# Fault Tolerant Distributed Consensus

## There is no conversation more boring than the one where everybody agrees.

Marco Aiello, Eirini Kaldeli

University of Groningen

Distributed Systems, 2009

# Chapter 14: Consensus and Agreement

Ajay Kshemkalyani and Mukesh Singhal

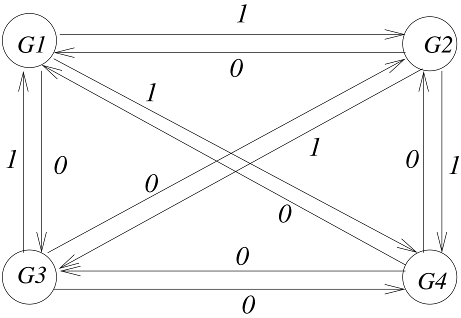
Distributed Computing: Principles, Algorithms, and Systems

Cambridge University Press

# Assumptions

System assumptions

- Failure models
- Synchronous/ Asynchronous communication
- Network connectivity
- Sender identification
- Channel reliability
- Authenticated vs. non-authenticated messages
- Agreement variable



# Problem Specifications

Byzantine Agreement (single source has an initial value)

- **Agreement:** All non-faulty processes must agree on the same value.
- **Validity:** If the source process is non-faulty, then the agreed upon value by all the non-faulty processes must be the same as the initial value of the source.
- **Termination:** Each non-faulty process must eventually decide on a value.

Consensus Problem (all processes have an initial value)

- **Agreement:** All non-faulty processes must agree on the same (single) value.
- **Validity:** If all the non-faulty processes have the same initial value, then the agreed upon value by all the non-faulty processes must be that same value.
- **Termination:** Each non-faulty process must eventually decide on a value.

Interactive Consistency (all processes have an initial value)

- **Agreement:** All non-faulty processes must agree on the same array of values $A[v_1 \ldots v_n]$.
- **Validity:** If process $i$ is non-faulty and its initial value is $v_i$, then all non-faulty processes agree on $v_i$ as the $i$th element of the array $A$. If process $j$ is faulty, then the non-faulty processes can agree on any value for $A[j]$.
- **Termination:** Each non-faulty process must eventually decide on the array $A$.

# Overview of Results

| Failure mode | Synchronous system (message-passing and shared memory) | Asynchronous system (message-passing and shared memory) |
|---|---|---|
| No failure | agreement attainable; common knowledge also attainable | agreement attainable; concurrent common knowledge attainable |
| Crash failure | agreement attainable $f < n$ Byzantine processes $\Omega(f+1)$ rounds | agreement not attainable |
| Byzantine failure | agreement attainable $f \leq \lfloor (n-1)/3 \rfloor$ Byzantine processes $\Omega(f+1)$ rounds | agreement not attainable |

Table: Overview of results on agreement. $f$ denotes number of failure-prone processes. $n$ is the total number of processes.

In a failure-free system, consensus can be attained in a straightforward manner

# Some Solvable Variants of the Consensus Problem in Async Systems

| Solvable Variants | Failure model and overhead | Definition |
|---|---|---|
| Reliable broadcast | crash failures, $n > f$ (MP) | Validity, Agreement, Integrity conditions |
| $k$-set consensus | crash failures. $f < k < n$. (MP and SM) | size of the set of values agreed upon must be less than $k$ |
| $\epsilon$-agreement | crash failures $n \geq 5f + 1$ (MP) | values agreed upon are within $\epsilon$ of each other |
| Renaming | up to $f$ fail-stop processes, $n \geq 2f + 1$ (MP) Crash failures $f \leq n - 1$ (SM) | select a unique name from a set of names |

Table: Some solvable variants of the agreement problem in asynchronous system. The overhead bounds are for the given algorithms, and not necessarily tight bounds for the problem.

# Solvable Variants of the Consensus Problem in Async Systems

**Circumventing the impossibility results for consensus in asynchronous systems**

- **Message−passing**
  - $k$ **set consensus**
  - *epsilon−* **consensus**
  - **Renaming**
  - **Reliable broadcast**
- **Shared memory**
  - $k$ **set consensus**
  - *epsilon−* **consensus**
  - **Renaming**

    *using atomic registers and atomic snapshot objects constructed from atomic registers*
  - **Consensus**

    *using more powerful objects than atomic registers. This is the study of universal objects and universal constructions.*

## Formal requirements for the consesus algorithm

An algorithm solves the consensus problem if it satisfies the following formal properties:

- ***Termination***: Eventually every non-faulty processor decides on a value $y_i$.
- ***Agreement***: The final decisions of all non-faulty processors are identical, i.e. if $y_i$, $y_j$ are assigned then $\forall\ p_i, p_j \in Nonfaulty\ :\ (y_i = y_j)$.
- ***Validity***: If all non-faulty $p_i$s have the same input then the decision of a non-faulty processor equals the common input, i.e. if $\forall\ p_i \in Nonfaulty\ :\ (x_i = v)$ then if $y_i$ is assigned for some non-faulty $p_j$ then $y_i = v$.

