DESIGN: Online Device Selection and Edge Association for Federated Synergy Learning-enabled AIoT

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APPENDICES

Proof of the Lemma 1

In order to reduce the training cost of device i, the training cost after edge association should not exceed the training cost of vanilla FL, i.e.,

$$\{T_i^{device}(\bar{\mathcal{X}},\mathcal{Y})+T_i^{trans}(\bar{\mathcal{X}},\mathcal{Y})+T_i^{server}(\bar{\mathcal{X}},\mathcal{Y})\}+\alpha_i\{E_i^{device}(\bar{\mathcal{X}},\mathcal{Y})+E_i^{trans}(\bar{\mathcal{X}},\mathcal{Y})\}\le C_i^{FL}$$

Then, $|\mathcal{D}_i|(\frac{fd_i}{R_i(\mathcal{Y})}+\frac{bd_i}{R^{max}})+\rho_i|D_i|\frac{fd_i}{R_i(\mathcal{Y})}\le C_i^{FL}-T_i^{device}(\bar{\mathcal{X}},\mathcal{Y})-T_i^{server}(\bar{\mathcal{X}},\mathcal{Y})-\alpha_iE_i^{device}(\bar{\mathcal{X}},\mathcal{Y})$.

Let $\Lambda(\bar{\mathcal{X}},\mathcal{Y})\triangleq\frac{|\mathcal{D}_i|(1+\rho_i)fd_i}{C_i^{FL}-T_i^{device}-T_i^{server}-\alpha_iE_i^{device}-|\mathcal{D}_i|\frac{bd_i}{R^{max}}}$, we have $R_i(\mathcal{Y})\ge\Lambda(\bar{\mathcal{X}},\mathcal{Y})$. Then, according to the function of transmission rate

$$\sum_{l\in\mathcal{N}\setminus\{i\}:y_l=y_i}\rho_i\sigma_{l,y_i}\le\frac{\mathcal{W}_{y_i}\log_2(1+\frac{\rho_i\cdot\sigma_{i,y_i}}{\omega_0})}{\Lambda(\overline{\mathcal{X}},\mathcal{Y})}-1\triangleq\delta_i$$

Lemma 1 holds.

Proof of the Theorem 2

It is assumed that changing the association decision of device i from y_i to y_i' can reduce the training cost, i.e., $C_i(y_i,\mathcal{Y}_{-i})>C_i(y_i',\mathcal{Y}_{-i})$. We will consider the following three cases: *i)* $y_i=0$ and $y_i'>0$; *ii)* $y_i>0$ and $y_i'=0$; *iii)* $y_i>0$ and $y_i'>0$.

For *Case i)*: Given $y_i=0$ and $y_i'>0$, there is

$$\Phi(y_i,\mathcal{Y}_{-i})-\Phi(y'_i,\mathcal{Y}_{-i})=\rho_i\sigma_{i,y_i}\delta_i-\rho_i\sigma_{i,y_i}\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y'_i=y_j\}}$$

If device i chooses to change the association decision from y_i to y_i', then Lemma 1 implies that

$$\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y_i=y_j\}}\le\delta_i$$

Hence,

$$\Phi(y_i,\mathcal{Y}_{-i})-\Phi(y'_i,\mathcal{Y}_{-i})=$$
$$\rho_i\sigma_{i,y_i}(\delta_i-\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y'_i=y_j\}})>\rho_i\sigma_{i,y_i}(\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y_i=y_j\}}-\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y_i=y_j\}})$$

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Because $\sum_{j\in\mathcal{N}\setminus\{i\}} \rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y_i=y_j\}} > \sum_{j\in\mathcal{N}\setminus\{i\}} \rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y'_i=y_j\}}$, hence $\Phi(y_i,\mathcal{Y}_{-i})-\Phi(y'_i,\mathcal{Y}_{-i})>0$.

