Classical Mechanics 2 Summary Notes

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Contents

1 Generalised Description of Mechanical Systems

Dynamical variables describe completely the configuration of a mechanical system. They can change under the action of forces, the equation of motion of a system specifies the dynamical variables as functions of time.

Degrees of freedom are the number of independent dynamical variables needed to describe the configuration of a system. A system of M point masses has $N = 3M$ degrees of freedom, but the existence of j independent constraints reduces this number to $N = 3M - j$ DoF.

If the constraints on a system can be expressed in the form

$$f(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_M, t) = 0, \tag{1}$$

they are *Holonomic*. *Rheonomic* constraints have an explicit time dependence, *Schleronomic* do not.

We only require as many coordinates q_k as there are DoF, if the constraints are Holonomic then we can express the position of the ith part of the system as

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, \ldots, q_N, t) \tag{2}$$

where N is the number of degrees of freedom as calculated above. We regard $\dot{q}_k$ and q_k as independent variables.

2 The Lagrangian

We assume Holonomic constraints, that constraining forces do no work and that applied forces are conservative.

2.1 D'Alembert's Principle

By Newton's 2nd law for particle i, and considering a small virtual displacement $\delta\boldsymbol{r}_i$, we find

$$\sum_i (\boldsymbol{F}_i^{(a)} - \dot{\boldsymbol{p}}_i) \cdot \delta\boldsymbol{r}_i = 0 \tag{3}$$

where $\boldsymbol{F}_i^{(a)}$ gives the applied force on particle i.

2.2 Generalised Equations of Motion

We define the Lagrangian as

$$L \equiv T - V \tag{4}$$

where T and V give the kinetic and potential energy of the system respectively.

Manipulating (Eq. 3) into a generalised equation of motion, we find the **Euler-Lagrange equations**,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0. \tag{5}$$

These are the *fundamental equations* of Lagrangian mechanics.

2.3 Ignorable Coordinates

If the time derivative of a coordinate appears in the Lagrangian, but the coordinate itself does not, then this is an *ignorable* coordinate as its canonically conjugate momentum is a constant of the motion.

$$\dot{p} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \qquad \Longrightarrow \qquad p \equiv \frac{\partial L}{\partial \dot{q}} \quad \text{conserved.} \tag{6}$$

3 Variational Calculus

3.1 Hamilton's Principle of Least Action

$$S[\boldsymbol{q}(t)] = \int_{t_1}^{t_2} L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t), \mathrm{d}t \tag{7}$$

is known as the action functional. Hamilton's Principle states that a mechanical system moves in such a way as to minimise the action functional. It can be shown that the Euler-Lagrange equations are the solution to this calculus of variations problem.

Note that an alternative method to finding the equation of motion for a mechanical system with applied constraints involves using Lagrange multipliers.

4 Linear Oscillators

A mechanical system that remains at rest is in equilibrium, this occurs at points in configuration space where all generalised forces

$$\mathcal{F}_k = \frac{\delta W}{\delta q_k} \tag{8}$$

vanish. For a conservative system, this corresponds to configurations where the potential energy is stationary in each q_k.

We can carry out a 2nd order Taylor series expansion of the Lagrangian around a stationary point $q = q_s$,

$$L = \frac{m}{2}\left(\dot{q}^2 + \frac{D}{F}q^2\right), \tag{9}$$

where D and F are the coefficients of q^2 and $\dot{q}^2$ in the expansion.

4.1 Simple Harmonic Oscillator

The Lagrangian for a SHO such as a mass m attached to a spring with spring constant k can be written

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2, \tag{10}$$

if $D/F < 0$, then this is equivalent to (Eq. 9), with $k = -m(D/F)$. Close to equilibrium, one observes simple harmonic motion with characteristic frequency $\omega = \sqrt{-D/F}$, the motion is unstable if this is imaginary.

A frictional damping force acts to suppress motion, and vanishes when the motion ceases. Such a force is $F_d = \gamma\dot{q} = -(m\omega/Q)\dot{q}$ where Q is the quality factor of the oscillator.

We can solve for the EoM of a damped SHO, by setting the RHS of the Euler-Lagrange equation equal to the damping force, resulting in a linear homogeneous ODE we can solve with trial functions. Different values of Q give different soltions.

