Macromechanical Analysis of a Lamina

Generalized Hooke's Law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{ij}$$

C_{ijkl} is a 9 x 9 matrix!

Hooke's Law

- Assume
  - linear elastic behavior
  - small deformations

$$\sigma = E\varepsilon$$

Uniaxial loading

Triaxial Stress State

Similarly for the deformations in the *y*- and *z*-directions

$$\boxed{\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon\left(\sigma_x + \sigma_z\right)\right]}$$

$$\boxed{\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right]}$$

Triaxial Stress State

Strain-Stress Relations

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon\left(\sigma_x + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right]$$

Triaxial Stress State

Stress-Strain Relations

$$\sigma_x = \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_x + \upsilon\left(\varepsilon_y + \varepsilon_z\right)\right]$$

$$\sigma_y = \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_y + \upsilon\left(\varepsilon_x + \varepsilon_z\right)\right]$$

$$\sigma_z = \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_z + \upsilon\left(\varepsilon_x + \varepsilon_y\right)\right]$$

Similarly for shear stresses

Shear Stress Strain Relationships

- Shear stresses are independent of each other and all axial stresses
- Shear strains are independent of each other and all axial strains
  - Each obeys a simple linear elastic model

$$\boxed{\tau = G\gamma}$$

G – Shear Modulus

$$\boxed{G = \frac{E}{2(1+\upsilon)}}$$

Shear Stress-Strain Relationships

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy}$$

$$\tau_{xz} = G\gamma_{xz} = \frac{E}{2(1+\upsilon)}\gamma_{xz}$$

$$\tau_{yz} = G\gamma_{yz} = \frac{E}{2(1+\upsilon)}\gamma_{yz}$$

Generalized Hooke's Law

Compliance matrix

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0 \\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

Strain-stress relations for an isotropic material/matrix form

Generalized Hooke's Law

Stiffness matrix

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E(1-\upsilon)}{(1-2\nu)(1+\nu)} & \frac{\upsilon E}{(1-2\nu)(1+\nu)} & \frac{\upsilon E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\upsilon E}{(1-2\nu)(1+\nu)} & \frac{E(1-\upsilon)}{(1-2\nu)(1+\nu)} & \frac{\upsilon E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\upsilon E}{(1-2\nu)(1+\nu)} & \frac{\upsilon E}{(1-2\nu)(1+\nu)} & \frac{E(1-\upsilon)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}$$

Stress-strain relations for an isotropic material /matrix form

Stress-Strain Relations in Composite Materials

Material	**No. of independent elastic constants**
General anisotropic material	81
Anisotropic material (considering symmetry of stress and strain tensors)	36
Anisotropic material with energy considerations	21
General orthotropic material	9
Transversely orthotropic material	5
Isotropic material	2

Stress-Strain Relations in Composite Materials

- Orthotropic material ~ has three mutually perpendicular plans of material symmetry
- Specially orthotropic material ~ when the reference system of coordinates is selected along principal planes of material symmetry
- Transversely isotropic material ~ one of its principal planes is a plane of isotropy (properties are the same in all directions.)

Contracted Notation

- Thanks to symmetry of the stress and strain tensors, the compliance matrix reduces to a 6 x 6 matrix, introduce a contracted notation.

$$\sigma_{11} = \sigma_1, \sigma_{22} = \sigma_2, \sigma_{33} = \sigma_3,$$

$$\sigma_{23} = \sigma_4, \sigma_{31} = \sigma_5, \sigma_{12} = \sigma_6$$

$$\varepsilon_{11} = \varepsilon_1, \varepsilon_{22} = \varepsilon_2, \varepsilon_{33} = \varepsilon_1,$$

$$2\varepsilon_{23} = \varepsilon_4, 2\varepsilon_{31} = \varepsilon_5, 2\varepsilon_{12} = \varepsilon_6$$

$$C_{1111} = C_{11}, C_{1122} = C_{12}, C_{1133} = C_{13}, C_{1123} = 2C_{14}, C_{1131} = 2C_{15}, C_{1112} = 2C_{16}$$

$$C_{2211} = C_{21}, C_{2222} = C_{22}, C_{2233} = C_{23}, C_{2223} = 2C_{24}, C_{2231} = 2C_{25}, C_{2212} = 2C_{26}$$

Stress-Strain Relations for Thin UD Lamina

Assumed to be under a state of plane stress

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_6 \end{bmatrix}$$

Fully characterized by 4 independent constants,

Qij ~ reduced stiffnesses

Reduced Stiffness Matrix?

- If the stiffness matrix is the inverse of the compliance matrix, what is the reduced stiffness matrix?

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \quad (i, j = 1, 2, 6)$$

Relations Between Mathematical and Engineering Constants

$$Q_{11} = \frac{E_1}{1-\upsilon_{12}\upsilon_{21}}$$

$$Q_{22} = \frac{E_2}{1-\upsilon_{12}\upsilon_{21}}$$

$$Q_{12} = \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} = \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}$$

$$Q_{66} = G_{12}$$

You Said Four Independent Constants?

- From symmetry of the compliance matrix

$$\frac{\upsilon_{ij}}{E_i} = \frac{\upsilon_{ji}}{E_j}$$

- The above can also be deduced from Betti's reciprocal law according to which transverse deformation due to a stress applied in the longitudinal direction is equal to the longitudinal deformation due to an equal stress applied in the transverse direction.

Stress-Strain Relations

- Also expressed in terms of compliances

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_6 \end{bmatrix}$$

Relations between Mathematical and Engineering Constants

$$S_{11} = \frac{1}{E_1}$$

$$S_{22} = \frac{1}{E_2}$$

$$S_{12} = -\frac{\upsilon_{12}}{E_1} = -\frac{\upsilon_{21}}{E_2}$$

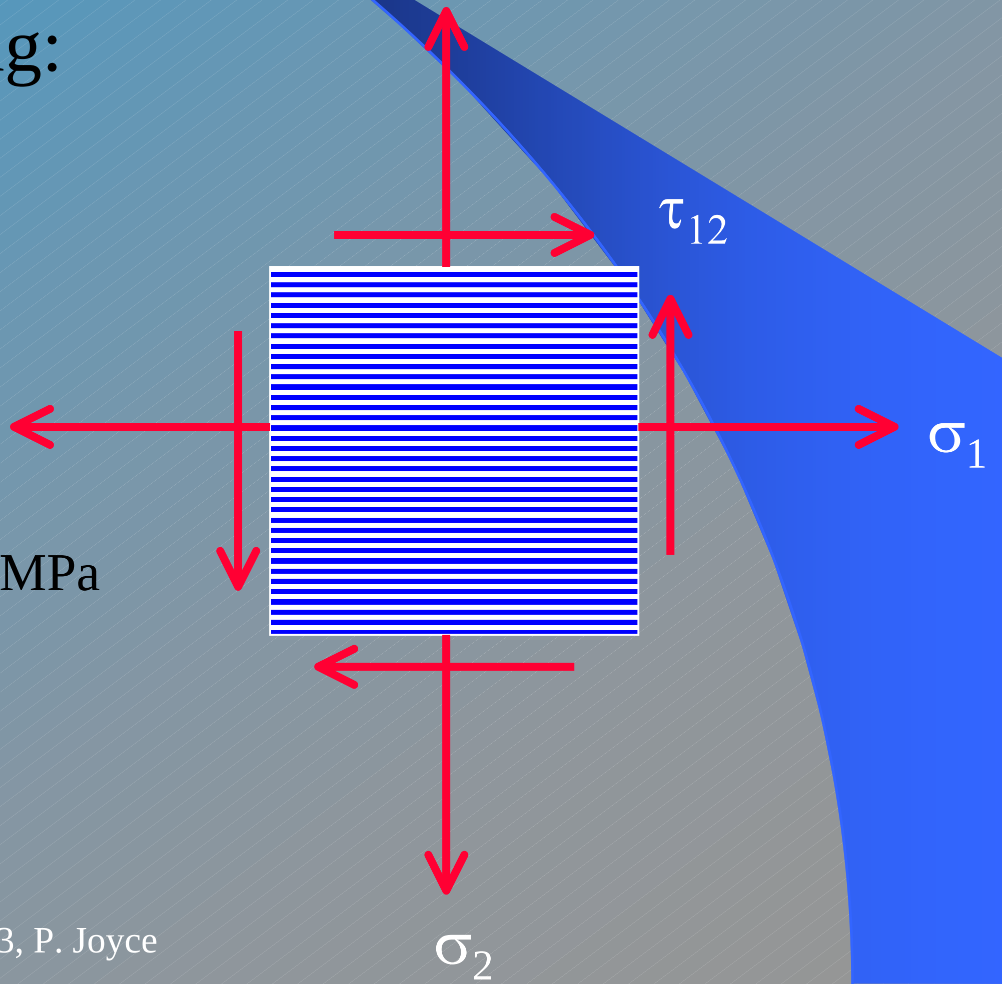
$$S_{66} = \frac{1}{G_{12}}$$

That's Better!

