



Supercomputing for Giga- and Tera-Pixel Image Scales, Part II: Connectivity and Connected Filters

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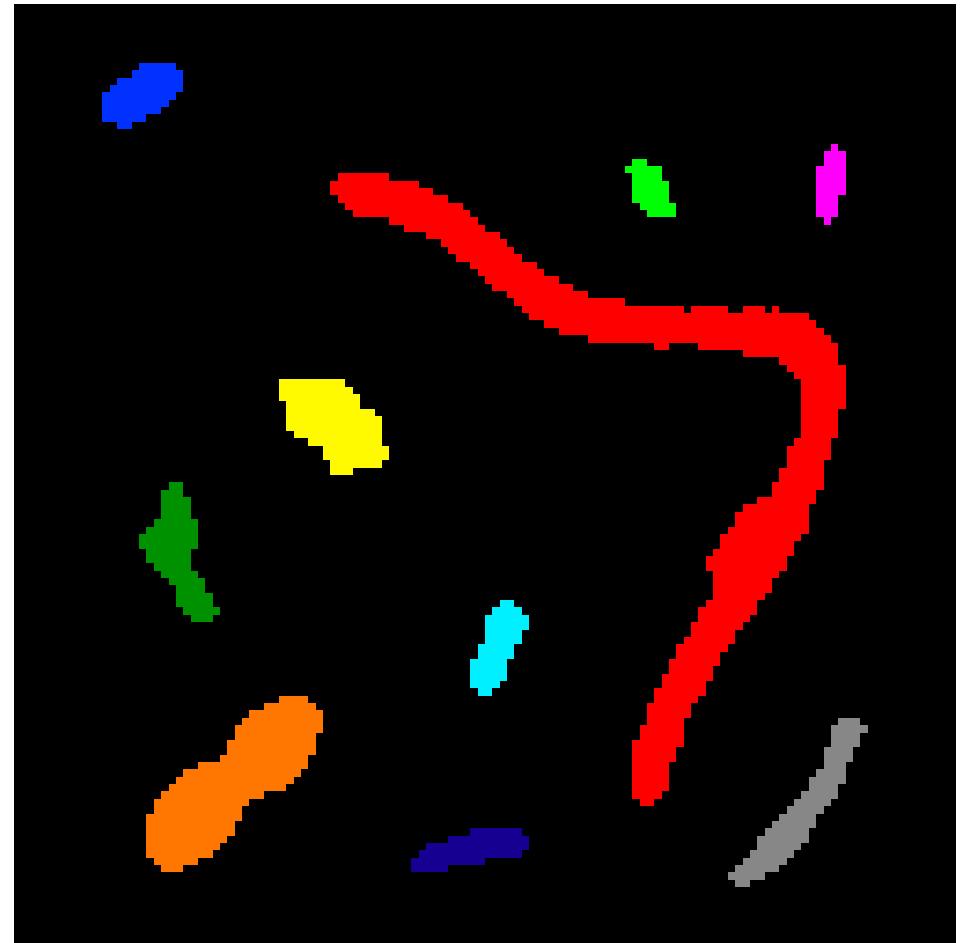
- Connected Component Labelling
 - Sequential algorithms
 - Parallel algorithms
- Connected Filters
 - Binary case
 - Grey scale
- Algorithms for connected filters
 - Sequential algorithms
 - Low dynamic range shared-memory
 - High dynamic range shared memory
 - Low dynamic range distributed memory



- Connected components are *connected sets* of pixels in the image of a constant value of *maximum extent*.
- This means that for every connected component C of image f
 - For ever pixel $p \in C$ there are no neighbours q such that $q \notin C$ and $f(p) = f(q)$
 - For ever pair of pixels $p, q \in C$ there exists a path $\{x_0, x_1, \dots, x_n\} \subseteq C$, such that $x_0 = p$, $x_n = q$, for all x_i $f(x_i) = f(p)$, and for all $i \in \{1, 2, \dots, n\}$ we have x_{i-1} and x_i are neighbours.
- Connected components partition the image domain into different *equivalence classes*



- Assigns labels to all pixels such that:
 - All the pixels from the *same* connected component have the *same* label
 - All pixels from *different* connected components have *different* labels
- Essential step in many segmentation methods





- Equivalence table methods
 - Earliest
 - Memory hungry!
- Flood filling
 - Simple and effective
- Union-Find
 - Efficient in both time and storage
 - Simple, but strange!



- We assume an input image f of N pixels
- We require a queue Q of size $O(N)$.
- We require a label image L of size N (typically unsigned int or long).
- Convention: label 0 means “unlabelled”



```
CurLabel ← 0;                                ▷ No current label in use
for all pixels  $p$  do
     $L[p] \leftarrow 0$ ;                         ▷ All pixels unlabelled
end for
for  $p \leftarrow 0$  to  $N - 1$  do
    if  $L[p] = 0$  then                      ▷ Pixels not labelled yet
        CurLabel ++;                          ▷ Create new label
        PUSH( $p, Q$ );
    while not IsEMPTY( $Q$ ) do                ▷ Start flooding
        curpix ← POP( $Q$ );
         $L[curpix] \leftarrow CurLabel$ ;
        for all neighbours  $n$  of curpix do
            if  $f[n] = f[curpix]$  then          ▷ Test for equivalence
                PUSH( $n, Q$ );                  ▷ Push into queue
            end if
        end for
    end while
    end if
end for
```



- A key issue is the use of a FIFO queue: cannot be parallelized
- Dividing the image into arbitrary sections does not work:
 - This will inevitably cut components into multiple sections
 - These will not get the same label
 - Some costly relabelling will be needed to correct for this
- Processing per grey level has problems with load balancing
- It will not work on binary image very well



- Designed to deal efficiently with disjoint sets
- Represents disjoint sets as a forest
- Items sharing the same root are in the same set
- Relies on three basic functions
 - $\text{MAKESET}(x)$: Initialises item x as a singleton set
 - $\text{UNION}(x, y)$: Merges trees containing x and y
 - $\text{FIND}(x)$: Find the root of the tree containing x



```
procedure MAKESET(x)
```

```
    x.parent  $\leftarrow$  x
```

```
end procedure
```

▷ Convention: the root is its own parent

```
function FIND( $x$ )
```

```
    if  $x.\text{parent} = x$  then
```

▷ x is the root

```
        return  $x$ ;
```

```
    else
```

```
        return FIND( $x.\text{parent}$ );
```

▷ Recursively seek the root

```
    end if
```

```
end function
```



procedure UNION(x, y)

$xr \leftarrow \text{FIND}(x);$

▷ Find the roots

$yr \leftarrow \text{FIND}(y);$

if $xr \neq yr$ **then**

▷ If not already in same set

$xr.\text{parent} \leftarrow yr;$

▷ Merge two sets

end if

end procedure



During find, it pays to flatten the tree, by letting all pointers along the root path point directly to the root:

function FIND(x)

if $x.\text{parent} \neq x$ **then**
 $x.\text{parent} \leftarrow \text{FIND}(x.\text{parent})$;
end if

▷ x is not the root
▷ Recursively seek the root

return $x.\text{parent}$

end function



```
function FIND( $x$ )
     $r \leftarrow x$ 
    while  $r.parent \neq r$  do
         $r \leftarrow r.parent;$ 
    end while

    while  $x.parent \neq r$  do
         $s \leftarrow x.parent;$ 
         $x.parent \leftarrow r;$ 
         $x \leftarrow s$ 
    end while

    return  $r;$ 

end function
```

- ▷ Iteratively seek the root
- ▷ r is now root
- ▷ Save current parent
- ▷ Update parent to root



A further improvement adds a *rank* to each item, to keep trees balanced:

```
procedure MAKESET(x)
    x.parent  $\leftarrow$  x
    x.rank  $\leftarrow$  0;
end procedure
```

▷ Convention: the root is its own parent
▷ Rank starts at zero

The rank maintains the depth of the tree (unless path-compression is also used)



```
procedure UNION(x,y)
    xr  $\leftarrow$  FIND(x);
    yr  $\leftarrow$  FIND(y);
    if xr  $\neq$  yr then
        if xr.rank  $>$  yr.rank then
            yr.parent  $\leftarrow$  xr;
        else
            if yr.rank  $>$  xr.rank then
                xr.parent  $\leftarrow$  yr;
            else
                xr.parent  $\leftarrow$  yr;
                yr.rank  $++$ ;
            end if
        end if
    end if
end procedure
```

- ▷ Find the roots
- ▷ If not already in same set
- ▷ Link *y* to *x*
- ▷ Link *x* to *y*
- ▷ You can choose any order



- The storage complexity of Union-Find is $O(N)$
- If we use path compression *or* union-by-rank we have $O(N \log N)$ computational complexity
- If we use path compression *and* union-by-rank we have $O(N\alpha(N))$ computational complexity
- α is the *inverse Ackermann function* which grows *extremely slowly*
- For certain processing orders, it becomes $O(N)$ in time complexity, even without union-by-rank



Phase 1: Union-Find

```
for  $p \leftarrow 0$  to  $N - 1$  do
    MAKESET( $p$ );
    for all neighbours  $q$  of  $p$  with  $q < p$  do
        if  $f(p) = f(q)$  then
            UNION( $p, q$ );
        end if
    end for
end for
```

▷ Initialize p

▷ Only previously
processed neighbours

▷ If same grey scale
▷ Merge

Note: The order of processing does not matter



Phase 2: Resolve, assume all $L[p] = 0$, and curlabel= 0

for $p \leftarrow 0$ **to** $N - 1$ **do**

$r \leftarrow p$

while $L[r] = 0$ **and** $r \neq r.\text{parent}$ **do**

$r \leftarrow r.\text{parent}$

end while

if $L[r] = 0$ **then**

 curlabel++;

$L[r] \leftarrow \text{curlabel};$

end if

$s \leftarrow p$

while $s \neq r$ **do**

$L[s] \leftarrow L[r]$

$s \leftarrow s.\text{parent}$

end while

end for

▷ Find root, or labelled
▷ pixel along root path

▷ We have an unlabelled root

▷ Create new label

▷ Assign to root

▷ $L[r]$ now contains the correct label

▷ Assign label to pixels along root path



- Because the order of the union operations does not matter, we can easily perform them in parallel
- Simply split the image into a number of sections
- Perform phase 1 of union-find CC on each section
- Hierarchically merge the sections, by processing the neighbours along the borders
- Once merged, perform resolving in parallel



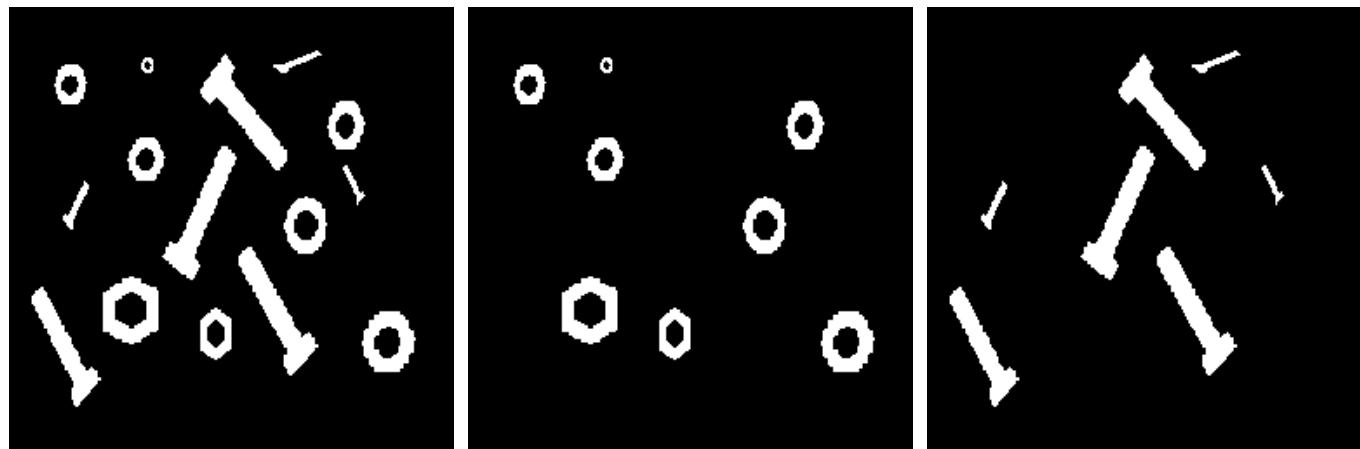
- In the binary case, the area opening removes all connected *foreground* components below a certain area
- We first compute all connected fore ground components
- Compute the area of each
- Assign each pixel of each foreground component a grey level equal to the area
- Threshold the resulting image at some threshold λ



- Introduced by Breen and Jones in 1996.
- Examples: area openings/closings, attribute openings, shape filters
- How do they work?

Binary image :

1. compute attribute for each connected component
2. keep components of which attribute value exceeds some threshold λ





- Let $T : \mathcal{P}(E) \rightarrow \{\text{false}, \text{true}\}$ be increasing, i.e. $C \subseteq D$ implies $T(C) \Rightarrow T(D)$.
- A binary *trivial opening* $\Gamma_T : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ using T as above is defined as

$$\Gamma_T(C) = \begin{cases} C & \text{if } T(C), \\ \emptyset & \text{otherwise.} \end{cases} \quad (1)$$

- A typical form of T is

$$T(C) = (\mu(C) \geq \lambda) \quad (2)$$

with μ an increasing scalar (i.e. $C \subseteq D \Rightarrow \mu(C) \leq \mu(D)$), and λ the *attribute threshold*.

- The binary *attribute opening* Γ^T is defined as

$$\Gamma^T(X) = \bigcup_{x \in X} \Gamma_T(\Gamma_x(X)), \quad (3)$$

 X  $T = A(C) \geq 11^2$  $T = I(C) \geq 11^4/6$

- An *area opening* is obtained if the criterion $T = A(C) \geq \lambda$, with A the area of the connected set C .
- A *moment-of-inertia opening* is obtained if the criterium is of the form $T = I(C) \geq \lambda$, with I the moment of inertia.



- In the case of attribute openings, generalization to grey scale is achieved through *threshold decomposition*.
- A threshold set X_h of grey level image (function) f is defined as

$$X_h(f) = \{x \in E | f(x) \geq h\}. \quad (4)$$

- The grey scale attribute opening γ^T based on binary counterpart Γ^T is given by

$$(\gamma^T(f))(x) = \sup\{h \leq f(x) | x \in \Gamma^T(X_h(f))\} \quad (5)$$

- Closings ψ^T are defined by duality:

$$\psi^T(f) = -\gamma^T(-f). \quad (6)$$

- The non-increasing case will be dealt with after discussing the algorithms.



Lenna with noise (left) structural open-close with square S.E. (middle)
area open-close (right)



- We need to build connected components starting at the highest grey levels
- We grow these components by adding new pixels, potentially at lower grey levels *until* the area is large enough.
- The root of such a component should “contain” the final grey level
- The root of each component should also contain the area of the CC
- Any connected component with a neighbour of larger grey level and area $\geq \lambda$ should remain unchanged!

