Formal Modelling of Communicating Systems - Homework 2

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March 30, 2017

Exercise 3.2

$$\mathrm{CA} \stackrel{\mathrm{def}}{=} \overline{\mathrm{coin}}.\mathrm{coffee}.\mathrm{CA}$$

$$\mathrm{CTM} \stackrel{\mathrm{def}}{=} \mathrm{coin}.(\overline{\mathrm{coffee}}.\mathrm{CTM} + \overline{\mathrm{tea}}.\mathrm{CTM})$$

$$\mathrm{CTM}' \stackrel{\mathrm{def}}{=} \mathrm{coin}.\overline{\mathrm{coffee}}.\mathrm{CTM}' + \mathrm{coin}.\overline{\mathrm{tea}}.\mathrm{CTM}'$$

1. No, the first process won't deadlock since it can always synchronize over coffee, hence it's set of completed traces is empty:

$$\mathrm{Traces}((\mathrm{CA}|\mathrm{CTM})\backslash\{\mathrm{coin}, \mathrm{coffee}, \mathrm{tea}\}) = \{\}$$

The second process, however can deadlock and has the following set of completed traces:

$$\begin{aligned}\mathrm{Traces}((\{\mathrm{CA}|\mathrm{CTM}')\backslash\{\mathrm{coin}, \mathrm{coffee}, \mathrm{tea}\}) = \{ \\ (\{\mathrm{CA}|\mathrm{CTM}')\backslash\{\mathrm{coin}, \mathrm{coffee}, \mathrm{tea}\} \rightarrow \\ (\{(\mathrm{coffee}.\mathrm{CA}|\overline{\mathrm{tea}}.\mathrm{CTM}')\backslash\{\mathrm{coin}, \mathrm{coffee}, \mathrm{tea}\} \nrightarrow\}\end{aligned}$$

2. Yes

Exercise 3.3

$$P \stackrel{\mathrm{def}}{=} a.P_1 \qquad Q \stackrel{\mathrm{def}}{=} a.Q_1$$

$$P_1 \stackrel{\mathrm{def}}{=} b.P + c.P \qquad Q_1 \stackrel{\mathrm{def}}{=} b.Q_2 + c.Q$$

$$Q_2 \stackrel{\mathrm{def}}{=} a.Q_3$$

$$Q_3 \stackrel{\mathrm{def}}{=} b.Q + c.Q_2$$

To show that $P \sim Q$ we create the following bisimulation:

$$\mathcal{R} = \{(P, Q), (P_1, Q_1), (P, Q_2), (P_1, Q_3)\}$$

- Consider the first pair (P, Q):
  - transitions from P
    - $P \xrightarrow{a} P_1$ can be matched by $Q \xrightarrow{a} Q_1$ and $(P_1, Q_1) \in \mathcal{R}$
    - this is the only transition from P.
  - transitions from Q
    - $Q \xrightarrow{a} Q_1$ can be matched by $P \xrightarrow{a} P_1$ and $(P_1, Q_1) \in \mathcal{R}$
    - this is the only transition from Q.
- Consider the next pair (P_1, Q_1):
  - transitions from P_1
    - $P_1 \xrightarrow{b} P$ can be matched by $Q_1 \xrightarrow{b} Q_2$ and $(P, Q_2) \in \mathcal{R}$
    - $P_1 \xrightarrow{c} P$ can be matched by $Q_1 \xrightarrow{c} Q$ and $(P, Q) \in \mathcal{R}$
    - these are all transitions from P.
  - transitions from Q
    - $Q_1 \xrightarrow{b} Q_2$ can be matched by $P_1 \xrightarrow{b} P$ and $(P, Q_2) \in \mathcal{R}$
    - $Q_1 \xrightarrow{c} Q$ can be matched by $P_1 \xrightarrow{c} P$ and $(P, Q) \in \mathcal{R}$
    - these are all transitions from Q.
- Consider the next pair (P, Q_2):
  - transitions from P
    - $P \xrightarrow{a} P_1$ can be matched by $Q_2 \xrightarrow{a} Q_3$ and $(P_1, Q_3) \in \mathcal{R}$
    - this is the only transition from P.
  - transitions from Q_2
    - $Q_2 \xrightarrow{a} Q_3$ can be matched by $P \xrightarrow{a} P_1$ and $(P_1, Q_3) \in \mathcal{R}$
    - this is the only transition from Q_3.
- Consider the final pair (P_1, Q_3):
  - transitions from P_1
    - $P_1 \xrightarrow{b} P$ can be matched by $Q_3 \xrightarrow{b} Q$ and $(P, Q) \in \mathcal{R}$
    - $P_1 \xrightarrow{c} P$ can be matched by $Q_3 \xrightarrow{c} Q_2$ and $(P, Q_2) \in \mathcal{R}$
    - these are all transitions from P.
  - transitions from Q_3
    - $Q_3 \xrightarrow{b} Q$ can be matched by $P_1 \xrightarrow{b} P$ and $(P, Q) \in \mathcal{R}$
    - $Q_3 \xrightarrow{c} Q_2$ can be matched by $P_1 \xrightarrow{c} P$ and $(P, Q_2) \in \mathcal{R}$
    - these are all transitions from Q.

