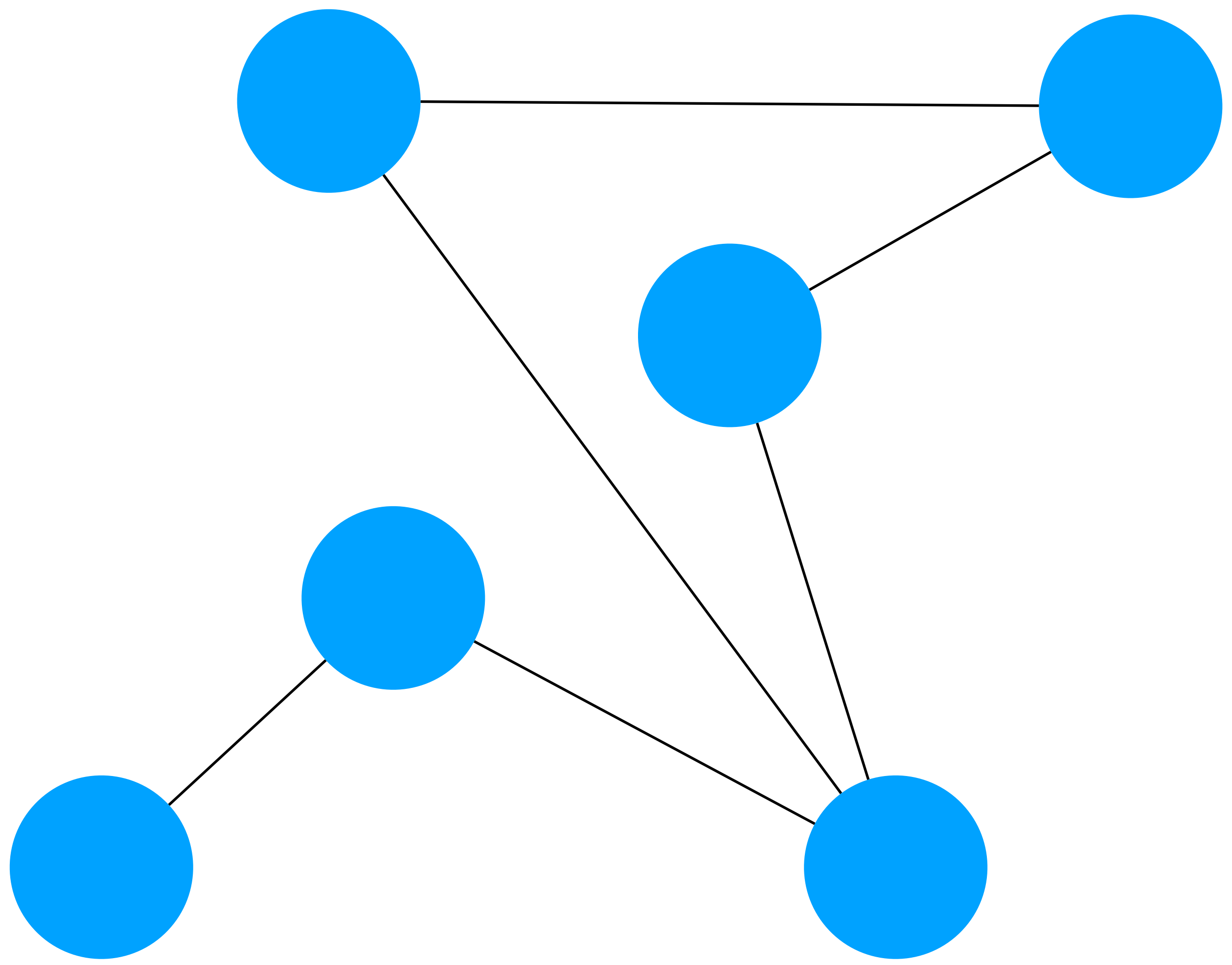
Algorithmics Part 4
Structural Decompositions and Algorithms

Friedrich Slivovsky

Elimination Orderings



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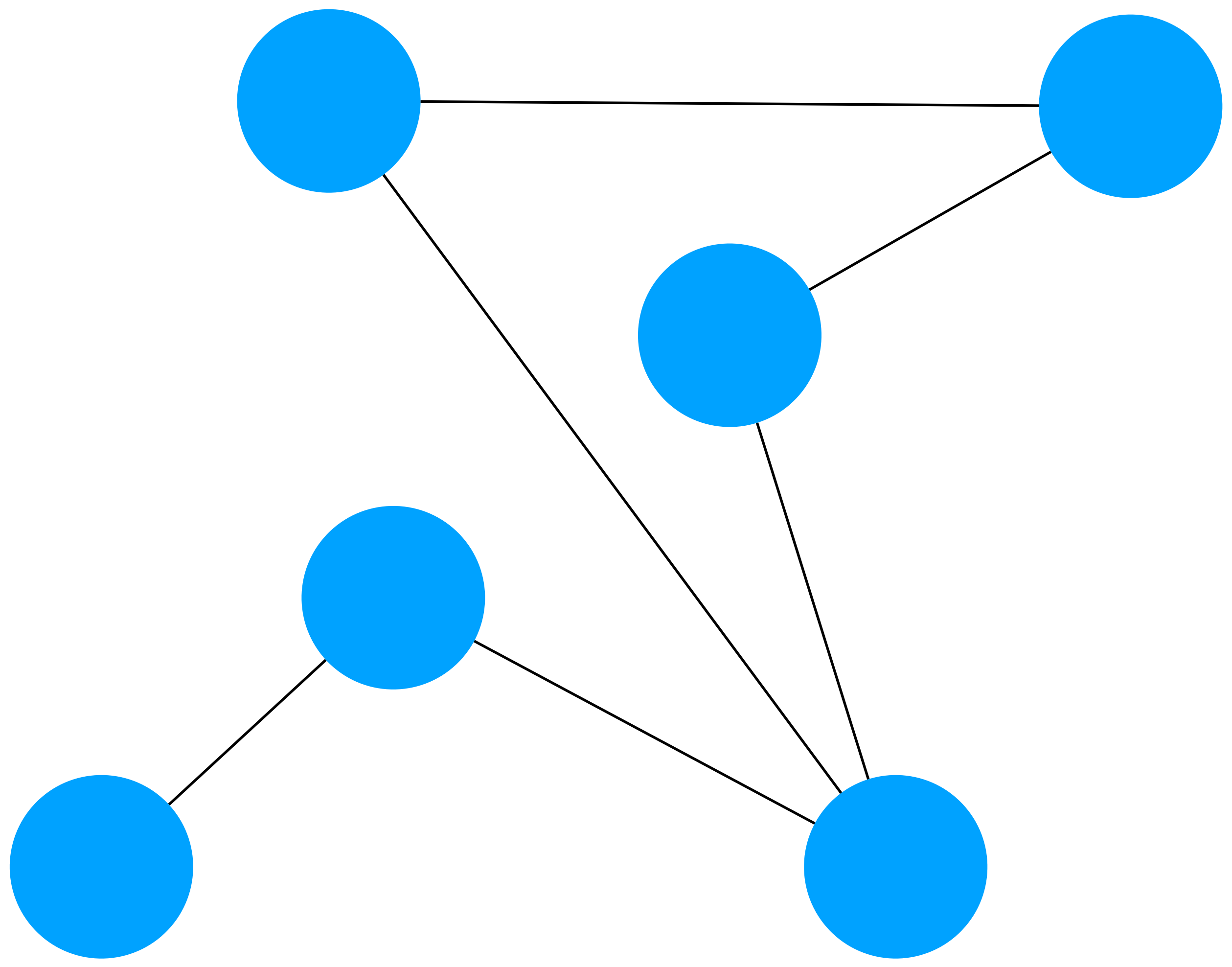

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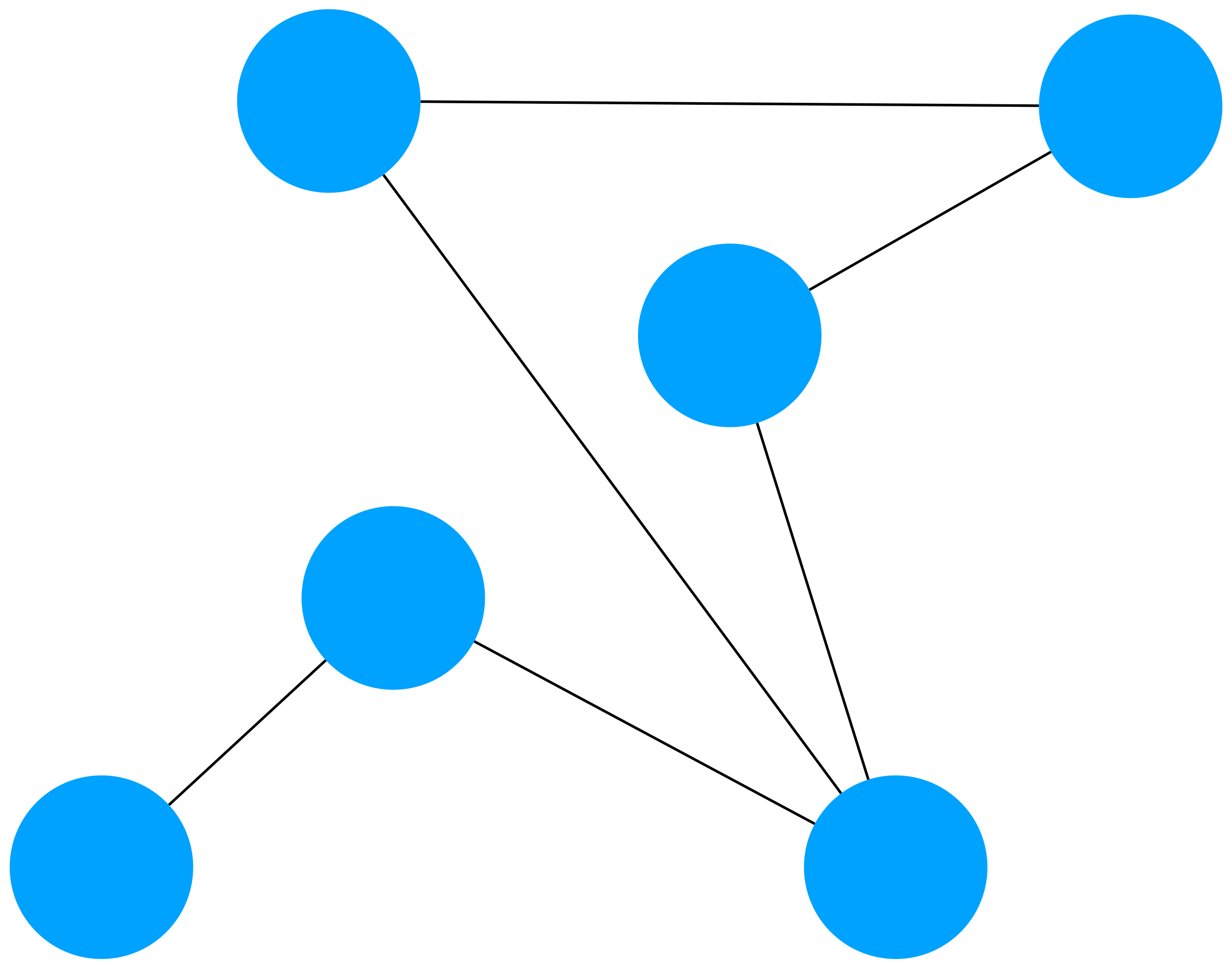

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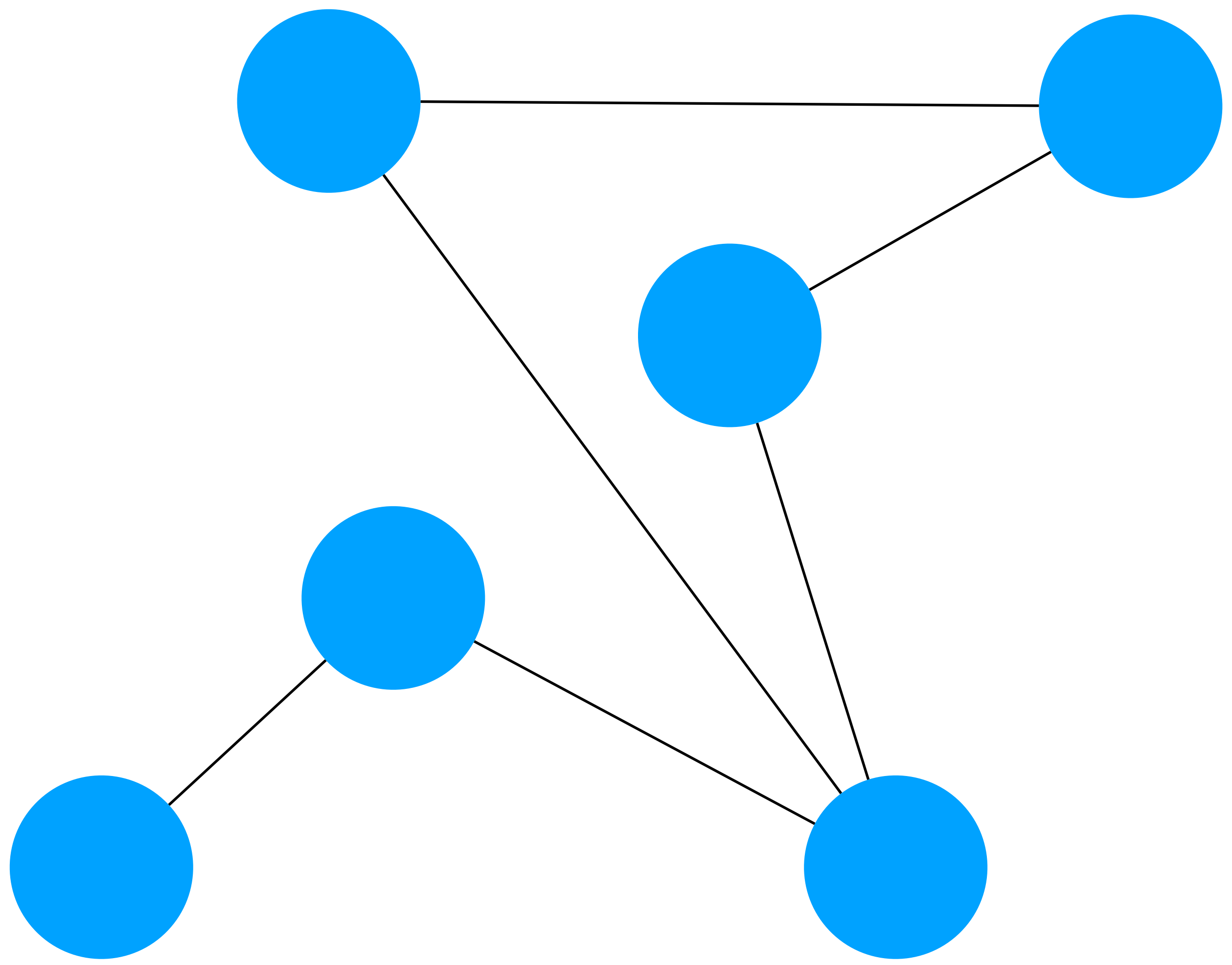

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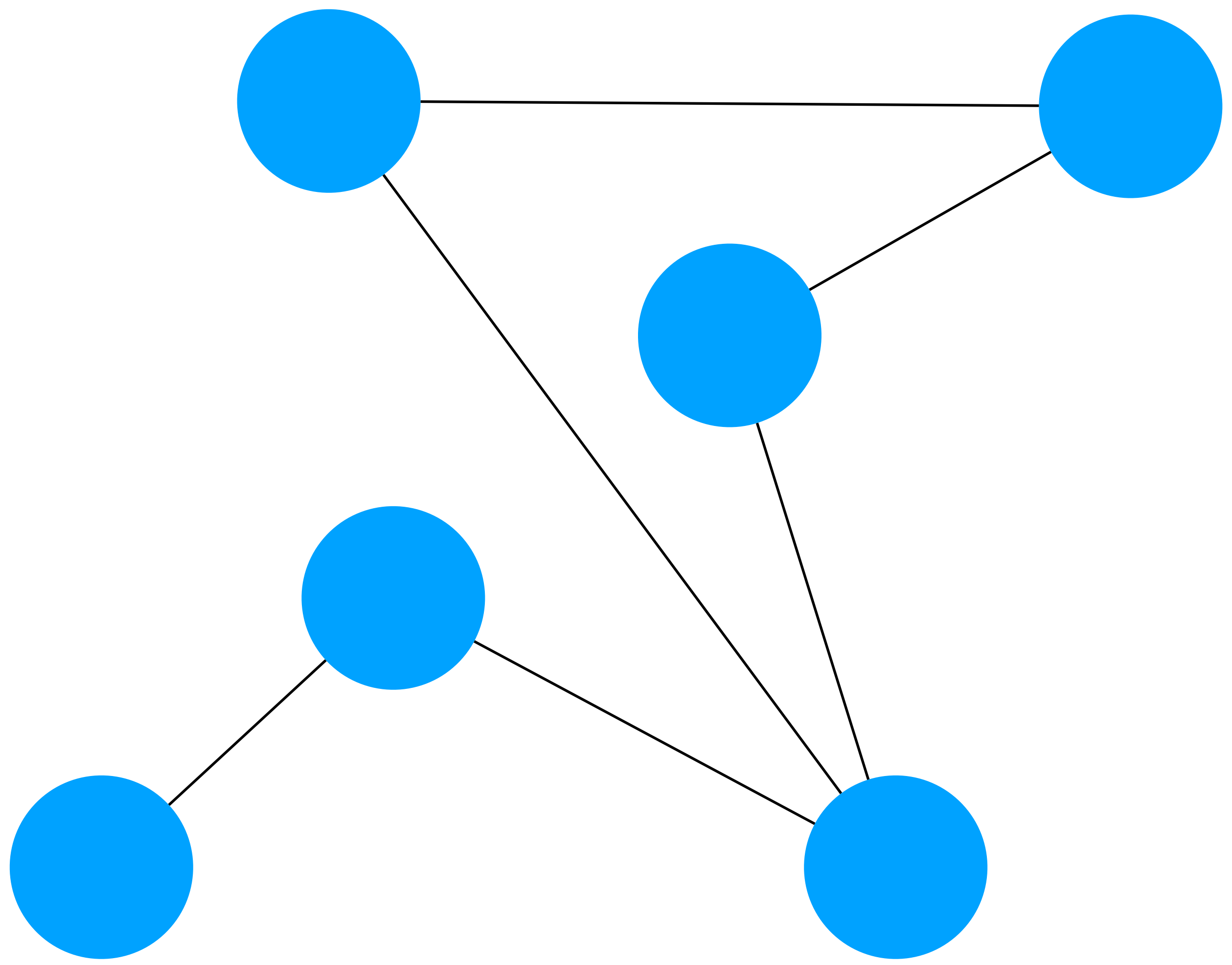

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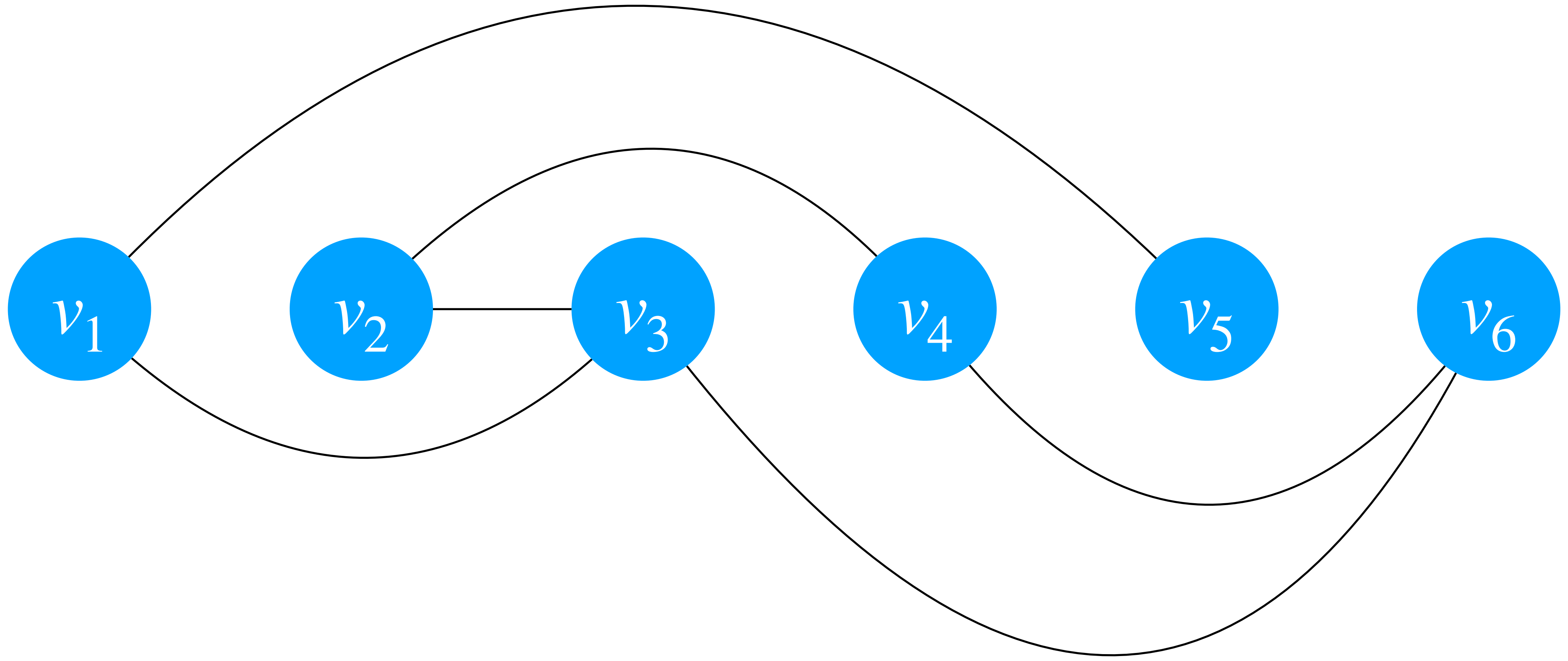