Systems with different levels of synchrony or different kinds of failures require different algorithms.

# Impossibility of Distributed Consensus with one faulty process [Fisher, Lynch and Peterson, 1985]

**Bad news**

The design of a consensus protocol that tolerates failures is impossible in asynchronous distributed systems.

- Impossibility holds for both shared memory and message passing systems:
  - even if we assume reliable communication channels.
  - even if considering only benign failures (crashes).
  - even if at most one processor fails.
- The problem is that in totally asynchronous systems we cannot distinguish a dead process from a merely slow one.
- We have to assume some level of synchrony on communication, processes or message order for consensus to become possible.

# Impossibility Result (MP, async)

## FLP Impossibility result

Impossible to reach consensus in an async MP system even if a single process has a crash failure

- In a failure-free async MP system, initial state is *monovalent* $\Longrightarrow$ consensus can be reached.
- In the face of failures, initial state is necessarily bivalent
- Transforming the input assignments from the all-0 case to the all-1 case, there must exist input assignments $\vec{I_a}$ and $\vec{I_b}$ that are 0-valent and 1-valent, resp., and that differ in the input value of only one process, say $P_i$. If a 1-failure tolerant consensus protocol exists, then:
  - Starting from $\vec{I_a}$, if $P_i$ fails immediately, the other processes must agree on 0 due to the termination condition.
  - Starting from $\vec{I_b}$, if $P_i$ fails immediately, the other processes must agree on 1 due to the termination condition.

  However, execution (2) looks identical to execution (1), to all processes, and must end with a consensus value of 0, a contradiction. Hence, there must exist at least one bivalent initial state.
- Consensus requires some communication of initial values.

# Impossibility Result (MP, async)

- To transition from bivalent to monovalent step, must exist a critical step which allows the transition by making a decision
- Critical step cannot be local (cannot tell apart between slow and failed process) nor can it be across multiple processes (it would not be well-defined)
- Hence, cannot transit from bivalent to univalent state.

Wider Significance of Impossibility Result

- By showing reduction from consensus to problem X, then X is also not solvable under same model (single crash failure)
- E.g., leader election, terminating reliable broadcast, atomic broadcast, computing a network-wide global function using BC-CC flows, transaction commit.

# Consensus in synchronous systems

Fault-tolerant consensus can be reached in synchronous systems under certain assumptions on the number of faulty processors and the connectivity of the communication graph.
We will consider a model that can accommodate for process failures:

- The system includes at most $f$ faulty processors: *f-resilient*.
- The subset $F$ of the faulty processors (maybe different in each execution) is not known in advance.
- Communication channels are reliable (compare with the two generals problem).
- The graph topology is a complete graph.

# Model of synchronous communication under failures

- In each of a synchronous execution every processor can send a message, all messages are delivered, and every processor takes a computation step.
- If a processor *crashes*, then an arbitrary subset of its outgoing messages are delivered, and in the subsequent rounds it takes no more steps.
- If a processor is *byzantine-faulty*, then it can take any arbitrary step, e.g. send different messages to different processors or not send messages at all.

# A solely crash tolerant consensus algorithm

Code for each processor $p_i$, $1 \leq i \leq n$:

```
V={x_i}                          /*set V constains p_i's input*/
for (k:=1 to f+1)                /*round k*/
 broadcast(u ∈ V : p_i has not already sent u)
 receive set of msgs S_j from p_j, 1 ≤ j ≤ n, j ≠ i
 V:=V ∪ ⋃_{j=1}^{n} S_j          /*update V by joining it with the received sets*/

y_i=majority(V)                  /*decide at f+1 round*/
```

Correctness of the algorithm:

- *Termination*: The algorithm requires exactly $f + 1$ rounds.
- *Validity*: The decision value is an input of some $p_i$, since no spurious messages are introduced: if all inputs have the same value, then that is the only one ever in circulation.

# Agreement condition of the crash tolerant consensus algorithm

*Agreement*: At the end of round $f+1$,
$\forall\, p_i, p_j \notin F :\ (x \in V_i \ \Rightarrow\ x \in V_j)$: prove by contradiction.

**Proof.**

- Suppose $\exists x :\ (x \in V_i) \bigwedge (x \notin V_j)$, where $p_i, p_j$ non-faulty.
- $p_i$ must have received $x$ for the first time at round $f+1$, otherwise it would have already sent it to $p_j$.
- There is a $p_{i_{f+1}}$ that sent $x$ to $p_i$ at round $f+1$. $p_{i_{f+1}}$ must have crashed in middle of this round, so $x$ was not sent to $p_j$.
- Similarly, there is a $p_{i_f}$ that sent $x$ to $p_{i_{f+1}}$.
- So, there is a chain of $f+1$ distinct faulty processors $p_{i_1}, \ldots, p_{i_{f+1}}$ (remember that after a crash there is no resurrection), that transfered $x$ to $p_i$ ➠ Contradiction.