$\mathcal{R}$ is a strong bisimulation and $(P, Q) \in \mathcal{R}$, therefore $P \sim Q$.

Exercise 3.5

The LTSs for s and t can be described by the following CCS expressions:

$$
\begin{array}{ll}
s \stackrel{\text{def}}{=} a.s_1 + a.s_2 & t \stackrel{\text{def}}{=} a.t_1 + a.t_3 \\
s_1 \stackrel{\text{def}}{=} a.s_3 + b.s_4 & t_1 \stackrel{\text{def}}{=} a.t_2 + b.t_2 \\
s_2 \stackrel{\text{def}}{=} a.s_4 & t_2 \stackrel{\text{def}}{=} a.t \\
s_3 \stackrel{\text{def}}{=} a.s & t_3 \stackrel{\text{def}}{=} a.t_4 \\
s_4 \stackrel{\text{def}}{=} a.s & t_4 \stackrel{\text{def}}{=} a.t
\end{array}
$$

We find the following bisimulation:

$$\mathcal{R} = \{(s,t),(s_1,t_1),(s_2,t_3),(s_3,t_2),(s_4,t_2),(s_4,t_4)\}$$

- Consider the first pair (s,t):
  - transitions from s
    - $s \xrightarrow{a} s_1$ can be matched by $t \xrightarrow{a} t_1$ and $(s_1,t_1) \in \mathcal{R}$
    - $s \xrightarrow{a} s_2$ can be matched by $t \xrightarrow{a} t_3$ and $(s_2,t_3) \in \mathcal{R}$
    - these are all transitions from s.
  - transitions from t
    - $t \xrightarrow{a} t_1$ can be matched by $s \xrightarrow{a} s_1$ and $(s_1,t_1) \in \mathcal{R}$
    - $t \xrightarrow{a} t_3$ can be matched by $s \xrightarrow{a} s_2$ and $(s_2,t_3) \in \mathcal{R}$
    - these are all transitions from t.
- Consider the next pair (s_1,t_1):
  - transitions from s_1
    - $s_1 \xrightarrow{a} s_3$ can be matched by $t_1 \xrightarrow{a} t_2$ and $(s_3,t_2,) \in \mathcal{R}$
    - $s_1 \xrightarrow{b} s_4$ can be matched by $t_1 \xrightarrow{b} t_2$ and $(s_4,t_2,) \in \mathcal{R}$
    - this is the only transition from s_1.
  - transitions from t_1
    - $t_1 \xrightarrow{a} t_2$ can be matched by $s_1 \xrightarrow{a} s_3$ and $(s_3,t_2) \in \mathcal{R}$
    - $t_1 \xrightarrow{b} t_2$ can be matched by $s_1 \xrightarrow{b} s_4$ and $(s_4,t_2) \in \mathcal{R}$
    - these are all transitions from t_1.
- Consider the next pair (s_2,t_3):
  - transitions from s_2

- * $s_2 \xrightarrow{a} s_4$ can be matched by $t_3 \xrightarrow{a} t_4$ and $(s_4, t_4) \in \mathcal{R}$
    * this is the only transition from s_2.
  - transitions from t_3
    * $t_3 \xrightarrow{a} t_4$ can be matched by $s_2 \xrightarrow{a} s_4$ and $(s_4, t_4) \in \mathcal{R}$
    * this is the only transition from t_3.

