# A Memory-Bounded, Deterministic and Terminating Semantics for the Synchronous Programming Language Céu

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## Abstract

Céu is a synchronous programming language for embedded soft real-time systems. It focuses on control-flow safety features in the presence of shared-memory concurrency and abortion of lines of execution, while enforcing memory-bounded, deterministic, and terminating reactions to the environment. In this work, we present a small-step structural operational semantics for Céu and a proof that reactions have the properties enumerated above: that for a given arbitrary timeline of input events, multiple executions of the same program always react in bounded time and arrive at the same final finite memory state.

***Keywords*** Determinism, Termination, Operational semantics, Synchronous languages

## 1 Introduction

Céu [19, 21] is a Esterel-based [10] programming language for embedded soft real-time systems that aims to offer a concurrent, safe, and expressive alternative to C with the characteristics that follow:

**Reactive:** code only executes in reactions to events.
**Structured:** programs use structured control mechanisms, such as await (to suspend a line of execution), and par (to combine multiple lines of execution).
**Synchronous:** reactions run atomically and to completion on each line of execution, i.e., there's no implicit preemption or real parallelism.

Structured reactive programming lets developers write code in direct style, recovering from the inversion of control imposed by event-driven execution [1, 15, 18]. Synchronous languages offer a simple run-to-completion execution model that enables deterministic execution and make formal reasoning tractable. For this reason, it has been successfully adopted in safety-critical real-time embedded systems [3].

Previous work in the context of embedded sensor networks evaluates the expressiveness of Céu in comparison to event-driven code in C and attests a reduction in source code size (around 25%) with a small increase in memory usage (around 5–10%) [21]. Céu has also been used in the context of multimedia systems [22] and games [20].

Céu inherits the synchronous and imperative mindset of Esterel but adopts a simpler semantics with fine-grained execution control. The list that follows summarizes the semantic peculiarities of Céu [19]:

- Fine-grained, intra-reaction deterministic execution, which makes Céu fully deterministic.
- Stack-based execution for internal events, which provides a limited but memory-bounded form of subroutines.
- Finalization mechanism for lines of execution, which makes abortion safe with regard to external resources.
- First-class timers with dedicated syntax, which provides automatic synchronization for multiple timers running simultaneously.

In this work, we present a formal small-step structural operational semantics for Céu and prove that it enforces memory-bounded, deterministic, and terminating reactions to the environment, i.e., that for a given arbitrary timeline of input events, multiple executions of the same program always react in bounded time and arrive at the same final finite memory state. Conceiving a formal semantics for Céu leads to a number of capabilities and outcomes as follows:

1. Understanding, finding corner cases, and stabilizing the language. After the semantics was complete and discussed in extent in our group, we found a critical bug in the order of execution of statements.
2. Explaining core aspects of the language in a reduced, precise, and unambiguous way. This is particularly important when comparing languages that are similar in the surface (e.g., Céu and Esterel).
3. Implementing the language. A small-step operational semantics describes an abstract machine that is close to a concrete implementation, which we concretized in Haskell for testing. Also, the current real-world implementation of the Céu scheduler is based on the formal semantics presented in this paper.

4. Proving properties for particular or all programs in the language. For instance, in this work, we prove that all programs in Céu are memory bounded, deterministic, and react in finite time.

The last item is particularly important in the context of constrained embedded systems:

**Memory Boundedness:** At compile time, we can ensure that the program fits in the device's restricted memory and that it will not grow unexpectedly during runtime.

**Deterministic Execution:** We can simulate an entire program execution providing an input history with the guarantee that it will always have the same behavior. This can be done in negligible time in a controlled development environment before deploying the application to the actual device (e.g., by multiple developers in standard desktops).

**Terminating Reactions:** Real-time applications must guarantee responses within specified deadlines. A terminating semantics enforces upper bounds for all reactions and guarantees that programs always progress with the time.

The rest of the paper is organized as follows: Section 2 is a review of the main characteristics of Céu based on previous work [19, 21], namely, deterministic event-driven execution, safe shared-memory concurrency, abortion and finalization for concurrent lines of execution, and first-class synchronized timers. Section 3 proposes a formal small-step operational semantics for Céu that encompasses all peculiarities of the language. Section 4 presents the proofs for the properties of memory boundedness, deterministic execution, and terminating reactions, which apply to all programs in Céu. Section 5 compares the semantics of Céu with related synchronous languages. Section 6 concludes the paper.

## 2 Céu

Céu [19, 21] is a synchronous reactive language in which programs evolve in a sequence of discrete reactions to external events. It is designed for control-intensive applications, supporting concurrent lines of execution, known as *trails*, and instantaneous broadcast communication through events. Computations within a reaction (such as expressions, assignments, and system calls) are also instantaneous considering the synchronous hypothesis [11]. Céu provides an `await` statement that blocks the current running trail allowing the program to execute its other trails; when all trails are blocked, the reaction terminates and control returns to the environment.

In Céu, every execution path within loops must contain at least a `break` or `await` statement to an external input event [6, 21]. This restriction, which is statically checked by the compiler, ensures that every reaction runs in bounded time, eventually terminating with all trails blocked in `await` statements. Céu has an additional restriction, which it shares with Esterel

```
// Declarations
input ⟨type⟩ ⟨id⟩;   // declares an external input event
event ⟨type⟩ ⟨id⟩;   // declares an internal event
var   ⟨type⟩ ⟨id⟩;   // declares a variable

// Event handling
⟨id⟩ = await ⟨id⟩;   // awaits event and assigns the received value
emit ⟨id⟩(⟨expr⟩);   // emits event passing a value

// Control flow
⟨stmt⟩ ; ⟨stmt⟩                              // sequence
if ⟨expr⟩ then ⟨stmt⟩ else ⟨stmt⟩ end        // conditional
loop do ⟨stmt⟩ end                           // repetition
every ⟨id⟩ in ⟨id⟩ do ⟨stmt⟩ end             // event iteration
finalize [⟨stmt⟩] with ⟨stmt⟩ end            // finalization

// Logical parallelism
par/or  do ⟨stmt⟩ with ⟨stmt⟩ end   // aborts if any side ends
par/and do ⟨stmt⟩ with ⟨stmt⟩ end   // terminates if all sides end

// Assignment & Integration with C
⟨id⟩ = ⟨expr⟩;    // assigns a value to a variable
_⟨id⟩(⟨exprs⟩)    // calls a C function (id starts with '_')
```

**Listing 1.** The concrete syntax of Céu.

and synchronous languages in general [4]: computations that take a non-negligible time to run (e.g., cryptography or image processing algorithms) violate the zero-delay hypothesis, and thus cannot be directly implemented.

Listing 1 shows a compact reference of Céu.

Listing 2 shows a complete example in Céu that toggles a LED whenever a radio packet is received, terminating on a button press always with the LED off. The program first declares the BUTTON and RADIO_RECV as input events (ln 1–2). Then, it uses a par/or composition to run two activities in parallel: a single-statement trail that waits for a button press before terminating (ln 4), and an endless loop that toggles the LED on and off on radio receives (ln 9–14). The finalize clause (ln 6–8) ensures that, no matter how its enclosing trail terminates, the LED will be unconditionally turned off (ln 7).

The par/or composition, which stands for a *parallel-or*, provides an orthogonal abortion mechanism [4] in which its composed trails do not know when and how they are aborted (i.e., abortion is external to them). The finalization mechanism extends orthogonal abortion to activities that use stateful resources from the environment (such as files and network handlers), as we discuss in Section 2.3.

In Céu, any identifier prefixed with an underscore (e.g., _led) is passed unchanged to the underlying C compiler. Therefore, access to C is straightforward and syntactically traceable. To ensure that programs operate under the synchronous hypothesis, the compiler environment should only provide access to C operations that can be assumed to be

instantaneous, such as non-blocking I/O and simple data structure accessors.[1]

```
1  input void BUTTON;
2  input void RADIO_RECV;
3  par/or do
4      await BUTTON;
5  with
6      finalize with
7          _led(0);
8      end
9      loop do
10         _led(1);
11         await RADIO_RECV;
12         _led(0);
13         await RADIO_RECV;
14     end
15 end
```

**Listing 2.** A Céu program that toggles a LED on receive, terminating on a button always with the LED off.

## 2.1 External and Internal Events

Céu defines time as a discrete sequence of reactions to unique external input events received from the environment. Each input event delimits a new logical unit of time that triggers an associated reaction. The life-cycle of a program in Céu can be summarized as follows [21]:

(i) The program initiates a "boot reaction" in a single trail (parallel constructs may create new trails).

(ii) Active trails execute until they await or terminate, one after another. This step is called a *reaction chain*, and always runs in bounded time.

(iii) When all trails are blocked, the program goes idle and the environment takes control.

(iv) On the occurrence of a new external input event, the environment awakes *all* trails awaiting that event, and the program goes back to step (ii).

A program must react to an event completely before handling the next one. By the synchronous hypothesis, the time the program spends in step (ii) is conceptually zero (in practice, negligible). Hence, from the point of view of the environment, the program is always idle on step (iii). In practice, if a new external input event occurs while a reaction executes, the event is saved on a queue, which effectively schedules it to be processed in a subsequent reaction.

### 2.1.1 External Events and Discrete Time

The sequential processing of external input events induces a discrete notion of time in Céu, as illustrated in Figure 1. The
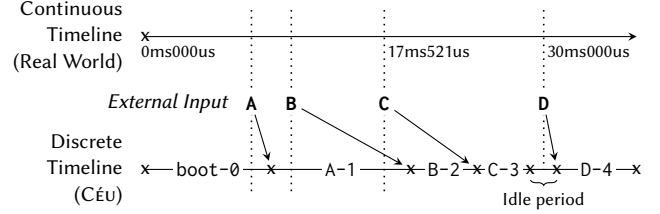


**Figure 1.** The discrete notion of time in Céu.

continuous timeline shows an absolute reference clock with "physical timestamps" for the event occurrences (e.g., event C occurs at 17ms521us). The discrete timeline shows how the same occurring events fit in the logical notion of time of Céu. The boot reaction boot-0 happens before any input, at program startup. Event A "physically" occurs during boot-0 but, because time is discrete, its corresponding reaction only executes afterwards, at logical instant A-1. Similarly, event B occurs during A-1 and its reaction is postponed to execute at B-2. Event C also occurs during A-1 but its reaction must also wait for B-2 to execute and so it is postponed to execute at C-3. Finally, event D occurs during an idle period and can start immediately at D-4.

Unique input events imply mutually exclusive reactions, which execute atomically and never overlap. Automatic mutual exclusion is a prerequisite for deterministic reactions as we discuss in Section 3.

In practice, the synchronous hypothesis for Céu holds if reactions execute faster than the rate of incoming input events. Otherwise, the program would continuously accumulate delays between physical occurrences and actual reactions for the input events. Considering the context of soft real-time systems, postponed reactions might be tolerated as long as they are infrequent and the application does not take too long to catch up with real time. Note that the synchronous semantics is also the norm in typical event-driven systems, such as event dispatching in UI toolkits, game loops in game engines, and clock ticks in embedded systems.

### 2.1.2 Internal Events as Subroutines

In Céu, queue-based processing of events applies only to external input events, i.e., events submitted to the program by the environment. Internal events, which are events generated internally by the program via emit statements, are processed in a stack-based manner. Internal events provide a fine-grained execution control, and, because of their stack-based processing, can be used to implement a limited form of subroutines, as illustrated in Listing 3.

In the example, the "subroutine" inc is defined as an event iterator (ln 4–6) that continuously awaits its identifying event (ln 4), and increments the value passed by reference (ln 5). A trail in parallel (ln 8–11) invokes the subroutine through two consecutive emit statements (ln 9–10). Given the stack-based execution for internal events, as the first

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[1] The actual implementation of Céu supports a command-line option that accepts a whitelist of library calls. If a program tries to use a function not in the list, the compiler raises an error.

emit executes, the calling trail pauses (ln 9), the subroutine awakes (ln 4), runs its body (yielding v=2), iterates, and awaits the next "call" (ln 4, again). Only after this sequence does the calling trail resumes (ln 9), makes a new invocation (ln 10), and passes the assertion test (ln 11).

```
1  event int* inc;        // declares subroutine "inc"
2  par/or do
3      var int* p;
4      every p in inc do  // implements "inc" as event iterator
5          *p = *p + 1;
6      end
7  with
8      var int v = 1;
9      emit inc(&v);       // calls "inc"
10     emit inc(&v);       // calls "inc"
11     _assert(v==3);      // asserts result after the two returns
12 end
```

**Listing 3.** A "subroutine" that increments its argument.

Céu also supports nested emit invocations, e.g., the body of the subroutine inc (ln 5) could emit an event targeting another subroutine, creating a new level in the stack. We can think of the stack as a record of the nested, fine-grained internal reactions that happen inside the same outer reaction to a single external event.

This form of subroutines has a significant limitation that it cannot express recursion, since an emit to itself is always ignored as a running trail cannot be waiting on itself. That being said, it is this very limitation that brings important safety properties to subroutines. First, they are guaranteed to react in bounded time. Second, memory for locals is also bounded, not requiring data stacks.

At first sight, the constructions "every e do ⟨···⟩ end" and "loop do await e; ⟨···⟩ end" seem to be equivalent. However, the loop variation would not compile since it does not contain an external input await (e is an internal event). The every variation compiles because event iterators have an additional syntactic restriction that they cannot contain break or await statements. This restriction guarantees that an iterator never terminates from itself and, thus, always awaits its identifying event, essentially behaving as a safe blocking point in the program. For this reason, the restriction that execution paths within loops must contain at least one external await is extended to alternatively contain an every statement.

## 2.2 Shared-Memory Concurrency

Embedded applications make extensive use of global memory and shared resources, such as through memory-mapped registers and system calls to device drivers. Hence, an important goal of Céu is to ensure a reliable behavior for programs with concurrent lines of execution sharing memory and interacting with the environment.

In Céu, when multiple trails are active in the same reaction, they are scheduled in lexical order, i.e., in the order they appear in the program source code. For instance, consider the examples in Figure 2, both defining shared variables (ln 2), and assigning to them in parallel trails (ln 5,8).

In Listing 4, the two assignments to x can only execute in reactions to different events A and B (ln 4,7), which cannot occur simultaneously by definition. Hence, for the sequence of events A->B, x becomes 4 ((1+1)*2), while for B->A, x becomes 3 ((1*2)+1).

