Complex Analysis Summary Notes

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Contents

0 Useful Formulae and Facts

0.1 Epsilon Sum

$$\frac{1}{1-\epsilon} = \sum_{n=0}^{\infty} \epsilon^n \quad \text{for} \quad |\epsilon| < 1 \tag{1}$$

0.2 Trigonometric and Hyperbolic Functions

$$\cosh(x) = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) \tag{2}$$

$$\sinh(x) = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right) \tag{3}$$

$$\cos(ix) = \cosh(x), \quad \text{and} \quad \sin(ix) = i\sinh(x). \tag{4}$$

0.3 Identities

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y) \tag{5}$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y) \tag{6}$$

$$\sinh(2x) = 2\sinh(x)\cosh(x) \tag{7}$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) \tag{8}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{9}$$

0.4 Common Taylor Expansions

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{10}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{11}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{12}$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \tag{13}$$

0.5 Common Branch Points

$z^{1/n}$ has a branch point at 0 of order n.

$\log(z)$ has a logarithmic branch point at 0 with infinite branches.

$(z^2-1)^{-1/2}$ has branch points at $z = \pm 1$

$z^{-y} = \mathrm{e}^{-y \log z}$ has a logarithmic branch point at $z = 0$.

1 Lecture 1 (pg. 4)

1.1 Branch Points

For a given complex function, a branch is a point in the complex plane where a rotation around it changes the value of the function. The order of the branch point is the number of revolutions after which it returns to the original value. A logarithmic branch point never returns to its original value.

To see if a point z_0 is a branch point, take $z = z_0 + \epsilon \mathrm{e}^{i\theta}$. Take $f(z)$ and Taylor expand it for small $\epsilon \in \mathbb{R}$ and $0 < \theta < 2\pi$. If $f(z)$ returns to its starting point when θ changes from 0 to 2π, it is not a branch point.

To find a branch point, consider where the function (or a part of the function) vanishes.

1.2 Complex Logarithm

$$\ln(z) = \ln|z| + i \arg(z) \tag{14}$$

multi-valued since $\arg(z)$ can take infinite values.

1.3 Branch Cuts

A branch cut makes a multi-valued function single-valued. We choose a branch cut such that it is not possible to make a full loop around a single branch point.

2 Lecture 2 (pg. 12)

2.1 Continuity

The function f is continuous if

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{15}$$

if and only if

$$\forall \epsilon > 0 \quad \exists \quad \delta > 0 \quad \text{s.t.} \quad |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon \tag{16}$$

2.2 Analytic Functions

A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is **analyic/holomorphic** at $z_0 \in \mathbb{C}$ if $\exists$ a neighbourhood U of z_0 on which f differentiable at all points. A complex function analytic at all points in $\mathbb{C}$ is **entire**.

2.3 Cauchy-Riemann Equations

For complex function $f : \mathbb{C} \longrightarrow \mathbb{C}$, we can write $f(x+iy) = u(x,y)+iv(x,y)$. The Cauchy-Riemann equations for f are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{17}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{18}$$

For f to be complex differentiable, the Cauchy-Riemann Equations must be satisfied. The derivative is given by

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{19}$$

3 Lecture 3 (pg. 18)

3.1 (Simply) Connected Regions

An open set is said to be **split** if

$$G = G_1 \cup G_2 \quad \text{s.t.} \quad G_1 \cap G_2 = \emptyset \quad G_1, G_2 \neq \emptyset. \tag{20}$$

G is **connected** if it does not split. G is **simply connected** if any pair of lines connecting two points can be deformed into eachother without leaving G.

3.2 Complex Integration

Path must be specified as there are many different curves from z_1 to z_2.