For *Case ii)*: Given $y_i>0$ and $y'_i=0$, similar to the proof in *Case i)*, there is $\Phi(y_i,\mathcal{Y}_{-i})-\Phi(y'_i,\mathcal{Y}_{-i})>0$

For *Case iii)*: Given $y_i>0$ and $y'_i>0$, according to (1), we know that the condition $C_i(y_i,\mathcal{Y}_{-i})>C_i(y'_i,\mathcal{Y}_{-i})$ implies that

$$\rho_i\sigma_{i,y_i}\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y_i=y_j\}} > \rho_i\sigma_{i,y_i}\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y'_i=y_j\}}$$

Then, we have

$$\Phi(y_i,\mathcal{Y}_{-i})-\Phi(y'_i,\mathcal{Y}_{-i}) = \rho_i\sigma_{i,y_i}\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y_i=y_j\}} - \rho_i\sigma_{i,y_i}\sum_{j\in\mathcal{N}\setminus\{i\}}\rho_j\sigma_{j,y_j}\cdot\mathcal{I}_{\{y'_i=y_j\}} > 0$$

Therefore, the original edge association issue can be modeled as a potential game describing resource competition among multiple devices. This proves the Theorem 2.

Proof of the Theorem 3

According to the definition of the potential function $\Phi(\mathcal{Y})$, there is

$$0\le\Phi(\mathcal{Y})\le\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{\max}^2+\sum_{i=1}^{N}\gamma_{\max}\delta_{\max}=\frac{1}{2}\gamma_{\max}^2N^2+\gamma_{\max}\delta_{\max}N$$

It is assumed that changing the association decision from y_i to y'_i can reduce the training time, i.e., $C_i(y_i,\mathcal{Y}_{-i})>C_i(y'_i,\mathcal{Y}_{-i})$. According to the nature of the potential game, the potential increases by at least $\gamma_{\min}$, i.e., $\Phi(y_i,\mathcal{Y}_{-i})\ge\Phi(y'_i,\mathcal{Y}_{-i})+\gamma_{\min}$. We will consider the following three cases: *i)* $y_i=0$ and $y'_i>0$; *ii)* $y_i>0$ and $y'_i=0$; *iii)* $y_i>0$ and $y'_i>0$.

For *Case i)*: Given $y_i=0$ and $y'_i>0$, there is $\Phi(y_i,\mathcal{Y}_{-i})-\Phi(y'_i,\mathcal{Y}_{-i})=\gamma_i(\delta_i-\sum_{k\neq i}\gamma_k\mathcal{I}_{\{y_k>0\}})>0$.

Since $\gamma_i>0$ is an integer for any $i\in\mathcal{N}$, there is $\delta_i\ge\sum_{k\neq i}\gamma_k\mathcal{I}_{\{y_k>0\}}+1$.

Hence, $\Phi(y_i,\mathcal{Y}_{-i})\ge\Phi(y'_i,\mathcal{Y}_{-i})+\gamma_i\ge\Phi(y'_i,\mathcal{Y}_{-i})+\gamma_{\min}$.

For *Case ii)*: Given $y_i>0$ and $y'_i=0$, similar to the proof in *Case i)*, there is $\Phi(y_i,\mathcal{Y}_{-i})\ge\Phi(y'_i,\mathcal{Y}_{-i})+\gamma_{\min}$.

For *Case iii)*: Given $y_i>0$ and $y'_i>0$, according to the proof of Theorem 3, we have

$$\Phi(y_i,\mathcal{Y}_{-i})-\Phi({y'}_i,\mathcal{Y}_{-i})=\gamma_i(\sum_{k\neq i}\gamma_k\cdot\mathcal{I}_{\{y_i=y_k\}}-\sum_{k\neq i}\gamma_k\cdot\mathcal{I}_{\{{y'}_i=y_k\}})>0$$

Since $\gamma_i>0$ is an integer for any $i\in\mathcal{N}$, there is $\sum_{k\neq i}\gamma_k\mathcal{I}_{\{y_i=y_k\}}\ge\sum_{k\neq i}\gamma_k\mathcal{I}_{\{{y'}_i=y_k\}}+1$.

Thus, we have $\Phi(y_i,\mathcal{Y}_{-i})\ge\Phi({y'}_i,\mathcal{Y}_{-i})+\gamma_i\ge\Phi({y'}_i,\mathcal{Y}_{-i})+\gamma_{\min}$. Combined with the above analysis, we can get

$$\mathcal{F}\le\frac{\gamma_{\max}^2}{2\gamma_{\min}}N^2+\frac{\gamma_{\max}\delta_{\max}}{\gamma_{\min}}N$$

Therefore, Theorem 3 holds.

Proof of the Theorem 4

For *Case i)*: the overall training cost corresponding to the optimal solution $\mathcal{Y}^*$ must be the lowest. Thus we have $PoA_{EAG}\ge1$.

