Bayesian Decision Theory

An Introduction

Dr Muhammad Sarim

Contents

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- Then, we will study the cases where the probabilistic structure is not completely known.

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  - May use different values depending on the fishing area, time of the year, etc.
- Assume there are no other types of fish

$$P(w_1) + P(w_2) = 1$$

(exclusivity and exhaustivity)

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$$P(error) = \min\{P(w_1), P(w_2)\}$$

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- Let x be a continuous random variable.
- Define $p(x|w_j)$ as the *class-conditional probability density* (probability of x given that the state of nature is w_j for $j = 1, 2$).
- $p(x|w_1)$ and $p(x|w_2)$ describe the difference in lightness between populations of sea bass and salmon.

Class-conditional Probabilities

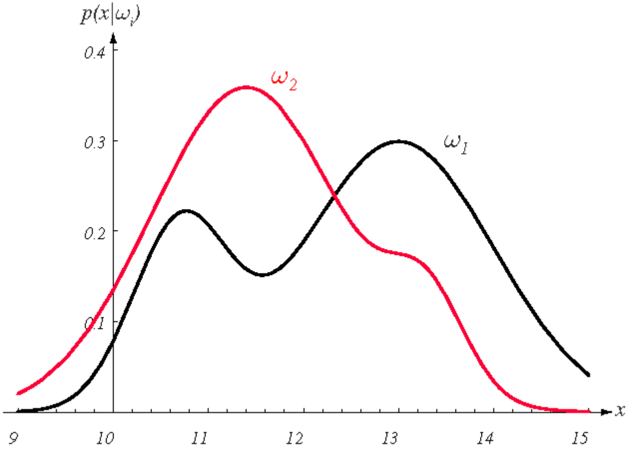

Figure: Hypothetical class-conditional probability density functions for two classes.

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Posterior Probabilities

- Suppose we know $P(w_j)$ and $p(x|w_j)$ for $j = 1, 2$, and measure the lightness of a fish as the value x.

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$$P(w_j|x) = \frac{p(x|w_j)P(w_j)}{p(x)}$$

where $p(x) = \sum_{j=1}^{2} p(x|w_j)P(w_j)$.

Posterior Probabilities

- $p(x|w_j)$ is called the *likelihood* and $p(x)$ is called the *evidence*.

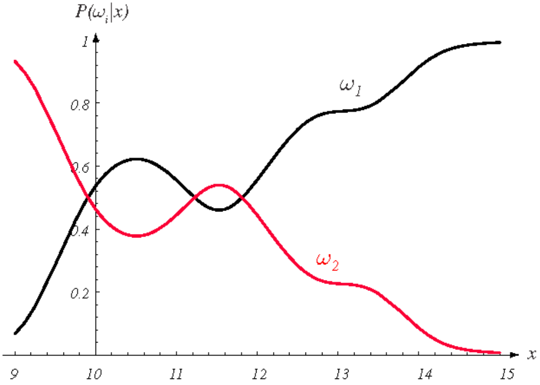

Figure: Posterior probabilities for the particular priors $P(w_1) = 2/3$ and $P(w_2) = 1/3$.

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- Rewriting the rule gives

$$\text{Decide} \quad \begin{cases} w_1 & \text{if } \frac{p(x|w_1)}{p(x|w_2)} > \frac{P(w_2)}{P(w_1)} \\ w_2 & \text{otherwise} \end{cases}$$

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- What is the average probability of error?

$$P(error) = \int_{-\infty}^{\infty} p(error, x)\, dx = \int_{-\infty}^{\infty} P(error|x)\, p(x)\, dx$$

- *Bayes decision rule* minimizes this error because

$$P(error|x) = \min\{P(w_1|x), P(w_2|x)\}$$

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  - allowing actions other than just decisions?
    - allow the possibility of rejection
  - different risks in the decision?
    - define how costly each action is

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- Let $\mathbf{x}$ be the d-component vector-valued random variable called the *feature vector*.

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- $P(w_j)$ is the prior probability that nature is in state w_j.
- The posterior probability can be computed as

$$P(w_j|\mathbf{x}) = \frac{p(\mathbf{x}|w_j)P(w_j)}{p(\mathbf{x})}$$

where $p(\mathbf{x}) = \sum_{j=1}^{c} p(\mathbf{x}|w_j)P(w_j)$.

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- The expected loss with taking action α_i is

$$R(\alpha_i|\mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i|w_j) P(w_j|\mathbf{x})$$

which is also called the *conditional risk*.

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- The resulting minimum overall risk is called the *Bayes risk* and is the best performance that can be achieved.