In Listing 5, the two assignments to y are simultaneous because they execute in reaction to the same event A (ln 4,7). Since Céu employs lexical order for intra-reaction statements, the execution is still deterministic, and y always becomes 4 ((1+1)*2). However, note that an apparently innocuous change in the order of trails modifies the behavior of the program. To mitigate this threat, Céu performs concurrency checks at compile time to detect conflicting accesses to shared variables: if a variable is written in a trail segment, then a concurrent trail segment cannot access that variable [21]. Nonetheless, the static checks are optional and are not a prerequisite for the deterministic semantics of the language.

```
   input void A, B;          1  input void A;
   var int x = 1;            2  var int y = 1;
   par/and do               3  par/and do
       await A;             4      await A;
       x = x + 1;           5      y = y + 1;
   with                     6  with
       await B;             7      await A;
       x = x * 2;           8      y = y * 2;
   end                      9  end
```

|     Listing 4.      |     Listing 5.      |

**Figure 2.** Shared-memory concurrency in Céu: Listing 4 is never concurrent because the trails access x atomically in different reactions; Listing 5 is concurrent, but still deterministic, because both trails access y atomically in the same reaction.

## 2.3 Abortion and Finalization

The par/or of Céu is an orthogonal abortion mechanism because the two sides in the composition need not be tweaked with synchronization primitives nor state variables to affect each other. In addition, abortion is *immediate* in the sense that it executes atomically inside the current micro reaction. Immediate orthogonal abortion is a distinctive feature of synchronous languages and cannot be expressed effectively in traditional (asynchronous) multi-threaded languages [4, 16].

However, aborting lines of execution that deal with resources may lead to inconsistencies. Therefore, Céu provides

a `finalize` construct to unconditionally execute a series of statements even if the enclosing block is externally aborted.

Céu also enforces, at compile time, the use of `finalize` for system calls that deal with pointers representing resources, as illustrated in the two examples of Figure 3. If Céu *passes* a pointer to a system call (Listing 6, ln 5), the pointer represents a *local* resource (ln 2) that requires finalization (ln 7). If Céu *receives* a pointer from a system call return (Listing 7, ln 4), the pointer represents an *external* resource (ln 2) that requires finalization (ln 6).

Céu tracks the interaction of system calls with pointers and requires finalization clauses to accompany them. In Listing 6, the local variable `msg` (ln 2) is an internal resource passed as a pointer to `_send` (ln 5), which is an asynchronous call that transmits the buffer in the background. If the block aborts (ln 11) before receiving an acknowledge from the environment (ln 9), the local `msg` goes out of scope and the external transmission now holds a *dangling pointer*. The finalization ensures that the transmission also aborts (ln 7). In Listing 7, the call to `_fopen` (ln 4) returns an external file resource as a pointer. If the block aborts (ln 12) during the `await` A (ln 9), the file remains open as a *memory leak*. The finalization ensures that the file closes properly (ln 6). In both cases, the code does not compile without the `finalize`.[2]

```
par/or do                      1   par/or do
   var _msg_t msg;             2      var _FILE* f;
   ⟨···⟩   // prepare msg      3      finalize
   finalize                    4         f = _fopen(...);
      _send(&msg);             5      with
   with                        6         _fclose(f);
      _cancel(&msg);           7      end
   end                         8      _fwrite(..., f);
   await SEND_ACK;             9      await A;
with                          10      _fwrite(..., f);
   ⟨···⟩                      11   with
end                           12      ⟨···⟩
                              13   end
```

**Listing 6.** Local resource.   **Listing 7.** External resource.

**Figure 3.** Céu enforces the use of finalization to prevent dangling pointers and memory leaks.

The finalization mechanism of Céu is fundamental to preserve the orthogonality of the `par/or` construct since the clean up code is encapsulated in the aborted trail itself.

### 2.4 First-Class Timers

Embedded systems typically rely on activities that react to *wall-clock time*[3], such as timeout watchdogs and periodic

sensor readings [8]. Céu supports wall-clock timers through the `await` statement [19], as illustrated in Listing 8. The `await` blocks the current running trail for the specified amount of time (ln 2,4).

However, underlying system timers do not activate programs immediately with zero delay due to intrinsic overhead, such as OS scheduling, interrupts with higher priorities, or even losses due to clock resolutions. We define the difference between a requested timeout and the actual expiring time as the *residual delta time (delta)*. Without explicit manipulation, the recurrent use of timed activities in a row (or in a loop) may accumulate a considerable amount of deltas that can lead to incorrect behavior in programs.

Céu handles deltas automatically, as illustrated in Listing 8. Suppose that after the first `await 10ms` request (ln 2), the underlying system gets busy and takes 15ms to notify Céu. The scheduler will notice that the `await` has not only already expired, but is delayed with `delta=5ms`. The awaiting trail awakes, sets `v=1` (ln 3), and then invokes `await 1ms` (ln 4). Notice that non-awaiting statements are considered to take no time in the synchronous model (e.g, ln 3). Since the current delta is still higher than the requested timeout (i.e. $5ms > 1ms$), the trail is rescheduled for execution, now with `delta=4ms`.

Delta compensation also makes timers in parallel to be perfectly synchronized. In the example in Listing 9, although the scheduler cannot guarantee that the first trail terminates exactly in 11ms (ln 2,4), it can at least ensure that it terminates before the second trail (yielding `v=1`), because $10 + 1 < 12$. A similar program in a language without first-class support for timers would depend on the execution timings for the code marked as <...> (ln 3), making the reasoning about the execution behavior more difficult.

```
var int v;           1   par/or do
await 10ms;          2      await 10ms;
v = 1;               3      <...> // non awaiting stmts
await 1ms;           4      await 1ms;
v = 2;               5      v = 1;
                     6   with
                     7      await 12ms;
                     8      v = 2;
                     9   end
```

**Listing 8.** Wall-clock   **Listing 9.** Synchronized timers.
await.

**Figure 4.** First-class timers in Céu.

## 3 Formal Semantics of Céu

We now introduce and formalize the semantics of a reduced version of Céu, called *basic Céu*. Although simpler than the full language presented in Section 2, basic Céu is expressive

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[2]The compiler only forces the programmer to write the finalization clause, but cannot check if it actually handles the resource properly.

[3] By wall-clock time we mean the passage of time from the real world, measured in hours, minutes, etc.

enough to capture all the essential characteristics of full Céu, in particular, the stack-based execution of internal events. Once basic Céu is defined, the more elaborate constructs of full Céu can be defined on top of it, as we will discuss shortly.

The statements of basic Céu are presented in Figure 5. In the figure, the metavariables $v$ (ln 1, 2, 16), $e$ (ln 3–5, 10), and $n$ (ln 15–16) range over variable identifiers, event identifiers, and integers. The metavariable *stmt* (ln 1, 7–13, 16–19) denotes a statement, i.e., any of the statements of Figure 5—complex statements are defined recursively in terms of simpler statements. Finally, the metavariable *expr* (ln 2, 7) denotes an expression.

| | | |
|---|---|---|
| 1 | var $v$ *stmt* | *variable declaration* |
| 2 | $v := expr$ | *assignment statement* |
| 3 | await$_{ext}(e)$ | *await external event* |
| 4 | await$_{int}(e)$ | *await internal event* |
| 5 | emit$_{int}(e)$ | *emit internal event* |
| 6 | break | *loop escape* |
| 7 | if *expr* then *stmt*$_1$ else *stmt*$_2$ | *conditional* |
| 8 | *stmt*$_1$ ; *stmt*$_2$ | *sequence* |
| 9 | loop *stmt* | *infinite loop* |
| 10 | every $e$ *stmt* | *event iteration* |
| 11 | *stmt*$_1$ par/and *stmt*$_2$ | *par/and statement* |
| 12 | *stmt*$_2$ par/or *stmt*$_2$ | *par/or statement* |
| 13 | fin *stmt* | *finalization statement* |
| 14 | @nop | *dummy statement* |
| 15 | @runat$(n)$ | *run at stack level $n$* |
| 16 | @var $v$ $n$ *stmt* | *expanded var* |
| 17 | *stmt*$_1$ @loop *stmt*$_2$ | *expanded loop* |
| 18 | *stmt*$_1$ @par/and *stmt*$_2$ | *expanded par/and* |
| 19 | *stmt*$_1$ @par/or *stmt*$_2$ | *expanded par/or* |

**Figure 5.** The statements of basic Céu.

For simplicity, we only consider integer expressions. These are build up from integer constants and variables by the usual mathematical operators $(+, -, \leq, \dots)$. We assume that expression evaluation takes zero time (in accordance with the synchronous hypothesis) and that it always produces an integer value. In places where a boolean value is expected, any nonzero value means true while zero means false.

We distinguish between three kinds of basic Céu statements. First, there are those statements which are common in imperative languages and behave as usual, namely, declarations, assignments, conditionals, sequences, loops, and breaks. The var statement of basic Céu introduces a single local variable whose scope is the statement immediately following it.

The statements of the second kind are those which are specific to Céu. These are the statements await$_{ext}$, await$_{int}$, emit$_{int}$, and every, which deal with events, the statements

par/and and par/or, which define parallel compositions, and the statement fin, which defines a finalization block. These basic Céu statements are more or less equivalent to their counterparts in full Céu. We defer a precise description of their behavior and entailed properties to Section 3.3.

Finally, the statements of the third kind are the remaining ones, namely, @nop, @runat, @var, @loop, @par/and, and @par/or. These are hidden internal statements used by the interpreter to encode in the program's text information about its execution. We will have more to say about these @-statements in Section 3.3. Before that, however, we need to present the syntactic restrictions of basic Céu and discuss the mapping of full Céu programs into basic Céu programs.

### 3.1 Syntactic Restrictions of Basic Céu

The syntax of basic Céu shown in Figure 5 can be seen as a schema for generating programs. Not all programs generated by this schema are well-formed though. To be considered a *well-formed* basic Céu program, the generated program must satisfy the following restrictions:

1. If variable $v$ occurs in an expression or assignment statement of the program, then this occurrence happens in the body of a var statement that declares $v$.
2. If a break occurs in the program, then this occurrence happens in the body of a loop.
3. If a statement of the form loop *stmt* occurs in the program, then all execution paths within *stmt* contain a matching break or an await$_{ext}$ or an every$_{ext}$ (i.e., an every that expects an external event.)
4. If a statement of the form every $e$ *stmt* or fin *stmt* occurs in the program, then *stmt* does not contain occurrences of loop, break, await$_{ext}$, await$_{int}$, every, or fin.

Restrictions 1 and 2 prevent the use of undeclared variables and orphan break's. Restriction 3 ensures that the program does not have an infinite loop with a body that runs in zero time (which would violate the synchronous hypothesis). And restriction 4 ensures that the body of every and fin statements always execute to completion within the same reaction. Similar restrictions exist in full Céu, as discussed in Section 2. All restrictions can be easily verified by inspecting the AST of programs.

From now on, whenever we speak of a basic Céu program we mean a well-formed basic Céu program.

### 3.2 From Full Céu to Basic Céu

Most statements of full Céu are also present in basic Céu. These statements in common, however, are not exactly equivalent. It is sometimes the case that a statement of full Céu (say await) has more features than its basic Céu counterpart (await$_{int}$). In this section, we discuss how the extra features of full Céu are implemented in basic Céu.

**Await and emit**

The `await` and `emit` primitives of full Céu are slightly more complex than those of basic Céu, as they support communication of values between them.

Figure 6 shows a translation that adds a variable to hold the value being communicated. The original full Céu code in Listing 10 declares an internal event `e` (ln 2) and has an `await` (ln 4) and an `emit` (ln 10) that communicate the value 1 between the trails in parallel. The translation to basic Céu in Listing 11 declares an additional shared variable `e_` (ln 1) to hold the emitted value (ln 9) and which can be accessed by the awaking trail (ln 5).

External events require a similar translation, i.e., each event needs a corresponding global variable shared between all awaiting trails.

```
                          1   var int e_;
event int e;              2   event int e;
par/or do                 3   par/or do
  var int v = await e;    4     await e;
                          5     var int v = e_;
  <...>                   6     <...>
with                      7   with
  <...>                   8     <...>
                          9     e_ = 1;
  emit e(1);              10    emit e;
end                       11  end
```

**Listing 10.**          **Listing 11.**

**Figure 6.** Full-to-Basic translation for `await` and `emit`.

**First-class timers**

To add support for first-class timers to basic Céu, we introduce a `TIMER` input event, which notifies the passage of time, and two global variables: `timer_`, which holds the elapsed time, and `delta_`, which holds the residual time (initially set to 0).

Listing 13 shows the basic Céu program resulting from the translation of the two timers shown in Listing 12 (ln 1,11). A timer first adds a (possibly) negative delta from the time it should await (ln 1). Then, it enters in a loop that awakes on every occurrence of `TIMER` (ln 7) and decrements the time it should await (ln 8). Each iteration of the loop checks if the timer has expired (ln 3), and sets the new delta, which may affect a timer in sequence. The check happens before the await because the timer may already start expired due to the residual time.

**Finalization**

The biggest mismatch between full Céu and basic Céu is in their support for finalization, i.e., between the statements `finalize` of full Céu and `fin` of basic Céu. Listing 14 shows a

```
await 10ms;      1   var int tot_ = 10000 + delta_;
                 2   loop do
                 3     if tot_ <= 0 then
                 4         delta_ = tot_;
                 5         break;
                 6     end
                 7     await TIMER;
                 8     tot_ = tot_ - timer_;
                 9   end
                 10
await 1ms;       11  var int tot_ = 1000 + delta_;
                 12  <...> // same loop as above
```

**Listing 12.**          **Listing 13.**

**Figure 7.** Full-to-Basic translation for timers.

full Céu program containing an explicit block (ln 1–8) that executes the statements in `<A>` (ln 3) immediately followed by `<C>` (ln 7), and unconditionally executes `<B>` (ln 5) when the block terminates or aborts. To simulate this behavior in basic Céu we need to perform the translation shown in Listing 15. The basic Céu code also executes `<A>` (ln 1) immediately followed by `<C>` (ln 5). The difference is that the basic `fin` `<B>` statement (ln 3) blocks the pending statement forever, which only awakes and executes when it is aborted. The `par/or` (ln 2–6) serves this purpose since it allows `<C>` to execute immediately and aborts the `fin` when terminating.

```
do                  1   <A>
    finalize        2   par/or do
        <A>         3       fin <B>
    with            4   with
        <B>         5       <C>
    end             6   end
    <C>             7
end                 8
```

**Listing 14.**          **Listing 15.**

**Figure 8.** Full-to-Basic translation for finalization.

### 3.3 Operational Semantics

We proceed to formalize the operation of the basic Céu interpreter. Our goal here is twofold. First, we want to define a function *reaction* that describes precisely the operation steps taken by the interpreter to compute a single reaction to an external event. Second, we want to establish (prove) some properties of this function. In particular, we want to establish that:

1. *reaction* is indeed a function: the same program will always react in the same way to a same external event (i.e., reactions are deterministic);

2. *reaction* is a total function: its computation always yields a result (i.e., reactions terminate); and

3. *reaction* can be computed by a linear bounded automaton: the amount of memory it uses never exceeds a fixed threshold which depends solely on the input program (i.e., reactions are memory-bounded).