Sample Calculation

- For a graphite/epoxy UD laminate, find the following:
  - 1.) Compliance matrix
  - 2.) Minor Poisson's ratio
  - 3.) Reduced Stiffness Matrix
  - 4.) Strains in the 1-2 coordinate system if the applied stresses are

    $\sigma_1 = 2$ MPa, $\sigma_2 = -3$MPa, $\tau_{12} = 4$ MPa



Sample Data

Property	Symbol	Units	Glass/ epoxy	Boron /epoxy	Graphite /epoxy
Fiber volume fraction	V_f	-	0.45	0.50	0.70
Long. elastic modulus	E_1	GPa	38.6	204	181
Trans. elastic modulus	E_2	GPa	8.27	18.50	10.30
Major Poisson's ratio	ν_{12}	-	0.26	0.23	0.28
Shear Modulus	G_{12}	GPa	4.14	5.59	7.17

Sample Calculation

The compliance matrix elements are calculated as follows:

$$S_{11} = \frac{1}{E_1} = \frac{1}{181(10^9)} = 0.5525(10^{-11})$$

$$S_{12} = \frac{\nu_{12}}{E_1} = \frac{0.28}{181(10^9)} = -0.1547(10^{-11})$$

$$S_{22} = \frac{1}{E_1} = \frac{1}{10.3(10^9)} = 0.9709(10^{-10})$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17(10^9)} = 0.1395(10^{-9})$$

(all terms have units of Pa^{-1})

Sample Calculation

From Betti's reciprocal law:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

$$\nu_{21} = \frac{(0.28)}{181(10^9)}(10.3)(10^9) = 0.01593$$

Sample Calculation

The stiffness matrix elements are calculated as follows:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = \frac{181(10^9)}{1-(0.28)(0.01593)} = 181.8(10^9)$$

$$Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} = \frac{(0.28)(10.3)(10^9)}{1-(0.28)(0.01593)} = 2.897(10^9)$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} = \frac{10.3(10^9)}{1-(0.28)(0.01593)} = 10.35(10^9)$$

$$Q_{66} = G_{12} = 7.17(10^9)$$

(all terms have units of Pa)

Sample Calculation

The stiffness matrix can also be calculated by inverting the compliance matrix of Part 1:

$$[Q] = \begin{bmatrix} 0.5525(10^{-11}) & -0.1547(10^{-11}) & 0 \\ -0.1547(10^{-11}) & 0.9709(10^{-10}) & 0 \\ 0 & 0 & 0.1395(10^{-9}) \end{bmatrix}^{-1}$$

$$[Q] = \begin{bmatrix} 181.8(10^{9}) & 2.897(10^{9}) & 0 \\ 2.897(10^{9}) & 10.35(10^{9}) & 0 \\ 0 & 0 & 7.17(10^{9}) \end{bmatrix} \text{Pa}$$

(all terms have units of Pa)

Sample Calculation

The strains in the 1-2 coordinate system are calculated as follows:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12}) & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.5525(10^{-11}) & -0.1547(10^{-11}) & 0 \\ -0.1547(10^{-11}) & 0.9709(10^{-10}) & 0 \\ 0 & 0 & 0.1395(10^{-9}) \end{bmatrix} \begin{bmatrix} 2(10^6) \\ -3(10^6) \\ 4(10^6) \end{bmatrix}$$

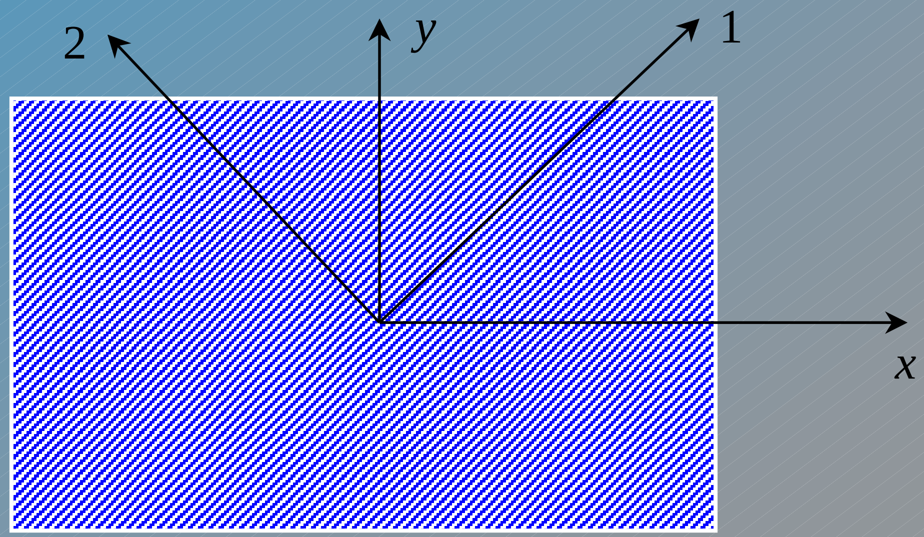
$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 15.69 \\ -294.4 \\ 557.9 \end{bmatrix} (10^{-6})$$

(10^{-6} is microstrain)

Stress-Strain Relations for Thin Angle Lamina

Generally, a laminate does not consist only of UD laminae because of their stiffness and strength properties in the transverse direction.

Hence, in most laminates, some laminae are placed at an angle.



Global and material axes of an angle lamina.

Stress-Strain Relations for Thin Angle Lamina

- The axes in the *x-y* coordinate system are called the global axes of the off-axes.
- The axes in the 1-2 coordinate system are called the material axes or the local axes, where direction 1 is parallel to the fibers (also called the longitudinal direction) and direction 2 is is perpendicular to the fibers (also called the transverse direction.)
- The angle between the two axes is denoted by the angle θ.

Stress-Strain Relations for Thin Angle Lamina

- The stress-strain relationship in the 1-2 coordinate system has already been established.
- From Mechanics of Materials, the stresses in the global and material axes are related to each other through the angle of the lamina, θ.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T(\theta)]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Where [T(θ)] is called the transformation matrix and is defined as

$$[T(\theta)] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \quad \text{thus} \quad [T(\theta)]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} = [\mathrm{T}(-\theta)]$$

Stress-Strain Relations for Thin Angle Lamina

- Using the stress-strain equation in the material axes together with the transformation equation we obtain:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1}[Q]\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

Stress-Strain Relations for Thin Angle Lamina

- Similarly, the strains in the global and material coordinate axes are related through the transformation matrix

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{bmatrix} = [R][T][R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}$$

where [R] is the Reuter matrix and is defined as

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Stress-Strain Relations for Thin Angle Lamina

- Multiplying out the first five matrices on the RHS of the previous equation we obtain the transformed reduced stiffness matrix, $\lfloor Q_{xy} \rfloor$
- Thus, $[\sigma]_{x,y} = [Q]_{x,y}[\varepsilon]_{x,y}$
- Summarizing,

$$Q_{xx} = c^4 Q_{11} + s^4 Q_{22} + 2c^2 s^2 Q_{12} + 4c^2 s^2 Q_{66}$$

$$Q_{yy} = s^4 Q_{11} + c^4 Q_{22} + 2c^2 s^2 Q_{12} + 4c^2 s^2 Q_{66}$$

$$Q_{xy} = c^2 s^2 Q_{11} + c^2 s^2 Q_{22} + (c^4 + s^4) Q_{12} - 4c^2 s^2 Q_{66}$$

$$Q_{xs} = c^3 s Q_{11} - cs^3 Q_{22} + (cs^3 - c^3 s) Q_{12} + 2(cs^3 - c^3 s) Q_{66}$$

$$Q_{ys} = cs^3 Q_{11} - c^3 s Q_{22} + (c^3 s - cs^3) Q_{12} + 2(c^3 s - cs^3) Q_{66}$$

$$Q_{ss} = c^2 s^2 Q_{11} + c^2 s^2 Q_{22} - 2c^2 s^2 Q_{12} + (c^2 - s^2)^2 Q_{66}$$

- The subscript s corresponds to shear stress or strain components in the x-y system, i.e., $\tau_s = \tau_{xy}$ and $\gamma_s = \gamma_{xy}$

Strain-Stress Relations for Thin Angle Lamina

- Similarly, the transformed compliance can be obtained:

$$S_{xx} = c^4 S_{11} + s^4 S_{22} + 2c^2 s^2 S_{12} + 4c^2 s^2 S_{66}$$
$$S_{yy} = s^4 S_{11} + c^4 S_{22} + 2c^2 s^2 S_{12} + 4c^2 s^2 S_{66}$$
$$S_{xy} = c^2 s^2 S_{11} + c^2 s^2 S_{22} + (c^4 + s^4) S_{12} - 4c^2 s^2 S_{66}$$
$$S_{xs} = 2c^3 s S_{11} - 2cs^3 S_{22} + 2(cs^3 - c^3 s) S_{12} + (cs^3 - c^3 s) S_{66}$$
$$S_{ys} = 2cs^3 S_{11} - 2c^3 s S_{22} + 2(c^3 s - cs^3) S_{12} + (c^3 s - cs^3) S_{66}$$
$$S_{ss} = 4c^2 s^2 S_{11} + 4c^2 s^2 S_{22} - 8c^2 s^2 S_{12} + (c^2 - s^2)^2 S_{66}$$

- Thus, $[\varepsilon]_{x,y} = [S]_{x,y} [\sigma]_{x,y}$

Strain-Stress Relations for Thin Angle Lamina

- How about in terms of Engineering Constants?
- If we imagine a series of simple experiments on an element with sides parallel to the *x*- and *y*-axes, we obtain:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_x} & -\dfrac{\nu_{yx}}{E_y} & \dfrac{\eta_{sx}}{G_{xy}} \\ -\dfrac{\nu_{xy}}{E_x} & \dfrac{1}{E_y} & \dfrac{\eta_{sy}}{G_{xy}} \\ \dfrac{\eta_{xs}}{E_x} & \dfrac{\eta_{ys}}{E_y} & \dfrac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix}$$

Strain-Stress Relations for Thin Angle Lamina

- What is η ? → **shear coupling coefficient**
- η_{xs} , the first subscript denotes normal loading in the x-direction; the second subscript denotes shear strain.

$$\eta_{xs} = \gamma_s / \varepsilon_x$$
$$\eta_{ys} = \gamma_s / \varepsilon_y$$
$$\eta_{sx} = \varepsilon_x / \gamma_s$$
$$\eta_{sy} = \varepsilon_y / \gamma_s$$