- To achieve these goals, sort pixels in descending grey level order.
- Pixels of the same grey level are stored in lexicographic order
- As we perform unions, we always maintain the root as *the last pixel added*
- We let

$$x.\text{parent} = -\text{AREA}(CC(x))$$

A negative parent flags a root!

- During unions, we simply add the parent values to maintain the area
- Merges of roots of differing grey levels only take place if the one with the larger grey level has an area smaller than λ



```
procedure MAKESET(x)
```

```
    x.parent  $\leftarrow$  -1
```

```
end procedure
```

▷ Area of singleton = 1

```
function FIND( $x$ )
```

```
    if  $x.\text{parent} \geq 0$  then
```

```
         $x.\text{parent} \leftarrow \text{FIND}(x.\text{parent});$ 
```

```
        return  $x.\text{parent}$ 
```

```
    else
```

```
        return  $x$ 
```

```
    end if
```

```
end function
```

▷ x is not the root

▷ Recursively seek the root

▷ x is the root, but $x.\text{parent}$ is not!



Require: $x \prec y$

▷ x earlier in processing order

Require: y is a root

procedure UNION(x, y)

$xr \leftarrow \text{FIND}(x);$

if $xr \neq y$ **then**

if $f(xr) > f(y) \wedge -xr.\text{parent} \geq \lambda$ **then**

$y.\text{parent} = -\lambda;$

else

$y.\text{parent} \leftarrow y.\text{parent} + xr.\text{parent};$

$xr.\text{parent} \leftarrow y;$

end if

end if

end procedure

▷ If not already in same set

▷ Block further merges

▷ Add areas

▷ Always point to last processed



Sort all pixels in descending grey level order

for All pixels p in sort order **do**

 MAKESET(p);

for All neighbours q of p such that $q \prec p$ **do**

 UNION(q, p);

end for

end for

for All pixels p in reverse sort order **do**

 ▷ Resolve

if $p.\text{parent} < 0$ **then**

 ▷ We have a root

 Out[p] $\leftarrow f[p]$

 ▷ $f[p]$ is the lowest grey level

else

 Out[p] \leftarrow Out[$p.\text{parent}$]; ▷ The parent has already been resolved

end if

end for

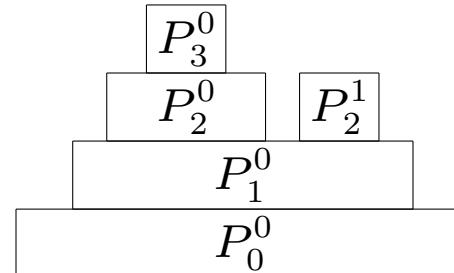


- A spatial division of labour has problems:
 - We cannot compute the areas of CCs correctly per section
 - We would need a barrier per grey level to link different sections first
 - Even if the number of grey levels $G = 256$ this kills performance
- Processing a series of G binary images and stacking them is possible, but:
 - Performing a sequential grey level area opening is as fast as a binary one
 - Overhead from computing thresholds would kill performance



0	1	2	3	2	1	2	1	0
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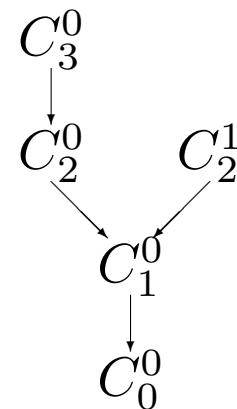
input signal



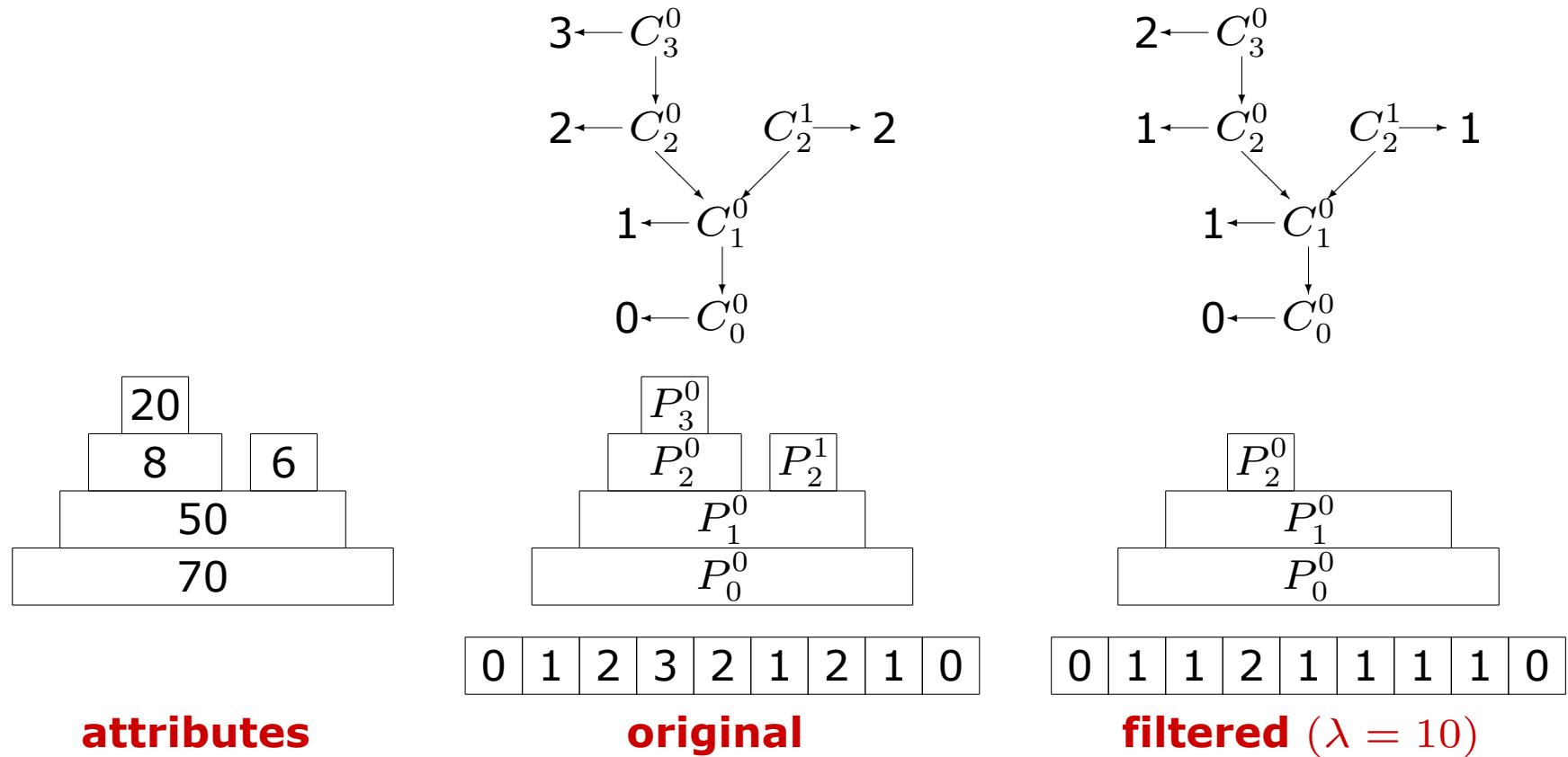
peak components

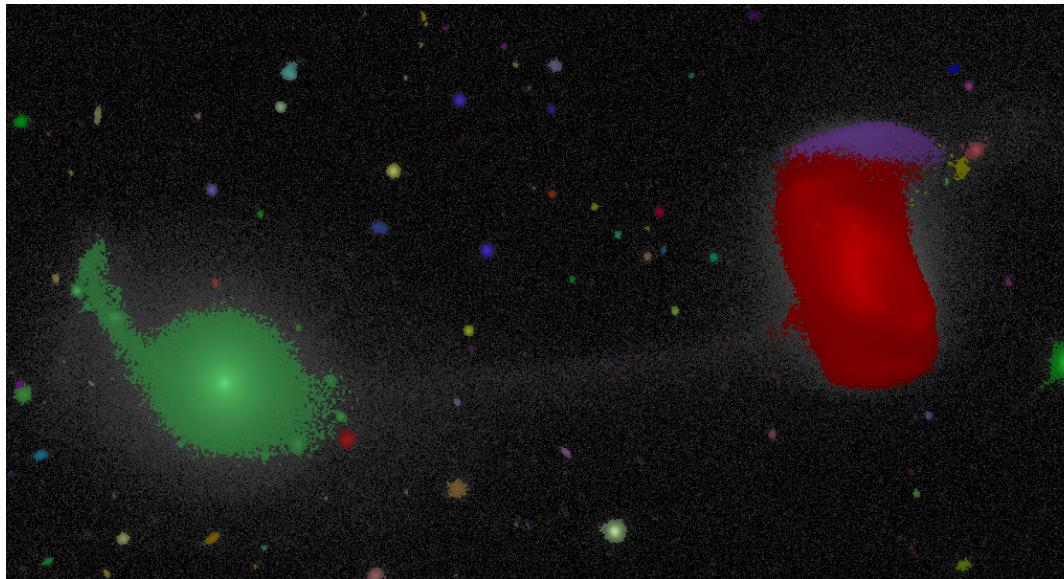
C_0^0	C_1^0	C_2^0	C_3^0	C_2^0	C_1^0	C_2^1	C_1^0	C_0^0
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labelling

