April 17, 2007

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \text{3-dim vectors} \right\}$$

3×3 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$v \in V$

$$A \cdot v \in V$$

$$V \longrightarrow V$$

$$v \longmapsto A \cdot v$$

- $$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$$

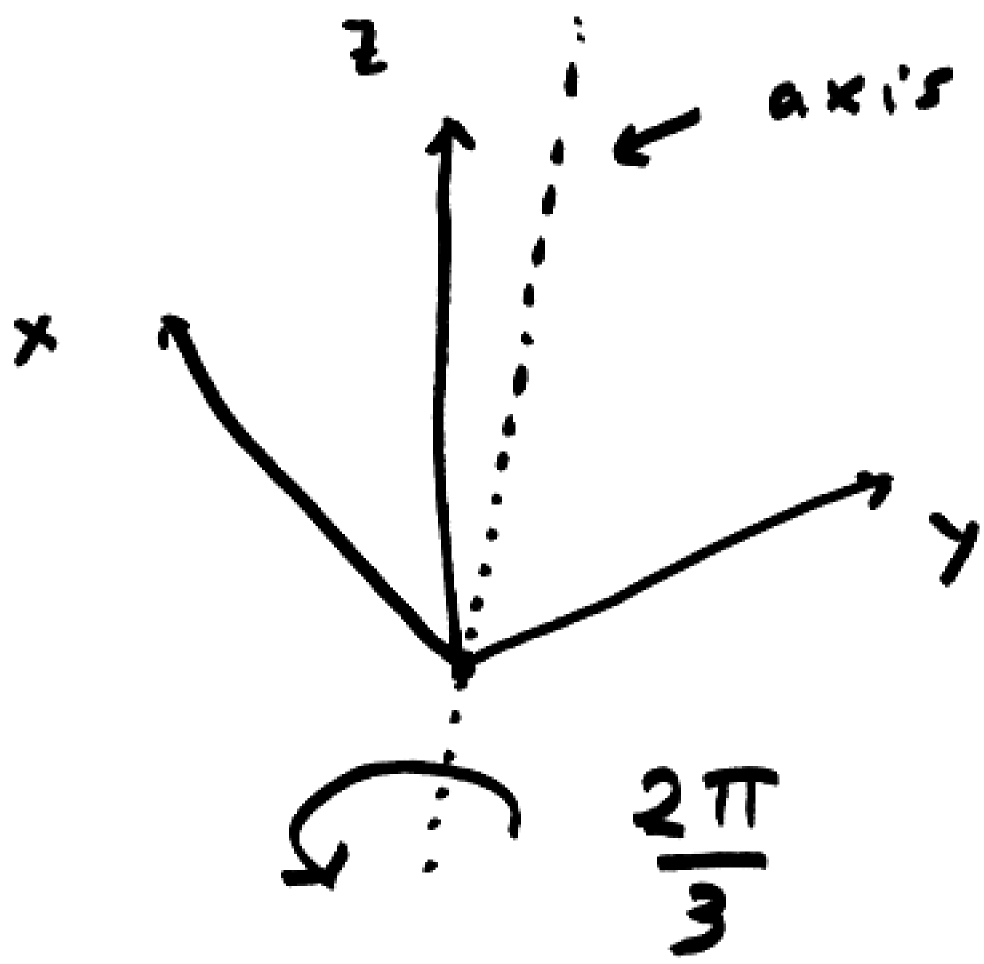
$$(x, y, z) \longmapsto (z, x, y)$$

Axis of rotation?

$$(a, a, a) = a(1, 1, 1)$$

fixed by A.