5 Driven Oscillators

Consider an undamped oscillator driven by a time dependent external force with no spacial dependence. Whatever supplies the driving force is not considered to be part of the dynamical system, we do not consider any effect of the forces the oscillator exerts on the source of the external force. The Lagrangian is given by

$$L = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2} - F(t)q, \tag{11}$$

which yields a second order linear inhomogeneous ODE

$$\ddot{q} + \omega^2 q = F(t)/m. \tag{12}$$

We can consider the driving force to be an infinite number of impulsive forces (represented by the Dirac delta) and sum up their resulting displacements.

5.1 Impulsive Force

We find there to be an instantaneous change in velocity due to an impulsive force

$$\dot{q}(t'_+) = \dot{q}(t'_-) + K/m \tag{13}$$

where K is the total impulse provided, and $t_\pm = \lim_{\epsilon\to 0} t' \pm \epsilon$ (the definition of an impulsive force).

The equation of motion for an SHO after an impulsive force is applied is given by

$$q(t) = q(t'_-)\cos(\omega[t - t']) + \frac{\dot{q}(t'_-) + K/m}{\omega}\sin(\omega[t - t']). \tag{14}$$

The Green's function for this system is the solution to the DE

$$\ddot{G}(t-t') + \omega^2 G(t-t') = \delta(t-t'). \tag{15}$$

It represents the response of the oscillator to a unit sized impulsive force. Considering the oscillator to be at rest prior to the application of the impulsive force, the solution for G is

$$G(t-t') = 0 \qquad t-t' \leq 0 \tag{16}$$

$$G(t-t') = \frac{1}{\omega}\sin(\omega[t-t']), \qquad t-t' \geq 0. \tag{17}$$

5.2 General Driving Force

We find the equation of motion for the driven oscillator by using the definition of the Dirac delta function,

$$\ddot{q} + \omega^2 q = \frac{1}{m}\int_{-\infty}^{\infty} \mathrm{d}t' F(t')\delta(t-t). \tag{18}$$

We are splitting the general driving force into infinite impulsive forces. Subbing in the differential equation for the Green's function, and substituting in the solution for $G(t-t')$, we find

$$q(t) = \frac{1}{m}\int_{-\infty}^{t} F(t')\sin(\omega[t-t'])\mathrm{d}t'. \tag{19}$$

6 Coupled Small Oscillations

We consider small oscillations around a stable equilibrium point of a mechanical system. If this motion does not depart too far from a stable equilibrium point, then the system resembles a set of *coupled oscillators.*

6.1 Two Coupled Pendulums

Consider a system with two pendulums of length l, connected half way down by a spring with constant k, and unstretched length equal to the horizontal separation of the two pivot points. There are 2 DoF, use θ_1 and θ_2 as generalised coordinates. If the rigid pendulums have all of their mass m contained in the bob and the spring is massless, then the Lagrangian is given by

$$\begin{aligned} L &= T - V \\ &= \frac{ml^2}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2}\left(mgl\left(\theta_1^2+\theta_2^2\right) + k\left(\frac{l}{2}\right)^2(\theta_2-\theta_1)^2\right) \\ &= ml^2\left[\left(\dot{\theta}_c^2 - \omega_0^2\theta_c^2\right) + \frac{1}{4}\left(\dot{(\theta)}_r^2 - \omega_0^2\left(1+2\eta\right)\theta_r^2\right)\right], \end{aligned} \tag{20}$$

where $\theta_c = (\theta_1+\theta_2)/2$, and $\theta_r = \theta_2 - \theta_1$ are more convenient coordinates, and $\omega_0 = \sqrt{g/l}$ and $\eta = \frac{kl}{4mg}$.

The Euler-Lagrange equation yields two separated harmonic oscillator EoMs,

$$\ddot{\theta}_c + \omega_0^2 \theta_c = 0 \tag{21}$$

$$\ddot{\theta}_r + \omega_0^2 (1 + 2\eta)\theta_r = 0. \tag{22}$$

In a system of N coupled oscillators, there always exists N generalised coordinates that separate, yielding N independent, homogeneous, linear DEs.

Generalised coordinates undergoing SHM are called **normal coordinates**.