- Consider the next pair (s_3, t_2):
  - transitions from s_3
    * $s_3 \xrightarrow{a} s$ can be matched by $t_2 \xrightarrow{a} t$ and $(s, t) \in \mathcal{R}$
    * this is the only transition from s_3.
  - transitions from t_2
    * $t_2 \xrightarrow{a} t$ can be matched by $s_3 \xrightarrow{a} s$ and $(s, t) \in \mathcal{R}$
    * this is the only transition from t_2.

- Consider the next pair (s_4, t_2):
  - transitions from s_4
    * $s_4 \xrightarrow{a} s$ can be matched by $t_2 \xrightarrow{a} t$ and $(s, t) \in \mathcal{R}$
    * this is the only transition from s_4.
  - transitions from t_2
    * $t_2 \xrightarrow{a} t$ can be matched by $s_4 \xrightarrow{a} s$ and $(s, t) \in \mathcal{R}$
    * this is the only transition from t_2.

- Consider the next pair (s_4, t_4):
  - transitions from s_4
    * $s_4 \xrightarrow{a} s$ can be matched by $t_4 \xrightarrow{a} t$ and $(s, t) \in \mathcal{R}$
    * this is the only transition from s_4.
  - transitions from t_4
    * $t_4 \xrightarrow{a} t$ can be matched by $s_4 \xrightarrow{a} s$ and $(s, t) \in \mathcal{R}$
    * this is the only transition from t_4.

$\mathcal{R}$ is a strong bisimulation and $(s, t) \in \mathcal{R}$, therefore $s \sim t$.

Exercise 3.13

No, given:

$$P \stackrel{\text{def}}{=} a.P \qquad Q \stackrel{\text{def}}{=} \overline{\text{a}}.Q$$

$$S \stackrel{\text{def}}{=} (P|Q)\backslash a \qquad T \stackrel{\text{def}}{=} (P\backslash a)|(Q\backslash a)$$

Process S exhibits an infinite series of τ-actions due to the synchronization over a. However, process T does nothing since a is restricted before P and Q can synchronize. Therefore T cannot match the τs from S and $S \not\approx T$.

Yes, lets define:

$$S \stackrel{\text{def}}{=} (P|Q)[f] \qquad\qquad T \stackrel{\text{def}}{=} (P[f])|(Q[f])$$

Any action outside the domain of f by S can be matched by T, since it would be as if the relabeling never happened on either side. For any action that is relabeled by f by S:

- $(\alpha.P'|Q)[\beta/\alpha] \xrightarrow{\beta} (P'|Q)[\beta/\alpha]$ can be matched by $((\alpha.P')[\beta/\alpha])|(Q[\beta/\alpha]) \xrightarrow{\beta} (P'[\beta/\alpha])|(Q[\beta/\alpha])$
- $(P|\alpha.Q')[\beta/\alpha] \xrightarrow{\beta} (P|Q')[\beta/\alpha]$ can be matched by $(P[\beta/\alpha])|((\alpha.Q')[\beta/\alpha]) \xrightarrow{\beta} (P[\beta/\alpha])|(Q'[\beta/\alpha])$
- $(\overline{\alpha}.P'|\alpha.Q')[\beta/\alpha] \xrightarrow{\tau} (P'|Q')[\beta/\alpha]$ can be matched by $((\overline{\alpha}.P')[\beta/\alpha])|((\alpha.Q')[\beta/\alpha]) \xrightarrow{\tau} (P'[\beta/\alpha])|(Q'[\beta/\alpha])$

Exercise 3.21

Definitions from exercise 2.11:

$$\begin{aligned}
\text{User} &\stackrel{\text{def}}{=} \overline{p}.\text{enter}.\text{exit}.\overline{v}.\text{User} \\
\text{Sem} &\stackrel{\text{def}}{=} p.v.\text{Sem} \\
\text{Mutex} &\stackrel{\text{def}}{=} (\text{User}|\text{Sem})\backslash\{p, v\} \\
\text{Mutex}_2 &\stackrel{\text{def}}{=} ((\text{User}|\text{Sem})|\text{User})\backslash\{p, v\} \\
\text{FUser} &\stackrel{\text{def}}{=} \overline{p}.\text{enter}.(\text{exit}.\overline{v}.\text{FUser} + \text{exit}.\overline{v}.\mathbf{0}) \\
\text{FMutex} &\stackrel{\text{def}}{=} ((\text{User}|\text{Sem})|\text{FUser})\backslash\{p, v\}
\end{aligned}$$