Strain-Stress Relations for Thin Angle Lamina

- Comparison of equivalent strain-stress relations yields the following relationships:

$$S_{xx} = \frac{1}{E_x} \qquad E_x = \frac{1}{S_{xx}}$$

$$S_{yy} = \frac{1}{E_y} \qquad E_y = \frac{1}{S_{yy}}$$

$$S_{ss} = \frac{1}{G_{xy}} \qquad G_{xy} = \frac{1}{S_{ss}}$$

$$S_{xy} = S_{yx} = -\frac{\nu_{xy}}{E_x} = -\frac{\nu_{yx}}{E_y} \qquad \nu_{xy} = -\frac{S_{yx}}{S_{xx}}; \; \nu_{yx} = -\frac{S_{xy}}{S_{yy}}$$

$$S_{xs} = S_{sx} = \frac{\eta_{xs}}{E_x} = \frac{\eta_{sx}}{G_{xy}} \qquad \eta_{xs} = \frac{S_{sx}}{S_{xx}}; \quad \eta_{sx} = \frac{S_{xs}}{S_{ss}}$$

$$S_{ys} = S_{sy} = \frac{\eta_{ys}}{E_y} = \frac{\eta_{sy}}{G_{xy}} \qquad \eta_{ys} = \frac{S_{sy}}{S_{yy}}; \quad \eta_{yx} = \frac{S_{ys}}{S_{ss}}$$

Sample Calculation

- Find the following for a 60° angle lamina of graphite/epoxy.
  - Transformed compliance matrix
  - Transformed reduced stiffness matrix
  - Global strains
  - Local strains
  - If the applied stresses are σ_x = 2 MPa, σ_y = -3MPa, τ_{xy} = 4 MPa

Sample Calculation

- From the previous example:

$$S_{11} = 0.5525(10^{-11})$$
$$S_{12} = -0.1547(10^{-11})$$
$$S_{22} = 0.9709(10^{-10})$$
$$S_{66} = 0.1395(10^{-9})$$

(all terms have units of Pa^{-1})

Sample Calculation

- The transformed compliance matrix elements are calculated as follows:

$$S_{xx} = c^4 S_{11} + s^4 S_{22} + 2c^2 s^2 S_{12} + 4c^2 s^2 S_{66} = 0.8053(10^{-10})$$

$$S_{yy} = s^4 S_{11} + c^4 S_{22} + 2c^2 s^2 S_{12} + 4c^2 s^2 S_{66} = -0.7878(10^{-11})$$

$$S_{xy} = c^2 s^2 S_{11} + c^2 s^2 S_{22} + (c^4 + s^4) S_{12} - 4c^2 s^2 S_{66} = -0.3234(10^{-10})$$

$$S_{xs} = 2c^3 s S_{11} - 2cs^3 S_{22} + 2(cs^3 - c^3 s) S_{12} + (cs^3 - c^3 s) S_{66} = 0.3475(10^{-10})$$

$$S_{ys} = 2cs^3 S_{11} - 2c^3 s S_{22} + 2(c^3 s - cs^3) S_{12} + (c^3 s - cs^3) S_{66} = -0.4696(10^{-10})$$

$$S_{ss} = 4c^2 s^2 S_{11} + 4c^2 s^2 S_{22} - 8c^2 s^2 S_{12} + (c^2 - s^2)^2 S_{66} = 0.1141(10^{-9})$$

(All terms are in Pa^{-1})

Sample Calculation

- Next, invert the transformed compliance matrix [S] to obtain the transformed reduced stiffness matrix [Q].

$$[Q]=[S]^{-1}=\begin{bmatrix} 0.8053(10^{-10}) & -0.7878(10^{-10}) & -0.3234(10^{-10}) \\ -0.7878(10^{-10}) & 0.3475(10^{-10}) & -0.4696(10^{-10}) \\ -0.3234(10^{-10}) & -0.4696(10^{-10}) & 0.1141(10^{-9}) \end{bmatrix}^{-1}$$

$$[Q]=\begin{bmatrix} 0.2365 & 0.3246 & 0.2005 \\ 0.3246 & 1.094 & 0.5419 \\ 0.2005 & 0.5419 & 0.3674 \end{bmatrix}(10^{11})$$

(all terms in Pa)

Sample Calculation

- The global strains in the x-y plane are given by $[\varepsilon]_{x,y} = [S]_{x,y}[\sigma]_{x,y}$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0.8053(10^{-10}) & -0.7878(10^{-10}) & -0.3234(10^{-10}) \\ -0.7878(10^{-10}) & 0.3475(10^{-10}) & -0.4696(10^{-10}) \\ -0.3234(10^{-10}) & -0.4696(10^{-10}) & 0.1141(10^{-9}) \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} (10^6)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0.5534(10^{-4}) \\ -0.3078(10^{-3}) \\ 0.5328(10^{-3}) \end{bmatrix}$$

Sample Calculation

- The local strains in the lamina can be calculated using the Transformation equation.

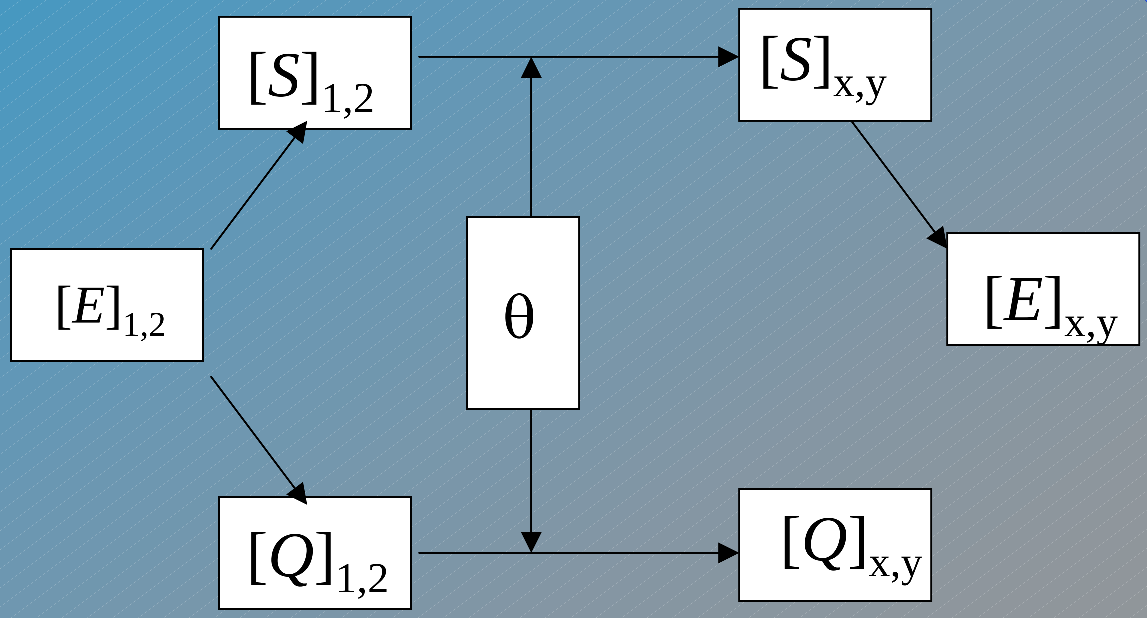
$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{bmatrix} = \begin{bmatrix} \cos^2 60 & \sin^2 60 & 2\cos 60 \sin 60 \\ \sin^2 60 & \cos^2 60 & -2\cos 60 \sin 60 \\ -\cos 60 \sin 60 & \cos 60 \sin 60 & \cos^2 60 - \sin^2 60 \end{bmatrix} \begin{bmatrix} 0.5534(10^{-4}) \\ -0.3078(10^{-3}) \\ 0.5328(10^{-3})/2 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.1367(10^{-4}) \\ -0.2662(10^{-3}) \\ -0.5809(10^{-3}) \end{bmatrix}$$

Transformation of Engineering Constants

- Flow chart for determination of transformed elastic constants of UD lamina.



Macromechanical Strength Parameters

- From a macromechanical POV, the strength of a lamina is an anisotropic property.
- It is desirable, for example, to correlate the strength along an arbitrary direction to some <u>basic strength parameters</u> (analogous to micromechanic definitions before.)

Strength Failure Theories of an Angle Lamina

- Various theories have been developed for studying the failure of an angle lamina.
- Generally based on the normal and shear strengths of a UD lamina.
- Need to consider tension and compression
- UD lamina has 2 material axes, 1-direction parallel to the fibers and 2-direction which is perpendicular to the fibers.
- Hence there are 4 normal strength parameters for UD lamina.
  - **Tensile strength in fiber direction**
  - **Transverse tensile strength**
  - **Compressive strength in fiber direction**
  - **Transverse compressive strength**
- The fifth strength parameter is the **shear strength**

Strength Failure Theories of an Angle Lamina

- Unlike the stiffness parameters, these strength parameters cannot be transformed directly for an angle lamina.
- Hence, the failure theories are based on first finding the stresses in the material axes and then using these five strength parameters of a UD lamina to find whether the lamina has failed.

