Proof Techniques

- Why we need proof methods
- Proof by contraposition
- Proof by contradiction
- Recursion: review
- Mathematical induction
- Fibonacci analysis

Some slides adapted from Fall 2010 CS251: Prof Luo Si

Why we need proof methods

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- Example: $1 + 2 \ldots + n-1 + n = n(n+1)/2$ for all n as an integer.

Intuition may not be correct.

- Example: $n! < n^2$ is easy to show for n=1,2,3, but what about for larger n?

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However, proof by direct derivation can be difficult

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- Everyone except the champion loses exactly 1 game.
- There are n – 1 non-champions.
- Thus, there are n – 1 games.

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- Contrapositive: Prove that if n is even, then n^2 is even.
  - Assume that n is even.
  - $n = 2k$
  - Then $n^2 = 4k^2 = 2(2k^2)$ which, by definition, is even.

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- Divide both sides by x (since $x \neq 0$)
- This implies 3=1, which contradicts existing knowledge
- Therefore, if $3x=x$ then $x=0$.

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- Therefore $(x_0-y_0)(x_0+y_0)=1$
- ……

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Example 3:

- Claim: For any integers a and b, $a+b \geq 15$ implies that $a \geq 8$ or $b \geq 8$
- If the claim is wrong, there are solutions a and b such that both are integers and both are smaller than 8
- ……

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- **Recursion: review**
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Many computations are naturally self-referential.

- Mergesort, FFT, gcd.
- Linked data structures.
- A folder contains files and other folders.

Recursive definitions

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Example recursive function S that defines the set $\{2, 4, 8, 16, 32, \ldots\}$:

- $S(1) = 2$ base case
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It is often convenient to define objects recursively because it is then easier to develop algorithms for such objects by using **recursion**.

- Recursive definitions lend themselves to proof by mathematical induction

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Base case. Reduction step.

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As a Java method:

```
public static int recursiveFactorial(int n) {
    if (n == 0) return 1;                        // base case
    else return n * recursiveFactorial(n- 1);    // recursive case
```

Linear recursion

Linear recursion

Recur once per method

- Perform a single recursive call
- This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
- Define each possible recursive call so that it makes progress towards a base case.

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- Define each possible recursive call so that it makes progress towards a base case.

Example:

- Reversing an array
  - Input: An array A and nonnegative integer indices i and j
  - Output: Reversal of the elements in A starting at index i and ending at j

```
public static void ReverseArray(A, i, j):
 if i < j then
   Swap A[i] and A[ j]
   ReverseArray(A, i + 1, j - 1)
 return
```

Defining arguments for recursion

In creating recursive methods, it is important to define the methods in ways that facilitate recursion.

This sometimes requires we define additional parameters that are passed to the method.

- For example, we defined the array reversal method as ReverseArray(A, i, j), not ReverseArray(A).

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- Input: An array A and nonnegative integer indices i and j
- Output: Reversal of the elements in A starting at index i and ending at j

```
public static void IterativeReverseArray(A, i, j ):
    while i < j do
        Swap A[i ] and A[ j ]
        i = i + 1
        j = j - 1
    return
```

Binary recursion

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Example

- Problem: add all the numbers in an integer array A:
  - Input: An array A and integers i and n
  - Output: The sum of the n integers in A starting at index i

```
public static int BinarySum(A, i, n):
    if n = 1 then
        return A[i ]
    return BinarySum(A, i, n/2) + BinarySum(A, i + n/2, n/2)
```

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- Show that the first among an infinite sequence of claims is true.
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- Conclusion: Since claim is true for the base case, must be true for all subsequent cases.

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For any domino presented in a long row of dominoes, it will fall if we assume:

- The first domino falls
- When a domino falls, the next one also falls

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- Conclusion:
  - Thus it must be true for all $n > 0$

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- Consider the Fibonacci function F(n), where:
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Note: when claim is assumed true for all values $\leq n'$ it is called **weak induction**; when assumed true for only n' it is called **strong induction**

Computing Fibonacci numbers

Recursive algorithm (first attempt):

- Input: Nonnegative integer k
- Output: The kth Fibonacci number F_k

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public static int BinaryFib(k):
  if k ≤ 1 then
    return k
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- $n_2 = n1 + n0 + 1 = 1 + 1 + 1 = 3$
- $n_3 = n2 + n1 + 1 = 3 + 1 + 1 = 5$

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- $n_4 = n3 + n2 + 1 = 5 + 3 + 1 = 9$

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- $n_5 = n4 + n3 + 1 = 9 + 5 + 1 = 15$

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- $n_5 = n4 + n3 + 1 = 9 + 5 + 1 = 15$
- $n_6 = n5 + n4 + 1 = 15 + 9 + 1 = 25$

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- $n_6 = n5 + n4 + 1 = 15 + 9 + 1 = 25$
- $n_7 = n6 + n5 + 1 = 25 + 15 + 1 = 41$

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- $n_7 = n6 + n5 + 1 = 25 + 15 + 1 = 41$
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Analysis

Let n_k be the number of recursive calls by BinaryFib(k)

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- That is, $n_k > 2^{k/2}$.
- This is **exponential** growth, which is even larger than quadratic

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- Input: A nonnegative integer k
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LinearFibonacci makes k–1 recursive calls

Algorithm development revisited

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Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm (and data structure) to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.