Requested: $\text{Mutex}_2 \approx \text{FMutex}$? First we add the following helper definitions to name the intermediate states:

$$
\begin{aligned}
\mathrm{User}_1 &\stackrel{\mathrm{def}}{=} \mathrm{enter}.\mathrm{exit}.\overline{v}.\mathrm{User} & \mathrm{FUser}_1 &\stackrel{\mathrm{def}}{=} \mathrm{enter}.(\mathrm{exit}.\overline{v}.\mathrm{FUser} + \mathrm{exit}.\overline{v}.\mathbf{0})\\
\mathrm{User}_2 &\stackrel{\mathrm{def}}{=} \mathrm{exit}.\overline{v}.\mathrm{User} & \mathrm{FUser}_2 &\stackrel{\mathrm{def}}{=} \mathrm{exit}.\overline{v}.\mathrm{FUser} + \mathrm{exit}.\overline{v}.\mathbf{0}\\
\mathrm{User}_3 &\stackrel{\mathrm{def}}{=} \overline{v}.\mathrm{User} & \mathrm{FUser}_3 &\stackrel{\mathrm{def}}{=} \overline{v}.\mathrm{FUser}\\
\mathrm{Sem}_1 &\stackrel{\mathrm{def}}{=} v.\mathrm{Sem} & \mathrm{FUser}_4 &\stackrel{\mathrm{def}}{=} \overline{v}.\mathbf{0}\\
\mathrm{Mutex}_{2,1} &\stackrel{\mathrm{def}}{=} ((\mathrm{User}_1|\mathrm{Sem}_1)|\mathrm{User})\backslash\{p,v\} & \mathrm{FMutex}_1 &\stackrel{\mathrm{def}}{=} ((\mathrm{User}_1|\mathrm{Sem}_1)|\mathrm{FUser})\backslash\{p,v\}\\
\mathrm{Mutex}_{2,2} &\stackrel{\mathrm{def}}{=} ((\mathrm{User}_2|\mathrm{Sem}_1)|\mathrm{User})\backslash\{p,v\} & \mathrm{FMutex}_2 &\stackrel{\mathrm{def}}{=} ((\mathrm{User}_2|\mathrm{Sem}_1)|\mathrm{FUser})\backslash\{p,v\}\\
\mathrm{Mutex}_{2,3} &\stackrel{\mathrm{def}}{=} ((\mathrm{User}_3|\mathrm{Sem}_1)|\mathrm{User})\backslash\{p,v\} & \mathrm{FMutex}_3 &\stackrel{\mathrm{def}}{=} ((\mathrm{User}_3|\mathrm{Sem}_1)|\mathrm{FUser})\backslash\{p,v\}\\
\mathrm{Mutex}_{2,4} &\stackrel{\mathrm{def}}{=} ((\mathrm{User}|\mathrm{Sem}_1)|\mathrm{User}_1)\backslash\{p,v\} & \mathrm{FMutex}_4 &\stackrel{\mathrm{def}}{=} ((\mathrm{User}|\mathrm{Sem}_1)|\mathrm{FUser}_1)\backslash\{p,v\}\\
\mathrm{Mutex}_{2,5} &\stackrel{\mathrm{def}}{=} ((\mathrm{User}|\mathrm{Sem}_1)|\mathrm{User}_2)\backslash\{p,v\} & \mathrm{FMutex}_5 &\stackrel{\mathrm{def}}{=} ((\mathrm{User}|\mathrm{Sem}_1)|\mathrm{FUser}_2)\backslash\{p,v\}\\
\mathrm{Mutex}_{2,6} &\stackrel{\mathrm{def}}{=} ((\mathrm{User}|\mathrm{Sem}_1)|\mathrm{User}_3)\backslash\{p,v\} & \mathrm{FMutex}_6 &\stackrel{\mathrm{def}}{=} ((\mathrm{User}|\mathrm{Sem}_1)|\mathrm{FUser}_3)\backslash\{p,v\}\\
& & \mathrm{FMutex}_7 &\stackrel{\mathrm{def}}{=} ((\mathrm{User}|\mathrm{Sem}_1)|\mathrm{FUser}_4)\backslash\{p,v\}
\end{aligned}
$$