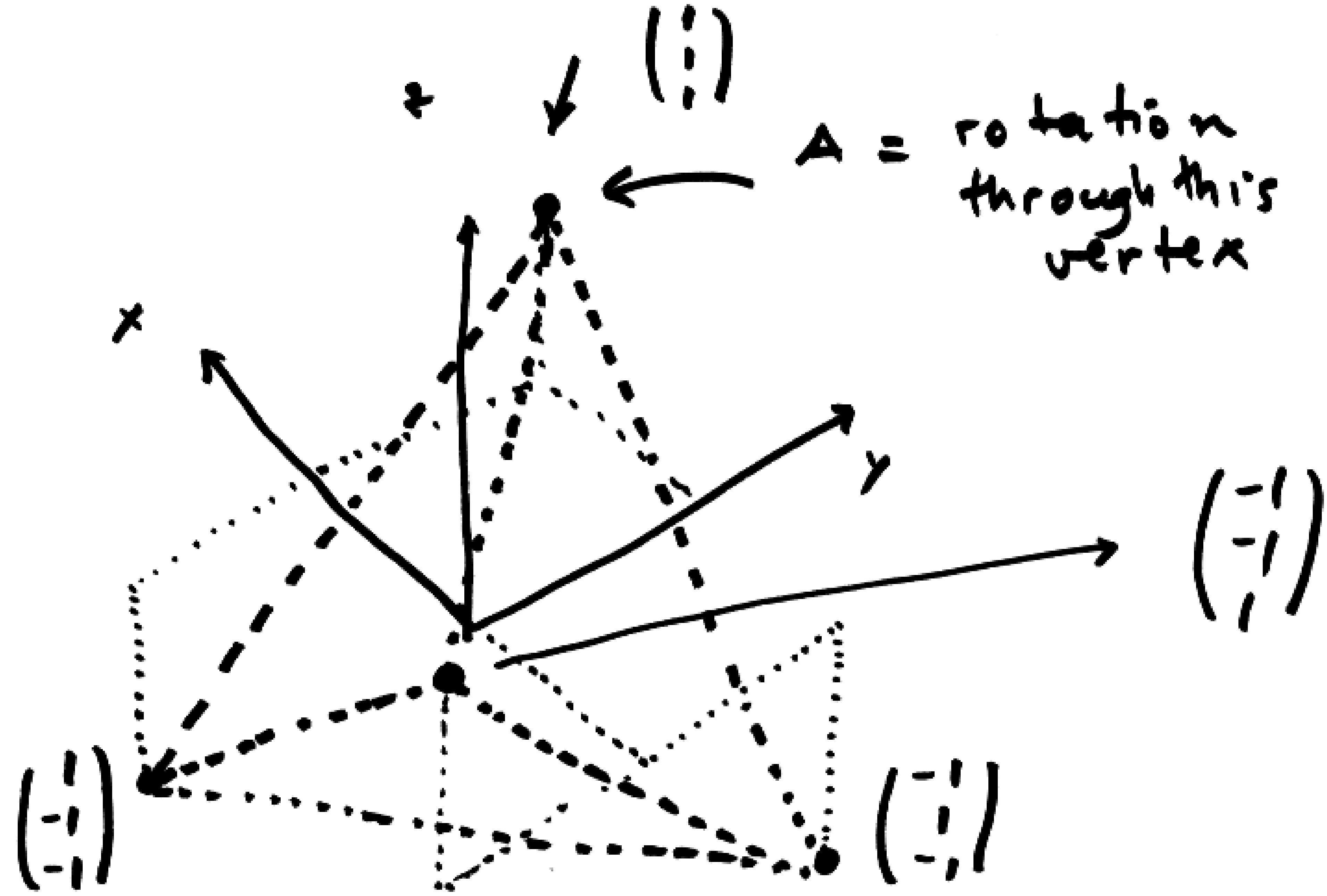
$x = y$
$y = z$



$$\begin{array}{c} (x, y, z) \\ \downarrow \\ (z, x, y) \\ \downarrow \\ (y, z, x) \end{array}$$

Four points

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$



$$\frac{1}{4}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\-1\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\\1\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right)$$

= center of mass.

$$= \begin{pmatrix}0\\0\\0\end{pmatrix}$$

R_V $(= R_F)$ rotations assoc. to vertices (faces)

R_E rotations assoc. to edges

- $$R_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$R_E \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

We can describe the action of the rotation group of (this) tetrahedron as follows:

cyclically permute (x, y, z) and change two signs.