$\square$

# Complexity of crash-tolerant consensus

- Number of messages sent: $O(n^2)$. There are at most $n$ different values and each of them is sent at most $n - 1$ times.
- Number of rounds: $f + 1$

➢ It can be proved that $f + 1$ is the lower bound on rounds for reaching fault-tolerant consensus (both for the benign and the severe case).

- Note that the algorithm is correct no matter how many the faulty processors are.

# Traitors trying to spoil consensus

- There are $n$ generals who head different divisions of the Byzantine army and have to agree whether to attack the enemy or not.
- Communication is reliable but $f$ of the generals are traitors and try to bring confusion by feeding incorrect information.
- How many traitors can a byzantine consensus protocol tolerate?

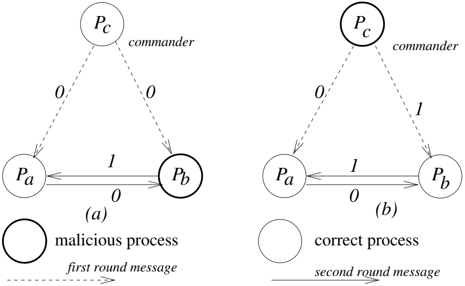
**Theorem**

*In a system with three processors one of which are byzantine, there is no algorithm that solves the consensus problem.*

☺ Let's see why.

# Upper Bound on Byzantine Processes (sync)

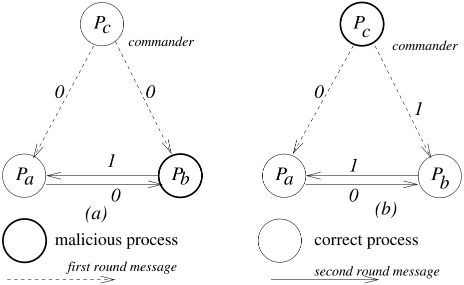
Agreement impossible when $f = 1, n = 3$.




- Taking simple majority decision does not help because loyal commander $P_a$ cannot distinguish between the possible scenarios (a) and (b);
- hence does not know which action to take.
- Proof using induction that problem solvable if $f \leq \lfloor \frac{n-1}{3} \rfloor$. See text.

# Upper Bound on Byzantine Processes (sync)

Agreement impossible when $f = 1, n = 3$.




- Taking simple majority decision does not help because loyal commander $P_a$ cannot distinguish between the possible scenarios (a) and (b);
- hence does not know which action to take.
- Proof using induction that problem solvable if $f \leq \lfloor \frac{n-1}{3} \rfloor$. See text.

## Lower bound on the ratio of faulty processors to achieve byzantine consensus

**Theorem**

*In a system with $n$ processors $f$ of which are byzantine, there is no algorithm that solves the consensus problem if $n \leq 3f$ (even if the network is synchronous and complete).*

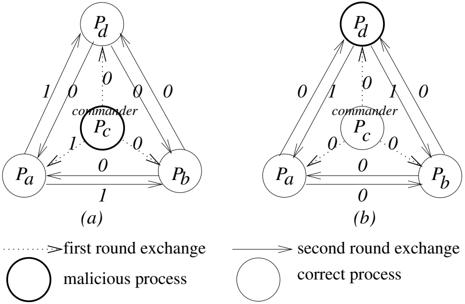
We can show that if we assume that there is such an algorithm, then we would be able to solve the problem for $n = 3$, $f = 1$ contradicting the previous theorem.

- What if the graph is not complete?
- It can be proved that for byzantine consensus to be possible the connectivity of the graph has to be at least $2f + 1$.

# An algorithm with exponential message size

- ✓ We will present a byzantine consenus algorithm that is optimal in terms of resilience ($f < \frac{n}{3}$) and number of rounds (f+1).
- ✗ However, the size of the exchanged messages is exponential.
- Each $p_i$ maintains a tree data structure of height $f + 1$ (levels 0 to $f + 1$).
- The algorithm consists of two phases:
  1. Information gathering: Values are filled in the tree level by level during the $f + 1$ rounds.
  2. Decision phase: Each $p_i$ calculates its decision based on the values in its tree.

# Consensus Solvable when $f = 1, n = 4$




- There is no ambiguity at any loyal commander, when taking majority decision
- Majority decision is over 2nd round messages, and 1st round message received directly from commander-in-chief process.

# Byzantine Generals (recursive formulation), (sync, msg-passing)

(variables)
**boolean**: $v \longleftarrow$ initial value;
**integer**: $f \longleftarrow$ maximum number of malicious processes, $\leq \lfloor (n-1)/3 \rfloor$;
(message type)
*Oral_Msg*($v$, *Dests*, *List*, *faulty*), where
$v$ is a boolean,
*Dests* is a set of destination process ids to which the message is sent,
*List* is a list of process ids traversed by this message, ordered from most recent to earliest,
*faulty* is an integer indicating the number of malicious processes to be tolerated.