We create the following relation:

$$
\begin{aligned}
\mathcal{R} = \{&(\mathrm{Mutex}_2, \mathrm{FMutex}), (\mathrm{Mutex}_{2,1}, \mathrm{FMutex}_1), (\mathrm{Mutex}_{2,2}, \mathrm{FMutex}_2),\\
&(\mathrm{Mutex}_{2,2}, \mathrm{FMutex}_2), (\mathrm{Mutex}_{2,3}, \mathrm{FMutex}_3), (\mathrm{Mutex}_{2,4}, \mathrm{FMutex}_4)\}
\end{aligned}
$$

For $\mathrm{Mutex}_{2,1} \approx \mathrm{FMutex}_1$, $\mathrm{Mutex}_{2,2} \approx \mathrm{FMutex}_2$ and $\mathrm{Mutex}_{2,3} \approx \mathrm{FMutex}_3$ both processes behave like an ordinary Mutex and their bisimilarity is obvious.

Since parallel composition is associative and commutative modulo strong bisimilarity[1], $\mathrm{Mutex}_{2,1} \sim \mathrm{Mutex}_{2,4}$, $\mathrm{Mutex}_{2,2} \sim \mathrm{Mutex}_{2,5}$ and $\mathrm{Mutex}_{2,3} \sim \mathrm{Mutex}_{2,6}$. For symmetry we simulate $\mathrm{Mutex}_{2,1-3}$ with FMutex_{4-7}.

$\mathrm{Mutex}_2 \xRightarrow{enter} \mathrm{Mutex}_{2,1}$ can be matched by $\mathrm{FMutex}_4 \xRightarrow{enter} \mathrm{FMutex}_5$

Exercise 3.25

The given LTS can be described with the following equations:

[1]Reactive Systems: Modelling, Specification and Verification, Page 53

$$s \stackrel{\text{def}}{=} \tau.s_1 + a.s_3 \qquad t \stackrel{\text{def}}{=} \tau.t_1 + a.t_2 + b.t_3$$

$$s_1 \stackrel{\text{def}}{=} \tau.s + \tau.s_2 + b.s_4 \qquad t_1 \stackrel{\text{def}}{=} \tau.t$$

$$s_2 \stackrel{\text{def}}{=} \tau.s_1 + \tau.s_5 \qquad t_2 \stackrel{\text{def}}{=} \mathbf{0}$$

$$s_3 \stackrel{\text{def}}{=} \mathbf{0} \qquad t_3 \stackrel{\text{def}}{=} \mathbf{0}$$

$$s_4 \stackrel{\text{def}}{=} \mathbf{0}$$

$$s_5 \stackrel{\text{def}}{=} \mathbf{0}$$

To prove weak bisimilarity we create the following relation:

$$\mathcal{R} = \{(s,t),(s_3,t_2),(s_1,t),(s_5,t_1),(s_4,t_3),(s_2,t)\}$$

- Consider the first pair (s,t):
  - transitions from s
    * $s \xrightarrow{a} s_3$ can be matched by $t \stackrel{a}{\Rightarrow} t_2$ and $(s_3,t_2) \in \mathcal{R}$
    * $s \xrightarrow{\tau} s_1$ can be matched by $t \stackrel{\tau}{\Rightarrow} t$ and $(s_1,t) \in \mathcal{R}$
    * these are all transitions from s.
  - transitions from t
    * $t \xrightarrow{\tau} t_1$ can be matched by $s \stackrel{\tau}{\Rightarrow} s_5$ and $(s_5,t_1) \in \mathcal{R}$
    * $t \xrightarrow{a} t_2$ can be matched by $s \stackrel{a}{\Rightarrow} s_3$ and $(s_3,t_2) \in \mathcal{R}$
    * $t \xrightarrow{b} t_3$ can be matched by $s \stackrel{b}{\Rightarrow} s_4$ and $(s_4,t_3) \in \mathcal{R}$
    * these are all transitions from t.
- Consider the next pair (s_3,t_2): They are both $\mathbf{0}$.
- Consider the next pair (s_1,t):
  - transitions from s_1
    * $s_1 \xrightarrow{b} s_4$ can be matched by $t \stackrel{b}{\Rightarrow} t_3$ and $(s_4,t_3) \in \mathcal{R}$
    * $s_1 \xrightarrow{\tau} s_2$ can be matched by $t \stackrel{\tau}{\Rightarrow} t$ and $(s_2,t) \in \mathcal{R}$
    * these are all transitions from s_2.
  - transitions from t
    * $t \xrightarrow{\tau} t_1$ can be matched by $s_1 \stackrel{\tau}{\Rightarrow} s_5$ and $(s_5,t_1) \in \mathcal{R}$
    * $t \xrightarrow{a} t_2$ can be matched by $s_1 \stackrel{a}{\Rightarrow} s_3$ and $(s_3,t_2) \in \mathcal{R}$
    * $t \xrightarrow{b} t_3$ can be matched by $s_1 \stackrel{b}{\Rightarrow} s_4$ and $(s_4,t_3) \in \mathcal{R}$
    * these are all transitions from t.