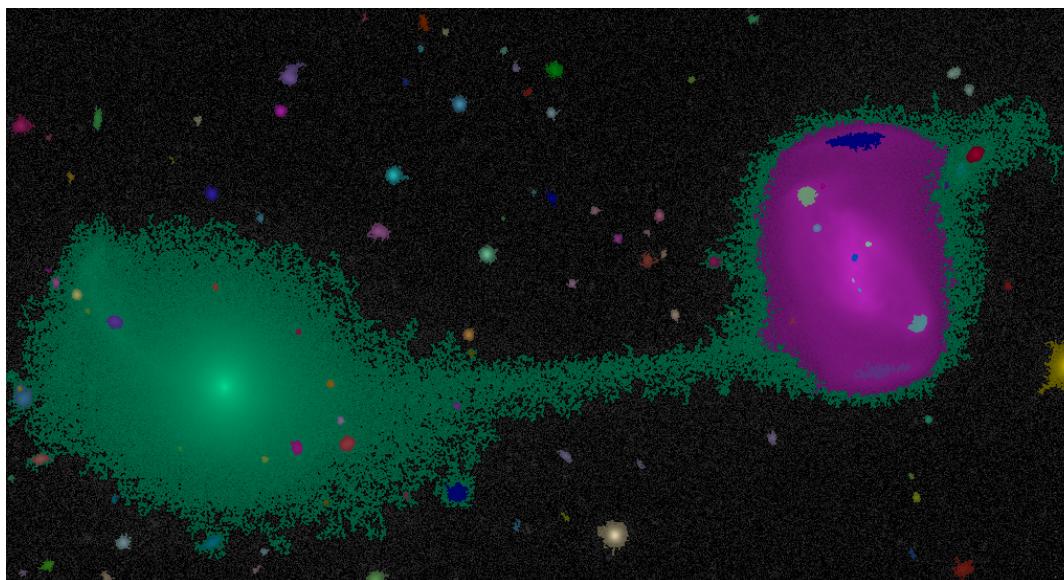


Max-Tree

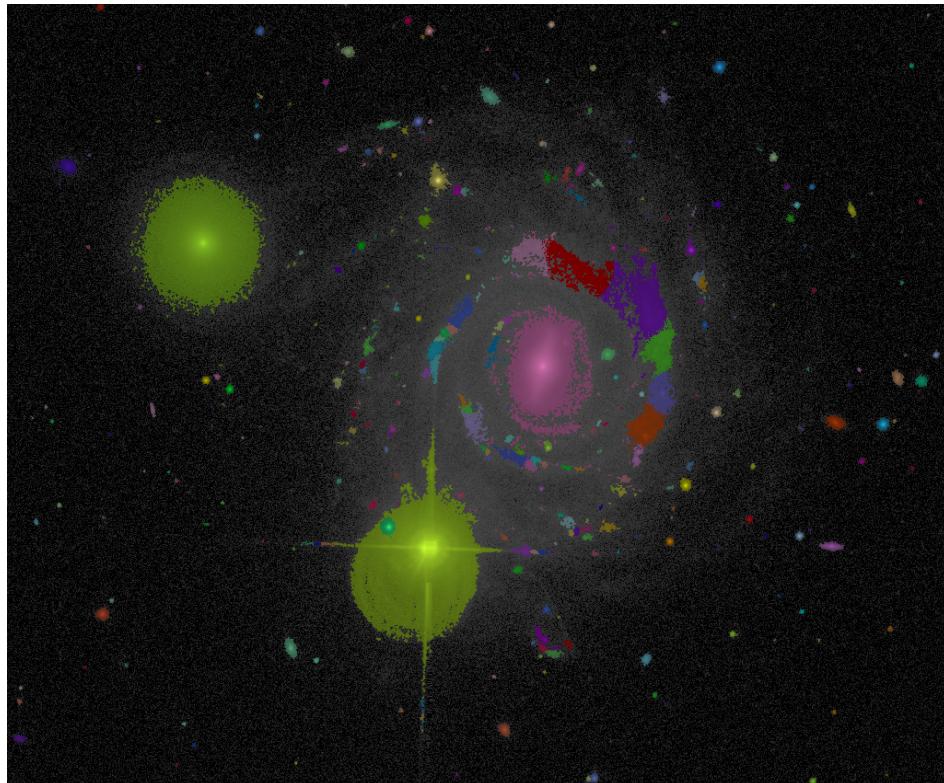




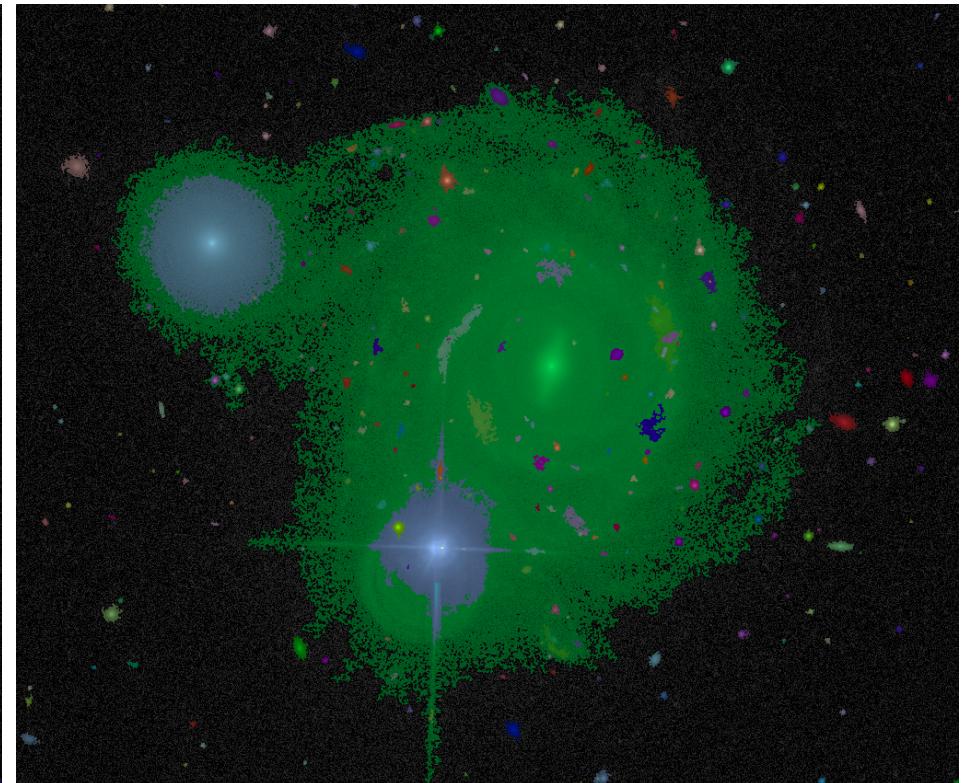
SExtractor



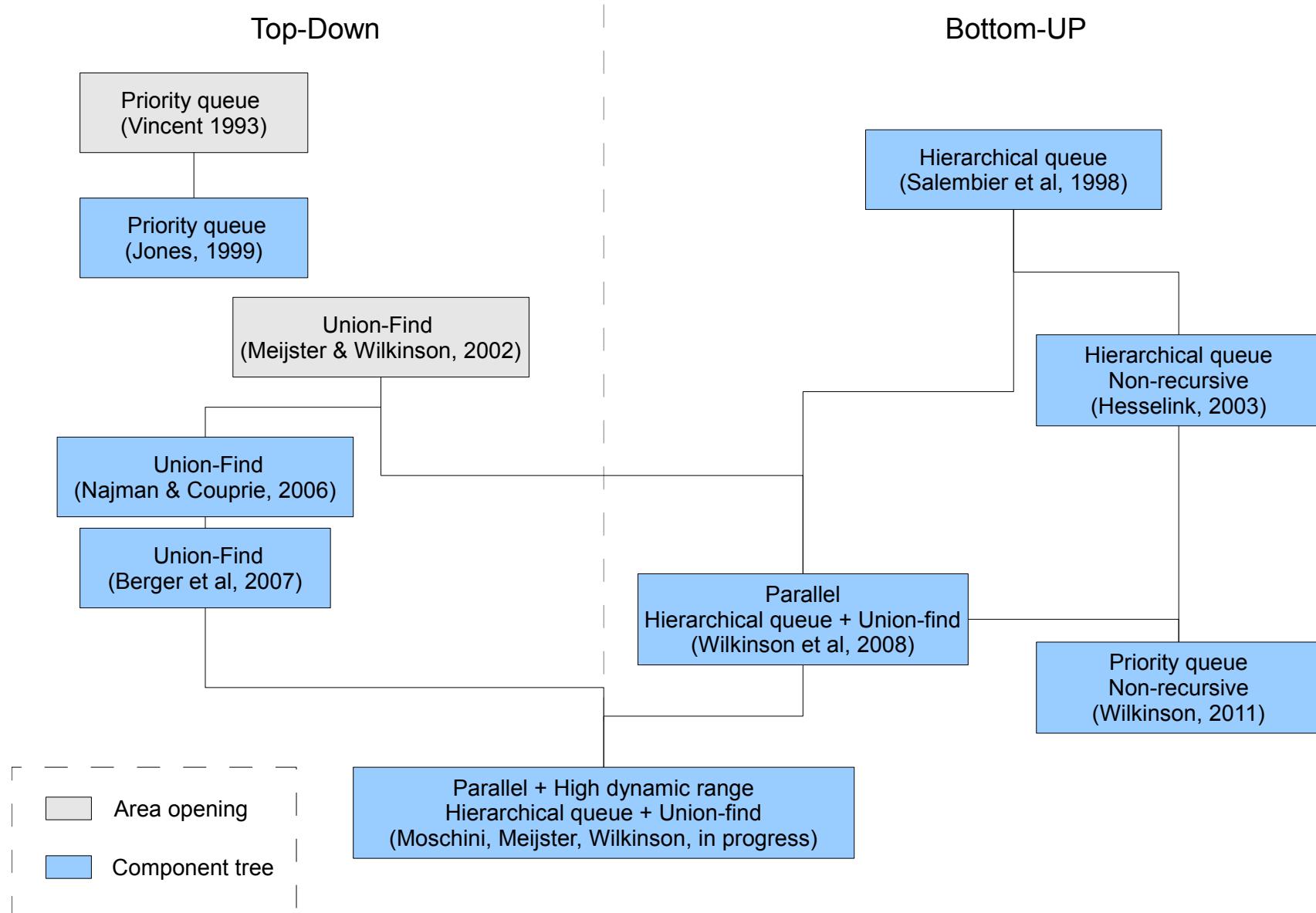
MTOBJECTS

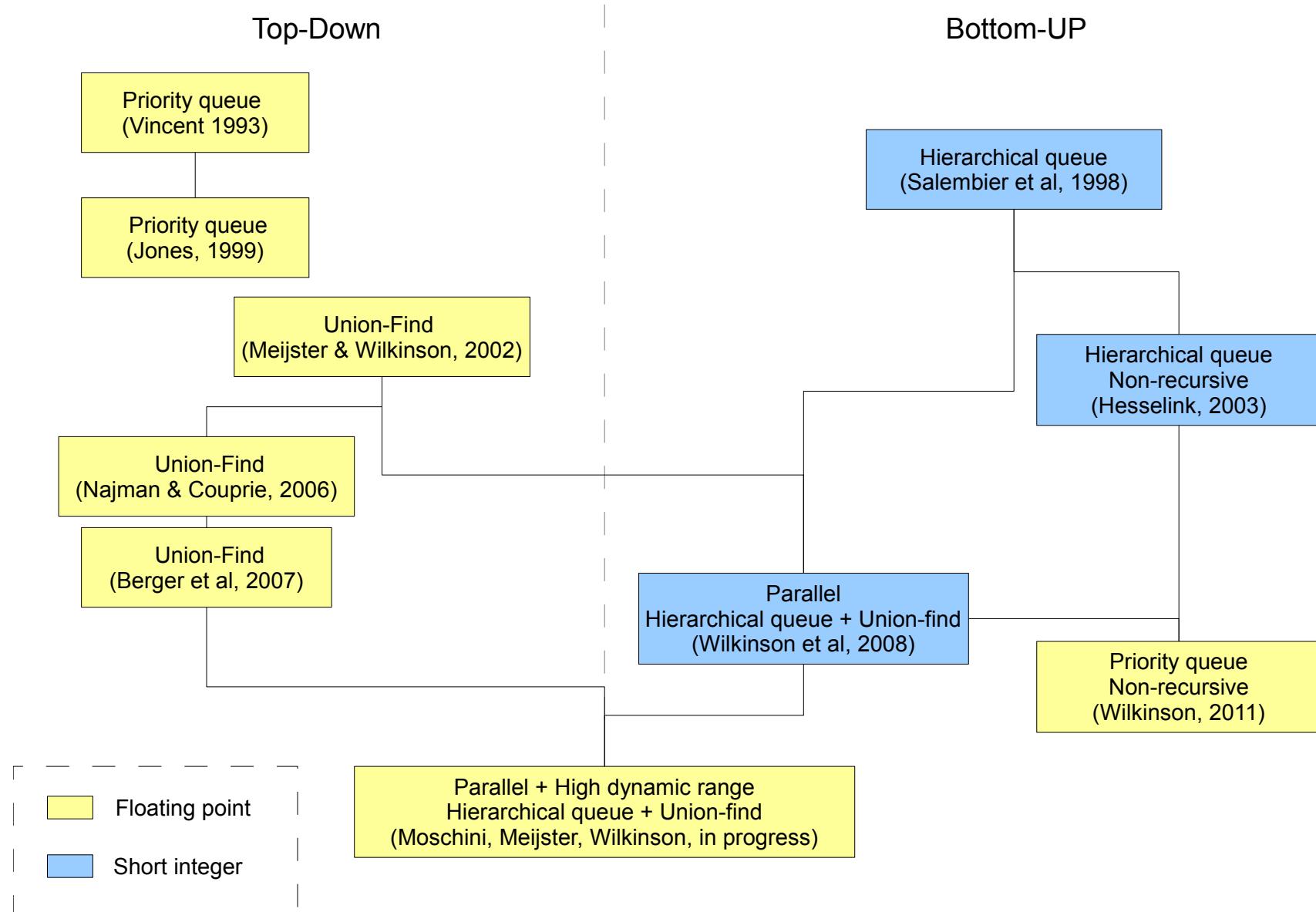


SExtractor



MTObjects







- Salembier et al. developed an efficient algorithm to compute the Max-Tree
- It requires:
 - hierarchical queue: one FIFO queue per grey level
 - a boolean array NodeAtLevel
 - an array NumNodesAtLevel
 - a integer array Status of the same size as the image
- All four structures are cleared before the algorithm starts



- We start by selecting a pixel at the lowest grey level h_{\min} in the image
- This is inserted at h_{\min} in the hierarchical queue
- A recursive, depth first flooding algorithm is called at level h_{\min}
- Attributes are computed as the tree is built
- Filtering is implemented as a simple look-up, where the combination of $\text{Status}[p]$ and $f[p]$ determine to which node pixel p belongs



- The algorithm is the fastest for building Max-Trees for low dynamic range images sequentially
- The fact that it is queue-based makes it hard to parallelize
- More importantly, the use of arbitrary labels means merging Max-Trees of image sections is costly
- The fact that a FIFO-queue is needed per grey level (and two arrays of length G means the algorithm will not work on extreme dynamic range images (>16 bits per pixel))
- A non-recursive alternative using a stack and a priority queue exists to solve the latter problem

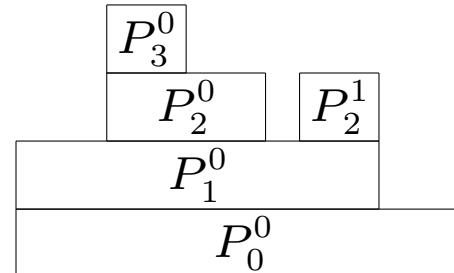


- We have developed a parallel algorithm:
 - divide the image (or volume) into strips
 - compute a Max-tree for each strip
 - merge the local Max-trees into a single tree
 - performing the filtering strip-wise
- To do this efficiently, we need to merge the Max-tree and union-find approaches.
- The big problem is keeping the attributes correct



1	1	3	2	2	1	2	0	0
---	---	---	---	---	---	---	---	---

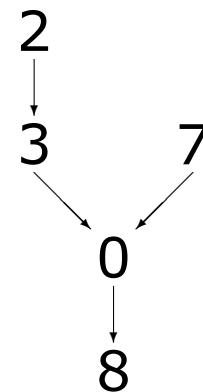
input signal



peak components

0	0	2	3	3	0	7	8	8
---	---	---	---	---	---	---	---	---

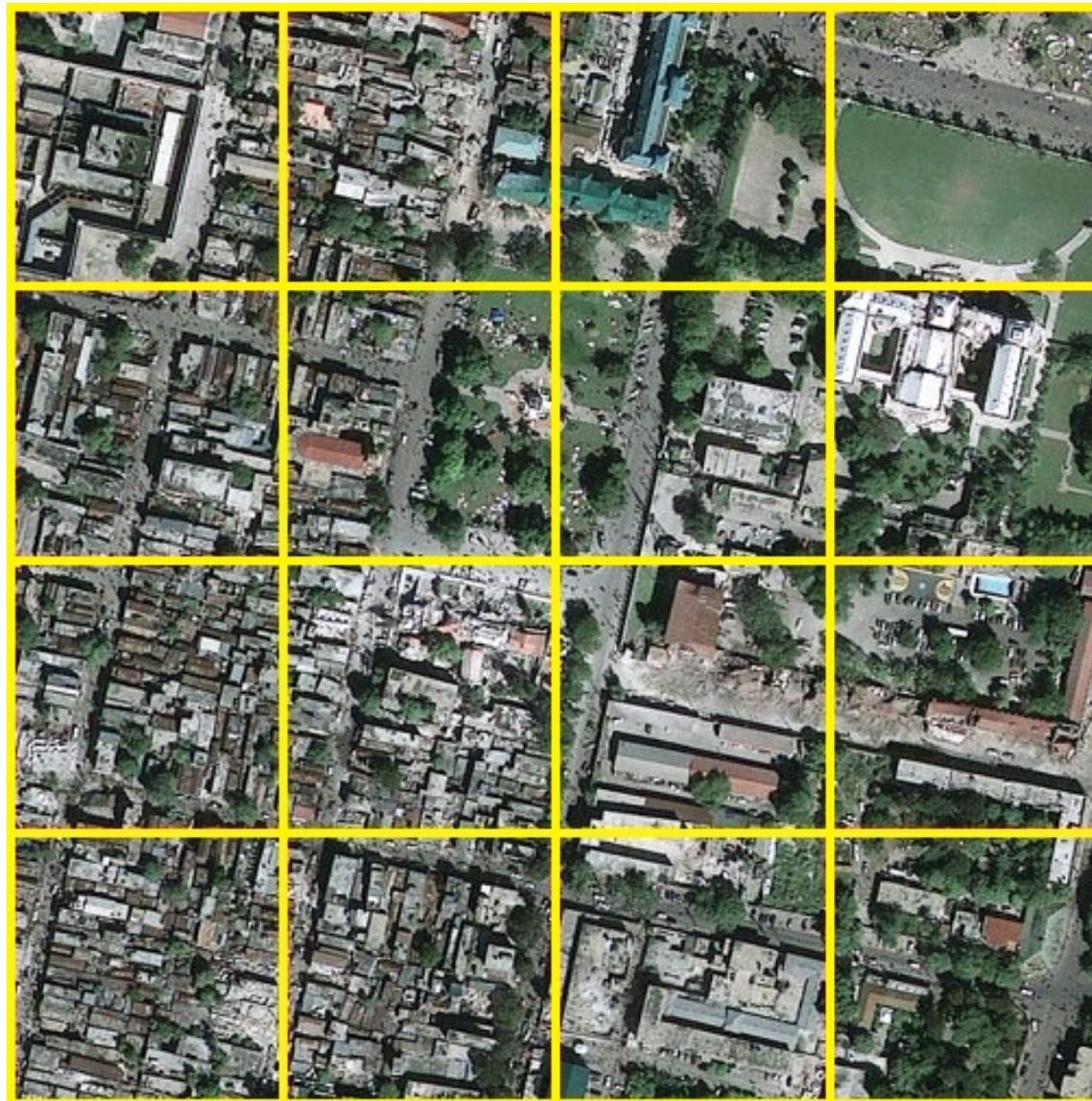
labelling



Max-Tree



- We store the Max-Tree as an array of nodes of the same size as the image
- Each node has a parent field which contains the index of its parent
- Use the first pixel p in each node found as *canonical element*
- Fill the parent field of all pixels of that node with the index p
- The parent pointer of p is set to the canonical element of its parent in the Max-Tree
- Such pixels are distinguished by the fact that $f[p] > f[p.\text{parent}]$
- We call these pixels *level roots*
- The root of an entire Max-Tree has a parent with special index \perp





procedure FLOOD(Level lev , Partition P , Attribute a)

