SF3580
HW 4

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Task 1

We implement the naive approach for solving the Lyapunov equation and compare it to the Bartels-Stewart method on random matrices. The dependence of the simulation time on n is illustrated in Figure 2. Theoretically, the complexity is as bad as $\mathcal{O}(n^6)$ for the naive approach and $\mathcal{O}(n^3)$ for the improved algorithm. However, the simulation time is in this test smaller due to the use of improved linear solvers in Matlab. Tests were done with the Matlab `timeit` command and the system was solved 10 times for each matrix size, such that an avarage computation time could be determined. From our investigations, we conclude that the theoretical complexities should be seen as upper bounds for the complexity. The measured complexity is $\mathcal{O}(n^{4.30})$ and $\mathcal{O}(n^{1.80})$ respectively (if fitted to a straight line in the loglog-plot) and $\mathcal{O}(n^{4.38})$ and $\mathcal{O}(n^{1.76})$ if the `cputime` command is used for the measurements instead.

Fredrik: kan du frklara komplexiteten bttre?

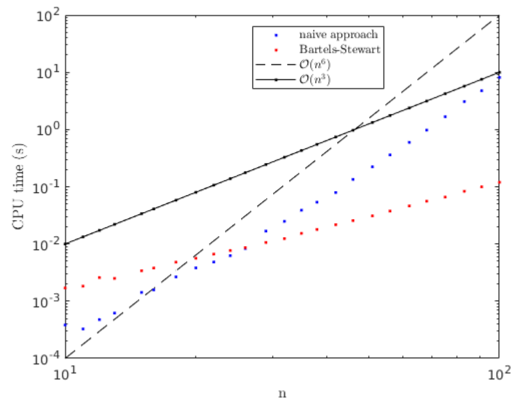

Figure 1: Measured CPU-time for solving the Lyapunov equation on random matrices of size n using a naive appoach compared to the Bartels-Stewart algorithm.

Task 3

We show that $\text{vec}(\mathbf{u}\mathbf{v}^T) = \mathbf{v} \otimes \mathbf{u}$, where $\mathbf{u}$ and $\mathbf{v}$ are two vectors of length n. Starting with the left hand side, we have

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1, & v_2, & \dots, & v_n \end{bmatrix} = \begin{bmatrix} u_1v_1, & u_1v_2, & \dots, & u_1v_n \\ u_2v_1 & u_2v_2, & \dots, & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \dots & u_nv_n \end{bmatrix}. \tag{1}$$

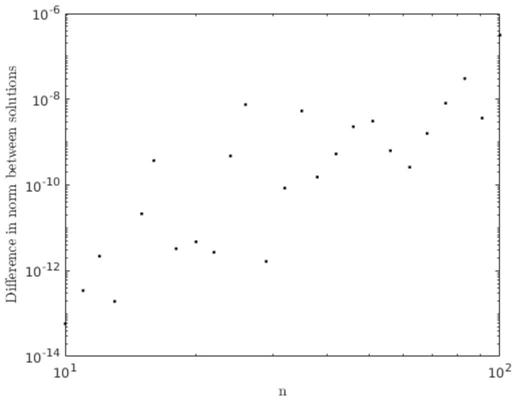

Figure 2: Difference in norm between the naive approach and the Bartels-Stewart algorithm for soving the Lyapunov equation on random matrices of size n.

Thus,

$$\text{vec}(\mathbf{u}\mathbf{v}^T) = \left[u_1v_1, \dots, u_nv_1, u_1v_2, \dots, u_nv_2, \dots, u_1v_n, \dots, u_nv_n\right]. \tag{2}$$

The right hand side expression can be written as

$$\mathbf{v} \otimes \mathbf{u} = \begin{bmatrix} v_1\mathbf{u} \\ v_2\mathbf{u} \\ \vdots \\ v_n\mathbf{u} \end{bmatrix}, \tag{3}$$

which is the same as the expression for the left hand side in (2).

Task 4

We consider

$$A = \begin{bmatrix} 0 & a \\ 1 & 0, \end{bmatrix} \; W = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{4}$$

and determine for which values of a that the Lyapunov equation has a unique solution.

The answer is given using theorem 4.1.1 from the lecture notes. The eigenvalues of the real matrix A is $\lambda = \pm\sqrt{a}$. The theorem states that the Lyapunov equation has a unique solution if and only if $\lambda_1 \neq -\lambda_2$. The only possibility for this to hold is if $a = 0$.

Task 5

(a)

Task 12

Consider the PDE

$$\begin{aligned} \Delta u + g(x,y)u = f(x,y), \text{ for } x, y \in \Omega, \\ u(x,y) = 0, \text{ for } x, y \in \partial\Omega, \end{aligned} \tag{5}$$

where Ω is the unit square.

(a)

Derive the 2nd order finite-difference discretization for the grid $x_i = hi, i = 1, \dots, m, y_j = hj, j = 1, \dots, m$ and $h = 1/(m+1)$. Also, derive matrices such that the discretizatioin can be expressed as

$$D_{xx}U + UD_{xx} + G \circ U = F, \tag{6}$$

where $U_{i,j} \approx u(x_i, y_j)$.