E.g.

$$(x, y, z) \longmapsto (-y, -z, x)$$

(<u>signed permutation</u>)

(x, y, z) $\quad (-x, -y, z)$ $\quad (-x, y, -z)$ $\quad (x, -y, -z)$

(z, x, y) $\quad (-z, -x, y)$ $\quad (-z, x, -y)$ $\quad (z, -x, -y)$

(y, z, x) $\quad (-y, -z, x)$ $\quad (-y, z, -x)$ $\quad (y, -z, -x)$

Total of 12.

group of rotations of tetrahedron $\cong A_4$ (alternating group)

Rotations of cube (octahedron)

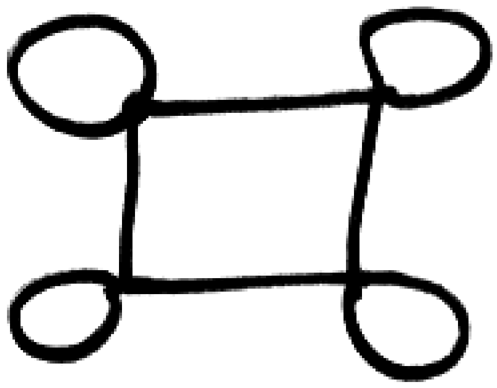
we can think of two inscribed tetrahedra inside cube.

Rotations of tetrahedron & swapping both tetrahedra.

get all signed permutations of 3-dim determinant = 1

group of rotations of cube $\simeq S_4$

April 19, 2007



$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Equation to solve

$$\boxed{S_I + At = 0}$$

$$t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} \qquad t_i = 0, 1$$

$\updownarrow$

dont / press button i

If A has an inverse A^{-1} then we can solve for t

$$\boxed{-A^{-1} S_I = t}$$

Row reduction algorithm.

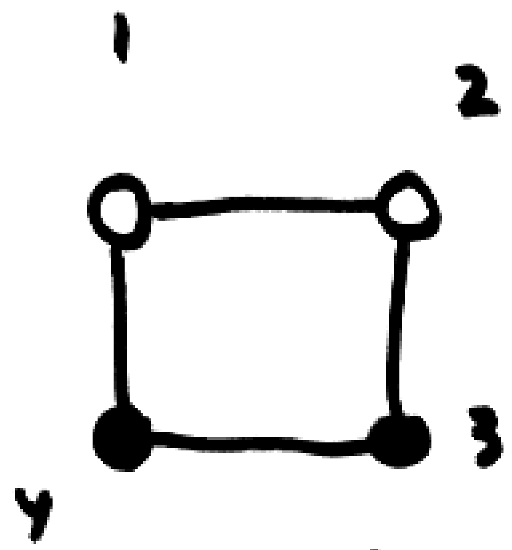
$$A^{-1} = A$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

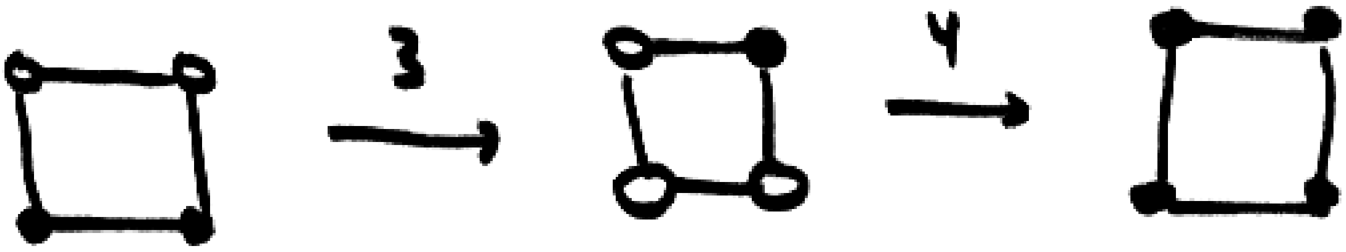
(mod 2
binary
sense)

$$\boxed{t = A s_I}$$

E.g.