For *Case ii)*: for any $n\in\mathcal{N}$ and $\widehat{\mathcal{Y}}\in\Omega$, there is $C_n(\widehat{\mathcal{Y}})\le C_n^{FL}(\mathcal{Y})$ and $\sum_{n\in\mathcal{N}}C_n(\widehat{\mathcal{Y}})\le\sum_{n\in\mathcal{N}}C_n^{FL}(\mathcal{Y})$.

For any n, if $y_n=0$, we have $C_n(\mathcal{Y})=C_n^{FL}(\mathcal{Y})$. Otherwise,

$$R_n(\mathcal{Y})=\frac{\mathcal{W}_m}{|\mathcal{I}_m|}\log_2(1+\frac{\rho_n\sigma_{n,m}}{\omega_0})\le\mathcal{W}_m\log_2(1+\frac{\rho_n\sigma_{n,m}}{\omega_0})$$

According to the above analysis

$C_n(\mathcal{Y}) = (T_n^{device}(\mathcal{Y}) + T_n^{trans}(\mathcal{Y}) + T_n^{server}(\mathcal{Y})) + \alpha_n(E_n^{device}(\mathcal{Y}) + E_n^{trans}(\mathcal{Y}))$
$= T_n^{device}(\mathcal{Y}) + |\mathcal{D}_n|(\frac{fd_n}{R_n(\mathcal{Y})} + \frac{bd_n}{R^{max}}) + T_n^{server}(\mathcal{Y}) + \alpha_n(\rho_n|D_n|\frac{fd_n}{R_n(\mathcal{Y})} + E_n^{trans}(\mathcal{Y})) \geq T_n^{device}(\mathcal{Y})$
$+|\mathcal{D}_n|(\frac{fd_n}{\mathcal{W}_m\log_2(1+\frac{\rho_n\sigma_{n,m}}{\omega_0})} + \frac{bd_n}{R^{max}}) + T_n^{server}(\mathcal{Y}) + \alpha_n(\rho_n|D_n|\frac{fd_n}{\mathcal{W}_m\log_2(1+\frac{\rho_n\sigma_{n,m}}{\omega_0})} + E_n^{trans}(\mathcal{Y})) = C_m^{\min}(\mathcal{Y})$

Further, it can be obtained that $C_n(\mathcal{Y}) \geq \min\{C_n^{FL}(\mathcal{Y}), C_n^{\min}(\mathcal{Y})\}$. Thus we have

$$1 \leq PoA_{EAG} \leq \frac{\sum_{n\in\mathcal{N}} C_n^{FL}(\mathcal{Y})}{\sum_{n\in\mathcal{N}} \min\{C_n^{FL}(\mathcal{Y}), C_n^{\min}(\mathcal{Y})\}}$$

This proves the Theorem 4.

Proof of the Theorem 5

The regret of online device selection can be written as

$$\mathcal{R}(t) = \mathbb{E}_\Omega[\sum_{\tau=1}^{t} r_\tau^* - r_\tau(\mathcal{X}_\tau)] = \mathbb{E}[\sum_{\tau=1}^{t} r_\tau(\Omega_\tau) - r_\tau(\mathcal{X}_\tau)]$$
$$= \mathbb{E}[\sum_{\tau=1}^{t} \Delta_\tau \cdot \mathcal{I}_{\{r_\tau(\Omega_\tau)\neq r_\tau(\mathcal{X}_\tau)\}}] \leq \Delta_{\max} \cdot \mathbb{E}[\sum_{\mathcal{N}_t} \mathcal{H}_t(\mathcal{N}_t)]$$

where Δ_τ is the reward difference between the highest reward and reward that could have been achieved at round τ (i.e., $\Delta_\tau = r_\tau(\Omega_\tau) - r_\tau(\mathcal{X}_\tau) > 0$). Note that $\Delta_{\max}$ is bounded by $\Delta_{\max} \leq 2\beta + C_{\max} - C_{\min}$. $\mathcal{H}_t$ is the number of rounds up until the t-th round in which selection $\mathcal{N}_t$ was selected, and the reward given from it was strictly less than the reward received by optimal selection in the same round. In addition, we define a N-dimensional counter vector $\mathcal{V}_t = (\mathcal{V}_{t,1}, \mathcal{V}_{t,2}, ..., \mathcal{V}_{t,N})$, corresponding to the N devices as follows. We use $\mathcal{V}_t$ to observe the device selected the least times in the selected devices in round t. For each round in which the selection $\mathcal{X}_t$ achieves a lower reward than the optimal one, i.e., $r_t(\mathcal{X}_t) < r_t(\Omega_t)$, then the counter of the device that has been selected the fewest number of times up to this round among all the selected devices that were selected in this round is incremented by 1.