We will define *reaction* using a set of rules for a small-step operational semantics [17]. These rules dictate how the internal state of the basic CÉU interpreter progresses while it is computing a reaction. A snapshot of this internal state is called a *description*, denoted $\delta$, and consists of a quadruple $\langle stmt, \ell, e, \theta \rangle$ where

- *stmt* is a well-formed basic CÉU program;
- $\ell$ is a nonnegative integer, called the stack level;
- $e$ is an event identifier or the empty identifier $\varepsilon$; and
- $\theta$ is a memory.

We will detail the precise meaning and purpose of the components of the description in due course. For now, the important thing is that the steps taken by the interpreter to compute a reaction can be viewed as transitions between descriptions. The transitions are dictated by rules hardcoded in the interpreter. Each rule establishes that when the interpreter is in a description $\delta$ and certain criteria are met, then it will *transition* to a modified description $\delta'$, in symbols,

$$\delta \longrightarrow \delta'.$$

We call the description on the left-hand side of the symbol $\longrightarrow$ the *input description*, and the one on its right-hand side the *output description*.

After transitioning to a modified description, the interpreter repeats the rule-evaluation/transition process, and continues to do so until a final description is reached. This final description, called an *irreducible description*, embodies the result of the reaction.

A full *reaction* is thus defined as a sequence of the transitions of the form

$$\delta_0 \longrightarrow \delta_1 \longrightarrow \cdots \longrightarrow \delta_f.$$

The initial description $\delta_0 = \langle stmt_0, 0, e, \theta_0 \rangle$ contains the three inputs of the reaction: the text of the program at the beginning of the reaction ($stmt_0$), the event $e$ to which the program must react, and the memory $\theta_0$ holding the values of the variables of $stmt_0$. The final description $\delta_f = \langle stmt_f, 0, \varepsilon, \theta_f \rangle$ contains the two outputs of the reaction: the text of the program at the end of the reaction ($stmt_f$) and the memory $\theta_f$ holding the values of the variables of $stmt_f$. The output program $stmt_f$ and memory $\theta_f$ are used by the interpreter as inputs in the next reaction.

We write $\delta_0 \xrightarrow{i} \delta_f$ to indicate that $\delta_0$ leads to $\delta_f$ after exactly $i$ transitions, and we write $\delta_0 \xrightarrow{*} \delta_f$ to indicate that it does so after an unspecified but finite number of transitions. Using this notation, we can define function *reaction* (the first

part of our goal) as follows:

$$reaction(stmt_0, \theta_0, e) = \langle stmt_f, \theta_f \rangle$$

if, and only if,

$$\langle stmt_0, 0, e, \theta_0 \rangle \xrightarrow{*} \langle stmt_f, 0, \varepsilon, \theta_f \rangle,$$

where $\langle stmt_f, 0, \varepsilon, \theta_f \rangle$ is an irreducible description. Under this definition, *reaction* will be deterministic, terminating, and memory-bounded (the second part of goal) if relation $\xrightarrow{*}$ happens to be so. That this is the case is a consequence of the way transitions are defined, as we will see in Section 4.

The next two sections, Sections 3.4 and 3.5, give the rules for transitions. There are two types of transitions: *outermost transitions* $\xrightarrow{out}$ and *nested transitions* $\xrightarrow{nst}$. The rules for $\xrightarrow{out}$ and $\xrightarrow{nst}$ transitions are listed all at once in Figure 9 (page 10). Each of these rules has the form

$$\frac{condition_1 \quad condition_2 \quad \cdots \quad condition_n}{\delta \longrightarrow \delta'}$$

and establishes that a transition $\delta \longrightarrow \delta'$ shall take place if $condition_1$, $condition_2$, ..., and $condition_n$ are all true. If the number of conditions is zero, then the line is omitted and the rule is called an axiom.

### 3.4 Outermost Transitions

The rules **push** and **pop** for $\xrightarrow{out}$ transitions (see Figure 9) are non-recursive definitions that apply to the program as a whole. These are the only rules that manipulate the stack level—the component of descriptions that determines the order of execution of nested reactions.

Rule **push** matches whenever there is a nonempty event in the input description; it instantly broadcasts the event to the program, which means it: (i) uses function *bcast* (defined below) to awake active $await_{ext}$, $await_{int}$, and every statements; (ii) creates a nested reaction by increasing the stack level ($\ell$ becomes $\ell + 1$); and (iii) consumes the input event ($e$ becomes $\varepsilon$).

Function *bcast* is defined as

$$bcast(await_{ext}(e), e) = @nop$$
$$bcast(await_{int}(e), e) = @nop$$
$$bcast(every\ e\ stmt, e) = stmt; every\ e\ stmt$$
$$bcast(stmt_1; stmt_2, e) = bcast(stmt_1, e); stmt_2$$
$$bcast(@var\ v\ n\ stmt) = @var\ v\ n\ bcast(stmt, e)$$
$$bcast(stmt_1\ @loop\ stmt_2, e) = bcast(stmt_1, e)\ @loop\ stmt_2$$
$$bcast(stmt_1\ @par/and\ stmt_2, e) =$$
$$bcast(stmt_1, e)\ @par/and\ bcast(stmt_2, e)$$
$$bcast(stmt_1\ @par/or\ stmt_2, e) =$$
$$bcast(stmt_1, e)\ @par/or\ bcast(stmt_2, e)$$
$$bcast(\_, e) = \_ \quad (otherwise)$$

where the underscore (_) denotes the omitted patterns.

Function *bcast* takes a statement *stmt* and an event $e$ and awakes any trails in *stmt* awaiting for $e$. It does so by converting any active $await_{ext}(e)$ and $await_{int}(e)$ statements in *stmt*

to @nop (and thus consuming them) and by unwinding once any active every $e$ $stmt'$ statements in $stmt$ (and thus allowing their body $stmt'$ to execute once).

Note that **push** is the only rule that uses function $bcast$. Moreover, it is the only rule in the semantics that matches a nonempty event ($e \neq \varepsilon$) in the input description.

Rule **pop**, the other $\xrightarrow[out]{}$ rule, simply decreases the stack level by one. Rule **pop** can be applied whenever the stack level is greater than zero ($\ell > 0$) and the program is blocked or terminated ($stmt$ = @nop). As we will see, these conditions ensure that an $emit_{int}$ only resumes after its nested reaction completes and blocks at the current stack level.

In order to determine whether the program is blocked, **pop** uses predicate $isblk$. This predicate evaluates to *true* if all parallel trails in the given program are blocked waiting for events, finalization clauses, or certain stack levels; otherwise, it evaluates to *false*. Predicate $isblk$ is defined as

$$isblk(\texttt{await}_{ext}(e), \ell) = true$$
$$isblk(\texttt{await}_{int}(e), \ell) = true$$
$$isblk(\texttt{every}\ e\ stmt, \ell) = true$$
$$isblk(\texttt{@runat}(\ell'), \ell) = (\ell > \ell')$$
$$isblk(\texttt{fin}\ stmt, \ell) = true$$
$$isblk(stmt_1; stmt_2, \ell) = isblk(stmt_1, \ell)$$
$$isblk(\texttt{@var}\ v\ \ell\ stmt, \ell) = isblk(stmt, \ell)$$
$$isblk(stmt_1\ \texttt{@loop}\ stmt_2, \ell) = isblk(stmt_1, \ell)$$
$$isblk(stmt_1\ \texttt{@par/and}\ stmt_2, \ell) =$$
$$isblk(stmt_1, \ell) \wedge isblk(stmt_2, \ell)$$
$$isblk(stmt_1\ \texttt{@par/or}\ stmt_2, \ell) =$$
$$isblk(stmt_1, \ell) \wedge isblk(stmt_2, \ell)$$
$$isblk(\_, \ell) = false \quad (otherwise)$$

where the underscore (_) denotes the omitted patterns.

Besides **pop**, predicate $isblk$ is also used as a condition in the $\xrightarrow[nst]{}$ rules for advancing the right-hand side of parallel compositions (which can only advance if the left-hand side is blocked). We detail these and the other $\xrightarrow[nst]{}$ rules next.

### 3.5 Nested Transitions

The $\xrightarrow[nst]{}$ transitions have the general form

$$\langle stmt, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{} \langle stmt', \ell, e, \theta' \rangle.$$

They do not affect the stack level and never have an emitted event as a precondition. The distinction between $\xrightarrow[out]{}$ and $\xrightarrow[nst]{}$ prevents rules **push** and **pop** from matching and, consequently, from inadvertently modifying the stack level before a nested reaction is over.

A full reaction follows the pattern

$$\langle stmt_0, 0, e_{ext}, \theta_0 \rangle \xrightarrow[out]{push} \left[ \xrightarrow[nst]{*} \xrightarrow[out]{} \right]* \xrightarrow[nst]{*} \xrightarrow[out]{pop} \langle stmt_f, 0, \varepsilon, \theta_f \rangle.$$

First, a $\xrightarrow[out]{push}$ starts a nested reaction at level 1. Then, a series of alternations between zero or more $\xrightarrow[nst]{}$ transitions and a single $\xrightarrow[out]{}$ transition (stack operation) takes place. Finally,

zero or more $\xrightarrow[nst]{}$ transitions followed by a last $\xrightarrow[out]{pop}$ transition, which decrements the stack level to 0, terminates the reaction.

We now describe the rules for nested transitions (listed in Figure 9). Since the $\xrightarrow[nst]{}$ rules do not affect the stack level, whenever convenient, we omit the "$\ell$" when discussing them. Thus, in Figure 9, we sometimes write descriptions as triples $\langle stmt, e, \theta \rangle$ with the tacit understanding that there is a hidden integer $\ell$ for the stack level between the components $stmt$ and $e$.

### Emissions

An $emit_{int}(e)$ generates event $e$ and becomes a @runat($\ell$) statement (rule **emit-int**); this @runat($\ell$) statement only resumes at stack level $\ell$ (rule **run-at**).

Since every $\xrightarrow[nst]{}$ rule expects the input event to be empty, an application of **emit-int** leads immediately to an application of **push** at the outer level, creating a new level $\ell + 1$ on the stack. With this new stack level, the @runat($\ell$) resulting from the $emit_{int}(e)$ can only transition after future applications of **pop** put the stack level back in $\ell$.

The interplay between **emit-int**, **push**, **pop**, and **run-at** provides the desired stack-based semantics for internal events.

### Variable declarations and assignments

There are five $\xrightarrow[nst]{}$ rules for dealing with variables: **var-expd**, **var-nop**, **var-brk**, **var-adv**, and **assign**. The last two, **var-adv** and **assign**, are the only rules in the semantics that modify the memory—the fourth component of descriptions.

A *memory* $\theta$ is a stack of bindings $[b_1, \ldots, b_k]$ where each binding $b_i$ is a pair $(v, n)$ associating the variable $v$ to the integer value $n$. For instance, the memory

$$[(v_1, 1), (v_2, \bot), (v_1, 0)]$$

has three bindings: $b_1 = (v_1, 1)$, which is the top-of-stack, $b_2 = (v_2, \bot)$, and $b_3 = (v_1, 0)$. From this memory, we can tell that, at some point, variable $v_1$ was declared and set to 0. Later, in the scope of $v_1$, variable $v_2$ was declared but not initialized, hence its value $\bot$ (undefined), and in the scope of $v_2$, variable $v_1$ was re-declared (shadowing the initial declaration) and set to 1.

The basic CÉU interpreter starts with the empty memory $\theta = []$. When it encounters a declaration var $v$ $stmt$, the interpreter immediately expands it to @var $v \bot stmt$ (rule **var-expd**) and proceeds to advance its body ($stmt$). At this point, three things can happen:

1. If $stmt$ is terminated (= @nop), the interpreter consumes the whole @var statement by transforming it into a @nop (rule **var-nop**).
2. If $stmt$ is a break (= break), the interpreter propagates the break outwards by transforming the whole @var statement into a break (rule **var-brk**).

*Outermost transitions*

**push:**
$$\frac{e \neq \varepsilon}{\langle stmt, \ell, e, \theta \rangle \xrightarrow{out} \langle bcast(stmt, e), \ell + 1, \varepsilon, \theta \rangle}$$

**pop:**
$$\frac{\ell > 0 \quad stmt = \text{@nop} \vee isblk(stmt, n)}{\langle stmt, \ell, \varepsilon, \theta \rangle \xrightarrow{out} \langle stmt, \ell - 1, \varepsilon, \theta \rangle}$$

*Assignments & conditionals*

**assign:** $\langle v := expr, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{@nop}, \varepsilon, updt(\theta, v, eval(\theta, expr)) \rangle$

**if-true:**
$$\frac{eval(\theta, expr) \neq 0}{\langle \text{if } expr \text{ then } stmt_1 \text{ else } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1, \varepsilon, \theta \rangle}$$

**if-false:**
$$\frac{eval(\theta, expr) = 0}{\langle \text{if } expr \text{ then } stmt_1 \text{ else } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_2, \varepsilon, \theta \rangle}$$

*Variable declarations*

**var-expd:** $\langle \text{var } v \ stmt, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{@var } v \perp stmt, \varepsilon, \theta \rangle$

**var-nop:** $\langle \text{@var } v \ n \ \text{@nop}, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{@nop}, \varepsilon, \theta \rangle$

**var-brk:** $\langle \text{@var } v \ n \ \text{break}, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{break}, \varepsilon, \theta \rangle$

**var-adv:**
$$\frac{\langle stmt, \varepsilon, (v, n){:}\theta \rangle \xrightarrow{nst} \langle stmt', e, (v, n'){:}\theta' \rangle}{\langle \text{@var } v \ n \ stmt, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{@var } v \ n' \ stmt', e, \theta' \rangle}$$

*Par/and*

**par/and-expd:**
$$\langle stmt_1 \text{ par/and } stmt_2, \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow{nst} \langle stmt_1 \text{ @par/and } (\text{@runat}(\ell); stmt_2), \ell, \varepsilon, \theta \rangle$$

**par/and-nop1:**
$$\langle \text{@nop @par/and } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_2, \varepsilon, \theta \rangle$$

**par/and-nop2:**
$$\frac{isblk(stmt_1, \ell)}{\langle stmt_1 \text{ @par/and @nop}, \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1, \ell, \varepsilon, \theta \rangle}$$

**par/and-brk1:**
$$\langle \text{break @par/and } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle clear(stmt_2); \text{break}, \varepsilon, \theta \rangle$$

**par/and-brk2:**
$$\frac{isblk(stmt_1, \ell)}{\langle stmt_1 \text{ @par/and break}, \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle clear(stmt_1); \text{break}, \ell, \varepsilon, \theta \rangle}$$

**par/and-adv1:**
$$\frac{\langle stmt_1, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1', e, \theta' \rangle}{\langle stmt_1 \text{ @par/and } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1' \text{ @par/and } stmt_2, e, \theta' \rangle}$$

**par/and-adv2:**
$$\frac{isblk(stmt_1, \ell) \quad \langle stmt_2, \ell, \varepsilon, \theta' \rangle \xrightarrow{nst} \langle stmt_2', \ell, e, \theta' \rangle}{\langle stmt_1 \text{ @par/and } stmt_2, \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1 \text{ @par/and } stmt_2', \ell, e, \theta' \rangle}$$

