MF

ACTSC 446 - Mathematics of Financial Markets

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3. Basic Stochastic Processes

3.1 Information and Filtration

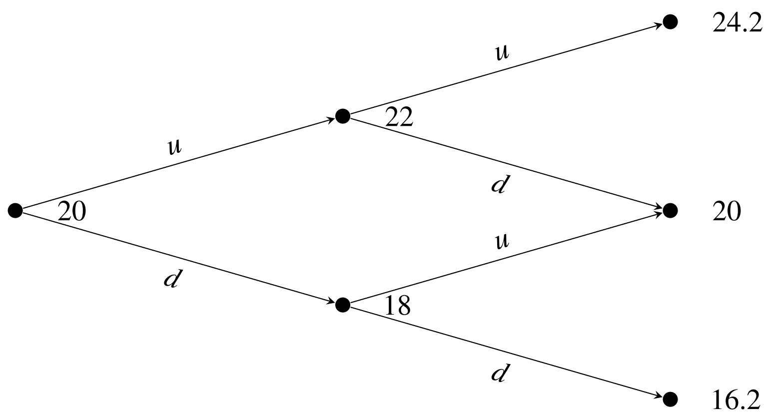
Definition 3.1.1 Let $\{X_t : t \in [0,\infty)\}$ be a continuous-time stochastic process over probability space $(\Omega, \mathcal{F}, P)$ (STAT330 (S) Remark 1.1.6). The **information set** at time t, denoted $\mathcal{F}_t$, represents everything we know about X_t's sample path.

We assume that

$$\mathcal{F}_s \subseteq \mathcal{F}_t$$

for all $0 \le s \le t$, i.e. no information is forgotten.

■ **Example 3.1.2** Consider the following 2-period binomial model.



Here $\Omega = \{\omega_i : 1 \le i \le 4\}$ where ω_1 is the *uu* path, ω_2 is the *ud* path, ω_3 is the *du* path and ω_4

is the dd path. We also have

$$\mathcal{F}_0 = \{\Omega, \emptyset\}$$
$$\mathcal{F}_1 = \{\Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \emptyset\}$$
$$\mathcal{F}_2 = \mathcal{P}(\Omega), \text{ the power set of } \Omega$$

■

Definition 3.1.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space. A collection $\{\mathcal{F}_t : t \in [0,\infty)\}$ of σ-algebras over Ω is called a **filtration** when $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t < \infty$. A probability space with such a filtration is a **filtered probability space** and is denoted $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$. A continuous time stochastic process over $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$ is **adapted** to $\{\mathcal{F}_t\}_t$ if every X_t (from the underlying stochastic process) is $\mathcal{F}_t$-measurable, i.e. for every possible values $r \in \mathbb{R}$, $X_t^{-1}(\{x \in \mathbb{R} : x \leq r\}) \in \mathcal{F}_t$.

Definition 3.1.4 A stochastic process $X = \{X_t : t \in [0,\infty)\}$ defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$ is called a **martingale** with respect to $\{\mathcal{F}_t\}_t$ if

1. X is adapted to $\{\mathcal{F}_t\}_t$.
2. $E(|X_t|) < \infty$ for all $t \in [0,\infty)$.
3. (**martingale property**) $E(X_t|\mathcal{F}_s) = X_s$ almost surely for all $0 \leq s < t < \infty$.

Remark 3.1.5 In measure theory terms, we have the concept "almost everywhere" with respect to a particular measure.

So $E(X_t|\mathcal{F}_s) = X_s$ almost surely means

$$\Pr(E(X_t|\mathcal{F}_s) \neq X_s) = 0.$$

■ **Example 3.1.6** Let $X := \{X_t : t \in [0,\infty)\}$ be a stochastic process based on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$. Define random variable

$$Z_t := E(X|\mathcal{F}_t).$$

Assume $E(|X|) < \infty$. Consider the stochastic process

$$Z := \{Z_t : t \in [0,\infty)\}.$$

Because each $\mathcal{F}_t$ is a σ-algebra in Ω, each Z_t is $\mathcal{F}_t$-measurable, hence Z is adapted to $\{\mathcal{F}_t\}_t$. Next,

$$E(|Z_t|) = E(|E(X|\mathcal{F}_t)|) \leq E(E(|X||\mathcal{F}_t)) = E(|X|)$$

by Law of Total Expectation. By assumption,

$$E(|Z_t|) \leq E(|X|) < \infty.$$

Finally if $s < t$, then

$$E(Z_t|\mathcal{F}_s) = E(E(X|\mathcal{F}_t)|\mathcal{F}_s) = E(X|\mathcal{F}_s) = Z_s$$

by another application of Law of Total Expectation.

Hence Z is a martingale. ■

Remark 3.1.7 Intuitively, a stochastic process behaves like a martingale if it follows no discernable pattern, i.e. the best forecast of a future value is the currently observed value.

Formally, for an arbitrary $u > 0$, if $\{X_t\}_{t \geq 0}$ is a martingale, then

$$E(X_{t+u} - X_t|\mathcal{F}_t) = E(X_{t+u}|\mathcal{F}_t) - E(X_t|\mathcal{F}_t) = E(X_t|\mathcal{F}_t) - E(X_t|\mathcal{F}_t) = 0.$$

A martingale is defined with respect to a filtration and a probability measure. A non-martingale process may be converted into a martingale through a change of measure.

3.2 Brownian Motion

Definition 3.2.1 A continuous-time stochastic process $\{W_t : t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$ is a **standard one-dimensional Brownian motion** if

1. $W_0(\omega) = 0$ for all $\omega \in \Omega$.
2. The sample paths $t \mapsto W(t, \omega)$ are continuous for all $\omega \in \Omega$.
3. For all $0 \leq s < t$, $W_t - W_s \sim N(0, t-s)$.
4. For all $0 = t_0 < t_1 < t_2 < \ldots < t_n < \infty$, we have the random variables $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, $\ldots$, $W_{t_n} - W_{t_{n-1}}$ to be independent.

Remark 3.2.2 Stock price movements are often modelled using Brownian motion due to the latter's frantal nature.

Definition 3.2.3 A Brownian motion with **drift** μ and **diffusion** coefficient σ is

$$X_t = \mu t + \sigma W_t, t \geq 0$$

where $\{W_t\}_t$ is a standard Brownian motion.

Proposition 3.2.4 A Brownian motion $\{X_t\}_t$ with drift μ and diffusion coefficient σ satisfies

$$X_t - X_s \sim N(\mu(t-s), \sigma^2(t-s))$$

for all $0 \leq s < t$.

Proof. We have $X_t - X_s = \mu t + \sigma W_t - \mu s - \sigma W_s = \mu(t-s) + \sigma(W_t - W_s)$ where $W_t - W_s \sim N(0, t-s)$. Hence

$$E(X_t - X_s) = \mu(t-s) + \sigma E(W_t - W_s) = \mu(t-s)$$

and $\mathrm{Var}(X_t - X_s) = \sigma^2 \mathrm{Var}(W_t - W_s) = \sigma^2(s-t)$ and follows a normal distribution. ■

Definition 3.2.5 A random variable X on a probability space $\Omega, \mathcal{F}, P)$ is **independent of a σ-algebra** $\mathcal{F}_0 \subseteq \mathcal{P}(\Omega)$ if for any event $A \in \Omega$ corresponding to $X \in \mathcal{B}$ where $\mathcal{B}$ is a Borel set in $\mathbb{R}$, and any $C \in \mathcal{F}_0$, we have

$$\Pr(A \cap C) = \Pr(A)\Pr(C).$$

Definition 3.2.6 Let $\{W_t\}_{t\geq 0}$ be a 1-dimensional standard Brownian motion on $(\Sigma, \mathcal{F}, P)$, then a **filtration for** $\{W_t\}_{t\geq 0}$ is a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on $(\Sigma, \mathcal{F}, P)$ such that

1. $\{W_t\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.
2. For all $0 \leq s < t$, the increment $W_t - W_s$ is independent of the σ-algebra $\mathcal{F}_s$

Remark 3.2.7 A filtration on a standard Brownian motion has the property that future increments do not depend on the information today.

Definition 3.2.8 Let $X_1, \ldots, X_n$ be rv's on $(\Omega, \mathcal{F}, P)$. A **filtration generated by** $X_1, \ldots, X_n$ is the σ-algebra over Ω generated by $(\Omega, \mathcal{F}, P)$, i.e. the collection of the inverse images of the Borel sets of $\mathbb{R}$:

$$\{X_i^{-1}(S) : S \in \mathcal{B}(\mathbb{R}), 1 \leq i \leq n\}.$$

Definition 3.2.9 For a standard Brownian motion $\{W_t\}_{t\geq 0}$ over $(\Omega, \mathcal{F}, P)$, the filtration generated by $\{W_t\}_{t\geq 0}$ is a filtration for $\{W_t\}_{t\geq 0}$.

Proof. By Def. 3.2.8, clearly each W_t is measurable because the σ-algebras are generated with the inverse images of W_t, so the filtration is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

For the second property, let $A \in \Omega$ such that A corresponds to $W_t - W_s \in \mathcal{B}(\mathbb{R})$, and let $C \in \mathcal{F}_s$, where $\mathcal{F}_s$ is the σ-algebra generated by W_s. Note that $W_s = W_s - W_0$ and by Def. 3.2.1(3), $W_t - W_s$ is independent of $W_s - W_0$. Hence A and C are independent. ■

Remark 3.2.10 Another way of saying Def. 3.2.5, given Def. 3.2.8, is to say that the σ-algebra generated by X is independent of $\mathcal{F}$.

Definition 3.2.11 Let $\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1, \mathcal{F}_2$ be sub-σ-algebras of F. F_1, F_2 are **independent σ-algebras** if

$$F_1 \perp F_2$$

for all $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$.

Proposition 3.2.12 Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration for $\{W_t\}_{t\geq 0}$, then

1. $E(W_t) = 0$ and $\mathrm{Var}(W_t) = t$ for all $t \geq 0$.
2. For all $0 \leq s \leq t$:
   (2.1) $E(W_t|\mathcal{F}_s) = W_s$.
   (2.2) $\mathrm{Var}(W_t|\mathcal{F}_s) = t - s$.
   (2.3) $\mathrm{Corr}(W_t, W_s) = \min(s,t) = s$.
3. $\{W_t\}_{t\geq 0}$ is a martingale with respect to $\{F_t\}_{t\geq 0}$.

