SF3580
HW 2

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1 Task 7

(a)

Given that $A = V^{-1}\Lambda V$ we want to show $A^k = A = V^{-1}\Lambda^k V$, which is done by induction. The initial stage is

$$A^2 = V^{-1}\Lambda V V^{-1}\Lambda V = V^{-1}\Lambda^2 V. \tag{1}$$

Assume $A^k = V^{-1}\Lambda^k V$ for some nonzero k, then

$$A^{k+1} = (V^{-1}\Lambda V)^k (V^{-1}\Lambda V) = V^{-1}\Lambda^k V\, V^{-1}\Lambda V = V^{-1}\Lambda^{k+1} V. \tag{2}$$

Thus $A^k = V^{-1}\Lambda^k V$. A simple consequence is that for $p \in P_n^0$ one has

$$p(A) = \sum_{k=1}^{n} a_k A^k = \sum_{k=1}^{n} a_k V^{-1}\Lambda^{k+1} V = V^{-1}\left(\sum_{k=1}^{n} a_k \Lambda^{k+1}\right) V = V^{-1} p\left(\Lambda^{k+1}\right) V \tag{3}$$

with $a_0 = 1$ for $p \in P_n^0$ and knowing that $A^0 = I$. We have

$$\min_{p\in P_n^0} \|p(A)\| \leq \|V\|\|V^{-1}\| \min_{p\in P_n^0} \|p(\Lambda)\| \tag{4}$$

as a consequence of norms being submultiplicative.

(b)

First we show by induction that

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{pmatrix}. \tag{5}$$

The initial stage is for $k = 2$:

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}^2 = \begin{pmatrix} \lambda_1^2 & 2\lambda_1 \\ 0 & \lambda_1^2 \end{pmatrix}. \tag{6}$$

Assume (5) holds for k, then

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}^{k+1} = \begin{pmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{pmatrix}\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1^{k+1} & (k+1)\lambda_1^k \\ 0 & \lambda_1^{k+1} \end{pmatrix}, \tag{7}$$

i.e. the proposition (5) holds for all nonzero k.

Introduce the monomial $p_k(z) = z^k$, then

$$p_k\left(\begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{pmatrix}\right) = \begin{pmatrix}\lambda_1^k & k\lambda_1^{k-1}\\ 0 & \lambda_1^k\end{pmatrix} = \begin{pmatrix}p_k(\lambda_1) & p_k'(\lambda_1)\\ 0 & p_k(\lambda_1)\end{pmatrix} \tag{8}$$

which holds for all nonzero k from the induction proof above. We now have

$$p\left(\begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{pmatrix}\right) = \sum_{k=1}^{n} a_k p_k\left(\begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{pmatrix}\right) = \sum_{k=1}^{n}\begin{pmatrix}a_k p_k(\lambda_1) & a_k p_k'(\lambda_1)\\ 0 & a_k p_k(\lambda_1)\end{pmatrix} = \begin{pmatrix}p(\lambda_1) & p'(\lambda_1)\\ 0 & p(\lambda_1)\end{pmatrix}. \tag{9}$$

(c)

Let A be a block diagonal matrix, such that

$$A = \begin{pmatrix}A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m\end{pmatrix} \tag{10}$$

where A_i are Jordan block matrices. Due to the block structure we have

$$p(A) = \begin{pmatrix}p(A_1) & & & \\ & p(A_2) & & \\ & & \ddots & \\ & & & p(A_m)\end{pmatrix}. \tag{11}$$

Each block $p(A_i)$ has a singular value decomposition $p(A_i) = U_i S_i V_H^*$, where U_i and V_i are unitary matrices. S_i is a diagonal matrix with the singular values σ as elements. We can now write $p(A)$ as follows.

$$p(A) = \underbrace{\begin{pmatrix}U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_m\end{pmatrix}}_{U:=}\underbrace{\begin{pmatrix}S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_m\end{pmatrix}}_{S:=}\underbrace{\begin{pmatrix}V_1^H & & & \\ & V_2^H & & \\ & & \ddots & \\ & & & V_m^H\end{pmatrix}}_{V^H:=} \tag{12}$$

due to the rules of multiplication for block diagonal matrices. The final result follows from the definition of the operator norm $\|\cdot\|_2$:

$$\begin{aligned}\|p(A)\|_2 &= \sigma_{\max}(p(A)) = \max S = \max_{i=1,\dots,m}(\max S_i))\\ &= \max_{i=1,\dots,m}(\sigma_{\max}(p(A_i))) = \max_{i=1,\dots,m}(\|p(A_i)\|_2)\\ &= \max\left(\left\|\begin{pmatrix}p(\lambda_1) & p'(\lambda_1)\\ 0 & p(\lambda_1)\end{pmatrix}\right\|_2, |p(\lambda_3)|, \dots, |p(\lambda_m)|\right)\end{aligned}$$

(d)

It is clear that

$$p(z) = (\alpha_n + \beta_n z)\frac{(c-z)^{n-1}}{c^{n-1}} \tag{13}$$

satisfies $p \in P_n$. This immediately gives $\alpha_n = 1$. We now study

$$p'(z) = \frac{c\left(1-\frac{z}{c}\right)^n(\alpha_n - \alpha_n n + \beta_n(c-nz))}{(c-z)^2} = \frac{c\left(1-\frac{z}{c}\right)^n(1-n+\beta_n(c-nz))}{(c-z)^2}. \tag{14}$$

Thus

$$p'(\lambda_1) = 0 \Leftrightarrow \frac{c\left(1-\frac{\lambda_1}{c}\right)^n(1-n+\beta_n(c-n\lambda_1))}{(c-\lambda_1)^2} = 0 \Leftrightarrow (1-n+\beta_n(c-n\lambda_1)) = 0, \tag{15}$$

that is

$$\beta_n = \frac{n-1}{c-n\lambda_1}. \tag{16}$$

In turn this assumes that $c \neq n\lambda_1$ for $n > 1$.