*Oral_Msg(f)*, where $f > 0$:

1. The algorithm is initiated by the Commander, who sends his source value $v$ to all other processes using a $OM(v, N, \langle i \rangle, f)$ message. The commander returns his own value $v$ and terminates.
2. **[Recursion unfolding:]** For each message of the form $OM(v_j, Dests, List, f')$ received in this round from some process $j$, the process $i$ uses the value $v_j$ it receives from the source, and using that value, acts as a *new* source. (If no value is received, a default value is assumed.)
   To act as a new source, the process $i$ initiates *Oral_Msg*$(f' - 1)$, wherein it sends
   $OM(v_j, Dests - \{i\}, concat(\langle i \rangle, L), (f' - 1))$
   to destinations not in $concat(\langle i \rangle, L)$
   in the next round.
3. **[Recursion folding:]** For each message of the form $OM(v_j, Dests, List, f')$ received in Step 2, each process $i$ has computed the agreement value $v_k$, for each $k$ not in *List* and $k \neq i$,corresponding to the value received from $P_k$ after traversing the nodes in *List*, at one level lower in the recursion. If it receives no value in this round, it uses a default value. Process $i$ then uses the value $majority_{k \not\in List, k \neq i}(v_j, v_k)$ as the agreement value and returns it to the next higher level in the recursive invocation.

*Oral_Msg(0)*:

1. **[Recursion unfolding:]** Process acts as a source and sends its value to each other process.
2. **[Recursion folding:]** Each process uses the value it receives from the other sources, and uses that value as the agreement value. If no value is received, a default value is assumed.

# Relationship between # Messages and Rounds

| round number | a message has already visited | aims to tolerate these many failures | and each message gets sent to | total number of messages in round |
|---|---|---|---|---|
| 1 | 1 | $f$ | $n-1$ | $n-1$ |
| 2 | 2 | $f-1$ | $n-2$ | $(n-1)\cdot(n-2)$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $x$ | $x$ | $(f+1)-x$ | $n-x$ | $(n-1)(n-2)\ldots(n-x)$ |
| $x+1$ | $x+1$ | $(f+1)-x-1$ | $n-x-1$ | $(n-1)(n-2)\ldots(n-x-1)$ |
| $f+1$ | $f+1$ | 0 | $n-f-1$ | $(n-1)(n-2)\ldots(n-f-1)$ |

Table: Relationships between messages and rounds in the Oral Messages algorithm for Byzantine agreement.

Complexity: $f+1$ rounds, exponential amount of space, and

$$(n-1)+(n-1)(n-2)+\ldots+(n-1)(n-2)..(n-f-1)\text{messages}$$

# Bzantine Generals (iterative formulation), Sync, Msg-passing

(variables)
**boolean**: $v \longleftarrow$ initial value;
**integer**: $f \longleftarrow$ maximum number of malicious processes, $\leq \lfloor \frac{n-1}{3} \rfloor$;
**tree of boolean**:

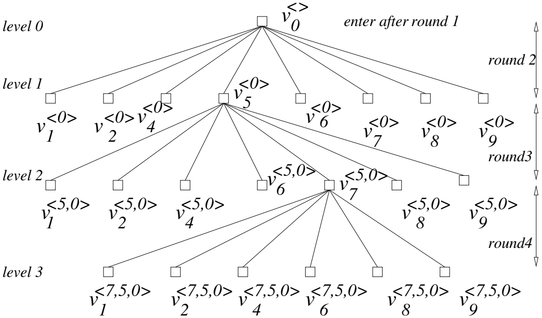
- level 0 root is $v^{L}_{init}$, where $L = \langle \rangle$;
- level $h (f \geq h > 0)$ nodes: for each $v^{L}_{j}$ at level $h - 1 = sizeof(L)$, its $n - 2 - sizeof(L)$ descendants at level $h$ are $v_k^{concat(\langle j \rangle, L)}$, $\forall k$ such that $k \neq j$, $i$ and $k$ is not a member of list $L$.

(message type)
OM($v$, $Dests$, $List$, $faulty$), where the parameters are as in the recursive formulation.

(1) Initiator (i.e., Commander) initiates Oral Byzantine agreement:
(1a) **send** $OM(v, N - \{i\}, \langle P_i \rangle, f)$ to $N - \{i\}$;
(1b) **return**($v$).

(2) (Non-initiator, i.e., Lieutenant) receives Oral Message $OM$:
(2a) **for** $rnd = 0$ **to** $f$ **do**
(2b) **for** each message OM that arrives in this round, **do**
(2c) **receive** $OM(v, Dests, L = \langle P_{k_1} \ldots P_{k_{f+1-faulty}} \rangle, faulty)$ from $P_{k_1}$;
// $faulty + round = f$; $|Dests| + sizeof(L) = n$
(2d) $v_{head(L)}^{tail(L)} \longleftarrow v$; // $sizeof(L) + faulty = f + 1$. fill in estimate.
(2e) **send** $OM(v, Dests - \{i\}, \langle P_i, P_{k_1} \ldots P_{k_{f+1-faulty}} \rangle, faulty - 1)$ to $Dests - \{i\}$ **if** $rnd < f$;
(2f) **for** $level = f - 1$ **down to** 0 **do**
(2g) **for** each of the $1 \cdot (n-2) \cdot \ldots (n - (level + 1))$ nodes $v^{L}_{x}$ in level $level$, **do**
(2h) $v^{L}_{x}(x \neq i, x \notin L) = majority_{y \notin concat(\langle x \rangle, L); y \neq i}(v^{L}_{x}, v_y^{concat(\langle x \rangle, L)})$;

# Tree Data Structure for Agreement Problem (Byzantine Generals)




Some branches of the tree at $P_3$. In this example, $n = 10, f = 3$, commander is $P_0$.