Proof. (1)By Def. 3.2.1(3), $W_t - W_0 \sim N(0,t)$, so

$$E(W_t) = E(W_t - W_0) = E(W_t) - E(W_0) = 0 - 0 = 0$$

where $\mathrm{Var}(W_t) = \mathrm{Var}(W_t - W_0) = t$.
(2.1)

$$\begin{aligned} E(W_t|\mathcal{F}_s) &= E(W_t - W_s + W_s|\mathcal{F}_s) \\ &= E(W_t - W_s|\mathcal{F}_s) + E(W_s|\mathcal{F}_s) \\ &= 0 + W_s \\ &= W_s. \end{aligned}$$

(2.2)

$$\begin{aligned} \mathrm{Var}(W_t|\mathcal{F}_s) &= \mathrm{Var}(W_t - W_s + W_s|\mathcal{F}_s) \\ &= \mathrm{Var}(W_t - W_s|\mathcal{F}_s) + \mathrm{Var}(W_s|\mathcal{F}_s) \text{ by independence} \\ &= t - s + 0 \\ &= t - s. \end{aligned}$$

(2.3)

$$\begin{aligned} \mathrm{Corr}(W_t, W_s) &= E(W_t W_s) - E(W_t)E(W_s) \\ &= E(W_t W_s) \\ &= E((W_t - W_s + W_s)W_s) \\ &= E((W_t - W_s)W_s) + E(W_s^2) \\ &= E((W_t - W_s)(W_s - W_0)) + \mathrm{Var}(W_s) \\ &= 0 + \mathrm{Var}(W_s) \text{ by independence} \\ &= s. \end{aligned}$$

(3) Clearly $\{W_t\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{f\geq 0}$ because $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by $\{W_t\}_{t\geq 0}$. Moreover, equation 2.1 proves the martingale property, so it suffices to show that each

$$E(|W_t|) < \infty.$$

By the Cauchy-Schwarz Inequality we have

$$E(|W_t|) \leq \left(E(|W_t|^2)\right)^{\frac{1}{2}} = \sqrt{t} < \infty.$$

This completes the proof. ■

Proposition 3.2.13 The sample paths $t \mapsto W_t(\omega)$ for a fixed $\omega \in \Omega$ are continuous but nowhere differentiable.

Proof. We omit the proof. ■

Definition 3.2.14 Let $f : [0,T] \to \mathbb{R}$ be a functionn and $\Pi = \{t_i : 0 \leq i \leq n, 0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0,T]$. The **total variation** of f is

$$TV(f) = \lim_{||\Pi||\to 0} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

the **quadratic variation** of f is

$$QV(f) = [f,f]_T = \lim_{||\Pi||\to 0} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^2$$

where

$$||\Pi|| = \max_{1\leq i\leq n} t_i - t_{i-1}$$

is the mesh of the partition.

f is **of bounded variation** if $TV(f) < \infty$, and of **unbounded variation** if otherwise.

Theorem 3.2.15 The sample paths of a Brownian motion $\{W_t\}_{t\geq 0}$, $t \mapsto W_t(\omega)$, are of unbounded variation but

$$QV(W) = (W,W)_T = T$$

for all $T \geq 0$ with probability 1.

Proof. We omit the proof. ■

Theorem 3.2.16 Continuously differentiable functions have quadratic variations of 0.

Proof. We omit the proof. ■

Theorem 3.2.17 Let $\{W_t\}_{t\geq 0}$ be a Brownian motion, then

1. $[W,t]_T = \lim_{||\Pi||\to 0} \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})(t_i - t_{i-1}) = 0.$
2. $[t,t]_T = \lim_{||\Pi||\to 0} \sum_{i=1}^{n} (t_i - t_{i-1})^2 = 0.$

Proof. We omit the proof. ■

Remark 3.2.18 Informally, we write the above results to be:
Thm. 3.2.15: $d(W,W)_t = dW_t dW_t = (dW_t)^2 = dt$.
Thm. 3.2.17(1): $d(W,T)_t = dW_t dt = 0$.
Thm. 3.2.17(2): $d(T,T)_t = dtdt = 0$.

Proposition 3.2.19 Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion, then the process

$$\{W_t^2 - t\}_{t\geq 0}$$

is a martingale with respect to the filtration generated by $\{W_t\}_{t\geq 0}$.

Proof. Since $\{W_t\}_{t\geq 0}$ is adapted, so is $\{W_t\}_{t\geq 0}$ and therefore so is $\{W_t - t\}_{t\geq 0}$.

Next, $E(|W_t^2 - t|) \leq E(|W_t^2|) + t$ by the triangle inequality, and so in turn

$$E(|W_t^2 - t|) \leq E(W_t^2) + t = t + t = 2t < \infty.$$

Finally,

$$\begin{aligned}
&E(W_t^2 - t|\mathcal{F}_s) \text{ for } 0 \leq s < t \\
=&E((W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 - t|\mathcal{F}_s) \\
=&E((W_t - W_s)^2|\mathcal{F}_s) + 2W_s E(W_t - W_s|\mathcal{F}_s) + E(W_s^2|\mathcal{F}_s) - t \\
=&\operatorname{Var}(W_t - W_s) + 2W_s \cdot 0 + W_s^2 - t \\
=&t - s + W_s^2 - t \\
=&W_s^2 - s
\end{aligned}$$

which satisfies the martingale property. ■

3.3 The Ito Integral and the Ito-Doeblin Lemma

Definition 3.3.1 Let $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$ be a filtered probability space such that $\{\mathcal{F}_t\}_t$ is a filtration for a standard Brownian motion $\{W_t\}_t$. Let f_t be a function of random variables depending on t such that

$$E\left(\int_0^t f_u^2 du\right) < \infty.$$

Then we say $\{f_t\}_t$ is a **square-integrable process** and we define the **Ito integral of** $\{f_t\}_t$ to be the random variable

$$I_t = \int_0^t f_u dW_u = \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in\Pi_n} H_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$$

where $\Pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ is an n-partition of the interval $[0,t]$ and the convergence of the limit is convergence in probability.

Proposition 3.3.2 Let $\{I_t\}_t$ be the Ito integral for $\{f_t\}_t$, then
1. I_t is a continuous function of t.
2. For each t, I_t is $\mathcal{F}_t$-measurable, i.e. for every Borel set $B \in \mathcal{B}(\mathbb{R})$, $I_t^{-1}(B) \in \mathcal{F}_t$.

3. For any constant $c \in \mathbb{R}$,

$$c\int_0^t f_u dW_u = \int_0^t c f_u dW_u.$$

4. For square-integrable processes $(f_t)_t$ and $(g_t)_t$,

$$\int_0^t f_u dW_u + \int_0^t g_u dW_u = \int_0^t (f_u + g_u) dW_u.$$

Proof. We omit the proof. ■

Theorem 3.3.3 Let $\{I_t\}_t$ be the Ito integral on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$. $\{I_t\}_t$ is a martingale process with respect to $\{\mathcal{F}_t\}_t$, where $\{\mathcal{F}_t\}_t$ is adapted to the underlying standard Brownian motion of $\{I_t\}_t$.

Proof. We omit the proof. ■

Corollary 3.3.4 With the same setup as Thm. 3.3.3, we have
1. $E(I_t) = 0$ for all t.
2. $E\left(\int_s^t f_u dW_u | \mathcal{F}_s\right) = 0$ for all $0 \leq s < t$.

Proof. 1. By the martingale property

$$E(I_t) = E(I_t|\mathcal{F}_0) = I_0 = \int_0^0 f_u dW_u = 0.$$

2. We have

$$\begin{aligned}
&E\left(\int_s^t f_u dW_u | \mathcal{F}_s\right) \\
=&E\left(\int_0^t f_u dW_u | \mathcal{F}_s\right) - E\left(\int_0^s f_u dW_u | \mathcal{F}_s\right) \\
=&E(I_t|\mathcal{F}_s) - I_s \\
=&I_s - I_s \text{ by part (1)} \\
=&0
\end{aligned}$$

■

Theorem 3.3.5 — Ito Isometry. Let $\{I_t\}_t$ be the Ito integral with respect to a process $\{f_t\}_t$, then

$$E(I_t^2) = E\left(\int_0^t f_u^2 du\right) < \infty.$$

Proof. The $< \infty$ part is a consequence of the fact that $\{f_t\}_t$ is square integrable. The proof of the first part, i.e.

$$E\left(\left(\int_0^t f_u dW_u\right)^2\right) = E\left(\int_0^t f_u^2 du\right),$$

is omitted. ■

Corollary 3.3.6 With the same setup as Thm. 3.3.5, we have

1. $\mathrm{Var}(I_t) = E(I_t^2) = E\left(\int_0^t f_u^2 du\right)$.
2. $\mathrm{Var}(I_t|\mathcal{F}_s) = E\left(\int_s^t f_u^2 du|\mathcal{F}_s\right)$ for $0 \le s < t$.

Proof. 1.

$$\mathrm{Var}(I_t) = E(I_t^2) - E(I_t)^2 = E\left(\int_0^t f_u^2 du\right) - 0^2 = E\left(\int_0^t f_u^2 du\right)$$

where the second step is by Thm. 3.3.5 and Corollary 3.3.4.

2.

$$\begin{aligned}
&\mathrm{Var}(I_t|\mathcal{F}_s)\\
=&\mathrm{Var}\left(I_s + \int_s^t f_u dW_u|\mathcal{F}_s\right)\\
=&E\left(\left(I_s + \int_s^t f_u dW_u\right)^2|\mathcal{F}_s\right) - E\left(I_s + \int_s^t f_u dW_u|\mathcal{F}_s\right)^2\\
=&E(I_s^2|\mathcal{F}_s) + 2E\left(I_s\int_s^t f_u dW_u|\mathcal{F}_s\right) + E\left(\left(\int_s^t f_u dW_u\right)^2|\mathcal{F}_s\right) - E(I_t|\mathcal{F}_s)^2\\
=&0 + 2(0) + E\left(\left(\int_s^t f_u dW_u\right)^2|\mathcal{F}_s\right)\\
=&E\left(\int_s^t f_u dW_u|\mathcal{F}_s\right)
\end{aligned}$$

as required.

■

Definition 3.3.7 Let $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$ be a filtered probability space where $\{\mathcal{F}_t\}_t$ is adapted to a standard Brownian motion $\{W_t\}_t$, then a stochastic process $\{X_t\}_t$ is an **Ito process** if X_t has the form

$$dX_t = \alpha_t dt + \sigma_t dW_t$$

where:

dX_t, dt, dW_t denote the infinitismal change of X_t, time and W_t respectively,

α_t is a process depending on t such that it is adapted to $\{\mathcal{F}_t\}_t$ and

$$E\left(\int_0^t |\alpha_u| du\right) < \infty,$$

σ_t is a stochastic process adaptd to $\{\mathcal{F}_t\}_t$.

We may write the Ito process in **integral form**

$$X_t - x_0 = \int_0^t \alpha_u du + \int_0^t \sigma_u dW_u$$

where x_0 is a non-random constant.

We call α_t the **drift**, and σ_t the **diffusion** or **volatility** of the Ito process.

We call

$$dX_t = \alpha_t dt + \sigma_t dW_t$$

to be the **differential form** of the Ito process.