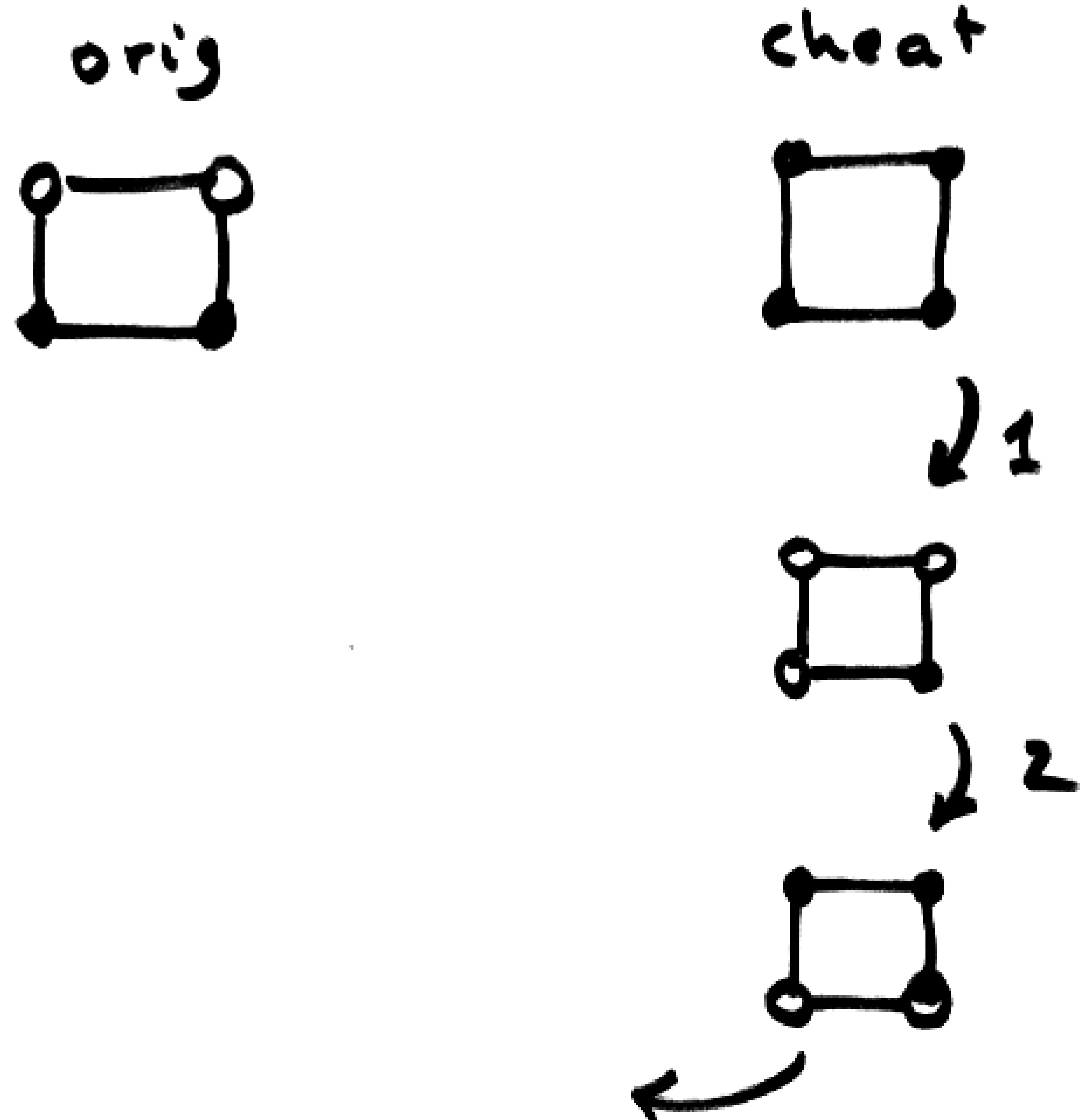


$$t = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

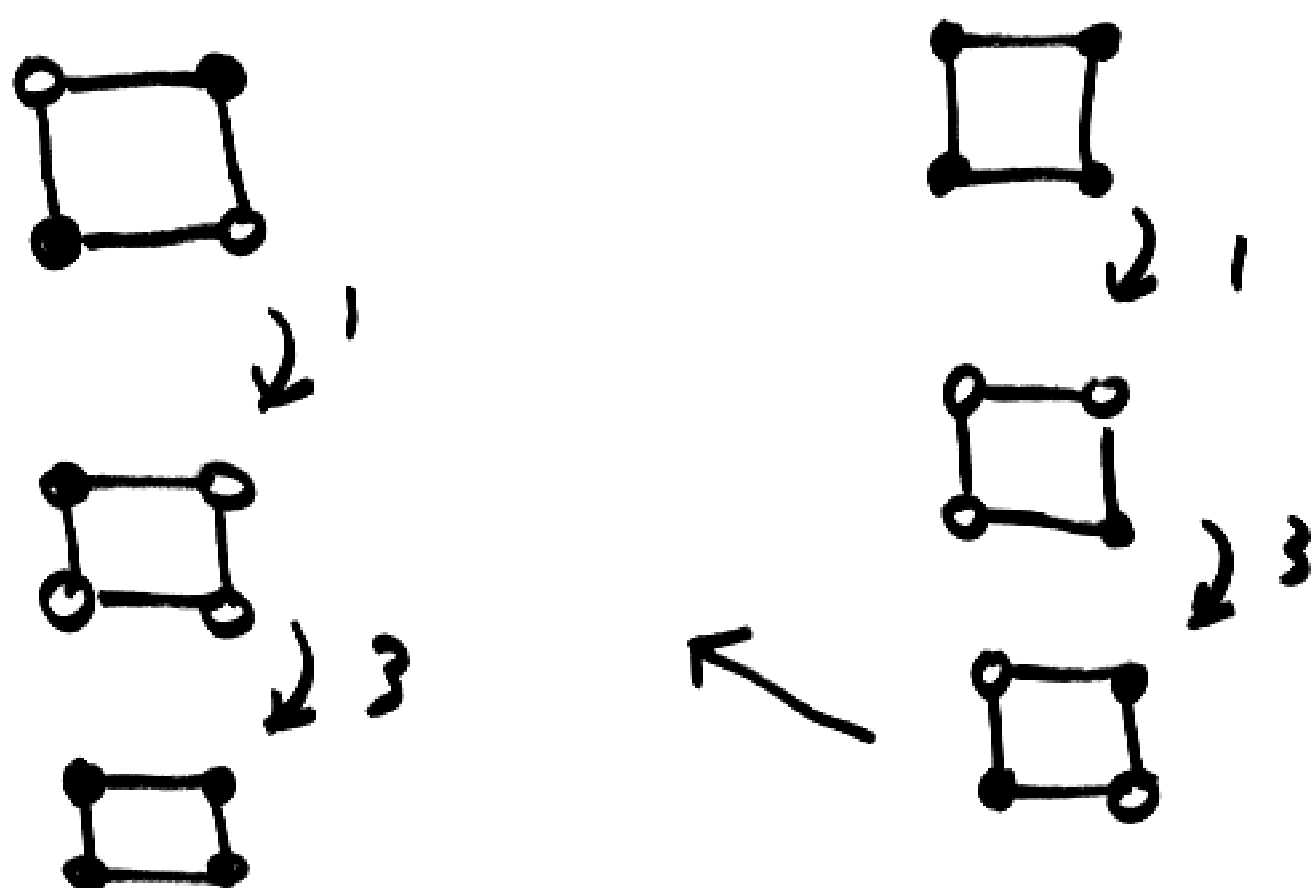


$-A^{-1} \rightsquigarrow$ cheat puzzle

In our case cheat puzzle = original puzzle b/c $-A^{-1} = A$.