- Consider the next pair (s_5, t_1):
  - transitions from s_5: s_5 is $\mathbf{0}$ and does nothing.
  - transitions from t_1
    - $t_1 \xrightarrow{\tau} t_1$ can be matched by $s_5 \overset{\tau}{\Rightarrow} s_5$ (by doing zero τ-actions) and $(s_5, t_1) \in \mathcal{R}$
    - this is the only transition from t_1.
- Consider the next pair (s_4, t_3): They are both $\mathbf{0}$.
- Consider the final pair (s_2, t):
  - transitions from s_2
    - $s_2 \xrightarrow{\tau} s_1$ can be matched by $t \overset{b}{\Rightarrow} t$ and $(s_1, t) \in \mathcal{R}$
    - $s_2 \xrightarrow{\tau} s_5$ can be matched by $t \overset{\tau}{\Rightarrow} t_1$ and $(s_5, t_1) \in \mathcal{R}$
    - these are all transitions from s_2.
  - transitions from t
    - $t \xrightarrow{\tau} t_1$ can be matched by $s_2 \overset{\tau}{\Rightarrow} s_5$ and $(s_5, t_1) \in \mathcal{R}$
    - $t \xrightarrow{a} t_2$ can be matched by $s_2 \overset{a}{\Rightarrow} s_3$ and $(s_3, t_2) \in \mathcal{R}$
    - $t \xrightarrow{b} t_3$ can be matched by $s_2 \overset{b}{\Rightarrow} s_4$ and $(s_4, t_3) \in \mathcal{R}$
    - these are all transitions from t.

$\mathcal{R}$ is a weak bisimulation containing (s, t), therefore $s \approx t$.

Exercise 3.27

With $P \overset{\tau}{\Rightarrow} Q$ and $Q \overset{\tau}{\Rightarrow} P$, we show that $P \approx Q$. To this end we start a weak bisimulation relation with (P, Q).

$$\mathcal{R} = \{(P, Q), (P', Q'), ...\}$$

- Consider the first pair (P, Q):
  - transitions from P: $P \overset{\tau}{\Rightarrow} Q$:
    - P can do nothing and become Q (i.e. $P = Q$). Q matches this by being itself.
    - P can perform $P \xrightarrow{\tau} P'$, where $P' \overset{\tau}{\Rightarrow} Q$. Q can either do nothing or perform $Q \xrightarrow{\tau} Q'$, where $Q' \overset{\tau}{\Rightarrow} P$.
  - transitions from Q
    - Q can do nothing and become P (i.e. $Q = P$). P matches this by being itself.

- $*$ Q can perform $Q \xrightarrow{\tau} Q'$, where $Q' \overset{\tau}{\Rightarrow} P$. P can either do nothing or perform $P \xrightarrow{\tau} P'$, where $P' \overset{\tau}{\Rightarrow} Q$.

- Consider the next pair (P', Q'): both have the same transitions as the first pair, with possibly fewer τ-transitions left before P becoming Q and vice versa.

Since both P and Q can only perform τ-actions, or do nothing, the other can always match the action. Eventually, the bisimulation ends up in (P, Q) again and thus $P \approx Q$.