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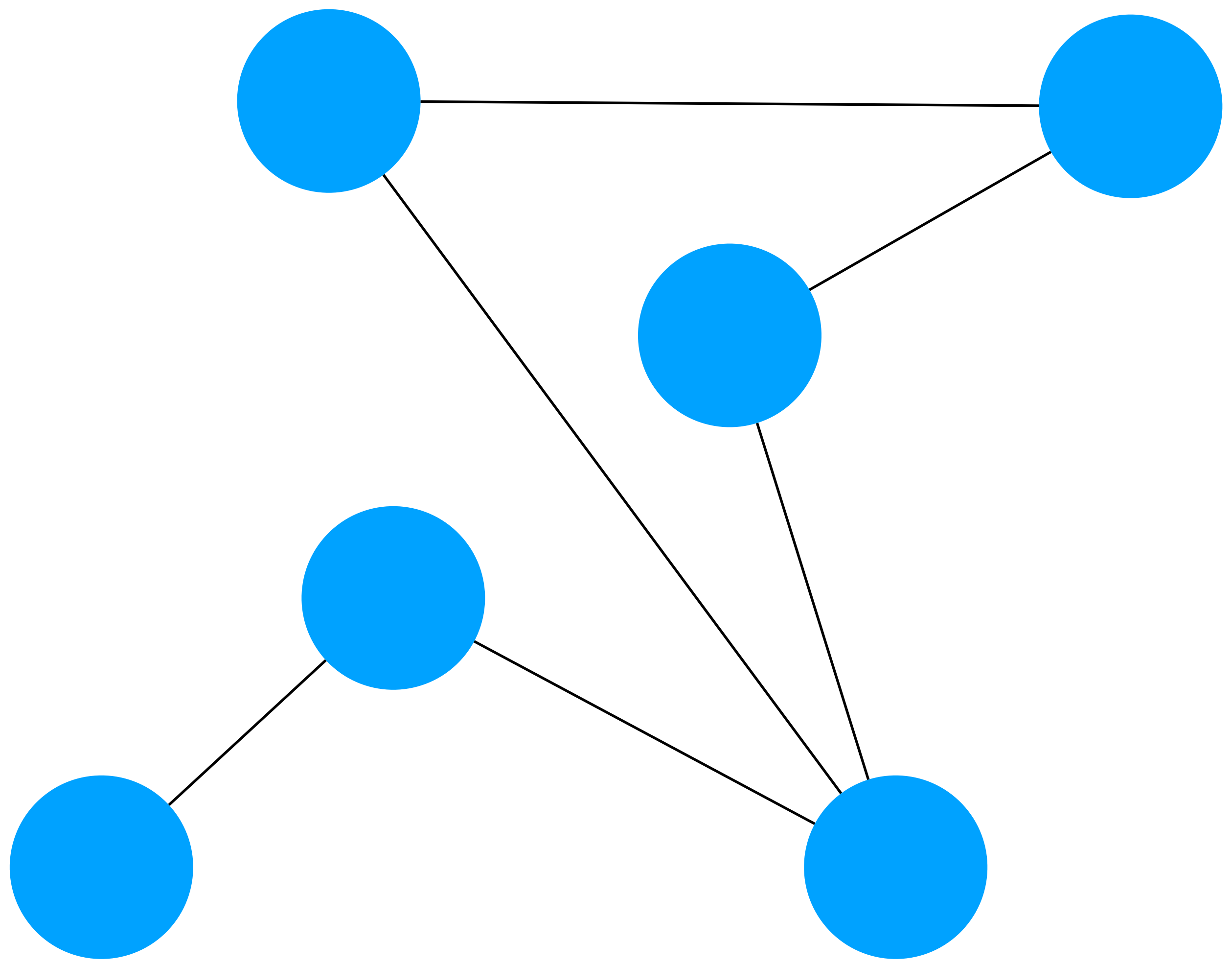

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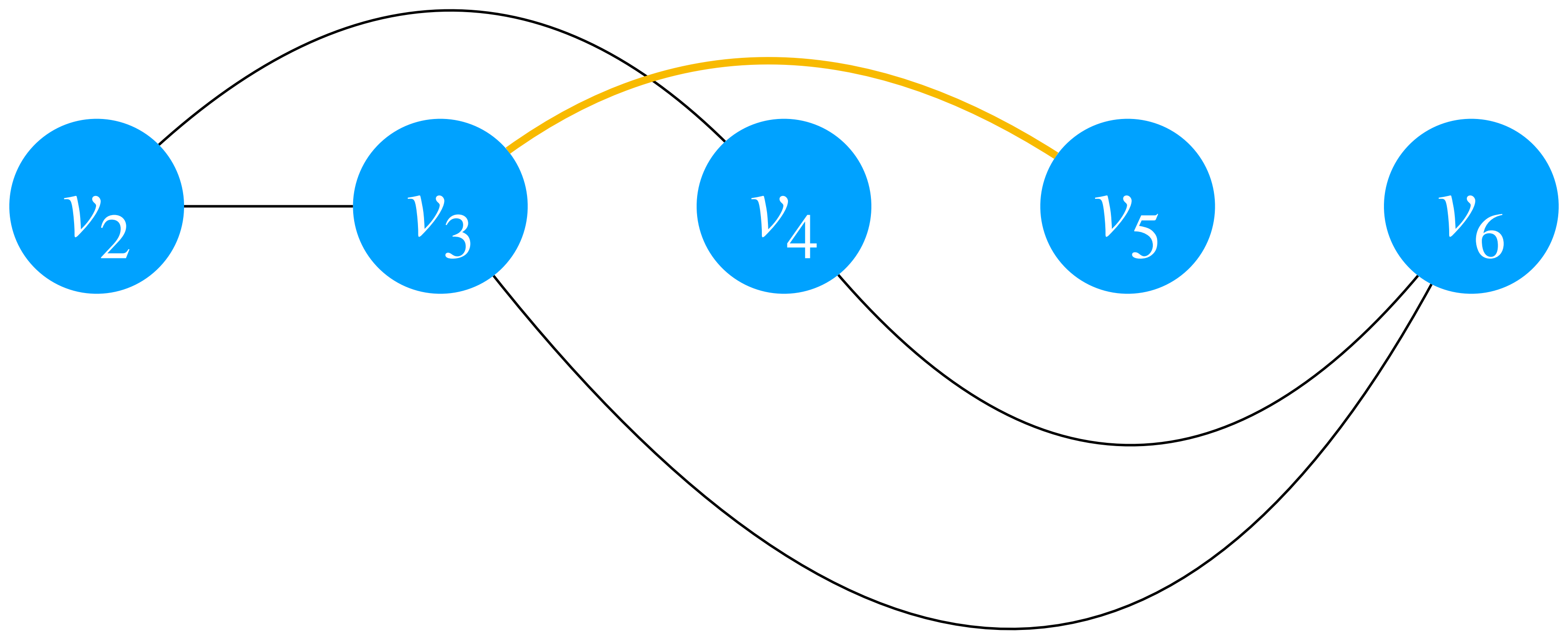

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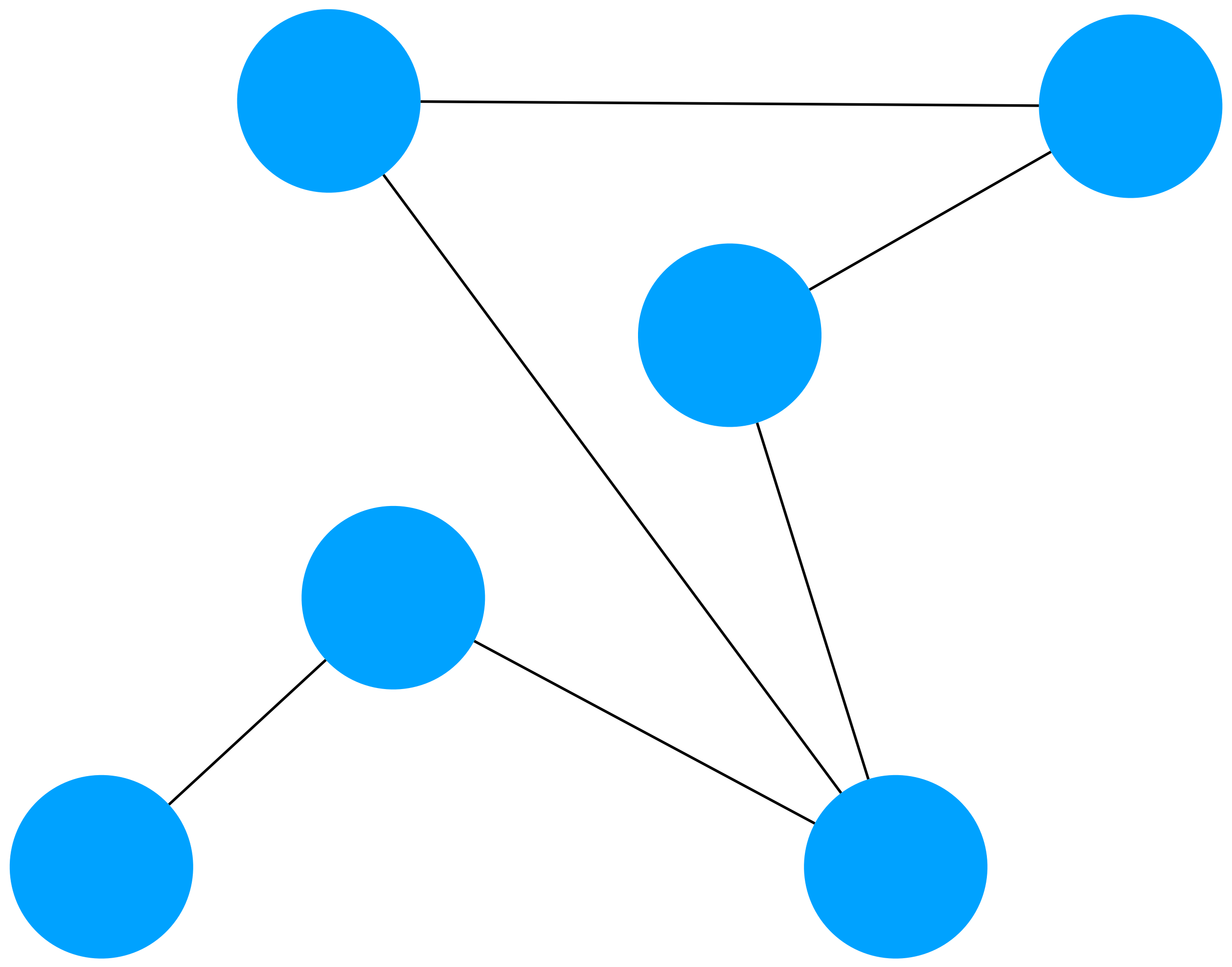

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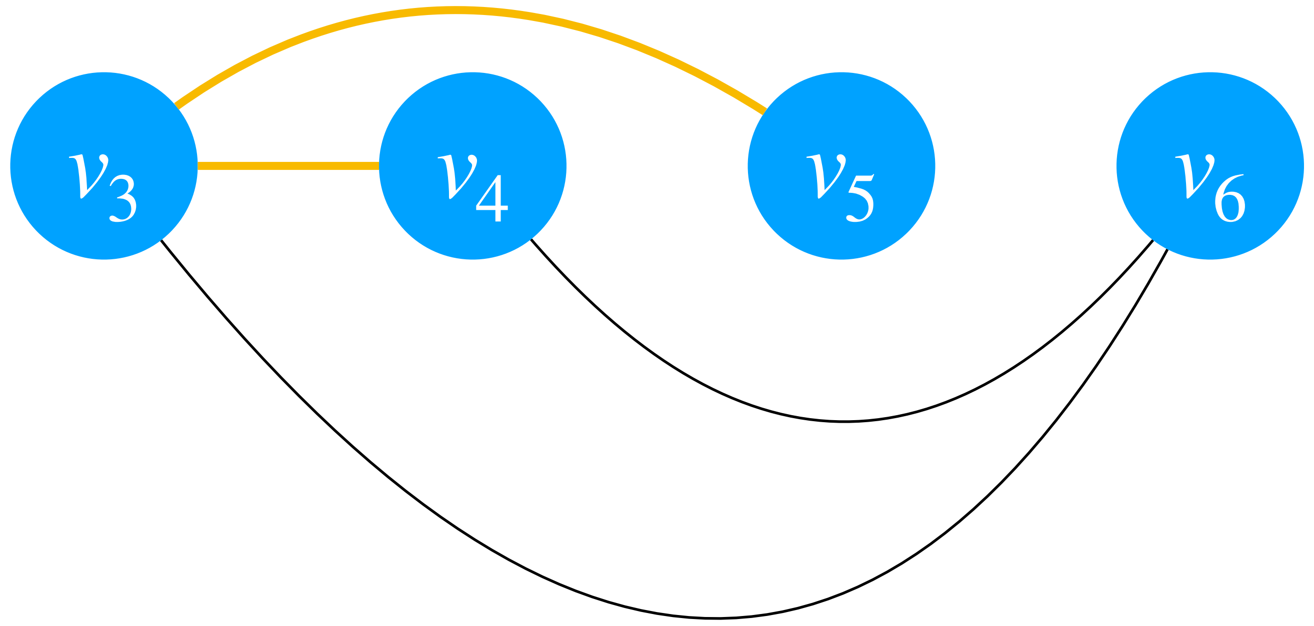

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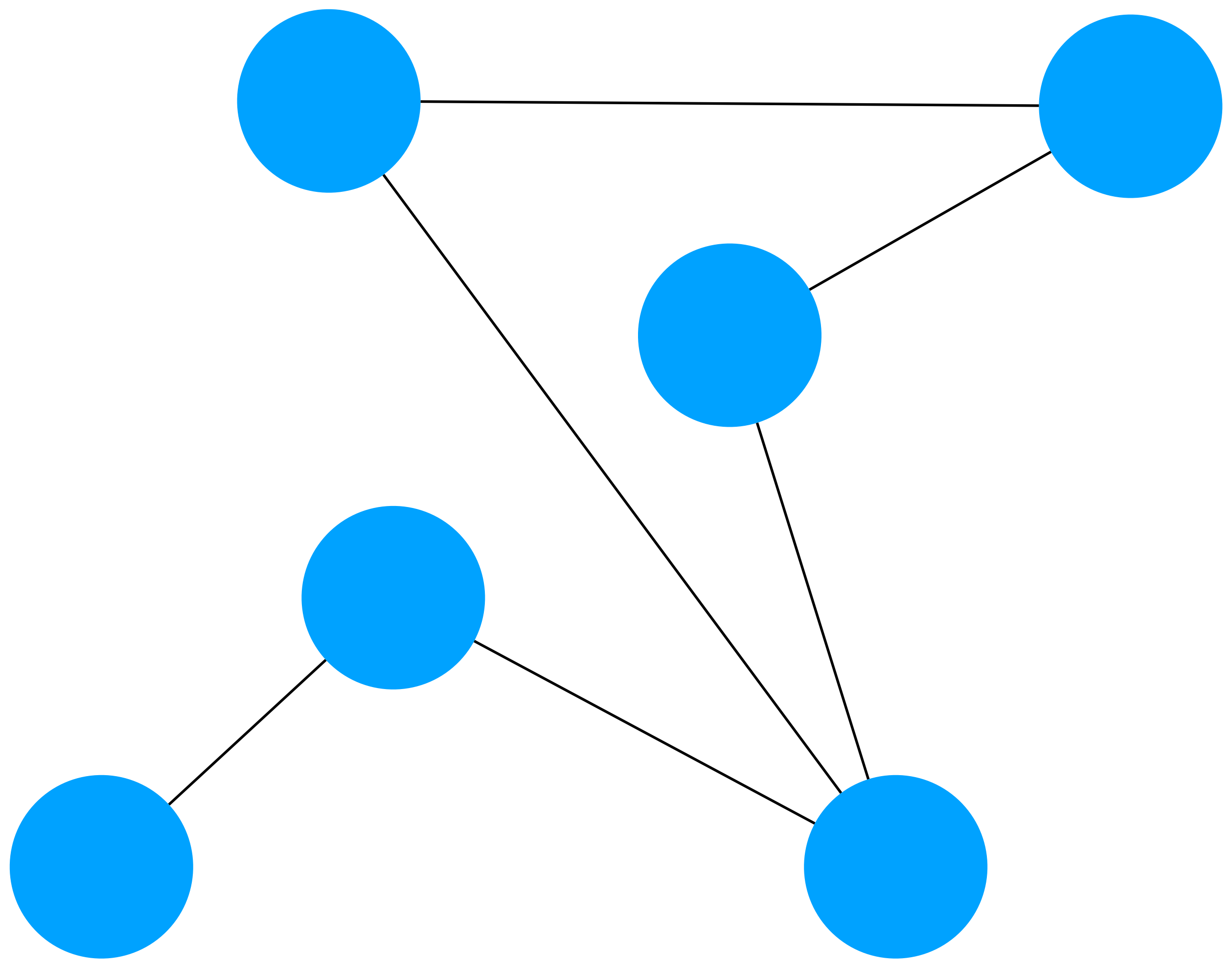

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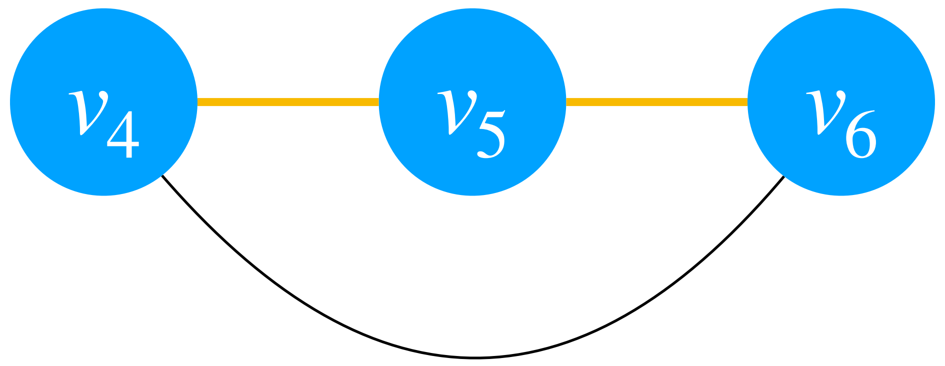