$at \leftarrow a;$

while $lev \neq 0$ **do**

 Extract pixel p from the queue; Update attribute in $levelroot[p]$;

for all neighbours $q \in P$ of p **do**

if $isVisited[q] = false$ **then**

$isVisited[q] \leftarrow true$; $atq \leftarrow 0$;

if $levelroot[f(q)] = \text{not set}$ **then**

$levelroot[f(q)] \leftarrow q$;

else

$q.parent \leftarrow levelroot[f(q)]$;

end if

 Add q to the queue;

while $f(q) > lev$ **do**

$flood(f(q), P, atq)$; $at \leftarrow a + atq$;

end while

end if

end for

end while



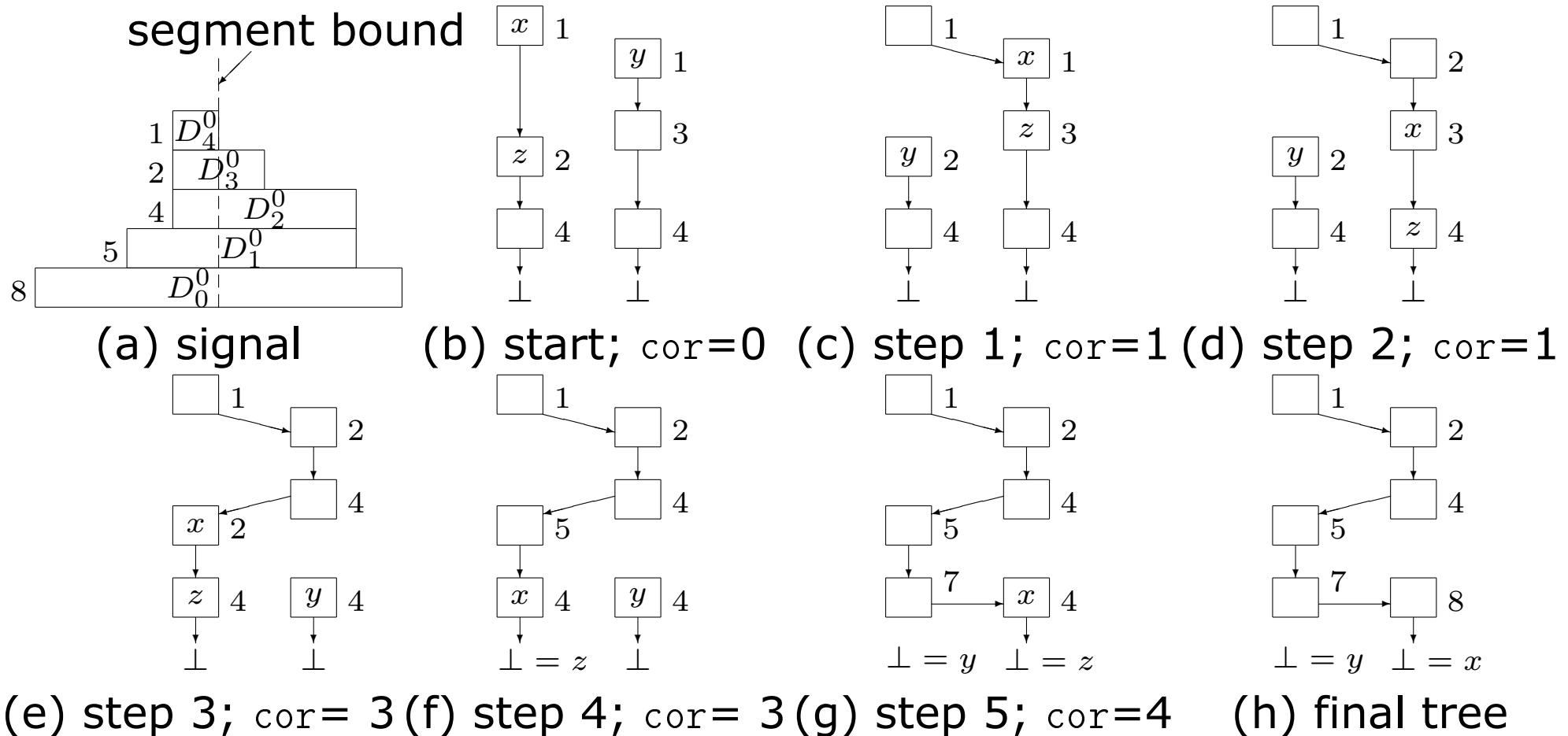
```
m ← m - 1;  
while m ≥ 0 ∧ levelroot[m] = not set do  
    m ← m - 1;  
end while  
if m ≥ 0 then  
    par(node_qu[levelroot[lev]]) ← levelroot[m];  
end if  
node_qu[levelroot[lev]].attribute = at;  
levelroot[lev] ← not set;  
a ← at;  
end procedure
```



```
procedure CONNECT(th, i, (x,y))
    area  $\leftarrow$  0; areat  $\leftarrow$  0;
    x  $\leftarrow$  GetLevelRootOf(x);    y  $\leftarrow$  GetLevelRootOf(y);
    if  $f(x) < f(y)$  then
        Swap(x,y);
    end if
    while  $x \neq y \wedge x \neq \perp$  do
        z  $\leftarrow$  GetLevelRootOf(node[x].parent);
        if  $f(z) \geq f(y) \wedge z \neq \perp$  then
            node[x].Area  $\leftarrow$  node[x].Area + area;
            x  $\leftarrow$  z;
        else
            areat  $\leftarrow$  node[x].Area + area;
            area  $\leftarrow$  node[x].Area;
            node[x].Area  $\leftarrow$  areat;
            node[x].parent  $\leftarrow$  y;
            x  $\leftarrow$  y; y  $\leftarrow$  z;
        end if
    end while
```

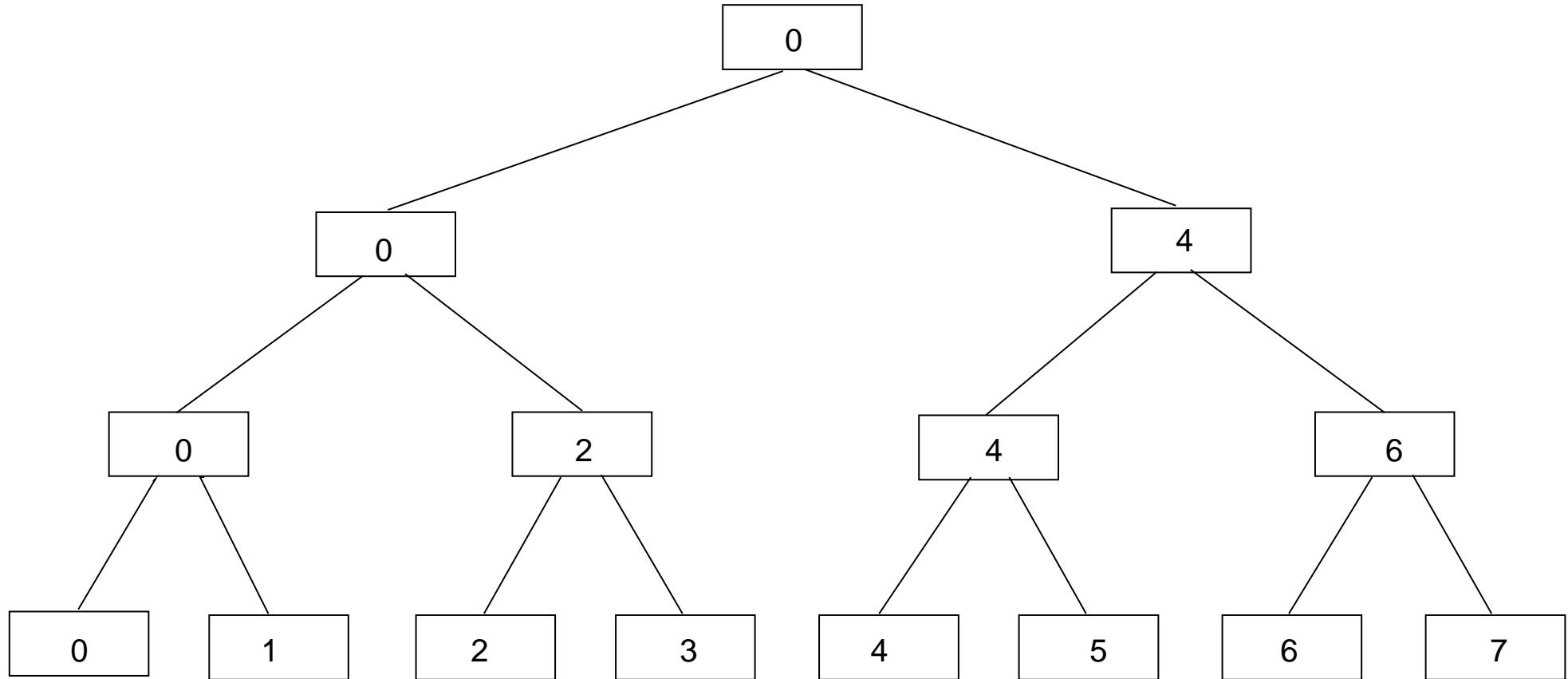


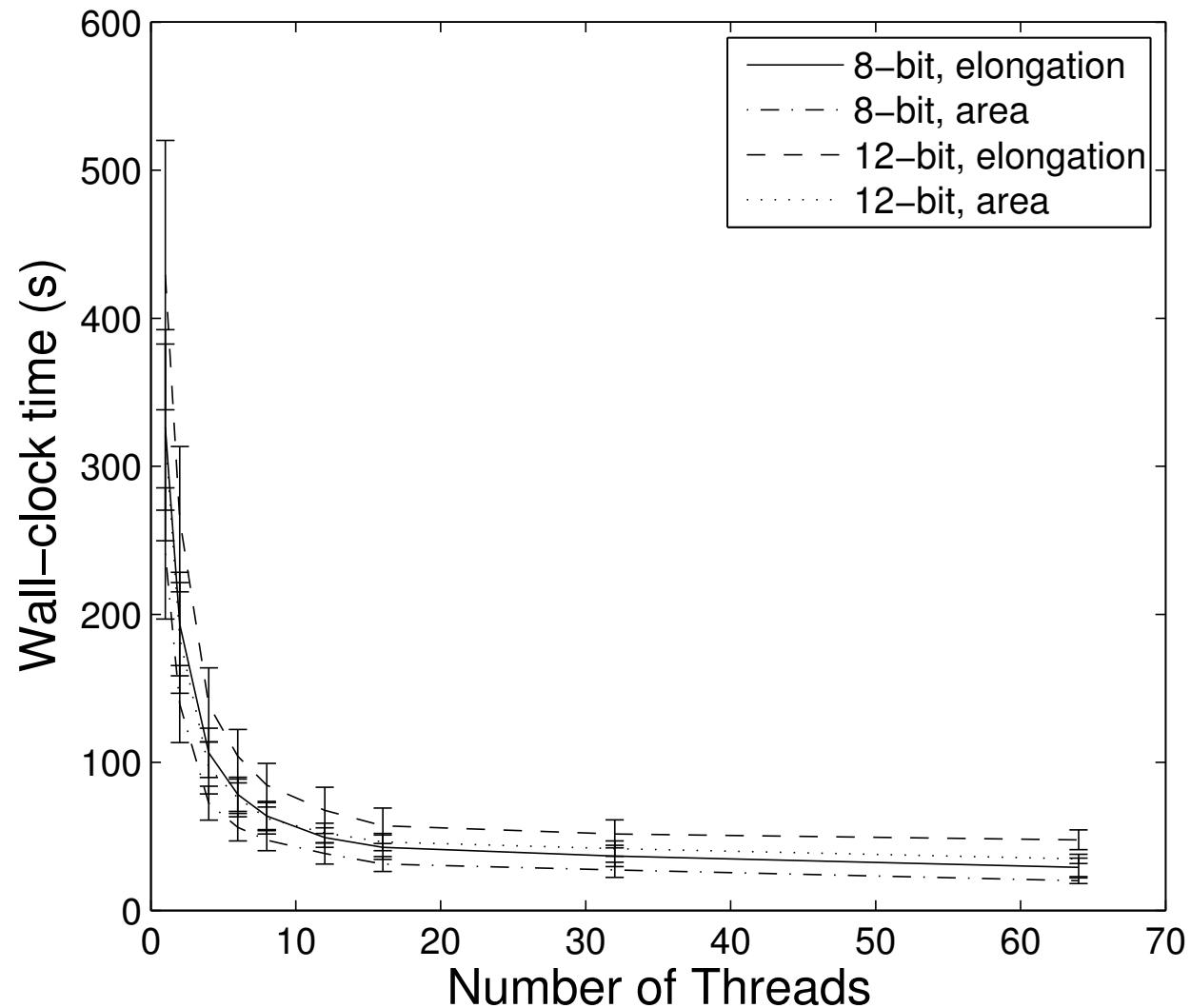
```
if  $y = \perp$  then
  while  $y \neq \perp$  do
     $node[x].Area \leftarrow node[x].Area + area;$ 
     $x \leftarrow GetLevelRootOf(node[x].parent);$ 
  end while
end if
end procedure
```