Lemma 3.3.8 Let $\{X_t\}_t$ be an Ito process with

$$dX_t = \alpha_t dt + \sigma_t dW_t,$$

then

$$(dX_t)^2 = \sigma_t^2 dt.$$

Proof. We have

$$(dX_t)^2 = \alpha_t^2 (dt)^2 + 2\alpha_t \sigma^2 dt dW_t + \sigma_t^2 (dW_t)^2 = \alpha_t^2 \cdot 0 + 2\alpha_t \sigma^2 \cdot 0 + \sigma_t^2 dt$$

by Remark 3.2.18. Consequently

$$(dX_t)^2 = \sigma_t^2 dt.$$

■

Theorem 3.3.9 — Ito-Doeblin Lemma. Let $\{X_t\}_t$ be an Ito process defined on $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$ with $dX_t = \alpha_t dt + \sigma_t dW_t$ and

$$f : [0,T] \times \mathbb{R} \to \mathbb{R}$$

be a function such that

$$f_t := \frac{\partial f}{\partial t}, f_x := \frac{\partial f}{\partial x}, f_{xx} := \frac{\partial f^2}{\partial x \partial x}$$

are well-defined and continuous, then

$$Y_t := f(t, X_t)$$

is also an Ito process with drift equal to

$$f_t(t,X_t) + f_x(t,X_t)\alpha_t + \frac{1}{2} f_{xx}(t,X_t)\sigma_t^2$$

and diffusion

$$f_x(t,X_t)\sigma_t.$$

In other words

$$\begin{aligned} dY_t &= df(t,X_t) \\ &= \left(f_t(t,X_t) + f_x(t,X_t)\alpha_t + \frac{1}{2} f_{xx}(t,X_t)\sigma_t^2 \right) dt + f_x(t,X_t) dt \sigma_t dW_t \\ &= f_t(t,X_t)dt + f_x(t,X_t)dX_t + \frac{1}{2} f_{xx}(t,X_t)(dX_t)^2 \text{ where } (dX_t)^2 = \sigma_t^2 dt \text{ by Lemma 3.3.8} \end{aligned}$$

Proof. We omit the proof. ■

■ **Example 3.3.10** Suppose we would like to compute

$$\int_0^T W_t dW_t$$

where $\{W_t\}_t$ is a standard Brownian motion.

Using Ito-Doeblin's Lemma, take $X_t = W_t$ and define

$$\begin{aligned} f : [0,T] \times \mathbb{R} &\to \mathbb{R} \\ (t,x) &\mapsto x^2 \end{aligned}$$

and $Y_t = f(t, X_t) = X_t^2$. It follows that

$$f_t(t,x) = 0, f_x(t,x) = 2x, f_{xx}(t,x) = 2.$$

By Ito-Doeblin's lemma,

$$dY_t = d(W_t)^2 = 0dt + 2W_t dW_t + \frac{1}{2}(2)(1)dt$$

since $\sigma_t = 1$ ($\{W_t\}_t$ is the standard Brownian motion).

Therefore

$$\int_0^T d(W_t)^2 = \int_0^T 2W_t dW_t + \int_0^T dt$$

and so

$$\begin{aligned} W_T^2 - W_0^2 &= 2\int_0^T W_t dW_t + T \\ \Leftrightarrow W_T^2 - T &= 2\int_0^T W_t dW_t \\ \Leftrightarrow \int_0^T W_t dW_t &= \frac{1}{2}(W_T^2 - T) \end{aligned}$$

■

3.4 Arithmetic and Geometric Brownian Motion Models

Definition 3.4.1 A stochastic process $\{X_t\}_t$ satisfying the stochastic differential equation

$$dX_t = \alpha dt + \sigma dW_t$$

where α, σ are constants, and $\{W_t\}_t$ is a standard Brownian motion, is called an **arithmetic Brownian motion (ABM)**.

Proposition 3.4.2 Let $\{X_t\}_t$ ba an ABM, then

1. $X_t = X_0 + \alpha t + \sigma W_t$.
2. $E(X_t) = X_0 + \alpha t$.
3. $\mathrm{Var}(X_t) = \sigma^2 t$.
4. $X_t \sim N(X_0 + \alpha t, \sigma^2 t)$.
5. For $s \leq t$, $E(X_t|\mathcal{F}_s) = X_s + \alpha(t-s)$.

Proof. 1. In differential form we have

$$dX_t = \alpha dt + \sigma dW_t$$

and integrating both sides yields

$$\int_0^t X_u du = \int_0^t \alpha dt + \int_0^t \sigma dW_u$$

where $\int_0^t \sigma dW_u$ is an Ito integral. This yields

$$X_t - X_0 = \alpha t + \sigma W_t$$

as required.

2. $E(W_t) = 0$ by Proposition 3.2.12(1).
3. Proposition 3.2.12(1) has $\text{Var}(W_t) = t$.
4. We have $W_t \sim N(0,t)$ by Def. 3.2.1. The rest follows.
5.

$$\begin{aligned} E(X_t|\mathcal{F}_s) &= E\left(X_s + \int_s^t \alpha du + \int_s^t \sigma dW_u|\mathcal{F}_s\right) \\ &= E(X_s) + \alpha(t-s) + \sigma E(W_t - W_s|\mathcal{F}_s) \\ &= X_s + \alpha(t-s) \end{aligned}$$

■

Corollary 3.4.3 It an ABM $\{X_t\}_t$ has drift 0, then $\{X_t\}_t$ is a martingale.

Proof. By Proposition 3.4.2(5), we have

$$E(X_t|\mathcal{F}_s) = X_s$$

if the drift $\alpha = 0$. By definition, $\{X_t\}_t$ is a martingale. ■

Remark 3.4.4 If we use ABM to model asset returns, we get

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where S_t is the price of the asset at time t. This motivates the following definition.

Definition 3.4.5 Suppose $\{S_t\}_t$ is a stochastic process that follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\{W_t\}_t$ is the standard Brownian motion, then $\{S_t\}_t$ is said to follow a **geometric Brownian motion (GBM)**.

Theorem 3.4.6 Let $\{S_t\}_t$ follow a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

then the unique solution to this stochastic differential equation is

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right),$$

and furthermore for $0 \le t \le T$,

$$S_T = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right).$$

Proof. Define function $f(t, S_t) = \log(S_t)$ and $X_t = S_t$, then

$$f_t = 0, f_s = \frac{1}{s}, f_{ss} = -\frac{1}{s^2}.$$

By the Ito-Doeblin Lemma, we get

$$\begin{aligned}
d\log(S_t) &= f_t dt + f_s dS_t + \frac{1}{2} f_{ss}(dS_t)^2 \\
&= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2 \\
&= \frac{1}{s}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2S_t^2}(\mu S_t dt + \sigma S_t dW_t)^2 \\
&= \mu dt + \sigma dW_t - \frac{1}{2S_t^2}(\mu^2 S_t^2 (dt)^2 + 2\mu S_t^2 \sigma dt dW_t + \sigma^2 S_t^2 (dW_t)^2) \\
&= \mu dt + \sigma dW_t - \frac{1}{2S_t^2}(0 + 0 + \sigma^2 S_t^2 dt) \text{ by Remark 3.2.18} \\
&= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t.
\end{aligned}$$

Integrating both sides gives

$$\int_0^t d\log(S_u) = \int_0^t \mu - \frac{\sigma^2}{2} du + \int_0^t \sigma dW_u$$

and consequently

$$\log(S_t) - \log(S_0) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

which gives

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), t \ge 0.$$

Integrating $d\log(S_t)$ above from t to T gives

$$\int_t^T d\log(S_u) = \int_t^T \mu - \frac{\sigma^2}{2} du + \int_t^T \sigma dW_u,$$

and similar to before,

$$\log(S_T) - \log(S_t) = \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)$$

and

$$S_T = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right), 0 \le t \le T,$$

which completes the proof. ■

Definition 3.4.7 If X is a random variable such that

$$\log(X) \sim N(\mu, \sigma^2)$$

for some μ, σ^2, then X is said to follow a **log-normal distribution** of mean *mu* and standard deviation σ. We write $X \sim \mathrm{LogN}(\mu, \sigma^2)$.

Proposition 3.4.8 If $X \sim \mathrm{LogN}(\mu, \sigma^2)$, then

$$E(X) = e^{\mu + \frac{1}{2}\sigma^2}$$

and

$$\mathrm{Var}(X) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}.$$

Proof. We omit the proof. ■

Proposition 3.4.9 If $\{S_t\}_t$ is a GBM with

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

then for all $t \geq 0$,

$$E(S_t) = S_0 e^{-\mu t}$$

and for all $0 \leq t \leq T$,

$$E(S_T | \mathcal{F}_t) = S_t e^{\mu(T-t)}.$$

Proof. By Thm. 3.4.6

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}.$$

Let

$$Z_t = \log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t,$$

then, since

$$W_t \sim N(0, t),$$

we have

$$Z_t \sim N\left(\log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right).$$

Note that by construction, $S_t = e^{Z_t}$, so

$$S_t \sim \mathrm{LogN}\left(\log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right),$$

and by Proposition 3.4.8,

$$E(S_t) = S_0 e^{\mu t - \frac{\sigma^2}{2}t + \frac{1}{2}\sigma^2 t} = S_0 e^{\mu t}$$

as desired.

On the other hand, Thm. 3.4.6 also has that

$$S_T = S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T - W_t)}.$$

Given $\mathcal{F}_t$, $t \leq T$, we have

$$\log(S_T) = \log(S_t) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma Z,$$

where $Z \sim N(0, T-t)$ by the property of Brownian motion. It follows that

$$S_T|\mathcal{F}_t \sim \mathrm{LogN}\left(\log(S_t) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right).$$

and subsequently

$$E(S_T|\mathcal{F}_t) = S_t e^{\mu(T-t) - \frac{\sigma^2}{2}(T-t) + \frac{1}{2}\sigma^2(T-t)} = S_t e^{\mu(T-t)}$$

as desired. ■

4. Continuous-Time Financial Models

4.1 The Black-Scholes Model

Definition 4.1.1 The **Black-Scholes model** has 2 assets, a risk-free asset with price process $\{B_t\}_t$, a risky asset with price process $\{S_t\}_t$ which does not pay dividend, from time $t \in [0,T]$, in a filtered probability space $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$, and a corresponding Brownian motion $\{W_t^P\}_t$ with respect to P, where P, called the **physical probability measure**, is such that

$$dB_t = rB_t dt$$

for a fixed risk-free rate r, and $\{S_t\}_t$ follows a geometric Brownian motion

$$dS_t = \alpha S_t dt + \sigma S_t dW_t^P$$

for some constants α and σ.

A **trading strategy** is a process $\{h_t\}_t$ where each $h_t := (h_t^B, h_t^S)$ represents h_t^B units of the risk-free asset held in interval $[t, t+\Delta t)$ and h_t^S units of the risky asset held in interval $[t, t+\Delta t)$. Each h_t has **valuation**

$$V_t^h = h_t^B B_t + h_t^S S_t.$$

The process $\{V_t^h\}_t$ of trading strategy $\{h_t\}_t$ is the **valuation process** of the trading strategy.