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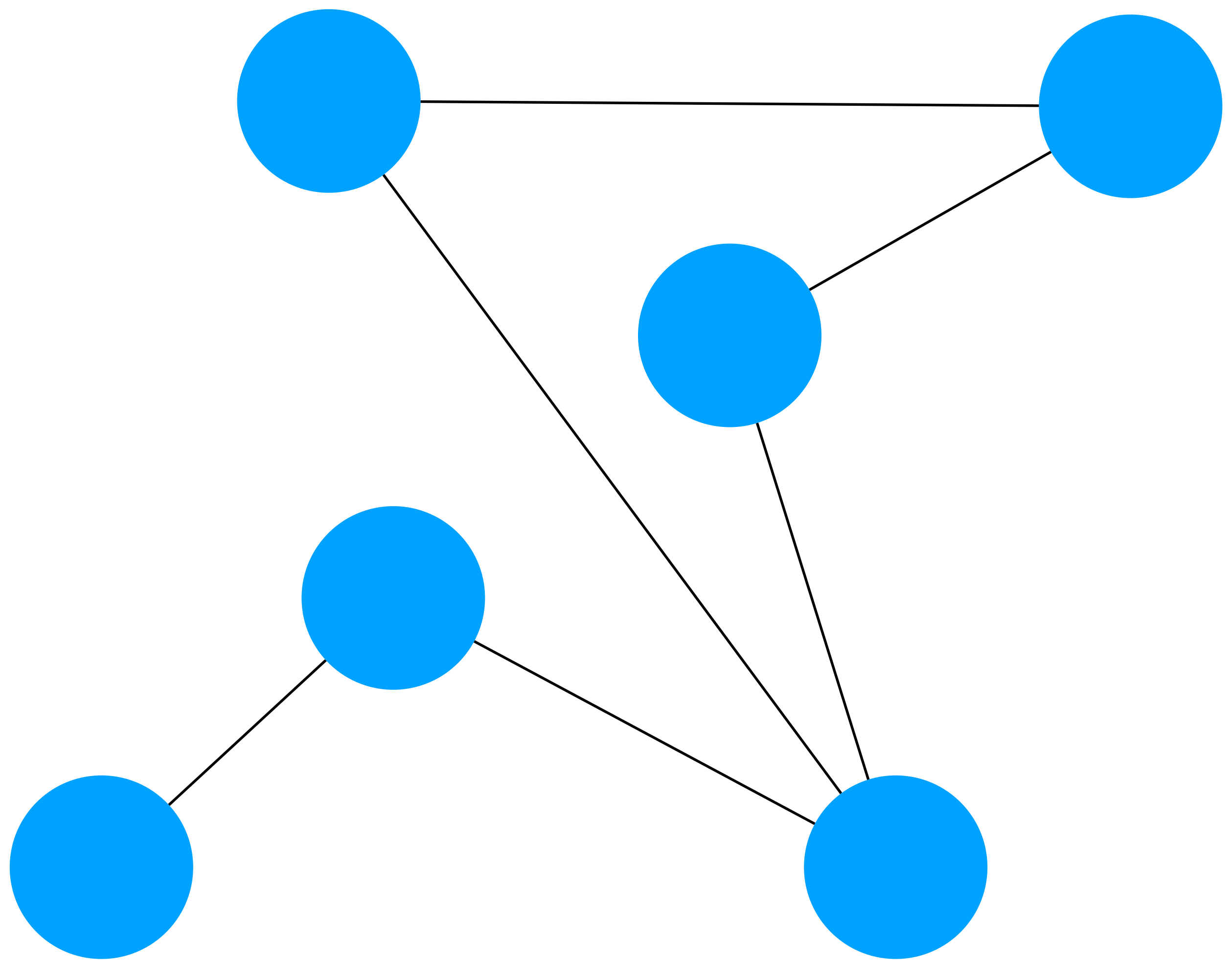

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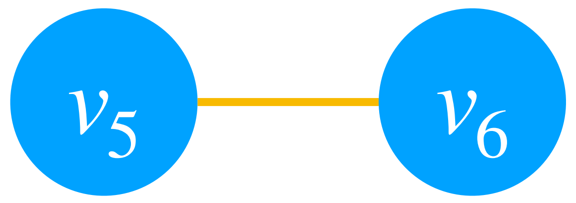

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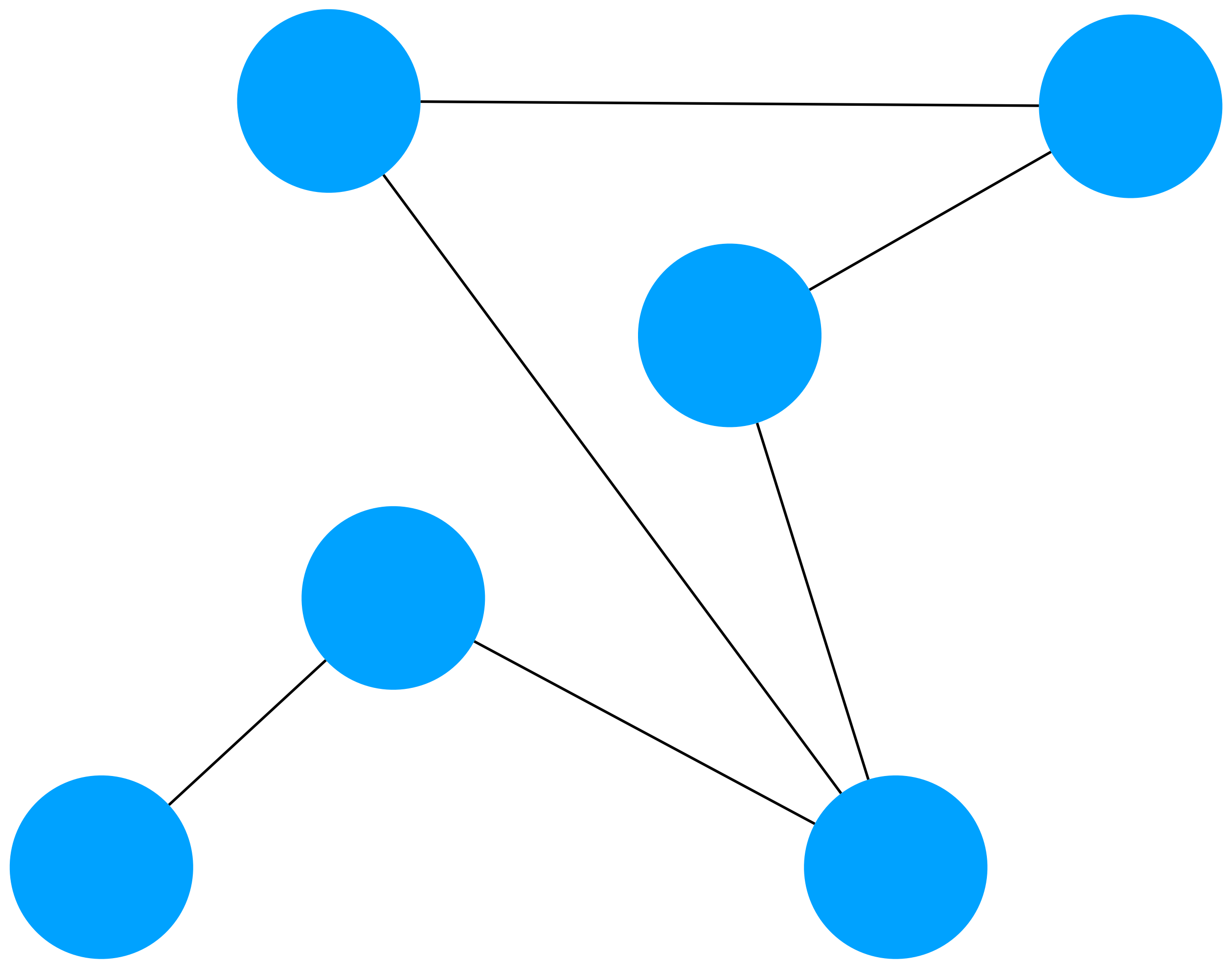

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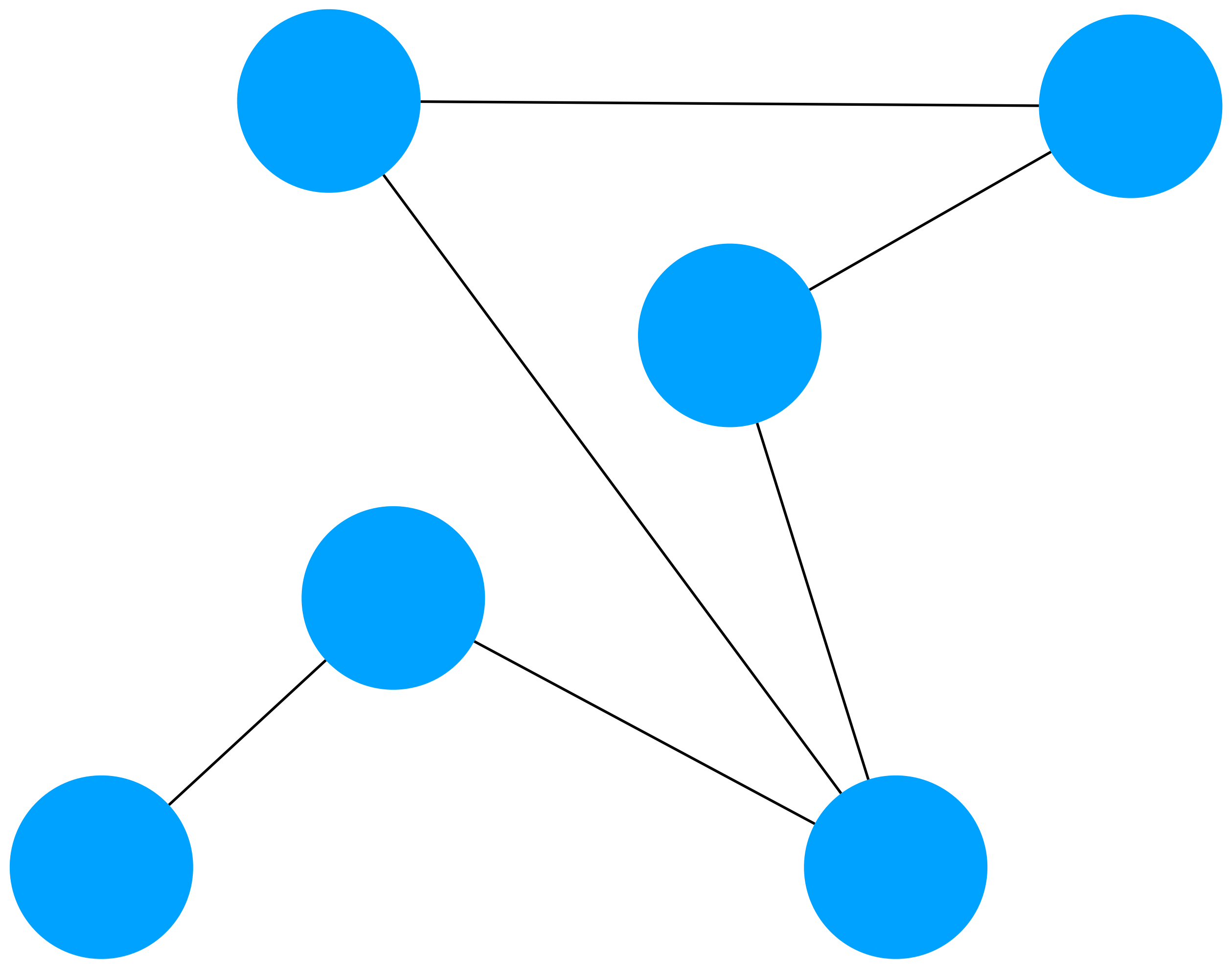

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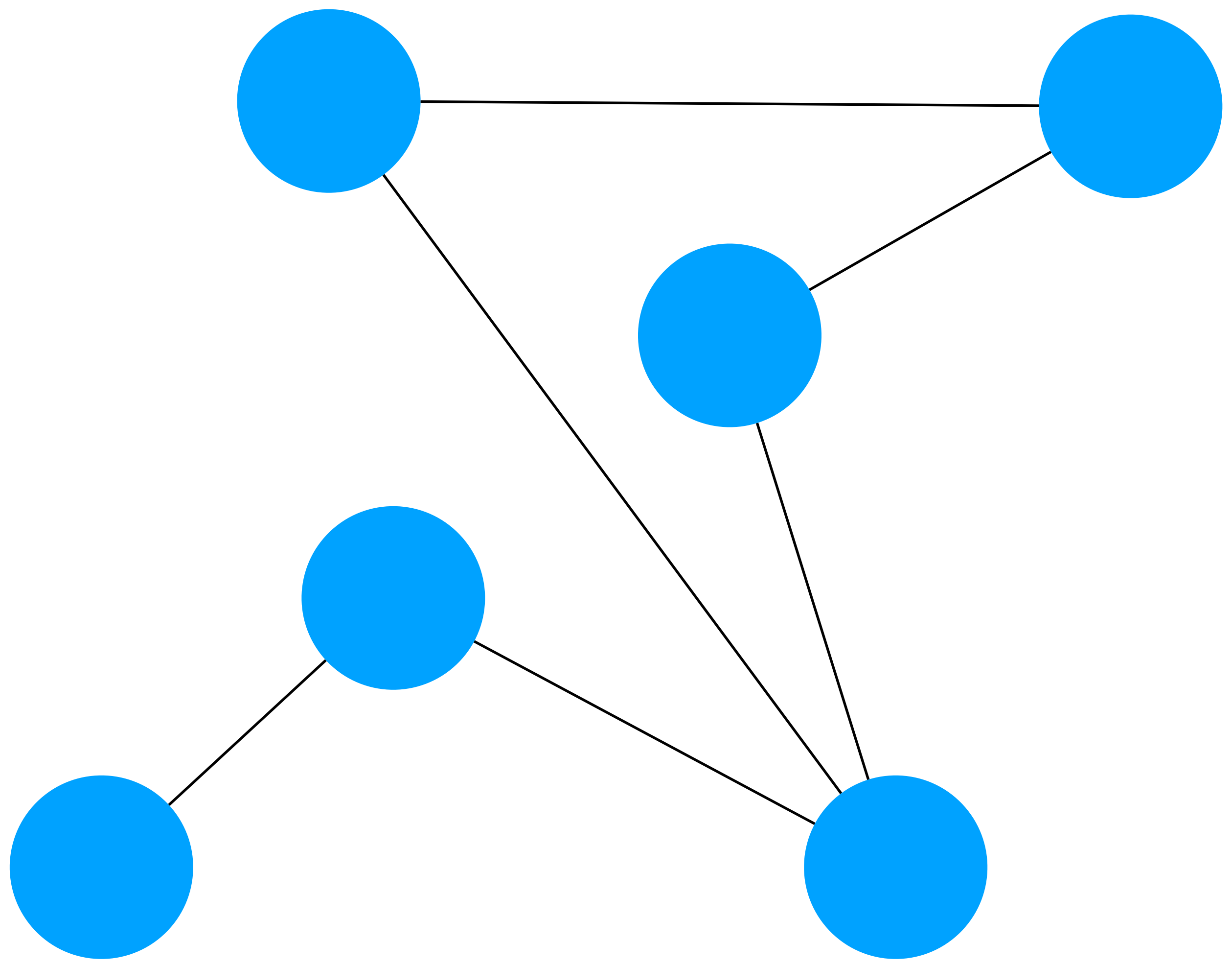

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fill-in edges

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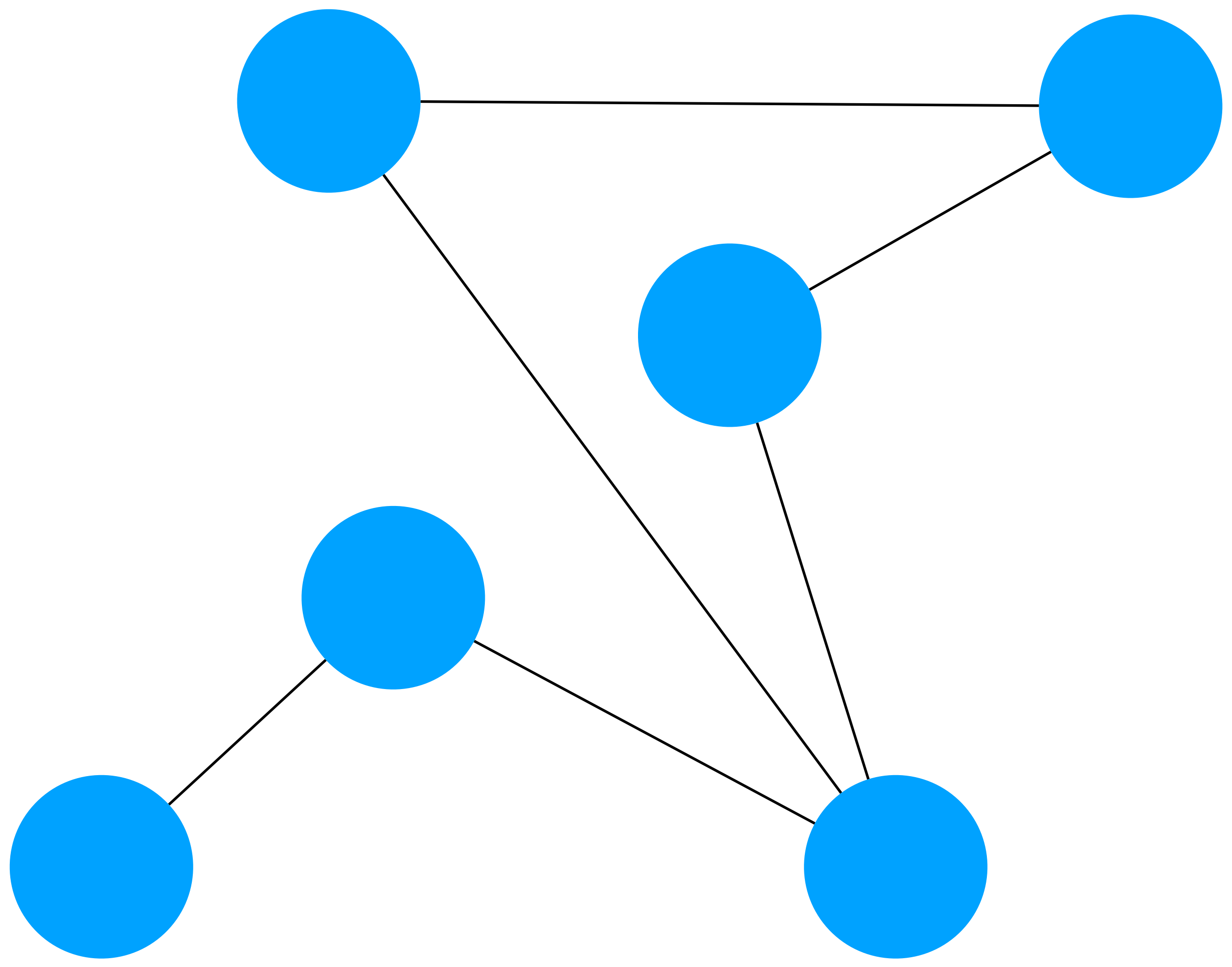

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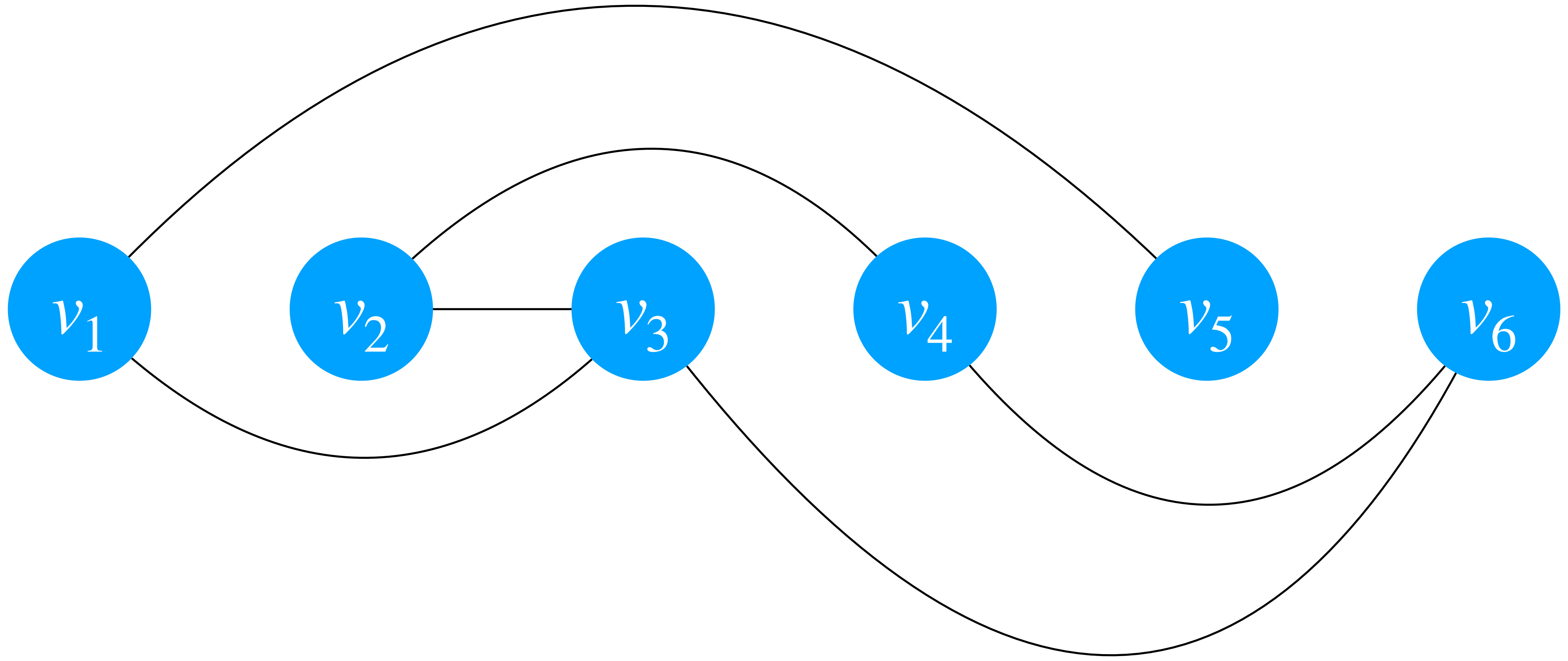

