SF3580
HW 3

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Task 2

(a)

The QR-algorithm is implemented and applied to the given matrix, see Julia code and Figure 1.

(b) and (c)

The eigenvalues of the matrix A are (by construction) $[2^{[0:7]}, \lambda_9, 2^9]$, where

$$\lambda^9 = 2^9 \left(0.99 - \frac{1}{5\alpha}\right). \tag{1}$$

After n iterations, the elements below the diagonal in the QR-iterates will be *proportional* to $|\lambda_i/\lambda_j|^n$, where $i < j$. The large α, the dominating eigenvalues are λ_9 and $\lambda_1 0$. Computing $|\lambda_i/\lambda_j|^n$, we obtain

$$|\lambda_i/\lambda_j|^n = \left(0.99 - \frac{1}{5\alpha}\right)^n. \tag{2}$$

Taking this as a meassure of the error and setting a tolerence of 10^{-10}, we obtain

$$\begin{aligned} \left(0.99 - \frac{1}{5\alpha}\right)^n &\leq 10^{-10} \\ \Leftrightarrow n \cdot \log(0.99 - \frac{1}{5\alpha}) &\leq -10\log(10) \\ n &\geq \frac{-10\log(10}{\log(0.99 - \frac{1}{5\alpha})}. \end{aligned} \tag{3}$$

The predicted number of iterations is plotted together with the obtained number of iterations in Figure 1. The predicted number of iterations is proportional to the true number of iterations, as expected.

Task 3

(a)

Given a vector $x \in \mathbb{R}^n$ with $y \neq 0$ and $x \neq 0$, we derive a formula for a Householder reflector (represented by a normal direction $u \in \mathbb{R}^n$) such that $Px = \alpha y$ for some value α.

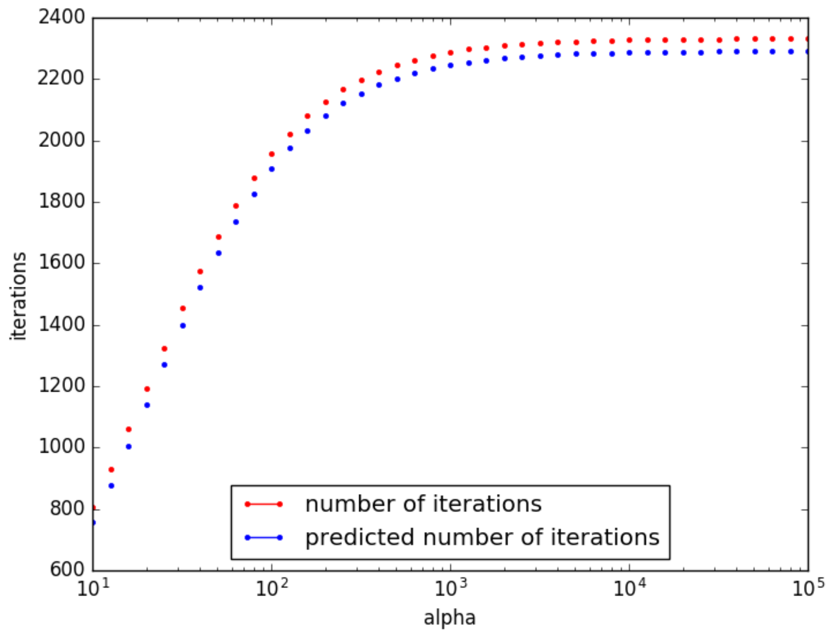

Figure 1: Task 2: obtained and predicted number of iterations for the QR-method applied to the given matrix.

Let $z = x - \frac{\|x\|}{\|y\|}y$ and $u = \frac{z}{\|z\|}$. Then, the matrix $P = I - 2uu^T$ is a Householder reflector since u is normalized. Further, Px can be computed as

$$Px = (I - 2uu^T)x. \tag{4}$$

Consider the product

$$uu^T x = \frac{zz^T x}{\|z\|\|z\|} = \frac{z(z^T x)}{z^T z}, \tag{5}$$

where

$$z^T x = \left(x - \frac{\|x\|}{\|y\|}y\right)^T x = \|x\|^2 - \frac{\|x\|}{\|y\|}y^T x \tag{6}$$

and

$$z^T z = \left(x - \frac{\|x\|}{\|y\|}y\right)^T \left(x - \frac{\|x\|}{\|y\|}y\right) = \|x\|^2 - 2x^T y\frac{\|x\|}{\|y\|} + \|x\|^2 = 2\left(\|x\|^2 - x^T y\frac{\|x\|}{\|y\|}\right). \tag{7}$$

Now, we easily identify that

$$Px = x - z = \frac{\|x\|}{\|y\|}y = \alpha y, \tag{8}$$

which is what we wanted to show.

(b) and (c)

Both a naive and an improved Hessenberg reduction algorithm is implemented in Julia. We compare the algorithms by computing a Hessenberg reduction of a given matrix of size $m \times m$. The result is found in Table 1.

Table 1: CPU-time in seconds for the naive and improved Hessenberg reduction algorithm applied on a specified matrix

	CPU-time naive algorithm	CPU-time improved algorithm
$m = 10$	$1.4 \cdot 10^{-4}$	$4.5 \cdot 10^{-5}$
$m = 100$	$3.3 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$
$m = 200$	$3.0 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$
$m = 300$	$1.0 \cdot 10^{0}$	$3.4 \cdot 10^{-1}$
$m = 400$	$4.6 \cdot 10^{-2}$	$9.8 \cdot 10^{-1}$

(d)

Let $\hat{H}$ be the result of one step of the shifted QR-method with shift σ for the matrix

$$A = \begin{bmatrix} 3 & 2 \\ \epsilon & 1 \end{bmatrix}. \tag{9}$$

Results are found in Table 2.

Table 2: Task 3(d)

ϵ	$\lvert\hat{h}_2, 1$—	$\lvert\hat{h}_2, 1\rvert$
	$\sigma = 0$	$\sigma = A_{2,2}$
0.4	0.0961	0.0769
0.1	$3.3 \cdot 10^{-3}$	$5.0 \cdot 10^{-5}$
0.01	$3.3 \cdot 10^{-4}$	$5.0 \cdot 10^{-7}$
10^{-3}	$3.3 \cdot 10^{-5}$	$5.0 \cdot 10^{-9}$
10^{-4}	$3.3 \cdot 10^{-6}$	$5.0 \cdot 10^{-11}$
10^{-5}	$3.3 \cdot 10^{-7}$	$5.0 \cdot 10^{-13}$
10^{-6}	$3.3 \cdot 10^{-8}$	$5.0 \cdot 10^{-15}$
10^{-7}	$3.3 \cdot 10^{-9}$	$5.0 \cdot 10^{-17}$
10^{-8}	$3.3 \cdot 10^{-10}$	$5.0 \cdot 10^{-19}$
10^{-9}	$3.3 \cdot 10^{-11}$	$5.0 \cdot 10^{-21}$
10^{-10}	$3.3 \cdot 10^{-12}$	$5.0 \cdot 10^{-23}$

The values in the table corresponds to the values on the off- diagonal which can be seen as the error while computing the eigenvalues of the matrix A. The error decreases much faster with ϵ when the shifted QR-method is used than when the unshifted QR-method is used (corresponding to $\sigma = 0$).