6.2 General Recipe

1. Find values of the generalised coordinates where all generalised forces $\mathcal{F}_k = -\partial V/\partial q_k = 0$.
2. Taylor expand L to second order in the generalised coordinates and velocities around the equilibrium values of the coordinates, and with the $\dot{q}_k$ set to 0.

6.3 Matrix form of L

We can define a new set of generalised coordinates which have equilibrium values 0. L takes the form

$$L = \dot{\boldsymbol{q}}^T \hat{\tau} \dot{\boldsymbol{q}} - \boldsymbol{q}^T \hat{v} \boldsymbol{q} + c \tag{23}$$

where

$$\tau_{jk} = \frac{1}{2} \left(\frac{\partial^2 T}{\partial \dot{q}_j \, \partial \dot{q}_k} \right)_{\dot{q}_j, \dot{q}_k = 0} \tag{24}$$

$$v_{jk} = \frac{1}{2} \left(\frac{\partial^2 V}{\partial q_j \, \partial q_k} \right)_{q_j, q_k = 0} . \tag{25}$$

The Euler-Lagrange equations yield a linear system of differential equations

$$\hat{\tau} \ddot{\boldsymbol{q}} + \hat{v} \boldsymbol{q} = 0. \tag{26}$$

7 Normal Modes

Inserting a trial solution $\boldsymbol{q} = \boldsymbol{b} e^{i\omega t}$ into the Euler-Lagrange equation above yields

$$\left(\hat{v} - \omega^2 \hat{\tau} \right) \boldsymbol{b} = 0 \tag{27}$$

which, for non-trivial $\boldsymbol{b}$ implies

$$\left| -\hat{\tau} \omega^2 + \hat{v} \right| = 0. \tag{28}$$

Calculating the determinant yields an Nth order polynomial in ω^2, hence there are N generalised values of ω_j^2, each with corresponding vectors $\boldsymbol{b}_j$ that solve (Eq. 27).

7.1 Normal Coordinates

It is common to normalise the mode vectors $\boldsymbol{b}_j$. We define a new set of separated generalised normal coordinates r_j via

$$r_j = \sum_{k=1} N \left(\boldsymbol{b}_j\right)_k q_k. \tag{29}$$

If $\omega_j^2 > 0$, the equilibrium configuration is stable, and the normal coordinate r_j evolves as

$$r_j(t) = r_j(0)\cos(\omega_j t) + (\dot{r}_j(0)/w_j)\sin(\omega_j t). \tag{30}$$

Such motion is called a normal mode, where all parts of the system oscillate with the same frequency. Motion in general can be described by a *superposition of normal modes.*

A normal mode is the motion of a normal coordinate.

7.2 Mode Orthogonality

The normal modes are orthogonal (as they are effectively eigenvectors of a real symmetric matrix).

If all w_i are different, then the $\boldsymbol{b}_i$ are unique.

The $\boldsymbol{b}_i$s are the natural axes in N-dimensional space. They diagonalise the transformed potential energy, and in these coordinates, the system undergoes SHO.

8 Central Forces

Consider two bodies with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ interacting via a central force. A central force is defined as a force depending only on the distance r between the two bodies, directed along the line between them.

8.1 Conservation of Momentum

Conservation of momentum implies that the centre of mass (CoM), with coordinate $\boldsymbol{R} = (m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2)/(m_1 + m_2)$, moves like a free particle. We can rewrite the Lagrangian for the two bodies, in the absence of external potentials, in terms of $\boldsymbol{R}$ and the relative coordinate $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$,

$$L = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 - V_{12}(r), \tag{31}$$

where $M = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$ denote the total and reduced masses.

Since R does not appear in the Lagrangian, it is an ignorable coordinate and hence

$$p_q \equiv \frac{\partial L}{\partial \dot{q}} \tag{32}$$

is conserved.

8.2 Conservation of Angular Momentum

If we transform to the centre of mass frame (a frame moving with the CoM), then $\boldsymbol{R} = \dot{\boldsymbol{R}} = 0$ and $L = \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - V_{12}(r)$. $V_{12}(r)$ is dependent on distance only, and hence L is *rotationally invariant.*

If we switch to spherical coordinates, we find ϕ is ignorable and therefore

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} \sin^2\theta = J_z \quad \text{constant.} \tag{33}$$

For any choice of z-axis, the angular momentum in that direction is always constant, therefore the total angular momentum $\boldsymbol{J} = \boldsymbol{r} \times \mu\dot{\boldsymbol{r}}$ is *always* constant. Without loss of generality, we can choose the coordinates such that $\boldsymbol{J}$ always lies on the z-axis.