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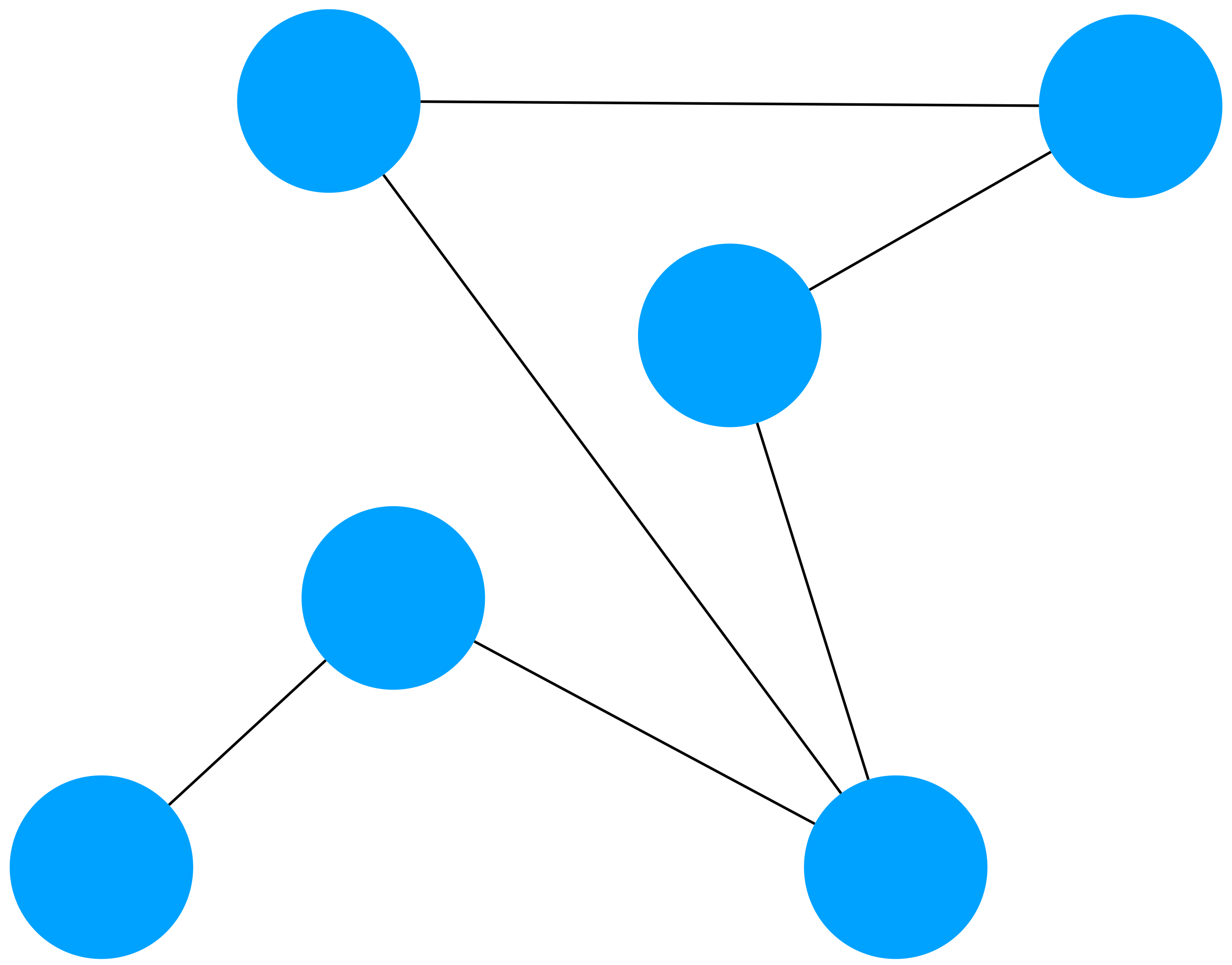

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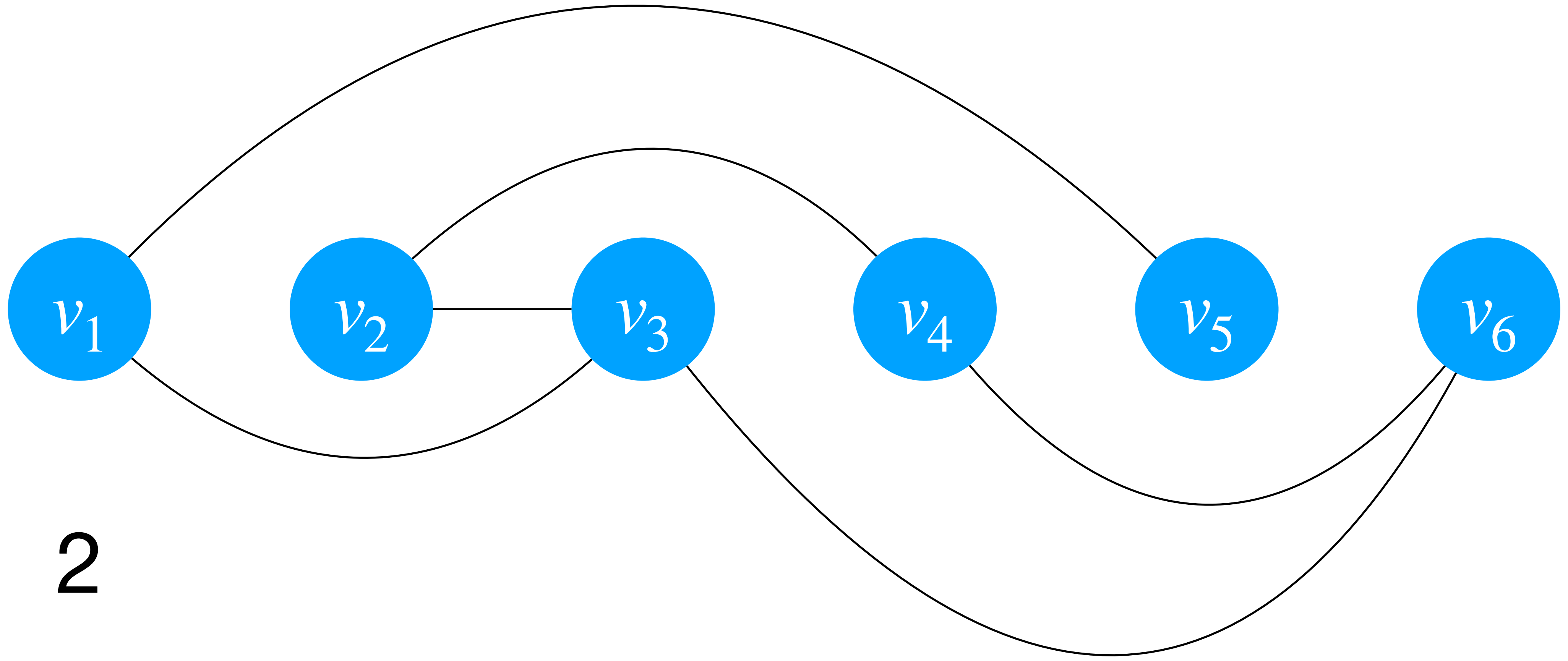

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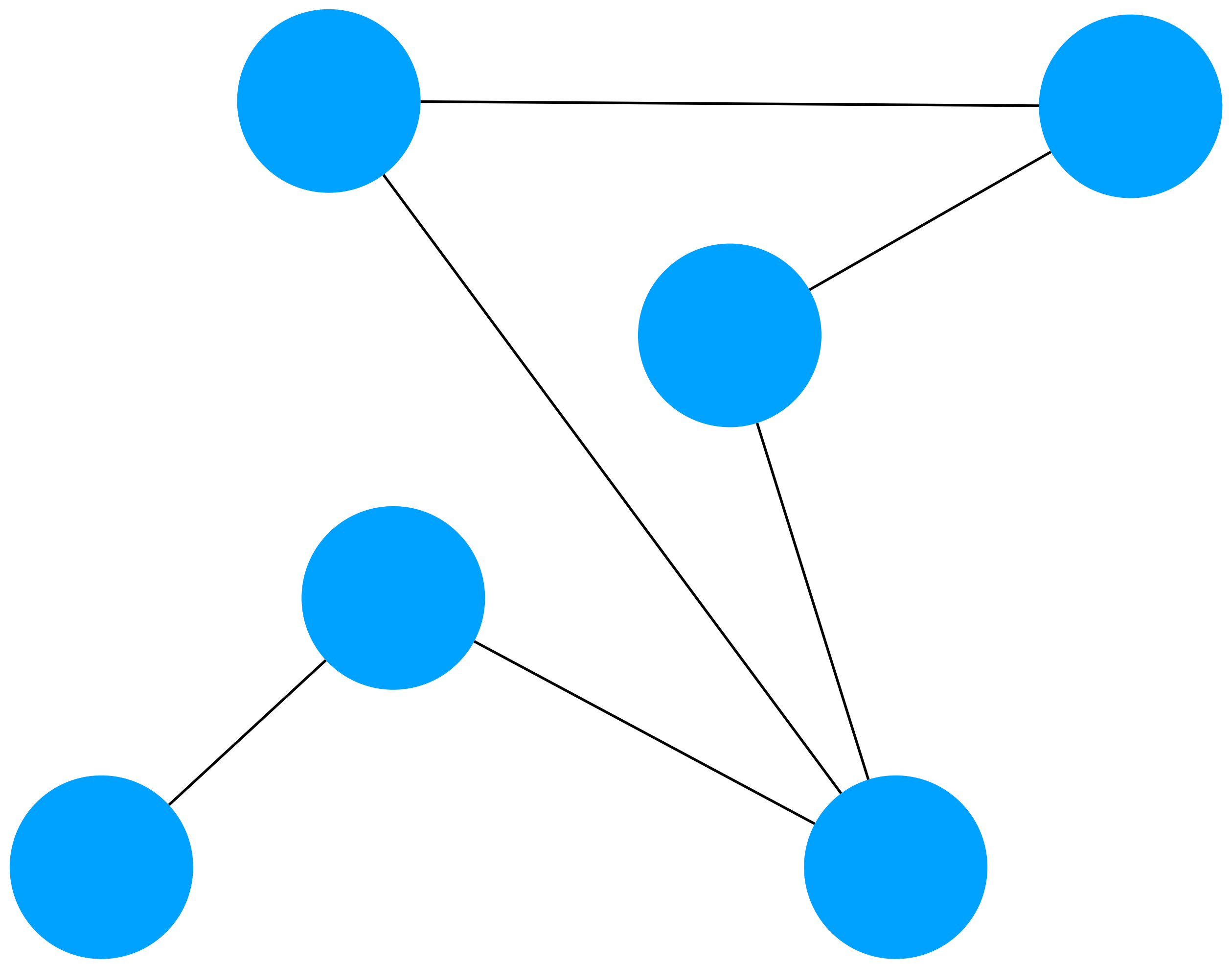

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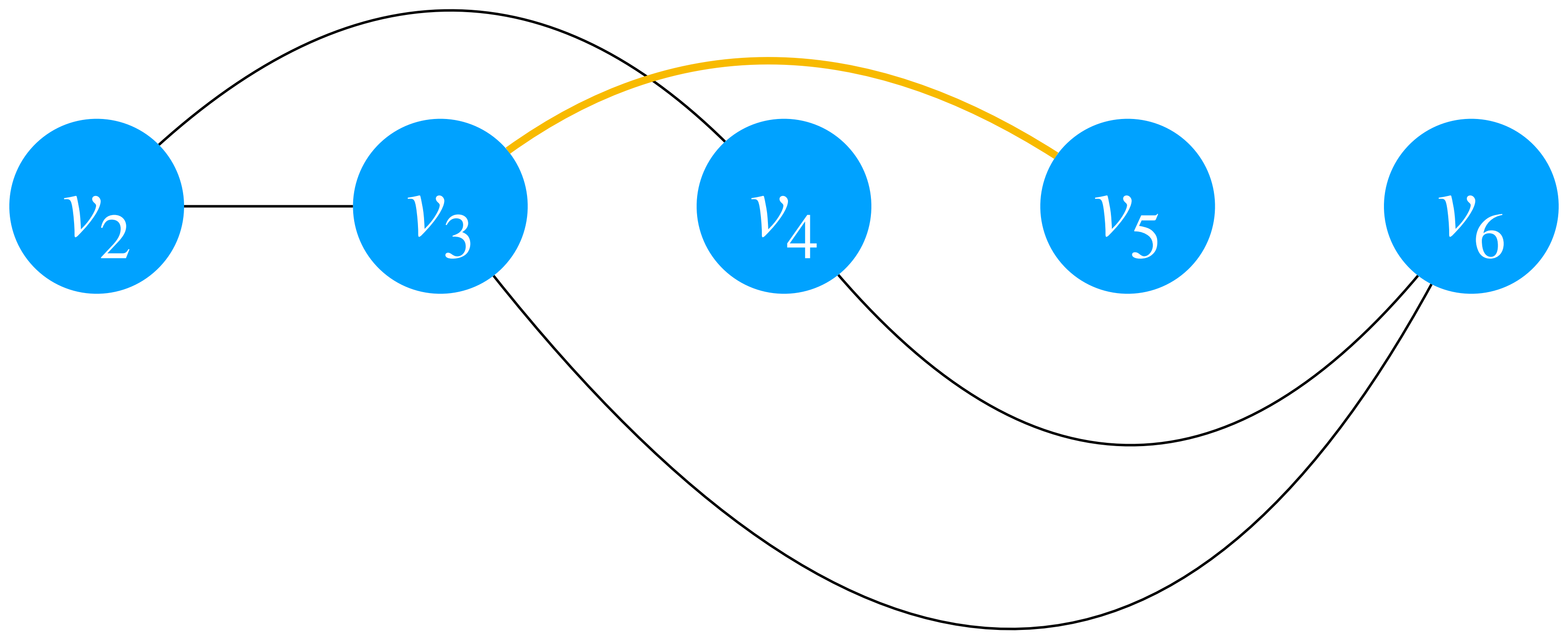

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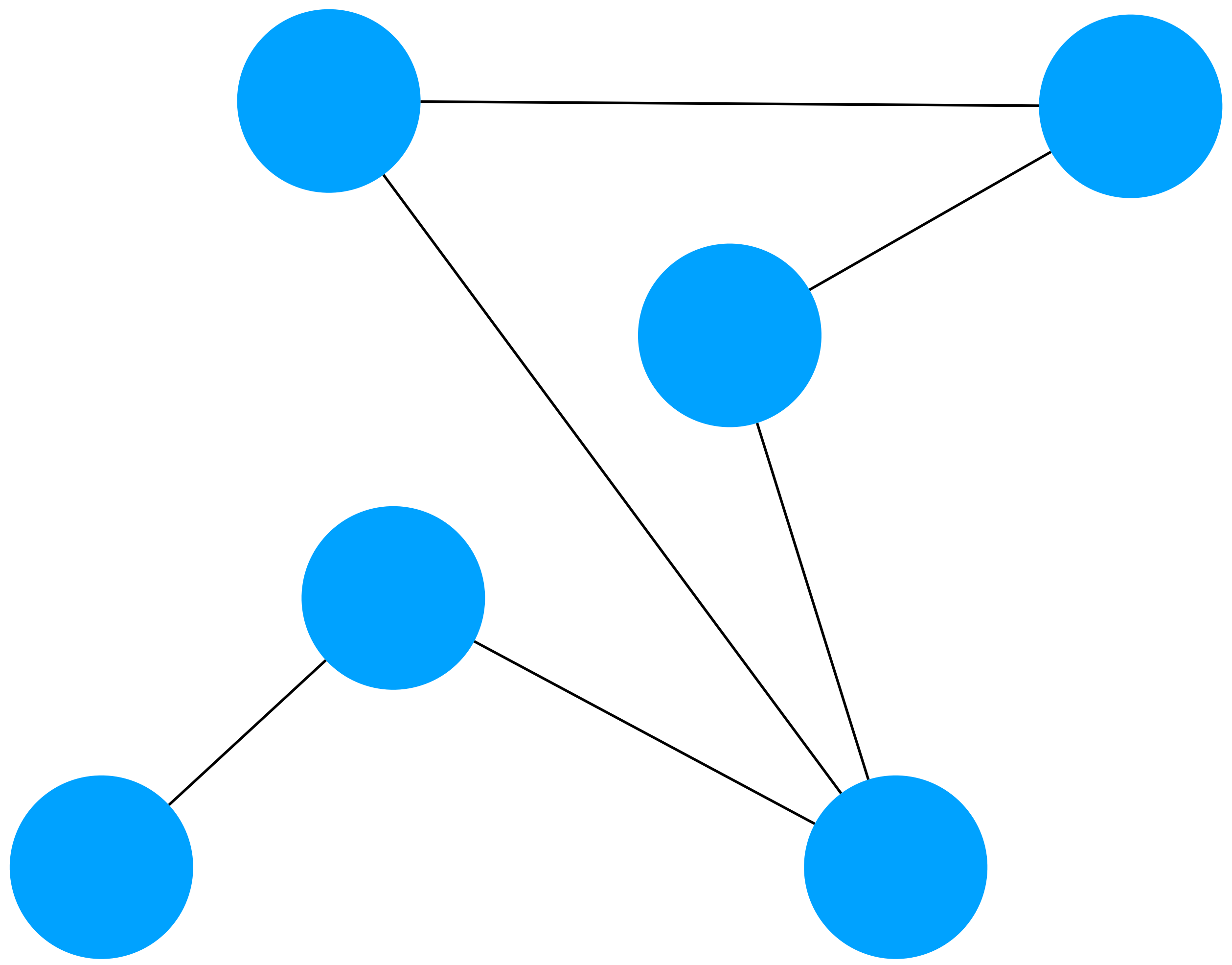

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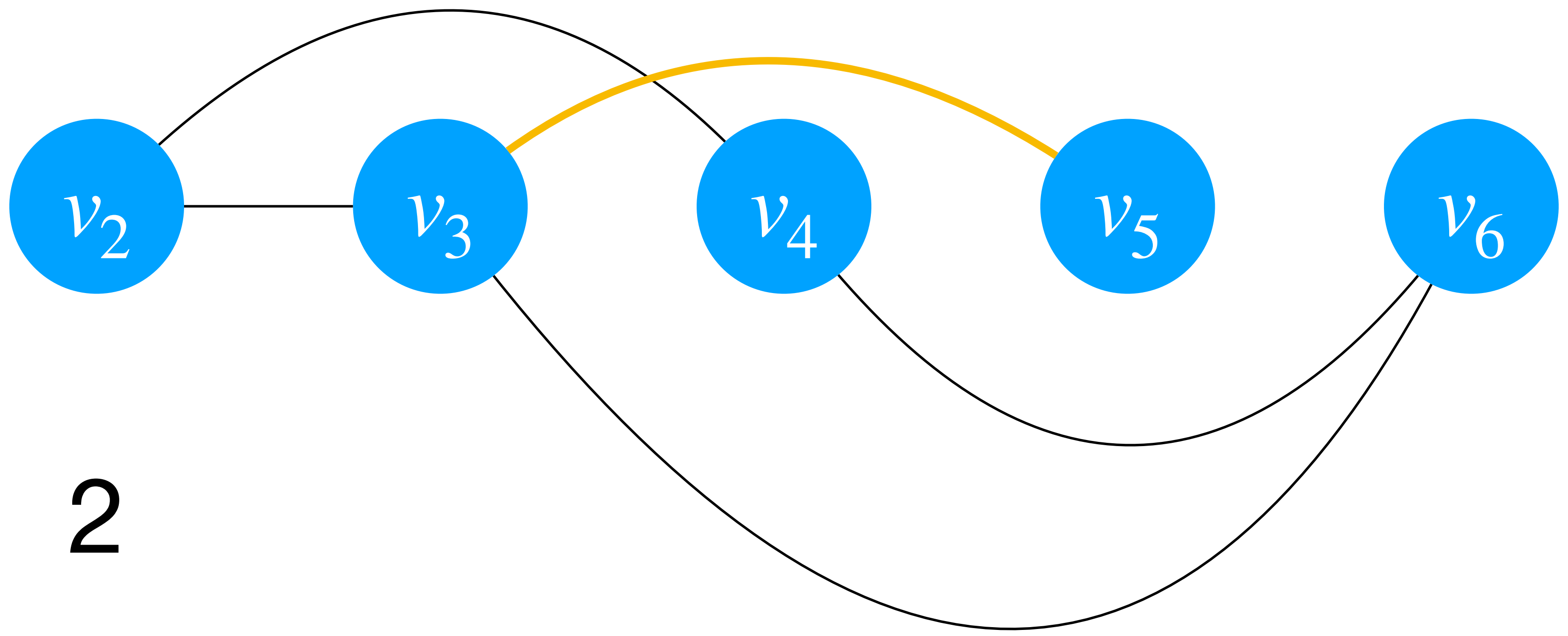