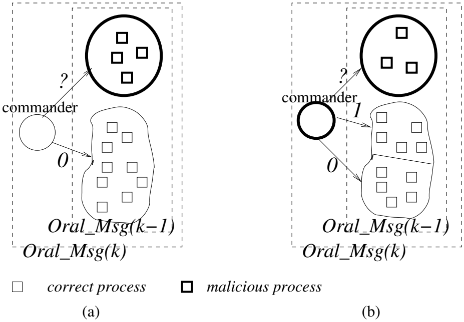
- (round 1) $P_0$ sends its value to all other processes using *Oral_Msg*(3), including to $P_3$.
- (round 2) $P_3$ sends 8 messages to others (excl. $P_0$ and $P_3$) using *Oral_Msg*(2). $P_3$ also receives 8 messages.
- (round 3) $P_3$ sends $8 \times 7 = 56$ messages to all others using *Oral_Msg*(1); $P_3$ also receives 56 messages.
- (round 4) $P_3$ sends $56 \times 6 = 336$ messages to all others using *Oral_Msg*(0); $P_3$ also receives 336 messages. The received values are used as estimates of the majority function at this level of recursion.

# Exponential Algorithm: An example

An example of the majority computation is as follows.

- $P_3$ revises its estimate of $v_7^{\langle 5,0\rangle}$ by taking $majority(v_7^{\langle 5,0\rangle}, v_1^{\langle 7,5,0\rangle}, v_2^{\langle 7,5,0\rangle}, v_4^{\langle 7,5,0\rangle}, v_6^{\langle 7,5,0\rangle}, v_8^{\langle 7,5,0\rangle}, v_9^{\langle 7,5,0\rangle})$. Similarly for the other nodes at level 2 of the tree.
- $P_3$ revises its estimate of $v_5^{\langle 0\rangle}$ by taking $majority(v_5^{\langle 0\rangle}, v_1^{\langle 5,0\rangle}, v_2^{\langle 5,0\rangle}, v_4^{\langle 5,0\rangle}, v_6^{\langle 5,0\rangle}, v_7^{\langle 5,0\rangle}, v_8^{\langle 5,0\rangle}, v_9^{\langle 5,0\rangle})$. Similarly for the other nodes at level 1 of the tree.
- $P_3$ revises its estimate of $v_0^{\langle\rangle}$ by taking $majority(v_0^{\langle\rangle}, v_1^{\langle 0\rangle}, v_2^{\langle 0\rangle}, v_4^{\langle 0\rangle}, v_5^{\langle 0\rangle}, v_6^{\langle 0\rangle}, v_7^{\langle 0\rangle}, v_8^{\langle 0\rangle}, v_9^{\langle 0\rangle})$. This is the consensus value.

# Impact of a Loyal and of a Disloyal Commander



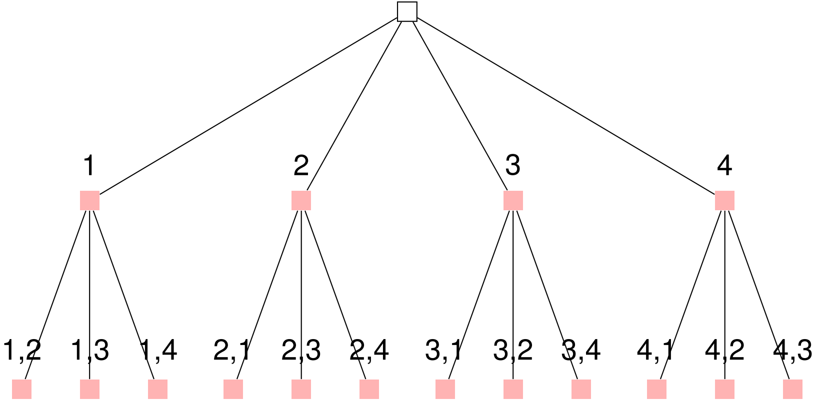

The effects of a loyal or a disloyal commander in a system with $n = 14$ and $f = 4$. The subsystems that need to tolerate $k$ and $k - 1$ traitors are shown for two cases. (a) Loyal commander. (b) No assumptions about commander.

(a) the commander who invokes *Oral_Msg(x)* is loyal, so all the loyal processes have the same estimate. Although the subsystem of $3x$ processes has $x$ malicious processes, all the loyal processes have the same view to begin with. Even if this case repeats for each nested invocation of *Oral_Msg*, even after $x$ rounds, among the processes, the loyal processes are in a simple majority, so the majority function works in having them maintain the same common view of the loyal commander's value.

