Solving the Maximal Independent Set Problem with Adiabatic Quantum Computing

Adiabatic Quantum Computing is a form of quantum computing based on the Adiabatic Theorem. We demonstrate that it can be used to solve a simple instance of the well known optimisation problem *Maximal Independent Set*. We find that the algorithm is more successful when incremental time steps are smaller.

Introduction

The Adiabatic Theorem of Quantum Mechanics states that a physical system prepared in the ground-state of a time-dependent Hamiltonian stays in the ground-state as time evolves and the Hamiltonian changes, as long as the change is sufficiently small and there is a sufficient energy gap between the ground and first-excited states**[1]**.

An optimisation problem is the problem of finding the best solution out of all possible solutions. Solutions to some optimisation problems can be found easily, such as the turning point of a quadratic function, but others require more complex algorithms to solve. Running time of these algorithms can scale dramatically with the size of the problem; we require more efficient ways to solve such problems**[2]**.

In this project, we encode our problem into a time-dependent Hamiltonian, such that the initial ground-state is known, and the final ground-state describes the optimal solution. We simulate time evolution, and find that, for sufficiently small increments in time, we are able to solve an instance of the Maximal Independent Set problem.

The Ising Model

For a given optimisation problem, we can construct a Hamiltonian encoding the problem such that its ground state describes the optimal solution. To do so, we use the Ising model. The Ising Hamiltonian is given by

$$H_{\text{Ising}} = \sum_{k=1}^{n} \sum_{j=k+1}^{n} J_{kj} \sigma_k^z \sigma_j^z + \sum_{j=1}^{n} h_j \sigma_j^z, \tag{1}$$

where

$$\sigma_j^z = \left(\bigotimes_{k=1}^{j-1} I_2 \right) \otimes \sigma^z \otimes \left(\bigotimes_{k=j+1}^{n} I_2 \right), \tag{2}$$

I_2 is the 2 x 2 identity matrix, σ^z the Pauli matrix, and h_j and J_{kj} encode the problem. Here, $\bigotimes_{k=1}^{j-1}$ indicates j - 1 sequential tensor products**[3]**.

The Maximal Independent Set Problem

The Maximal Independent Set problem is the problem of colouring as many vertices on a graph as possible with no two adjacent vertices (connected by an edge) being coloured. **Fig. 1** shows the example we investigated. The graph can be described by an adjacency matrix M, see **Fig. 2**.
Here, $M_{ij} = 1$ if there is an edge connecting vertices i and j, and 0 if not. Numbering of vertices, as in **Fig 1.**, is arbitrary, but it is important to be consistent.
To encode the problem into the Ising Hamiltonian, we set $J = M$, and

$$h_k = -\sum_{j=1}^{n} (M_{kj} + M_{jk}) + \kappa, \tag{3}$$

where $\kappa \in (0, 1)$ is added to reward colouring more vertices.

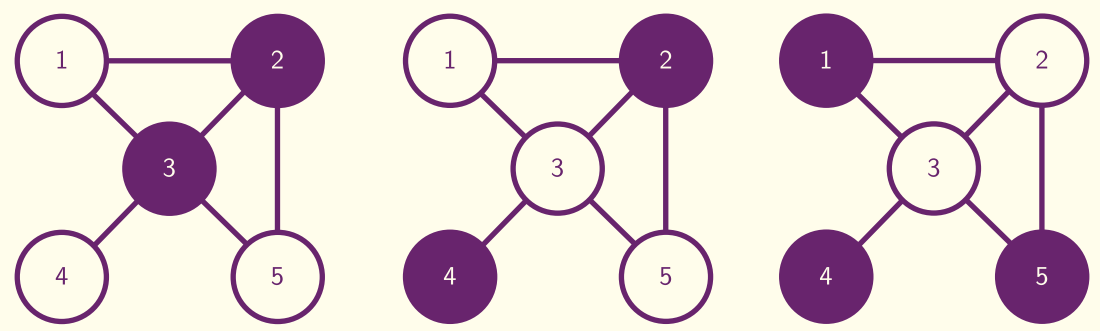

Fig.1: The instance of Maximal Independent Set we investigated. From left to right: a set which isn't independent, a set which is independent but not the maximum, the maximal independent set.

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Fig. 2: Adjacency matrix for the graph in **Fig. 1**.