Definition 4.1.2 A trading strategy $\{h_t\}_t$ in the Black-Scholes model is **self-financing** if

$$dV_t^h = h_t^S dS_t + h_t^B dB_t,$$

and it is an **arbitrage opportunity** if it is self-financing with $V_0^h \leq 0$, $P(V_T^h \geq 0) = 1$, and $P(V_T^h > 0) > 0$.

The model is **arbitrage free** if there does not exist arbitrage opportunities.

Lemma 4.1.3 Suppose a contingency claim in the Black-Scholes model has pricing process $\{\Pi_t\}_t$ where each Π_t is a function of t and the risky asset price S_t: $\Pi_t = F(t, S_t)$, and that there exists a replicating portfolio $\{h_t\}_t$ that is also self-financing, then we have

1.
$$\alpha h_t^S S_t + r h_t^B B_t = \frac{\partial F}{\partial t} + \alpha S_t \frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2.$$

2.
$$\sigma h_t^S S_t = \frac{\partial F}{\partial S}\sigma S_t$$

and consequently $h_t^S = \frac{\partial F}{\partial S}$.

Proof. Since $\{h_t\}_t$ is self-financing, we have

$$dV_t^h = h_t^S dS_t + h_t^B dB_t$$

where $dS_t = \alpha S_t dt + \sigma S_t dW_t^P$ and $dB_t = rB_t dt$ by definition. Expand to get

$$\begin{aligned} dV_t^h &= h_t^S(\alpha S_t dt + \sigma S_t dW_t^P) + h_t^B r B_t dt \\ &= (\alpha h_t^S S_t dt + h_t^B r B_t)dt + \sigma S_t h_t^S dW_t^P. \end{aligned}$$

On the other hand, $\{S_t\}_t$ is an Ito process, so by the Ito-Doeblin Lemma, $\{\Pi_t\}_t = \{F(t, S_t)\}_t$ is an Ito process with

$$d\Pi_t = dF(t, S_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS_t)^2.$$

Substitute in $dS_t = \alpha S_t dt + \sigma S_t dW_t^P$ to get

$$d\Pi_t = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}(\alpha S_t dt + \sigma S_t dW_t^P) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(\sigma^2 S_t^2 dt)$$

where the last term above is a consequence of Remark 3.2.18. Simplify this further to get

$$d\Pi_t = \left(\frac{\partial F}{\partial t} + \alpha S_t \frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2\right)dt + \frac{\partial F}{\partial S}\sigma S_t dW_t^P.$$

Now, $\{h_t\}_t$ replicates $\{\Pi_t\}_t$, so $dV_t^h = d\Pi_t$, and we equate their expanded forms to get

$$\begin{aligned} &(\alpha h_t^S S_t dt + h_t^B r B_t)dt + \sigma S_t h_t^S dW_t^P = dV_t^h \\ =&d\Pi_t = \left(\frac{\partial F}{\partial t} + \alpha S_t \frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2\right)dt + \frac{\partial F}{\partial S}\sigma S_t dW_t^P. \end{aligned}$$

Equating drift yields

$$\alpha h_t^S S_t + r h_t^B B_t = \frac{\partial F}{\partial t} + \alpha S_t \frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2$$

which is equation 1, and equating diffusion yields

$$\sigma S_t h_t^S = \frac{\partial F}{\partial S}\sigma S_t,$$

which is equation 2. This completes the proof. ■

Theorem 4.1.4 — The Black-Scholes Partial Differential Equation. Suppose a contingency claim in the Black-Scholes model has pricing process $\{\Pi_t\}_t$ where each Π_t is a function of t and the risky asset price S_t: $\Pi_t = F(t, S_t)$, and that there exists a replicating portfolio $\{h_t\}_t$ that is also self-financing, then we have

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 = rF$$

where $F(T, S_T) = \Pi_T$ is the payoff of the contingency at time T.

Proof. Lemma 4.1.3(1) gives $h_t^S = \frac{\partial F}{\partial S}$, but via the definition of the valuation process we also have

$$h_t^B B_t = V_t^h - h_t^S S_t.$$

Hence $h_t^B B_t = V_t^h - \frac{\partial F}{\partial S} S_t$. Moreover, because $\{h_t\}_t$ replicates $\{\Pi_t\}_t$, we have

$$h_t^B B_t = \Pi_t - \frac{\partial F}{\partial S} S_t = F(t, S_t) - \frac{\partial F}{\partial S} S_t.$$

Take Lemma 4.1.3(1) and substitute $h_t^B B_t = \Pi_t - \frac{\partial F}{\partial S} S_t$ and $h_t^S = \frac{\partial F}{\partial S}$ to get

$$\alpha\frac{\partial F}{\partial S}S_t + rF - r\frac{\partial F}{\partial S}S_t = \frac{\partial F}{\partial t} + \alpha S_t\frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S_t^2.$$

This in turn yields

$$rF = \frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2$$

as required. ■

Corollary 4.1.5 In a replicating portfolio $\{h_t\}_t$ of a contingency claim $\{\Pi_t\}_t$ in a Black-Scholes model, we have

$$h_t = (h_t^B, h_t^S) = \left(\frac{1}{B_t}\left(F - \frac{\partial F}{\partial S}S_t\right), \frac{\partial F}{\partial S}\right)$$

where $F(t, S_t) = \Pi_t$ for all $t \in [0, T]$.

Proof. This is in the proof of Thm. 4.1.4. ■

Theorem 4.1.6 — Feynman-Kac Theorem. Consider the partial differential equation

$$\frac{\partial}{\partial t}F(t,x) + \mu(t,x)\frac{\partial}{\partial t}F(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}F(t,x) = V(t,x)F(t,x) - f(t,x)$$

defined for all $x \in \mathbb{R}$ and $t \in [0, T]$ subject to the boundary condition

$$F(T,x) = \Phi(x)$$

where μ, σ, Φ, V, f are known functions, $T \in \mathbb{R}^+$ is known, and

$$F : [0, T] \times \mathbb{R} \to \mathbb{R}$$

is the unknown function, then the solution F^* can be written as the conditional expectation

$$F^*(t,x) = E^P\left(\int_t^T e^{-\int_t^r V(u,X_u)du} f(r,X_r)dr + e^{-\int_t^T V(u,X_u)du}\Phi(X_T)\middle| X_t = x\right)$$

under the probability measure P such that X is an Ito process driven by

$$dX = \mu(X,t)dt + \sigma(t,X)dW_t^P$$

where $\{W_t^P\}$ is a Brownian motion under P and the initial condition $X(t)$ is $X(t) = x$.

Proof. We omit the proof. ■

Corollary 4.1.7 If F is the solution to the partial differential equation

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}F(t,x) = rF(t,x)$$

subjected to the boundary condition

$$F(T,x) = \Phi(x)$$

where μ, Φ, σ are known functions, $r, T \in \mathbb{R}^+$, and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$, then F has representation

$$F(t,x) = e^{-r(T-t)}E^P(\Phi(X_T)|X_t = x)$$

where X satisfies the partial differential equation

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s^P, X_t = x$$

where $\{W_s\}_s$ is a Brownian motion under P.

Proof. Let $f(t,x) = 0$ and $V(t,x) = r$ in the Feynman-Kac Theorem, and we yield the result. ■

Corollary 4.1.8 The solution to the Black-Scholes PDE in Thm. 4.1.4 has the form

$$\Pi_t = F(t,S_t) = e^{-r(T-t)}E^Q(\Phi(S_T)|\mathcal{F}_t)$$

where S_t is the spot price at time t that follows the Ito process

$$dS_t = rS_t dt + \sigma S_t dW_t^Q$$

and $\{\mathcal{F}_t\}_t$ is the filtration adapted to the Brownian motion $\{W_t\}_t$ with respect to probability measure Q.

Proof. Define functions $\mu(t,x) = rx$, $\sigma(t,x) = \sigma x$, and fix S_t to be a price at a certain fixed time t. Re-write the Black-Scholes PDE as

$$\frac{\partial F}{\partial t}(t,S_t) + \mu(t,S_t)\frac{\partial F}{\partial S}(t,S_t) + \frac{1}{2}\sigma^2(t,S_t)\frac{\partial^2}{\partial S^2}F(t,S_t) = rF(t,S_t)$$

with boundary condition

$$F(T,S_T) = \Pi_T = \Phi(S_T)$$

and then apply Corollary 4.1.7 to get

$$F(t,S_t) = e^{-r(T-t)}E^Q(\Phi(S_T)|S_t = S_t).$$

Note that the condition in the conditional expectation above is simply $\mathcal{F}_t$, the information set at time t, so

$$F(t,S_t) = e^{-r(T-t)}E^Q(\Phi(S_T)|\mathcal{F}_t)$$

as required, where

$$dS_t = rS_t dt + \sigma S_t W_t^Q.$$

■

Theorem 4.1.9 — Solution to Black-Scholes PDE. Under the Black-Scholes model, the arbitrage-free price at time t of the derivative instrument ξ with maturity T and payoff $\xi_T = \Phi(S_T)$ is given by

$$\Pi_t = e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi(e^y) f_{Y_T}(y) dy,$$

where

$$Y_T|\mathcal{F}_t \sim N\left(\log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right).$$

Proof. From Corollary 4.1.8 we inferred the Ito process of the spot price $\{S_t\}_t$ if $F(t,S_t)$ is a solution to the Black-Scholes PDE. By Def. 3.4.5, $\{S_t\}_t$ follows a geometric Brownian motion with drift rS_t and diffusion σS_t. By Thm. 3.4.6 we have

$$S_T = S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T - W_t)}.$$

Write $Y_T = \log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)$ and consequently $S_T = e^{Y_T}$. Moreover because $W_T - W_t \sim N(0, T-t)$, we have

$$Y_T|\mathcal{F}_t \sim N\left(\log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right).$$

For convenience, denote

$$\tilde{\mu} = \log(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \tilde{\sigma}^2 = \sigma^2(T-t).$$

It follows that

$$S_T|\mathcal{F}_t \sim \text{LogN}(\tilde{\mu}, \tilde{\sigma}^2).$$

With the distribution of $S_T|\mathcal{F}_t$ known, we can evaluate the expression in Corollary 4.1.8 to get

$$\begin{aligned}
\Pi_t &= e^{-r(T-t)}E^Q(\Phi(S_T)|\mathcal{F}_t) \\
&= e^{-r(T-t)}E^Q(\Phi(e^{Y_T})|\mathcal{F}_t) \\
&= e^{-r(T-t)}E^Q(\Phi(e^{Y_T})|S_t) \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi(e^y) f_{Y_T}(y) dy
\end{aligned}$$

by the definition of expectation, $f_{Y_T}(y)$ being the density function of Y_T. This completes the proof. ■

Lemma 4.1.10 If $X \sim \text{LogN}(\mu, \sigma^2)$, then for all $K > 0$,

$$E(X\mathbf{1}_{\{X>K\}}) = E(X)N\left(\frac{\mu+\sigma^2-\log(K)}{\sigma}\right)$$

where $N(\cdot)$ is the distribution function of the standard normal distribution.