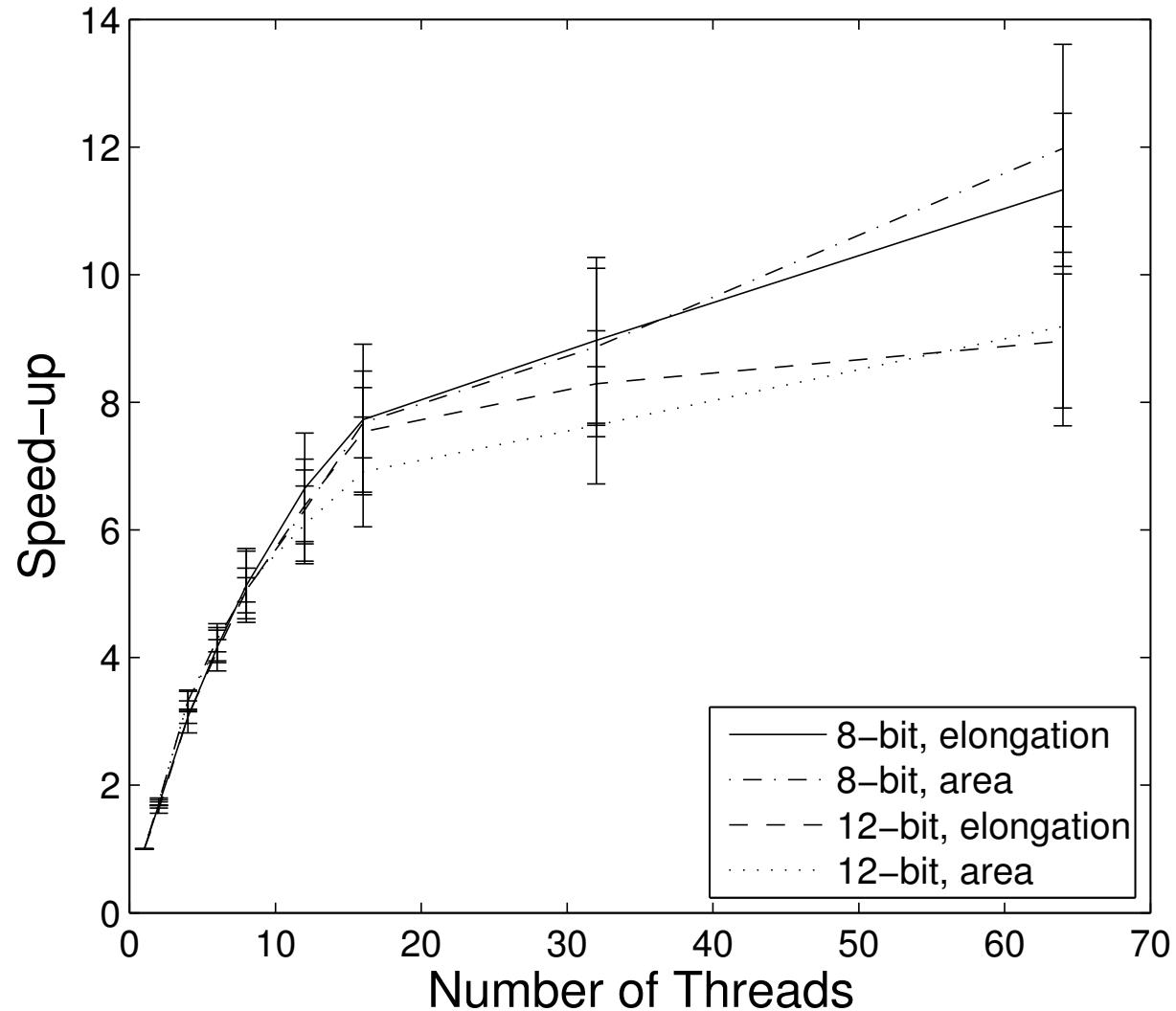




- Build local Max-Trees using FLOOD
- When finished, signal appropriate neighbour
- Receiving thread merges two max trees by calling CONNECT for each pair of pixels spanning the border
- When pairs of section have merged, repeat merging hierarchically





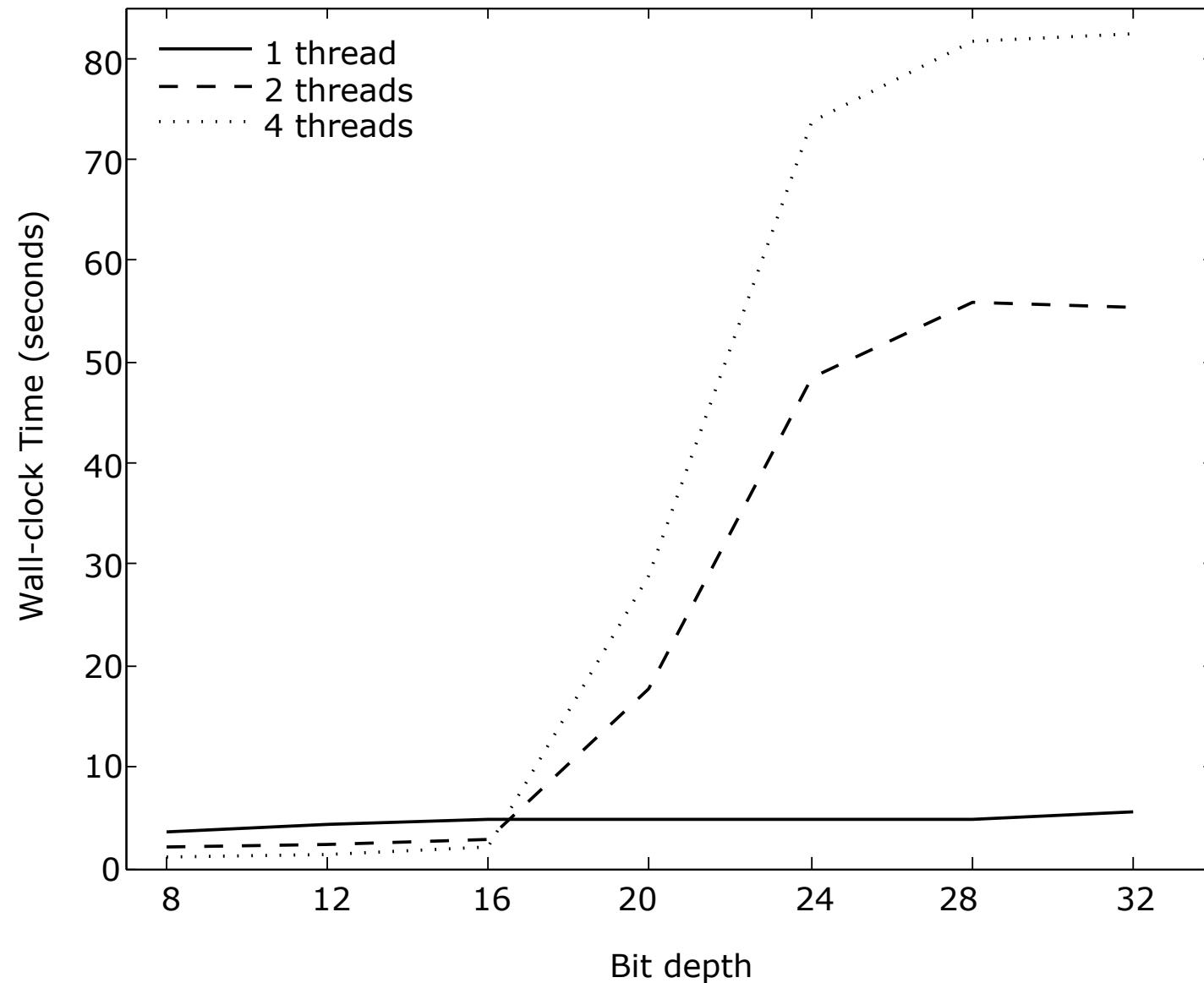


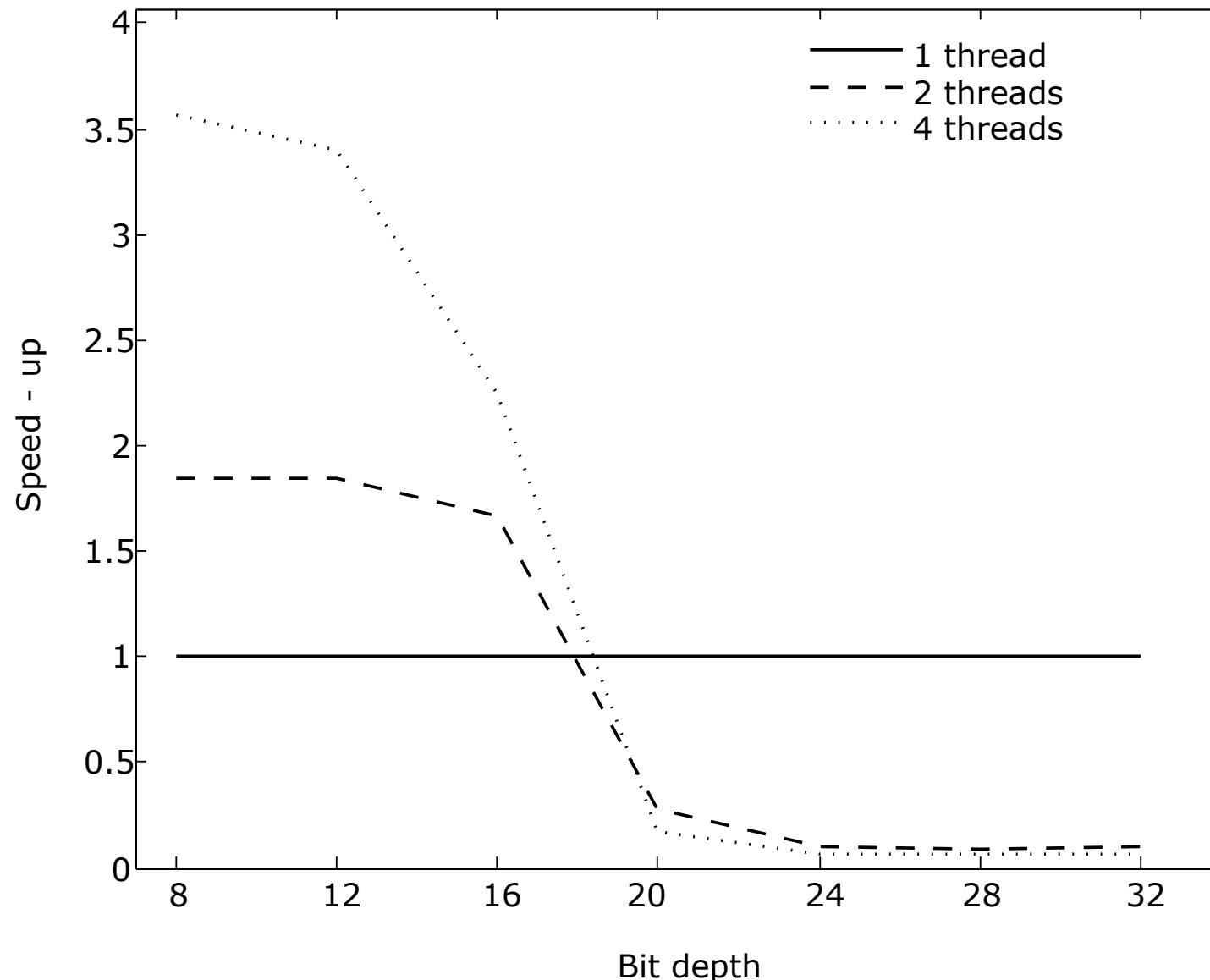


- Each merge process has complexity

$$O(A_{interface} \min(G, N_{pixels}/N_{blocks})) \quad (7)$$

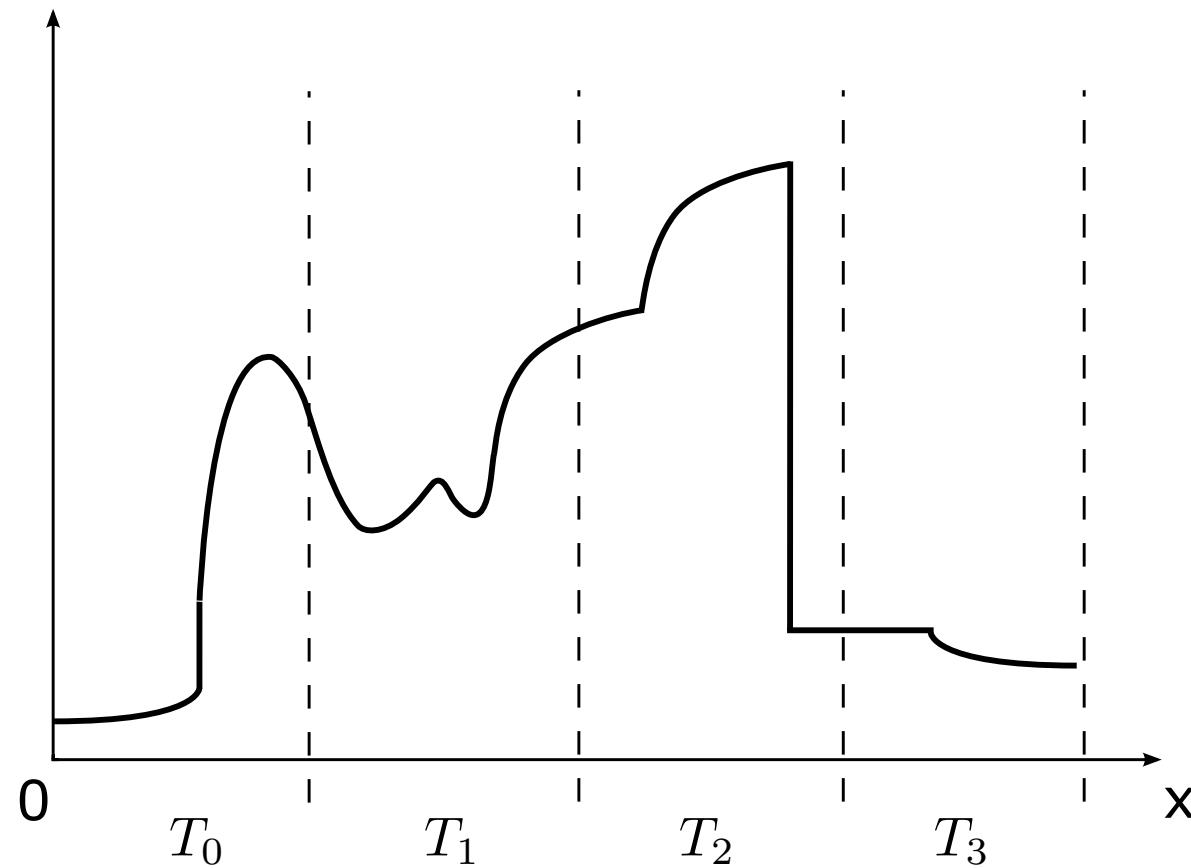
- N_{blocks} ranges from 2 to P , with P the number of processors
- $A_{interface}$ is the area of the interface:
 - $O(\sqrt{N_{pixels}})$ in 2D
 - $O(N_{pixels}^{2/3})$ in 3D.
- The number of merge processes is $\log P$