As $\boldsymbol{J} = \boldsymbol{r} \times \mu\dot{\boldsymbol{r}} = \boldsymbol{r} \times \boldsymbol{p}$, both $\boldsymbol{r}$ and $\boldsymbol{p}$ must be perpendicular to $\boldsymbol{J}$, hence $\boldsymbol{r}$ and $\boldsymbol{p}$ must be in a plane perpendicular to $\boldsymbol{J}$ and remain in that plane if $\boldsymbol{J}$ is to remain constant.

We choose $\boldsymbol{J}$ to lie along the z-direction, such that the angle of elevation $\theta = \pi/2, \dot{\theta} = 0$, and $J = J_z = \mu r^2\dot{\phi}$. We can therefore remove θ and ϕ from the Lagrangian entirely, yielding

$$L = \frac{1}{2}\mu\dot{r}^2 - \left(\frac{J^2}{2\mu r^2} + V_{12}(r)\right) = \frac{1}{2}\mu\dot{r}^2 - V_{\text{eff}}(r). \tag{34}$$

8.3 Kepler's Second Law

Equal areas are swept out in equal times.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\phi} = \frac{J}{2\mu}. \tag{35}$$

J is conserved, so the rate at which area is swept out is constant.

9 Gravitational Attraction

The potential between two gravitating bodies of masses m_1 and m_2 is $V_{12}(r) = -k/r$ where $k = Gm_1m_2$. The ODE to solve is then (from the Euler-Lagrange equation)

$$\mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2}, \tag{36}$$

solving this equation determines the orbits of the bodies.

Making the transformation $u = 1/r$ allows us to easily solve, giving

$$1/r \equiv u = (\mu k/J^2) + B\cos\phi \tag{37}$$

where B is a constant of integration.

9.1 Elliptical Orbits (Kepler's First Law)

Defining $p \equiv J^2/\mu k$ and $\epsilon \equiv pB$, we find that if $0 \leq \epsilon < 1$ the energy is negative, implying a bound state. The solution above can be written

$$p = r + \epsilon r \cos\phi \tag{38}$$

which defines an ellipse with semi-major axis $a = p/(1-\epsilon^2)$ and semi-minor axis $b = p/\sqrt{1-\epsilon^2}$. ϵ is the eccentricity of the elliptical orbit.

- $\epsilon = 0$ for a circular orbit
- $0 \leq \epsilon < 1$ for an elliptical orbit
- $\epsilon = 1$ for a parabolic trajectory
- $\epsilon > 1$ for a hyperbola

9.2 Kepler's Third Law

From Kepler's Second Law above, the period follows from the total area $A = \pi ab$ of the orbit, $\tau = A/\mathrm{d}/\mathrm{d}t = 2A\mu/J$, yielding

$$\tau = 2\pi\sqrt{\frac{1}{G(m_1+m_2)}}a^{\frac{3}{2}} \implies \tau^2 \propto a^3. \tag{39}$$

Kepler's law states $\tau^2 \propto a^3$ with the same constant of proportionality for all planets in the solar system, which is approximately true as $M_\odot \gg M_{\text{planet}}$ for all planets.

9.3 Bertrand's Theorem

For force laws of the form

$$F \propto -cr^\alpha, \tag{40}$$

only $\alpha \in -2, 1$ give rise to closed, non-circular orbits (note that a circular orbit is possible for any α).

A closed orbit is one for which the radius at a fixed azimuthal angle, θ, is described by a function $r = r(\theta)$.

9.4 Solving 1D systems by Quadrature

A problem solved as an integral is said to be solved by quadrature. For 1D systems, we can solve for the time taken to move from $q = 0$ to $q = q(t)$ by integrating $\dot{q}$ with respect to time, leaving the solution in the form of an integral.

10 Noether's Theorem and Hamiltonian Mechanics

10.1 Noether's Theorem

If the Lagrangian is invariant under a continuous symmetry transformation, then there are conserved quantities associated with that symmetry, one for each parameter of the transformation. These can be found by differentiating each coordinate with respect to the parameters of the transformation in the immediate neighbourhood of the identity transformation, multiplying by the conjugate momentum, and summing over the degrees of freedom.