Proof. Write $X = e^Y$ where $Y \sim N(\mu, \sigma^2)$. We have

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}, y \in \mathbb{R},$$

to be the density function of Y. It follows that

$$E(X\mathbf{1}_{\{X>K\}}) = E(e^Y\mathbf{1}_{\{Y>\log(K)\}}) = \int_{\log(K)}^{\infty} e^y \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}dy.$$

Let $z = \frac{y-\mu}{\sigma}$. It follows that $dz = \frac{1}{\sigma}dy$ and $y = \sigma z + \mu$. Re-write the above to be

$$\begin{aligned}
E(X\mathbf{1}_{\{X>K\}}) &= \int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} e^{\mu+\sigma z}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\sigma dz \\
&= e^{\mu}\int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{2\sigma z-z^2}{2}}dz \\
&= e^{\mu}\int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{\sigma^2-(z-\sigma)^2}{2}}dz \\
&= e^{\mu+\frac{\sigma^2}{2}}\int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\sigma)^2}{2}}dz.
\end{aligned}$$

Let $u = z - \sigma$, and thus $du = dz$, $z = u + \sigma$, we get

$$E(X\mathbf{1}_{\{X>K\}}) = e^{\mu+\frac{\sigma^2}{2}}\int_{\frac{\log(K)-\mu}{\sigma}-\sigma}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du.$$

Note that $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)$ is the density function of $N(0,1)$, so

$$\begin{aligned}
E(X\mathbf{1}_{\{X>K\}}) &= e^{\mu+\frac{\sigma^2}{2}}\left(1-N\left(\frac{\log(K)-\mu}{\sigma}-\sigma\right)\right) \\
&= e^{\mu+\frac{\sigma^2}{2}}N\left(\sigma-\frac{\log(K)-\mu}{\sigma}\right) \\
&= e^{\mu+\frac{\sigma^2}{2}}N\left(\frac{\mu+\sigma^2-\log(K)}{\sigma}\right)
\end{aligned}$$

as required. ■

Lemma 4.1.11 If $X \sim \text{LogN}(\mu, \sigma^2)$, then for all $K > 0$,

$$E(X\mathbf{1}_{\{X<K\}}) = E(X)N\left(\frac{\log(K)-\mu-\sigma^2}{\sigma}\right)$$

where $N(\cdot)$ is the distribution function of the standard normal distribution.

Proof. To be completed. ■

Lemma 4.1.12 If $X \sim \text{LogN}(\mu, \sigma^2)$, then for all $K > 0$,

1.
$$E(K\mathbf{1}_{\{X>K\}}) = KN\left(\frac{\mu - \log(K)}{\sigma}\right)$$

2.
$$E(K\mathbf{1}_{\{X<K\}}) = KN\left(\frac{\log(K) - \mu}{\sigma}\right)$$

where $N(\cdot)$ is the distribution function of the standard normal distribution.

Proof. To be completed. ■

Theorem 4.1.13 In the Black-Scholes model, the price of a European call option with strike K and maturity T at time t when spot is S_t is

$$c(t, S_t, K, T) = S_t N(d_1(t, S_t)) - e^{-r(T-t)} KN(d_2(t, S_t))$$

where $N(\cdot)$ is the distribution function of the standard normal distribution, r is the risk-free rate, and

$$d_1(t, S_t) = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2(t, S_t) = d_1(t, S_t) - \sigma\sqrt{T-t}.$$

Proof. To be completed. ■

Corollary 4.1.14 In the Black-Scholes model, the price of a European put option with strike K and maturity T at time t when spot is S_t is

$$p(t, S_t, K, T) = e^{-r(T-t)} KN(-d_2(t, S_t)) - S_t N(-d_1(t, S_t))$$

where $N(\cdot)$ is the distribution function of the standard normal distribution, r is the risk-free rate, and

$$d_1(t, S_t) = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2(t, S_t) = d_1(t, S_t) - \sigma\sqrt{T-t}.$$

Proof. To be completed. ■

4.2 Risk-Neutral Pricing and Girsanov Theorem

Proposition 4.2.1 Suppose the risky asset in the Black-Scholes model $\{S_t\}_t$ satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t^Q$$

where Q is the probability measure in the solution of the Black-Scholes PDE in Corollary 4.1.8, then the series

$$\{S_t e^{-rt}, t \geq 0\}$$

is a martingale under Q.

Proof. Define $Y_t = f(t, S_t) = e^{-rt} S_t$ where $f(t,x) \mapsto e^{-rt} x$ is a function, then

$$f_t(x) = \frac{\partial f}{\partial t}(x) = -re^{-rt}x, f_x(t) = \frac{\partial f}{\partial x}(t) = e^{-rt}, f_{xx}(t) = \frac{\partial^2 f}{\partial x^2}(t) = 0.$$

Since $\{S_t\}_t$ is an Ito process (Corollary 4.1.8), we have, by the Ito-Doeblin Lemma,

$$\begin{aligned} dY_t &= -re^{-rt}xdt + e^{-rt}dS_t + \frac{1}{2}(0) \\ &= -re^{-rt}xdt + e^{-rt}dS_t \\ &= -re^{-rt}xdt + e^{-rt}(rS_t dt + \sigma S_t dW_t^Q) \\ &= \sigma S_t e^{-rt} dW_t^Q \end{aligned}$$

to be an Ito process as well. Put $\{g_t\}_t$ to be

$$g_t = \sigma S_t e^{-rt}$$

we note that

$$E\left(\int_0^t g_u^2 du\right) = E\left(\int_0^t \sigma^2 S_u^2 e^{-2ru} du\right) < \infty,$$

so $\{Y_t\}_t$ is a well-defined Ito process.

By Theorem 3.3.3, $\{Y_t\}_t$ is a martingale with respect to $\{\mathcal{F}_t\}_t$, which is adapted to (generated by, in fact) the Brownian motion $\{W_t^Q\}_t$. This completes the proof. ■

Proposition 4.2.2 If $\{V_t\}_t$ is the value process of a self-financing portfolio, then the discounted portfolio values

$$\{e^{-rt} V_t, t \geq 0\}$$

is a martingale under Q, the probability measure in the solution to to the Black-Scholes PDE.

Proof. Because $\{V_t\}_t$ is self-financing, we have

$$dV_t = h_t^B dB_t + h_t^S dS_t$$

where $dB_t = re^{rt}dt$.

Let $Y_t = e^{-rt} V_t$ and define function $f : (t,x) \mapsto e^{-rt}x$ and get f_t, f_x, and f_{xx} in similar fashion to Proposition 4.2.1. By the Ito-Doeblin Lemma, we get

$$dY_t = -re^{-rt} V_t dt + e^{-rt} dV_t$$

where $V_t = h_t^B B_t + h_t^S S_t$. Substitute this and dV_t in yields

$$\begin{aligned} dY_t &= -re^{-rt}(h_t^B B_t + h_t^S S_t)dt + e^{-rt}(h_t^B dB_t + h_t^S dS_t) \\ &= -re^{-rt}(h_t^B B_t + h_t^S S_t)dt + e^{-rt}(h_t^B re^{rt} dt + h_t^S dS_t) \\ &= -re^{-rt} h_t^S S_t dt + e^{-rt} h_t^S dS_t \\ &= -re^{-rt} h_t^S S_t dt + e^{-rt} h_t^S (rS_t dt + \sigma S_t dW_t^Q) \\ &= h_t^S e^{-rt} \sigma S_t dW_t^Q \end{aligned}$$

which is, using similar reasoning as Proposition 4.2.1, an Ito process and consequently a martingale with respect to Q. ■

Definition 4.2.3 In a continuous-time financial model, a contingent claim with payoff $\Phi(S_T)$ at time T is **attainable** if there exists a self-financing portfolio strategy $\{h_t\}_{t\geq 0} = \{(h_t^B, h_t^S) : t \geq 0\}$ with valuation process

$$V_t = h_t^B B_t + h_t^S S_t$$

such that $S_T = \Phi(S_T)$.

If all contingency claims are attainable, the market is said to be **complete**.

Remark 4.2.4 Note that we already used the content of Def. 4.2.3 in the statement of Lemma 4.1.3.

Definition 4.2.5 In a model with a riskless asset and a risky asset, a **martingale measure** is a probability measure under which the discounted expectation of the risky asset price is equal to the current risky asset price.

Theorem 4.2.6 — First Fundamental Theorem of Asset Pricing. A market model in continuous-time is arbitrage-free if and only if there exists a martingale measure.

Proof. We omit the proof. ■

Theorem 4.2.7 — Second Fundamental Theorem of Asset Pricing. A continuous-time arbitrage-free market model is complete if and only if the martingale measure is complete (in a measure space $(X, \mathcal{B}, \mu)$, μ is a complete measure if for all $E \in \mathcal{B}$ such that $\mu(E) = 0$, and $F \subset E$, then $F \in \mathcal{B}$).

Proof. We omit the proof. ■

Theorem 4.2.8 — Risk-Neutral Valuation. Suppose we have riskless asset with price process $\{B_t\}_t$ and a risk asset with price process $\{S_t\}_t$, such that

$$dS_t = \alpha(t, S_t)dt + \sigma S_t dW_t$$
$$dB_t = r_t B_t dt, B_0 = 1$$

in an arbitrage-free model with martingale measure Q, then the price of an attainable contingency claim at time t is given by

$$F(t, S_t) = E^Q\left(\Phi(S_T)\frac{B_t}{B_T}\middle|\mathcal{F}_t\right).$$

Proof. Since the contingency claim is attainable, let $\{V_t\}_t$ be the valuation process of the replicating portfolio. Because the model is arbitrage-free, we have

$$V_t = F(t, S_t)$$

for all $t \in [0, T]$.

Because Q is a martingale measure, the process

$$\left\{Z_t = \frac{S_t}{B_t} : t \in [0, T]\right\}$$

is a martingale, and in particular

$$E(Z_t|\mathcal{F}_s) = E\left(\frac{S_t}{B_t}\middle|\mathcal{F}_s\right) = \frac{S_s}{B_s} = Z_s = \frac{1}{B_t}E(S_t|\mathcal{F}_s).$$

Re-write $\{Z_t\}_t$ in its differential form

$$dZ_t = g_t dW_t^Q$$

for some function g_t.