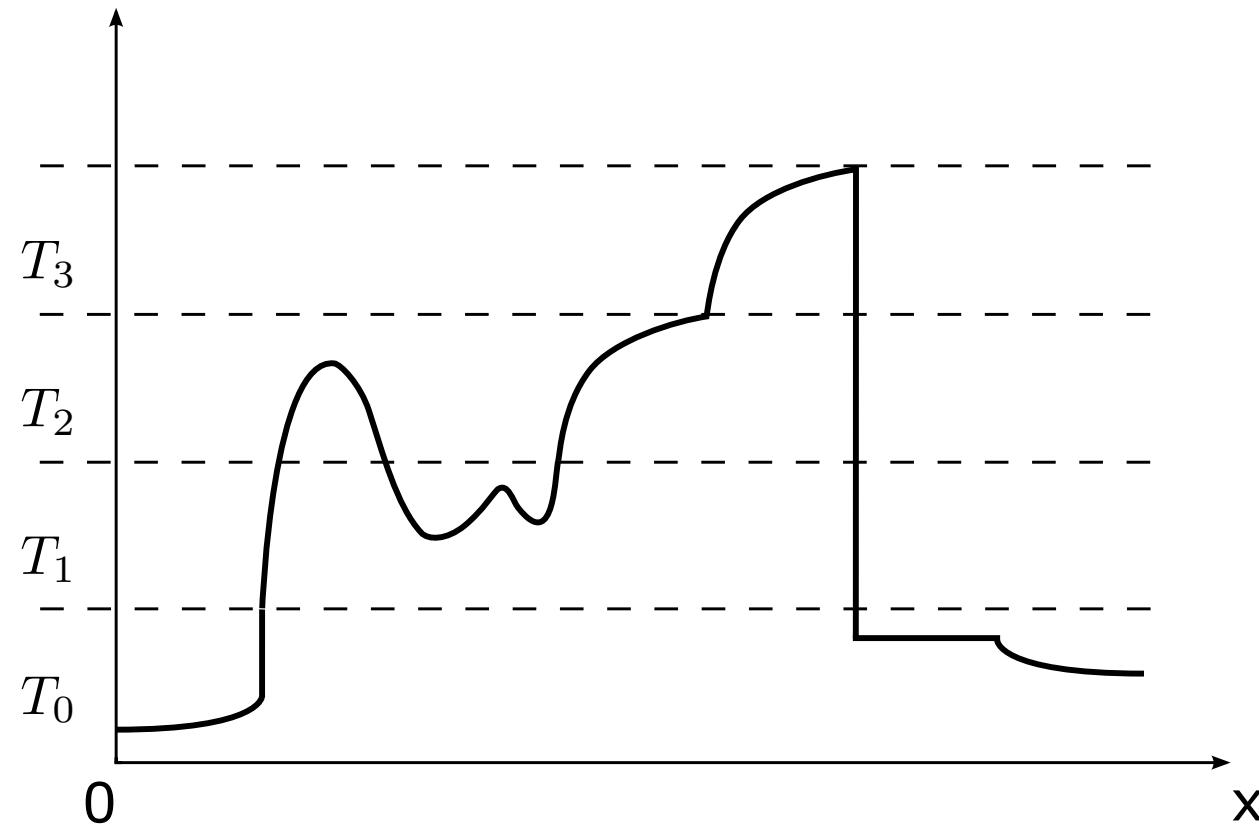






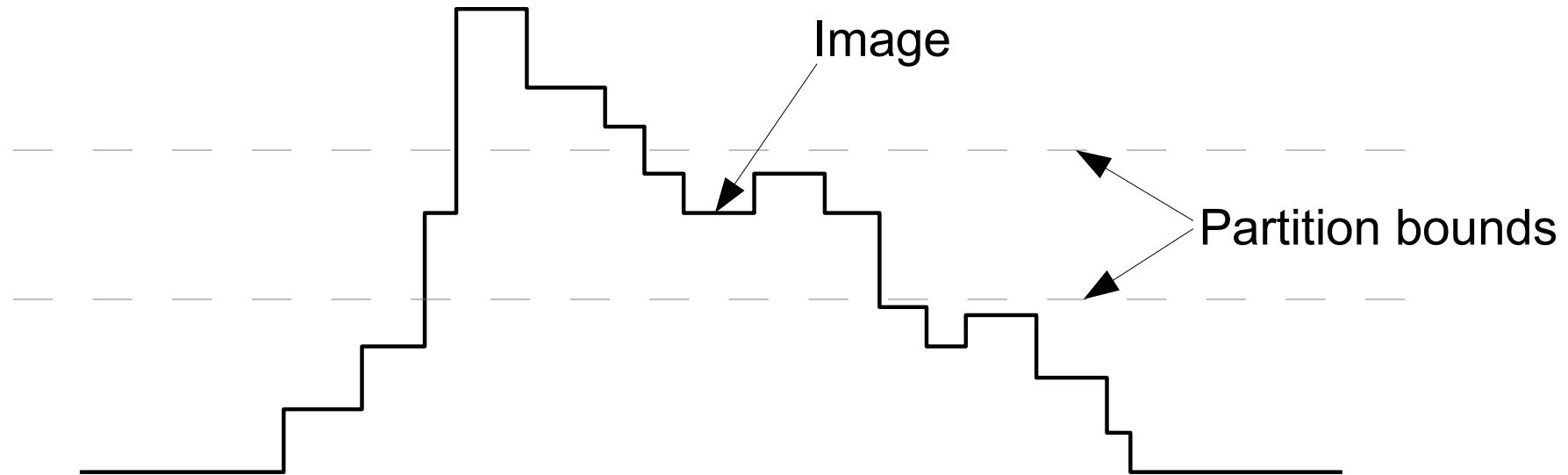
- Two options:
 1. Start out with one thread per maximum
 - Flood using priority queue.
 - Flooding waits when next pixel has lower grey level than current
 - Problem: mutual exclusion
 - Semaphore per pixel may be needed
 2. Classic spatial division per grey level
 - Region growing waits when new grey level reached
 - Problem: One barrier per grey level needed.
 - Worst case: one barrier per pixel
- Conclusion: does not look promising

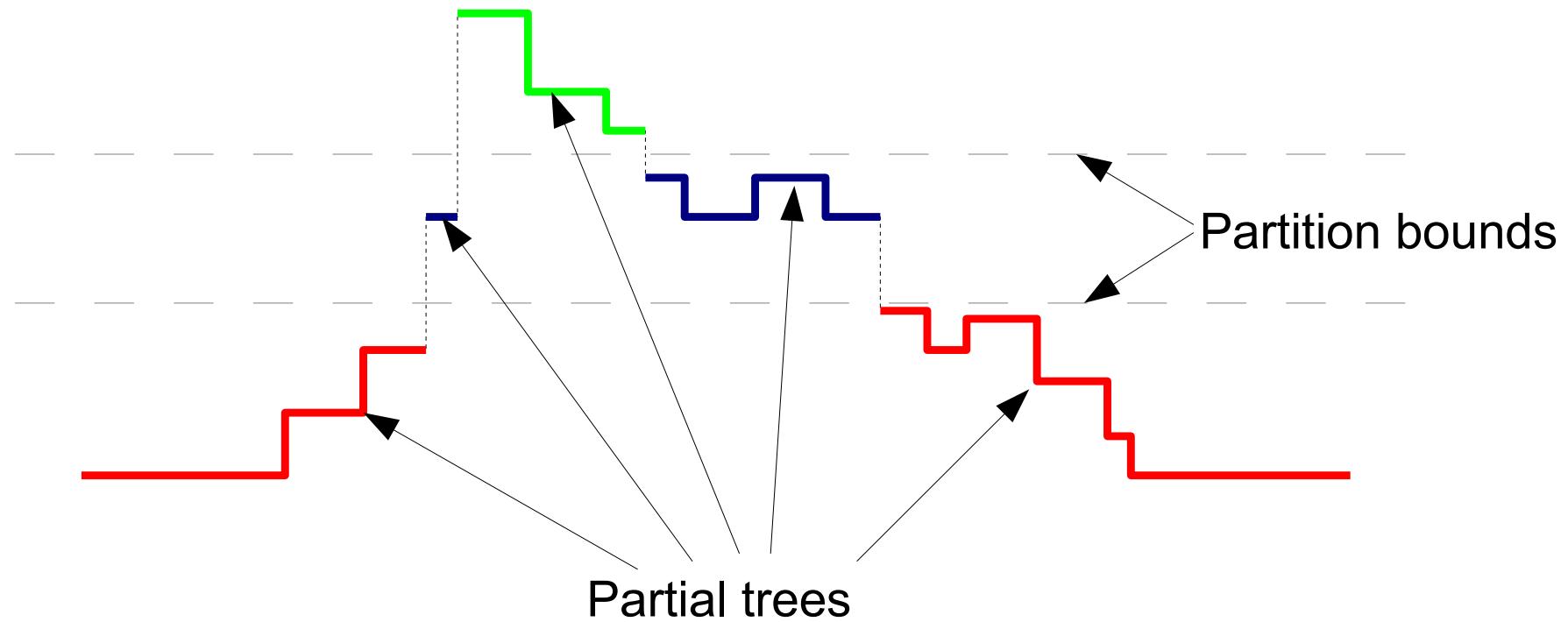




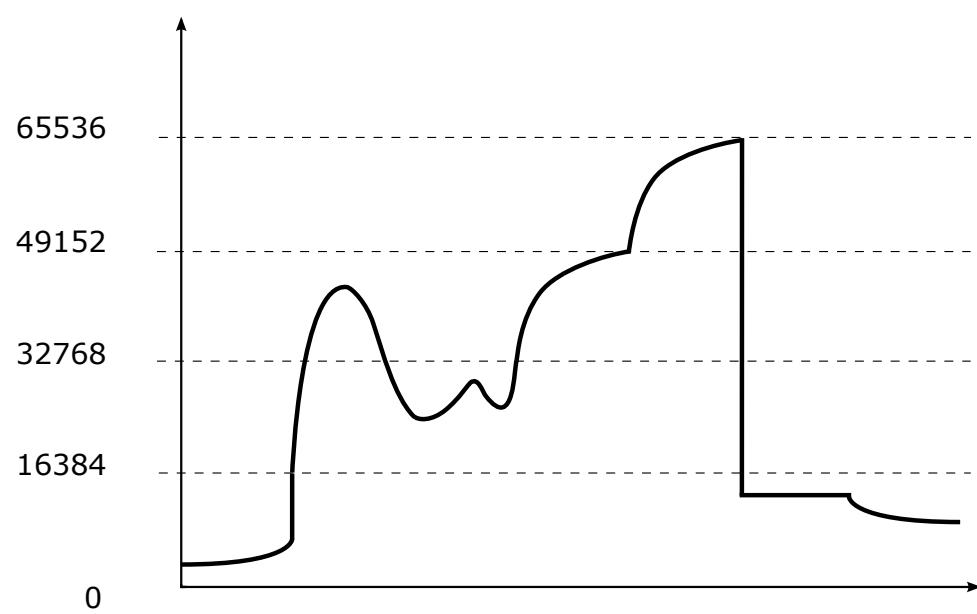


- Divide image in data chunks *by grey level*
- Compute forest of Max-Trees per grey level
- Merge these into full tree
 - Pro: Most single merges are $O(1)$
 - Con: some “zipping” still required
- These costly merges arise because the topology of the upper neighbours unknown
- Does not look like whole answer

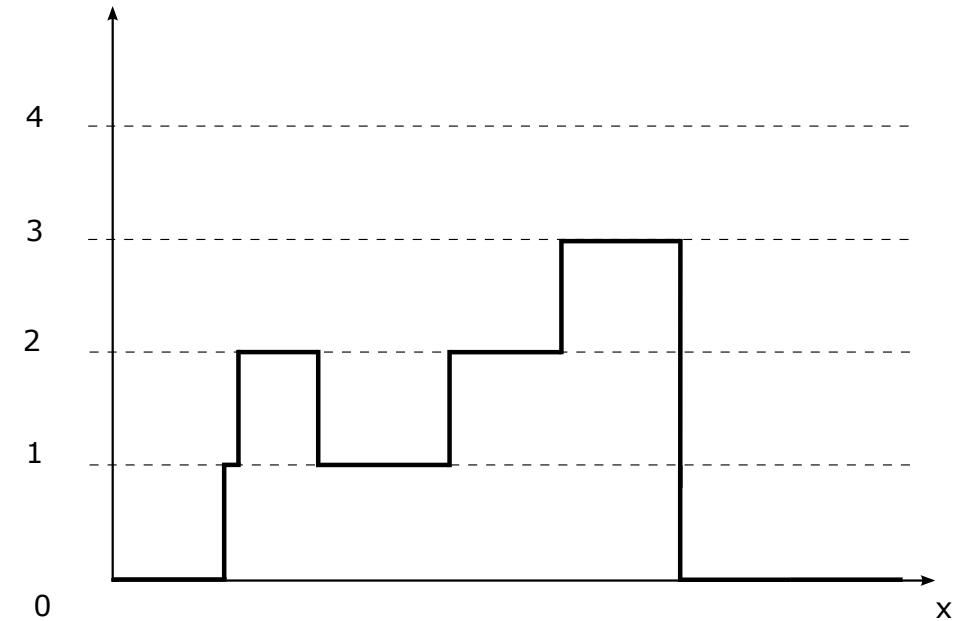




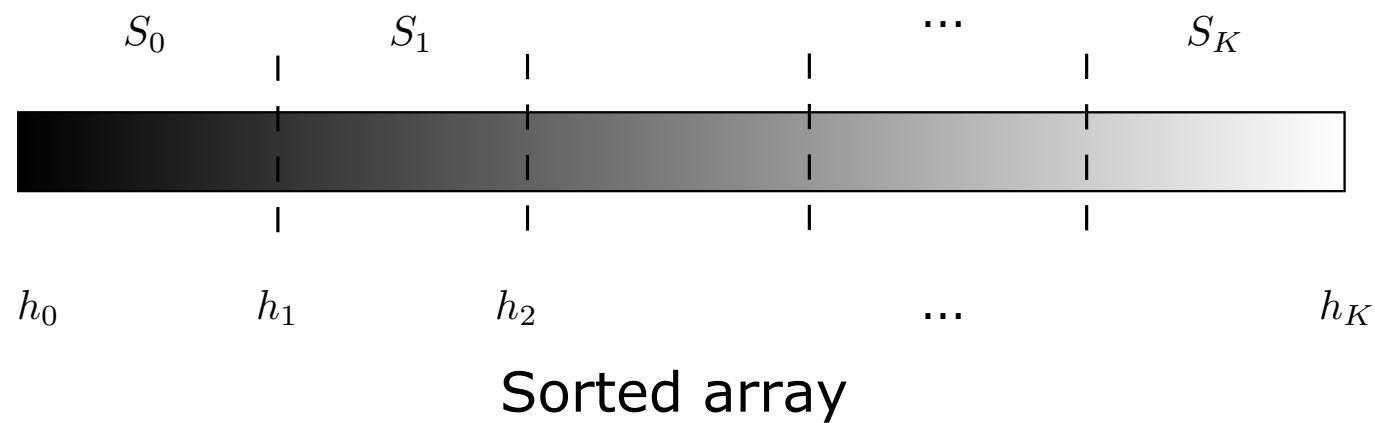
- Sort data according to grey level (Parallel Radix Sort)
- Compute image g quantised to P grey levels with uniform histogram (in parallel)
- Build “pilot” Max-Tree of using parallel low-dynamic-range algorithm
- Use modification of Berger et al per quantisation bin to refine this pilot tree
- filter in parallel as usual



original



quantised





- At the moment our quantisation is uniform
- It is not clear that this gives optimal load balancing
- Lower quantisation bins are expected to have larger numbers of nested maxima
- This means that the refinement phase of lower bins has to consult the quantised tree more often
- The depth of the branch above them also plays a role
- Inspecting a nested maximum is $O(P - i)$ worst case, with P the number of processors, and i the thread id (quantisation bin).



procedure FLOOD(Level lev , Partition P , Attribute a)