(b) the commander who invokes *Oral_Msg(x)* may be malicious and can send conflicting values to the loyal processes. The subsystem of $3x$ processes has $x - 1$ malicious processes, but all the loyal processes do not have the same view to begin with.

# EIG algorithm: The tree structure

The EIG tree data structure with $n = 4$ and $f = 1$: the labels of nodes

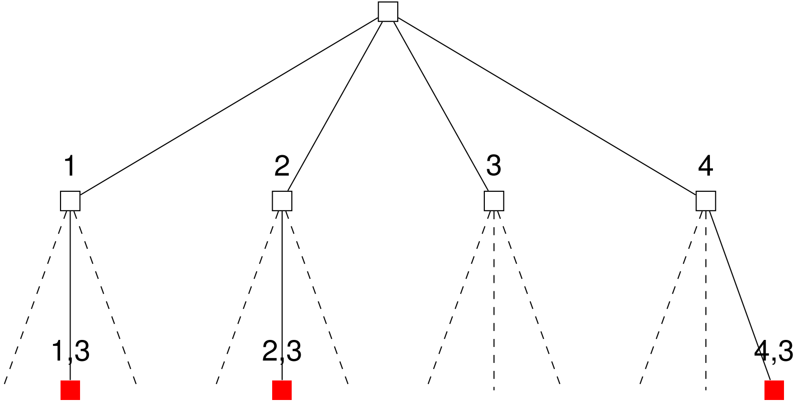


# The EIG algorithm: Information gathering

- Initially each $p_i$ stores its input in the root (level 0).
- Round $1 \leq r \leq f + 1$, each $p_i$:
  - broadcasts all nodes of the $r - 1$th level of its tree.
  - fills in level $r$: when it receives a message from $p_j$ with the value of the node labeled $v = i_1, \ldots, i_k$, it stores it to the node labeled $v,j$ of its tree (if a value for a node is not received, then default $u_\perp$ is stored).
- So, $p_i$ stores in node $i_1, \ldots, i_k, j$ the value that "$p_j$ says that $p_{i_k}$ says that . . . that $p_{i_1}$ said".

# The EIG algorithm: Information gathering

The tree is filled in from the root to the leaves, level by level.
Information received from $p_3$ in round 2 is stored at level 2:

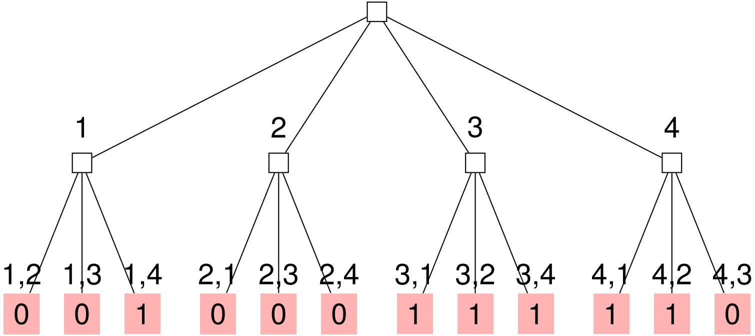


- At round $f+1$ the entire tree has been filled in. Node labeled with sequence $\pi$ has value $tree(\pi)$.
- Each $p_i$ applies to each subtree with root $\pi$ a recursive reduction function (usually majority vote) $resolve_i(\pi)$.
- The decision value is the resolved value of the root, $resolve()$, which is computed recursively based on the following definition:

$$resolve(\pi) = \begin{cases} tree(\pi) \text{ if } \pi \text{ is a leaf} \\ \\ majority\{resolve(\pi'),\ \pi' : child\ of\ \pi\} \text{ otherwise} \\ (u_\perp \text{ if no majority exists}) \end{cases}$$

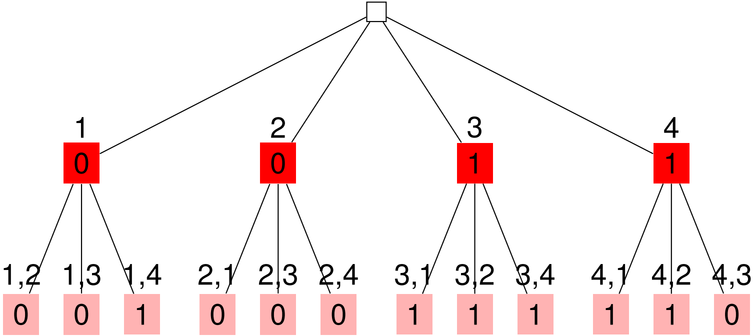
# The EIG algorithm: Decision phase

Start from the leaves, and compute the resolved value level by level till the root. Assume binary consensus (input $\in \{0, 1\}$). Default value is 0.