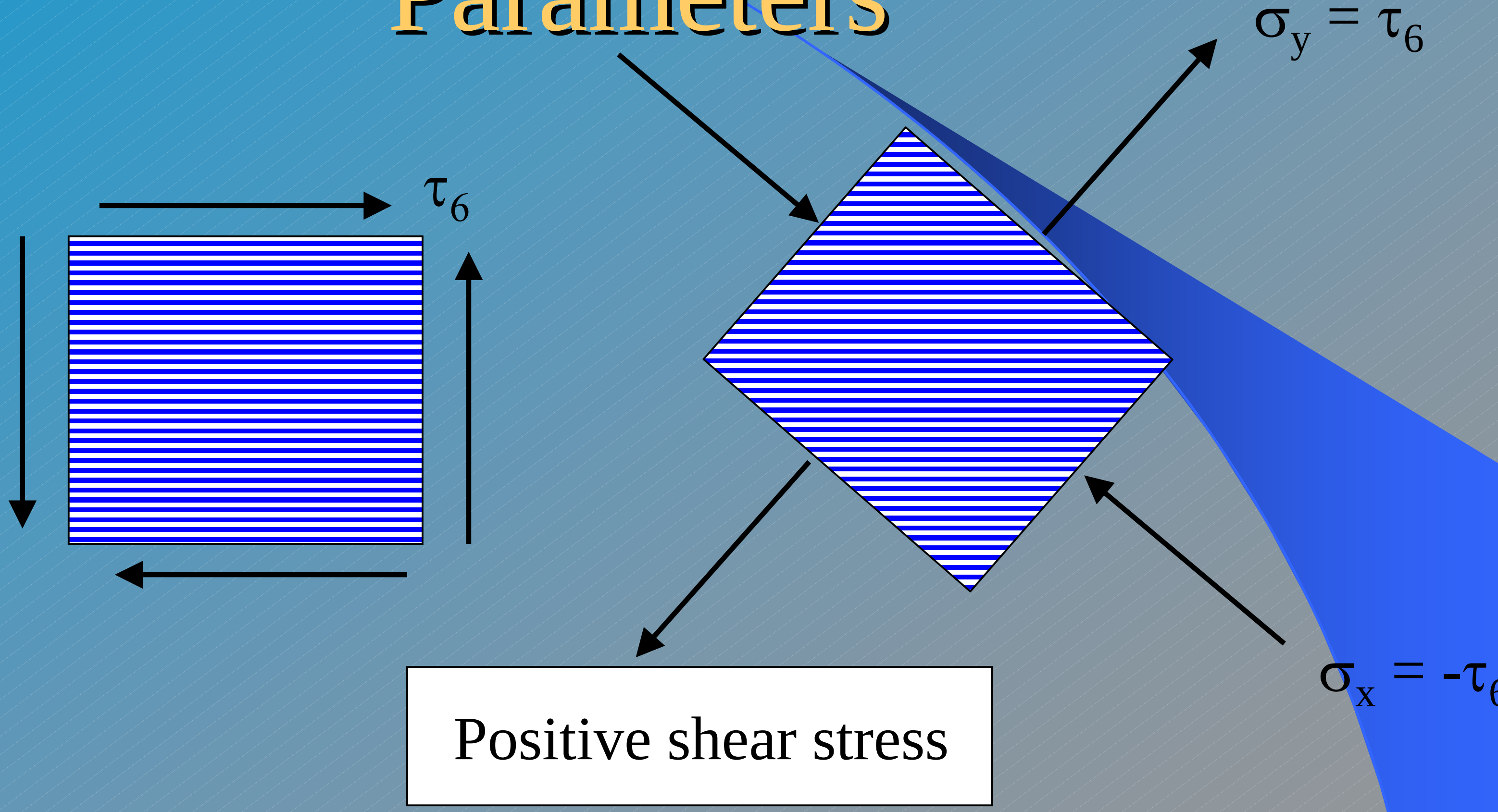
Macromechanical Strength Parameters

- Also predict transverse compressive strength and in-plane shear strength using micromechanics. . .
- Failure mechanisms vary greatly with material properties and type of loading.
- Even when predictions are accurate with regard to failure initiation at critical points, they are only approximate as far as global failure of the lamina is concerned.
- Furthermore, the possible interaction of failure mechanisms makes it difficult to obtain reliable strength predictions under a general type of loading.
- A macromechanical or phenomological approach to failure analysis may be preferable.

Macromechanical Strength Parameters

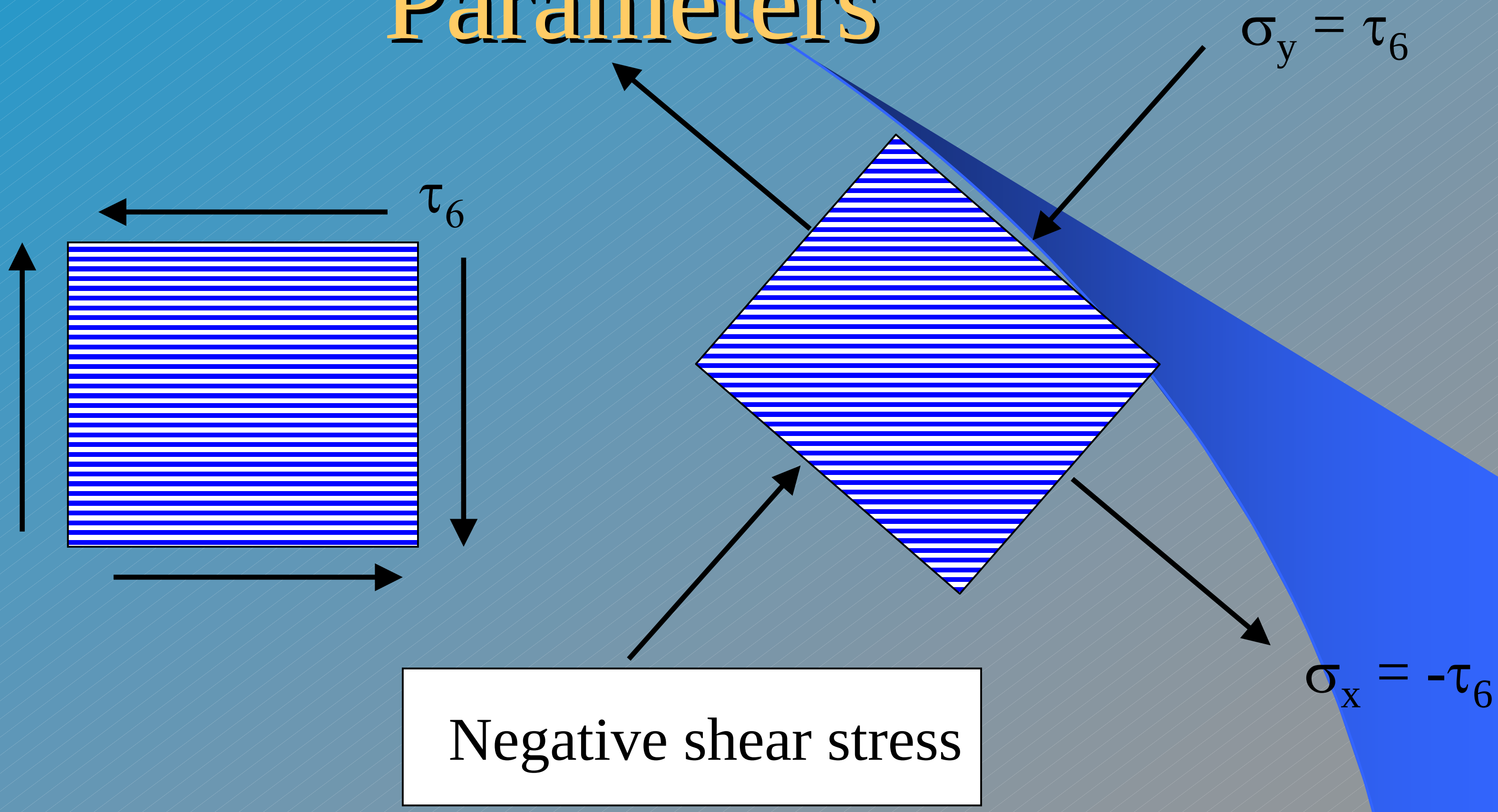
- This characterization recognizes the fact that most composite materials have different strengths in tension and compression.
- By convention the sign of the shear stress is immaterial, as long as the shear strength is referred to the principal material directions.
- Exception, refers to the case when the shear stress is applied at an angle wrt the principal material directions.
- Since most composites have different tensile and compressive strengths and they are weakest in transverse tension, it follows that in this case the lamina would be stronger under positive shear.

Macromechanical Strength Parameters



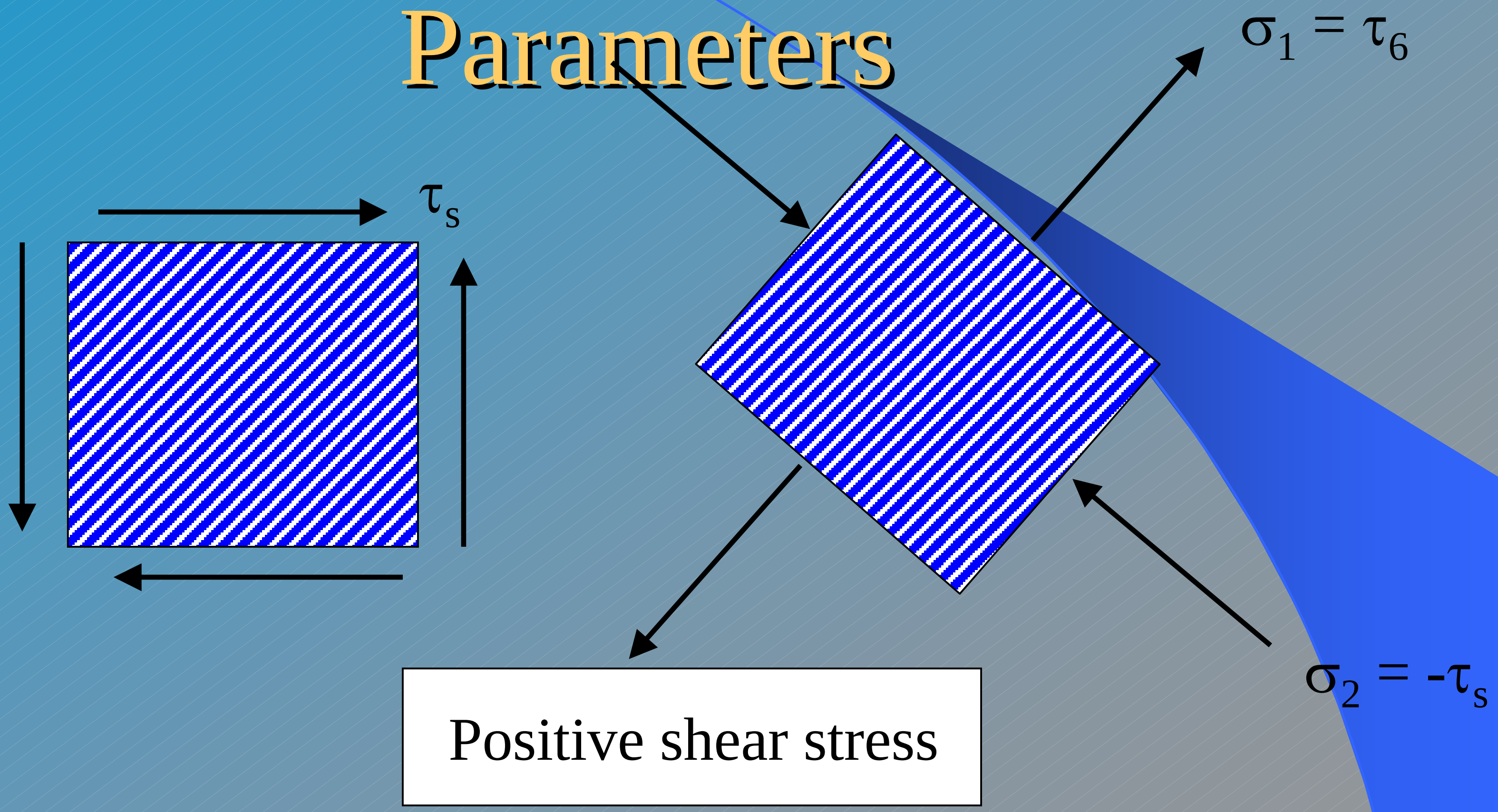
Shear stress acting along principal material axes.