$at \leftarrow a;$

while $lev \neq 0$ **do**

 Extract pixel p from the queue; Update attribute in $levelroot[p]$;

for all neighbours $q \in P$ of p **do**

if $isVisited[q] = false$ **then**

$isVisited[q] \leftarrow true$; $atq \leftarrow 0$;

if $levelroot[\mathbf{g}(q)] = \text{not set}$ **then**

$levelroot[\mathbf{g}(q)] \leftarrow q$;

else

KeepLowestLevelRoot();

end if

 Add q to the queue;

while $\mathbf{g}(q) > lev$ **do**

$flood(\mathbf{g}(q), P, atq)$; $at \leftarrow a + atq$;

end while

end if

end for

end while



```
m ← m - 1;  
while m ≥ 0 ∧ levelroot[m] = not set do  
    m ← m - 1;  
end while  
if m ≥ 0 then  
    par(node_qu[levelroot[lev]]) ← levelroot[m];  
end if  
node_qu[levelroot[lev]].attribute = at;  
levelroot[lev] ← not set;  
a ← at;  
end procedure
```



```
procedure KEEPLOWESTLEVELROOT( )
  if  $q \prec levelroot[g(q)]$  then
     $par(node\_qu[levelroot[g(q)]])) \leftarrow q;$                                 ▷ exchange level roots
     $levelroot[g(q)] \leftarrow q;$ 
  end if
   $par(node\_qu[q]) \leftarrow levelroot[g(q)];$ 
end procedure
```

With

$$q \prec levelroot[g(q)] \equiv (f(q) < f(levelroot[g(q)])) \vee \quad (8)$$
$$((f(q) = f(levelroot[g(q)])) \wedge (q < levelroot[g(q)]))$$

with $g(\cdot)$ the quantised image and $f(\cdot)$ the original image value.



```
procedure REFINEMENT( $i, S_i, H_i$ )
     $lwb \leftarrow \min(S_i); upb \leftarrow \max(S_i);$ 
    for  $j = upb$  to  $lwb$  do
         $p \leftarrow SORTED[j];$ 
         $zpar[p] = p;$ 
        for all neighbours  $q$  of  $p$  do
            if  $g(q) > i$  then
                 $anc \leftarrow DESCENDROOTS(q, i);$ 
                if  $par(node\_ref[desc]) = \text{not set}$  then
                     $par(node\_ref[desc]) \leftarrow p;$ 
                    merge attribute of  $desc$  with  $p$ ;
                else
                     $z \leftarrow FINDROOT(par(node\_ref[desc]));$ 
                    if  $z \neq p$  then
                         $par(node\_ref[z]) \leftarrow p;$ 
                         $zpar[z] \leftarrow p;$ 
                        merge attribute of  $z$  with  $p$ ;
                    end if
                end if
            end if
```

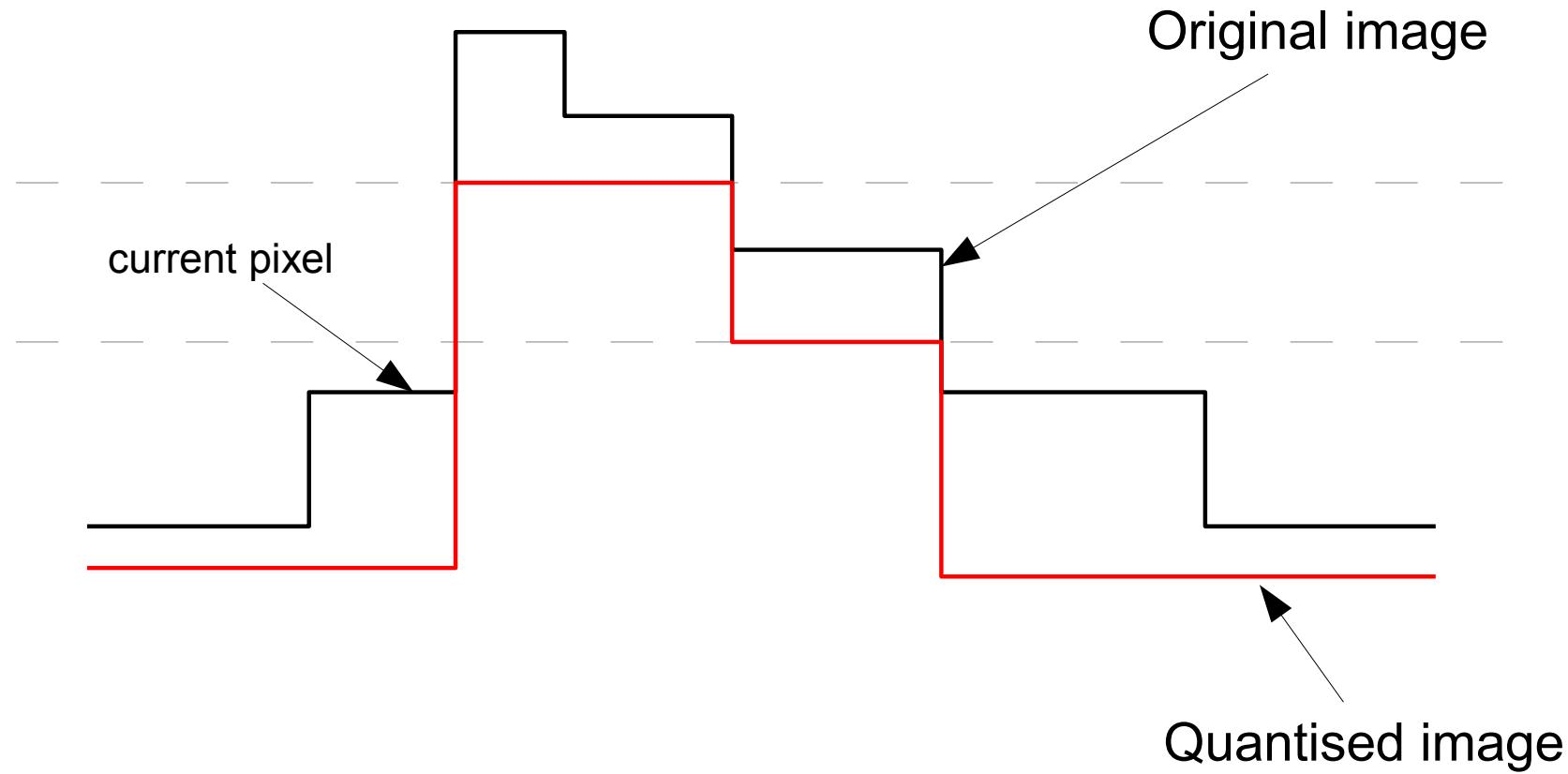


```
else if g(q) = i then
    if zpar[q] ≠ -1 then
        r ← FINDROOT(q);
        if r ≠ p then
            par(node_ref[r]) ← p;
            zpar[r] ← p;
            merge attribute of r with p;
        end if
    end if
    end if
end for
end for
end procedure
```



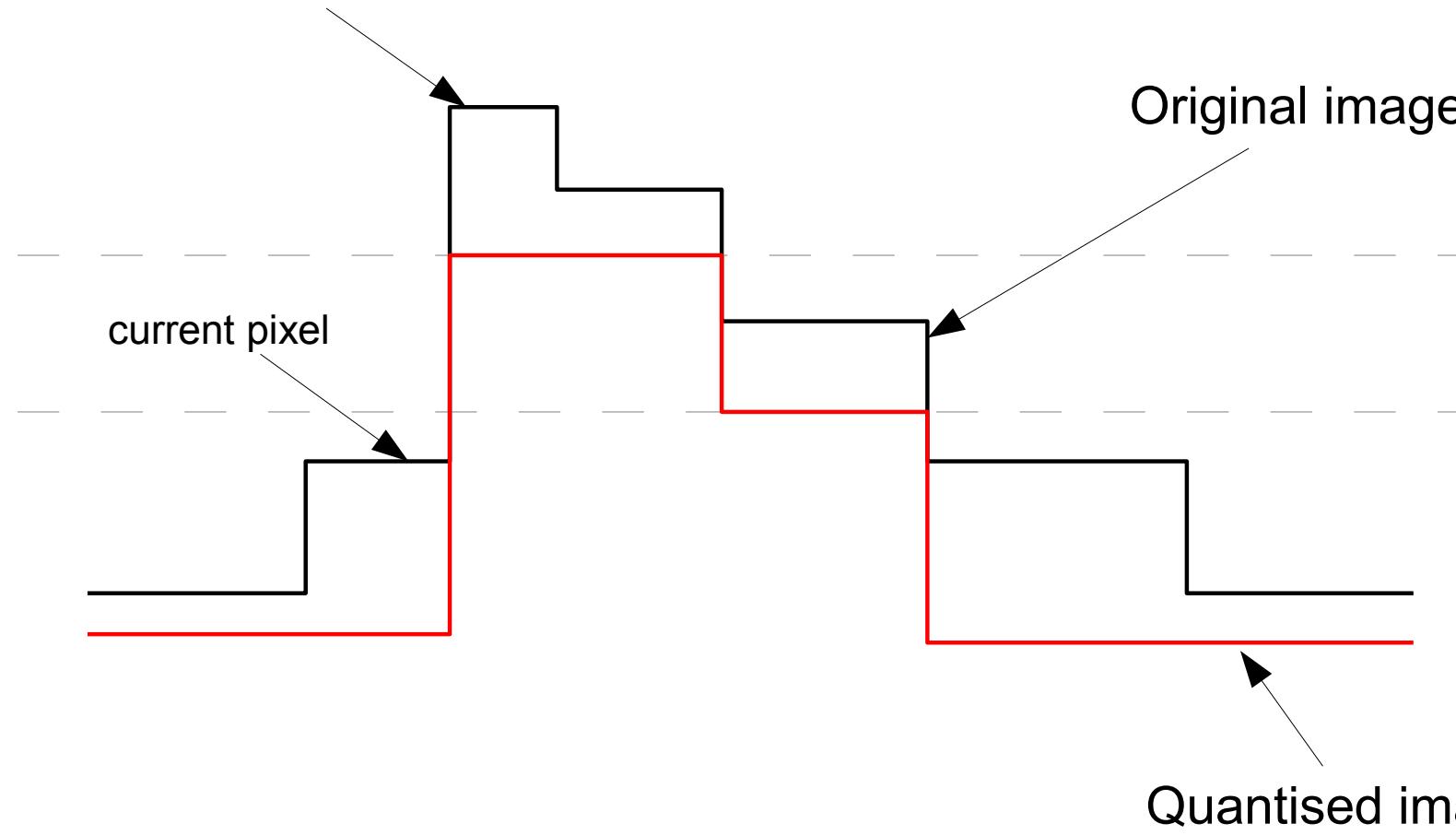
```
procedure DESCENDROOTS(Pixel  $q$ , int  $i$ )
     $c \leftarrow q$ ;
    while  $g(\text{par}(\text{node\_qu}[c])) > i$  do
         $c \leftarrow \text{par}(\text{node\_qu}[c])$ ;
    end while
    return  $c$ ;
end procedure
```

With i the thread number (= quantised grey level)