Using a 2nd order approximation of the second derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{7}$$

and similarly for $\frac{\partial^2 f}{\partial y^2}$, we can approximate the laplacian by the five point stencil

$$\Delta u_{i,j} \approx \frac{u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}}{h^2}, \tag{8}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$. From the boundary condition, we have

$$u_{0,j} = u_{m+1,j} = u_{i,0} = u_{m+1,0} = 0. \tag{9}$$

The structure for the Laplace operator alone is thus the $m^2 \times m^2$ matrix

$$\begin{bmatrix} T & I & 0 & \dots & 0 \\ I & T & I & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & I & T \end{bmatrix} \tag{10}$$

with T being the $m \times m$ matrix with -4 on the diagonal, 1 on the sub- and super-diagonals and zero otherwise. The entire system for the PDE can be written on the form $Au = b$, where A has the same structure as the matrix in (10), but where the diagonal is given by matrices $B_1, B_2, \dots, B_m$, where $B_j = T + h^2 \cdot GG_j$, with GG_j having values only on the main diagonal taking the values $g(x_i, y_j)$, $i = 1, \dots, m$. The right hand side vector b is given by

$$b = h^2 \begin{bmatrix} f(x_i, y_1) \\ f(x_i, y_2) \\ \vdots \\ f(x_i, y_m) \end{bmatrix}. \tag{11}$$

Observe that this is the vecorized form of the equation (6), where D_{xx} is a $m \times m$ matrix with -2 on the main diagonal and -1 on the sub- and superdiagonals, $G = g(x_i, y_j)$ and $F = f(x_i, y_j)$.

(b)

Derive explicit expressions for the eigenvalues of $I \otimes D_{xx} + D_{xx} \otimes I$ in the limit $m \to \infty$ and show that the matrix $I \otimes D_{xx} + D_{xx} \otimes I$ is non-singular in the limit.

First, we identify that the matrix $I \otimes D_{xx} + D_{xx} \otimes I$ is the finite difference approximation of the laplace operator. In the limit $m \to \infty$, the approximation approaches the continuous laplace operator. Therefore, we seek eigenvalues

$$\Delta u = \lambda u. \tag{12}$$

Writing u as $u(x, y) = X(x)Y(y)$, we get the equation

$$X'' + Y'' = \lambda XY \Leftrightarrow \frac{X''}{X} + \frac{Y''}{Y} = \lambda, \tag{13}$$

enforcing that for some constants α and β,

$$X'' = \alpha X \text{ and } Y'' = \beta Y. \tag{14}$$

Solving these equations along with the homogeneous boundary conditions, we obtain that

$$\lambda = (k\pi)^2 + (l\pi)^2, \quad k, l = 1, 2, 3, ... \tag{15}$$

The equations for $X(x)$ and $Y(y)$ has no nontrivial solutions for $\lambda = 0$ and thus it can be concluded the matrix is non-singular.

(c)

Let

$$g(x, y) = \alpha\sqrt{(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2} \tag{16}$$

and $f(x, y) = |x - y|$.

1. Let $\alpha = 0$ and solve the sparse linear system $Au = b$ using \.
2. We compare the above with solving the equation using the matlab command `lyap`.
3. Let $\alpha = 1$ and solve with \
4. Again, let $\alpha = 1$, but use instead `gmres` to solve the system.
5. Using $\alpha = 1$ and `gmres`, use `lyap` as a left pre-conditioner.
6. Using $\alpha = 0.1$ and `gmres`, use `lyap` as a left pre-conditioner.

(d)

Explain the performance in (c) of the the preconditioned gmres. Consider $A - E$ to be the matrix A obtained by choosing $\alpha = 0$. Then, from the hint in the lecture notes,

$$(A - E)^{-1} = A^{-1} - A^{-1}EA^{-1} + \mathcal{O}(\|E\|^2), \tag{17}$$

for sufficiently small $\|E\|$. The matrix E is diagonal, and therefore, $\|E\|_2 = \max(g(x_i, y_2)) < \alpha\sqrt{2}/2$. Thus, using the solution from the lyap command (with $\alpha = 0$) as a pre-conditioner, the preconditioner is very close to being the inverse of the matrix A where e.g. $\alpha = 0.1$. The consequence is clearly that gmres converges in few iterations.

(e)

Suppose all elements of the matrix G are zero except $G_{m/4,m/2} = 1/h$, where $m \in 4\mathbb{Z}$. Solve the equation efficiently by using the `lyap` command.

We consider the problem as a rank-one modification and use the Sherman-Morrison-Woodbury formula, stating that if C is an invertible square matrix and u, v are column vectors of the same size, then $C+uv^T$ is invertible iff $1 + v^TC^{-1}u \neq 0$. The inverse is given by

$$(C + uv^T)^{-1} = C^{-1} - \frac{C^{-1}uv^TC^{-1}}{1 + v^TC^{-1}u}. \tag{18}$$

Using $u = e_{m/4}$ and $v = (1/h)e_{m/2}$, we can then compute

A naive approach representing G as a matrix to form the matrix A described in task (a), and then using \ to solve yields