Macromechanical Strength Parameters



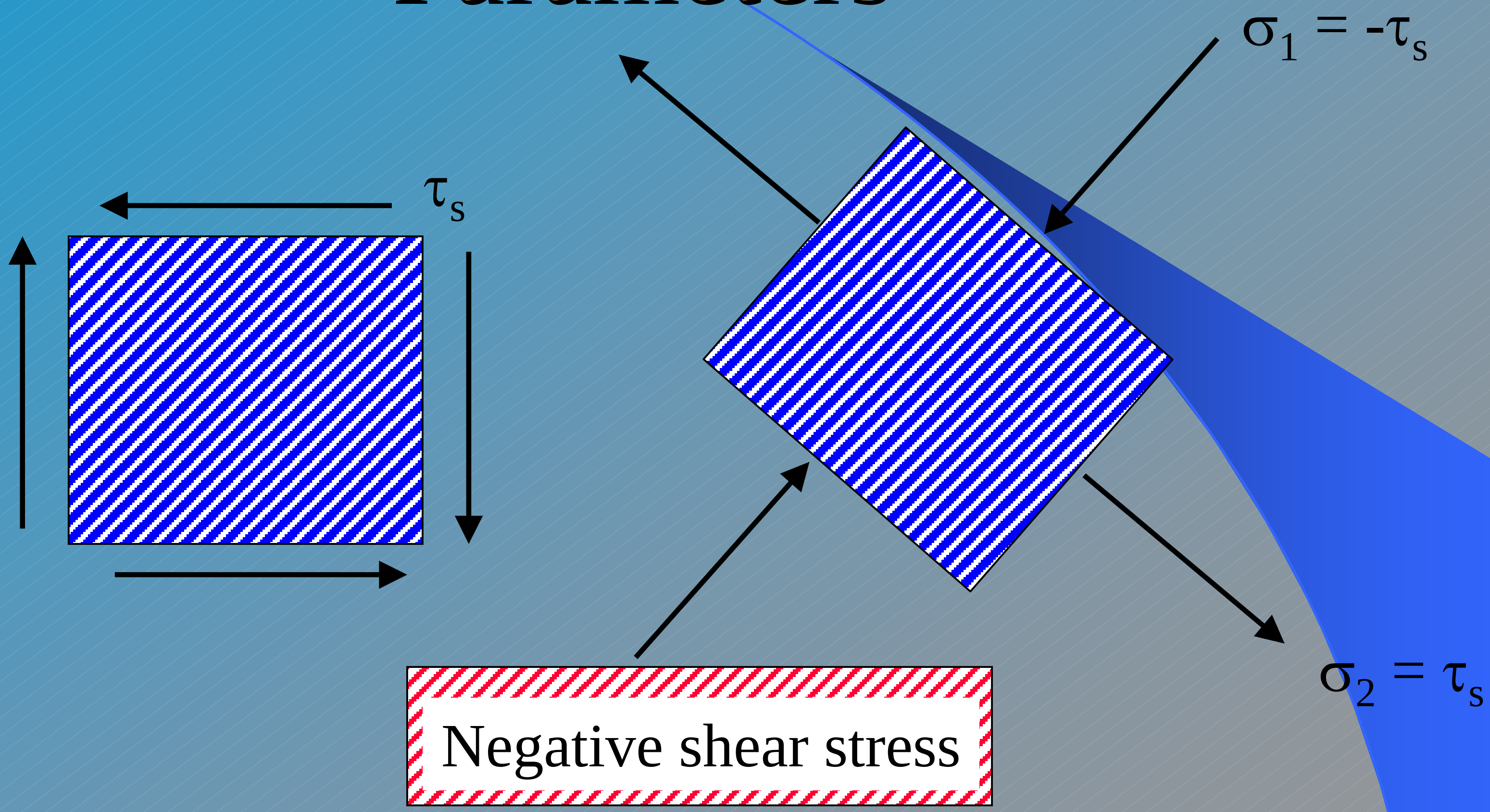
Shear stress acting along principal material axes.

Macromechanical Strength Parameters



Shear stress acting at 45° wrt principal material axes.

Macromechanical Strength Parameters



Shear stress acting at 45° wrt principal material axes.

Macromechanical Failure Theories

- Given a state of stress, the principal stresses and their directions are obtained by stress transformation (independent of material properties.)
- The principal strains and their directions are obtained by using the appropriate anisotropic stress-strain relations and strain transformation.
- In general, the principal stress, principal strain, and material symmetry directions do not coincide.
- Since strength varies with orientation, maximum stress alone is not the critical factor in failure.

Macromechanical Failure Theories

- An anisotropic failure theory is needed.
- Failure criteria for homogeneous isotropic materials, such as
  - Maximum normal stress (Rankine),
  - Maximum shear stress (Tresca),
  - Maximum distortional energy (von Mises),
  - and so forth are well established.
- More than 40 anisotropic theories have been proposed – look at the four most widely used.

Maximum Stress Failure Theory

- Related to the Maximum Normal stress theory by Rankine and the Maximum Shear stress theory by Tresca.
- The stresses acting on a lamina are resolved into the normal and shear stresses in the material axes.
- Failure is predicted in a lamina, if any of the normal or shear stresses in the material axes are equal to or greater than the corresponding ultimate strengths of a UD lamina.

$$-\left(\sigma_1^C\right)_{ult} < \sigma_1 < \left(\sigma_1^T\right)_{ult}, \quad -\left(\sigma_2^C\right)_{ult} < \sigma_2 < \left(\sigma_2^T\right)_{ult}, \quad -\left(\tau_{12}\right)_{ult} < \tau_{12} < \left(\tau_{12}\right)_{ult}$$

- Each component of stress is compared with the corresponding strength and hence does not have an interaction with the others.

Sample Calculation

- Find the off-axis shear strength of a 60° graphite/epoxy lamina using the Maximum Stress failure criteria.
- Assume the following stress state

$$\sigma_x = 0, \sigma_y = 0, \tau_{xy} = \tau,$$

- Find the stresses along the principal material axes, using the Transformation Equation.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.866\tau \\ -0.866\tau \\ -0.500\tau \end{bmatrix}$$

Sample Calculation

- Applying the Maximum Stress Failure Criteria together with strength data for graphite/epoxy composites from the Data Sheet,we have

$$-1500 < 0.866\tau < 1500$$

$$-246 < -0.866\tau < 40$$

$$-68 < 0.500\tau < 68$$

or

$$-1732 < \tau < 1732$$

$$-46.19 < \tau < 284.1$$

$$-136.0 < \tau < 136.0$$

Sample Calculation

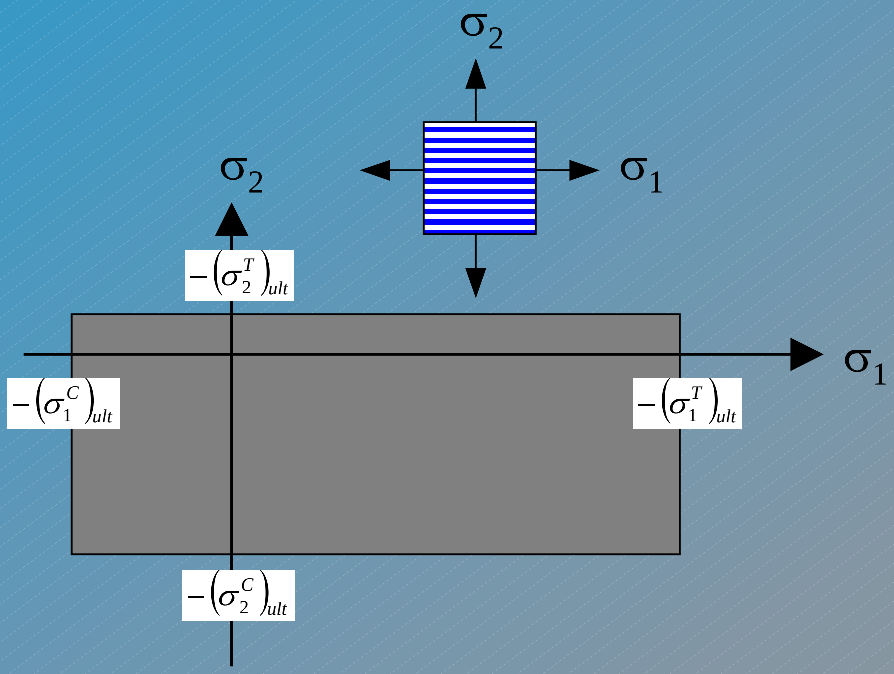
- The off-axis shear strength of a lamina is defined as <u>the minimum of the positive and negative shear stress</u> which can be applied to an angle lamina before failure.
- Calculations show that τ_{xy} = 46.19 MPa is the largest magnitude of shear stress one can apply to the 60° graphite/epoxy composite.
- However, the largest positive shear stress one could apply is 136.0 MPa, and the largest negative shear stress one could apply is –46.19 MPa.
- This shows that the maximum magnitude of allowable shear stress in other than the material axes direction depends on the sign of the shear stress.
- This is because the tensile strength perpendicular to the fiber direction is much lower than the compressive strength perpendicular to the fiber direction.

