by A?

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} y \\ x \end{pmatrix}$$



I.e. A is the reflection through the $y=x$ line.

- Identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$I_n \cdot v = v$$

- $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2 I_2$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

scaling by two.

<u>Scalars</u> a, $\quad a v = \begin{pmatrix} a v_1 \\ a v_2 \\ \vdots \\ a v_n \end{pmatrix}$

- Rotations can also be written in terms of matrices.

Features of transformation

$$v \longmapsto A v$$

= <u>linear transformation</u>

- $A \cdot (a v) = a (A v)$
- $A \cdot (u + v) = A u + A v$

- $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

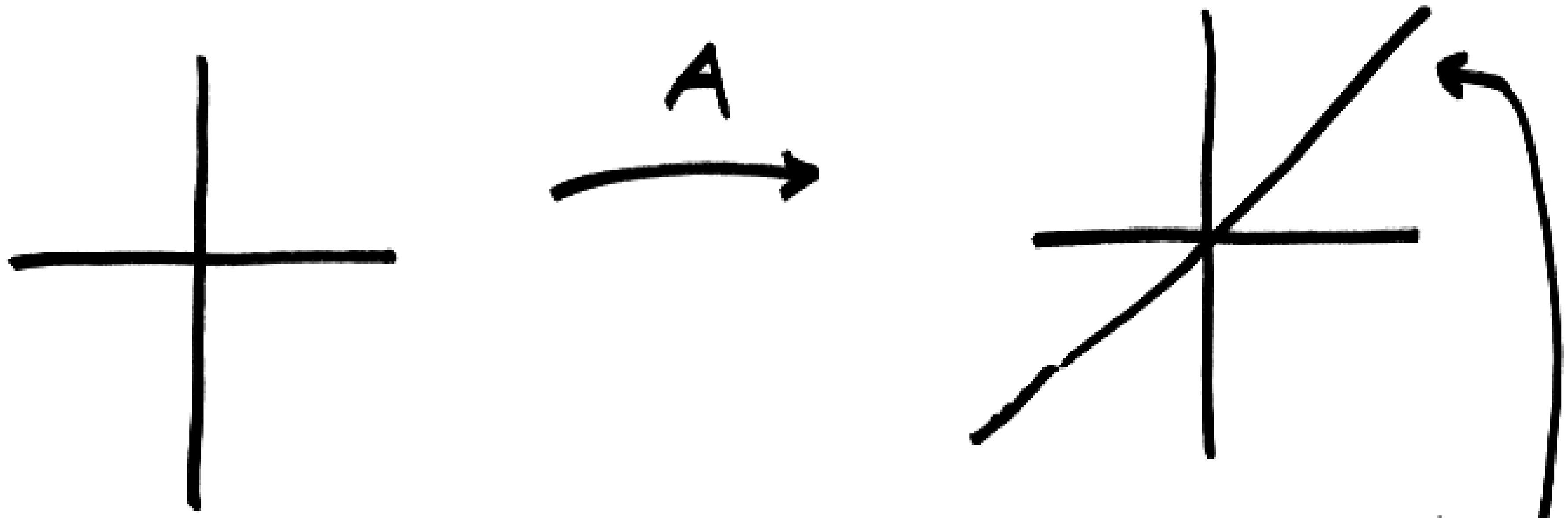


Image of points is on this line

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

this system of equations not always has a solution. If $a \neq b$ we have no solution. If $a = b$ we can solve it

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}$$

$$\left.\begin{array}{l} x+y=a \\ x+y=a \end{array}\right\} \longrightarrow \boxed{x+y=a}$$

To solve: Pick any x and set $y = a - x$

system $\boxed{Av = u}$

$v = \begin{pmatrix} x \\ y \end{pmatrix}$

$u = \begin{pmatrix} a \\ b \end{pmatrix}$

- <u>NO</u> solution unless $u = \begin{pmatrix} a \\ a \end{pmatrix}$

- If $u = \begin{pmatrix} a \\ a \end{pmatrix}$ then it has a lot of solutions.

all solutions : $\begin{pmatrix} x \\ a-x \end{pmatrix}$

For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ situation is different.

$$\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$y = a$$
$$x = b$$

Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$

Every $\begin{pmatrix} a \\ b \end{pmatrix}$ has a <u>unique</u> solution.

Say A is <u>invertible</u> if it has an inverse

$$A^{-1} \cdot A = A \cdot A^{-1} = I_n$$

In this case

$$A v = u$$

can be solved by multiplying by A^{-1}.

$$A^{-1}(Av) = A^{-1} \cdot u$$

$$(A^{-1}A)v = A^{-1}u$$

$$\boxed{v = A^{-1}u}$$

Unique solution for each choice of u.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^{-1} = A$$

- How do we determine if a matrix A has an inverse?

There is a number called the <u>determinant</u> of A $\quad \det(A)$

A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}^{= A^*} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \det(A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- If $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A A^* = \det(A) \cdot I_2$$

- If $\det(A) = 0$ then $\quad A A^* = 0$

if A had an inverse

$$A^{-1}(A A^*) = 0$$

$$(A^{-1}A) A^* = A^* \quad \text{not the case}$$