Formally, for each device i let $\Gamma_{t,i}$ denote the set of time indices up to time t that satisfy the following conditions:

Case i): Device i was selected, i.e., $x_{\tau,i} = 1$;

Case ii): Based on *i)*, the counter $ct_{\tau,i}$ is the minimal among all selected devices, i.e., $i = \arg\min_{j\in\mathcal{N}_\tau} ct_{\tau,j}$;

Case iii): The selection $\mathcal{X}_\tau$ achieves a lower reward than optimal selection, i.e., $r_\tau(\mathcal{X}_\tau) < r_\tau(\Omega_\tau)$.

Then, the value of the counter $\mathcal{V}_{t,i}$ is the cardinality of the index indicator $\Gamma_{t,i}$, i.e., $\mathcal{V}_{t,i} = |\Gamma_{t,i}|$.

Next, we aim at upper bounding $\mathcal{H}_t(\mathcal{X}_\tau)$ for each $\mathcal{X}_\tau \in \mathcal{P}(\mathcal{X})$. Note that based on the definition of $\mathcal{V}_t$, for every round in which the selection $\mathcal{N}_t$ has a lower reward than optimal selection, one of the coordinates in the vector $\mathcal{V}_t$ is incremented by 1. Therefore, $\sum_{\mathcal{X}_\tau\in\mathcal{P}(\mathcal{X})} \mathcal{H}_t(\mathcal{X}_\tau) = \sum_n \mathcal{V}_{t,i}$, which implies $\mathbb{E}[\sum_{\mathcal{X}_\tau\in P(\mathcal{X})} \mathcal{H}_t(\mathcal{X}_\tau)] = \sum_n \mathbb{E}[\mathcal{V}_{t,i}]$.

Let $\zeta_{t,i}$ be the indicator for the event that $\mathcal{V}_{t,i}$ is incremented by 1 at round t. Hence, we obtain

$$\mathcal{V}_{t,i} = \sum_{\tau=1}^{t} \mathcal{I}_{\{\zeta_{\tau,i}=1\}} \leq 1 + \sum_{\tau=\lceil\frac{N}{n}\rceil+1}^{t} \mathcal{I}_{\{\zeta_{\tau,i}=1\}}$$
$$\leq L + \sum_{\tau=\lceil\frac{N}{n}\rceil+1}^{t} \mathcal{I}_{\{\zeta_{\tau,i}=1, \mathcal{V}_{\tau,i}\geq L\}} \leq L + \sum_{\tau=\lceil\frac{N}{n}\rceil+1}^{t} \mathcal{I}_{\{\mathcal{J}(\Omega_\tau)<\mathcal{J}(\mathcal{X}_\tau), \mathcal{V}_{\tau,i}\geq L\}}$$

Let $\mathcal{J}(\mathcal{X}_\tau) = \sum_{k\in\Omega_\tau} \mathcal{U}_{\tau,k} + \beta D_{\tau,k}$. Note that according to the definition of $\mathcal{V}_t$, for all $k \in \mathcal{N}_t(\mathcal{X}_t)$ we have: $\mathcal{V}_{t,k} \leq ct_{t,k}$. Therefore, since in the indicator function there is an intersection with the event that $\mathcal{V}_{t,k} \geq L$ we have for all $k \in \mathcal{N}_t(\mathcal{X}_t)$ that: $L \leq \mathcal{V}_{t,k} \leq ct_{t,k}$. Denote $\psi(ct_{t,i}) = \sqrt{\frac{(n+1)\ln t}{ct_{t,i}}}$, and $\widetilde{\mu}(ct_i)$ denotes the sampled mean of the $\frac{C_i^{FL} - C_{t,i}}{C_i^{FL}}$ of device i after ct_i observations. Let $\mathcal{K}(ct_{t,i}) = \widetilde{\mu}(ct_i) + \psi(ct_{t,i}) + \beta D_{\tau,i}$. Using these notations, we can upper bound $\mathcal{V}_{t,i}$ by