Failure Envelopes

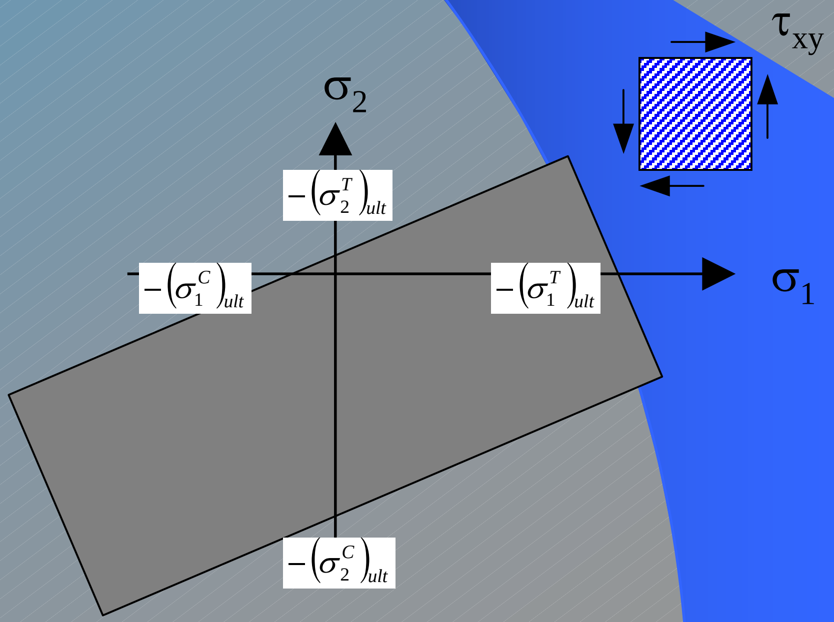
- A failure envelope is a 3D plot of the combinations of normal and shear stresses which can be applied to an angle lamina before failure.
- Drawing 3D graphs is time consuming. . .
- One may develop failure envelopes for constant shear stress, τ_{xy}, and then use the 2 normal stresses σ_x and σ_y as the 2 axes.
- If the applied stress is within the failure envelope, the lamina is safe; otherwise it has failed.

Failure Envelopes

- For a UD lamina at a given shear stress loading, the failure envelope takes the form of a rectangle as shown.



- For a 60° lamina at a given shear stress loading, the failure envelope takes the form of a rectangle as shown.



Maximum Strain Failure Theory

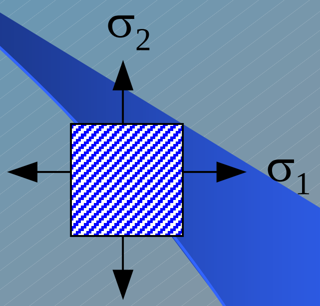
- Based on the Maximum Normal Strain Theory by St. Venant and the Maximum Shear Stress Theory by Tresca.
- The strains applied to a lamina are resolved into the normal and shear stresses in the material axes.
- Failure is predicted in a lamina, if any of the normal or shear strains in the material axes are equal to or greater than the corresponding ultimate strains of a UD lamina.

$$-\left(\varepsilon_1^C\right)_{ult} < \varepsilon_1 < \left(\varepsilon_1^T\right)_{ult}, \quad -\left(\varepsilon_2^C\right)_{ult} < \varepsilon_2 < \left(\varepsilon_2^T\right)_{ult}, \quad -\left(\gamma_{12}\right)_{ult} < \gamma_{12} < \left(\gamma_{12}\right)_{ult}$$

- The ultimate strains can be found directly from the ultimate strength parameters and the elastic moduli, *assuming the stress-strain response is linear until failure*.
- Each component of strain is compared with the corresponding ultimate strain and hence does not have an interaction with the others.
- Yields different results from Maximum Stress Failure Theory, because the local strains in a lamina include the Poisson's ratio effect (allows some interaction of stress components.)

Maximum Strain Failure Theory

- Assume a general biaxial state of stress on an angle lamina.



- Obtain the stress components along the principal material axes by stress transformation.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \left[T(\theta) \right]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Maximum Strain Failure Theory

- Then the corresponding strain components can be calculated by means of the lamina stress-strain relations:

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \nu_{21}\frac{\sigma_2}{E_2}$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} - \nu_{12}\frac{\sigma_1}{E_1}$$

$$\gamma_6 = \frac{\tau_6}{G_{12}}$$

- Next calculate the ultimate strains as follows

$$\left(\varepsilon_1^T\right)_{ult} = \frac{\left(\sigma_1^T\right)_{ult}}{E_1},\ \left(\varepsilon_1^C\right)_{ult} = \frac{\left(\sigma_1^C\right)_{ult}}{E_1},\ \left(\varepsilon_2^T\right)_{ult} = \frac{\left(\sigma_2^T\right)_{ult}}{E_2},\ \left(\varepsilon_2^C\right)_{ult} = \frac{\left(\sigma_2^C\right)_{ult}}{E_2},\ \left(\gamma_{12}\right)_{ult} = \frac{\left(\tau_{12}\right)_{ult}}{G_{12}}$$

Maximum Strain Failure Theory

- Failure subcriteria restated in terms of the stresses:

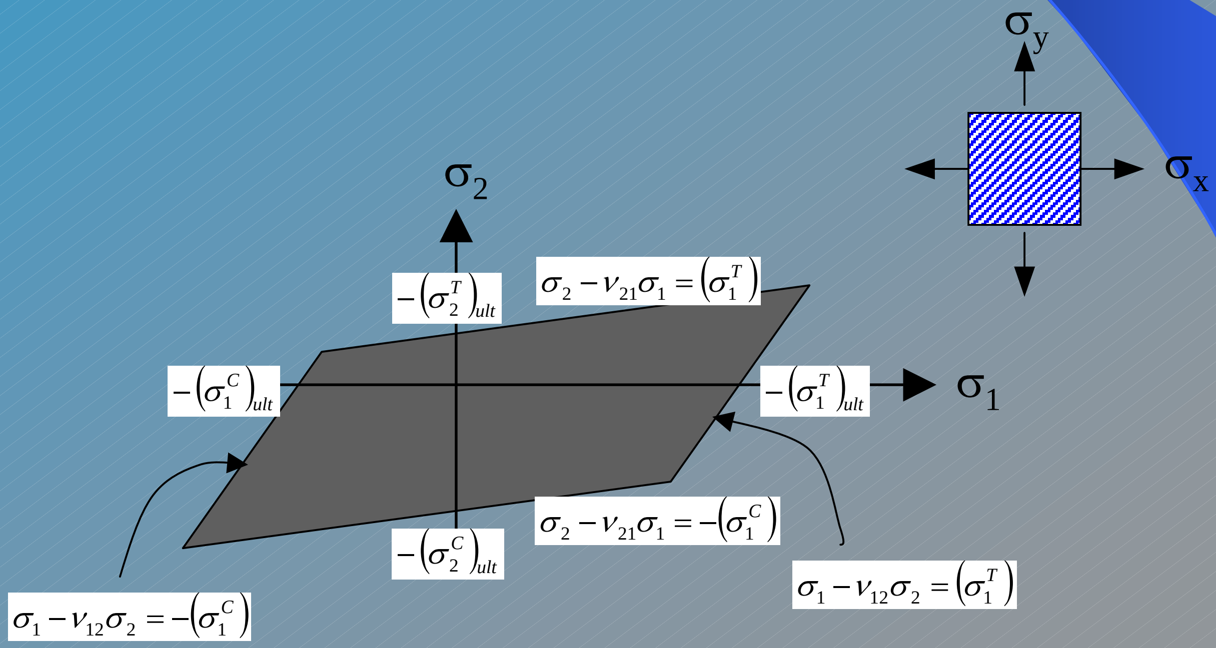
$$\sigma_1 - \nu_{12}\sigma_2 = \begin{cases} \left(\sigma_1^T\right)_{ult} \text{ when } \varepsilon_1 > 0 \\ -\left(\sigma_1^C\right)_{ult} \text{ when } \varepsilon_1 < 0 \end{cases}$$

$$\sigma_2 - \nu_{21}\sigma_1 = \begin{cases} \left(\sigma_2^T\right)_{ult} \text{ when } \varepsilon_2 > 0 \\ -\left(\sigma_2^C\right)_{ult} \text{ when } \varepsilon_2 < 0 \end{cases}$$

$$-\left(\tau_{12}\right)_{ult} < \tau_{12} < \left(\tau_{12}\right)_{ult}$$

Maximum Strain Failure Theory

- For a 2D state of stress with $\tau_6 = 0$, the failure envelope takes the form of a parallelogram with its center off the origin.