If we parameterise the path of integration γ connecting points a and b in terms of a real parameter $t \in \mathbb{R}$,

$$f(t) = u(t) + iv(t) \tag{21}$$

$$\int_a^b f(t)\mathrm{d}t = \int_a^b u(t)\mathrm{d}t + i\int_a^b v(t)\mathrm{d}t. \tag{22}$$

3.3 Cauchy's Theorem

Let $D \subset \mathbb{C}$ be a simply connected open subset of $\mathbb{C}$ and let f be a function which is analytic in D and $f : D \to \mathbb{C}$. For any closed curve $C \in D$,

$$\oint_C f(z)\mathrm{d}z = 0. \tag{23}$$

For two curves γ_1, γ_2 from z_1 to z_2 where $f(z)$ is analytic on simply connected domain connecting z_1, z_2,

$$\int_{\gamma_1} f(z) = \int_{\gamma_2} f(z). \tag{24}$$

If C_1 and C_2 are closed contours enclosing simply connected domains D_1 and D_2, and if $f(z)$ is analytic on $D = D_2 - D_1$, then

$$\oint_{C_1} f(z)\mathrm{d}z = \oint_{C_2} f(z)\mathrm{d}z. \tag{25}$$

4 Lecture 4 (pg. 25)

4.1 Cauchy's Integral Formula

If $f(z)$ is **analytic** on a **simply connected** domain D that is contained by the closed curve C, then, for $a \in D$,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\mathrm{d}z. \tag{26}$$

The behaviour of the analytic function inside the boundary is entirely determined by the function's behaviour at the boundary.

4.2 Differentiating Cauchy's Formula

We can differentiate Cauchy's formula with respect to a point a n times,

$$\frac{\mathrm{d}^n f(a)}{\mathrm{d}a^n} = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z, \quad n = 0, 1, 2, \ldots \tag{27}$$

We can use this to solve complex integrals where the integrand can be put in the form of the integrand in Equation (27).

4.3 Taylor Series

Taylor's theorem for complex analysis is essentially the same as Taylor's theorem for real analysis.

To determine if the series determines to a finite value, we use the ratio test. For series

$$P(z) = \sum_{n=1}^{\infty} a_n z^n \tag{28}$$

then $P(z)$ converges if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| z| < 1. \tag{29}$$

The radius of convergence R of a series is given by

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}. \tag{30}$$

5 Lecture 5 (pg. 31)

5.1 Liouville's Theorem

Bounded ($|f(z)| \leq M$), entire functions are constant. Note that $\sin(z)$ is not bounded for $z \in \mathbb{C}$.

5.2 Zeroes

If $f(z) = (z - z_0)^n g(z)$ where f and g are analytic in D and $g(z_0) \neq 0$, then f has a zero of order n at z_0.

Zeroes of an analytic function are isolated.

5.3 Singularities

Let f be analytic in a domain $D \setminus \{z_0\}$, then f has a pole at z_0 if an only if we can write

$$f(z) = \frac{g(z)}{(z - z_0)^m} \tag{31}$$

where $g(z)$ is analytic in D and $g(z_0) \neq 0$. We say m is the order of the pole.

6 Lecture 6 (pg. 36)

A function is **meromorphic** in domain D if it is analytic except for at the poles of the function.

6.1 Laurent Series

A generalisation of the Taylor series of a function expanding around a singular point.

Let $f(z)$ be analytic within the annulus $C_1 < |z - z_0| < C_2$ with f undefined at z_0. The Laurent series around z_0 can be expressed as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{32}$$

redifining the variable, we find

$$f(z) = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} + \sum_{n=1}^{\infty} a_n (z - z_0)^n. \tag{33}$$

The first sum is the principal part and the second is the analytic part. The coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z \tag{34}$$

$$a_{-n} = \frac{1}{2\pi i} \oint_C f(z)(z - z_0)^{n-1} \mathrm{d}z \tag{35}$$

where C is a closed curve contained within the annulus that winds around z_0 once.

Some important results following from the Laurent coefficients:

- If all a_{-n}'s are non-zero then z_0 is an essential pole/ singularity
- The coefficient a_{-a} is the residue
- The analytic part converges to an analytic function for all z inside C_2.
- The principal part converges to an analytic function for all z outside C_1.

When integrating a meromorphic function, consider which parts of the Laurent series are analytic, then apply Cauchy's theroem to set analytic parts to zero.

6.2 Calculating Residues

For a meromorphic function f with a pole of order n at z_0,

$$f(z) = \frac{g(z)}{(z - z_0)^n} \tag{36}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. The residue is given by

$$\mathrm{Res}\,[f, z_0] = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}((z - z_0)^n f). \tag{37}$$

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7.1 Residue Theorem

Let f be a meromorphic function in domain D with a set of poles $\{z_i\}$, then

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_i \mathrm{Res}[f, z_i]. \tag{38}$$

Here, C is some closed anti-clockwise path. If integrating clockwise, the integral is multiplied by -1.

