Risk Measurement Exercises: Value-at-Risk (VaR) and Expected Shortfall (ES)

Context

This document contains exercises designed to build intuition and technical skill for **Value-at-Risk (VaR)** and **Expected Shortfall (ES)**. No prior knowledge of ERM is required.

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1 Exercise 1 — Interpreting Value-at-Risk (Conceptual)

A trading desk reports a **1-day VaR at 99% confidence** of **€10 million**.

a) Explain in plain language what this number means.

b) Out of 100 trading days, on how many days should losses exceed €10 million?

c) Does this measure tell you how large losses can be beyond €10 million?

d) Can the loss exceed €10 million? Explain.

Answer in words; no calculations required.

2 Exercise 2 — Mathematical Definition of VaR

Let L be a real-valued random variable denoting the **loss** of a portfolio over one day (so larger L means worse outcomes).

a) Write a mathematical definition of $\text{VaR}_\alpha(L)$ for a confidence level $\alpha \in (0, 1)$ using the CDF of L.

b) If q_α denotes the α-quantile of L, explain the relationship between $\text{VaR}_\alpha(L)$ and q_α.

c) In one sentence, explain why VaR is a **quantile** and not an **expected value**.

3 Exercise 3 — Parametric VaR under Normal Returns (with Numbers)

Assume daily **returns** R are normally distributed:

$$R \sim \mathcal{N}(\mu, \sigma^2), \quad \mu = 0.10\% = 0.001, \quad \sigma = 2.0\% = 0.02.$$

Assume the portfolio value is €100 million and define daily **loss** as

$$L = -VR, \quad V = 100{,}000{,}000.$$

a) Compute the **95%** 1-day VaR in euros (use $z_{0.95} = 1.645$).

b) Compute the **99%** 1-day VaR in euros (use $z_{0.99} = 2.326$).

c) Briefly explain why increasing the confidence level increases VaR.

4 Exercise 4 — Expected Shortfall under Normality (Formula + Calculation)

Continue with Exercise 3 and assume L is normal.

For a standard normal $Z \sim \mathcal{N}(0, 1)$, you may use:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad \text{ES}_\alpha(Z) = \frac{\phi(z_\alpha)}{1 - \alpha},$$

where z_α is the α-quantile of Z (e.g., $z_{0.95} = 1.645$, $z_{0.99} = 2.326$).

a) Write $\text{ES}_\alpha(L)$ in terms of μ, σ, V and z_α.

b) Compute $\text{ES}_{0.95}(L)$ in euros (use $\phi(1.645) \approx 0.103$).

c) Compute $\text{ES}_{0.99}(L)$ in euros (use $\phi(2.326) \approx 0.026$).

d) Explain (intuitively) why ES captures information that VaR does not.

5 Exercise 5 — Historical VaR (Order Statistics)

You observe the following daily portfolio returns (in %) over 10 days:

-3.2	-2.5	-1.8	-1.2	-0.9	-0.4	0.1	0.4	0.7	1.0

Assume all observations are equally likely.

a) Sort the returns from worst to best.

b) Compute the **95% historical VaR** for **losses** if portfolio value is €50 million.

c) State clearly which observation(s) determine the 95% historical VaR in this small sample.

6 Exercise 6 — Historical Expected Shortfall (Tail Average)

Use the same return sample as in Exercise 5 and portfolio value €50 million.

a) Identify the tail observations used to compute ES$_{0.95}$ (based on your VaR rule from Exercise 5).

b) Compute the **95% historical Expected Shortfall** in euros.

c) Explain in one sentence how ES reacts if the single worst return becomes even worse (e.g. from -3.2% to -8%).

7 Exercise 7 — Scaling VaR Over Time (Square-root-of-time)

Assume daily returns are i.i.d. normal with mean 0 and volatility $\sigma_{1d} = 1.5\%$. Let $V =$ €200 million.

a) Compute the 1-day 99% VaR in euros (use $z_{0.99} = 2.326$).

b) Under the square-root-of-time rule, compute the 10-day 99% VaR:

$$\sigma_{10d} = \sigma_{1d}\sqrt{10}.$$

c) Give two reasons why this scaling can fail in practice (no calculations required).

8 Exercise 8 — Normal vs Fat Tails (Concept + Implications)

Two portfolios have the same mean and variance of returns. Portfolio A has normally distributed returns. Portfolio B has Student-t returns with low degrees of freedom (fat tails).

a) Which portfolio is likely to have higher VaR$_{0.99}$? Why?

b) Which portfolio is likely to have higher ES$_{0.99}$? Why?

c) Explain why ES is generally more sensitive to tail thickness than VaR.

9 Exercise 9 — Properties of Risk Measures (Coherence)

A risk measure $\rho(\cdot)$ is called **coherent** if it satisfies: monotonicity, translation invariance, positive homogeneity, and subadditivity.

a) State each of the four coherence properties in words.

b) Explain what **subadditivity** means for diversification.

c) True/False (justify briefly): VaR is always subadditive.

d) True/False (justify briefly): Expected Shortfall is subadditive.

10 Exercise 10 — VaR vs ES in a Simple Discrete Example (Math)

Consider a loss variable L (in €m) with the following distribution:

Loss L (in €m)	0	1	2	10	50
Probability	0.90	0.05	0.03	0.015	0.005

a) Compute VaR$_{0.95}(L)$.

b) Compute VaR$_{0.99}(L)$.

c) Compute ES$_{0.95}(L)$.

d) Compute ES$_{0.99}(L)$.

e) Compare VaR and ES: which one reacts more strongly to the rare €50m loss, and why?

Key Takeaways

- VaR is a **quantile**: it answers "how bad can it get, most of the time?"
- ES is a **tail average**: it answers "how bad is it when it goes really wrong?"
- Two portfolios can have the same VaR but very different tail risk (captured by ES).