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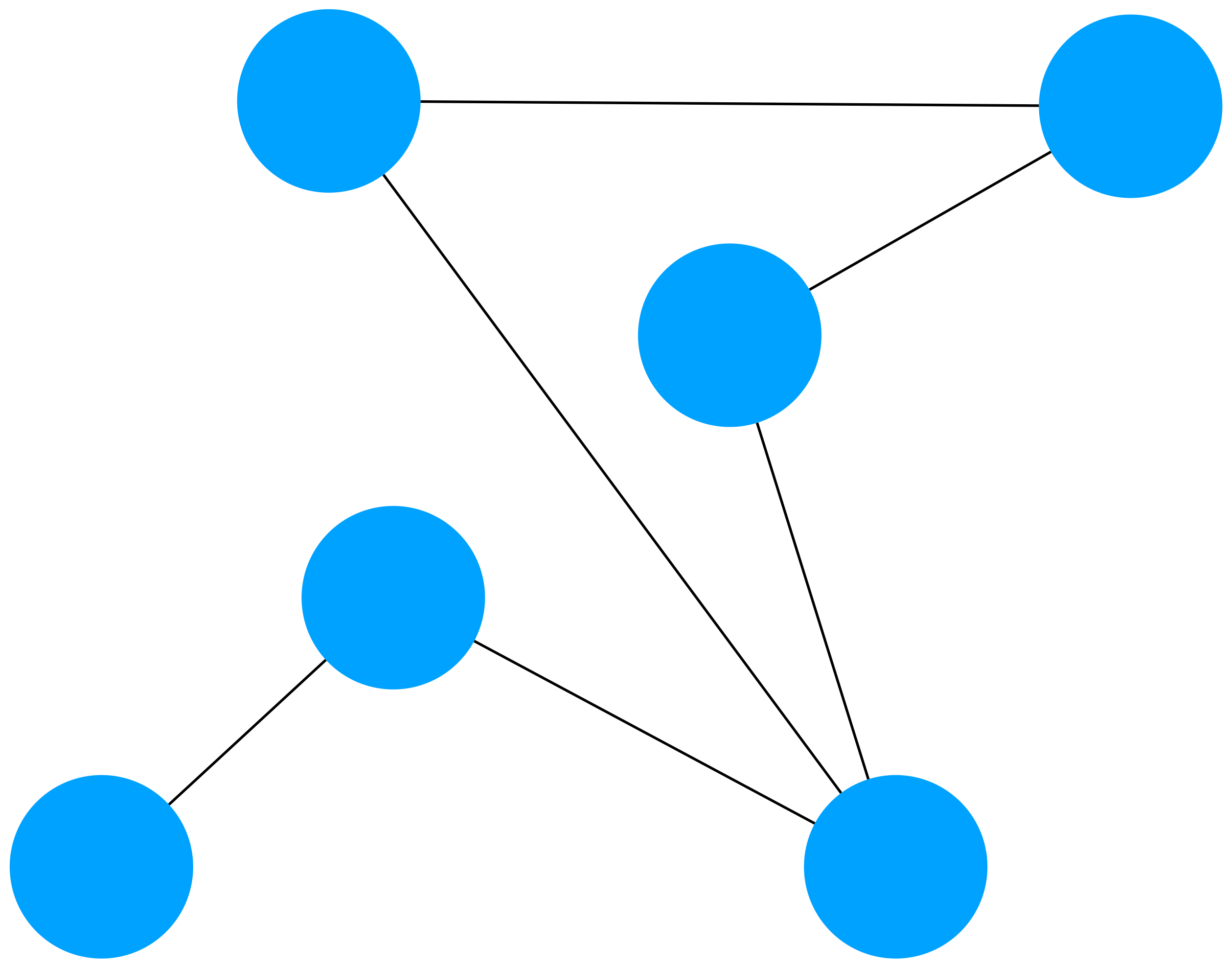

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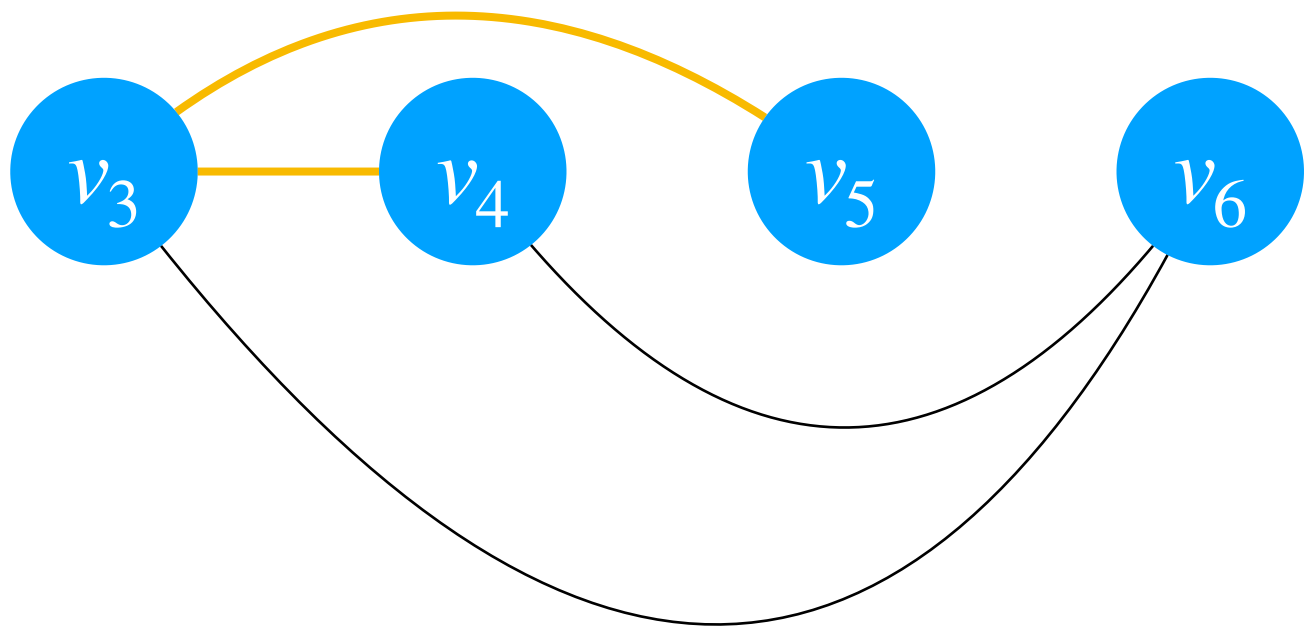

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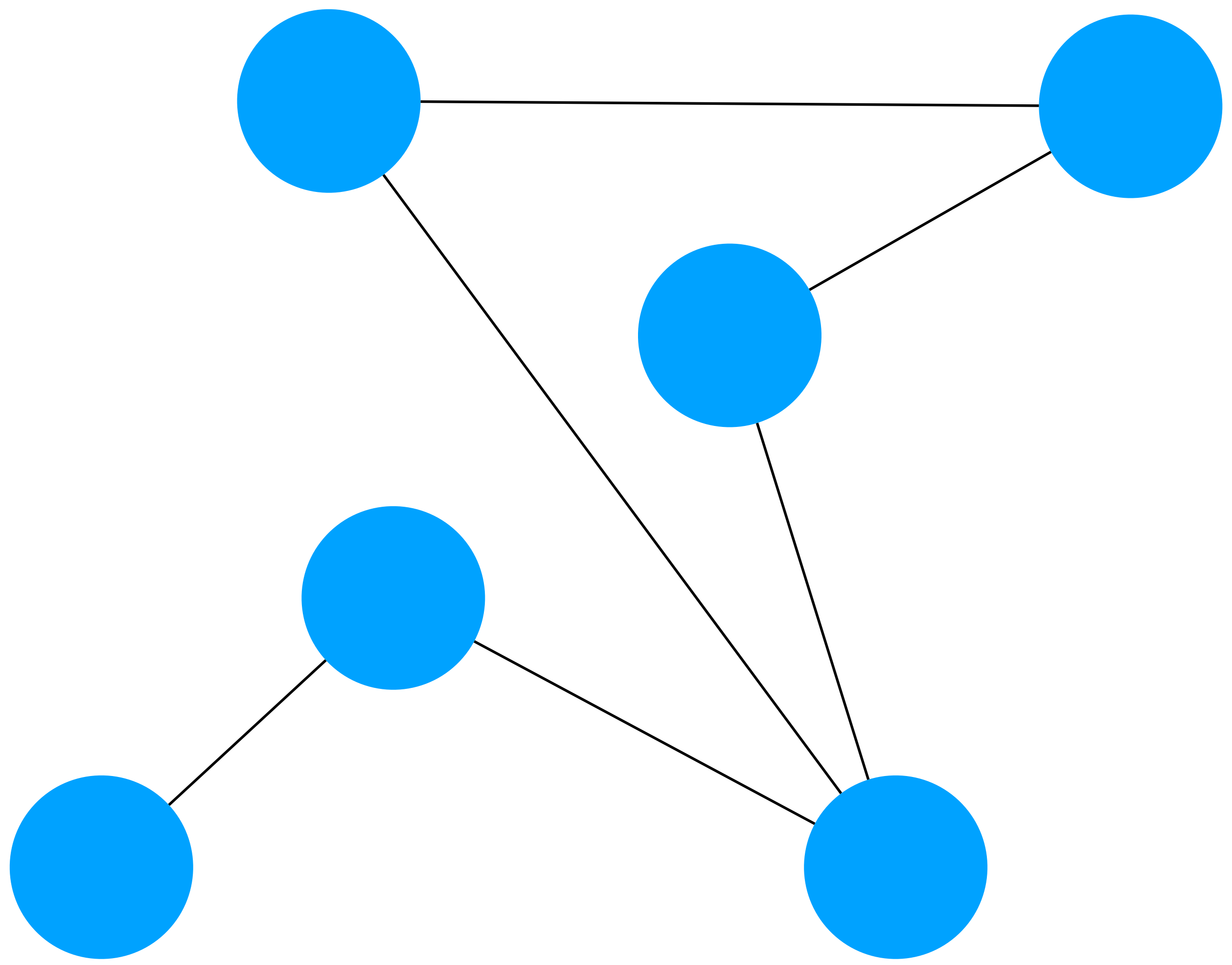

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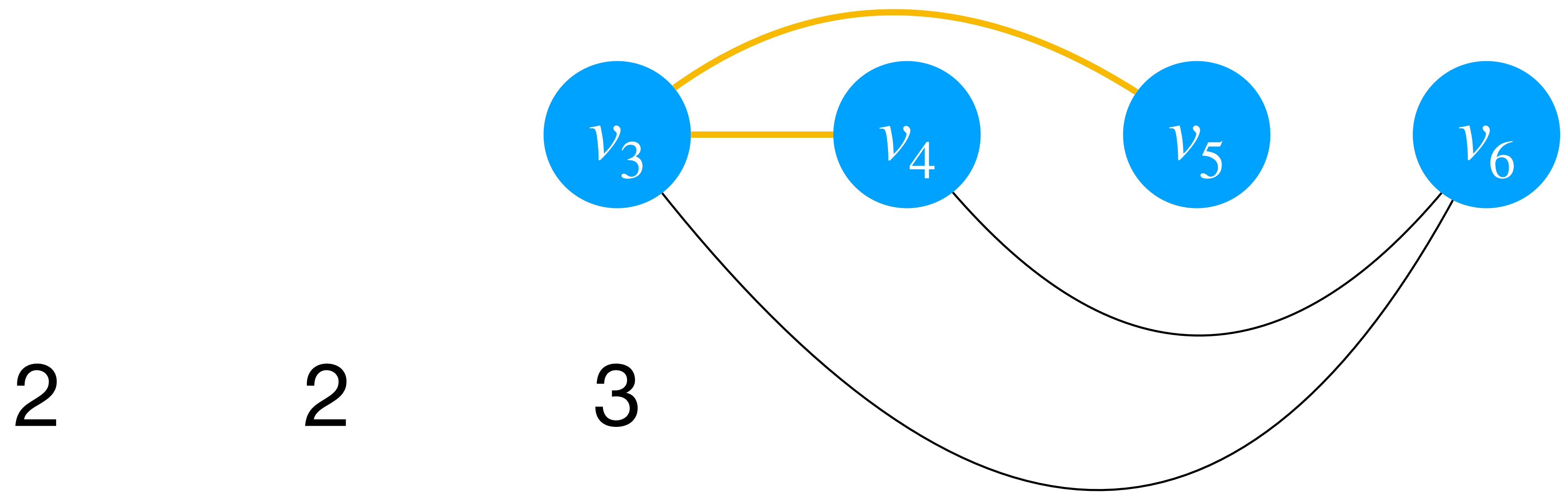

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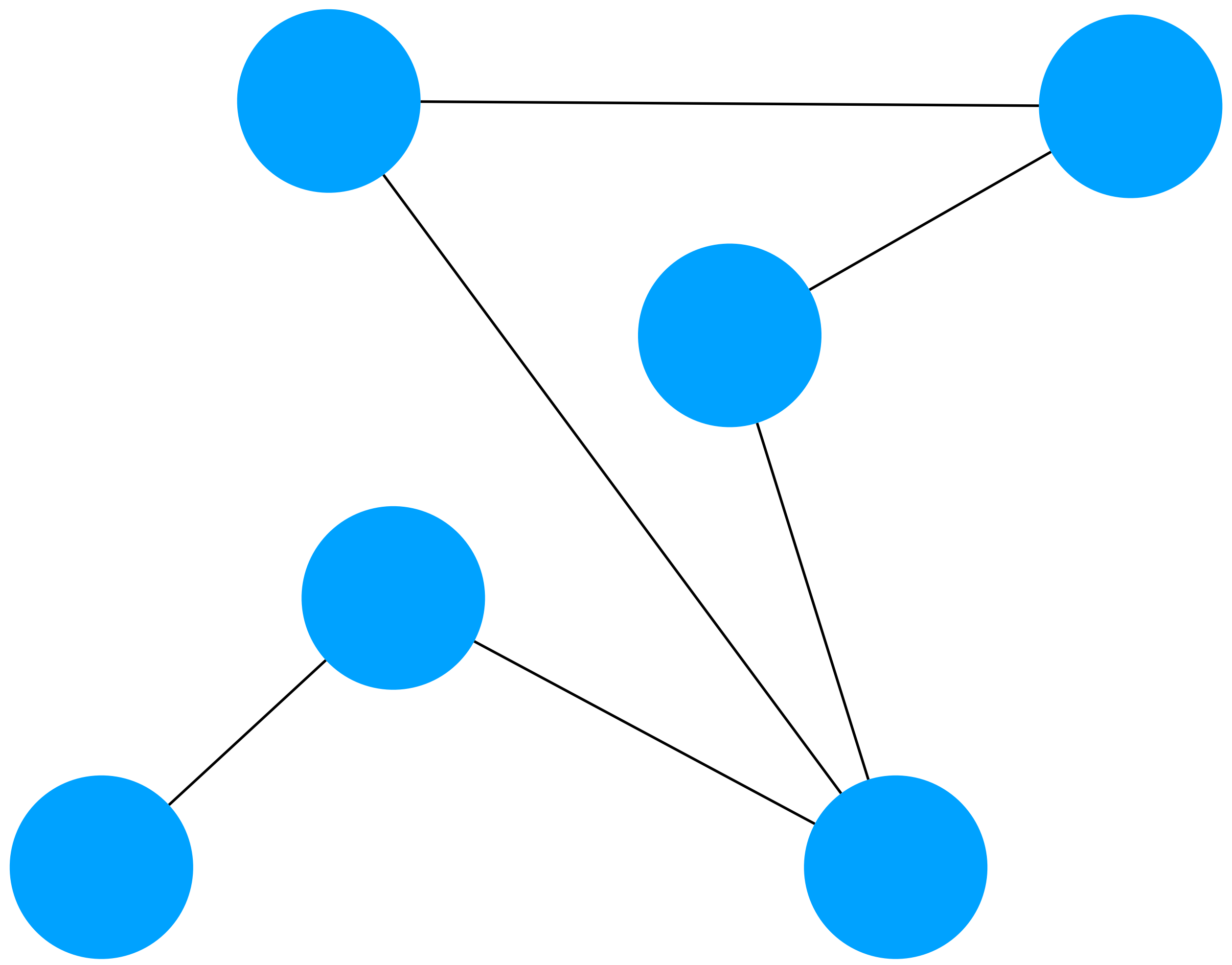

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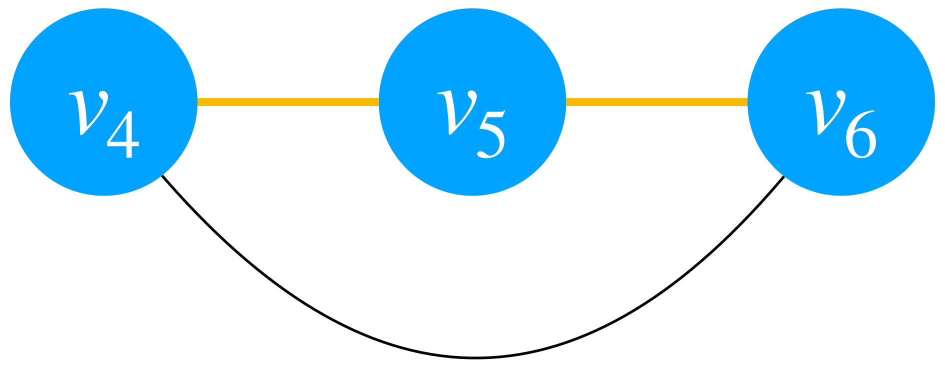