For example, a point mass m moving in free space with $L = \frac{1}{2}m\dot{q}^2$ with transformed coordinates $Q = q + s$ gives

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}s}\right|_{s=0} = 1 \implies I(q) = m\dot{q}, \tag{41}$$

where $I(q)$ is a conserved quantity.

10.2 The Hamiltonian

A Legendre transformation of a time-independent, 1 DoF Lagrangian $L = L(q, \dot{q})$, where the active variable is the velocity, the passive variable is the coordinate, and the third variable is the canonically conjugate momentum p, yields the Hamiltonian

$$H(q, p) \equiv p\dot{q} - L(q, \dot{q}), \quad \text{where} \quad p \equiv \left(\frac{\partial L}{\partial \dot{q}}\right)_q. \tag{42}$$

Note that the Lagrangian formulation of mechanics produces N 2nd order ODEs in q_k, the Hamiltonian formulation produces $2N$ 1st order ODES in p_k and q_k.

10.3 Hamilton's Equations of Motion

For N DoF,

$$H \equiv \left(\sum_{k=1}^{N} p_k \dot{q}_k\right) - L, \tag{43}$$

Hamilton's equations in complete generality are

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad \frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t}. \tag{44}$$

11 Canonical Transformations and the Poisson Brackets

11.1 Canonical Transformations

Changing to new generalised coordinates and their associated generalised velocities is called a point transformation.

Contact transformations are transformations to new coordinates that are functions of the original coordinates, their momenta (and possibly the time).

A transformation is canonical if it *preserves the structure of Hamilton's equations for all dynamical systems.*

11.2 The Generating Function

Consider a starting Lagrangian $L(q, \dot{q}, t)$, and a second Lagrangian in transformed coordinates $L'(Q, \dot{Q}, t)$,

$$L'(Q, \dot{Q}, t) = L(q, \dot{q}, t) - \frac{\mathrm{d}F(q, Q, t)}{\mathrm{d}t}. \tag{45}$$

$F(q, Q, t)$ is called a generating function, and we can find its time derivative from the chain rule. By definition, $\dot{q}$ doesn't appear in L', and $\dot{Q}$ doesn't appear in L, differentiating with respect to these variables gives

$$p = \frac{\partial F}{\partial q} \quad \text{and} \quad P = -\frac{\partial F}{\partial Q}. \tag{46}$$

These are known as **implicit transformation equations**. To find an explicit form for the transformation, we solve the left equation to get $Q = Q(q, p, t)$ and substitute into the right equation to get $P = P(q, p, t)$.

Note that there are four different forms of generating functions, they need not be written in terms of q and Q. Different forms have different implicit transformation equations, see page 36 in course notes.

11.3 The Transformed Hamiltonian

We find the new Hamiltonian by returning to the original definition on the Hamiltonian as a Legendre transformation, and find, with help from the implicit transformation equations,

$$H'(Q, P, t) = H(q, p, t) + \frac{\partial F(q, Q, t)}{\partial t}. \tag{47}$$

Clearly, if the generating function has no explicit time dependence, then the new Hamiltonian is simply the original rewritten by inserting the inverse of the transformation equations to express P and Q in terms of p and q.

11.4 Poisson Brackets

For N DoF, a Posisson bracket for two arbitrary functions F and G is defined as

$$\{F, G\} \equiv \sum_{k=1}^{N} \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \tag{48}$$

Poisson brackets of conjugate variables are

$$\{q, q\} = 0, \quad \{p, p\} = 0, \quad \{p, q\} = 1. \tag{49}$$

The value of any Poisson bracket is independent of coordinate system, and $\{Q, P\} = 1$ is a necessary and sufficient condition for a transformation to be canonical.