*Emissions*

**emit-int:** $\langle \text{emit}_{int}(e), \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{@runat}(\ell), \ell, e, \theta \rangle$

**run-at:**
$$\frac{\ell' = \ell}{\langle \text{@runat}(\ell'), \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{@nop}, \ell, \varepsilon, \theta \rangle}$$

*Sequences*

**seq-nop:** $\langle \text{@nop}; stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_2, \varepsilon, \theta \rangle$

**seq-brk:** $\langle \text{break}; stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{break}, \varepsilon, \theta \rangle$

**seq-adv:**
$$\frac{\langle stmt_1, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1', e, \theta' \rangle}{\langle stmt_1; stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1'; stmt_2, e, \theta' \rangle}$$

*Loops*

**loop-expd:** $\langle \text{loop } stmt, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt \text{ @loop } stmt, \varepsilon, \theta \rangle$

**loop-nop:** $\langle \text{@nop @loop } stmt, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{loop } stmt, \varepsilon, \theta \rangle$

**loop-brk:** $\langle \text{break @loop } stmt, \varepsilon, \theta \rangle \xrightarrow{nst} \langle \text{@nop}, \varepsilon, \theta \rangle$

**loop-adv:**
$$\frac{\langle stmt_1, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1', e, \theta' \rangle}{\langle stmt_1 \text{ @loop } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1' \text{ @loop } stmt_2, e, \theta' \rangle}$$

*Par/or*

**par/or-expd:**
$$\langle stmt_1 \text{ par/or } stmt_2, \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow{nst} \langle stmt_1 \text{ @par/or } (\text{@runat}(\ell); stmt_2), \ell, \varepsilon, \theta \rangle$$

**par/or-nop1:**
$$\langle \text{@nop @par/or } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle clear(stmt_2), \varepsilon, \theta \rangle$$

**par/or-nop2:**
$$\frac{isblk(stmt_1, \ell)}{\langle stmt_1 \text{ @par/or @nop}, \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle clear(stmt_1), \ell, \varepsilon, \theta \rangle}$$

**par/or-brk1:**
$$\langle \text{break @par/or } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle clear(stmt_2); \text{break}, \varepsilon, \theta \rangle$$

**par/or-brk2:**
$$\frac{isblk(stmt_1, \ell)}{\langle stmt_1 \text{ @par/or break}, \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle clear(stmt_1); \text{break}, \ell, \varepsilon, \theta \rangle}$$

**par/or-adv1:**
$$\frac{\langle stmt_1, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1', e, \theta' \rangle}{\langle stmt_1 \text{ @par/or } stmt_2, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1' \text{ @par/or } stmt_2, e, \theta' \rangle}$$

**par/or-adv2:**
$$\frac{isblk(stmt_1, \ell) \quad \langle stmt_2, \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_2', \ell, e, \theta' \rangle}{\langle stmt_1 \text{ @par/or } stmt_2, \ell, \varepsilon, \theta \rangle \xrightarrow{nst} \langle stmt_1 \text{ @par/or } stmt_2', \ell, e, \theta' \rangle}$$

**Figure 9.** Rules for $\xrightarrow{out}$ and $\xrightarrow{nst}$ transitions.

3. Otherwise, the interpreter advances *stmt* recursively (rule **var-adv**) and updates the @var statement with the resulting body *stmt'*, event *e*, memory $\theta'$, and binding $(v, n')$.

In rule **var-adv**, the recursive execution of the body (*stmt*) of the @var statement happens in a memory $(v, n) : \theta$, i.e., the memory resulting from pushing binding $(v, n)$ to the current memory $\theta$, which effectively declares $v$ with value $n$ in the scope of *stmt*. Initially, $n = \bot$ (undefined), but any assignments in *stmt* may change that. Thus, after *stmt* is advanced, the value $n'$ associated with variable $v$ in the (possibly) updated binding $(v, n')$ is stored back in the enclosing @var statement. (The updated binding itself is popped from the resulting memory because at this point the @var statement is no longer in effect.)

We borrowed the technique used in rule **var-adv** to implement lexical scoping from the operational semantics of Esterel [7]. Our version is simpler though. While in Esterel the memories resulting from advancing parallel trails need to be merged into a single memory, in Céu this is not necessary: parallel trails are always advanced from left to right and can thus safely operate over the same memory.

The last $\xrightarrow[nst]{}$ rule for dealing with variables is **assign**. Rule **assign** transforms $v := expr$ into @nop and performs the assignment. That is, it evaluates *expr* to an integer *n* and updates the most recent binding of variable $v$ to *n*.

Expression evaluation is carried out by function *eval*. For simplicity, we assume that *eval* always returns an integer. In practice, any error in the evaluation, such as access to an uninitialized variable or division by zero, will cause the interpreter to abort.

Memory updates are carried out by function *updt*, which simply updates the most recent binding of the given variable to the given value:

$$updt((v, n) : \theta, v', n') = \begin{cases} (v, n') : \theta & \text{if } v = v' \\ (v, n) : updt(\theta, v', n') & \text{otherwise.} \end{cases}$$

The first syntactic restriction listed in Section 3.1 ensures that *updt* is always defined.

### Conditionals

Rules **if-true** and **if-false** transform a conditional statement into its if-part ($stmt_1$) or else-part ($stmt_2$) depending on the value to which the associated expression (*expr*) evaluates in the current memory. If it evaluates to a nonzero value, then **if-true** applies and the conditional becomes $stmt_1$; otherwise, **if-false** applies and the conditional becomes $stmt_2$.

These are the only rules that use the contents of the memory in a way that affects the control flow.

### Sequences

A sequence whose first part is terminated (@nop) becomes its second part (rule **seq-nop**). A sequence whose first part is a break, propagates the break outwards by dropping its second part (rule **seq-brk**). And a sequence whose first part is neither @nop nor break advances to the sequence, event, and memory resulting from advancing just its first part (rule **seq-adv**).

### Loops

When the interpreter encounters a loop *stmt*, it unwinds it once, expanding it to *stmt* @loop *stmt* (rule **loop-expd**).

The remaining rules for loops are similar to those for sequences but use "@" as separators to bind the current code of the loop to its next iteration. Rules **loop-nop** and **loop-adv** are analogous to rules **seq-nop** and **seq-adv**. They advance the current code of the loop until it becomes a @nop. When this happens the loop is restarted (**loop-nop**). (Note that if we had used ";" as a separator in loops, rules **loop-brk** and **seq-brk** would conflict.) Finally, rule **loop-brk** escapes the enclosing loop, transforming everything into a @nop.

### Par/and and par/or compositions

When the interpreter encounters par/and and par/or, it expands them to @par/and and @par/or prefixing a @runat($\ell$) to the right-hand side of the resulting compositions (rules **par/and-expd** and **par/or-expd**). The @runat($\ell$) ensures that any reachable emit$_{int}$ on the left-hand side executes to completion before the right-hand side starts.

There are twelve rules for advancing the sides of @par/and and @par/or statements. Each of these rules ensures that the left-hand side always advances before the right-hand side. That is, in each rule the right-hand side is advanced only when the left-hand side is blocked. The twelve rules are divided into three groups: (i) those that apply when one of the sides is @nop; (ii) those that apply when one of the sides is break; and (iii) those that apply when neither side is @nop or break.

We begin by presenting the rules of the last group, which do not involve the termination of one of the sides.

Rules **par/and-adv1** and **par/or-adv1** match whenever the left-hand side of the composition is not blocked (see *isblk* in Section 3.4). They advance the composition the to the statement, event, and memory resulting from advancing just its left-hand side. Rules **par/and-adv2** and **par/or-adv2** are similar, but they act on the right-hand side and match only when the left-hand side is blocked.

The next rules apply whenever one of the sides of the @par/and or @par/or terminates (becomes a @nop).

In a @par/and, if one of the sides terminates, the composition becomes the other side (rules **par/and-nop1** and **par/and-nop2**). In a @par/or, however, if one of the sides terminates, the whole composition terminates and function *clear* is called to finalize the aborted side (rules **par/or-nop1** and **par/or-nop2**).

Function *clear* takes a statement *stmt* and extracts the bodies of all active fin statements in *stmt*, returning them

in a sequence:

$$clear(\text{await}_{ext}(e)) = \text{@nop}$$
$$clear(\text{await}_{int}(e)) = \text{@nop}$$
$$clear(\text{every } e \text{ } stmt) = \text{@nop}$$
$$clear(\text{@runat}(\ell)) = \text{@nop}$$
$$clear(\text{fin } stmt) = stmt$$
$$clear(stmt_1; stmt_2) = clear(stmt_1)$$
$$clear(\text{@var } v \text{ } n \text{ } stmt) = clear(stmt)$$
$$clear(stmt_1 \text{ @loop } stmt_2) = clear(stmt_1)$$
$$clear(stmt_1 \text{ @par/and } stmt_2) = clear(stmt_1); clear(stmt_2)$$
$$clear(stmt_1 \text{ @par/or } stmt_2) = clear(stmt_1); clear(stmt_2)$$
$$clear(\_) = \bot \quad \text{(undefined)}$$

When applied to the aborted side of a @par/or, *clear* returns all finalization code associated with it (if any). Thus, by expanding to a *clear* call on the aborted side, rules **par/or-nop1** and **par/or-nop2**, ensure that the composition is properly finalized.

Note that as there are no transition rules for fin statements, once reached, a fin halts and will only be consumed if its trail is aborted. At this moment, its body will execute as a result of the *clear* call. Note also that the syntactic restrictions of Section 3.1 ensure that the body of a fin statement always execute within the same reaction.

Finally, a break in one of the sides of a @par/and or @par/or is propagated outwards, terminating the composition and properly finalizing the aborted side with a *clear* call (rules **par/and-brk1**, **par/and-brk2**, **par/or-brk1**, and **par/or-brk2**).

## 3.6 Using the Rules to Compute Reactions

We conclude the discussion of the rules for $\xrightarrow{out}$ and $\xrightarrow{nst}$ transitions with an example of their application in the computation of reactions.

Consider the the following basic CÉU program:

$$\text{var } led$$
$$\left(\text{await}_{ext}(e_{bt})\right) \text{ par/or } \left(\text{loop } \left(led := 1; \text{await}_{ext}(e_{rd});\right.\right.$$
$$led := 0; \text{await}_{ext}(e_{rd}));$$
$$\left.\text{fin } led := 0\right)$$

This is a simplified basic CÉU version of the full CÉU program presented in Listing 2 (page 2). As its full CÉU counterpart, this program toggles a LED whenever a radio packet (event $e_{rd}$) is received, terminating on a button press (event $e_{bt}$) always with the LED off—the state of the led is represented by variable *led*.

We will compute three reactions of the above program. First, to the bootstrap event $e_0$ (the bootstrap reaction) and then to the external events $e_{rd}$ and $e_{bt}$ (in this order). We will execute the reactions in series, which means that the outputs of a reaction will become inputs of the next reaction.

The bootstrap reaction starts with a description

$$\langle \text{var } led \ldots, 0, e_0, [] \rangle,$$

where var *led* … is the original program, $e_0$ is the bootstrap event, and $[]$ is the empty memory, and proceeds as follows:[4]

$$\xrightarrow{\text{push}} \text{var } led \ldots \qquad\qquad \ell = 1, e = \varepsilon, \theta = []$$
$$\xrightarrow[\text{expd}]{\text{var}} \text{@var } led \perp \ldots \text{ par/or } \ldots$$
$$\hookrightarrow \ldots \text{ par/or } \ldots \qquad\qquad \theta = [(led, \perp)]$$
$$\xrightarrow[\text{expd}]{\text{par/or}} \text{await}_{ext}(e_{bt}) \text{ @par/or } (\text{loop } \ldots)$$
$$\hookrightarrow \text{loop} \ldots$$
$$\xrightarrow[\text{expd}]{\text{loop}} led := 1; \ldots \text{@loop} \ldots$$
$$\hookrightarrow led := 1; \text{await}_{ext}(e_{rd}); \ldots$$
$$\hookrightarrow led := 1$$
$$\xrightarrow{\text{assign}} \text{@nop} \qquad\qquad \theta = [(led, 1)]$$
$$\xrightarrow[\text{nop}]{\text{seq}} \text{await}_{ext}(e_{rd}); \ldots$$
$$\xrightarrow[\text{adv}]{\text{loop}} \text{await}_{ext}(e_{rd}); \ldots \text{@loop} \ldots$$
$$\xrightarrow[\text{adv2}]{\text{par/or}} \text{await}_{ext}(e_{bt}) \text{ @par/or } (\text{await}_{ext}(e_{rd}); \ldots \text{@loop} \ldots)$$
$$\xrightarrow[\text{adv}]{\text{var}} \text{@var } led \text{ } 1 \ldots \text{ @par/or } \ldots \qquad \theta = []$$
$$\xrightarrow{\text{pop}} \text{@var } led \text{ } 1 \ldots \text{ @par/or } \ldots \qquad \ell = 0$$

So, the bootstrap reaction terminates with variable *led* set to 1 and with both sides of the @par/or blocked on $\text{await}_{ext}$ statements: the left-hand side waiting on $e_{bt}$ and the right hand-side waiting on $e_{rd}$.

This is a general property of reactions. Every reaction eventually blocks in $\text{await}_{ext}$, $\text{await}_{int}$, every, fin, or @runat statements in parallel trails. If none of these trails is blocked in a @runat, then the program can no longer advance and the reaction is terminated.

The second reaction is a reaction to event $e_{rd}$. It starts with the description produced by the previous reaction but with $\varepsilon$ replaced by $e_{rd}$, i.e.,

$$\langle \text{@var } led \text{ } 1 \text{ await}_{ext}(e_{bt})$$
$$\text{@par/or}(\text{await}_{ext}(e_{rd}); led := 0; \ldots \text{@loop} \ldots), 0, e_{rd}, [] \rangle$$

and after three transitions in the outer level (**push**, **var-adv**, and **pop**) results in:

$$\langle \text{@var } led \text{ } 0 \text{ await}_{ext}(e_{bt})$$
$$\text{@par/or}(\text{await}_{ext}(e_{rd}); led := 1; \ldots \text{@loop} \ldots), 0, \varepsilon, [] \rangle$$

where *led* is set to 0 and the event is consumed.