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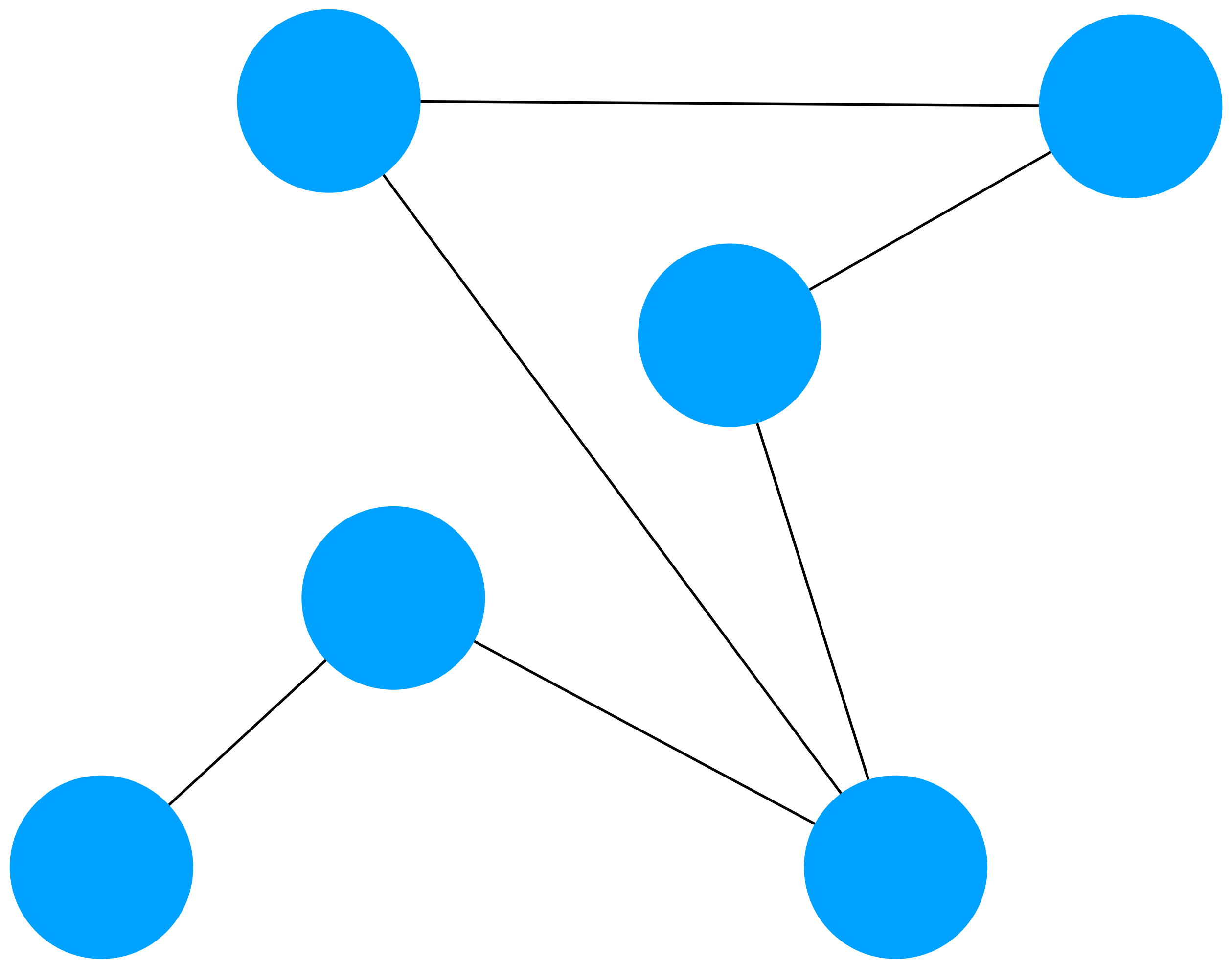

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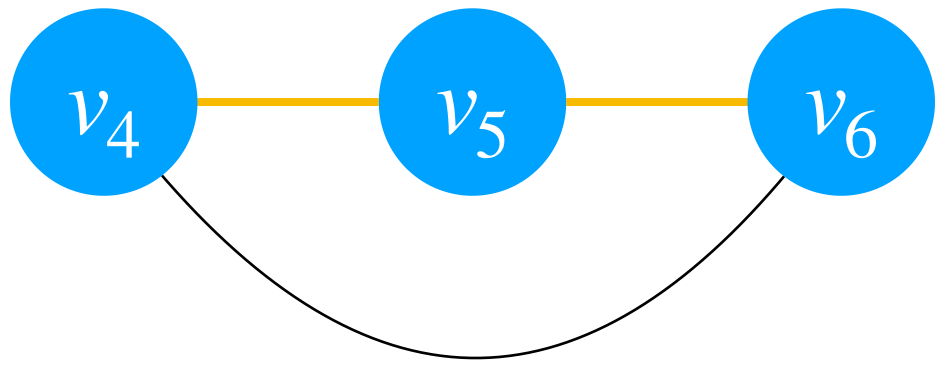

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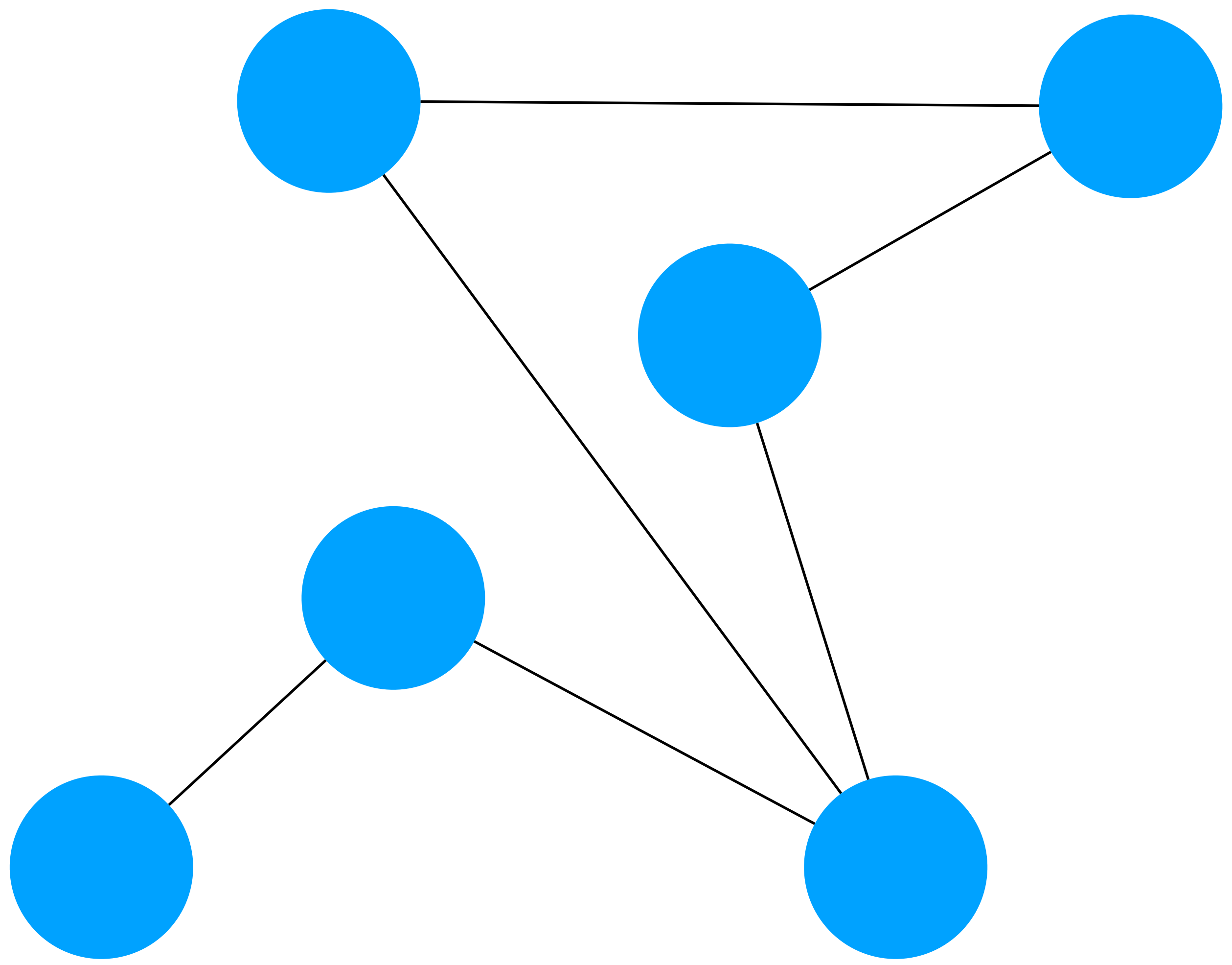

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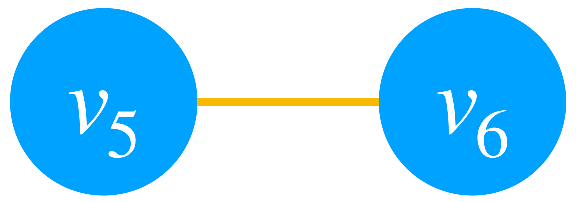

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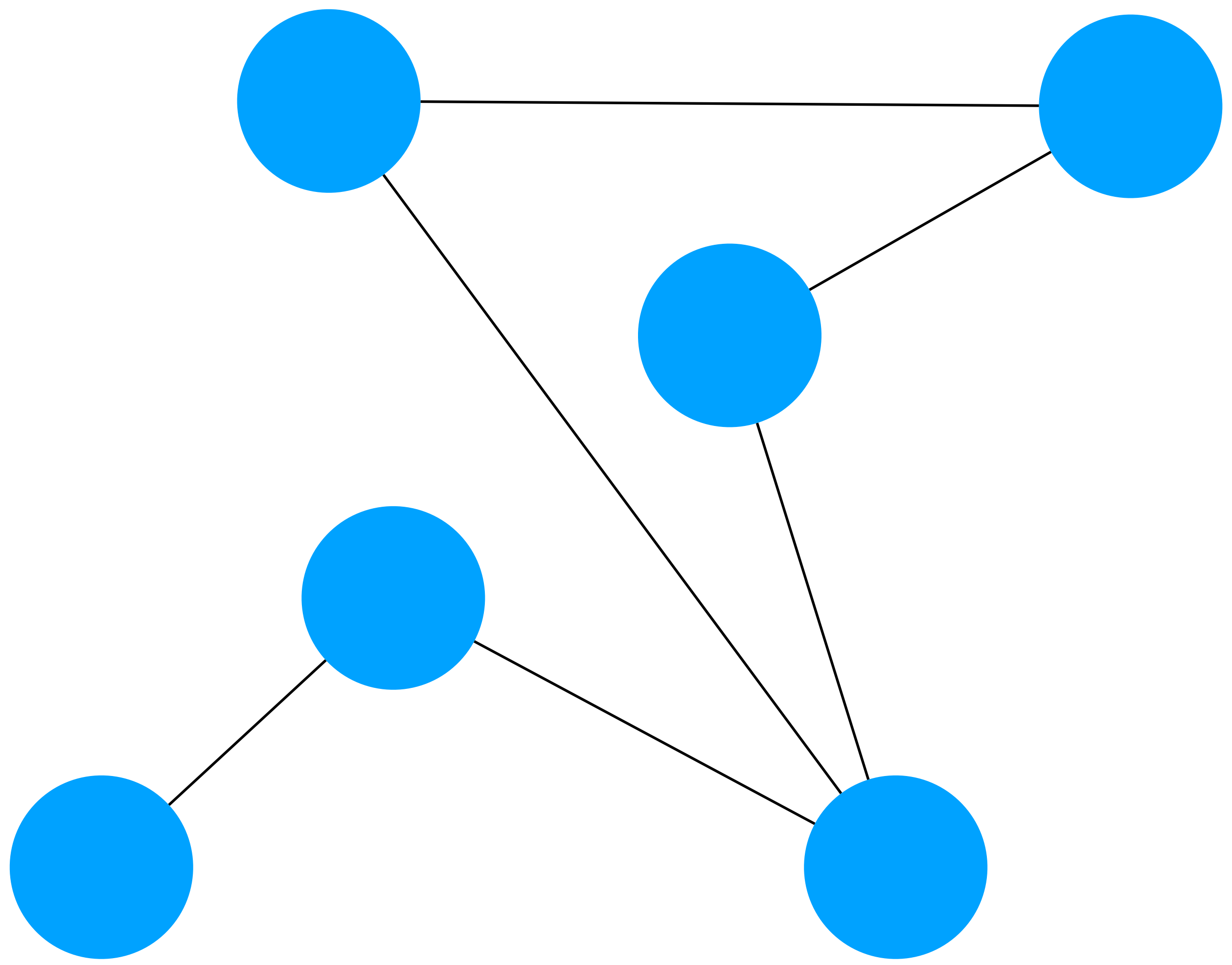

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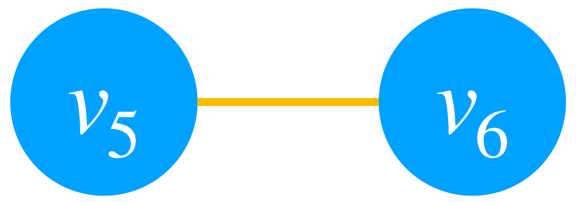

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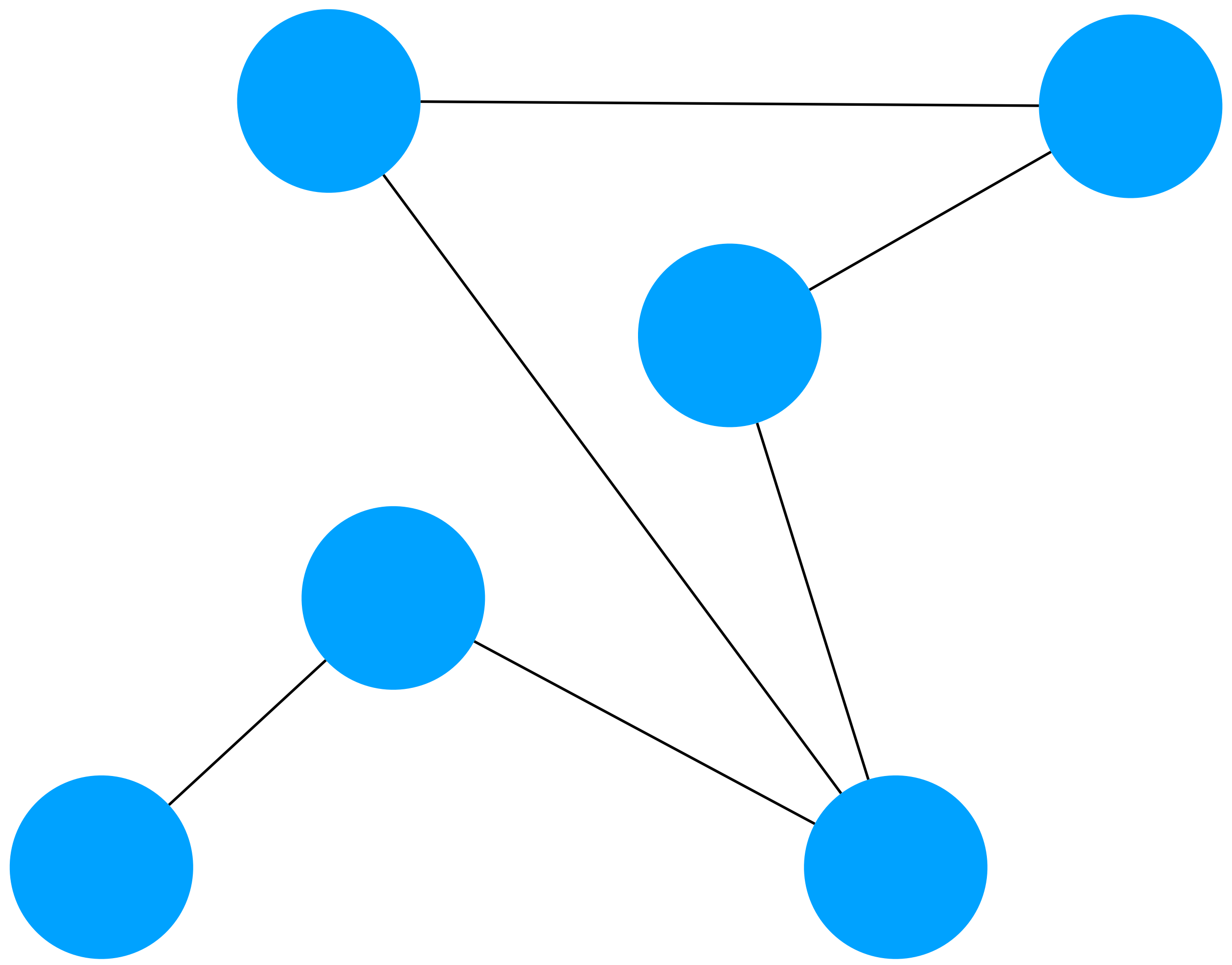

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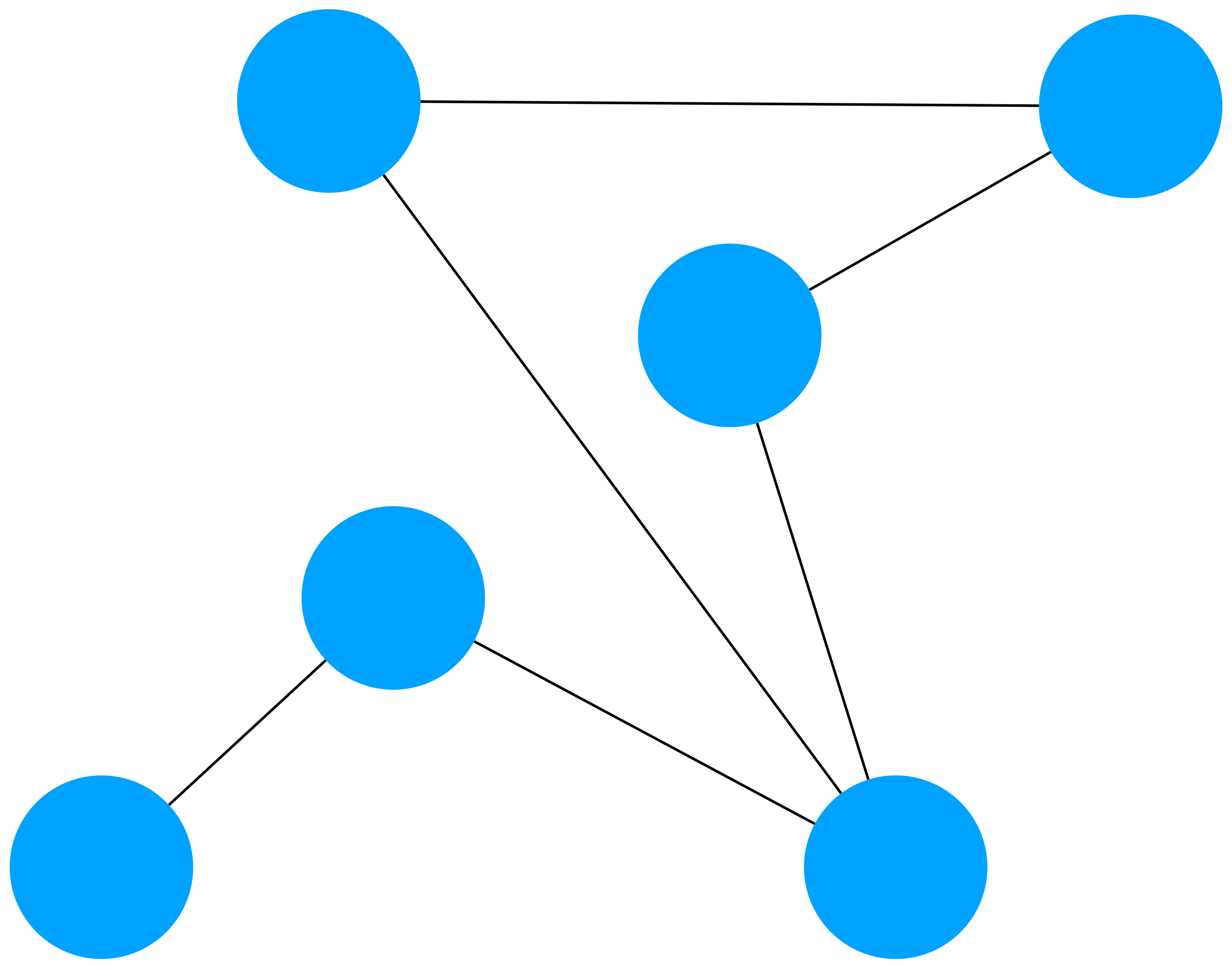