We can rewrite Hamilton's first two equations as

$$\dot{q} = \{q, H\}, \qquad \dot{p} = \{p, H\}. \tag{50}$$

12 Hamilton-Jacobi Theory

12.1 Hamilton-Jacobi Equation

Assume there exists a canonical transformation for which $H'(Q_1, \ldots, Q_N, P_1, \ldots, P_N, t) = 0$, then Hamilton's equations of motion imply $\dot{Q}_k = \dot{P}_k = 0$ for all k. Let $S \equiv F(q_1, \ldots, q_N, p_1, \ldots, p_N, t)$, the implicit transformation equations give us the Hamilton-Jacobi equation,

$$H\left(q_1, \ldots, q_N, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_N}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{51}$$

S is Hamilton's Principle Function, rather than $2N$ 1st order ODEs, we now have a single first order PDE with $N + 1$ independent variables, $\{q_1, \ldots, q_N, t\}$.

Note that $\dfrac{\mathrm{d}S}{\mathrm{d}t} = L$, implying the generating function S is just the action, by $S = \int L \mathrm{d}t$.

12.2 Action-angle variables in 1D

For a system undergoing periodic motion in 1D, consider a canonical transformation $(p, q) \to (I, \psi)$, where ψ is an ignorable coordinate in the new Hamiltonian $H'(I)$. If ψ ignorable, then I and $H'(I)$ constant. Hamilton's EoM implies

$$\dot{\psi} = \frac{\partial H'}{\partial I} = \omega(I), \tag{52}$$

where ω is a constant. Hence, $\psi = \omega(t - t_0)$ increases linearly with time. I is called the action variable, and ψ the angle variable.

12.3 Integrability

A system with N DoF moves within a phase space of dimensionality $2N$. Such a system is said to be *integrable* if there are N constants of the motion, and that these constants are in *involution*. Two constant functions are in involution if their Poisson bracket is 0.

13 Rotating Reference Frames

13.1 Accelerating Reference Frames

Let $\boldsymbol{r}_S$ be the position of a point mass with respect to chosen reference frame S. Let $\boldsymbol{r}_B$ be its position in another frame, B. $\boldsymbol{r}_B = \boldsymbol{r}_S - \boldsymbol{R}$, where B is displaced from S by $\boldsymbol{R}$.

If S is an inertial frame where Newton's laws hold, but $\ddot{\boldsymbol{R}} \neq 0$, then B is not an inertial frame, and

$$m\ddot{\boldsymbol{r}}_B = m(\ddot{\boldsymbol{r}}_S - \ddot{\boldsymbol{R}}) = \boldsymbol{F} - m\ddot{\boldsymbol{R}}. \tag{53}$$

Here, $\boldsymbol{F}$ denotes the 'true force' on the point mass in both frames, and $m\ddot{\boldsymbol{R}}$ the 'fictitious force' felt only in the non-inertial B frame.

13.2 Infinitesimal Rotations

Consider a point fixed in frame S at $\boldsymbol{r}_S = \boldsymbol{r}$, if frame B is infinitesimally rotated with respect to S by $\mathrm{d}\theta$, then $\boldsymbol{r}_B = \boldsymbol{r} + \mathrm{d}\boldsymbol{r} = \boldsymbol{r} + (\boldsymbol{r} \times \boldsymbol{n})\mathrm{d}\theta$ (comes from applying the small angle approximation to the rotation formula). This gives the velocity of the point in reference frame B relative to that in frame S,

$$\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]_{\mathrm{in}\,B} = -\left(\boldsymbol{n}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) \times \boldsymbol{r} = -\boldsymbol{\omega} \times \boldsymbol{r}. \tag{54}$$

13.3 Velocity and Acceleration

If the vector $\boldsymbol{r}$ does have a time-dependence in S, then this must be accounted for,

$$\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]_{\mathrm{in}\,B} = \left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]_{\mathrm{in}\,S} - \boldsymbol{\omega} \times \boldsymbol{r}. \tag{55}$$

In essence, velocity in B is the difference between the velocity in S and the velocity due to motion of B relative to S.

