Sep 13, 2007

- $\mathrm{Hom}(V, W) \simeq V^* \otimes W$
- $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W$
- $\underset{\substack{\uparrow \\ \text{fixed homom.}}}{\mathrm{Hom}(V, W)^G} = \underset{\substack{\uparrow \\ G\text{-linear}}}{\mathrm{Hom}_G(V, W)}$
- $(\chi_V, 1) = \dim V^G$
- $(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V, W)$

Inner product is canonical

- <u>Schur's Lemma</u>

  $\{\chi_V\}$ V irred.
  are orthonormal
  $\Rightarrow$ lin. indep.

<u>Cor</u>

1) A repn is uniquely characterized (up to isom) by its character.

$$V = \bigoplus \text{irred.}$$
$$= \bigoplus_j U_j \qquad U_j \text{ irred.}$$

$$\Rightarrow \quad \chi_V = \sum_j \chi_{U_j}$$

Let U be an irred. repn of G.

$$(\chi_V, \chi_U) = \sum_j (\chi_{U_j}, \chi_U)$$
$$= \#\{j \mid U \simeq V_j\}$$
$$=: a_U$$

In any decomp of V as a sum of irreducibles the number of factors isom to U is a_U.
This is called the <u>multiplicity</u> of U in V.

Conversely

$$\chi_V = \sum_U a_U \chi_U$$

U runs over irreducibles $a_U \in \mathbb{C}$

then $a_U = (\chi_V, \chi_U)$

$$\Leftrightarrow \quad V \simeq \bigoplus_U U^{a_U}$$

$$U^{\oplus a_U} = \underbrace{U \oplus \cdots \oplus U}_{a_U}$$

Two repn with same character are isom.

2) V is irreducible iff

$$(\chi_V, \chi_V) = 1$$

$$1 = (\chi_V, \chi_V) = \sum_U a_U^2$$

$$a_U \neq 0 \Rightarrow a_U \geq 1$$

$\Rightarrow a_U = 0$ except for one U.

3) $\#$ irred repns of G $\leq$ $\#$ conj classes in G

$\langle \chi_U \mid U \text{ irred} \rangle \leq$ class functions

[In partic. $\#$ irred repns $< \infty$

dim ... $\leq$ dim ...

(we'll see we have $=$ actually)

4) $$(\chi_V, \chi_{reg}) = \chi_V(1) \cdot \frac{|G|}{|G|} = \chi_V(1) = \dim V$$

$$Reg = \bigoplus_U U^{\dim U}$$

Take dim on both sides

$$\boxed{|G| = \sum_U (\dim U)^2}$$

$$\delta_1(g) := \begin{cases} 1 & g = 1 \\ 0 & g \neq 1 \end{cases}$$

$$\boxed{\delta_1 = \sum_U \frac{\chi_U(1)}{|G|} \cdot \chi_U}$$

In the abelian case

$$\delta_1 = \frac{1}{|G|} \sum_{\chi} \chi$$

$$\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$$

$\underline{G = S_4}$

	1	6	8	6	3
	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	−1	1	−1	1
V	3	1	0	−1	−1
V'	3	−1	0	1	−1
W	2	0	−1	0	2

defining = trivial ⊕ V

defining	4	2	1	0	0
V	3	1	0	−1	−1

$$(\chi_V, \chi_V) = \frac{1}{24}\left(3^2 \times 1 + 1^2 \times 6 + 0^2 \times 8 + (-1)^2 \times 6 + (-1)^2 \times 3\right)$$

$$= 1$$

$\Rightarrow$ V irred.

V standard repn

W irred U 1-diml $\Rightarrow$ $W \otimes U$ irred.

So far we have: U, U', V, V'

$$1^2 + 1^2 + 3^2 + 3^2 = 20$$

There is at most one other repn since # conjugacy classes is 5

$\Rightarrow$ missing repn has dim 2

Ideas for "finding" W

- cosets of a subgp e.g. $A_4, S_3, \ldots$
- Λ, Sym of V or V'.
- permutation repns of S_4

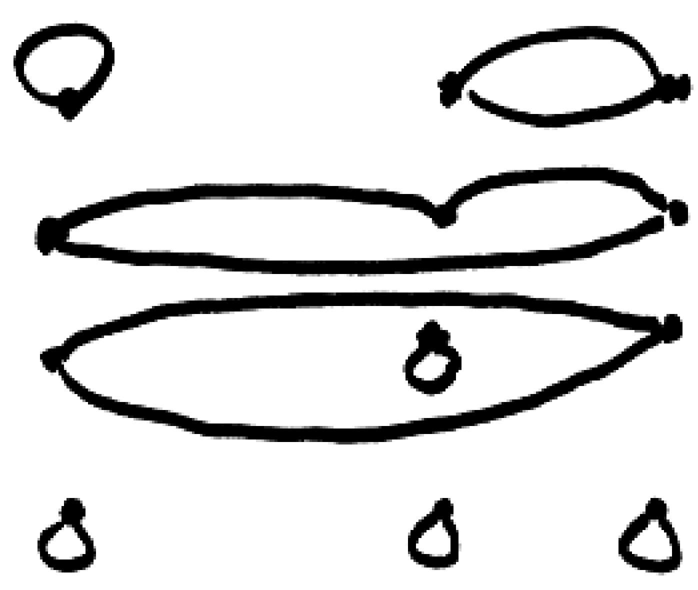
e.g. on a given conjugacy class

$S_4 \circlearrowright \{(12)(34), (13)(24), (14)(23)\}$ acting by conjugation

- $$\begin{tikzcd} S_4 \arrow[r] \arrow[dr, dotted] & S_3 \arrow[d, "\text{stand}"] \\ & Gl_2(\mathbb{C}) \end{tikzcd}$$

$\rightarrow$ has to be W as standard of S_3 is irred hence so is as an S_4 repn.

- S_4 as rotations of a cube acting on pairs of opposite faces



	$(\cdot)(\cdot\cdot)$	1
(12)	$(\cdot\cdot\cdot)$	0
(123)	$(\cdot)(\cdot\cdot)$	1
(1234)		
$(12)(34)$	$(\cdot)(\cdot)(\cdot)$	3

3-dim'l rep'n of S_4

	3	1	0	1	3	
− triv	1	1	1	1	1	
	2	0	−1	0	2	$\to W$

$\underline{G = A_4}$

	1	4	4	3
	1	(123)	(132)	(12)(34)
1	1	1	1	1
	1	ω	ω^2	1
	1	ω^2	ω	1
standard	3	0	0	−1

ω = primitive cubic root of 1

$$\begin{array}{ccc} S_4 & \to & S_3 \\ \cup\!\! \mid & & \cup\!\! \mid \\ A_4 & \to & A_3 \cong \mathbb{Z}/3\mathbb{Z} \\ & \searrow & \downarrow \\ & & \mathbb{C}^\times \end{array}$$

$1^2 + 1^2 + 1^2 = 3$

$12 - 3 = 9 = \dim^2$

$\Rightarrow \dim = 3$