Tsai-Hill Failure Theory

- Based on the deviatoric or distortional energy failure theory of von Mises.
- Adapted to anisotropic materials by Hill.
- Then adapted to a UD lamina by Tsai.

$$\left[\frac{\sigma_1}{\left(\sigma_1^T\right)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{\left(\sigma_1^T\right)_{ult}^2}\right] + \left[\frac{\sigma_2}{\left(\sigma_2^T\right)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{\left(\tau_{12}\right)_{ult}}\right]^2 < 1$$

- Given the global stresses in a lamina, one can find the local stresses in a lamina and apply the above failure theory to determine whether or not the lamina has failed.

Tsai-Hill Failure Theory

- The failure envelope described by the Tsai-Hill criterion is a closed surface in the σ_1, σ_2, τ_{12} space.
- Failure envelopes for constant values of $k = \tau_{12} / \left(\tau_{12}\right)_{ult}$ have the form $\dfrac{\sigma_1^2}{F_1^2} + \dfrac{\sigma_2^2}{F_2^2} - \dfrac{\sigma_1 \sigma_2}{F_1^2} = 1 - k^2$
- Where:

$$F_1 = \begin{cases} \left(\sigma_1^T\right)_{ult} & \text{when } \sigma_1 > 0 \\ \left(\sigma_1^C\right)_{ult} & \text{when } \sigma_1 < 0 \end{cases}$$

$$F_2 = \begin{cases} \left(\sigma_2^T\right)_{ult} & \text{when } \sigma_2 > 0 \\ \left(\sigma_2^C\right)_{ult} & \text{when } \sigma_2 < 0 \end{cases}$$

(Modified Tsai-Hill Criterion)

- Graphically represents 4 different elliptical arcs joined at the σ_1, σ_2 axes.

Tsai-Hill Failure Theory

- Considers the interaction between the 3 UD lamina strength parameters, unlike the Maximum Stress and Maximum Strain Theories.
- Tsai-Hill Failure Theory is a Unified Theory and hence does not give the mode of failure like the Maximum Stress and Maximum Strain Theories.

Tsai-Wu Failure Theory

- Based on a general failure theory for anisotropic materials first proposed by Gol'denblat and Kopnov (1965).
- Capable of predicting strength under general states of stress for which no experimental data are available.
- Uses the concept of strength tensors.
- Has the form of an invariant formed from stress and strain tensor components
- Has the capability to account for the difference between tensile and compressive strengths

Tsai-Wu Failure Theory

- Tsai and Wu (1971) proposed a modified tensor polynomial theory by assuming the existence of a failure surface in the stress space of the form —

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\sigma_1{}^2 + H_{22}\sigma_2{}^2 + H_{66}\tau_{12}{}^2 + 2H_{12}\sigma_1\sigma_2 < 1$$

- The coefficients are obtained by applying elementary loading conditions to the lamina. Thus —

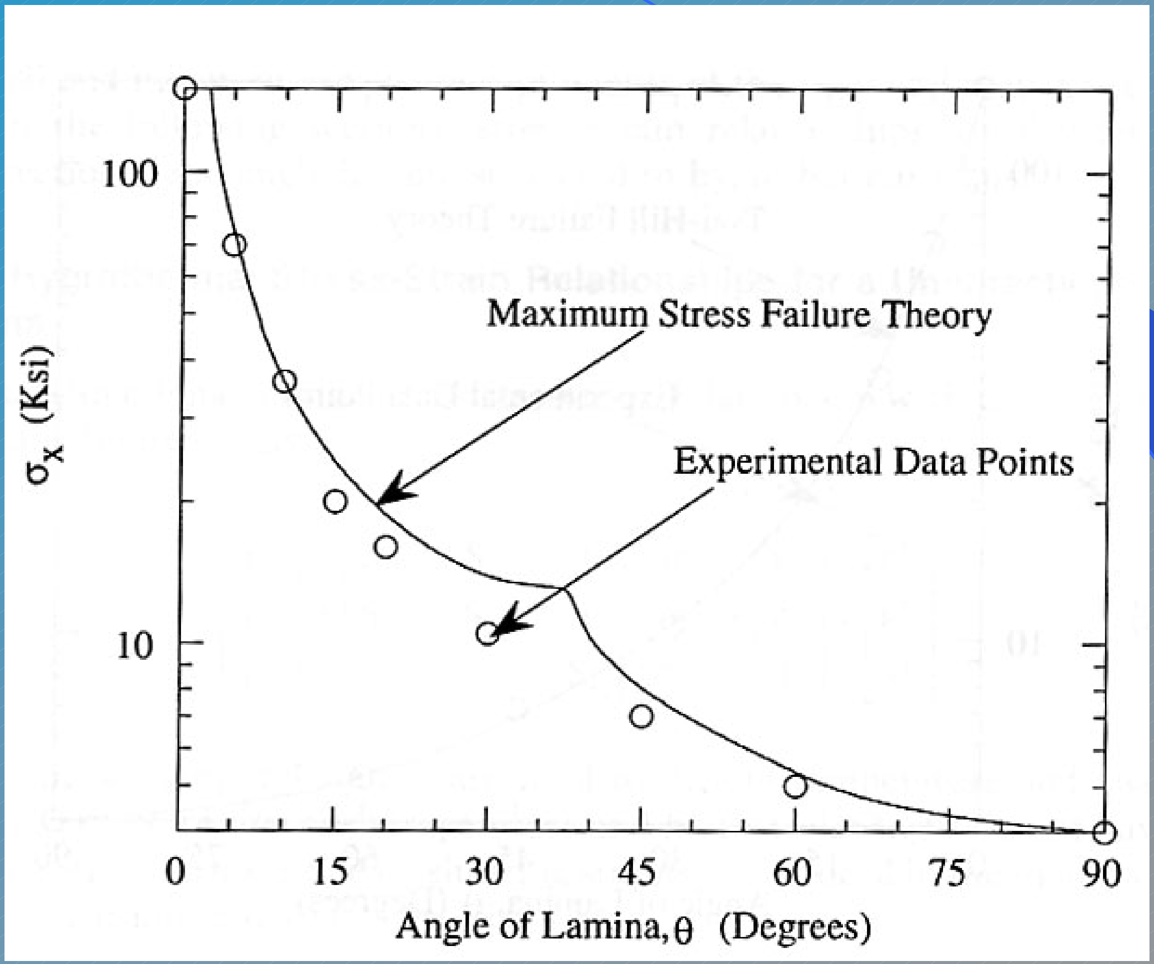
$$H_1 = \frac{1}{\left(\sigma_1^T\right)_{ult}} - \frac{1}{\left(\sigma_1^C\right)_{ult}} \quad H_2 = \frac{1}{\left(\sigma_2^T\right)_{ult}} - \frac{1}{\left(\sigma_2^C\right)_{ult}} \quad H_6 = 0$$

$$H_{11} = \frac{1}{\left(\sigma_1^T\right)_{ult}\left(\sigma_1^C\right)_{ult}} \quad H_{22} = \frac{1}{\left(\sigma_2^T\right)_{ult}\left(\sigma_2^C\right)_{ult}} \quad H_{66} = \frac{1}{\left(\tau_{12}\right)_{ult}{}^2}$$