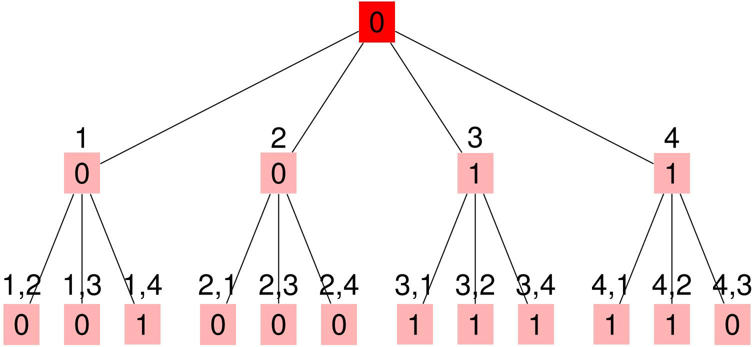


# The EIG algorithm: Decision phase

Start from the leaves, and compute the resolved value level by level till the root. Assume binary consensus (input $\in \{0, 1\}$). Default value is 0.



Start from the leaves, and compute the resolved value level by level till the root. Assume binary consensus (input $\in \{0, 1\}$). Default value is 0.



## Correctness of the EIG algorithm: Validity

Validity can be proved based on the following lemma that says that the resolved values are consistent.

**Lemma 1**

For each non-faulty $p_i$, the resolved value of a node $\pi = \pi' j$ corresponding to a non-faulty $p_j$ is the value that $p_j$ had stored in its $\pi'$ node: $resolve_i(\pi) = tree_j(\pi')$.

☺ Let's proove it (induction on the height of the tree).

- Suppose all non-faulty $p_i$s start with initial value $v$.
- The decision value of $p_i$ is $resolve_i() = majority\{resolve_i(j),\ j : child\ of\ root\}$
- By lemma 1 we have that $\forall\ non\text{-}faulty\ j :\ (resolve_i(j) = tree_j() = v)$.
- Since the majority of $p_i$s are non-faulty $resolve_i() = v$.

To proove agreement we need to introduce some additional terms.

- A node $\pi$ is ***common*** if
  $\forall p_i, p_j \in \mathit{Nonfaulty} : \ (\mathit{resolve}_i(\pi) = \mathit{resolve}_j(\pi))$.
- a node $\pi$ has a ***common frontier*** if every path from $\pi$ to a leaf contains a common node.

**Lemma 2**

If a node $\pi$ has a common frontier then $\pi$ is common.

☺ Let's proove it (induction on height of the tree, by contradiction).

# Correctness of the EIG algorithm: Agreement

Agreement can be proven based on lemma 2.

- The nodes on each path from a node at level 1 to a leaf correspond to $f + 1$ different processors.
- So, at least one such node $\pi$ corresponds to a non-faulty $p_j$, and by lemma 1 its resolved values are consistent (equal to $tree_j(\pi')$, where $\pi = \pi' j$), and thus it is common.
- Thus, the root has a common frontier since every path from the root to the leaves includes a common node.
- By lemma 2 the root is common, meaning that all non-faulty processors resolve the same decision value.

The EIG algorithm uses:

- $f + 1$ rounds (optimal)
- $n \geq 3f + 1$ processors (optimal)
- exponential size messages (sub-optimal):
  - At each round $r$ every process bradcasts the whole level $r$ of its tree.
  - At $r = 1$ each $p_i$ broadcasts one value, at $r = 2$ $n$ values, at $r = 3$ $n(n-1)$ values and at $r = k$ $n(n-1)(n-2)\dots(n-(r-2))$ values.
  - The largest message corresponds to $r = f + 1$: $n(n-1)\dots(n-(f+1)) = O(n^f)$.
- $n^2(f + 1)$ number of messages (sub-optimal)

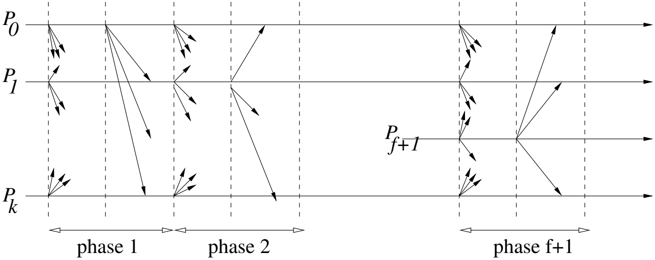
## The Berman-Garray algorithm

- ✓ We will present an algorithm that uses message of $O(1)$ size.
- ✗ However, at the cost of $2(f+1)$ rounds and with the requirement that $n > 4f$.
- The algorithm contains $f+1$ phases, each taking 2 rounds.
- Each $p_i$ has a *preference* at each phase, which is initially its input value and becomes its decision at phase $f+1$.
- At each phase $k$ $p_k$ is said to be the *king* of the phase.

# The Phase King Algorithm

Operation

- Each round has a unique "phases king" derived, say, from PID.
- Each round has two phases:
- in 1st phase, each process sends its estimate to all other processes.
- in 2nd phase, the "Phase king" process arrives at an estimate based on the values it received in 1st phase, and broadcasts its new estimate to all others.



