SF3580
HW 4

Anna Broms & Fredrik Fryklund

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Task 1

We implement the naive approach for solving the Lyapunov equation and compare it to the Bartels-Stewart method on random matrices. The dependence of the simulation time on n is illustrated in Figure 2. Theoretically, the complexity is as bad as $\mathcal{O}(n^6)$ for the naive approach and $\mathcal{O}(n^3)$ for the improved algorithm. However, the simulation time is in this test smaller due to the use of improved linear solvers in Matlab. Tests were done with the Matlab `timeit` command and the system was solved 10 times for each matrix size, such that an avarage computation time could be determined. From our investigations, we conclude that the theoretical complexities should be seen as upper bounds for the complexity. The measured complexity is $\mathcal{O}(n^{4.30})$ and $\mathcal{O}(n^{1.80})$ respectively (if fitted to a straight line in the loglog-plot) and $\mathcal{O}(n^{4.38})$ and $\mathcal{O}(n^{1.76})$ if the `cputime` command is used for the measurements instead.

Fredrik: kan du frklara komplexiteten bttre?

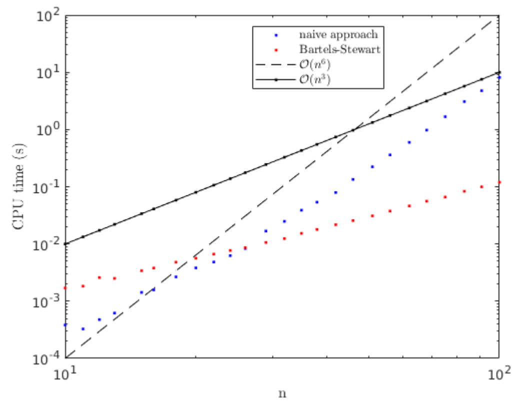

Figure 1: Measured CPU-time for solving the Lyapunov equation on random matrices of size n using a naive appoach compared to the Bartels-Stewart algorithm.

Task 3

We show that $\text{vec}(\mathbf{u}\mathbf{v}^T) = \mathbf{v} \otimes \mathbf{u}$, where $\mathbf{u}$ and $\mathbf{v}$ are two vectors of length n. Starting with the left hand side, we have

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1, & v_2, & \dots, & v_n \end{bmatrix} = \begin{bmatrix} u_1v_1, & u_1v_2, & \dots, & u_1v_n \\ u_2v_1 & u_2v_2, & \dots, & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \dots & u_nv_n \end{bmatrix}. \tag{1}$$

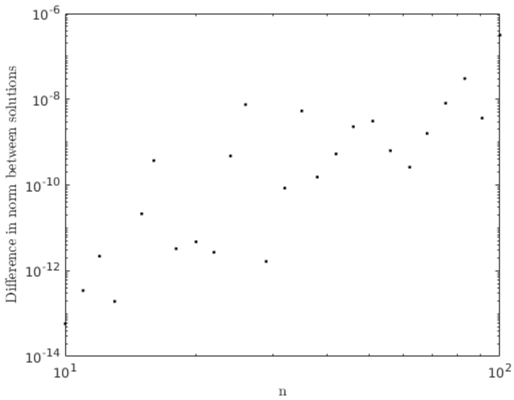

Figure 2: Difference in norm between the naive approach and the Bartels-Stewart algorithm for soving the Lyapunov equation on random matrices of size n.

Thus,

$$\text{vec}(\mathbf{u}\mathbf{v}^T) = \left[u_1v_1, \ldots, u_nv_1, u_1v_2, \ldots, u_nv_2, \ldots, u_1v_n, \ldots, u_nv_n\right]. \tag{2}$$

The right hand side expression can be written as

$$\mathbf{v} \otimes \mathbf{u} = \begin{bmatrix} v_1\mathbf{u} \\ v_2\mathbf{u} \\ \vdots \\ v_n\mathbf{u} \end{bmatrix}, \tag{3}$$

which is the same as the expression for the left hand side in (2).

Task 4

We consider

$$A = \begin{bmatrix} 0 & a \\ 1 & 0, \end{bmatrix} W = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{4}$$

and determine for which values of a that the Lyapunov equation has a unique solution.

The answer is given using theorem 4.1.1 from the lecture notes. The eigenvalues of the real matrix A is $\lambda = \pm\sqrt{a}$. The theorem states that the Lyapunov equation has a unique solution if and only if $\lambda_1 \neq -\lambda_2$. The only possibility for this to hold is if $a = 0$.

Task 5

Let $A \in \mathbb{R}^{m_A \times k}$, $B \in \mathbb{R}^{m_B \times p}$, $C \in \mathbb{R}^{k \times n_C}$ and $D \in \mathbb{R}^{p \times n_D}$. Then the matrices A, B, C and D are of compatible size such that they satisfy the left hand side of

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{5}$$

is satisfied. This is shown by first computing $(A \otimes B)$ and $(C \otimes D)$:

$$(A \otimes B) = \begin{bmatrix} a_{11}B & \cdots & a_{1k}B \\ \vdots & \ddots & \vdots \\ a_{m_A1}B & \cdots & a_{m_Ak}B \end{bmatrix} \in \mathbb{R}^{(m_Am_B)\times(kp)}, \quad (C \otimes D) = \begin{bmatrix} c_{11}D & \cdots & c_{1n_C}D \\ \vdots & \ddots & \vdots \\ c_{k1}D & \cdots & c_{kn_C}D \end{bmatrix} \in \mathbb{R}^{(kp)\times(n_Cn_D)}.$$

We now have

$$
\begin{aligned}
(A \otimes B)(C \otimes D) &= \begin{bmatrix} a_{11}B & \cdots & a_{1k}B \\ \vdots & \ddots & \vdots \\ a_{m_A 1}B & \cdots & a_{m_A k}B \end{bmatrix} \begin{bmatrix} c_{11}D & \cdots & c_{1n_C}D \\ \vdots & \ddots & \vdots \\ c_{k1}D & \cdots & c_{kn_C}D \end{bmatrix} \\
&= \begin{bmatrix} \sum_k^{i=1} a_{1i}c_{i1}BD & \cdots & \sum_k^{i=1} a_{1i}c_{in_C}BD \\ \vdots & \ddots & \vdots \\ \sum_k^{i=1} a_{m_A i}c_{i1}BD & \cdots & \sum_k^{i=1} a_{m_A i}c_{in_C}BD \end{bmatrix}
\end{aligned}
$$

where the matrix product CD is well defined. Finally consider the right hand side of (5)

$$
\begin{aligned}
(AC) \otimes (BD) &= \begin{bmatrix} \sum_k^{i=1} a_{1i}c_{i1} & \cdots & \sum_k^{i=1} a_{1i}c_{in_C} \\ \vdots & \ddots & \vdots \\ \sum_k^{i=1} a_{m_A i}c_{i1} & \cdots & \sum_k^{i=1} a_{m_A i}c_{in_C} \end{bmatrix} \otimes (BD) \\
&= \begin{bmatrix} \sum_k^{i=1} a_{1i}c_{i1}BD & \cdots & \sum_k^{i=1} a_{1i}c_{in_C}BD \\ \vdots & \ddots & \vdots \\ \sum_k^{i=1} a_{m_A i}c_{i1}BD & \cdots & \sum_k^{i=1} a_{m_A i}c_{in_C}BD \end{bmatrix} = (A \otimes B)(C \otimes D).
\end{aligned}
$$

Task 12