$$
\begin{aligned}
\mathcal{V}_{t,i} \leq & L + \sum_{\tau=\lceil \frac{N}{n} \rceil+1}^{t} \mathcal{I}_{\{\sum_{k\in\Omega_\tau} \mathcal{K}(ct_{\tau,k}) < \sum_{k\in\mathcal{N}_\tau} \mathcal{K}(ct_{\tau,k}), \mathcal{V}_{\tau,i}\geq L\}} \\
\leq & L + \sum_{\tau=\lceil \frac{N}{n} \rceil+1}^{t} \mathcal{I}_{\{\min_{ct^*_{\tau,i}} \sum_{k\in\Omega_\tau} \mathcal{K}(ct_{\tau,k}) < \max_{ct^{\times}_{\tau,i}} \sum_{k\in\mathcal{N}_\tau} \mathcal{K}(ct_{\tau,k}), \mathcal{V}_{\tau,i}\geq L\}} \\
\leq & L + \sum_{\tau=\lceil \frac{N}{n} \rceil+1}^{t} \sum_{ct^*_{\tau,1}=1}^{t} \cdots \sum_{ct^{\times}_{\tau,N}=1}^{t} \mathcal{I}_{\{\sum_{k\in\Omega_\tau} \mathcal{K}(ct_{\tau,k}) < \sum_{k\in\mathcal{N}_\tau} \mathcal{K}(ct_{\tau,k})\}}
\end{aligned}
$$

where $\{ct^*_{t,k} : 1 \leq k \leq N\}$ and $\{ct^{\times}_{t,k} : 1 \leq k \leq N\}$ are the devices in optimal selection and the CMAB's selection at round t, respectively. Let ct^*_t and $ct^{\times}_t$ respectively, be the devices in optimal selection and the CMAB's selection at round t that minimizes the expression in the upper bound

$$
\begin{aligned}
ct^*_t &= \underset{ct^*_{t,i}\in\{ct^*_{t,1},\ldots,ct^*_{t,N}\}}{\arg\min} \{\widetilde{\mu}(ct_{t,i}) + \psi(ct_{t,i})\} \\
ct^{\times}_t &= \underset{ct^{\times}_{t,i}\in\{ct^{\times}_{t,1},\ldots,ct^{\times}_{t,N}\}}{\arg\min} \{\widetilde{\mu}(ct_{t,i}) + \psi(ct_{t,i})\}
\end{aligned}
$$

Now, we claim that the event $\{\widetilde{\mu}(ct^*_t) + \psi(ct^*_t)\} + \beta D_{t,k}(ct^*_t) < \{\widetilde{\mu}(ct^{\times}_t) + \psi(ct^{\times}_t)\} + \beta D_{t,k}(ct^{\times}_t)$ implies that at least one of three following events must occur:
I: $\widetilde{\mu}(ct^*_t) + \psi(ct^*_t) + \beta D_{t,k}(ct^*_t) \leq \widetilde{\mu}(ct^{\times}_t) + \beta D_{t,k}(ct^{\times}_t)$;
II: $\widetilde{\mu}(ct^{\times}_t) + \beta D_{t,k}(ct^{\times}_t) \geq \widetilde{\mu}(ct^{\times}_t) + \psi(ct^{\times}_t) + \beta D_{t,k}(ct^{\times}_t)$;
III: $\widetilde{\mu}(ct^*_t) + \beta D_{t,k}(ct^*_t) < \widetilde{\mu}(ct^{\times}_t) + 2\psi(ct^{\times}_t) + \beta D_{t,k}(ct^{\times}_t)$;