Define a function $f : (t,x) \mapsto \frac{x}{B_t}$ and we get

$$\begin{aligned}
f_t(x) &= \frac{\partial f}{\partial t}(t,x) = \left(-\frac{x}{B_t^2}\right)\frac{\partial}{\partial t}B_t = \left(\frac{-x}{B_t^2}\right)r_tB_t \\
f_x(t) &= \frac{\partial f}{\partial x}(t,x) = \frac{1}{B_t} \\
f_{xx}(t) &= \frac{\partial^2 f}{\partial x^2}(t,x) = 0.
\end{aligned}$$

Applying the Ito-Doeblin lemma, we have

$$dZ_t = \frac{-1}{B_t}S_tr_tB_tdt + \frac{1}{B_t}dS_t = \frac{-r_t}{B_t}S_tdt + \frac{1}{B_t}dS_t.$$

On the other hand, if we consider $Y_t := V_t/B_t$ for $t \geq 0$, using the same f, we apply the Ito-Doeblin Lemma again to get

$$\begin{aligned}
dY_t &= f_t(t,Y_t)dt + f_x(t,V_t)dV_t \\
&= \frac{-V_t}{B_t^2}r_tB_tdt + \frac{1}{B_t}dV_t \\
&= -\frac{r_t}{B_t}(h_t^BB_t + h_t^SS_t)dt + \frac{1}{B_t}(h_t^SdS_t + h_t^BdB_t) \\
&= -\frac{r_t}{B_t}h_t^SS_tdt + \frac{1}{B_t}h_t^SdS_t - r_th_t^Bdt + \frac{1}{B_t}h_t^B(r_tB_tdt) \\
&= \frac{-r_t}{B_t}h_t^SS_tdt + \frac{1}{B_t}h_t^SdS_t \\
&= h_t^S\left(\frac{-r_t}{B_t}S_tdt + \frac{1}{B_t}dS_t\right) \\
&= h_t^SdZ_t \text{ by the previous identity} \\
&= h_t^Sg_tdW_t^Q.
\end{aligned}$$

Hence $\{Y_t\}_t$ is a martingale under Q, and by the martingale property (Def. 3.1.4(3), we have

$$E(Y_t|\mathcal{F}_s) = Y_s \text{ for all } s \leq t$$

and in particular $Y_t = E(Y_T|\mathcal{F}_t)$ for any $t \in [0,T]$. Thus

$$\frac{V_t}{B_t} = E(Y_T|\mathcal{F}_t) = \frac{F(t,S_t)}{B_t} \Rightarrow F(t,S_t) = B_tE(Y_T|\mathcal{F}_t).$$

Now $Y_T = \frac{1}{B_T}\Phi(S_T)$ because $V_T = \Phi(S_T)$, so substitution yields

$$F(t,S_t) = B_tE\left(\frac{\Phi(S_T)}{B_T}\middle|\mathcal{F}_t\right) = E\left(\Phi(S_T)\frac{B_t}{B_T}\middle|\mathcal{F}_t\right)$$

as required. ■

Theorem 4.2.9 — Girsanov's Theorem. Let $\{W_t\}_t$ be a standard Brownian motion on the filtered probability space $(\Omega, \{\mathcal{F}_t\}_t, \mathcal{F}, P)$ and $\{\Psi_t\}_t$ be a stochastic process such that

$$E^P\left(e^{\frac{1}{2}\int_0^T \Psi_t^2 dt}\right) < \infty$$

for some fixed $T > 0$. Define a process $\{L_t\}_t$ on $t \in [0,T]$ where

$$L_0 = 1, L_t = e^{\int_0^t \Phi_s dW_s^P - \frac{1}{2}\int_0^t \Phi_s^2 ds},$$

i.e. $dL_t = \Phi_t L_t dW_t^P$, then there exists a probability measure Q on Ω such that

$$L_T = \frac{dQ}{dP}$$

and

$$dW_t^P = \Phi_t dt + dW_t^Q,$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative between measures Q and P and $\{W_t^Q\}_t$ is the Brownian motion under Q.

Proof. We omit the proof. ■

Remark 4.2.10 Since $dL_t = \Psi_t L_t dW_t^P$ is a Brownian motion with no drift, it is a martingale with respect to P, and

$$L_t = E^P\left(\frac{dQ}{dP}\bigg|\mathcal{F}_t\right).$$

We sometimes write the process as

$$L_t = \left(\frac{dQ}{dP}\right)_t$$

i.e. the Radon-Nikodym derivative is a random variable.

Corollary 4.2.11 Consider a Black-Scholes model with price processes $dB_t = rB_t dt$ and $dS_t = \mu S_t dt + \sigma S_t dW_t^P$ for the riskless and the risky assets respectively over a filtered probability space $(\Omega, (\mathcal{F}_t)_t, \mathcal{F}, P)$, then there exists a probability measure Q such that the Brownian motion with respect to Q satisfies

$$dW_t^Q = dW_t^P - \frac{r-\mu}{\sigma}.$$

Proof. By Corollary 4.1.8, the solution to the Black-Scholes PDE has the spot price following the process

$$dS_t = rS_t dt + \sigma S_t dW_t^Q$$

where Q is a result of the Feynman-Kac Theorem.

Define the process $(\Psi_t)_t$ such that $\Psi_t = \frac{r-\mu}{\sigma}$ for all $t > 0$. Note that

$$E^P\left(e^{\frac{1}{2}\int_0^T \frac{(r-\mu)^2}{\sigma^2} dt}\right) = E^P\left(e^{\frac{1}{2}T(t-\mu)^2/\sigma^2}\right) < \infty$$

as long as $\sigma > 0$, which is true in any risky asset prices, so by Girsanov's Theorem, there exists a probability measure Q such that

$$\frac{dQ}{dP} = e^{\int_0^t \frac{r-\mu}{\sigma} dW_s^P - \frac{1}{2}\int_0^t \frac{(r-\mu)^2}{\sigma^2} ds}$$

and $dW_t^P = \frac{r-\mu}{\sigma} dt + dW_t^Q$, as required. ■

Corollary 4.2.12 The Black-Scholes model is arbitrage-free.

Proof. Direct consequence of Corollary 4.2.11, Proposition 4.2.1, and the First Fundamental Theorem of Asset Pricing. ■

Theorem 4.2.13 The Black-Scholes model is complete.

Proof. We omit the proof. ■

Proposition 4.2.14 Let P be the physical probability measure in the Black-Scholes model, Q be the risk-neutral probability obtained in Corollary 4.2.11, then the expectation of S_T given S_0 and riskless rate r under Q is

$$E^Q(S_T) = S_0 e^{rT}.$$

Proof. From the proof of Corollary 4.2.11, we let $(\Psi_t)_t$ be

$$\Psi_t = \frac{r-\mu}{\sigma}.$$

From the statement of Girsanov's Theorem, we define $(L_t)_t = (dQ/dP)_t$, where

$$\left(\frac{dQ}{dP}\right)_T = L_T = \exp\left(\int_0^T \Psi_s dW_s^P - \frac{1}{2}\int_0^T \Psi_s^2 ds\right).$$

Now

$$E^Q(S_T) = E^P\left(\left(\frac{dQ}{dP}\right)_T S_T\right)$$

where $E^P(S_T) = S_0 e^{\mu T}$ since dS_t follows a geometric Brownian motion and we invoke Thm. 3.4.6 to get

$$\begin{aligned} E^P(S_T) &= E^P\left(S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma(W_T - W_t)}\right) \\ &= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T} E^P(e^{\sigma(W_T - W_t)}) \end{aligned}$$

where $W_T - W_0 \sim N(0,T)$ and the result follows from the moment-generating function of the normal distribution.

Substitute in L_T to get

$$\begin{aligned}
E^Q(S_T) &= E^P\left(\left(\frac{dQ}{dP}\right)_T S_T\right)\\
&= E^P\left(\exp\left(\int_0^T \frac{r-\mu}{\sigma}dW_s^P - \frac{1}{2}\int_0^T\left(\frac{r-\mu}{\sigma}\right)^2 ds\right)\cdot S_T\right)\\
&= E^P\left(\exp\left(\frac{r-\mu}{\sigma}W_T^P - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right)\cdot S_0\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T^P\right)\right)\\
&= e^{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T}\cdot S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)T} E^P\left(e^{\left(\frac{r-\mu}{\sigma}+\sigma\right)W_T^P}\right)\\
&= e^{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T} S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)T}\cdot e^{\frac{1}{2}T\left(\frac{r-\mu}{\sigma}+\sigma\right)^2} \text{ by moment-generating function of } N(0,T)\\
&= S_0 e^{rT}
\end{aligned}$$

as required. ■

■ **Example 4.2.15** Consider a continuous-time model and a forward contract with forward price K on an asset which has price process $(S_t)_{t=0}^T$.

The payoff at time T is $S_T - K$ and by definition of a forward, the value of the forward at time 0 is 0. Suppose the riskless asset has price B_T at time T, then a risk-neutral measure Q must satisfy, by Thm. 4.2.8,

$$0 = E^Q\left(S_T - \frac{K}{B_T}\right).$$

Now $\frac{S}{B}$ is a martingale under Q by the proof of Thm. 4.2.8, thus

$$0 = E^Q\left(S_T - \frac{K}{B_T}\right) = S_0 - \frac{K}{B_T}$$

and so $K = S_0 B_T$.

If there is a riskless rate r such that $B_t = e^{rt}$, then $K = S_0 e^{rt}$, which is the same result as Proposition 1.5.4.

On the other hand, the value of the contract itself is given by

$$\begin{aligned}
f_{t,T} &= E^Q\left((S_T - K)\frac{B_t}{B_T}\middle|\mathcal{F}_t\right)\\
&= B_t E^Q\left(\frac{S_T}{B_T}\middle|\mathcal{F}_t\right) - E^Q\left(K\frac{B_t}{B_T}\right)\\
&= B_t\frac{S_t}{B_t} - S_0 B_T\frac{B_t}{B_T} \text{ by martingale property}\\
&= S_t - S_0 B_t
\end{aligned}$$

at time $t \in [0,T]$. ■

4.3 Monte-Carlo Method for Pricing

4.4 Implies Volatility

4.5 The Greeks

4.6 Hedging

5. Continuous-Time Interest Rate Models

5.1 Bonds and Interest Rates

Definition 5.1.1 A zero-coupon bond with maturity T is called a **T-bond**. The price of the T-bond at time t is denoted $p(t,T)$, $0 \leq t \leq T$. We assume $p(t,T)$ is a differentiable function with respect to t.

Definition 5.1.2 The **short rate** at time $t \in [0,T]$, denoted $r(t)$, is a random variable which represents the continuously compounded interest rate at which one can borrow or lend for an infinitesimal amount of time Δt at time t.

Definition 5.1.3 At time $t \in [0,T]$, the $[S,T]$ **LIBOR forward rate** for some $S \in [t,T]$, denoted $L(t;S,T)$, is the simple interest on \$1 invested at time S till time T.