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$$R(\alpha_1|\mathbf{x}) = \lambda_{11}\,P(w_1|\mathbf{x}) + \lambda_{12}\,P(w_2|\mathbf{x})$$
$$R(\alpha_2|\mathbf{x}) = \lambda_{21}\,P(w_1|\mathbf{x}) + \lambda_{22}\,P(w_2|\mathbf{x})$$

Two-category Classification

- The *minimum-risk decision rule* becomes

$$\text{Decide} \begin{cases} w_1 & \text{if } (\lambda_{21} - \lambda_{11}) P(w_1|\mathbf{x}) > (\lambda_{12} - \lambda_{22}) P(w_2|\mathbf{x}) \\ w_2 & \text{otherwise} \end{cases}$$

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- This corresponds to deciding w_1 if

$$\frac{p(\mathbf{x}|w_1)}{p(\mathbf{x}|w_2)} > \frac{(\lambda_{12} - \lambda_{22})}{(\lambda_{21} - \lambda_{11})} \frac{P(w_2)}{P(w_1)}$$

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$\Rightarrow$ comparing the *likelihood ratio* to a threshold that is independent of the observation $\mathbf{x}$

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- We want to find a decision rule that minimizes the probability of error.
- Define the *zero-one loss function*

$$\lambda(\alpha_i|w_j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} \qquad i,j = 1,\dots,c$$

(all errors are equally costly)

Minimum-error-rate Classification

- Conditional risk becomes

$$
\begin{aligned}
R(\alpha_i|\mathbf{x}) &= \sum_{j=1}^{c} \lambda(\alpha_i|w_j)\, P(w_j|\mathbf{x}) \\
&= \sum_{j \neq i} P(w_j|\mathbf{x}) \\
&= 1 - P(w_i|\mathbf{x})
\end{aligned}
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- Minimizing the risk requires maximizing $P(w_i|\mathbf{x})$ and results in the *minimum-error decision rule*

$$\text{Decide } w_i \text{ if } P(w_i|\mathbf{x}) > P(w_j|\mathbf{x}) \quad \forall j \neq i$$

Minimum-error-rate Classification

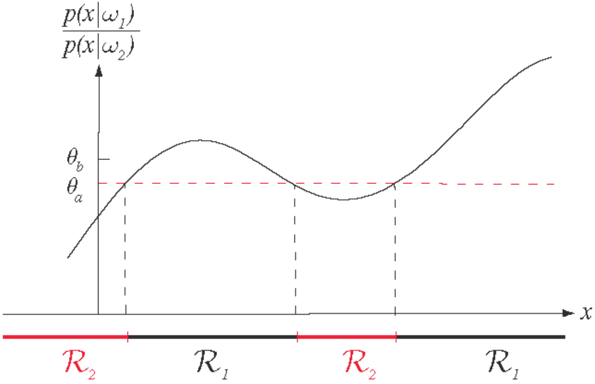

Figure: The likelihood ratio $p(\mathbf{x}|w_1)/p(\mathbf{x}|w_2)$. The threshold θ_a is computed using the priors $P(w_1) = 2/3$ and $P(w_2) = 1/3$, and a zero-one loss function. If we penalize mistakes in classifying w_2 patterns as w_1 more than the converse, we should increase the threshold to θ_b.

Discriminant Functions

- A useful way of representing classifiers is through *discriminant functions* $g_i(\mathbf{x}), i = 1, \ldots, c$, where the classifier assigns a feature vector $\mathbf{x}$ to class w_i if

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- For the classifier that minimizes error

$$g_i(\mathbf{x}) = P(w_i|\mathbf{x})$$

Discriminant Functions

- These functions divide the feature space into c *decision regions* $(\mathcal{R}_1, \ldots, \mathcal{R}_c)$, separated by *decision boundaries*.

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- These functions divide the feature space into c *decision regions* $(\mathcal{R}_1, \ldots, \mathcal{R}_c)$, separated by *decision boundaries*.
- Note that the results do not change even if we replace every $g_i(\mathbf{x})$ by $f(g_i(\mathbf{x}))$ where $f(\cdot)$ is a monotonically increasing function (e.g., logarithm).

Discriminant Functions

- These functions divide the feature space into c *decision regions* $(\mathcal{R}_1, \ldots, \mathcal{R}_c)$, separated by *decision boundaries*.
- Note that the results do not change even if we replace every $g_i(\mathbf{x})$ by $f(g_i(\mathbf{x}))$ where $f(\cdot)$ is a monotonically increasing function (e.g., logarithm).
- This may lead to significant analytical and computational simplifications.