The third and last reaction is a reaction to event $e_{bt}$. It starts with the description produced in the previous reaction (with $e_{bt}$ replaced for $\varepsilon$) and proceeds as follows:

---

[4]The arrow ($\rightarrow$) denotes an $\xrightarrow{out}$ or $\xrightarrow{nst}$ transition and the hook ($\hookrightarrow$) introduces a recursive $\xrightarrow{nst}$ transition. We only show the affected part of statements, and any updates on the level $\ell$, event $e$, or memory $\theta$ appear on the right-hand side.

$$\xrightarrow{\text{push}} \texttt{@var } led\ 0\ \texttt{@nop @par/or}\dots \qquad \ell = 1, e = \varepsilon, \theta = []$$

$$\xhookrightarrow{\phantom{aaa}} \texttt{@nop @par/or}\dots \qquad \theta = [(led, 0)]$$

$$\xrightarrow[\text{nop1}]{\text{par/or}} led := 0 \qquad (= clear(\dots))$$

$$\xrightarrow{\text{assign}} \texttt{@nop} \qquad \theta = [(led, 0)]$$

$$\xrightarrow[\text{adv}]{\text{var}} \texttt{@var } led\ 0\ \texttt{@nop} \qquad \theta = []$$

$$\xrightarrow[\text{nop}]{\text{var}} \texttt{@nop}$$

$$\xrightarrow{\text{pop}} \texttt{@nop} \qquad \ell = 0$$

After the third reaction the program is terminated (consists of a single @nop) and no longer responds to the environment.

# 4 Properties of Céu Reactions

In this section, we establish three properties of Céu reactions. First, that reactions are deterministic. Second, that they always produce a result and terminate. And third, that the maximum amount of memory they consume never exceeds a fixed threshold which can be derived statically from the input program.

The proofs of determinism (Section 4.1) and termination (Section 4.2) are constructed from the bottom up. We first prove the determinism and termination of single $\xrightarrow{out}$ and $\xrightarrow{nst}$ transitions, then of their finite iterations $\xrightarrow{*}{out}$ and $\xrightarrow{*}{nst}$, and finally of the finite iteration of transitions in general (relation $\xrightarrow{*}$). Since *reaction* is formally defined in terms of relation $\xrightarrow{*}$ (see Section 3.3), by proving that $\xrightarrow{*}$ is deterministic and that it always terminates (with an irreducible description), we establish that both properties also hold for *reaction*.

The remaining property of reactions, memory-boundedness (Section 4.3), follows from termination and from the fact that only a limited amount of expansions, declarations, and emissions can occur during a reaction.

## 4.1 Determinism

Transitions $\xrightarrow{out}$ and $\xrightarrow{nst}$ are defined in such a way that given an input description either no rule is applicable or exactly one of them can be applied (no choice involved). This coupled with the fact that the output of every rule is a function of its input implies that transitions are deterministic: the same input description, if it can transition, will always result in the same output description. Thus the transition relation $\longrightarrow$ is in fact a (partial) function.

The next two lemmas establish the determinism of a single application of $\xrightarrow{out}$ and $\xrightarrow{nst}$. Lemma 4.1 follows from the inspection of rules **push** and **pop**. The proof of Lemma 4.2 is by induction on the structure of the derivation trees produced by the rules for $\xrightarrow{nst}$. Both lemmas are used in the proof of the Theorem 4.3, the main result of this section.

**Lemma 4.1.** *If $\delta \xrightarrow{out} \delta_1$ and $\delta \xrightarrow{out} \delta_2$ then $\delta_1 = \delta_2$.*

**Lemma 4.2.** *If $\delta \xrightarrow{nst} \delta_1$ and $\delta \xrightarrow{nst} \delta_2$ then $\delta_1 = \delta_2$.*

Theorem 4.3 establishes that any number $i \geq 0$ of applications of arbitrary transition rules, starting from the same input description, will always lead to the same output description. In other words, any finite sequence of transitions behave deterministically.

**Theorem 4.3** (Determinism). *$\delta \xrightarrow{i} \delta_1$ and $\delta \xrightarrow{i} \delta_2$ implies $\delta_1 = \delta_2$.*

*Proof.* By induction on $i$. The theorem is trivially true if $i = 0$ and follows directly from the previous lemmas if $i = 1$. Suppose

$$\delta \xrightarrow{1} \delta'_1 \xrightarrow{i-1} \delta_1 \quad \text{and} \quad \delta \xrightarrow{1} \delta'_2 \xrightarrow{i-1} \delta_2,$$

for some $i > 1$, $\delta'_1$, and $\delta'_2$. Then, by Lemma 4.1 or 4.2, depending on whether the first transition is $\xrightarrow{out}$ or $\xrightarrow{nst}$ (it cannot be both), $\delta'_1 = \delta'_2$, and by the induction hypothesis, $\delta_1 = \delta_2$. $\square$

## 4.2 Termination

We now turn to the problem of termination. We want to show that any sufficiently long sequence of applications of arbitrary transition rules will eventually lead to an irreducible description, i.e., one that cannot be modified by further transitions. Before doing that, however, we need to introduce some notation and establish some basic properties of the transition relations $\xrightarrow{nst}$ and $\xrightarrow{out}$.

**Definition 4.4.** A description $\delta = \langle stmt, \ell, e, \theta \rangle$ is *nested-irreducible* if, and only if, $e \neq \varepsilon$ or $stmt = $ @nop or $stmt = $ break or $isblk(stmt, \ell)$.

Nested-irreducible descriptions serve as normal forms for $\xrightarrow{nst}$ transitions: they embody the result of an exhaustive number of $\xrightarrow{nst}$ applications. We will write $\delta_{\#nst}$ to indicate that description $\delta$ is nested-irreducible.

The use of word "irreducible" in the previous definition is justified by Proposition 4.5, which states that if a finite number of applications of $\xrightarrow{nst}$ results in an irreducible description, then that occurs exactly once, at some specific number $i$. Its proof follows directly from the definition of $\xrightarrow{nst}$ by contradiction on the hypothesis that there is such $k \neq i$.

**Proposition 4.5.** *If $\delta \xrightarrow{i}{nst} \delta'_{\#nst}$ then, for all $k \neq i$, there is no $\delta''_{\#nst}$ such that $\delta \xrightarrow{k}{nst} \delta''_{\#nst}$.*

The next lemma establishes that sequences of $\xrightarrow{nst}$ transitions behave as expected regarding the internal evaluation of compound statements. Its proof is by induction on $i$.

**Lemma 4.6.**

1. If $\langle stmt_1, \varepsilon, \theta \rangle \xrightarrow[nst]{i} \langle stmt_1', e, \theta' \rangle$ then, for any $stmt_2$,

$$\langle stmt_1;\ stmt_2, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{i} \langle stmt_1';\ stmt_2, e, \theta' \rangle,$$
$$\langle stmt_1 \,\texttt{@loop}\, p_2, stmt_2, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{i} \langle stmt_1' \,\texttt{@loop}\, stmt_2, e, \theta' \rangle,$$
$$\langle stmt_1 \,\texttt{@par/and}\, stmt_2, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{i} \langle stmt_1' \,\texttt{@par/and}\, stmt_2, e, \theta' \rangle,$$
$$\langle stmt_1 \,\texttt{@par/or}\, stmt_2, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{i} \langle stmt_1' \,\texttt{@par/or}\, stmt_2, e, \theta' \rangle.$$

2. If $\langle stmt_2, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{i} \langle stmt_2', \ell, e, \theta' \rangle$ then, for any $stmt_1$ such that $isblk(stmt_1, \ell)$,

$$\langle stmt_1 \,\texttt{@par/and}\, stmt_2, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{i} \langle stmt_1 \,\texttt{@par/and}\, stmt_2', e, \theta' \rangle,$$
$$\langle stmt_1 \,\texttt{@par/or}\, stmt_2, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{i} \langle stmt_1 \,\texttt{@par/or}\, stmt_2', e, \theta' \rangle.$$

The syntactic restrictions presented in Section 3.1 regarding the bodies of loops, every statements, and fin statements give rise to the following propositions.

**Proposition 4.7.** *If stmt is the body of a loop, then*

$$\langle stmt, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt', \ell, e, \theta' \rangle,$$

*where stmt′ =* break @loop *stmt or isblk(stmt′, ℓ).*

**Proposition 4.8.** *If stmt is the body of an every or fin, then*

$$\langle clear(stmt), \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt', \ell, \varepsilon, \theta' \rangle,$$

*where stmt′ =* @nop *or isblk(stmt′, ℓ).*

The next theorem, Theorem 4.9, establishes that a finite (possibly zero) number of $\xrightarrow[nst]{}$ transitions eventually leads to a nested-irreducible description. Hence, for any input description $\delta$, it is always possible to transform $\delta$ in a nested-irreducible description $\delta'$ by applying to it a sufficiently long sequence of $\xrightarrow[nst]{}$ transitions. The proof of the theorem follows by induction on the structure of statements and depends on Lemma 4.6 and Propositions 4.7 and 4.8.

**Theorem 4.9.** *For any $\delta$ there is a $\delta'_{\#nst}$ such that $\delta \xrightarrow[nst]{*} \delta'_{\#nst}$.*

The main result of this section, Theorem 4.16, is similar to Theorem 4.9 but applies to transitions $\longrightarrow$ in general. In the statement of Theorem 4.9 we use the notion of a general irreducible description $\delta_\#$, i.e., one that cannot be advanced by $\xrightarrow[out]{}$ and $\xrightarrow[nst]{}$. This notion is defined with the help of the auxiliary notions of potency and rank, introduced below.

**Definition 4.10.** The *potency* of a statement *stmt* in reaction to event $e$, denoted $pot(stmt, e)$, is the maximum number of $\texttt{emit}_{int}$ statements that can be executed in a reaction of *stmt* to $e$, i.e.,

$$pot(stmt, e) = pot'(bcast(stmt, e)),$$

where $pot'$ is a function that counts the maximum number of reachable $\texttt{emit}_{int}$'s in the given statement (see Figure 10).

**Definition 4.11.** The *rank* of a description $\delta = \langle stmt, \ell, e, \theta \rangle$, denoted $rank(\delta)$, is a pair $(i, j)$ of nonnegative integers where

$$i = pot(stmt, e) \quad \text{and} \quad j = \begin{cases} \ell & \text{if } e = \varepsilon \\ \ell + 1 & \text{otherwise} \end{cases}.$$

Intuitively, the rank of a description $\delta$ is a measure of the effort required to transform $\delta$ into an irreducible description, i.e., into one that satisfies the next definition.

**Definition 4.12.** A description $\delta$ is *irreducible* (in symbols, $\delta_\#$) if, and only if, it is nested-irreducible and its $rank(\delta)$ is $(i, 0)$, for some $i \geq 0$.

An irreducible description $\delta_\# = \langle stmt, \ell, e, \theta \rangle$ serves as a normal form for transitions $\longrightarrow$ in general. Such description cannot be advanced by $\xrightarrow[nst]{}$, as it is nested-irreducible, and neither by $\xrightarrow[out]{push}$ nor $\xrightarrow[out]{pop}$, as the second coordinate of its rank is 0, which implies $e = \varepsilon$ and $\ell = 0$.

The next two lemmas establish that a single application of $\xrightarrow[out]{}$ or $\xrightarrow[nst]{}$ either preserves or decreases the rank of the input description. All rank comparisons use the lexicographic order, i.e., if $rank(\delta) = (i, j)$ and $rank(\delta') = (i', j')$ then

$$rank(\delta) > rank(\delta')$$

if, and only if,

$$i > i' \quad \text{or} \quad (i = i \text{ and } j > j').$$

The proof of Lemma 4.13 follows directly from **push** and **pop** and from the definitions of *pot* and *rank*. The proof of Lemma 4.14, however, is by induction on the structure of $\xrightarrow[nst]{}$ derivations.

**Lemma 4.13.**

1. If $\delta \xrightarrow[out]{push} \delta'$ then $rank(\delta) = rank(\delta')$.
2. If $\delta \xrightarrow[out]{pop} \delta'$ then $rank(\delta) > rank(\delta')$.

**Lemma 4.14.** *If $\delta \xrightarrow[nst]{} \delta'$ then $rank(\delta) \geq rank(\delta')$.*

The next theorem is a generalization of Lemma 4.14 for $\xrightarrow[nst]{*}$. Its proof follows from the lemma by induction on $i$.

**Theorem 4.15.** *If $\delta \xrightarrow[nst]{*} \delta'$ then $rank(\delta) \geq rank(\delta')$.*

We are finally in a position to state and prove the main result of this section, Theorem 4.16, which is the termination theorem for $\longrightarrow$. The idea of the proof is that a sufficiently large sequence of $\xrightarrow[nst]{}$ and $\xrightarrow[out]{}$ transitions eventually decreases the rank of the current description until an irreducible description is reached. This irreducible description is the final result of the reaction.

**Theorem 4.16** (Termination). *For any $\delta$, there is a $\delta'_\#$ such that $\delta \overset{*}{\longrightarrow} \delta'_\#$.*

$$pot'(\mathtt{emit}_{int}(e)) = 1$$

$$pot'(\mathtt{var}\ v\ stmt) = pot'(stmt)$$

$$pot'(\mathtt{if}\ expr\ \mathtt{then}\ stmt_1\ \mathtt{else}\ stmt_2) = \max\{pot'(stmt_1), pot'(stmt_2)\}$$

$$pot'(stmt_1; stmt_2) = \begin{cases} 0 & \text{if } stmt_1 = \mathtt{break}\ \text{or}\ stmt_1 = \mathtt{await}_{ext} \\ & \quad \text{or}\ stmt_1 = \mathtt{every}_{ext} \\ pot'(stmt_1) + pot'(stmt_2) & \text{otherwise} \end{cases}$$

$$pot'(\mathtt{loop}\ stmt) = pot'(stmt)$$

$$pot'(\mathtt{every}\ e\ stmt) = \begin{cases} 0 & \text{if } e \text{ is an external event} \\ pot'(stmt) & \text{otherwise} \end{cases}$$

$$pot'(stmt_1\ \mathtt{par/and}\ stmt_2) = pot'(stmt_1) + pot'(stmt_2)$$

$$pot'(stmt_1\ \mathtt{par/or}\ stmt_2) = pot'(stmt_1) + pot'(stmt_2)$$

$$pot'(\mathtt{fin}\ stmt) = pot'(stmt)$$

$$pot'(\mathtt{@var}\ v\ n\ stmt) = pot'(stmt)$$

$$pot'(stmt_1\ \mathtt{@loop}\ stmt_2) = \begin{cases} pot'(stmt_1) & \text{if all execution paths in } stmt_1 \text{ have a} \\ & \quad \text{matching break or } \mathtt{await}_{ext} \text{ or } \mathtt{every}_{ext} \\ pot'(stmt_1) + pot'(stmt_2) & \text{otherwise} \end{cases}$$

$$pot'(stmt_1\ \mathtt{@par/and}\ stmt_2) = \begin{cases} 0 & \text{if } stmt_1 = \mathtt{break} \\ pot'(stmt_1) + pot'(stmt_2) & \text{otherwise} \end{cases}$$

$$pot'(stmt_1\ \mathtt{@par/or}\ stmt_2) = \begin{cases} 0 & \text{if } stmt_1 = \mathtt{@nop}\ \text{or}\ stmt_1 = \mathtt{break} \\ pot'(stmt_1) + pot'(stmt_2) & \text{otherwise} \end{cases}$$

$$pot'(\_) = 0 \quad \textit{(otherwise)}$$

**Figure 10.** Function $pot'$ counts the maximum number of reachable $\mathtt{emit}_{int}$'s in the given statement.

*Proof.* By induction on induction on $rank(\delta)$. Let

$$\delta = \langle stmt, \ell, e, \theta \rangle \quad \text{and} \quad rank(\delta) = (i, j).$$

For the basis, suppose $(i, j) = (0, 0)$. Then $\delta$ cannot be advanced by $\xrightarrow{out}$, as $j = 0$ implies $e = \varepsilon$ and $\ell = 0$. If $\delta$ is nested-irreducible, the theorem is trivially true, as $\delta \xrightarrow[nst]{0} \delta_{\#nst}$ and $\delta_\#$. If $\delta$ is not nested-irreducible then, by Theorem 4.9, $\delta \xrightarrow[nst]{*} \delta'_{\#nst}$, for some $\delta'_{\#nst}$. By Theorem 4.15, $rank(\delta) \geq rank(\delta')$, which implies $rank(\delta') = (0, 0)$, and so $\delta'_\#$.