7.2 Poles and Zeroes Theorem

Let f be a meromorphic function inside a closed contour C, then

$$\frac{1}{2\pi i} \oint_c \frac{f'(z)}{f(z)}\mathrm{d}z = n_{\text{zeroes}} - n_{\text{poles}}. \tag{39}$$

7.3 Trigonometric Integrals

An integral of the form

$$I = \int_0^{2\pi} \mathrm{d}\theta R(\cos(m\theta), \sin(n\theta)) \tag{40}$$

where R is a rational function i.e., a ratio of polynomials in $\cos(m\theta)$ and $\sin(n\theta)$ can be solved by turning it into a complex integral. We write $z = e^{i\theta}$ on the unit circle. As θ goes from $\theta \to 2\pi$ we go around the origin of the complex plane.

$$z = e^{i\theta} \implies \frac{\mathrm{d}z}{iz} = \mathrm{d}\theta : \quad \int_0^{2\pi} \mathrm{d}\theta \to \oint_{C_1} \frac{\mathrm{d}z}{iz} \tag{41}$$

where C_1 is the unit circle centred at the origin of the complex plane. We can rewrite sin and cos in exponential form and change variables, then solve the integral using the residue theorem.

8 Lecture 8 (pg. 49)

8.1 Integrals Over the Real Numbers

Integrals on the real line of the form

$$I = \int_{-\infty}^{\infty} f(x)\mathrm{d}x \tag{42}$$

can be converted into complex integrals

$$I = \oint_C f(z)\mathrm{d}z \tag{43}$$

where C is a semi-circle with radius $R \to \infty$ contour in $\mathbb{C}$ if

$$|f(z)| \quad \text{tends to zero faster than} \quad \frac{1}{|z|} \quad \text{for} \quad |z| \to \infty. \tag{44}$$

This can be solved using the residue theorem, but note that only the residues in the half of the complex plane where the semi-circle is defined contribute to I.

8.2 Branch Cuts and Contours

Sometimes the integrand may contain a branch point, in which case we need to introduce a branch cut to make it analytic.

We need to integrate over a contour C that excludes the branch cut. Showing that the integral of the function over the curved sections of C, we can study the integral over the remaining sections and equate it to the real integral we wanted to solve.

8.3 Closing Contours

Sometimes it is not clear which of the UHP or LHP to integrate in, especially if $f(z)$ $1/|z|$ and our integrand isn't a decaying function. In this case, we need to split $f = f_+ + f_-$ where the components of f are decaying in the UHP/LHP respectively. Take care to factor in the - sign for the curve in the LHP being traversed clockwise.

9 Lecture 9 (pg. 58)

We can use complex analysis to sum infinite series by turning the entries of the series into residues and then applying the Residue theorem.

9.1 Solving Infinite Series

The Riemann zeta is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \in \mathbb{N}. \tag{45}$$

To solve this and other similar sums, we want a function which has simple poles whose residues are the elements of the infinite series. Such a function is

$$f(z) = \frac{\pi \cot(\pi z)}{z^s}. \tag{46}$$

Consider this meromorphic function to be enclosed by a family of contours that form a square around the poles. It can be shown that $|\cot(\pi z)|$ is bounded by 1 along this family of contours, and thus that $f(z)$ is a decaying function as we take $R \to \infty$. Therefore

$$\oint_{C_R} f(z)\mathrm{d}z = 0 \implies \sum_i \mathrm{Res}[f, z_i] = 0 \tag{47}$$

where f has residuals $\{z_i\}$.

Depending on the sum in question, we need to put the right hand side in terms of the sum and rearrange. For a sum S, this could look like

$$\sum_i \mathrm{Res}[f, z_i] = 2S + \mathrm{Res}[f, 0] = 0 \implies S = -\frac{1}{2}\mathrm{Res}[f, 0]. \tag{48}$$

9.2 Altenating Series

Consider alternating series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \tag{49}$$

we use the function

$$f(z) = \frac{\pi}{z^2 \sin(\pi z)} \tag{50}$$

instead.

9.3 Series Not Centred at the Origin

If our series is not centred at the origin, more care must be taken to construct the equation for the sum in terms of the other residuals.