# The Phase King Algorithm: Code

(variables)
**boolean**: $v \longleftarrow$ initial value;
**integer**: $f \longleftarrow$ maximum number of malicious processes, $f < \lceil n/4 \rceil$;

(1) Each process executes the following $f+1$ phases, where $f < n/4$:
(1a) **for** *phase* $= 1$ **to** $f+1$ **do**
(1b)   Execute the following Round 1 actions:   // actions in round one of each phase
(1c)     **broadcast** $v$ to all processes;
(1d)     **await** value $v_j$ from each process $P_j$;
(1e)     *majority* $\longleftarrow$ the value among the $v_j$ that occurs $> n/2$ times (default if no maj.);
(1f)     *mult* $\longleftarrow$ number of times that *majority* occurs;
(1g)   Execute the following Round 2 actions:   // actions in round two of each phase
(1h)     **if** $i =$ *phase* **then** // only the phase leader executes this send step
(1i)       **broadcast** *majority* to all processes;
(1j)     **receive** *tiebreaker* from $P_{phase}$ (default value if nothing is received);
(1k)     **if** *mult* $> n/2 + f$ **then**
(1l)       $v \longleftarrow$ *majority*;
(1m)     **else** $v \longleftarrow$ *tiebreaker*;
(1n)     **if** *phase* $= f+1$ **then**
(1o)       output decision value $v$.

# The Phase King Algorithm

- $(f+1)$ rounds, $(f+1)[(n-1)(n+1)]$ messages, and can tolerate up to $f < \lceil n/4 \rceil$ malicious processes

Correctness Argument

- Among $f+1$ rounds, at least one round $k$ where phase-king is non-malicious.
- In round $k$, all non-malicious processes $P_i$ and $P_j$ will have same estimate of consensus value as $P_k$ does.
  - $P_i$ and $P_j$ use their own majority values (Hint: $\Longrightarrow P_i$'s $mult > n/2 + f$)
  - $P_i$ uses its majority value; $P_j$ uses phase-king's tie-breaker value.
  - $P_i$ and $P_j$ use the phase-king's tie-breaker value. (Hint: In the round in which $P_k$ is non-malicious, it sends same value to $P_i$ and $P_j$)

  In all 3 cases, argue that $P_i$ and $P_j$ end up with same value as estimate
- If all non-malicious processes have the value $x$ at the start of a round, they will continue to have $x$ as the consensus value at the end of the round.

```
pref[i]=x_i, pref[j = v⊥] for any 0 ≤ j ≤ n−1, j ≠ i

Round 2k−1, 1 ≤ k ≤ f+1:
 broadcast(pref[i])
 receive v_j from p_j
 ∀j : 0 ≤ j ≤ n−1, j ≠ i pref[j]:=v_j
 maj:=majority{pref[0],...,pref[n−1]}
 mult=multiplicity of maj   /*#procs that voted for maj*/

Round 2k, 1 ≤ k ≤ f+1:
 if i = k then broadcast(maj)   //p_i is the king
 receive(king_maj) from p_k
 if (mult > n/2 + f)
  then pref[i]:=maj
 else pref[i]:=king_maj
 if (k = f+1) then decision:=pref[i]
```

# Correctness of the Berman-Garray algorithm: Validity

**Lemma 3**

If all nonfaulty processes prefer $v$ at the beginning of phase $k$, then they all prefer $v$ at the end of phase $k$, for all $1 \leq k \leq f + 1$.

Proof sketch:

- Each $p_i$ receives at least $n - f$ copies of $v$ (including its own) in the first round of phase $k$.
- Because of the assumption $n > 4f \Rightarrow n/2 > 2f \Rightarrow n > n/2 + 2f \Rightarrow n - f > n/2 + f$.
- Thus, all nonfaulty $p_i$s will prefer $v$ at the end of phase $k$.

Validity follows by lemma 3: If all nonfaulty $p_i$s start with $v$, they continue to prefer $v$ throughout the phase.

- Observation: There are at most $f$ faulty $p_i$s, and $f + 1$ phases, so at least one phase has a nonfaulty king.

**Lemma 4**

Let $g$ be a phase whose king $p_g$ is nonfaulty. Then all nonfaulty $p_i$s finish phase $g$ with the same preference.

Proof sketch:

- Case 1: Suppose all nonfaulty $p_i$s use king's majority for their preference. Since the king is nonfaulty it sends everyone the same value.

Proof sketch of lemma 4 continued:

- Case 2: Suppose that some nonfaulty $p_j$ uses its own majority value $v$ for its preference. This means that $p_j$ has received more than $n/2 + f$ votes for $v$ in the first round of $g$. Thus, every processor, including the king $p_g$, has received more than $n/2$ votes for $v$ in the first round of $g$, and sets its majority value to $v$.

➢ Agreement follows by lemma 4: At phase $g + 1$ all nonfaulty $p_i$s start with the same preference and by lemma 3 this agreement persists.