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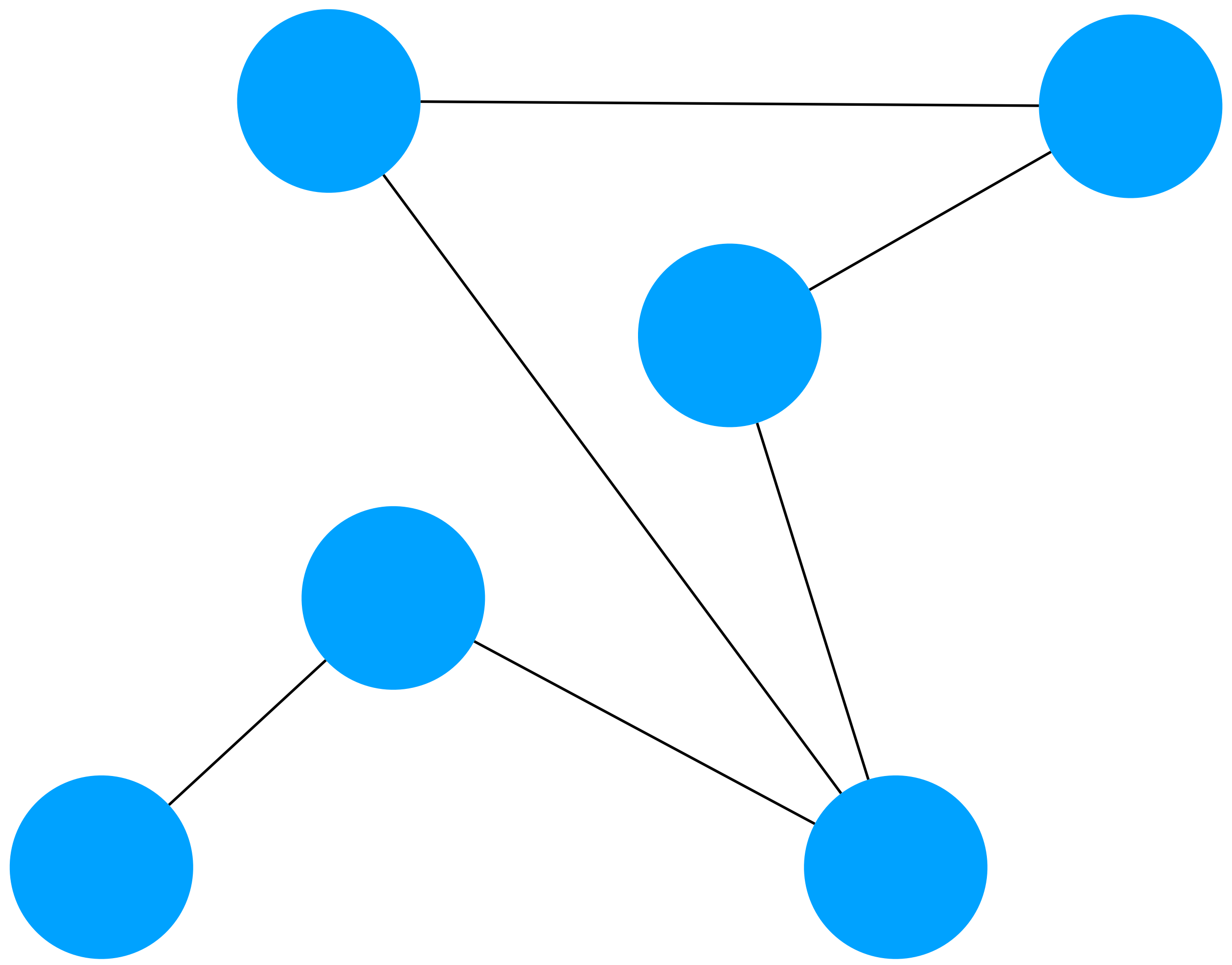

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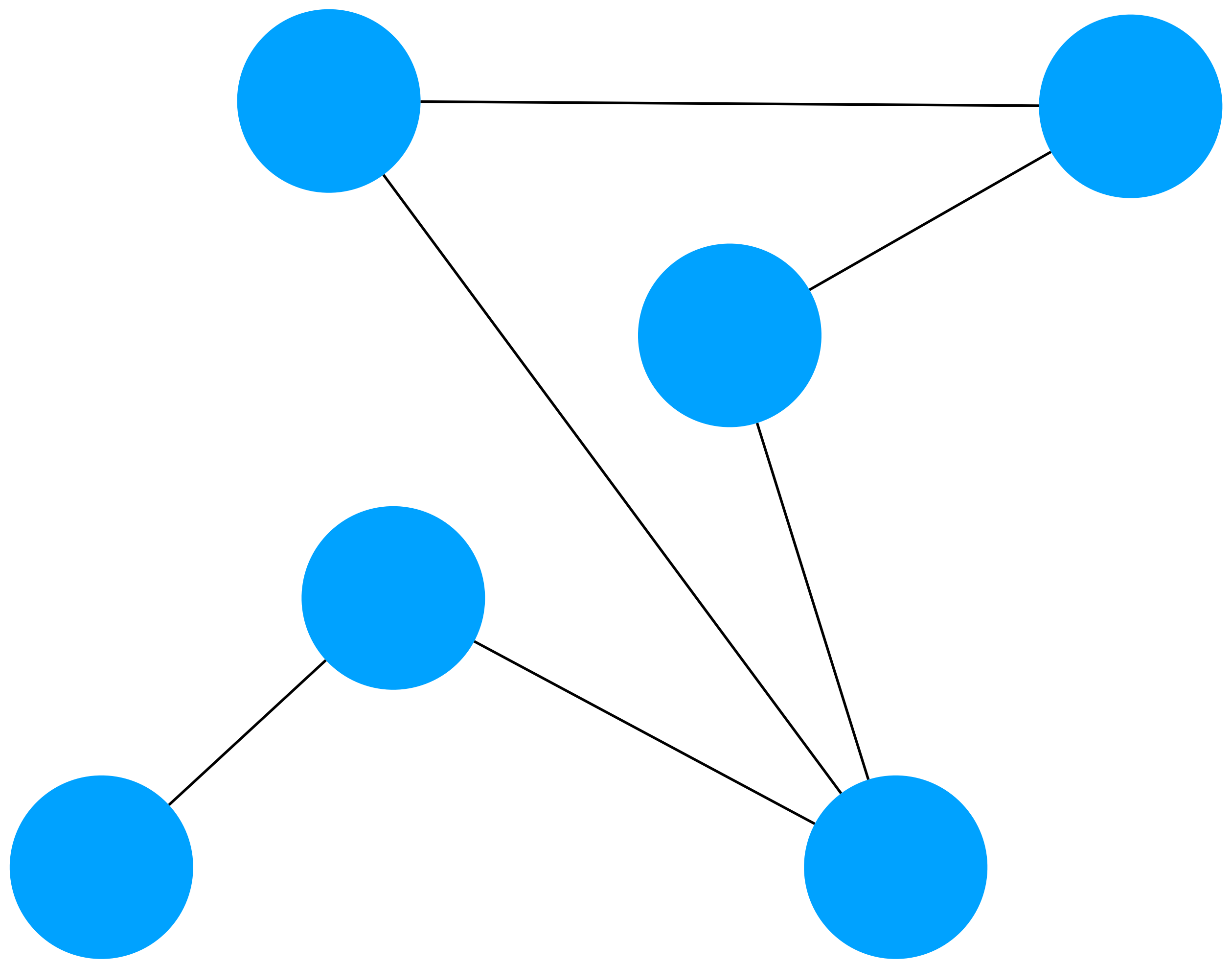

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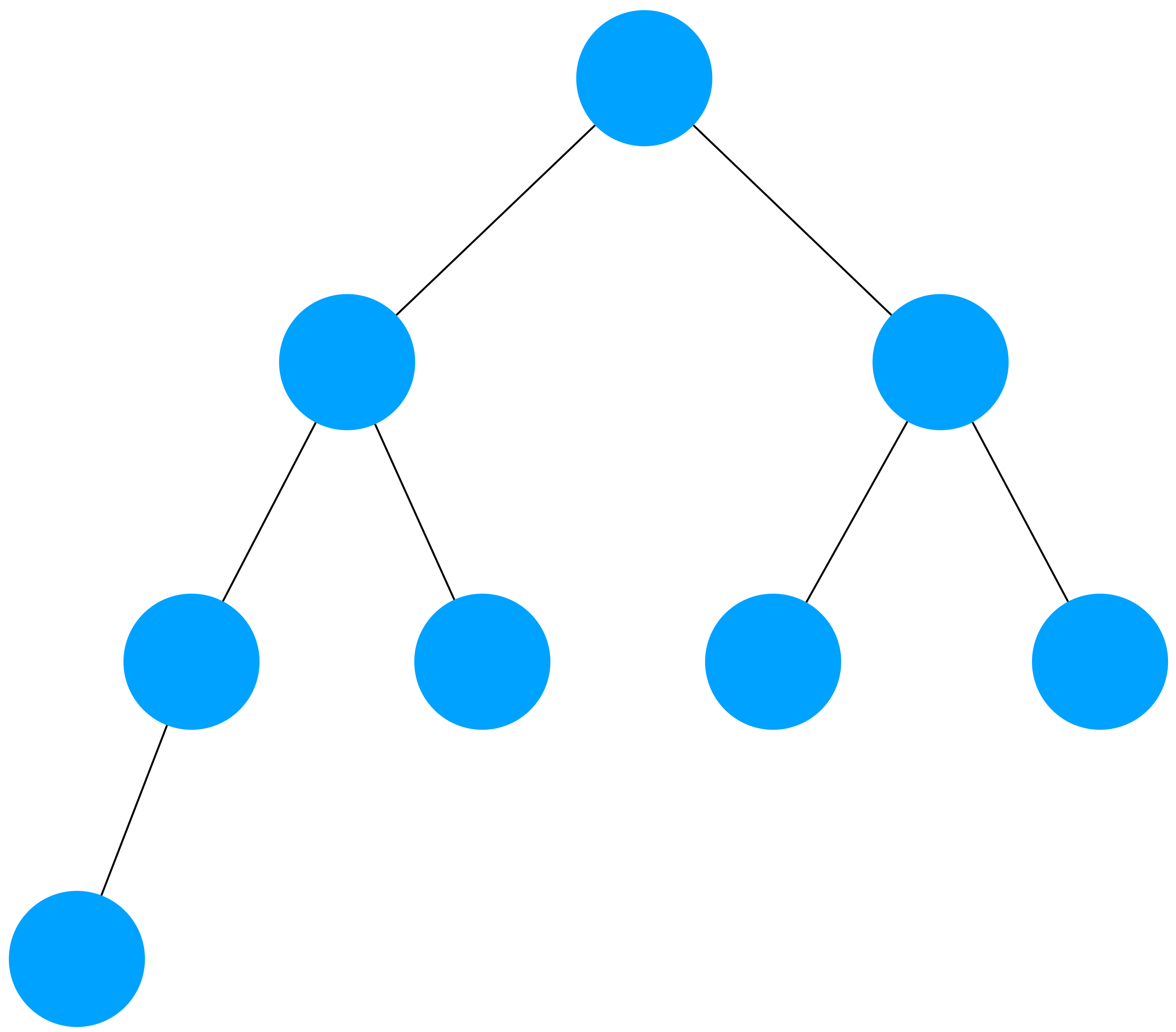

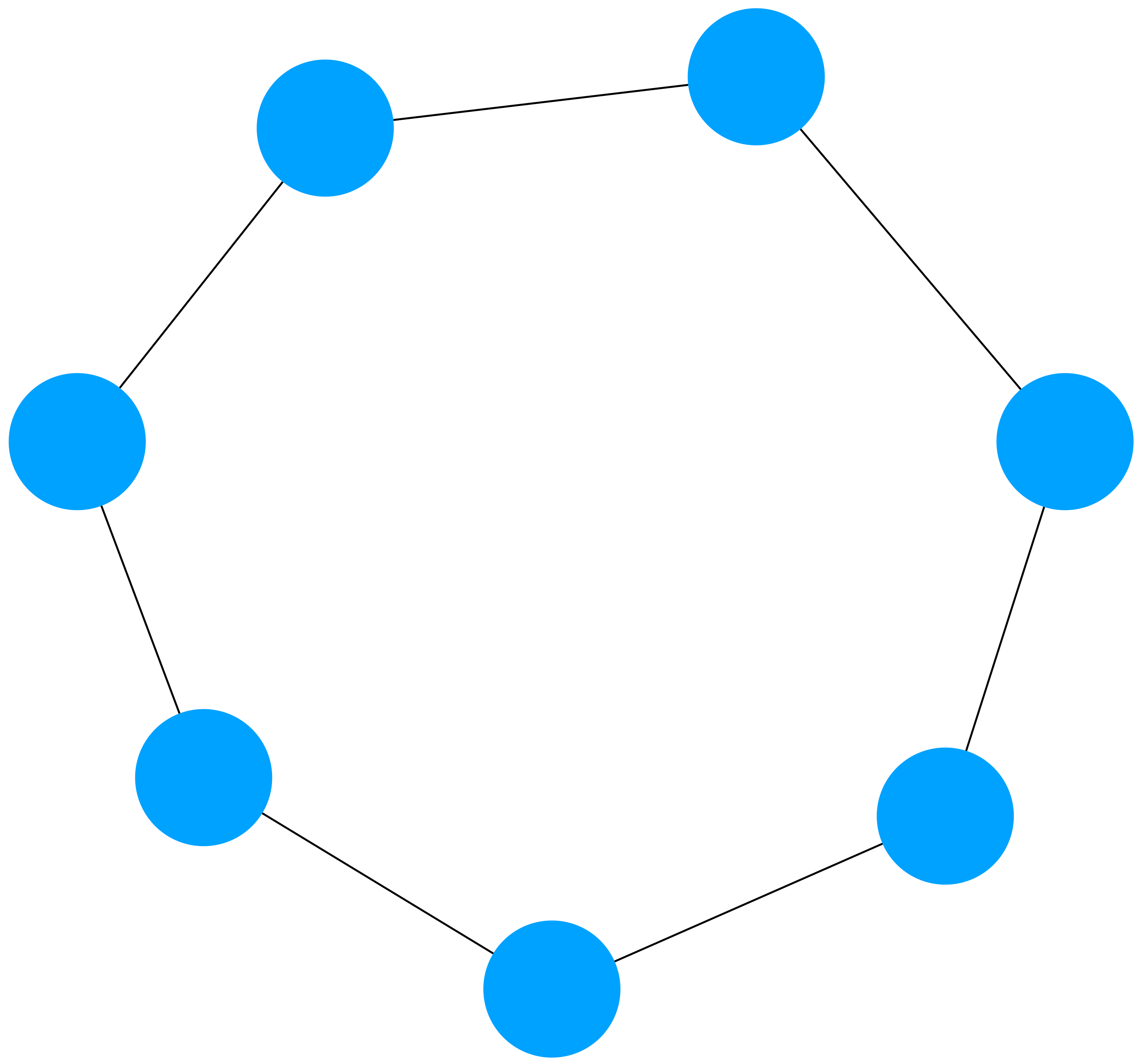

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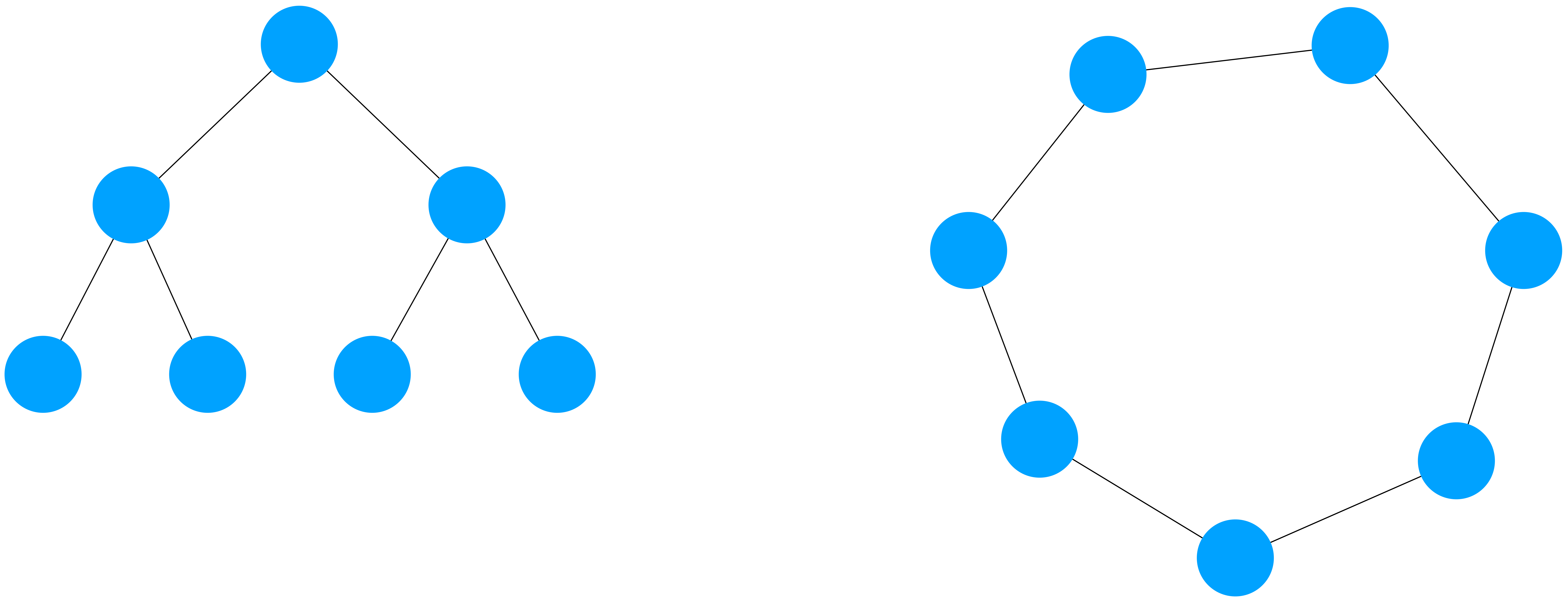

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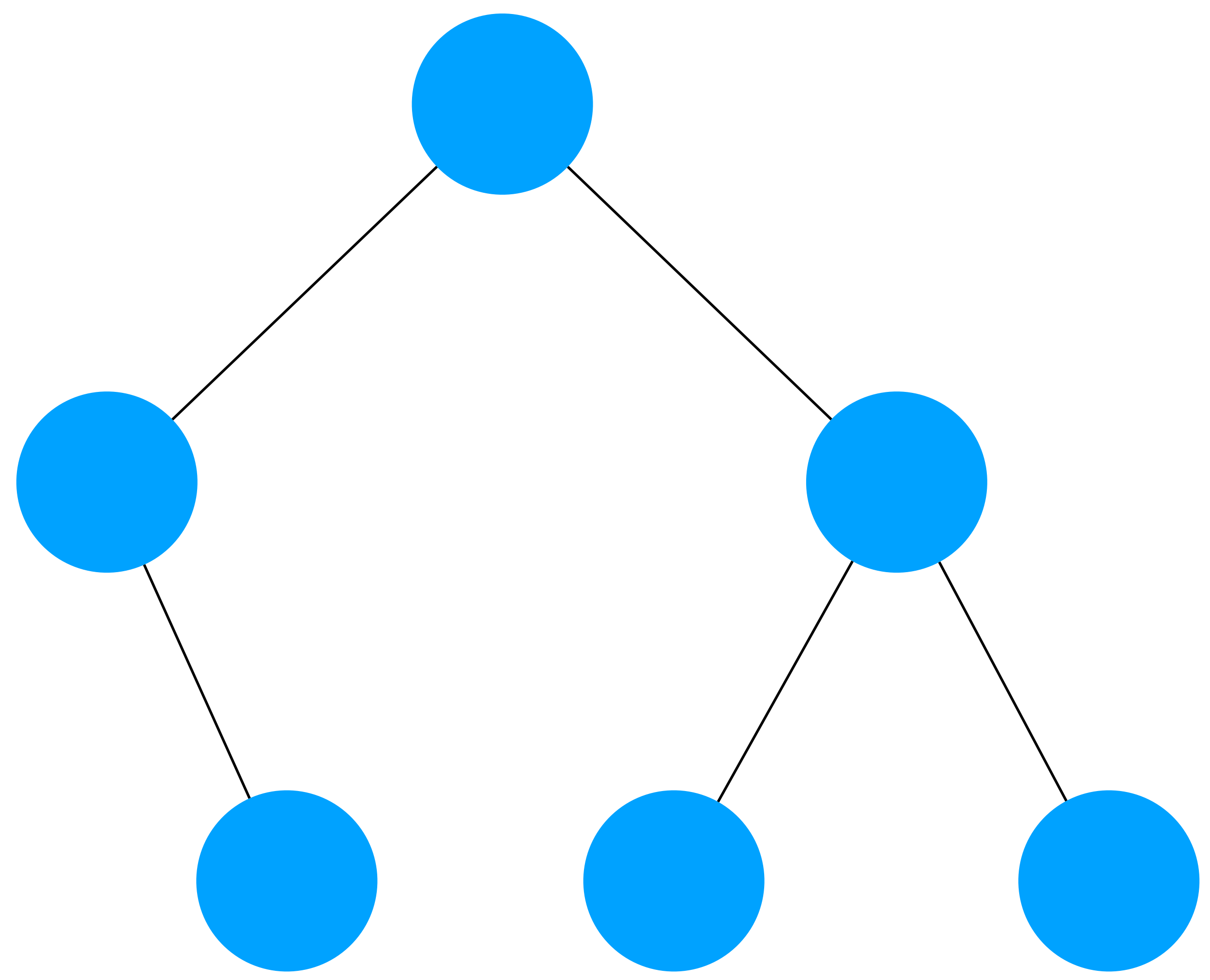