For the inductive step, suppose $(i, j) > (0, 0)$. There are two cases.

*Case* [$\delta$ is nested-irreducible]. If $j = 0$, by Definition 4.12, $\delta_\#$, and so $\delta \xrightarrow{0} \delta_\#$. If $j > 0$, there are two subcases:

*Case* [$e \neq \varepsilon$]. Then, by **push** and by Theorem 4.9, there are $\delta'_1$ and $\delta'_{\#nst} = \langle stmt', \ell + 1, e', \theta' \rangle$ such that

$$\delta \xrightarrow[out]{push} \delta'_1 \xrightarrow[nst]{*} \delta'_{\#nst}.$$

Thus, by Lemma 4.13 and by Theorem 4.15,

$$rank(\delta) = rank(\delta'_1) = (i, j) \geq rank(\delta') = (i', j').$$

If $e' = \varepsilon$, then $i = i'$ and $j = j'$, and the rest of this proof is similar to that of the case for $e = \varepsilon$ below. Otherwise, if $e' \neq \varepsilon$, then $i > i'$, as at least one reachable

$\mathtt{emit}_{int}$ was consumed by the nested transitions. Thus, $rank(\delta) > rank(\delta')$. By the induction hypothesis, $\delta' \xrightarrow{*} \delta''_\#$, for some $\delta''_\#$. Therefore, $\delta \xrightarrow{*} \delta''_\#$.

*Case* [$e = \varepsilon$]. Then, as $j > 0$, $\delta \xrightarrow[out]{pop} \delta'$, for some $\delta'$. By Lemma 4.13, $rank(\delta) > rank(\delta')$. Hence, by the induction hypothesis, $\delta' \xrightarrow{*} \delta''_\#$, for some $\delta''_\#$. And so, $\delta \xrightarrow{*} \delta''_\#$.

*Case* [$\delta$ is not nested-irreducible]. Then $e = \varepsilon$ and, by Theorems 4.9 and 4.15, there is a $\delta'_{\#nst}$ such that $\delta \xrightarrow[nst]{*} \delta'_{\#nst}$ with $rank(\delta) \geq rank(\delta'_{\#nst})$. The rest of this proof is similar to that of case when $\delta$ is nested-irreducible above. □

### 4.3 Memory Boundedness

Céu has no mechanism for heap allocation, unbounded iteration, or general recursion. Consequently, the memory (or space) usage of a given Céu program never exceeds a fixed threshold, which depends solely on the program's text. We can define this threshold for a single reaction as follows.

The maximum space used by the Céu interpreter to compute a reaction of program $stmt$ to external event $e$, denoted $react_{spc}(stmt, e)$, corresponds to the maximum space it uses

when computing

$$\langle stmt, 0, e, [] \rangle \xrightarrow{*} \langle stmt', 0, \varepsilon, [] \rangle_{\#}. \quad (1)$$

If we assume that to compute (1) the interpreter only keeps information about the current description $\delta_c$, and that recursive $\xrightarrow[nst]{}$ transitions are computed inline (by rewriting $\delta_c$), then $react_{spc}(stmt, e)$ is the maximum space taken by:

1. the statement part $stmt_c$ of $\delta_c$;
2. the memory part $\theta_c$ of $\delta_c$; and
3. stack level part $\ell_c$ of $\delta_c$.

The three objects, $stmt_c$, $\theta_c$, and $\ell_c$, can only grow in proportion to the part of $stmt_c$ that can be advanced during the reaction, which we will call *reachable part* of $stmt_c$.

The maximum size for $stmt_c$ is linearly proportional to the maximum number of $*$-**expd** applications that can occur in the reachable part of $stmt$ in (1). These expansions are the only rule applications that increase the size of $stmt_c$.

The maximum size for $\theta_c$ is linearly proportional to the maximum number of variables simultaneously declared (in the same scope) in the reachable part of $stmt$ in (1). Each of these declarations, if advanced, will cause $\theta_c$ to grow by a constant.

Finally, the maximum value for $\ell_c$ is equal to the maximum number of $\xrightarrow[out]{}$ transitions that can occur in (1), which in turn is equal to the maximum number of $\mathtt{emit}_{int}$'s that can be consumed in the reachable part of $stmt$ in (1), i.e., $pot(stmt, e)$, see $pot$ in Section 4.2.

We can extend the definition of maximal space to an arbitrary reaction as follows. Let $stmt$ be a program that can react to external events $e_1, \ldots, e_n$ then the maximal space taken by any reaction of $stmt$ is

$$\max\{react_{spc}(stmt, e_1), \ldots, react_{spc}(stmt, e_n)\}.$$

Similarly, we can extend the definition of maximal space to histories of external events (or sequences of reactions), and lastly, to the worst-case prefix of such a history, i.e., to the prefix which would trigger the worst possible $react_{spc}$ call. But this is out of the scope of this paper.

## 5    Related Work

Céu follows the lineage of imperative synchronous languages initiated by Esterel [10]. These languages typically define time as a discrete sequence of logical "ticks" in which multiple simultaneous input events can be active [19]. The presence of multiple inputs requires careful static analysis to detect and reject programs with *causality cycles* and *schizophrenia problems* [5]. In contrast, Céu defines time as a discrete sequence of reactions to unique input events, which is a prerequisite for the concurrency checks that enable safe shared-memory concurrency, as discussed in Section 2.2.

In most synchronous languages, the behavior of external and internal events is equivalent. However, in Céu, internal events introduce stack-based micro reactions within external reactions, providing more fine-grained control for intra-reaction execution. This allows for memory-bounded subroutines that can execute multiple times during the same external reaction. The synchronous languages Statecharts [23] and Statemate [13] also distinguish internal from external events. Although the descriptions suggest a stack-based semantics, we are not aware of formalizations or more precision for a deeper comparison with Céu.

Like Céu, many other synchronous languages [2, 9, 12, 14, 24] rely on lexical scheduling to preserve determinism. In contrast, in Esterel, the execution order for operations within a reaction is non-deterministic: *"if there is no control dependency, as in* `(call f1()||call f2())`, *the order is unspecified and it would be an error to rely on it"* [6]. For this reason, Esterel, does not support shared-memory concurrency: *"if a variable is written by some thread, then it can neither be read nor be written by concurrent threads"* [6]. Considering the constant and low-level interactions with the underlying architecture in embedded systems (e.g., direct port manipulation), we believe that it is advantageous to embrace lexical scheduling as part of the language specification as a pragmatic design decision to enforce determinism. However, since Céu statically detects trails not sharing memory, an optimized scheduler could exploit real parallelism in such cases.

Regarding the integration with *C* language-based environments, Céu supports a finalization mechanism for external resources. In addition, Céu also tracks pointers representing resources that cross *C* boundaries and forces the programmer to provide associated finalizers. As far as we know, this extra safety level is unique to Céu among synchronous languages.

## 6    Conclusion

The programming language Céu aims to offer a concurrent, safe, and realistic alternative to C for embedded soft real-time systems, such as sensor networks and multimedia systems. Céu inherits the synchronous and imperative mindset of Esterel but adopts a simpler semantics with fine-grained execution control, which makes the language fully deterministic. In addition, its stack-based execution for internal events provides a limited but memory-bounded form of subroutines. Céu also provides a finalization mechanism for resources when interacting with the external environment.

In this paper, we proposed a small-step operational semantics for Céu and proved that under it reactions are deterministic, terminate in finite time, and use bounded memory, i.e., that for a given arbitrary timeline of input events, multiple executions of the same program always react in bounded time and arrive at the same final finite memory state.

## References

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## A   Detailed Proofs

**Lemma 4.1.** *If $\delta \xrightarrow[out]{} \delta_1$ and $\delta \xrightarrow[out]{} \delta_2$ then $\delta_1 = \delta_2$.*

*Proof.* The lemma is vacuously true if $\delta$ cannot be advanced by $\xrightarrow[out]{}$ transitions. Suppose that is not the case, and let $\delta$, $\delta_1$, and $\delta_2$ be descriptions such that

$$\delta = \langle stmt, \ell, e, \theta \rangle,$$
$$\delta_1 = \langle stmt_1, \ell_1, e_1, \theta_1 \rangle,$$
$$\delta_2 = \langle stmt_2, \ell_2, e_2, \theta_2 \rangle.$$

Then, two cases are possible.

*Case* $[e \neq \varepsilon]$. Both transitions are applications of **push**. Hence, $stmt_1 = stmt_2 = bcast(stmt, e)$, $\ell_1 = \ell_2 = \ell + 1$, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$.

*Case* $[e = \varepsilon]$. Both transitions are applications of **pop**. Hence, $stmt_1 = stmt_2 = stmt$, $\ell_1 = \ell_2 = \ell - 1$, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$. □

**Lemma 4.2.** *If $\delta \xrightarrow[nst]{} \delta_1$ and $\delta \xrightarrow[nst]{} \delta_2$ then $\delta_1 = \delta_2$.*

*Proof.* By induction on the structure of $\xrightarrow[nst]{}$ derivations. The lemma is vacuously true if $\delta$ cannot be advanced by $\xrightarrow[nst]{}$ transitions. Suppose that is not the case and let $\delta$, $\delta_1$, and $\delta_2$ be descriptions such that

$$\delta = \langle stmt, \ell, e, \theta \rangle,$$
$$\delta_1 = \langle stmt_1, \ell, e_1, \theta_1 \rangle,$$
$$\delta_2 = \langle stmt_2, \ell, e_2, \theta_2 \rangle.$$

Then, by the hypothesis of the lemma, there are derivations $\pi_1$ and $\pi_2$ such that

$$\pi_1 = \frac{\vdots}{\delta \to \delta_1} \quad \text{and} \quad \pi_2 = \frac{\vdots}{\delta \to \delta_2}.$$

That is, the conclusion of derivation $\pi_1$ is $\delta \xrightarrow[nst]{} \delta_1$ and the conclusion of derivation $\pi_2$ is $\delta \xrightarrow[nst]{} \delta_2$.

By definition of $\xrightarrow[nst]{}$, $e = \varepsilon$. It remains to be shown that $stmt_1 = stmt_2$, $e_1 = e_2$, and $\theta_1 = \theta_2$. Depending on the structure of program $stmt$, the following cases are possible. (Note that these are the only values for $stmt$ that permit an application of $\xrightarrow[nst]{}$.)

*Case* $[\text{var } v \; stmt']$. Then $\pi_1$ and $\pi_2$ are instances of **var-expd**, i.e., their conclusions are obtained by an application of this rule. Hence, $stmt_1 = stmt_2 = @\text{var } v \perp stmt'$, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$.

*Case* $[v := expr]$. Then $\pi_1$ and $\pi_2$ are instances of **assign**. Hence, $stmt_1 = stmt_2 = @\text{nop}$, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = updt(\theta, v, eval(\theta, expr))$.

*Case* $[\text{emit}_{int}(e')]$. Then $\pi_1$ and $\pi_2$ are instances of **emit-int**. Hence, $stmt_1 = stmt_2 = @\text{runat}(\ell)$, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$.

*Case* $[\text{if } expr \text{ then } stmt' \text{ else } stmt'']$. If $eval(\theta, expr) \neq 0$, then $\pi_1$ and $\pi_2$ are instances of **if-true**. Hence, $stmt_1 = stmt_2 = stmt'$, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$. The case for $eval(\theta, expr) = 0$ is similar.

*Case* $[stmt'; stmt'']$. There are three subcases.

*Case* [$stmt' = $ @nop]. Then $\pi_1$ and $\pi_2$ are instances of **seq-nop**. Hence, $stmt_1 = stmt_2 = stmt''$, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$.

*Case* [$stmt' = $ break]. Then $\pi_1$ and $\pi_2$ are instances of **seq-brk**. Hence, $stmt_1 = stmt_2 = $ break, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$.

*Case* [$stmt' \neq $ @nop, break]. Then $\pi_1$ and $\pi_2$ are instances of **seq-adv**. Hence, there are derivations $\pi'_1 \prec \pi_1$ and $\pi'_2 \prec \pi_2$ (i.e., structurally smaller than $\pi_1$ and $\pi_2$) such that

$$\pi'_1 = \cfrac{\vdots}{\langle stmt', \varepsilon, \theta \rangle \xrightarrow[nst]{} \langle stmt'_1, e'_1, \theta'_1 \rangle}$$

and

$$\pi'_2 = \cfrac{\vdots}{\langle stmt', \varepsilon, \theta \rangle \xrightarrow[nst]{} \langle stmt'_2, e'_2, \theta'_2 \rangle}$$

By the induction hypothesis, $stmt'_1 = stmt'_2$, $e'_1 = e'_2$ and $\theta'_1 = \theta'_2$. Therefore,

$$stmt_1 = stmt'_1; stmt'' = stmt'_2; stmt'' = stmt_2,$$

$e_1 = e'_1 = e'_2 = e_2$, and $\theta_1 = \theta'_1 = \theta'_2 = \theta_2$.

*Case* [loop $stmt'$]. Similar to the case for variable declarations (var).

*Case* [$stmt'$ par/and $stmt''$]. Ditto.

*Case* [$stmt'$ par/or $stmt''$]. Ditto.

*Case* [@runat($\ell$)]. Then $\pi_1$ and $\pi_2$ are instances of **run-at**. Hence, $stmt_1 = stmt_2 = $ @nop, $e_1 = e_2 = \varepsilon$, and $\theta_1 = \theta_2 = \theta$.

*Case* [@var $v$ $n$ $stmt'$]. Similar to the case for sequences.

*Case* [$stmt'$ @loop $stmt''$]. Ditto.

*Case* [$stmt'$ @par/and $stmt''$]. There are two subcases.
    *Case* [$\neg$ $isblk(stmt', \ell)$]. Then $\pi_1$ and $\pi_2$ are instances of **par/and-nop1**, **par/and-brk1**, or **par/and-adv1**. The rest of the proof of is similar to that of the case for sequences.