Contents

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  - Many processes are asymptotically Gaussian (Central Limit Theorem)
  - Uncorrelatedness implies independence

Univariate Gaussian

- For $x \in \mathbb{R}$:

$$p(x) = N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

where

$$\mu = E[x] = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$\sigma^2 = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2\, p(x)\, dx$$

Univariate Gaussian

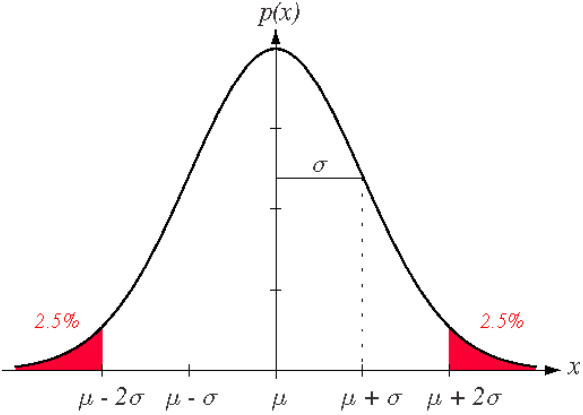

Figure: A univariate Gaussian distribution has roughly 95% of its area in the range $|x - \mu| \leq 2\sigma$.

Multivariate Gaussian

- For $\mathbf{x} \in \mathbb{R}^d$:

$$p(\mathbf{x}) = N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$
$$= \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

where

$$\boldsymbol{\mu} = E[\mathbf{x}] = \int \mathbf{x}\, p(\mathbf{x})\, d\mathbf{x}$$
$$\boldsymbol{\Sigma} = E[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T] = \int (\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\, p(\mathbf{x})\, d\mathbf{x}$$

Multivariate Gaussian

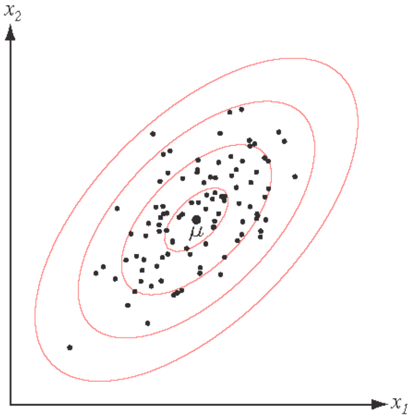

Figure: Samples drawn from a two-dimensional Gaussian lie in a cloud centered on the mean $\boldsymbol{\mu}$.

Linear Transformations

- Recall that, given $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{A} \in \mathbb{R}^{d \times k}$, $\mathbf{y} = \mathbf{A}^T \mathbf{x} \in \mathbb{R}^k$, if $x \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $y \sim N(\mathbf{A}^T \boldsymbol{\mu}, \mathbf{A}^T \boldsymbol{\Sigma} \mathbf{A})$.

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gives a covariance matrix equal to the identity matrix $\mathbf{I}$.

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$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{i})^T \boldsymbol{\Sigma}_\mathbf{i}^{-1}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{i}) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\boldsymbol{\Sigma}_\mathbf{i}| + \ln P(w_i)$$

Case 1: $\mathbf{\Sigma_i} = \sigma^2 \mathbf{I}$

- Discriminant functions are

$$g_i(\mathbf{x}) = \mathbf{w_i}^T \mathbf{x} + w_{i0} \quad \text{(linear discriminant)}$$

where

$$\mathbf{w_i} = \frac{1}{\sigma^2} \boldsymbol{\mu_i}$$
$$w_{i0} = -\frac{1}{2\sigma^2} \boldsymbol{\mu_i}^T \boldsymbol{\mu_i} + \ln P(w_i)$$

(w_{i0} is the threshold or bias for the i'th category)

Case 1: $\mathbf{\Sigma_i} = \sigma^2 \mathbf{I}$

- Decision boundaries are the hyperplanes $g_i(\mathbf{x}) = g_j(\mathbf{x})$, and can be written as

$$\mathbf{w}^T(\mathbf{x} - \mathbf{x_0}) = 0$$

where

$$\mathbf{w} = \boldsymbol{\mu}_\mathbf{i} - \boldsymbol{\mu}_\mathbf{j}$$

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- Hyperplane separating $\mathcal{R}_i$ and $\mathcal{R}_j$ passes through the point $\mathbf{x_0}$ and is orthogonal to the vector $\mathbf{w}$.

