SF3580
HW 3

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Task 2

Task 4

Task 5

(a)

The matrix A is diagonalizable with the eigendecomposition $A = QDQ^{-1}$, where D is a diagonal matrix. For such structures it holds that $\sin(A) = Q\sin(D)Q^{-1}$. Thus we can validate the result for the Schur-Parlett method, which is

$$\sin(A) = \sin\left(\begin{bmatrix} 1 & 4 & 4 \\ 3 & -1 & 3 \\ -1 & 4 & 4 \end{bmatrix}\right) \approx \begin{bmatrix} 0.846192 & 0.0655435 & -0.187806 \\ 0.33476 & 0.385017 & -0.141244 \\ -0.190921 & 0.192478 & 0.848269 \end{bmatrix}. \tag{1}$$

which in norm differs $4.28e-16$ from $Q\sin(D)Q^{-1}$.

```
using LinearAlgebra
function schur_parlett(A,f)
    T,Q,ev=schur(A)
    n = size(A,1)
    F = zeros(n,n)
    for i=1:n
        F[i,i]=f(T[i,i])
    end
    for p=1:n-1
        for i=1:n-p
            j=i+p
            s=T[i,j]*(F[j,j]-F[i,i])
            for k=i+1:j-1
                s = s + T[i,k]*F[k,j]-F[i,k]*T[k,j];
            end
            F[i,j]=s/(T[j,j]-T[i,i])
        end
    end
    F=Q*F*Q';
    return F
end
```

(b) & (c)

It is clear from Figure 1 that the number of flops required for Schur–Parlett is not discernibly affected by N, at leat for $N \in 10, 50, 100, 150, 200, 250, 300$. This is not suprsining, as often the most computationally demanding part

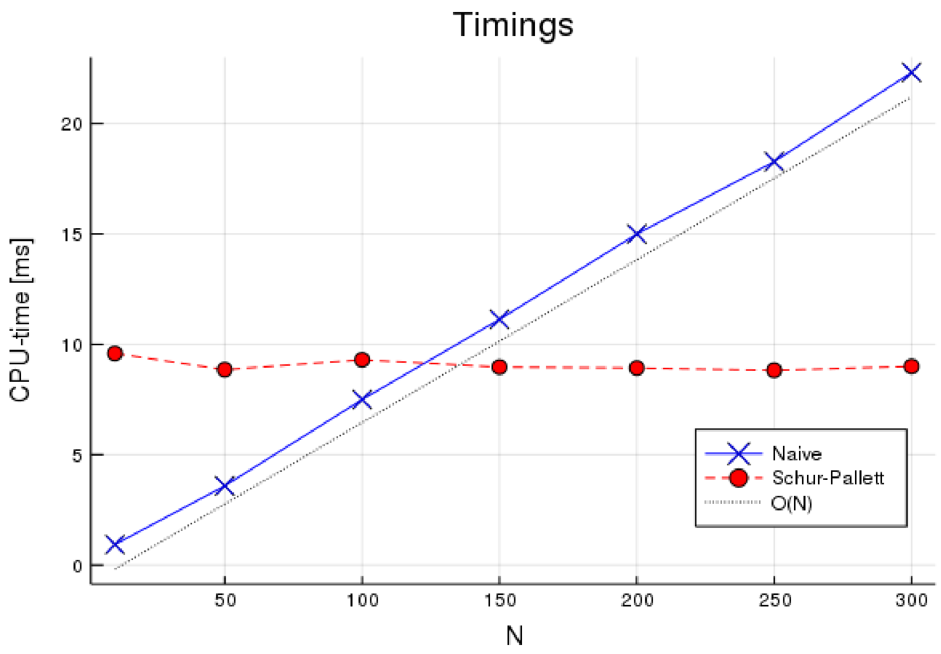

Figure 1: Task 5, (b) & (c): CPU–time in milliseconds, as a function of N.

of the Schur–Parlett method, in is performing the Schur decomposition, which scales like $O(n^3)$. Once obtained, the function f is only applied to the diagonal elements, which are scalars.

For the naive appraoch the number of flops is proportional to N. A matrix multiplication is of $O(n^3)$, thus performing N matrix gives $O(Nn^3)$, which we read from Figure 1. The black line corresponds to the line $0.08 + 0.07\,N$.

Task 6

(a)

The matrix

$$A = \begin{bmatrix} \pi & 1 \\ 0 & \pi + \varepsilon \end{bmatrix}, \tag{2}$$

with $\varepsilon > 0$, has two eigenvalues: $\lambda_1 = \pi$ and $\lambda_2 = \pi + \varepsilon$. Let $A = X\,\mathrm{diag}(J_1, J_2)\,X^{-1}$, be the Jordan canonical form, with $J_1 = \lambda_1$ and $J_2 = \lambda_2$.

The Jordan canonical form definition gives that

$$p(A) = X\,\mathrm{diag}(p(J_1), p(J_2))\,X^{-1}. \tag{3}$$

A simple consequence is

$$g(A) = X\,\mathrm{diag}(g(\lambda_1), g(\lambda_2))\,X^{-1} = X\,\mathrm{diag}(p(\lambda_1), p(\lambda_2))\,X^{-1} = p(A), \tag{4}$$

since the polynomial p interpolates the function g in the eigenvalues of A, i.e. $p(\lambda_1) = g(\lambda_1)$ and $p(\lambda_2) = g(\lambda_2)$.