    *Case* [$isblk(stmt', \ell)$]. Similar but with $\pi_1$ and $\pi_2$ as instances of **par/and-nop2**, **par/and-brk2**, and **par/and-adv2**.

*Case* [$stmt'$ @par/or $stmt''$]. Similar to the case for @par/and. $\qquad\square$

**Proposition 4.5.** *If $\delta \xrightarrow[nst]{i} \delta'_{\#nst}$ then, for all $k \neq i$, there is no $\delta''_{\#nst}$ such that $\delta \xrightarrow[nst]{k} \delta''_{\#nst}$.*

*Proof.* By contradiction on the hypothesis that there is such $k$. Let $\delta \xrightarrow[nst]{i} \delta'_{\#nst}$, for some $i \geq 0$. There are two cases.

*Case* 1. Suppose there are $k > i$ and $\delta''_{\#nst}$ such that $\delta \xrightarrow[nst]{k} \delta''$. Then, by definition of $\xrightarrow[nst]{k}$,

$$\delta \xrightarrow[nst]{i} \delta' \xrightarrow[nst]{i+1} \delta'_1 \xrightarrow[nst]{i+2} \cdots \xrightarrow[nst]{k} \delta''. \qquad (2)$$

Since $\delta' = \langle stmt', \ell, e', \theta' \rangle$ is nested-irreducible, $e' = \varepsilon$ or $stmt' = $ @nop or $stmt' = $ break or $isblk(stmt', \ell)$. In any of

these cases, by the definition of $\xrightarrow[nst]{}$, there is no $\delta'_1$ such that $\delta' \xrightarrow[nst]{1} \delta'_1$, which contradicts (2). Therefore, no such $k$ can exist.

*Case* 2. Suppose there are $k < i$ and $\delta''_{\#nst}$ such that $\delta \xrightarrow[nst]{k} \delta''$. Then, since $i > k$, by Case 1, $\delta'$ cannot exist, which is absurd. Therefore, the assumption that there is such $k$ is false. $\qquad\square$

**Lemma 4.6.**

1. *If $\langle stmt_1, \varepsilon, \theta \rangle \xrightarrow[nst]{i} \langle stmt'_1, e, \theta' \rangle$ then, for any $stmt_2$,*

   $$\langle stmt_1; stmt_2, \varepsilon, \theta \rangle$$
   $$\xrightarrow[nst]{i} \langle stmt'_1; stmt_2, e, \theta' \rangle,$$
   $$\langle stmt_1 \text{ @loop } p_2, stmt_2, \varepsilon, \theta \rangle$$
   $$\xrightarrow[nst]{i} \langle stmt'_1 \text{ @loop } stmt_2, e, \theta' \rangle,$$
   $$\langle stmt_1 \text{ @par/and } stmt_2, \varepsilon, \theta \rangle$$
   $$\xrightarrow[nst]{i} \langle stmt'_1 \text{ @par/and } stmt_2, e, \theta' \rangle,$$
   $$\langle stmt_1 \text{ @par/or } stmt_2, \varepsilon, \theta \rangle$$
   $$\xrightarrow[nst]{i} \langle stmt_1' \text{ @par/or } stmt_2, e, \theta' \rangle.$$

2. *If $\langle stmt_2, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{i} \langle stmt'_2, \ell, e, \theta' \rangle$ then, for any $stmt_1$ such that $isblk(stmt_1, \ell)$,*

   $$\langle stmt_1 \text{ @par/and } stmt_2, \varepsilon, \theta \rangle$$
   $$\xrightarrow[nst]{i} \langle stmt_1 \text{ @par/and } stmt'_2, e, \theta' \rangle,$$
   $$\langle stmt_1 \text{ @par/or } stmt_2, \varepsilon, \theta \rangle$$
   $$\xrightarrow[nst]{i} \langle stmt_1 \text{ @par/or } stmt'_2, e, \theta' \rangle.$$

*Proof.* By induction on $i$. We will only prove the case for the @par/or composition in item (1). The proof of the other cases is similar.

So, considering the case for the @par/or in item (1), the lemma is trivially true for $i = 0$, as $stmt_1 = stmt'_1$, and follows directly from **par/or-adv1** for $i = 1$. Suppose

$$\langle stmt_1, \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle stmt''_1, e'', \theta'' \rangle$$
$$\xrightarrow[nst]{i-1} \langle stmt'_1, e, \theta' \rangle,$$

for some $i > 1$. Then $\langle stmt''_1, e, \theta'' \rangle$ is not nested-irreducible, and so, by **par/or-adv1**,

$$\langle stmt_1 \text{ @par/or } stmt_2, \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle stmt''_1 \text{ @par/or } stmt_2, e'', \theta'' \rangle.$$

Since

$$\langle stmt''_1, e'', \theta'' \rangle \xrightarrow[nst]{i-1} \langle stmt'_1, e, \theta' \rangle,$$

by the induction hypothesis,

$$\langle stmt''_1 \text{ @par/or } stmt_2, e'', \theta'' \rangle \xrightarrow[nst]{i-1} \langle stmt'_1 \text{ @par/or } stmt_2, e, \theta' \rangle.$$

Therefore,

$$\langle stmt_1 \text{ @par/or } stmt_2, \varepsilon, \theta \rangle \xrightarrow[nst]{i} \langle stmt'_1 \text{ @par/or } stmt_2, e, \theta' \rangle.$$
$\qquad\square$

**Proposition 4.7.** *If stmt is the body of a* loop*, then*

$$\langle stmt, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt', \ell, e, \theta' \rangle,$$

*where stmt′ =* break @loop *stmt or isblk(stmt′, ℓ).*

**Proposition 4.8.** *If stmt is the body of an* every *or* fin*, then*

$$\langle clear(stmt), \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt', \ell, \varepsilon, \theta' \rangle,$$

*where stmt′ =* @nop *or isblk(stmt′, ℓ).*

**Theorem 4.9.** *For any δ there is a $\delta'_{\#nst}$ such that $\delta \xrightarrow[nst]{*} \delta'_{\#nst}$.*

*Proof.* By induction on the structure of statements. The theorem is trivially true if $\delta$ is nested-irreducible, as by definition $\delta \xrightarrow[nst]{0} \delta$.

Suppose $\delta = \langle stmt, \ell, \varepsilon, \theta \rangle$ is not nested-irreducible. Then, depending on the structure of *stmt*, the following cases are possible (these are the only values for *stmt* that do not make $\delta$ nested-irreducible). In each of the cases, we show that the required description $\delta'_{\#nst}$ exists.

*Case* [var $v$ *stmt′*]. By **var-expd**,

$$\langle \text{var } v \text{ } stmt', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle \text{var } v \perp stmt', \varepsilon, \theta \rangle.$$

The rest of the proof is similar to that of the case for @var.

*Case* [$v := expr$]. By **assign**,

$$\langle v := expr, \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{1} \langle @\text{nop}, \ell, \varepsilon, updt(\theta, v, eval(\theta, expr)) \rangle_{\#nst}.$$

*Case* [$\text{emit}_{int}(e)$]. By **emit-int**,

$$\langle \text{emit}_{int}(e), \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle @\text{runat}(\ell), \ell, e, \theta \rangle_{\#nst}.$$

*Case* [if $expr$ then *stmt′* else *stmt″*]. If $eval(\theta, expr) \neq 0$ then, by **if-true** and by the induction hypothesis, for some $\delta'$,

$$\langle \text{if } expr \text{ then } stmt' \text{ else } stmt'', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle stmt', \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{*} \delta'_{\#nst}.$$

The case for $eval(\theta, expr) = 0$ is similar.

*Case* [*stmt′*; *stmt″*]. There are three subcases.

*Case* [*stmt′* = @nop]. By **seq-nop** and by the induction hypothesis, for some $\delta'$,

$$\langle @\text{nop}; stmt'', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle stmt'', \varepsilon \text{ } theta \rangle$$
$$\xrightarrow[nst]{*} \delta'_{\#nst}.$$

*Case* [*stmt′* = break]. By **seq-brk**,

$$\langle \text{break}; stmt'', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle \text{break}, \varepsilon, \theta \rangle_{\#nst}.$$

*Case* [*stmt′* ≠ @nop, break]. By induction hypothesis, for some $stmt'_1$,

$$\langle stmt', \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt'_1, e, \theta' \rangle_{\#nst}.$$

Then, by Lemma 4.6,

$$\langle stmt'; stmt'', \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt'_1; stmt'', e, \theta' \rangle = \delta'.$$

It remains to be shown that $\delta'$ is nested-irreducible or leads to a nested-irreducible description. This follows from the fact that the simpler description $\langle stmt'_1, \ell, e, \theta' \rangle$ is nested-irreducible: (i) if $e \neq \varepsilon$ then $\delta'_{\#nst}$; (ii) if $stmt'_1 =$ @nop or $stmt'_1 =$ break then the rest of the proof is similar to the cases for $stmt =$ @nop; $stmt''$ and $stmt =$ break; $stmt''$; and (iii) if $isblk(stmt'_1, \ell)$ then, by definition, $isblk(stmt'_1; stmt_2, \ell)$, and so $\delta'_{\#nst}$.

*Case* [loop *stmt′*]. By **loop-expd**,

$$\langle \text{loop } stmt', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle stmt' \text{ @loop } stmt', \varepsilon, \theta' \rangle.$$

The rest of the proof is similar to that of the case for @loop.

*Case* [*stmt′* par/and *stmt″*]. By **par/and-expd**,

$$\langle stmt' \text{ par/and } stmt'', \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{1} \langle stmt1 \text{ @par/and } (@\text{runat}(\ell); stmt''), \ell, \varepsilon, \theta \rangle.$$

The rest of the proof is similar to that of the case for @par/and.

*Case* [*stmt′* par/or *stmt″*]. By **par/or-expd**,

$$\langle stmt' \text{ par/or } stmt'', \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{1} \langle stmt1 \text{ @par/or } (@\text{runat}(\ell); stmt''), \ell, \varepsilon, \theta \rangle.$$

The rest of the proof is similar to that of the case for @par/or.

*Case* [@runat($\ell$)]. By **run-at**,

$$\langle @\text{runat}(\ell), \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle @\text{nop}, \ell, \varepsilon, \theta \rangle_{\#nst}.$$

*Case* [@var $v$ $n$ *stmt′*]. Trivial if *stmt′* is @nop or break. Suppose that is not the case. Then, by the induction hypothesis,

$$\langle stmt', \varepsilon, (v, n) : \theta \rangle \xrightarrow[nst]{*} \langle stmt'', e, (v, n') : \theta' \rangle_{\#nst}.$$

And so, by **var-adv**,

$$\langle @\text{var } v \text{ } n \text{ } stmt', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle @\text{var } v \text{ } n', stmt'', e, \theta' \rangle_{\#nst}.$$

*Case* [*stmt′* @loop *stmt″*]. There are three subcases.

*Case* [*stmt′* = @nop]. By **loop-nop** and **loop-expd**,

$$\langle @\text{nop @loop } stmt'', \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle \text{loop } stmt'', \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{1} \langle stmt'' \text{ @loop } stmt'', \ell, \theta \rangle$$

By Proposition 4.7,

$$\langle stmt'', \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt''', \ell, e, \theta' \rangle.$$

where either $stmt''' =$ break or $isblk(stmt''', \ell)$. By Lemma 4.6,

$$\langle stmt'' \text{ @loop } stmt'', \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{*} \langle stmt''' \text{ @loop } stmt'', \ell, e, \theta' \rangle = \delta'.$$

If $stmt''' =$ break, then this case becomes break @loop $stmt''$. Otherwise, $isblk(stmt''', \ell)$, and so, $\delta'_{\#nst}$.

*Case* [*stmt′* = break]. By **loop-brk**,

$$\langle \text{break @loop } stmt'', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle @\text{nop}, \varepsilon, \theta \rangle_{\#nst}.$$

*Case* [$stmt' \neq$ @nop, break]. Similar to the case for $stmt'; stmt''$ with $stmt' \neq$ @nop, break.

*Case* [$stmt = stmt'$ @par/and $stmt''$]. The proof depends on whether or not $stmt'$ is blocked. We will prove the case for $\neg\ isblk(stmt', \ell)$; the proof of the case for $isblk(stmt', \ell)$ is similar. So, assuming $stmt'$ is not blocked, there are three subcases.

*Case* [$stmt' =$ @nop]. By **par/and-nop1** and by the induction hypothesis, for some $\delta'$,

$$\langle \text{@nop @par/and } stmt'', \varepsilon, \theta \rangle \xrightarrow[nst]{1} \langle stmt'', \varepsilon, \theta \rangle \xrightarrow[nst]{*} \delta'_{\#nst}.$$

*Case* [$stmt' =$ break]. By **par/and-brk1**,

$$\langle \text{break @par/and} stmt'', \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{1} \langle clear(stmt''); \text{break}, \varepsilon, \theta \rangle.$$

By Proposition 4.8,

$$\langle clear(stmt''), \ell, \varepsilon, \theta \rangle . \xrightarrow[nst]{*} \langle stmt''', \ell, e, \theta' \rangle_{\#nst}.$$

And so, by Lemma 4.6,

$$\langle clear(stmt''); \text{break}, \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{*} \langle stmt'''; \text{break}, \ell, e, \theta' \rangle_{\#nst}.$$

*Case* [$stmt' \neq$ @nop, break]. By the induction hypothesis,

$$\langle stmt', \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{*} \langle stmt'_1, \ell, e, \theta' \rangle_{\#nst}.$$

Hence, by Lemma 4.6,

$$\langle stmt' \text{ @par/and } stmt'', \ell, \varepsilon, \theta \rangle$$
$$\xrightarrow[nst]{*} \langle stmt'_1 \text{ @par/and } stmt'', \ell, e, \theta \rangle = \delta'.$$

It remains to be shown that description $\delta'$ is nested-irreducible or leads to a nested-irreducible description. This proof is similar to that of the case for $stmt'; stmt''$ with $stmt' \neq$ @nop, break. That is, (i) if $e \neq \varepsilon$ then $\delta'_{\#nst}$; (ii) if $stmt'_1 =$ @nop, break, then this case becomes @nop @par/and$stmt''$ or break @par/and$stmt''$; and (iii) if $isblk(stmt'_1)$, then this case becomes $stmt'$ @par/and $stmt''$ with $isblk(stmt', \ell)$ evaluating to true.

*Case* [$stmt = stmt'$ @par/or $stmt''$]. Similar to the case for $stmt'$ @par/and $stmt''$. □

### Lemma 4.13.

1. If $\delta \xrightarrow[out]{push} \delta'$ then $rank(\delta) = rank(\delta')$.
2. If $\delta \xrightarrow[out]{pop} \delta'$ then $rank(\delta) > rank(\delta')$.