Case 1: $\Sigma_i = \sigma^2 \mathbf{I}$

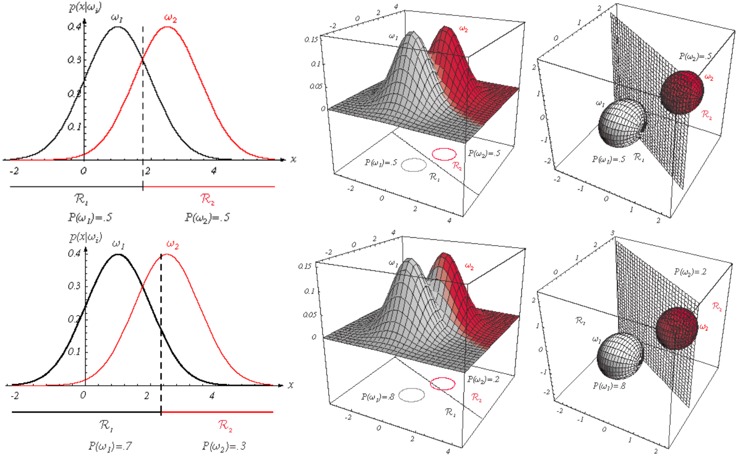

Figure: If the covariance matrices of two distributions are equal and proportional to the identity matrix, then the distributions are spherical in d dimensions.

Case 1: $\mathbf{\Sigma_i} = \sigma^2\mathbf{I}$

- Special case when $P(w_i)$ are the same for $i = 1, \ldots, c$ is the *minimum-distance classifier* that uses the decision rule

$$\text{assign } \mathbf{x} \text{ to } w_{i^*} \text{ where } i^* = \arg\min_{i=1,\ldots,c} \|\mathbf{x} - \boldsymbol{\mu}_\mathbf{i}\|$$

Case 2: $\mathbf{\Sigma_i} = \mathbf{\Sigma}$

- Discriminant functions are

$$g_i(\mathbf{x}) = \mathbf{w_i}^T \mathbf{x} + w_{i0} \quad \text{(linear discriminant)}$$

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- Hyperplane passes through $\mathbf{x_0}$ but is not necessarily orthogonal to the line between the means.

Case 2: $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}$

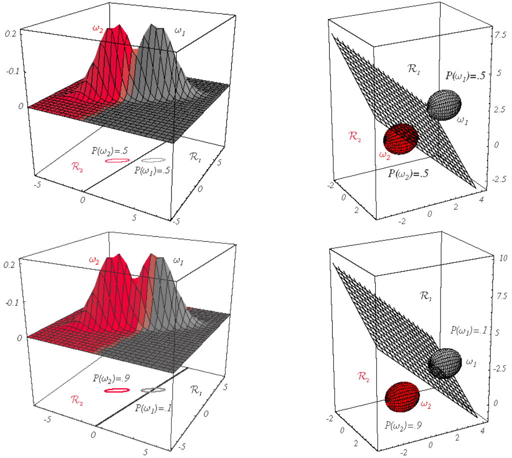

Figure: Probability densities with equal but asymmetric Gaussian distributions.

Case 3: $\mathbf{\Sigma_i}$ = arbitrary

- Discriminant functions are

$$g_i(\mathbf{x}) = \mathbf{x}^T \mathbf{W_i} \mathbf{x} + \mathbf{w_i}^T \mathbf{x} + w_{i0} \quad \text{(quadratic discriminant)}$$

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- Decision boundaries are hyperquadrics.

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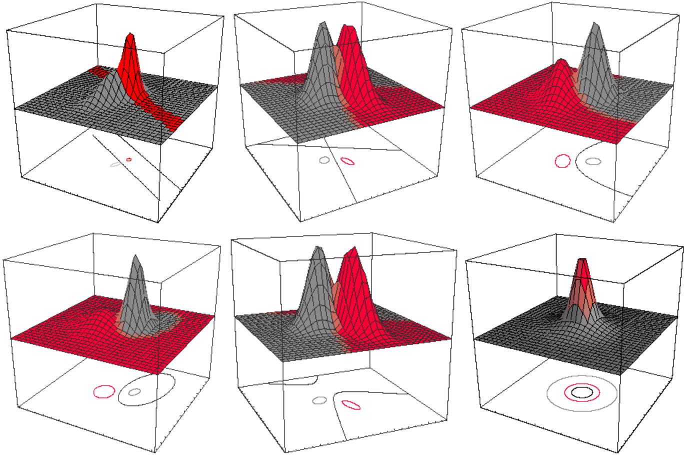

Figure: Arbitrary Gaussian distributions lead to Bayes decision boundaries that are general hyperquadrics.