This equation is completely general for any vector, hence we can use the operator

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_{\mathrm{in}\,B} = \left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_{\mathrm{in}\,S} - \boldsymbol{\omega}\times\right) \tag{56}$$

in conjunction with $\boldsymbol{v}_B = \boldsymbol{v}_S - \boldsymbol{\omega} \times \boldsymbol{r}$ to determine the acceleration in the non inertial frame B,

$$\begin{aligned}\left[\frac{\mathrm{d}\boldsymbol{v}_B}{\mathrm{d}t}\right]_{\mathrm{in}\,B} &= \left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_{\mathrm{in}\,S} - \boldsymbol{\omega}\times\right)\boldsymbol{v}_B \\ &= \boldsymbol{a}_S - 2\boldsymbol{\omega}\times\boldsymbol{v}_B - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) - \dot{\boldsymbol{\omega}}\times\boldsymbol{r}.\end{aligned} \tag{57}$$

If considering the motion of a point mass, then multiplying the above by m and noticing $m\boldsymbol{a}_S = \boldsymbol{F}$, we deduce

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} - 2m\boldsymbol{\omega}\times\boldsymbol{v}_B - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) - m\dot{\boldsymbol{\omega}}\times\boldsymbol{r}. \tag{58}$$

The second, third and fourth terms give the **coriolis** force, the **centrifugal** force, and the **Euler** force.

14 Inertial Forces on Earth

14.1 Local Coordinate System

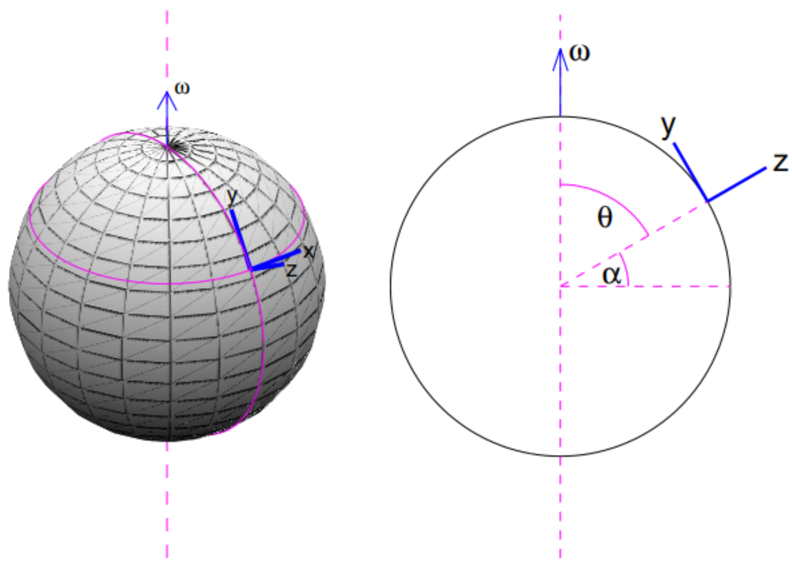

Figure 1: **Left**: Overall view of a local coordinate system in the region of a point on the surface of the spherical earth. The positive x direction is east, the positive y direction is north, the positive z direction is out from the centre of the earth. **Right**: Cross-sectional view, showing the local y and z axes only (the x axis points perpendicularly into the page). α is the latitude, and θ the colatitude (in spherical coordinates, the angle of elevation).

Angular velocity in local frame is given by

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ \omega\sin\theta \\ \omega\sin\theta \end{pmatrix}. \tag{59}$$

14.2 Inertial Forces

Note that our derivation of the coriolis, centrifugal and Euler forces above assume $\boldsymbol{r}$ to be defined with respect to a point on the axis of rotation. We must therefore substitute $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{R}$ into the inertial force terms, where $\boldsymbol{r}'$ is the position vector associated with the local coordinate system.

The centrifugal force points directly away from the axis of rotation, it can counteract gravity.

The coriolis force is proportional to velocity, so only comes into play when there is motion.

The Euler force is not zero on Earth, but is commonly taken to be negligible.

In a rotating reference frame that is also accelerating translationally with acceleration $\ddot{\boldsymbol{R}}$, the general EoM is

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} - 2m\boldsymbol{\omega} \times \boldsymbol{v}_B - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r} - m\ddot{\boldsymbol{R}}. \tag{60}$$

15 Rotational Inertia, Angular Momentum, and Kinetic Energy

Any rigid body more complex than two connected point masses has 6 DoF; 3 translation, 3 rotation (two define the axis and 1 defines the rotation).