*Proof.* Let

$$\delta = \langle stmt, \ell, e, \theta \rangle \qquad rank(\delta) = (i, j)$$
$$\delta' = \langle stmt', \ell', e', \theta' \rangle \qquad rank(\delta') = (i', j')$$

Then

1. Suppose $\delta \xrightarrow[out]{push} \delta'$. Then, by **push**, $e \neq \varepsilon$, $stmt' = bcast(stmt, e)$, $\ell' = \ell + 1$, $e' = \varepsilon$, and $\theta = \theta'$. By definition of $rank$, $j = \ell + 1$, as $e \neq \varepsilon$, and $j' = \ell' = \ell + 1$.

Hence, $j = j'$. It remains to be shown that $i = i'$:

$$
\begin{aligned}
i &= pot(stmt, e) && \text{by definition of } rank \\
&= pot'(bcast(stmt, e)) && \text{by defintion of } pot \\
&= pot'(stmt') && \text{as } stmt' = bcast(stmt, e) \\
&= pot'(bcast(stmt', e')) && \text{as } e' = \varepsilon \text{ (nothing changes)} \\
&= pot(stmt', e') && \text{by definition of } pot \\
&= i' && \text{by definition of } rank
\end{aligned}
$$

Therefore, $(i, j) = (i', j')$.

2. Suppose $\delta \xrightarrow[out]{pop} \delta'$. Then, by **pop**, $stmt = stmt'$, $\ell > 0$, $\ell' = \ell - 1$, $e = e' = \varepsilon$, and $\theta = \theta'$. By definition of $pot$,

$$pot(bcast(stmt, e)) = pot(bcast(stmt', e')),$$

as $stmt = stmt'$ and $e = e' = \varepsilon$. Hence, $i = i'$. And by definition of $rank$, $j = \ell$, as $e = \varepsilon$, and $j' = \ell - 1$, as $e' = \varepsilon$ and $\ell' = \ell - 1$. Hence, $j > j'$. Therefore, $(i, j) > (i', j')$. □

### Lemma 4.14. If $\delta \xrightarrow[nst]{} \delta'$ then $rank(\delta) \geq rank(\delta')$.

*Proof.* By induction on the structure of $\xrightarrow[nst]{}$ derivations. In this proof, we will write $pot_e(stmt)$ for $pot(stmt, e)$, and we will often use (implicitly) the fact that $pot_\varepsilon = pot'$, i.e, broadcasting the empty event is a no-op ($bcast(stmt, \varepsilon) = stmt$).
Let

$$\delta = \langle stmt, \ell, e, \theta \rangle \qquad rank(\delta) = (i, j)$$
$$\delta' = \langle stmt', \ell', e', \theta' \rangle \qquad rank(\delta') = (i', j')$$

By the hypothesis of the lemma, there is derivation $\pi$ such that

$$\pi = \cfrac{\vdots}{\delta \xrightarrow[nst]{} \delta'}$$

By definition of $\xrightarrow[nst]{}$, $e = \varepsilon$ and $\ell = \ell'$. Depending on the structure of program $stmt$, the following cases are possible. In each one of them we show that $rank(\delta) \geq rank(\delta')$.

*Case* [$\text{var } v\ stmt_1$]. Then $\pi$ is an instance of **var-expd**. Hence, $stmt' = \text{var } v \perp stmt_1$ and $e' = \varepsilon$. Thus,

$$
\begin{aligned}
rank(\delta) &= (pot_\varepsilon(\text{var } v\ stmt_1), \ell) \\
&= (pot'(stmt_1), \ell) \\
&= (pot_\varepsilon(stmt_1), \ell) = rank(\delta').
\end{aligned}
$$

*Case* [$v := expr$]. Then $\pi$ is an instance of **assign**. Hence, $stmt' =$ @nop and $e' = \varepsilon$. Thus, $rank(\delta) = rank(\delta') = (0, \ell)$.

*Case* [$\text{emit}_{int}(e_1)$]. Then $\pi$ is an instance of **emit-int**. Hence, $stmt' = $ @runat$(\ell)$ and $e' = e_1 \neq \varepsilon$. Thus,

$$rank(\delta) = (1, \ell) > (0, \ell + 1) = rank(\delta').$$

*Case* [$\text{if } expr \text{ then } stmt_1 \text{ else } stmt_2$]. Suppose $eval(\theta, expr) \neq 0$. (The case for $eval(\theta, expr) = 0$ is similar.) Then $\pi$ is an

instance of **if-true**. Hence, $stmt' = stmt_1$ and $e' = \varepsilon$.

$$rank(\delta) = (pot_\varepsilon(\texttt{if } expr \texttt{ then } stmt_1 \texttt{ else } stmt_2, \ell)$$
$$= (\max\{pot'(stmt_1), pot'(stmt_2)\}, \ell)$$
$$\geq (pot'(stmt_1), \ell)$$
$$= pot_\varepsilon(stmt_1, \ell) = rank(\delta').$$

*Case* $[stmt_1; stmt_2]$. There are three subcases.

*Case* $[stmt_1 = \texttt{@nop}]$. Then $\pi$ is an instance of **seq-nop**. Hence, $stmt' = stmt_2$ and $e' = \varepsilon$. Thus,

$$rank(\delta) = (pot_\varepsilon(\texttt{@nop}; stmt_2), \ell)$$
$$= (0 + pot'(stmt_2), \ell)$$
$$= (pot_\varepsilon(stmt_2) = rank(\delta').$$

*Case* $[stmt_1 = \texttt{break}]$. Then $\pi$ is an instance of **seq-brk**. Hence, $stmt' = \texttt{break}$ and $e' = \varepsilon$. Thus, $rank(\delta) = rank(\delta') = (0, \ell)$.

*Case* $[stmt_1 \neq \texttt{@nop}, \texttt{break}]$. Then $\pi$ is an instance of **seq-adv**. Hence, there is a derivation $\pi'$ such that

$$\pi' = \frac{\begin{array}{c} \vdots \\ \langle stmt_1, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{} \langle stmt_1', \ell, e', \theta' \rangle \end{array}}{\langle stmt_1; stmt_2, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{} \langle stmt_1'; stmt_2, \ell, e', \theta' \rangle}$$

By the induction hypothesis,

$$rank\langle stmt_1, \ell, \varepsilon, \theta \rangle \geq rank\langle stmt_1', \ell, e', \theta' \rangle. \qquad (3)$$

There are two possibilities: either $e' = \varepsilon$ or $e' \neq \varepsilon$. If $e' = \varepsilon$, then

$$rank(\delta) = (pot_\varepsilon(stmt), \ell)$$
$$= (pot'(stmt_1) + pot'(stmt_2), \ell)$$

and

$$rank(\delta') = (pot_\varepsilon(stmt'), \ell)$$
$$= (pot'(stmt_1') + pot'(stmt_2), \ell)$$

By (3), $pot'(stmt_1) \geq pot'(stmt_1')$. Thus,

$$rank(\delta) \geq rank(\delta').$$

If however $e' \neq \varepsilon$, then $\pi'$ contains one application of **emit-int**, which consumes at least one reachable $\texttt{emit}_{int}$ statement from $stmt_1$, and so $pot'(stmt_1) > pot'(stmt_1')$. Thus,

$$rank(\delta) = (pot_\varepsilon(stmt), \ell)$$
$$= (pot'(stmt_1) + pot'(stmt_2), \ell)$$
$$> (pot'(stmt_1') + pot'(stmt_2), \ell + 1)$$
$$= (pot_e(stmt'), \ell + 1) = rank(\delta').$$

Because $e$ is necessarily an internal event, as it was generated by an $\texttt{emit}_{int}$, broadcasting it to $stmt'$ cannot modify the potency of the resulting statement.

*Case* $[\texttt{loop } stmt_1]$. Then $\pi$ is an instance of **loop-expd**. Hence, $stmt' = stmt_1 \texttt{ @loop } stmt_1$ and $e' = \varepsilon$. By the syntactic restriction for loop bodies, all execution paths within $stmt_1$ contain a matching $\texttt{break}$ or $\texttt{await}_{ext}$ or $\texttt{every}_{ext}$. Thus,

$$rank(\delta) = rank(\delta') = (pot_\varepsilon(stmt_1), \ell).$$

*Case* $[stmt_1 \texttt{ par/and } stmt_2]$. Then $\pi$ is an instance of **par/and-expd**. Hence, $stmt' = stmt_1 \texttt{ @par/and } (\texttt{@runat}(\ell); stmt_2)$ and $e' = \varepsilon$. Thus,

$$rank(\delta) = (pot_\varepsilon(stmt_1 \texttt{ par/and } stmt_2), \ell)$$
$$= (pot'(stmt_1) + pot'(stmt_2), \ell)$$
$$\geq (pot_\varepsilon(stmt'), \ell) = rank(\delta').$$

*Case* $[stmt_1 \texttt{ par/or } stmt_2]$. Similar to the case for par/and.

*Case* $[\texttt{@runat}(\ell)]$. Then $\pi$ is an instance of **run-at**. Hence, $stmt' = \texttt{@nop}$ and $e' = \varepsilon$. Thus, $rank(\delta) = rank(\delta') = (0, \ell)$.

*Case* $[\texttt{@var } v \, n \, stmt_1]$. Similar to the case for sequences.

*Case* $[stmt_1 \texttt{ @loop } stmt_2]$. There are three subcases.

*Case* $[stmt_1 = \texttt{@nop}]$. Then $\pi$ is an instance of **loop-nop**. Hence, $stmt' = \texttt{loop } stmt_2$ and $e' = \varepsilon$. Thus,

$$rank(\delta) = (pot_\varepsilon(\texttt{@nop @loop } stmt_2), \ell)$$
$$= (0 + pot'(stmt_2), \ell)$$
$$= (pot_\varepsilon(stmt_2), \ell) = rank(\delta').$$

*Case* $[stmt_1 = \texttt{break}]$. Then $\pi$ is an instance of **loop-brk**. Hence, $stmt' = \texttt{@nop}$ and $e' = \varepsilon$. Thus,

$$rank(\delta) = (pot_\varepsilon(\texttt{break @loop } stmt_2), \ell)$$
$$= (0, \ell)$$
$$= (pot'(\texttt{@nop}), \ell) = rank(\delta').$$

*Case* $[stmt_1 \neq \texttt{@nop}, \texttt{break}]$. Then $\pi$ is an instance of **loop-adv**. Hence, there is a derivation $\pi'$ such that

$$\pi' = \frac{\begin{array}{c} \vdots \\ \langle stmt_1, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{} \langle stmt_1', \ell, e', \theta' \rangle \end{array}}{\langle stmt, \ell, \varepsilon, \theta \rangle \xrightarrow[nst]{} \langle stmt_1' \texttt{ @loop } stmt_2, \ell, e', \theta' \rangle}$$

There are two possibilities: either (i) every execution path in $stmt_1$ contains a matching $\texttt{break}$, $\texttt{await}_{ext}$, or $\texttt{every}_{ext}$; or (ii) some execution path in $stmt_1$ does not contain a matching $\texttt{break}$ or $\texttt{await}_{ext}$ or $\texttt{every}_{ext}$.

If (i) holds, then $pot'(stmt) = pot'(stmt_1)$. Moreover, a single $\xrightarrow[nst]{}$ cannot terminate the loop, as by hypothesis $stmt_1 \neq \texttt{break}$, nor can it consume an $\texttt{await}_{ext}$ or $\texttt{every}_{ext}$ (which could increase $pot'(stmt_1')$). This means that all execution paths in $stmt_1'$ still contain a matching $\texttt{break}$, $\texttt{await}_{ext}$ or $\texttt{every}_{ext}$. And so,

$$pot'(stmt') = pot'(stmt_1').$$

If (ii) holds then, since by hypothesis $stmt_1 \neq \texttt{@nop}$, a single $\xrightarrow[nst]{}$ cannot restart the loop which means that $stmt_1'$ still

contain some execution path in which a matching break, $await_{ext}$, or $every_{ext}$ does not occur. And so,

$$pot'(stmt') = pot'(stmt'_1) + pot'(stmt_2).$$

In either case, the rest of the proof is similar to that of the case for sequences $stmt_1; stmt_2$ with $stmt_1 \neq$ @nop, break.

*Case* [$stmt_1$ @par/and $stmt_2$]. Similar to the case for @par/or.

*Case* [$stmt_1$ @par/or $stmt_2$]. The proof depends on whether or not $stmt_1$ is blocked. We will prove the case for $isblk(stmt_1, \ell)$; the proof of the case for $\neg\, isblk(stmt_1, \ell)$ is similar. So, assuming $stmt_1$ is blocked, there are three subcases.

*Case* [$stmt_2 =$ @nop]. Then $\pi$ is an instance of **par/or-nop2**. Hence, $stmt' = clear(stmt_1)$ and $e' = \varepsilon$. Thus,

$$
\begin{aligned}
rank(\delta) &= (pot_\varepsilon(stmt_1 \text{ @par/or @nop}), \ell) \\
&= (pot'(stmt_1) + 0, \ell) \\
&\geq (pot'(clear(stmt_1)), \ell) \\
&= (pot_\varepsilon(clear(stmt_1)), \ell) = rank(\delta').
\end{aligned}
$$

Since it only returns the body of *active* fin statements (i.e., statements which are already being considered by $pot$), function *clear* cannot increase the potency of the resulting statement.

*Case* [$stmt_2 =$ break]. Then $\pi$ is an instance of **par/or-brk2**. Hence, $stmt' = clear(stmt_1);$ break and $e' = \varepsilon$. Thus,

$$
\begin{aligned}
rank(\delta) &= (pot_\varepsilon(stmt_1 \text{ @par/or break}), \ell) \\
&= (pot'(stmt_1) + 0, \ell) \\
&\geq (pot'(clear(stmt_1); \text{break}), \ell) \\
&= (pot_\varepsilon(clear(stmt_1); \text{break}), \ell) = rank(\delta').
\end{aligned}
$$

*Case* [$stmt_2 \neq$ @nop, break]. Then $\pi$ is an instance of **par/or-adv2**. The rest of the proof is similar to that of the case for sequences $stmt_1; stmt_2$ with $stmt_1 \neq$ @nop, break. $\square$

**Theorem 4.15.** *If $\delta \xrightarrow[nst]{*} \delta'$ then $rank(\delta) \geq rank(\delta')$.*

*Proof.* If $\delta \xrightarrow[nst]{*} \delta'$ then $\delta \xrightarrow[nst]{i} \delta'$, for some $i$. We proceed by induction on $i$. The theorem is trivially true for $i = 0$ and follows directly from Lemma 4.14 for $i = 1$. Suppose

$$\delta \xrightarrow[nst]{1} \delta'_1 \xrightarrow[nst]{i-1} \delta',$$

for some $i > 1$. Then, by Lemma 4.14 and by the induction hypothesis,

$$rank(\delta) \geq rank(\delta'_1) \geq rank(\delta').\qquad\square$$