Proposition 5.1.4 Suppose $0 \leq t \leq S \leq T$, then the LIBOR forward rate is

$$L(t;S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)}$$

and in particular

$$L(S;S,T) = -\frac{p(t,T) - 1}{(T-S)p(t,T)}.$$

Proof. By the definition of $L(t;S,T)$ we have the payoff of \$1, invested at time S to be

$$1 + (T-S)L(t;S,T)$$

at time T. On the other hand, we can pay $p(t,S)$ at time t to enter into a T-bond. This yields \$1 at time S, and becomes $p(t,S)/p(t,T)$ at time T. Thus we have the equation

$$1 + (T-S)L(t;S,T) = \frac{p(t,S)}{p(t,T)}$$

and equivalently

$$L(t;S,T) = \frac{\frac{p(t,S)}{p(t,T)} - 1}{T-S} = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)}.$$

For the second result, simply substitute in $p(S,S) = 1$ and get

$$L(S;S,T) = -\frac{p(t,T) - 1}{(T-S)p(t,T)}.$$

■

Definition 5.1.5 The **simple spot rate** or the **LIBOR spot rate** for $[S,T]$ is the $L(S;S,T)$ in Proposition 5.1.4 above.

Definition 5.1.6 At time $t \in [0,T]$, the **continuously compounded forward rate for** $[S,T]$ **contracted at** $t \leq S$ is the interest on \$1 invested at S till time T, assuming that interest is continuously compounded, denoted $R(t;S,T)$.

The **continuously compounded spot rate**, or the **zero-coupon yield**, is $R(S;S,T)$, or denoted $R(S,T)$.

Proposition 5.1.7 We have

$$R(t;S,T) = -\frac{\log(p(t,T)) - \log(p(t,S))}{T-S}$$

and

$$R(S;S,T) = R(S,T) = -\frac{\log(p(t,T))}{T-S}.$$

Proof. Consider two strategies at time t. The first strategy enters a contract that invests \$1 at time S. By definition, the \$1 will grow at rate $R(t;S,T)$ for the period of $T-S$. The second strategy invests $p(t,S)$ at time t to enter a T-bond. This yields \$1 at time S, and becomes $p(t,S)/p(t,T)$ at time T. This these two strategies both effectively have investment of \$1 at time S, their payoff at time T are equal:

$$e^{R(t;S,T)(T-S)} = \frac{p(t,S)}{p(t,T)}.$$

It follows that

$$\begin{aligned}
R(t;S,T) &= \frac{1}{T-S}\log\left(\frac{p(t,S)}{p(t,T)}\right) \\
&= \frac{1}{T-S}(\log(p(t,S)) - \log(p(t,T))) \\
&= -\frac{\log(p(t,T)) - \log(p(t,S))}{T-S}
\end{aligned}$$

as required. If $t = S$, then $p(S,S) = 1$ and $\log p(S,S) = 0$, and the rest is immediate. ■

Definition 5.1.8 The **instantaneous forward rate with maturity T contracted at time t** is

$$f(t,T) = -\frac{\partial}{\partial T}\log p(t,T),$$

while the **instantaneous short rate** at time $t \in [0,T]$ is

$$r(t) = f(t,t) = -\lim_{\Delta\to 0}\frac{\log(p(t,t+\Delta)) - \log(p(t,t))}{\Delta}.$$

Remark 5.1.9 For an infinitismal Δ we have

$$e^{r(t)\Delta} \approx \frac{p(t,t)}{p(t,t+\Delta)} = \frac{1}{p(t,t+\Delta)}.$$

Definition 5.1.10 The **money market process**, or the **risk-free asset**, for $t \in [0,T]$, is

$$\beta(t) = \exp\left(\int_0^t r(s)ds\right)$$

where $r(\cdot)$ is the instantaneous short rate function.

Proposition 5.1.11 A zero-coupon T-bond at time t has price

$$p(t,T) = p(t,S)\exp\left(-\int_S^T f(t,u)du\right) \text{ for all } S \leq T$$

and

$$p(t,T) = \exp\left(-\int_t^T f(t,s)ds\right).$$

Proof. The instantaneous forward rate $f(t,u)$ can be written as

$$f(t,u) = -\frac{\partial}{\partial u}\log p(t,u).$$

Hence, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
&\int_S^T f(t,u)du = -\log p(t,T) + \log p(t,S) \\
\Rightarrow &\int_S^T f(t,u)du = -\log\left(\frac{p(t,T)}{p(t,S)}\right)
\end{aligned}$$

and we have the result.

The second equation is the consequence of the fact that $p(t,t) = 1$. ■

Definition 5.1.12 Define a function

$$\begin{aligned}
y : [0,\infty) \times \mathbb{R} &\to \mathbb{R} \\
(t,T) &\mapsto R(t,T)
\end{aligned}$$

where

$$R(t,T) = -\frac{\log p(t,T)}{T-t}.$$

$R(t,T)$ is the **zero-coupon bond yield**. The map $(t, y(t,T))$ is the **yield curve of zero-coupon bonds**, and the function y is the **yield function**.

Remark 5.1.13 Note that

$$p(t,T) = e^{-y(t,T)(T-t)}$$

and the payoff, at time T, of \$1 invested in a zero-coupon bond with maturity T, is

$$\frac{1}{p(t,T)} = e^{y(t,T)(T-t)}.$$

This shows that $y(\cdot)$ is the rate of guaranteed return on the zero-coupon bond.

5.2 Zero-Coupon Bond Pricing and the Term Structure Equation

Definition 5.2.1 The **short rate model** has the following setup:

1. The short rate at time t follows the process
$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t^P$$
for some drift function μ, diffusion function σ, and W_t^P, the standard Brownian motion with respect to a physical probability measure P.
2. There is a risk-free asset whose price follows
$$d\beta(t) = r(t)\beta(t)dt$$
where $\beta(t)$ is a money-market process.
3. For all $T \geq 0$, there exists a market for the T-bond.
4. For all $T > 0$, the price of the T-bond at time t is of the form
$$p(t,T) = F(t, r(t), T),$$
which can also be written as
$$p(t,T) = F^T(t,r), t \in [0,T]$$
where F is a smooth function and satisfies the boundary condition $F^T(T,r) = 1$.

Proposition 5.2.2 Under the short rate model, the dynamics of the T-bond price is

$$dF^T = F^T \alpha_T(t)dt + F^T \sigma_T(t)dW_t^P$$

where

$$\alpha_T(t) = \frac{1}{F^T}\left(F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T\right)$$
$$F_t^T = \frac{\partial F^T}{\partial t}, F_r^T = \frac{\partial F^T}{\partial r}, F_{rr}^T = \frac{\partial^2 F^T}{\partial r \partial r},$$
$$\sigma_T(t) = \frac{\sigma F_r^T}{F^T},$$

μ and σ being the drift and diffusion of the short rate process $r(t)$ respectively.

Proof. Write $Y_t = F^T(t, r(t))$ where

$$dr(t) = \mu dt + \sigma dW_t^P$$

as defined in the short rate model and use the Ito-Doeblin Lemma to get

$$\begin{aligned} dY_t &= dF^T(t, r(t)) \\ &= \left(F_t^T + F_r^T \mu + \frac{1}{2} F_{rr}^T \sigma^2\right) dt + F_r^T \sigma dW_t^P \\ &= F^T \left(\frac{1}{F^T}\right) \left(F_t^T + F_r^T \mu + \frac{1}{2} F_{rr}^T \sigma^2\right) dt + F^T \left(\frac{F_r^T \sigma}{F^T}\right) dW_t^P \\ &= F^T \alpha_T(t) dt + F^T \sigma_T(t) dW_t^P \end{aligned}$$

as required. ■

Proposition 5.2.3 Let α_T and σ_T be the drift and diffusion of the T-bond price process in the short rate model, then, if the bond market is arbitrage-free, then there exists a stochastic process $\{\lambda(t) : t \geq 0\}$ such that

$$\lambda(t) = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} \text{ for all } t \geq 0.$$

Proof. We omit the proof. ■

Definition 5.2.4 In an arbitrage-free bond market under the short rate model, the stochastic process

$$\lambda(t) = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} \text{ for all } t \geq 0$$

where $\alpha_T(t)$ and $\sigma_T(t)$ are the drift and diffusion of the T-bond price process, is called the **market price of risk**.

Remark 5.2.5 In an arbitrage-free bond market under the short rate model, T-bonds of all maturities should have the same market price of risk. The substitution of $\alpha_T(t)$ and $\sigma_T(t)$ from Proposition 5.2.2 into Def. 5.2.4 yields the following formal definition.

Definition 5.2.6 Let $F^T(t, r(t))$ be a T-bond price process in a short rate model with the usual definitions of μ and σ, then the **term structure equation** is as follows:

$$\begin{cases} F_t^T + (\mu - \lambda\sigma) F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T & = 0 \\ F^T(T, r) & = 1 \end{cases}$$

for some real-valued process $\{\lambda(t) : t \geq 0\}$. Note that both μ and σ are also functions of t.

Theorem 5.2.7 In an arbitrage-free bond market under the short rate model, the zero-coupon bond prices $p(t, T)$ are given by

$$p(t, T) = F^T(t, r) = E^Q\left(\exp\left(-\int_t^T r(s) ds\right) \Big| \mathcal{F}_t\right)$$

where Q is a probability measure such that

$$dr(s) = (\mu - \sigma\lambda)ds + \sigma dW_s^Q$$

with initial condition $r(t) \in \mathbb{R}$.

Proof. Direct application of Feynman-Kac Theorem. ■

Corollary 5.2.8 The time-t-risk-neutral price of an interest-rate-contingent claim X paying $X(T) = \Phi(r(T))$ at maturity T is given by

$$X(t) = E^Q\left(\Phi(r(T))\exp\left(-\int_t^T r(s)ds\right)\middle|\mathcal{F}_t\right)$$

where the Q-dynamic of the short rate is given by

$$dr(s) = (\mu = \lambda\sigma)ds + \sigma dW_s^Q$$
$$r(t) = r \in \mathbb{R}.$$

Proof. We omit the proof. ■

Proposition 5.2.9 The relationship between probability measures P and Q in the short rate model and the risk-neutral valuation formula is

$$dW^Q(t) = dW^P(t) + \lambda dt$$

where λ is the market price of risk.

Proof. Under P we have

$$dr(t) = \mu dt + \sigma dW_t^P$$

by Def. 5.2.1, while under Q we have

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma dW_t^Q.$$

By equating the two expressions, we have

$$dW_t^P + \lambda dt = dW_t^Q$$

as required. ■

Corollary 5.2.10 Under the probability measure Q yielded in Corollary 5.2.8, the price of T-bonds under the short rate model follows the dynamic

$$dF^T = r(t)F^T dt + F^T \sigma_T(t) dW_t^Q.$$

Proof. By Proposition 5.2.2 we have

$$dF^T = \alpha_T(t)F^T dt + F^T \sigma_T(t) dW_t^P.$$

Substitute in Proposition 5.2.9 to get

$$dF^T = F^T \alpha_T(t)dt + F^T \sigma_T(t)(dW_t^Q - \lambda dt).$$

Now $\lambda(t) = (\alpha_T(t) - r(t))/\sigma_T(t)$, so

$$\begin{aligned} dF^T &= F^T \alpha_T(t)dt + F^T \sigma_T(t) \left(dW_t^Q - \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} dt \right) \\ &= F^T \alpha_T(t)dt + F^T \sigma_T(t) dW_t^Q - F^T(\alpha_T(t) - r(t))dt \\ &= F^T r(t)dt + F^T \sigma_T(t) dW_t^Q \end{aligned}$$

as required. ■

Corollary 5.2.11 Under the risk-neutral probability Q, the process

$$\left\{ \frac{F^T(t)}{\beta(t)}, t \geq 0 \right\}$$

in the short rate model is a martingale.

