Bielliptic Shimura curves $X_0^D(N)$ with nontrivial level

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Infinitude of degree d points

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What if we range over *all* number fields of specified degree?

Definition

The **arithmetic degree of irrationality** of X over F is the positive integer

$$\mathrm{a.irr}_F(X) := \min \left\{ d : \left(\bigcup_{[L:F]=d} X(L) \right) \text{ is infinite} \right\}.$$

Modular curves

The classical modular curves $X_1(N)$ and $X_0(N)$ over $\mathbb{Q}$ parameterize elliptic curves with extra torsion structure.

Complete lists of levels N for which these curves have arithmetic degree of irrationality d are known for low values of d:

degree d	$\text{a.irr}_{\mathbb{Q}}(X_1(N)) = d$	$\text{a.irr}_{\mathbb{Q}}(X_0(N)) = d$
2	Jeon–Kim 2004	Bars 1999
3	Jeon–Kim–Schweizer 2004	Jeon 2021
4	Jeon–Kim–Park 2006	Hwang–Jeon 2024 Derickx–Orlić 2024
$5, 6$	Derickx–Sutherland 2017	

Shimura curves

Fix $B/\mathbb{Q}$ an indefinite quaternion algebra, determined by its discriminant $D = \mathrm{Disc}(B)$. Let $\mathcal{O}_N$ be an Eichler order of level N in B.

An **$\mathcal{O}_{\mathbf{N}}$-QM abelian surface** over F is a pair (A, ι) consisting of:

- A, an abelian surface over F, and
- $\iota : \mathcal{O}_N \hookrightarrow \mathrm{End}(A)$, a QM-structure over F.

Shimura curves are generalizations of modular curves, which parameterize QM abelian surfaces with extra structure. The $D = 1$ case of $B = M_2(\mathbb{Q})$ recovers the modular curve setting.

Modular curves $\mathbf{X_0(N)}$	**Shimura curves $\mathbf{X_0^D(N)}$, $\mathbf{D > 1}$**
Parameterize elliptic curves with a cyclic order N subgroup	Parameterize $\mathcal{O}_N$-QM abelian surfaces
Have a canonical model over $\mathbb{Q}$	Have a canonical model over $\mathbb{Q}$
Have rational cuspidal points; $X_0(N)(\mathbb{Q}) \neq \emptyset$	Have no cusps, in fact $X_0^D(N)(\mathbb{R}) = \emptyset$

Another point of contrast: our knowledge of explicit equations for $X_0^D(N)$ is relatively limited for $D > 1$ compared to the $D = 1$ case.

Infinitude of quadratic points

Theorem (Padurariu–S.)

$\mathrm{a.irr}_{\mathbb{Q}}(X_0^D(N)) = 2$ *with* $D > 1$ *and* $\gcd(D, N) = 1$ *if and only if* (D, N) *is in the following set of 73 pairs:*

$$\{(6,1),(6,5),(6,7),(6,11),(6,13),(6,17),(6,19),(6,23),(6,29),(6,31),$$
$$(6,37),(6,41),(6,71),(10,1),(10,3),(10,7),(10,11),(10,13),(10,17),$$
$$(10,23),(10,29),(14,1),(14,5),(15,1),(15,2),(21,1),(22,1),(22,3),$$
$$(22,5),(22,7),(22,17),(26,1),(33,1),(34,1),(35,1),(38,1),(39,1),$$
$$(39,2),(46,1),(51,1),(55,1),(57,1),(58,1),(62,1),(65,1),(69,1),$$
$$(74,1),(77,1),(82,1),(86,1),(87,1),(94,1),(95,1),(106,1),(111,1),$$
$$(118,1),(119,1),(122,1),(129,1),(134,1),(143,1),(146,1),(159,1)$$
$$(166,1),(194,1),(206,1),(210,1),(215,1),(314,1),(330,1),(390,1)$$
$$(510,1),(546,1)\}.$$

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We call the curve X/F

- **hyperelliptic** (over F) if X admits a degree 2 map to $\mathbb{P}^1_F$
- **geometrically hyperelliptic** if X is hyperelliptic over $\overline{\mathbb{Q}}$
- **bielliptic** (over F) if X admits a degree 2 map to an elliptic curve E over F
- **geometrically bielliptic** if it is bielliptic over $\overline{\mathbb{Q}}$

Theorem (Harris–Silverman 1991)

If X/F has infinitely many quadratic points and $g(X) \geq 2$, then either

- *X is hyperelliptic, or*
- *X is bielliptic and admits a degree 2 map to an elliptic curve of positive rank over F.*

Prior work

The curves $X_0^D(N)$ with $D > 1$ of genus 0 and 1 were listed by Voight.

The hyperelliptic curves $X_0^D(N)$ with $D > 1$ were determined by work of Ogg and of Guo–Yang.

The bielliptic Shimura curves $X_0^D(1)$, and those with infinitely many quadratic points, were determined by Rotger.