(e)

Assuming x_n is the n:th iterate generated by GMRES-iterate, we have by lemma 2.1.3 from that lecture notes that

$$\|Ax_n - b\|_2 = \min_{x\in\mathcal{K}n(A,b)} \|Ax-b\|_2 = \min_{p\in P_n^0} \|p(A)b\| \leq \|V\|\|V^{-1}\| \min_{p\in P_n^0} \|p(\Lambda)\|\|b\|$$
$$\Leftrightarrow \frac{\|Ax_n-b\|_2}{\|b\|} \leq \|V\|\|V^{-1}\| \min_{p\in P_n^0} \|p(\Lambda)\|$$

due to the result in 7 (a). Let

$$q(z) = \left(1 + z\frac{n-1}{c-n\lambda_1}\right)\frac{(c-z)^{n-1}}{c^{n-1}}. \tag{17}$$

from the previous task, which by construction is an element of P_n^0. Thus

$$\min_{p\in P_n^0} \|p(\Lambda)\| \leq \|q(\Lambda)\| = \max\left(\left\|\begin{pmatrix} q(\lambda_1) & q'(\lambda_1) \\ 0 & q(\lambda_1)\end{pmatrix}\right\|_2, |q(\lambda_3)|, \ldots, |q(\lambda_m)|\right). \tag{18}$$

Recall that $q'(\lambda_1) = 0$ and that the matrix 2-norm of diagonal matrix is the largest element in modulus. The expression above can be simplified as

$$\min_{p\in P_n^0} \|p(\Lambda)\| \leq \max\left(|q(\lambda_1)|, |q(\lambda_3)|, \ldots, |q(\lambda_m)|\right) = \max_{\lambda_i}\left(1 + \lambda_i\frac{n-1}{c-n\lambda_1}\right)\frac{(c-\lambda_i)^{n-1}}{c^{n-1}}. \tag{19}$$

It is given that all eigenvalues are contained in the disc centered at c with radius ρ. By taking the modulus the inequality (19) and assumin $\lambda_1 \neq 0$ we get

$$\min_{p\in P_n^0} \|p(\Lambda)\| \leq \max_{\lambda_i}\left|1 + \lambda_i\frac{n-1}{c-n\lambda_1}\right|\frac{\rho^{n-1}}{|c^{n-1}|} \leq \max_{\lambda_i}\frac{\overbrace{|c-\lambda_i|}^{\leq\rho} + n\overbrace{|\lambda_i-\lambda_1|}^{\leq 2\rho}}{|c-n\lambda_1|}\frac{\rho^{n-1}}{|c^{n-1}|} \leq \gamma_n\frac{\rho^n}{|c^n|}$$

with

$$\gamma_n = \frac{\frac{1}{n}+2}{\left||\frac{1}{n}| - |\frac{\lambda_1}{c}|\right|}. \tag{20}$$

We already claimed that $c \neq n\lambda_1$, thus the denominator is nonzero for all n. In the limit we have

$$\lim_{n\to\infty} \gamma_n = 2\frac{|c|}{|\lambda_1|} \tag{21}$$

which is bounded. Combining all the results above gives

$$\frac{\|Ax_n - b\|_2}{\|b\|} \leq \|V\|\|V^{-1}\|\gamma_n \frac{\rho^n}{|c^n|}.$$

If $\lambda_1 = 0$ then $\beta_n = (n-1)/c$ and the corresponding bound for (19) is

$$\min_{p \in P_n^0} \|p(\Lambda)\| \leq \max_{\lambda_i} \left| 1 + \lambda_i \frac{n-1}{c} \right| \frac{\rho^{n-1}}{|c^{n-1}|} \leq \max_{\lambda_i} \left(\overbrace{|c - \lambda_i|}^{\leq \rho} + n|\lambda_i| \right) \frac{\rho^{n-1}}{|c^n|} \leq \gamma_n \frac{\rho^n}{|c^n|}.$$

However, now

$$\gamma_n = \max_{\lambda_i} \left(1 + n\frac{|\lambda_i|}{\rho} \right), \tag{22}$$

which is not a bounded sequence.

(f)

For nonzero λ_1 we have convergence, but the speed is influenced by γ_n. Roughly, the further the centre c is from λ_1 the better. For many iterations we approximately get

$$\gamma_n \frac{\rho^n}{|c^n|} \approx \frac{2\rho}{|\lambda_1|} \frac{\rho^{n-1}}{|c^{n-1}|}. \tag{23}$$

Thus the rate of convergence is the same, but the factor $\frac{2\rho}{|\lambda_1|}$ may be large. So if the double eigenvalues lie close to zero and the other eigenvalues lies far away from the origin then the factor will be large.

For $\lambda_1 = 0$ the sequence γ_n is not bounded. Note that this does not mean that GMRES will diverge, only that the estimate gives no information.

(e)

We discussed with Aku Kammonen and Parikshit Upadhyaya.