Exercise 3.37

The LTS can be described with the following equations:

$$
\begin{array}{llll}
s \overset{\text{def}}{=} a.s_1 & t \overset{\text{def}}{=} a.t_1 & u \overset{\text{def}}{=} a.u_1 & v \overset{\text{def}}{=} a.v_1 \\
s_1 \overset{\text{def}}{=} b.s_2 & t_1 \overset{\text{def}}{=} b.t_1 + b.t_2 & u_1 \overset{\text{def}}{=} b.u_2 & v_1 \overset{\text{def}}{=} b.v_2 + b.v_3 \\
s_2 \overset{\text{def}}{=} a.s + b.s_2 & t_2 \overset{\text{def}}{=} a.t & u_2 \overset{\text{def}}{=} a.u + b.u_2 & v_2 \overset{\text{def}}{=} a.v \\
 & & u_3 \overset{\text{def}}{=} a.u + b.u_2 & v_3 \overset{\text{def}}{=} b.v_2 + b.v_3
\end{array}
$$

To show $s \nsim t$, we give a universal winning strategy for the attacker:

- The start configuration is (s, t). The attacker plays $s \xrightarrow{a} s_1$ on the left, the defender responds with $t \xrightarrow{a} t_1$ on the right.
- The new configuration is (s_1, t_1). The attacker plays $t_1 \xrightarrow{b} t_1$ on the right (sabotage the defender for the next round). The defender plays $s_1 \xrightarrow{b} s_2$ on the left.
- The final configuration is (s_2, t_1). The attacker plays $s_2 \xrightarrow{a} s$ on the left. The defender cannot respond and loses.

To show $s \sim u$, we give a universal winning strategy for the defender:
The start configuration is (s, u).

- The attacker can play:
  - $s \xrightarrow{a} s_1$. The defender can only respond with $u \xrightarrow{a} u_1$.
  - $u \xrightarrow{a} u_1$. The defender can only respond with $s \xrightarrow{a} s_1$.

  Either way, the next configuration is (s_1, u_1).
- The attacker can play:

  - $s_1 \xrightarrow{b} s_2$. The defender can only respond with $u_1 \xrightarrow{b} u_3$.
  - $u_1 \xrightarrow{b} u_3$. The defender can only respond with $s_1 \xrightarrow{b} s_2$.

  Either way, the next configuration is (s_2, u_3).

- The attacker can play:
  - $s_2 \xrightarrow{a} s$. The defender can only respond with $u_3 \xrightarrow{a} u$.
  - $u_3 \xrightarrow{a} u$. The defender can only respond with $s_2 \xrightarrow{a} s$.

  In both cases, the next configuration is (s, u): a cycle, so the defender wins. Alternatively, the attacker could play:

  - $s_2 \xrightarrow{b} s_2$. The defender can only respond with $u_3 \xrightarrow{b} u_2$.
  - $u_3 \xrightarrow{b} u_2$. The defender can only respond with $s_2 \xrightarrow{b} s_2$.

  In these cases the next configuration is (s_2, u_2).

- The attacker can play:
  - $s_2 \xrightarrow{a} s$. The defender can only respond with $u_3 \xrightarrow{a} u$.
  - $u_3 \xrightarrow{a} u$. The defender can only respond with $s_2 \xrightarrow{a} s$.

  In both cases, the next configuration is (s, u): a cycle, so the defender wins. Alternatively, the attacker could play:

  - $s_2 \xrightarrow{b} s_2$. The defender can only respond with $u_3 \xrightarrow{b} u_2$.
  - $u_3 \xrightarrow{b} u_2$. The defender can only respond with $s_2 \xrightarrow{b} s_2$.

  In these cases the next configuration is (s_2, u_2): another cycle, so the defender wins.

In all cases, the defender wins. The relation is

$$\mathcal{R} = \{(s, u), (s_1, u_1), (s_2, u_3), (s_2, u_2)\}$$

Finally, we show that $s \not\sim v$ with a universal winning strategy for the attacker:

- The start configuration is (s, v). The attacker plays $s \xrightarrow{a} s_1$ on the left. The defender can only respond with $v \xrightarrow{v}_1$ on the right. (Attacking on the other side has the same result).

- The next configuration is (s_1, v_1). The attacker plays $v_1 \xrightarrow{b} v_2$ on the right. The defender can only respond with $s_1 \xrightarrow{b} s_2$ on the left.

- The final configuration is (s_2, v_2). The attacker plays $s_2 \xrightarrow{b} s_2$. The defender cannot respond and loses.