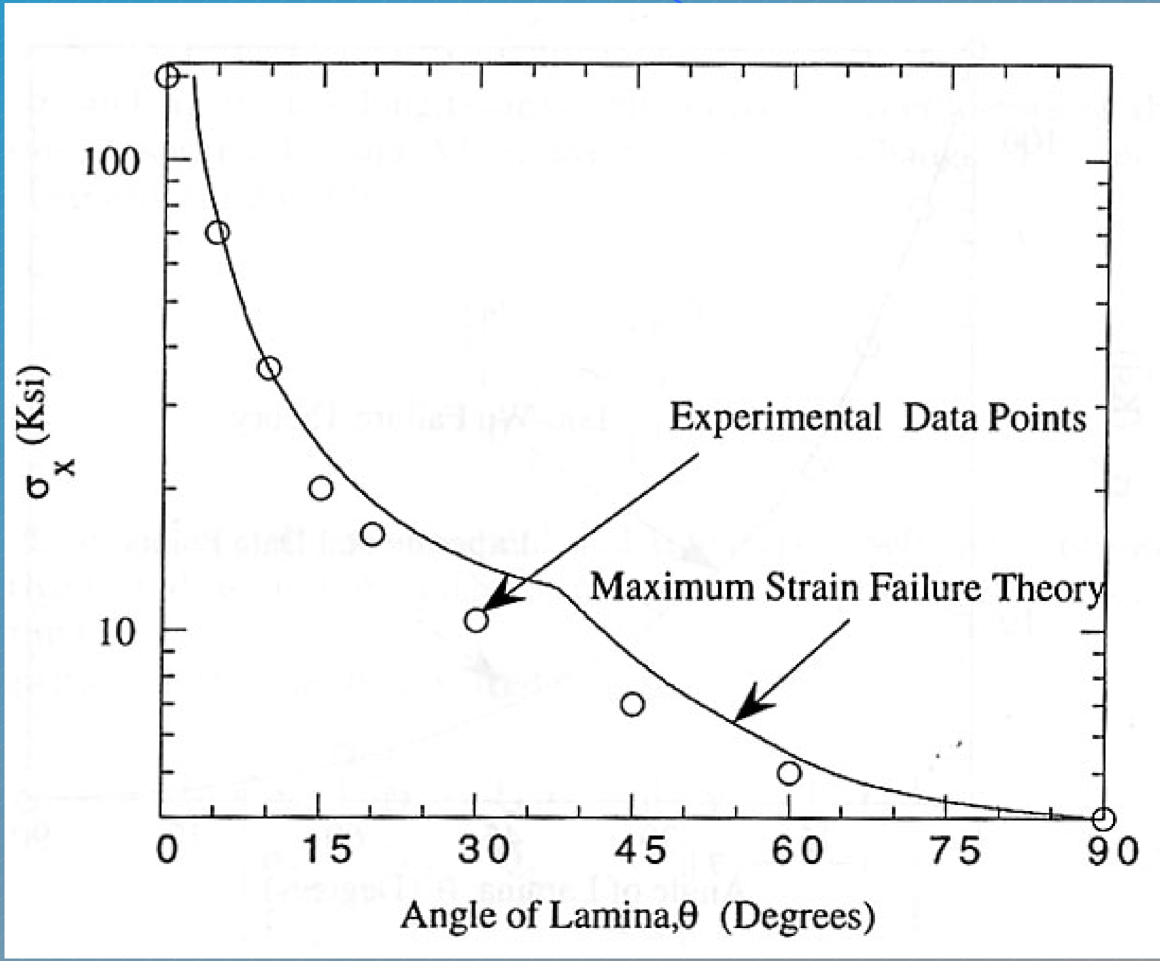
Tsai-Wu Failure Theory

- The remaining coefficient H_{12} must be obtained by some type of biaxial testing.
- Direct biaxial testing is not easy or practical to perform.
- An easier test producing a biaxial state of stress is the off-axis tensile test.
- For $\theta = 45°$ we can measure the off-axis tensile strength, σ.
- Then, $H_{12} = \dfrac{2}{\sigma} - \dfrac{(H_1 + H_2)}{\sigma} - \dfrac{1}{2}(H_{11} + H_{22} + H_{66})$
- Again produces an elliptical failure envelope.

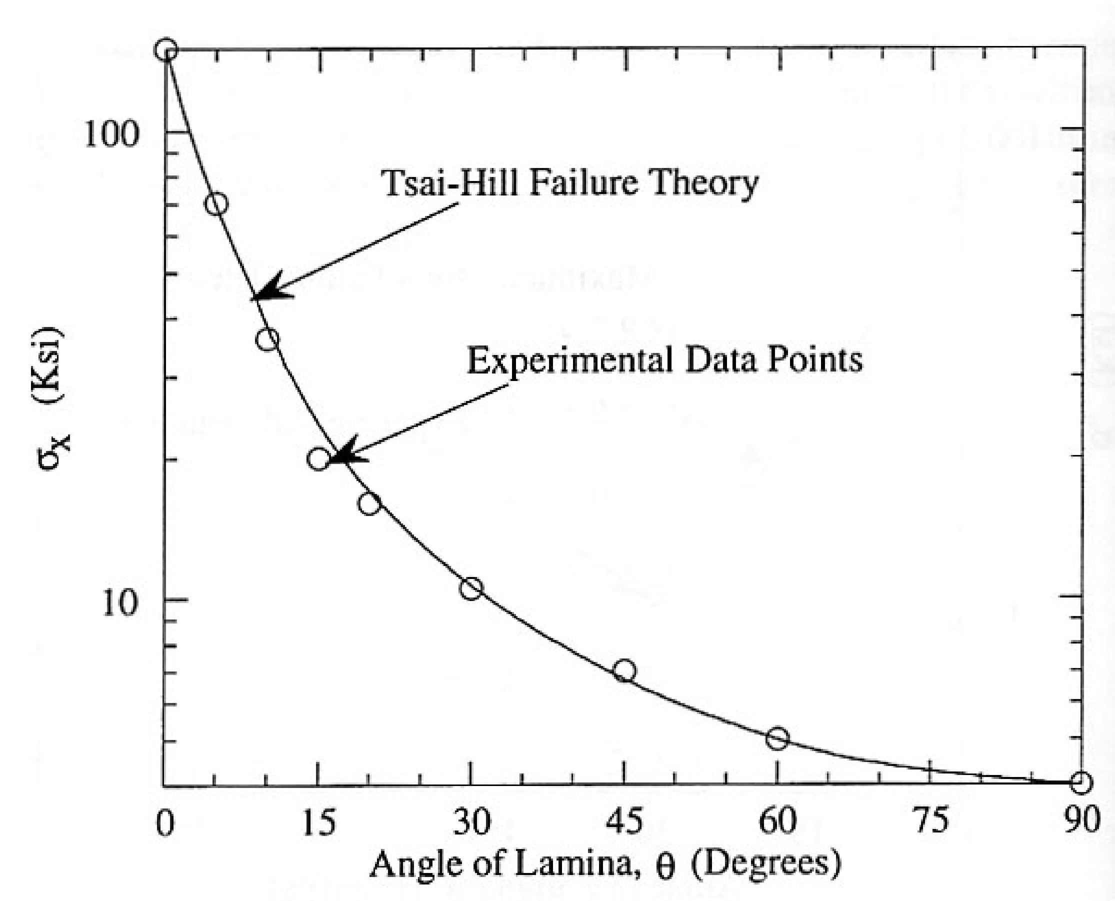
Comparison with Experimental Results



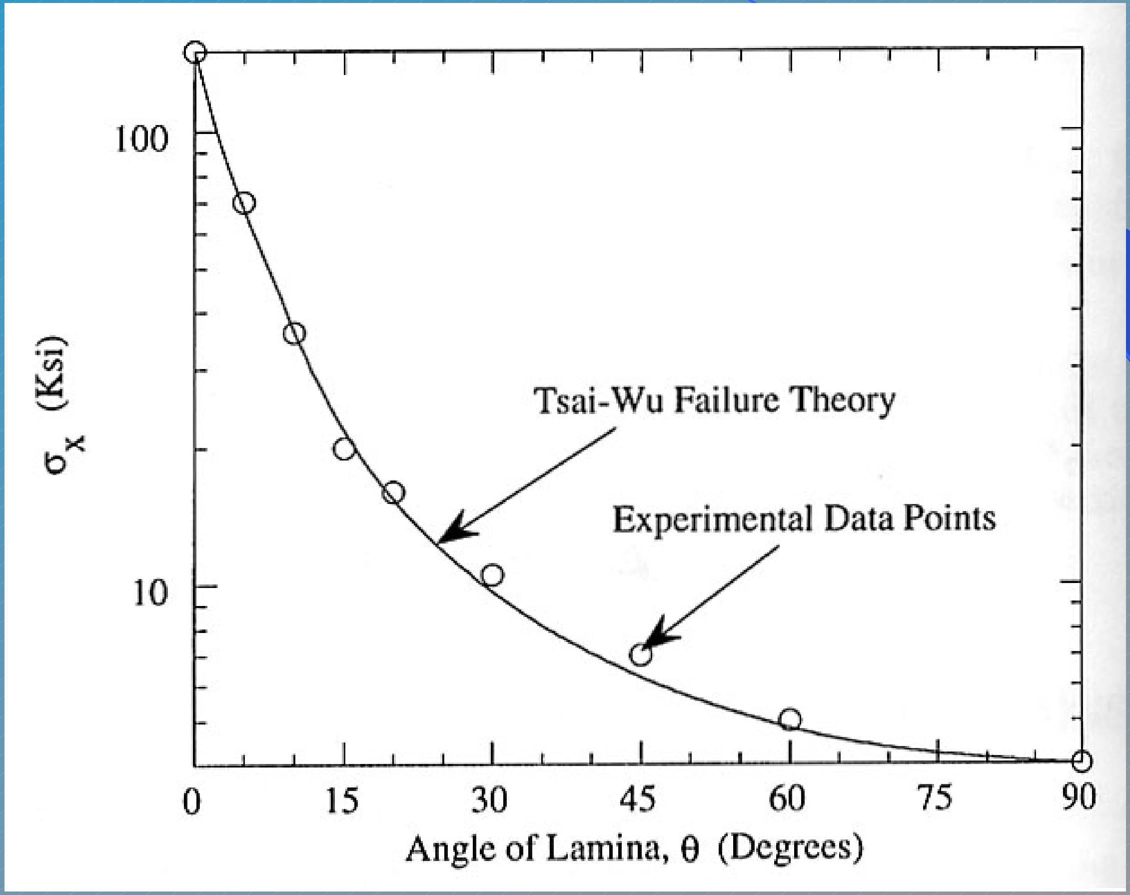
Comparison with Experimental Results



Comparison with Experimental Results



Comparison with Experimental Results



Comparison with Experimental Results

Observations —

- The difference between the Maximum Stress and Maximum Strain Failure Theories and the experimental results is quite pronounced.
- Tsai-Hill and Tsai-Wu Failure Theories are in good agreement with experimentally obtained results.
- The cusps observed in the Maximum Stress and Maximum Strain Failure Theories correspond to the change in failure mode.
- The variation of the strength as a function of angle is smooth in the Tsai-Hill and Tsai-Wu Failure Theories.

Macromechanical Failure Theories

Theory	Physical Basis	Operational Convenience	Req'd experimental characterization
Maximum Stress	Tensile behavior of brittle material	Inconvenient	Few parameters By simple testing
	No stress interaction		
Maximum Strain	Tensile behavior of brittle material	Inconvenient	Few parameters by simple testing
	Some stress interaction		
Deviatoric strain energy	Ductile behavior of anisotropic materials	Can be programmed	Biaxial testing is needed in addition to uniaxial testing
(Tsai-Hill)	"Curve fitting" for heterogeneous brittle composites	Different functions required for tensile and compressive strengths	
Interactive tensor polynomial	Mathematically consistent	General and comprehensive; operationally simple	Numerous parameters
Tsai-Wu	Reliable "curve fitting"		Comprehensive experimental program needed.

References

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- **Mechanics of Composite Materials**, Kaw, A.K., 1997.
- **Introduction to Composite Materials**, Tsai, S. W. and Hahn, H. T., 1980.
- "Application of Advanced Composites in Mechanical Engineering Designs," Zweben, C., **Proceedings of the 31st International SAMPE Technical Conference**, 1999.