Assume these three inequalities do not hold, it follows that:

$$
\begin{aligned}
\widetilde{\mu}(ct^*_t) + \psi(ct^*_t) + \beta D_{t,k}(ct^*_t) > \mu(ct^*_t) + \beta D_{t,k}(ct^*_t) \\
\geq \mu(ct^{\times}_t) + 2\psi(ct^{\times}_t) + \beta D_{t,k}(ct^{\times}_t) > \widetilde{\mu}(ct^{\times}_t) + \psi(ct^{\times}_t) + \beta D_{t,k}(ct^{\times}_t)
\end{aligned}
$$

where the first transition is derived from **I**, the second from **II**, the last from **III**, and all three together contradict the event. Now, we aim at upper bounding the probabilities $\mathcal{P}r(\mathbf{I})$, $\mathcal{P}r(\mathbf{II})$ that events **I** and **II** will occur:

$$
\begin{aligned}
\mathcal{P}r(\mathbf{I}) &= \mathcal{P}r\{\widetilde{\mu}(ct^*_t) + \psi(ct^*_t) + \beta D_{t,k}(ct^*_t) \leq \mu(ct^*_t) + \beta D_{t,k}(ct^*_t)\} \\
&= \mathcal{P}r\{\widetilde{\mu}(ct^*_t) + \psi(ct^*_t) \leq \mu(ct^*_t)\} \leq e^{-2ct_t^{*2} \cdot \frac{(n+1)\ln t}{ct^*_t} \cdot \frac{1}{ct^*_t}} = t^{-2(n+1)}
\end{aligned}
$$

where the inequality is due to Hoeffding's inequality. Similarly, we can upper bound $\mathcal{P}r(\mathbf{II})$ by the same upper bound and obtain $\mathcal{P}r(\mathbf{II}) \leq t^{-2(n+1)}$.

To ensure that **III** will not occur we need to put a lower bound on L (i.e., the minimum number of times a device should be selected when it has the minimum number of selections so far among the devices in the current selection):

$$
\begin{aligned}
&\mu(ct^*_t) + \beta D_{t,k}(ct^*_t) < \mu(ct^{\times}_t) + 2\psi(ct^{\times}_t) + \beta D_{t,k}(ct^{\times}_t) \\
\Leftrightarrow\ &\mu(ct^*_t) + \beta D_{t,k}(ct^*_t) - \mu(ct^{\times}_t) - \beta D_{t,k}(ct^{\times}_t) < 2\psi(ct^{\times}_t)
\end{aligned}
$$

Denote the left-hand side of the last inequality by $\Delta(\Omega_t, \mathcal{X}_t, \beta)$. Then, for the last inequality to hold, we can demand that for every Ω_t and $\mathcal{X}_t$ selection, respectively (which satisfy $r_t(\Omega_t) < r_t(\mathcal{X}_t)$):

$$\Delta(\Omega_t, \mathcal{X}_t, \beta) \geq 2\sqrt{\frac{(n+1)\ln t}{ct_t^*}}$$

and since we have already shown that $\forall k \in N_t(\mathcal{X}) : L \leq ct_{t,i}$ and $t \leq \tau$, it is sufficient to demand

$$\Delta(\Omega_t, \mathcal{X}_t, \beta) \geq 2\sqrt{\frac{(n+1)\ln \tau}{L}}$$

Therefore, for $L \geq \frac{4(n+1)\ln \tau}{\Delta^2(\Omega_t, \mathcal{X}_t, \beta)}$ for every t, or alternatively, we can choose $L = \left\lceil \frac{4(n+1)\ln \tau}{\Delta^2_{\min}} \right\rceil$ and obtain that inequality **III** will not be met. Hence, only one of the first two inequalities must occur, and we obtain

$$\begin{aligned}
&\mathbb{E}[\mathcal{V}_{t,i}] \leq L + \sum_{\tau=\lceil \frac{N}{n} \rceil+1}^{t} \sum_{ct^*_{\tau,1}=1}^{t} \cdots \sum_{ct^{\times}_{\tau,N}=1}^{t} (\mathcal{P}r(\mathbf{I}) + \mathcal{P}r(\mathbf{II})) \\
&\leq \left\lceil \frac{4(n+1)\ln t}{\Delta^2_{\min}} \right\rceil + \sum_{\tau=\lceil \frac{N}{n} \rceil+1}^{\infty} \sum_{ct^*_{\tau,1}=1}^{t} \cdots \sum_{ct^{\times}_{\tau,N}=1}^{t} 2t^{-2(n+1)} \leq \frac{4(n+1)\ln t}{\Delta^2_{\min}} + 1 + \frac{\pi^2}{3}
\end{aligned}$$

Finally, we can upper bound the regret by

$$\mathcal{R}(t) \leq \Delta_{\max} N \cdot \left(\frac{4(n+1)\ln t}{\Delta^2_{\min}} + 1 + \frac{\pi^2}{3}\right)$$

Therefore, Theorem 5 holds.