neighbour at higher level





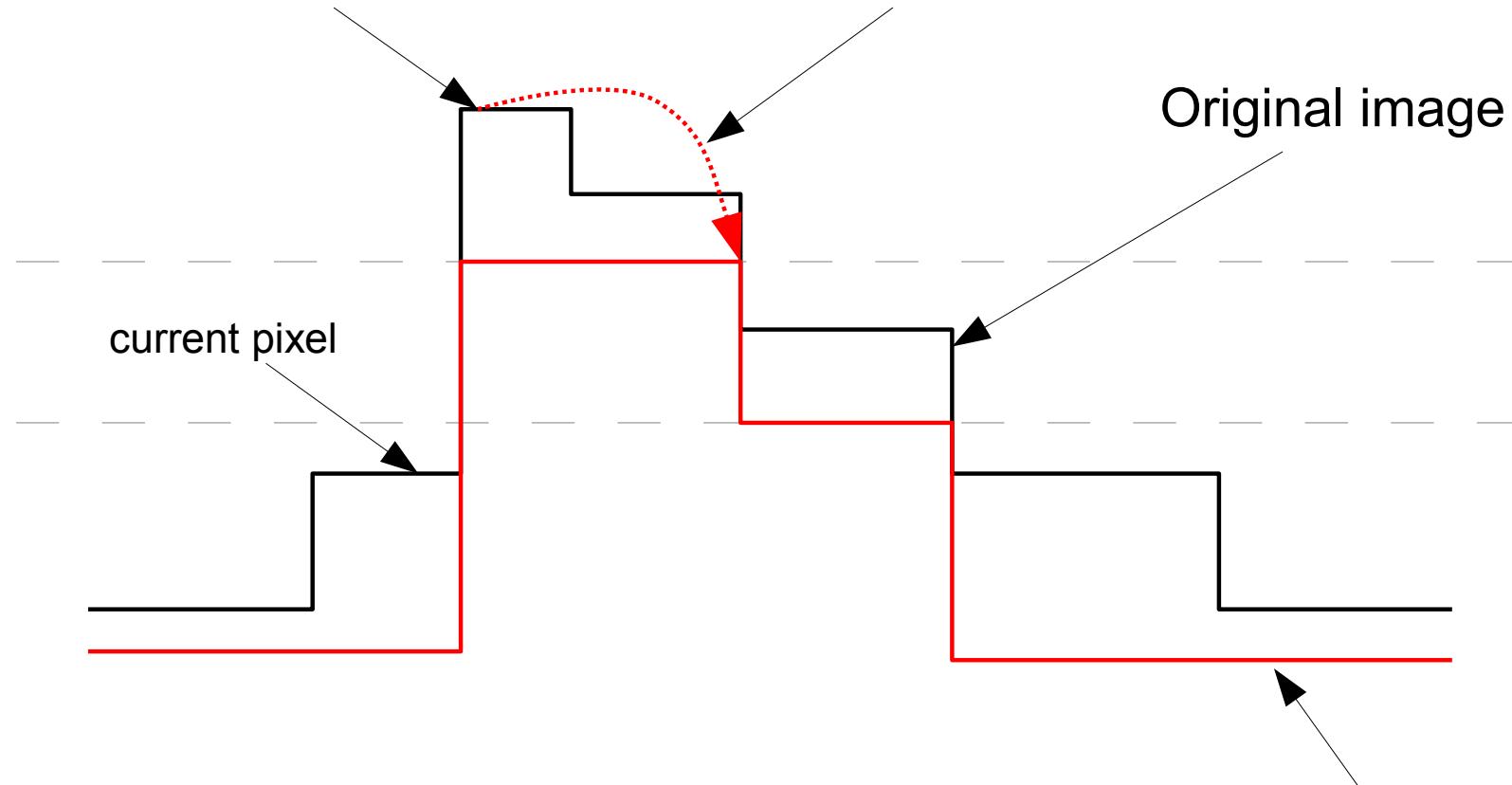
neighbour at higher level

link to quantized levelroot

Original image

current pixel

Quantised image





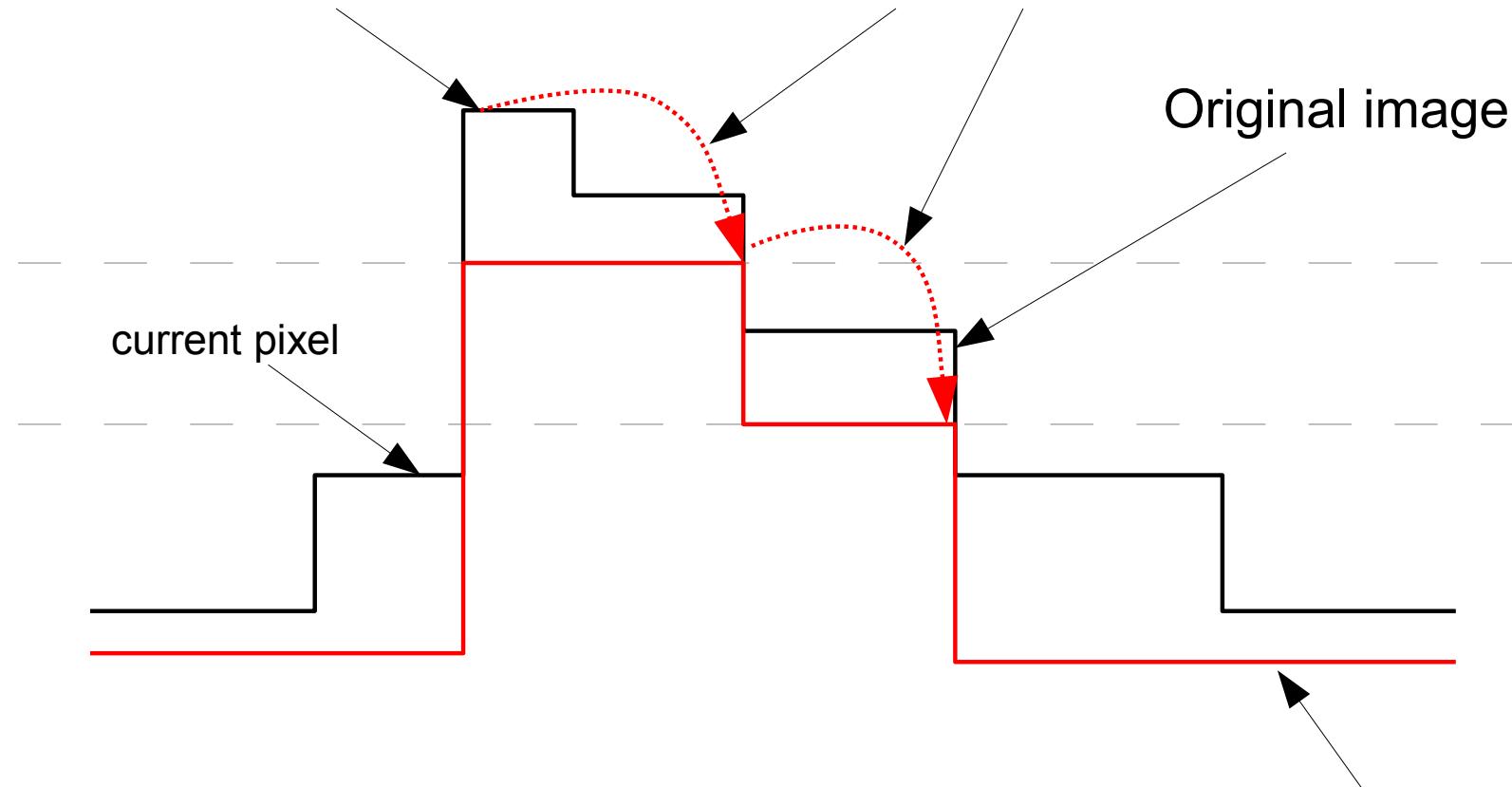
neighbour at higher level

link to quantized levelroot

Original image

current pixel

Quantised image



neighbour at higher level

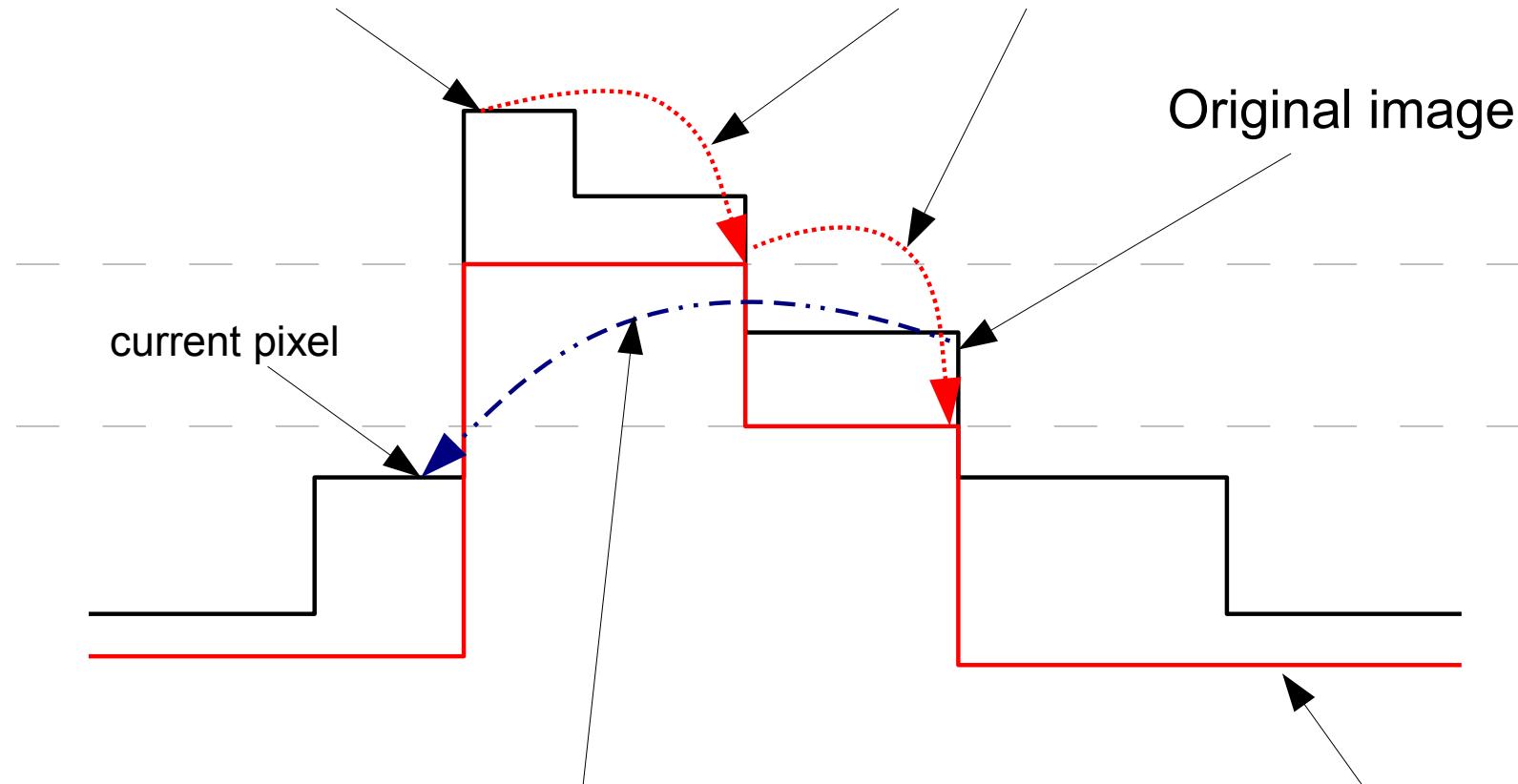
link to quantised levelroot

Original image

current pixel

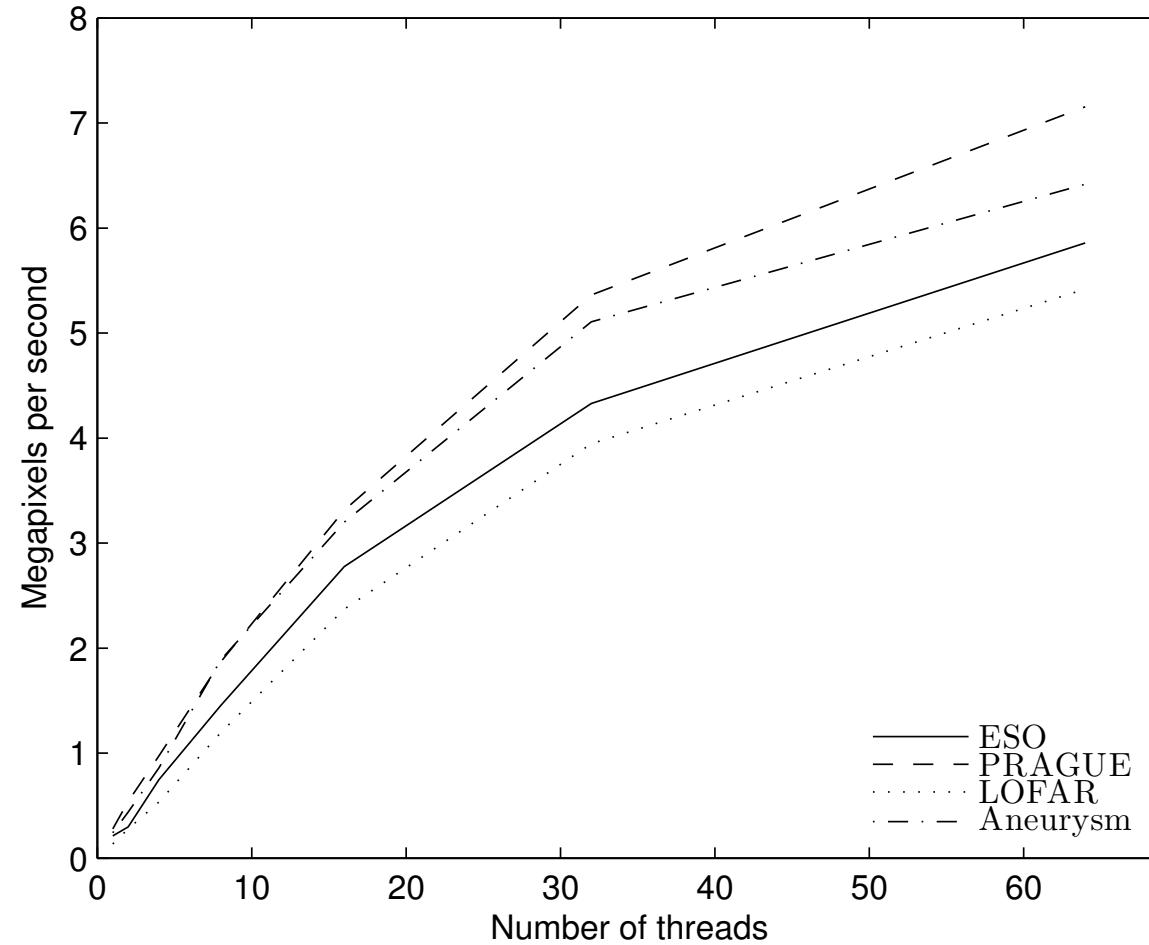
link in refined tree

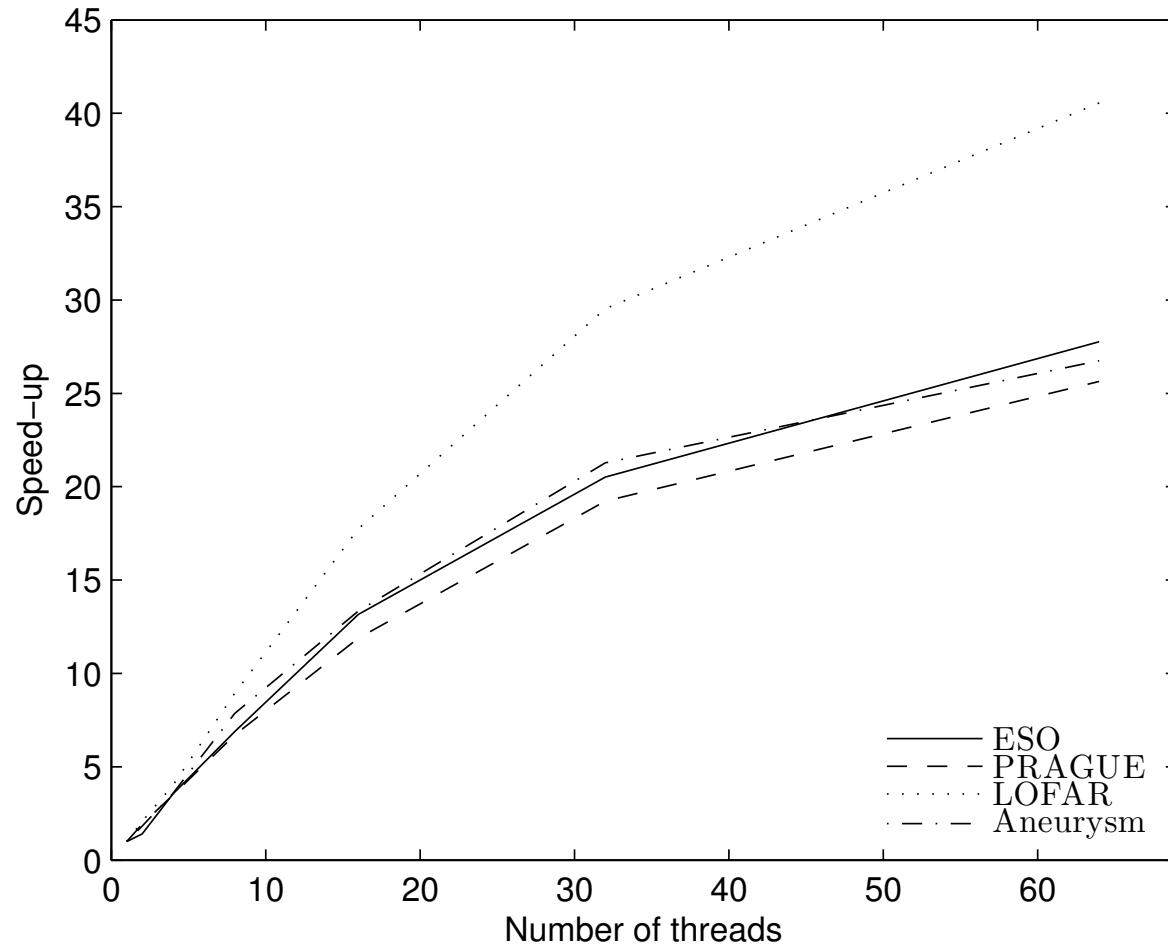
Quantised image





- All code was implemented in C using POSIX-threads
- Parallel radix sort was used because it is simple and stable
- Same data structure as in (Wilkinson et al 2008) was used.
- This means that the filtering stage of the algorithm could be left unchanged
- Performance test were carried out on several images and volumes on a Dell R815 compute server with 4 16 core Opteron processors and 512 GB of RAM.







- Parallel computation of CC-labelling and connected filters is possible
- CC-Labelling is fairly straightforward with Union-Find
- Connected filtering requires extra thought
- Shared-memory algorithms for low and extreme dynamic range (XDR) images Max-Trees have been developed
- The former algorithm has a speed-up of up to $50\times$ on 64 cores
- The XDR algorithm $25 - 40\times$ speed-up on 64 cores
- Roughly $14 - 20\times$ faster than fastest sequential algorithm (Berger et al. 2007)



- Multi-scale analysis using Max-Trees
- Moving to distributed memory