Proof outline

First, we work to determine which curves $X_0^D(N)$ are *geometrically* bielliptic.

Next, for each geometrically bielliptic curve we ask:

- Is it bielliptic over $\mathbb{Q}$?
- If so, is a bielliptic quotient an elliptic curve of positive rank over $\mathbb{Q}$?

Geometrically bielliptic $X_0^D(N)$

Theorem (Padurariu–S.)

There are exactly 41 *pairs* (D, N) *with* $D, N > 1$ *and* $\gcd(D, N) = 1$ *for which* $X_0^D(N)$ *is geometrically bielliptic.*

For these 41 pairs, we determine all of the bielliptic involutions, which are Atkin–Lehner involutions, **except possibly for** the two genus 5 curves with

$$(D, N) \in \{(6, 25), (10, 9)\}.$$

Generating candidates

If X is bielliptic, then geometrically $X_{\overline{\mathbb{Q}}}$ has a degree 4 map to $\mathbb{P}^1_{\overline{\mathbb{Q}}}$. So X has geometric **gonality** $\mathrm{gon}_{\mathbb{C}}(X) \leq 4$.

Theorem (Abramovich 1996)

For a Shimura curve $X_0^D(N)$, we have

$$g(X_0^D(N)) \leq \frac{200}{21}\mathrm{gon}_{\mathbb{C}}(X_0^D(N)) + 1.$$

With $D > 1$, using an explicit bound on the genus we get

$$\mathrm{gon}_{\mathbb{C}}(X_0^D(N)) \leq 4 \Longrightarrow g(X_0^D(N)) \leq 39,$$

from which we can reduce to finitely many "bielliptic candidate pairs" (D, N).

Atkin–Lehner involutions

There is a natural source of involutions on $X_0^D(N)$, the **Atkin–Lehner involutions**, $W_0(D,N) \leq \mathrm{Aut}(X_0^D(N))$, with

$$W_0(D,N) = \{w_m \mid m \parallel DN\} \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(DN)} .$$

We can calculate the genus of an Atkin–Lehner quotient $X_0^D(N)/\langle w_m \rangle$ using fixed point counts due to Ogg. So, we list all geometrically bielliptic quotients by Atkin–Lehner involutions.

The hope is that these are *all* the geometrically bielliptic involutions.

We prove that there are no geometrically bielliptic involutions on $X_0^D(N)$ other than w_m for $m \parallel DN$ through various means, including:

- results of Kontogeorgis–Rotger on $\mathrm{Aut}(X_0^D(N))$ for N squarefree
- Castelnuovo–Severi arguments: e.g., if $g(X) \geq 6$ then X has at most one bielliptic involution
- various other geometric results on bielliptic curves, such as restrictions on fixed point counts
- Ribet's isogeny and explicit isogeny decompositions of Jacobians of modular curves

We fail only for the genus 5 curves $X_0^6(25)$ and $X_0^{10}(9)$ which each have one bielliptic Atkin–Lehner involution.

Bielliptics over $\mathbb{Q}$

Knowing all geometrically bielliptic quotients $X_0^D(N)/\langle w_m \rangle$, we then ask: which are elliptic curves (have a $\mathbb{Q}$-rational point).

In some cases, we do have explicit models to help answer this question, by work of Guo–Yang, Nualart Riera, and Padurariu–Schembri.

Otherwise...

- Answering positively: existence of $\mathbb{Q}$-rational CM points
- Answering negatively: non-existence of points over $\mathbb{R}$ or $\mathbb{Q}_p$ (results of Ogg and Clark)

Ranks

By work of Ribet and others, we know that the Jacobian of $X_0^D(N)$ is isogenous to an abelian subvariety of the Jacobian of a modular curve:

$$\mathrm{Jac}(X_0^D(N)) \sim \mathrm{Jac}(X_0(DN))^{D-\mathrm{new}}.$$

If $X_0^D(N)/\langle w_m \rangle$ is an elliptic curve over $\mathbb{Q}$, then it is isogenous to an elliptic curve in $\mathrm{Jac}(X_0(DN))^{D-\mathrm{new}}$.

We use explicit isogeny decompositions of $\mathrm{Jac}(X_0(DN))^{D-\mathrm{new}}$ and rank computations of elliptic curves in Magma to determine when bielliptic quotients have positive rank.

Sporadic points on $X_0^D(N)$

A point x on $X/\mathbb{Q}$ is called **sporadic** if $\deg(x) < \text{a.irr}_{\mathbb{Q}}(X)$.

Theorem (Padurariu–S.)

For all but at most 129 *explicit pairs* (D, N) *with* $D > 1$*, the Shimura curve* $X_0^D(N)_{/\mathbb{Q}}$ *has a sporadic* CM *point.*

For at least 73 *of these pairs,* $X_0^D(N)_{/\mathbb{Q}}$ *has no sporadic points.*

There are 56 explicit pairs (D, N) with $D > 1$ for which we remain uncertain about the existence of a sporadic point on $X_0^D(N)$