SF3580
HW 2

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1 Task 2

2 Task 4

The linear system of equations

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

is considered.

2.1 (a)

We determine the constants α, β and γ such that for the iterates $x_0, \dots x_3$ of the conjugate gradient method we obtain $\text{span}(x_0, x_1, x_2, x_3) = \text{span}(c_0, c_1, c_2, c_3)$, where

$$C = [c_0, c_1, c_2, c_3] = \begin{bmatrix} 1 & \alpha & 0 & \gamma \\ 1 & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

For this purpose, we use Lemma 2.2.4 from the lecture notes, stating that

$$\text{span}(b, Ab, \dots A^{m-1}b) = \text{span}(x_0, x_1, \dots, x_m). \tag{3}$$

Thus, we want to compute α, β and γ such that

$$\text{span}(b, Ab, A^2b, A^3b) = \text{span}(c_0, c_1, c_2, c_3). \tag{4}$$

We can directly conclude that $\text{span}(b) = \text{span}(c_0)$. Next, we want to make sure that $\text{span}(b, Ab) = \text{span}(c_0, c_1)$. By column reduction we thus find that $\alpha = 0$. Using that $\text{span}(b, Ab, A^2b) = \text{span}(c_0, c_1, c_2)$, we can similarly find that $\beta = -1$ and finally, using (4), and again reduce columns, we identify that $\gamma = 6$.

2.2 (b)

See the implemented Julia code. We have replaced `???` with $\|Ax - b\|_{A^{-1}}$. Comparing $\texttt{x}_{\texttt{opt}}$ and $\texttt{x}_{\texttt{cg}}$, we obtain a difference of $1.51 \cdot 10^{-11}$.

2.3 (c)

For GMRES, `???` is replaced by $\|Ax - b\|_2$. Now, comparing $\texttt{x}_{\texttt{opt}}$ with $\texttt{x}_{\texttt{gmres}}$, we obtain the difference $1.06 \cdot 10^{-11}$.

3 Task 5

Given a real symmetric matrix A with eigenvalues 10, 10.5 and 100 eigenvalues in the interval $[2, 3]$, we prove a bound for then number of steps needed for CG to reduce the error measured in $\|Ax_n - b\|_{A^{-1}} = \|x_n - x_*\|$ by a factor 10^7. We assume exact arithmetic and no breakdown.

4 Task 6

4.1 (a)

We compare GMRES and CGN for a given matrix B and right hand side b. The result of the comparison is visualised in Figure **??**.

4.2 (b)

The result can be related to the convergence theory for CGN and GMRES.

5 Task 7