Solution: press buttons 3 & 4.



- If A has an inverse then every initial state can be solved in only one way.

- If A has no inverse then some initial states will not be solvable. And when solvable there will be more than one way to do it.

This dicothomy on A depends on how many states each light could be in.

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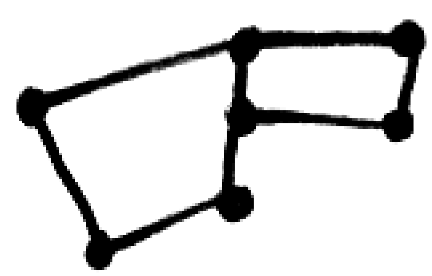
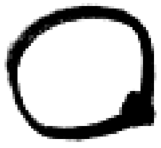
general 15 puzzle

Can play it on a <u>simple</u> graph

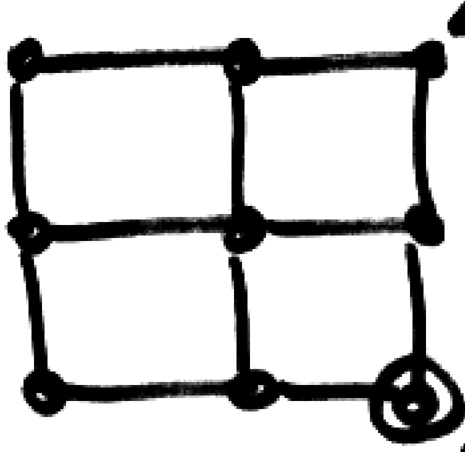
it has no loops or multiple edges




loop


multiple edge

#'s


Blank

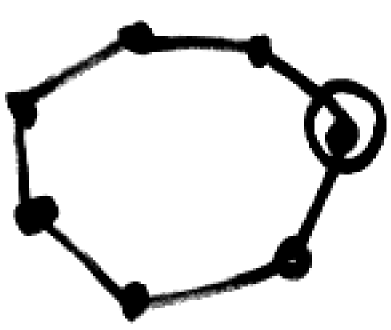
<u>Move</u>: Exchange Blank w/ nbh. #.

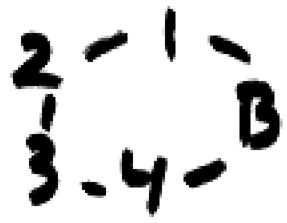
Move will take a blank on a grand tour of graph.

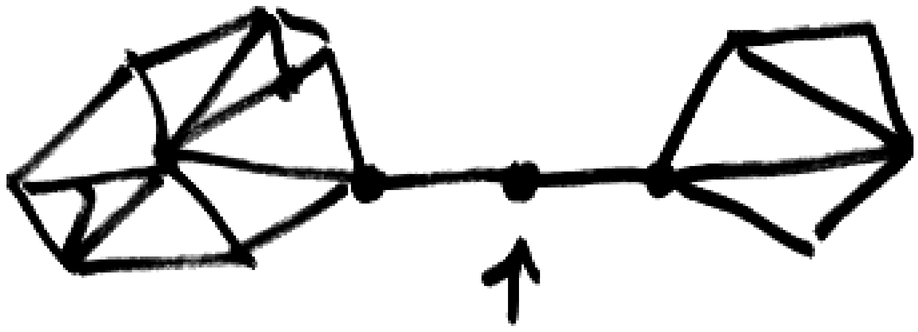
Each such path gives a permutation of the #'s.

<u>Question</u>: What permutations do we get?

Theorem of Wilson gives answer.