We define the moment of inertia for N point masses rotating with angular velocity ω anticlockwise about an axis of rotation through the origin to be

$$I = \sum_{k=1}^{N} m_k \left|(\boldsymbol{r}_k \cdot \boldsymbol{n})\boldsymbol{n} - \boldsymbol{r}_k\right|^2 \tag{61}$$

15.1 The Inertia Tensor

The inertia tensor $\hat{I}$is defined as

$$\boldsymbol{J} = \hat{I}\boldsymbol{\omega}, \tag{62}$$

with

$$I_{\alpha\beta} = \sum_{k}^{N} m_k (r_k^2 \delta_{\alpha\beta} - r_{k,\alpha} r_{k,\beta}) \tag{63}$$

For continuous rigid bodies, the sum is replaced with an integral.

Rotational Kinetic Energy can be phrased generally in terms of the inertia tensor,

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot (\hat{I}\boldsymbol{\omega}) = \frac{\omega^2}{2}\boldsymbol{n}^T \hat{I}\boldsymbol{n}, \tag{64}$$

where $\boldsymbol{n}^T \hat{I}\boldsymbol{n}$ is the moment of inertia about the axis defined by $\boldsymbol{n}$.

15.2 Centre of Mass

The centre of mass position and velocity is defined by

$$\boldsymbol{R}_C = \frac{\sum_{k=1}^{N} m_k \boldsymbol{r}_k}{M}, \qquad \dot{\boldsymbol{R}}_C = \frac{\sum_{k=1}^{N} m_k \dot{\boldsymbol{r}}_k}{M}. \tag{65}$$

It immediately follows that the total momentum $\boldsymbol{P}$ is equivalent to the momentum of a single particle of mass M moving with the CoM.

We can separate the total angular momentum into angular momentum about the CoM and angular momentum of the CoM about the origin, and likewise for kinetic energy. In general, if the external forces acting on a rigid body act through the CoM, then the CoM motion can be considered separately.

16 The Parallel and Principal Axis Theorems

16.1 Displaced Axis Theorem

It can be shown (see notes) that the inertia tensor of a rigid body, defined with respect to rotations about the origin can be related to the inertia tensor of the same rigid body, defined with respect to rotations about it CoM, through

$$\hat{I} = \hat{I}_C + M\hat{A}, \tag{66}$$

where $\hat{A}$ can be represented by a matrix, $A_{\alpha\beta} = R_C^2 \delta_{\alpha\beta} - R_{C,\alpha} R_{C,\beta}$.

16.2 Parallel Axis Theorem

Consider a moment of inertia I_C about an axis of rotation passing through the CoM, this is equivalent to considering a diagonal element of some representation of the inertia tensor. From the displaced axis theorem, the moment of inertia about a parallel axis is

$$\hat{I} = \hat{I}_C + Md^2, \tag{67}$$

where d is the distance of the parallel axis from the axis of rotation passing thought the CoM.

In any representation, the inertia tensor is symmetric, so can be represented by a symmetric matrix.

16.3 Principle Axis Theorem

The inertia tensor is always represented by a 3×3 symmetric matrix, therefore $\hat{I}$ has three eigenvalues, the principle moments of inertia I_1, I_2, I_3.

It must also have three orthonormal eigenvectors, which point in directions determining the principle axes, 1, 2 and 3.

It is always possible to find a rotated coordinate system such that the representation of $\hat{I}$ is diagonal, the diagonal elements are the principle moments of inertia. The coordinate system is set by the principle axis, and a principle moment of inertia I_k is the moment of inertia associated with rotation about principle axis k.

Rotating the coordinate system such that the representation of $\hat{I}$ is diagonal is principle axis transformation.

17 Rigid Body Dynamics and Stability

17.1 Euler's Equations of Motion

Consider N mass points subject to rigid-body constraints, and take the time derivative of the angular momentum vector

$$\dot{\boldsymbol{J}} = \sum_{k=1}^{N} \boldsymbol{r}_k \times \boldsymbol{F}_k. \tag{68}$$

Viewing the rigid body from outside, $\left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_S = \boldsymbol{N}$, the total torque.

Viewing the rigid body while travelling with its rotational motion, we can say that

$$\left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_B + \boldsymbol{\omega} \times \boldsymbol{J} = \boldsymbol{N}. \tag{69}$$

We assume a coordinate system defined by the principle axes, and thus derive Euler's equations of motion

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= N_1, \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= N_2, \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= N_3. \end{aligned} \tag{70}$$

If there are no external torques, then we can set $N_1 = N_2 = N_3 = 0$, and total angular momentum and kinetic energy are conserved.