Proof. Define a function

$$\begin{aligned} f : [0,T] \times \mathbb{R} &\to \mathbb{R} \\ (t,x) &\mapsto \frac{x}{t} \end{aligned}$$

and define $Y_t := f(t, F^T)$.

It follows that

$$f_t = (-1)(\beta(t))^{-2} \left(\frac{d\beta}{dt} \right) (F^T)$$

where $d\beta(t) = r(t)\beta(t)dt$ by the assumption of the short rate model, so

$$f_t(t, F^T) = \frac{-1}{\beta(t)^2} r(t)\beta(t)F^T = -\frac{r(t)}{\beta(t)} F^T.$$

Also,

$$f_x(t, F^T) = \frac{1}{\beta(t)}, f_{xx}(t, F^T) = 0.$$

By the Ito-Doeblin Lemma,

$$\begin{aligned} dY_t &= f_t(t, F^T)dt + f_x(t, F^T)dF^T \\ &= -\frac{-r(t)}{\beta(t)} F^T dt + \frac{1}{\beta(t)} dF^T \\ &= -\frac{r(t)}{\beta(t)} F^T dt + \frac{1}{\beta(t)} (r(t)F^T dt + F^T \sigma_T(t) dW_t^Q) \\ &= F^T \sigma_T(t) dW_t^Q \end{aligned}$$

dY_t has no drift term, so by Thm. 3.3.3, Y_t is a martingale. By the construction of $\{Y_t\}_t$, this completes the proof. ■

5.3 Martingale Models for the Short Rate

Definition 5.3.1 Let Q be the risk-neutral measure under a short rate model. The following are models for the short rate under Q:

1. The **Varsicek model** states
$$dr = (b - ar)dt + \sigma dW_t^Q, a > 0.$$
2. The **Cox-Ingersoll-Ross (CIR) model** states
$$dr = a(b - r)dt + \sigma\sqrt{r}dW_t^Q.$$
3. The **Dothan model** states
$$dr = ardt + \sigma r dW_t^Q.$$
4. The **Ho-Lee model** states
$$dr = \Omega(t)dt + \sigma dW_t^Q$$
where Ω is a function of time t chosen such that the model fits the original term structure.
5. The **Black-Derman-Toy model** states
$$dr = \Omega(t)rdt + \sigma(t)rdW_t^Q$$
where Ω and σ are both functions of time t.
6. The **Hull-White one-factor model** states
$$dr = (\Omega(t) - a(t)r)dt + \sigma(t)dW_t^Q, a(t) > 0,$$
where Ω, a, and σ are functions of time t.
7. The **Hull-White two-factor model** states
$$dr = (\Omega(t) - a(t)r)dt + \sigma(t)\sqrt{r}dW_t^Q, a(t) > 0,$$
where Ω, a, and σ are functions of time t.

In general, these models specify the dynamic for
$$dr = \tilde{\mu}(t, r(t); \alpha)dt + \sigma(t, r(t); \alpha)dW_t^Q$$
under probability measure Q for some functions $\tilde{\mu}$ and σ of time t, where α can be a vector or a function, and contains additional parameters for $\tilde{\mu}$.

Definition 5.3.2 Given a particular model for the short rate r:
$$dr = \tilde{\mu}(t, r(t); \alpha)dt + \sigma(t, r(t); \alpha)dW^Q,$$
the **"inverting the yield curve" procedure** for estimating α is as follows:

1. Solve, for every maturity T, the term structure equation
$$\begin{cases} F_t^T + \tilde{\mu}F_r^T + \frac{1}{2}F_{rr}^T - rF^T & = 0 \\ F^T(T, r) & = 1 \end{cases}$$
to obtain the zero-coupon prices $p(0, T; \alpha)$ for $T \geq 0$. Denote the results as $\{T, p(0, T; \alpha)\}$, the **theoretical term structure**.

2. Collect empirical data from the bond market for all maturities T. Denote this $\{T, p^*(0,T)\}$, the **empirical term structure**.
3. Choose α such that $\{T, p(0,T;\alpha)\}$ fits $\{T, p^*(0,T)\}$ based on some fitting criteria. Denote the optimal parameter to be α^*.
4. Insert α^* into $\tilde{\mu}$ and σ, and denote the estimated functions to be μ^* and σ^*.
5. Use

$$dr = \mu^*(t, r(t); \alpha^*)dt + \sigma^*(t, r(t); \alpha^*)dW^Q$$

to compute prices of interest rate derivatives with the methodology of Corollary 5.2.8.

Definition 5.3.3 Suppose $\{p(t,T) : t \in [0,T], T \geq 0\}$ is a term structure of zero-coupon bond prices and each $p(t,T)$ has the form

$$p(t,T) = F(t, r(t), T)$$

where

$$F(t, r(t), T) = e^{A(t,T) - B(t,T)r}$$

and A and B are deterministic functions, then the model of the bond prices is said to possess an **affine term structure (ATS)**.

Definition 5.3.4 For a short rate model

$$dr = \tilde{\mu}dt + \sigma dW_t^Q$$

under some probability measure Q, the model is **mean-reverting** is the mean of r tends to a constant level in the long run.

Proposition 5.3.5 Suppose the short rate follows a dynamic

$$dr = \tilde{\mu}(t, r(t); \alpha)dt + \sigma(t.r(t); \alpha)dW^Q$$

where $\tilde{\mu}$ and σ have the form

$$\begin{cases} \tilde{\mu}(t, r(t)) & = \alpha(t)r + \beta(t) \\ \sigma(t, r(t)) & = \sqrt{\gamma(t)r + \delta(t)}, \end{cases}$$

then the model admits an affine term structure where A and B satisfy the system

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) & = -1 \\ B(T,T) & = 0, \end{cases}$$

$$\begin{cases} A_t(t,T) & = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) & = 0. \end{cases}$$

Proof. We omit the proof. ■

Corollary 5.3.6 The Vasicek model for short rate

$$dr = (b - ar)dt + \sigma dW_t^Q$$

possess affine term structure

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)}$$

where

$$A(t,T) = \frac{(B(t,T) - T + t)\left(ab - \frac{1}{2}\sigma^2\right)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}$$
$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}.$$

Proof. By Proposition 5.3.5, we solve for A and B in systems

$$\begin{cases} B_t(t,T) + (-a)B(t,T) &= -1 \\ B(T,T) &= 0, \end{cases}$$

$$\begin{cases} A_t(t,T) &= bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) &= 0. \end{cases}$$

The first system easily yields

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

and integrating the second system yields

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds - b\int_t^T B(s,T)ds,$$

and substitution yields

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T \frac{1}{a^2}(1 - e^{-a(T-s)})^2 ds - b\int_t^T \frac{1}{a}(1 - e^{-a(T-s)})ds.$$

Integrating this yields the result. ∎

Remark 5.3.7 1. The Vasicek model is mean-reverting with mean of r to be $\frac{b}{a}$.
2. In the Vasicek, Ho-Lee, and Hull-White one-factor models, the integral $\int r(s)ds$ has a normal distribution.
3. Hence, the short rate r in the Vasicek model might become negative.

Corollary 5.3.8 Suppose the short rate follows a Ho-Lee model

$$dr = \Omega(t)dt + \sigma dW_t^Q$$

where, at $t = 0$, $\Omega(t)$ follows

$$p(0,T) = p^*(0,T), \text{ the observed } T\text{-bond price,}$$

and

$$\Omega(t) = \frac{\partial f^*}{\partial T}(0,t) + \sigma^2 t$$

where $f^*(0,t)$ denotes the observed forward rates. Then, the T-bond price has closed-form expression

$$p(t,T) = \frac{p^*(0,T)}{p(0,t)} \exp\left((T-t)f^*(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)\right).$$

Proof. We omit the proof. ■

Corollary 5.3.9 Suppose the short rate follows a CIR model

$$dr = a(b-r)dt + \sigma\sqrt{r}dW_t^Q,$$

then the bond prices are given by

$$F^T(t,r) = A_0(T-t)e^{-B(T-t)r},$$

where

$$B(x) = \frac{2(e^{\gamma x}-1)}{(\gamma+\alpha)(e^{\gamma x}-1)+2\gamma}$$

$$A_0(x) = \left(\frac{2\gamma e^{(a+\gamma)(x/2)}}{(\gamma+a)(e^{\gamma x}-1)+2\gamma}\right)^{2ab/\sigma^2}$$

$$\gamma = \sqrt{a^2+2\sigma^2}.$$

Proof. We omit the proof. ■

Remark 5.3.10 In the CIR model, there is mean reversion for $p(t,T)$, r is always positive, the volatility of r depends on r, and r follows a chi-squared distribution, but we are unable to fir the time-0 observed prices of bonds.

Definition 5.3.11 A **European bond call option** with strike K and expiration S on a T-bond where $S<T$ gives the option holder the right but not the obligation to buy a T-bond at K at time S.
A **European bond put option** with strike K and expiration S on a T-bond where $S<T$ gives the option holder the right but not the obligation to sell a T-bond at K at time S.

Proposition 5.3.12 Suppose the short rate follows the Vasicek model

$$dr = (b-ar)dt + \sigma dW_t^Q, a>0,$$

then the price of a European bond call option with strike K, expiration S, on a T-bond at time $t\in[0,S]$ is

$$c(t,S,K,T) = p(t,T)N(d) - p(t,S)KN(d-\sigma_p)$$

where

$$d = \frac{1}{\sigma_p} \log\left(\frac{p(t,T)}{p(t,S)K}\right) + \frac{1}{2}\sigma_p,$$
$$\sigma_p = \frac{1}{a}(1 - e^{-a(T-S)})\sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a(T-t)})},$$

and $N(\cdot)$ is the distribution function of the standard normal distribution.

Proof. We omit the proof. ■

Proposition 5.3.13 Suppose the short rate follows the Ho-Lee model

$$dr = \Omega(t)dt + \sigma dW_t^Q,$$

where

$$\Omega(t) = \frac{\partial f^*}{\partial T}(0,t) + \sigma^2 t,$$

then the price of a European call option with strike K, expiration S on a T-bond at time $t \in [0,S]$ is

$$c(t,S,K,T) = p(t,T)N(d) - p(t,S)KN(d - \sigma_p),$$

where

$$d = \frac{1}{\sigma_p} \log\left(\frac{p(t,T)}{p(t,S)K}\right) + \frac{1}{2}\sigma_p,$$
$$\sigma_p = \sigma(T-S)\sqrt{S-t},$$

and $N(\cdot)$ is the distribution function of the standard normal distribution.

Proof. We omit the proof. ■

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