-  ⇝ cycle permutation



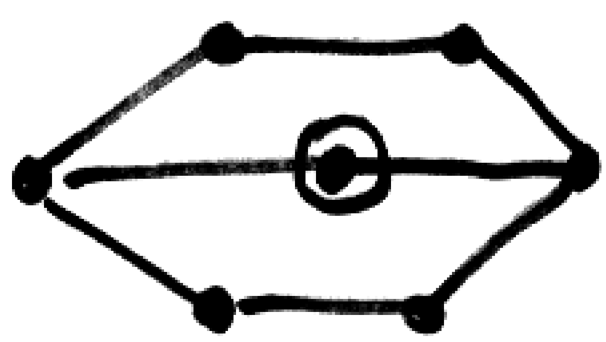
- 

take away this vertex
graph disconnects

two separate puzzle one per
component

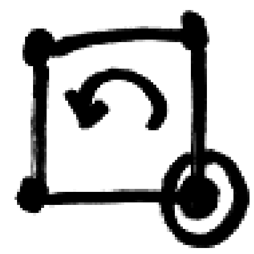
In all other cases the group
is either the symmetric group OR
the alternating group (even
permutation)

<u>Except</u> for this graph(!)

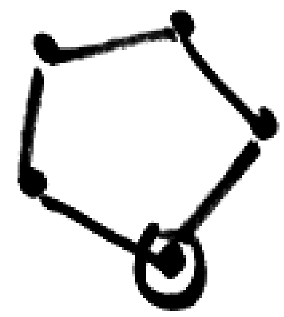


group has 120 elements.
A priori our group could be as
large as $6! = 720$.

How can we tell symmetric/alternating apart?

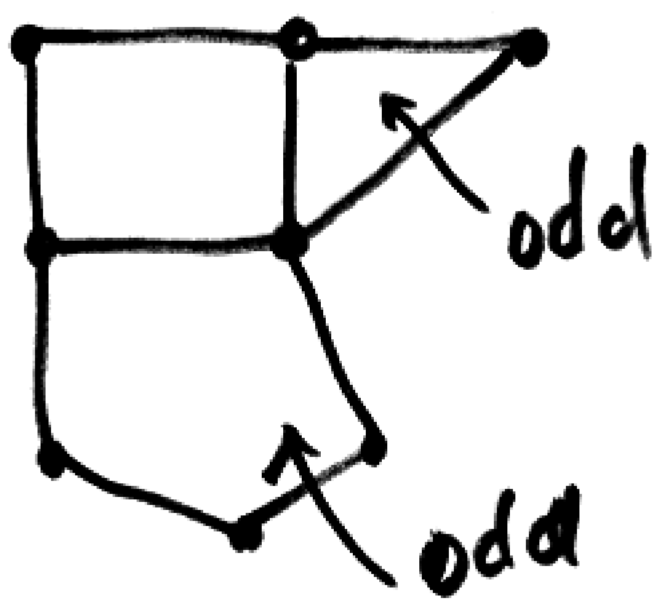


$\rightsquigarrow$ 3-cycle (even)



$\rightsquigarrow$ 4-cycle (odd)

As soon as we have an k-cycle with k even the group is all permutations.

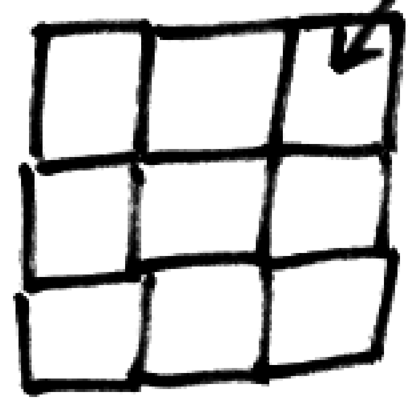


group $\simeq S_7$

A_n (even permutations only) $\longleftrightarrow$ all cycles in graph are even

E.g. 15-puzzle

4-vertices



$\rightsquigarrow$ group $\simeq A_8$

$\leftrightarrow$ graph bipartite

((color vertices •/o such that no two nbhd. vertices have same color