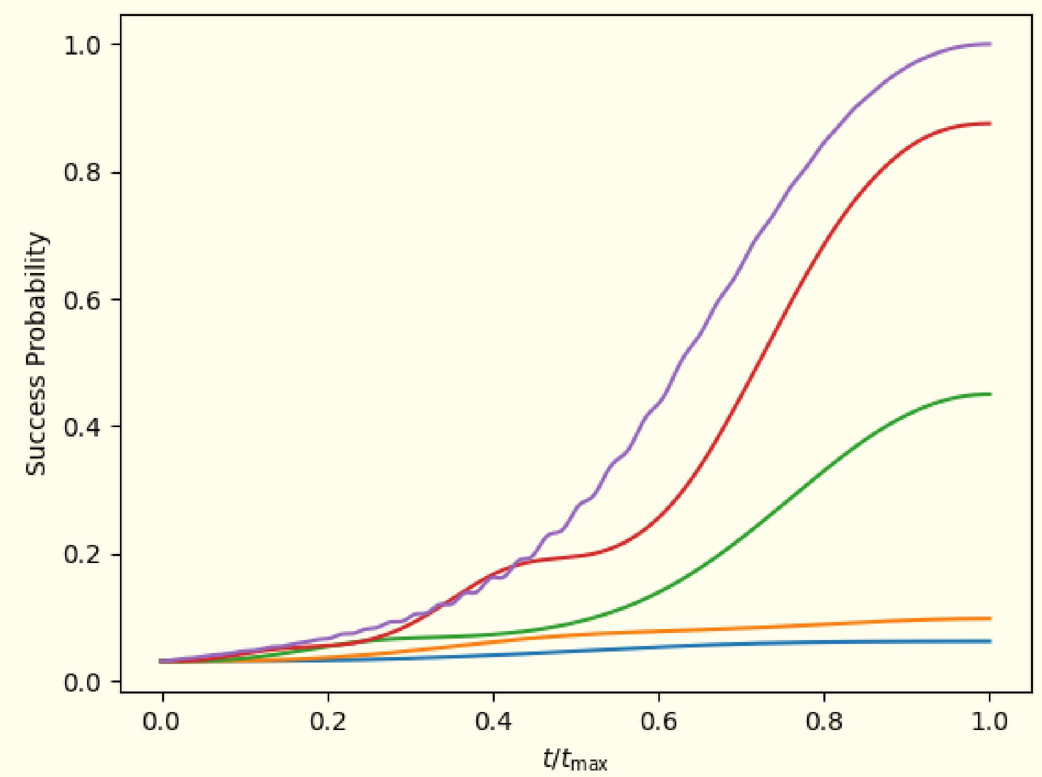

Fig. 3: Success probability against time for t_{max} of 1 (blue), 2 (orange), 5 (green), 10 (red), 100 (purple).

References

[1] Born, M. and Fock, V. (1928) 'Beweis des Adiabatensatzes', *Zeitschrift für Physik*, 51(3), pp. 165–180. doi: 10.1007/BF01343193.
[2] Chancellor, N. (2023) Computing Project Booklet, Ch. 6.
[3] Baierlein, R. (1999) *Thermal Physics*. Cambridge: Cambridge University Press. doi: 10.1017/CBO9780511840227.

Time-Dependent Hamiltonian

The time-dependent Hamiltonian is defined as

$$H(t) = -A(t) \sum_j \sigma_j^x + B(t) H_{\text{Ising}}\,. \tag{4}$$

In this project, we set the time-dependent controls as

$$A(t) = 1 - t/t_{\text{max}} \quad \text{and} \quad B(t) = t/t_{\text{max}}, \tag{5}$$

and $0 \le t \le t_{\text{max}}$. As t evolves from 0 to t_{max}, $H(t)$ evolves from the simple Hamiltonian $H(t{=}0)\ \sum_j \sigma_j^x$ to our problem Hamiltonian H_{Ising}. By the Adiabatic theorem, if we prepare a system in the ground-state of $H(t = 0)$ and evolve t sufficiently slowly, at $t = t_{\text{max}}$ the system will be in the ground-state of H_{Ising}, giving us the optimal solution.

Simulation

Adiabatic Quantum Computers are commercially available, but sell for millions of dollars. Instead, we simulated the process in Python; we prepared the system in the ground state of $H(t = 0)$, $|\psi(t = 0)\rangle$, then evolved it in time. The wavefunction at some time t was given by

$$\left|\psi\left(t = \frac{k t_{\text{max}}}{q}\right)\right\rangle \approx \mathcal{T} \prod_{j=1}^{k} \exp\left[-i \frac{t_{\text{max}}}{q} H\left(\frac{j t_{\text{max}}}{q}\right)\right] |\psi(t = 0)\rangle \tag{6}$$

where $\mathcal{T}$ indicates that the left-most product term has $j = k$, and q the number of time steps the algorithm takes, we set $q = 1000$.

Results

At each time step, the system is in a superposition of different states. This example problem was small, we knew the expected solution and thereforethe ground-state of H_{Ising}, ψ_0.We calculated at each time step the probability that the state of the system gave the solution to our optimisation problem,

$$P = |\langle \psi(t) | \psi_0 \rangle|^2. \tag{7}$$

These probabilities were plotted for five different values of t_{max} in **Fig. 3**. We can see that for low values of t_{max} there was limited success, and for higher values the probability approaches 1. In practice, we would want to limit t_{max} as much as possible to reduce running time, whilst keeping it high enough that results are useful.

Future Work

We intend to implement path integral quantum annealing (PIQA), and investigate possible applications. PIQA can be applied to much larger problems than maximal independent set, so we could solve something more interesting such as the travelling salesman problem.