Two points defines a unique polynomial of order 1, thus we may choose a p in $\mathbb{P}^1$ and write $p(z) = \alpha + \beta z$. The unknown coefficents are obtianed by solving

$$\begin{cases} \alpha + \beta\lambda_1 & = g(\lambda_1) \\ \alpha + \beta\lambda_2 & = g(\lambda_2) \end{cases} \Leftrightarrow \begin{cases} \alpha & = \frac{g(\lambda_1)\lambda_2 - g(\lambda_2)\lambda_1}{\lambda_2 - \lambda_1} \\ \beta & = \frac{g(\lambda_2) - g(\lambda_1)}{\lambda_2 - \lambda_1} \end{cases} \tag{5}$$

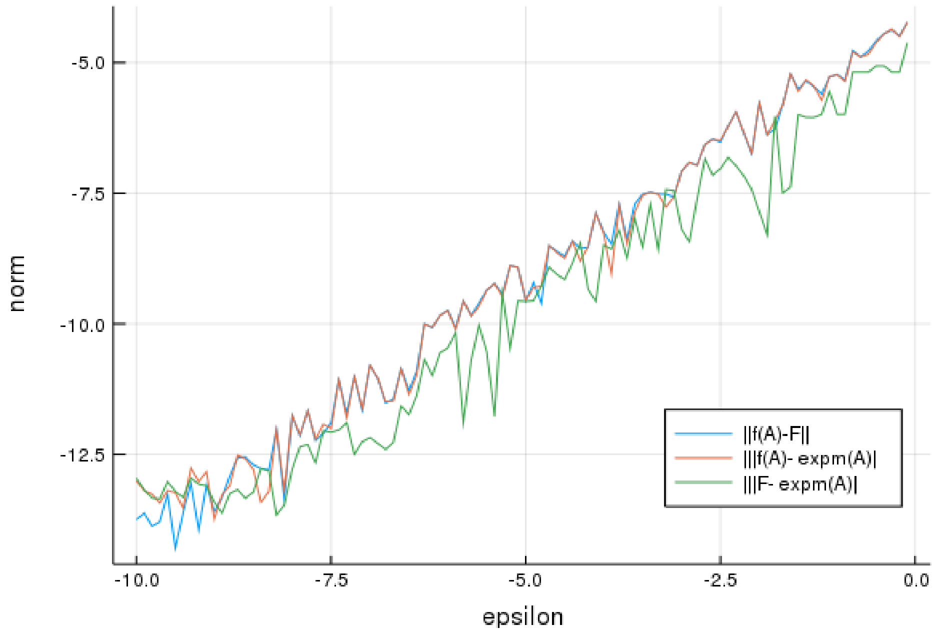

Figure 2: Task 5, (b) & (c): CPU–time in milliseconds, as a function of N.

(b)

Given $g := \exp$ we have from (a) and (b) that

$$p(A) = \alpha I + \beta A = \frac{\exp(\pi)(\pi + \varepsilon) - \exp(\pi + \varepsilon)\pi}{\varepsilon} I + \frac{\exp(\pi + \varepsilon) - \exp(\pi)}{\varepsilon} A \tag{6}$$

$$= \frac{\exp(\pi)}{\varepsilon}(\varepsilon I + (1 - \exp(\varepsilon))(\pi I - A)) \tag{7}$$

(c)

It is known that the Jordan decomposition is unstable for non-symmetric matrices, as the eigenvalues may lie close to each other. For the given matrix A this can be tuned artificially by setting $\varepsilon = |\lambda_1 - \lambda_2|$. As we see in Figure (to be added, some layout to fix). However, the function `exp` in Julia is analogous to `expm` in Matlab, (we compared the resluts for $\varepsilon = 1e-1, \dots, 1e-10$). Using Matlabs $\texttt{expm}(A)$ as reference value, the error from the exact result in (b) also increases as ε. This is most likely due to cancellation.

Task 7

(a)

We start by observing that if λ and v are an eigenpair for A, then $t\lambda$ is an eigenvalue of tA since $tAv = t\lambda v$.

Let $\mu \in \mathbb{C}$ be an expansion point, then

$$f(tA) = \sum_{i=0}^{\infty} \frac{f^{(i)}(\mu)}{i!}(tA - \mu I)^i. \tag{8}$$

If $A \in \mathbb{C}^{n\times n}$, then $f : C^{n\times n} \to C^{n\times n}$. Since exp is analytic everywhere, we can without loss of generality chose

$\mu = 0$. We can without loss of generality chose $\mu = 0$. Then

$$\frac{d}{dt}\exp(tA) = \frac{d}{dt}\left(\sum_{i=0}^{\infty}\frac{\exp(0)}{i!}(tA - 0I)^i\right) = \sum_{i=0}^{\infty}\frac{1}{i!}\frac{d}{dt}(t^i A^i) = \sum_{i=1}^{\infty}\frac{1}{(i-1)!}(tA)^{i-1}A = \sum_{i=0}^{\infty}\frac{1}{i!}(tA)^i A = \exp(tA)A, \tag{9}$$

since d/dt is the elementwise differential operator. Commutativity is shown by observing that

$$i(tA)^{i-1}A = \frac{d}{dt}(t^i A^i) = \frac{d}{dt}(A^i t^i) = A(At)^{i-1}. \tag{10}$$

The rest follows.

(b)