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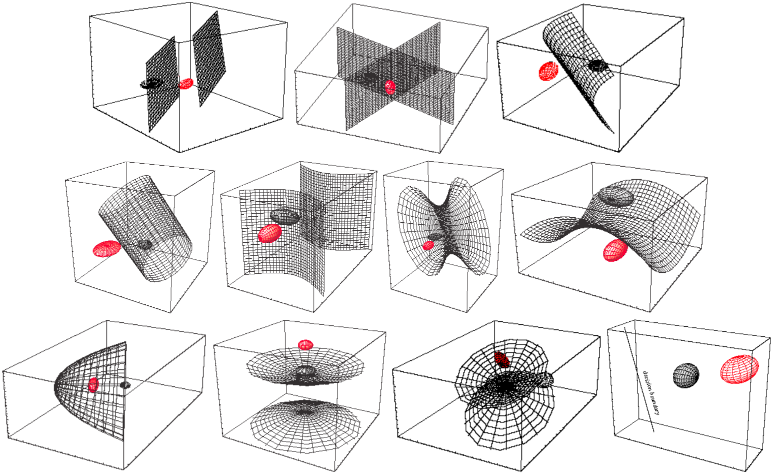

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Contents

Error Probabilities and Integrals

- For the two-category case

$$\begin{aligned} P(error) &= P(\mathbf{x} \in \mathcal{R}_2, w_1) + P(\mathbf{x} \in \mathcal{R}_1, w_2) \\ &= P(\mathbf{x} \in \mathcal{R}_2 | w_1) P(w_1) + P(\mathbf{x} \in \mathcal{R}_1 | w_2) P(w_2) \\ &= \int_{\mathcal{R}_2} p(\mathbf{x}|w_1)\, P(w_1)\, d\mathbf{x} + \int_{\mathcal{R}_1} p(\mathbf{x}|w_2)\, P(w_2)\, d\mathbf{x} \end{aligned}$$

Error Probabilities and Integrals

- For the multicategory case

$$
\begin{aligned}
P(error) &= 1 - P(correct) \\
&= 1 - \sum_{i=1}^{c} P(\mathbf{x} \in \mathcal{R}_i, w_i) \\
&= 1 - \sum_{i=1}^{c} P(\mathbf{x} \in \mathcal{R}_i | w_i) P(w_i) \\
&= 1 - \sum_{i=1}^{c} \int_{\mathcal{R}_i} p(\mathbf{x}|w_i)\, P(w_i)\, d\mathbf{x}
\end{aligned}
$$

Error Probabilities and Integrals

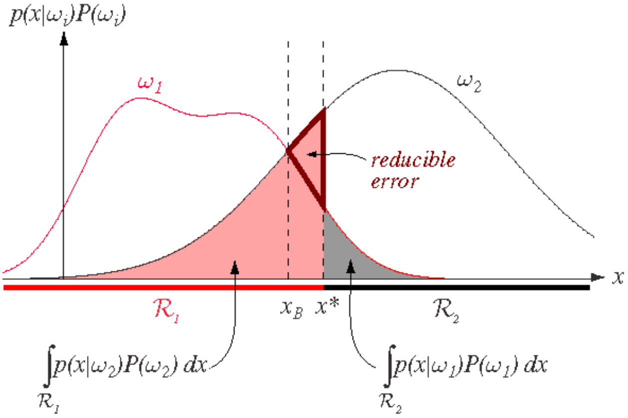

Figure: Components of the probability of error for equal priors and the non-optimal decision point x^*. The optimal point x_B minimizes the total shaded area and gives the Bayes error rate.

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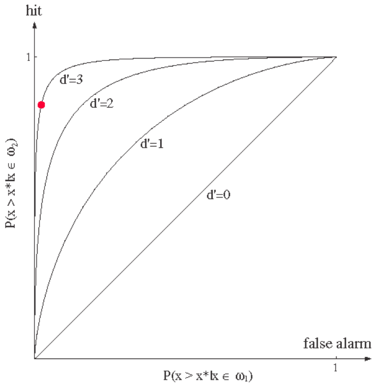
Table: *Confusion matrix*.

		Assigned	
		w_1	w_2
True	w_1	correct detection	mis-detection
	w_2	false alarm	correct rejection

Receiver Operating Characteristics

- If we use a parameter (e.g., a threshold) in our decision, the plot of these rates for different values of the parameter is called the *receiver operating characteristic* (ROC) curve.



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- To minimize the probability of error in a classification problem, choose the class that maximizes the posterior probability $P(w_j|\mathbf{x})$.
- If there are different penalties for misclassifying patterns from different classes, the posteriors must be weighted according to such penalties before taking action.
- Do not forget that these decisions are the optimal ones under the assumption that the "true" values of the probabilities are known.