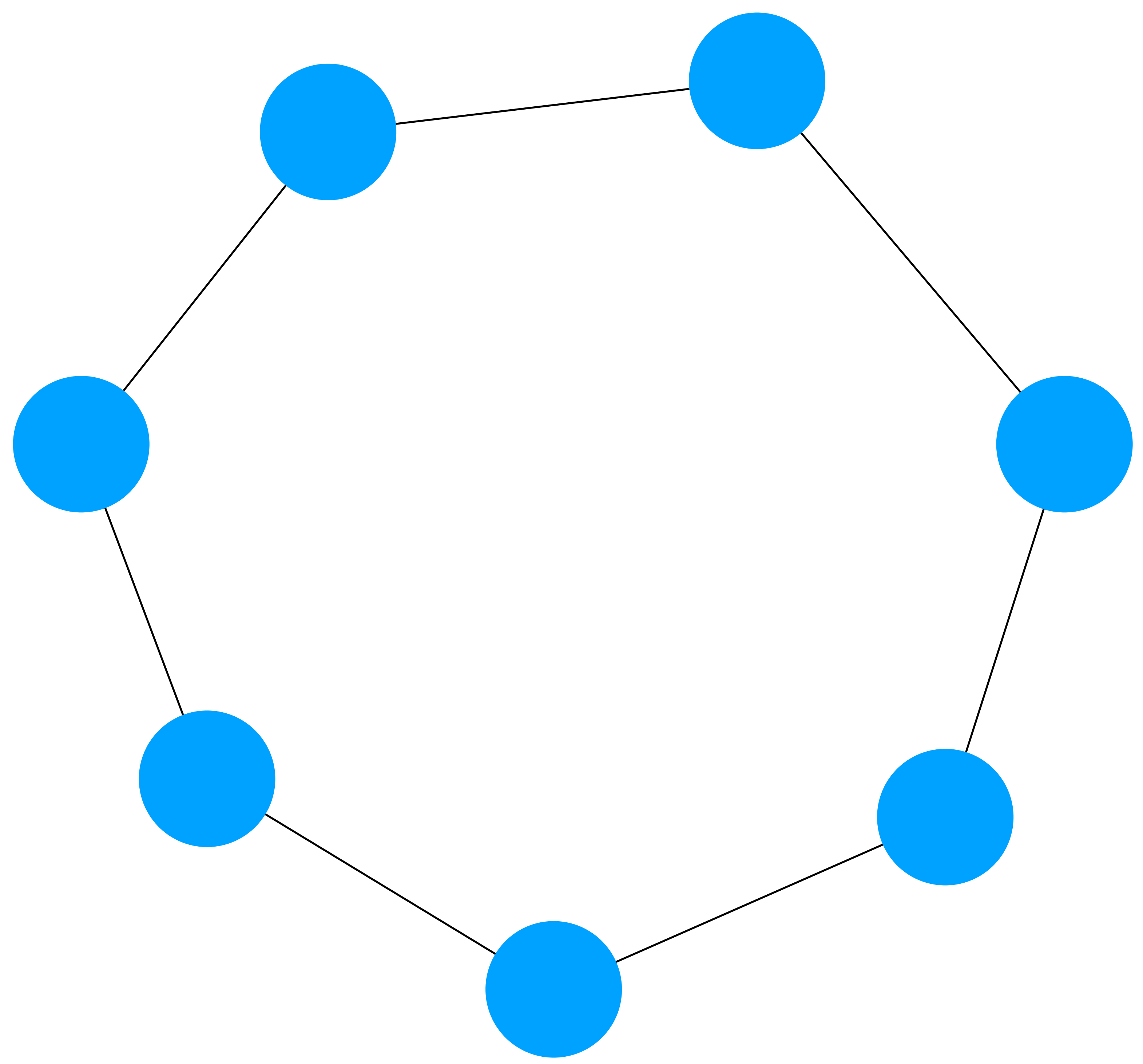

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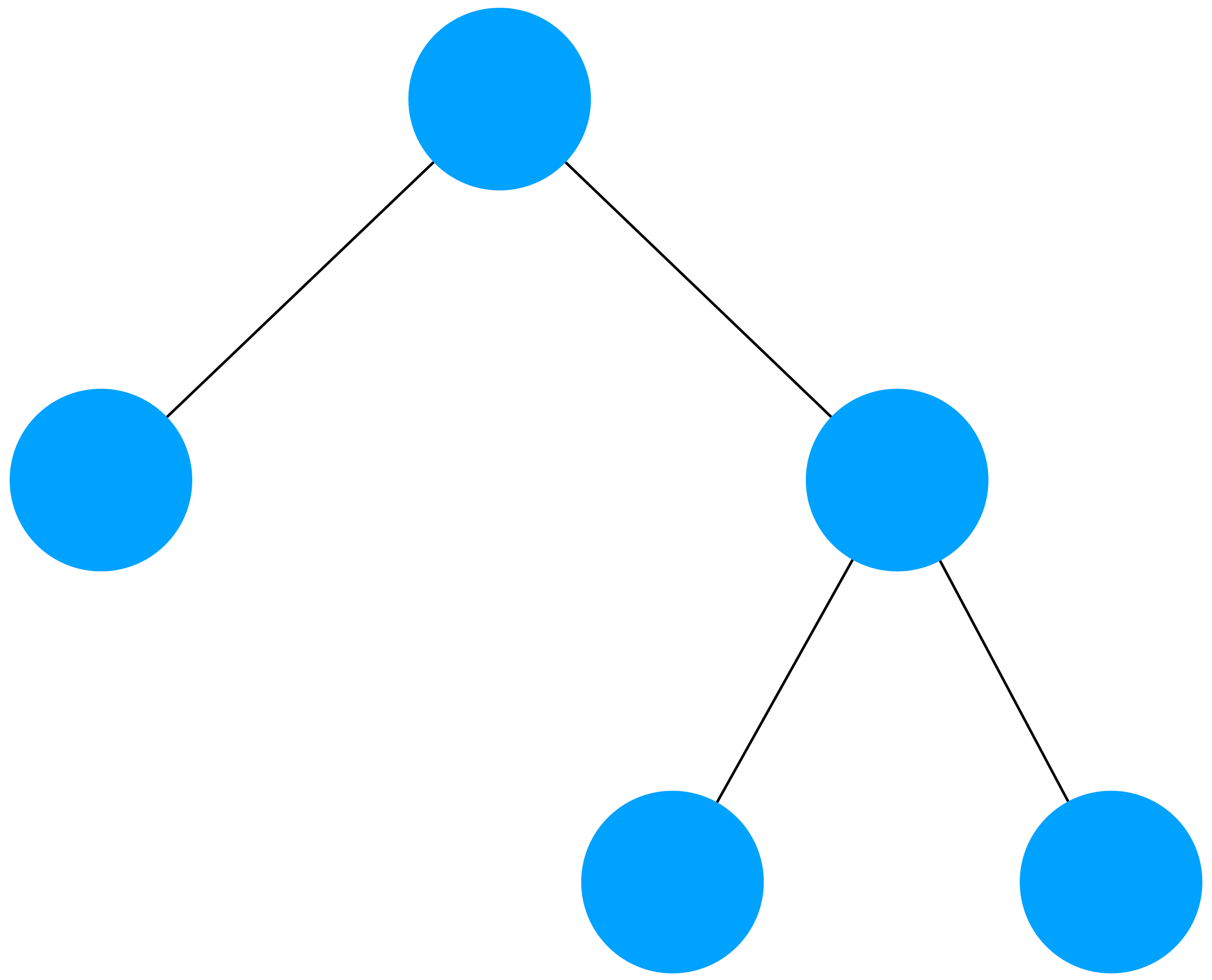

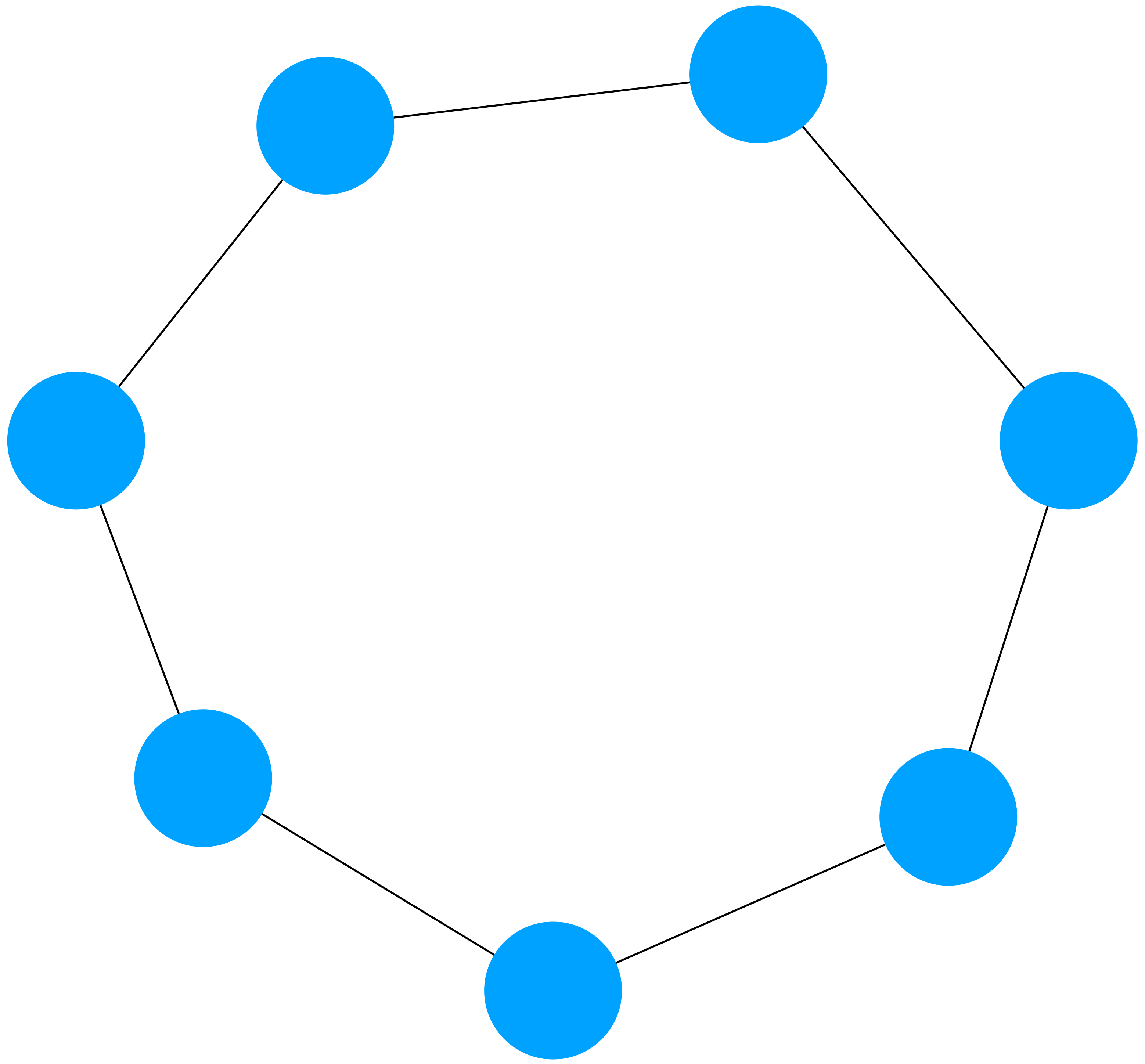

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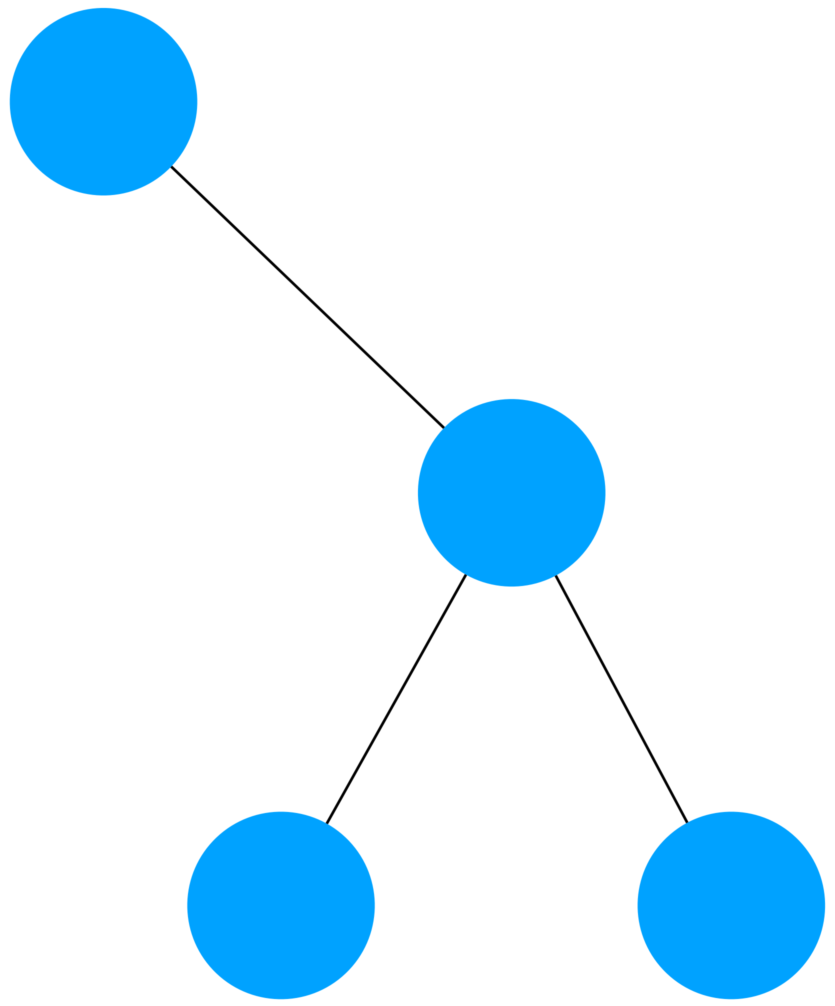

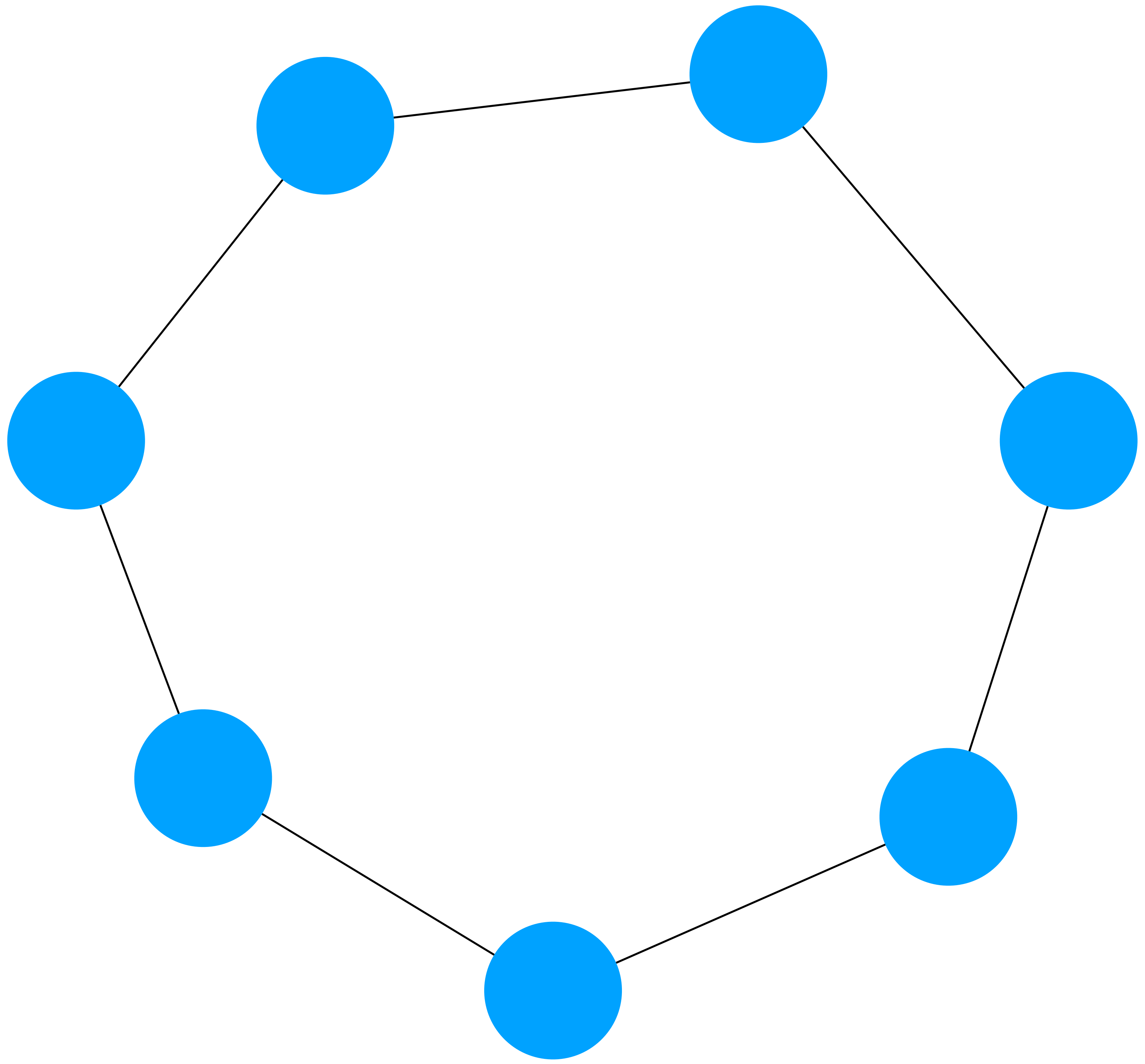

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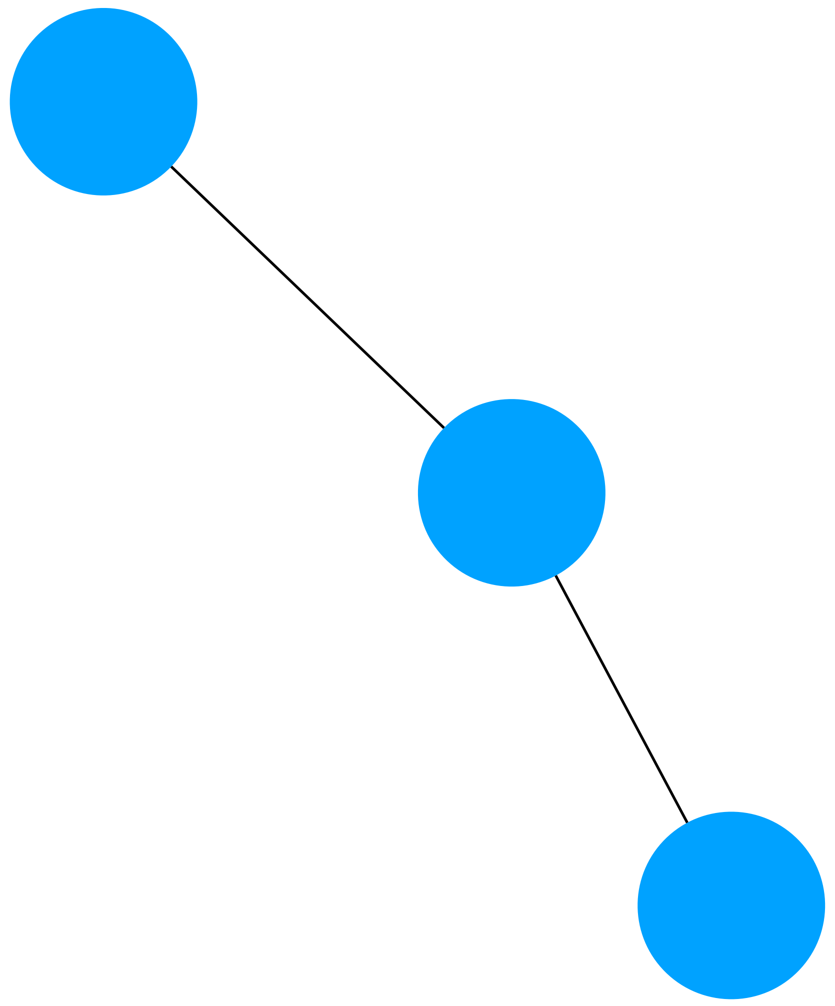

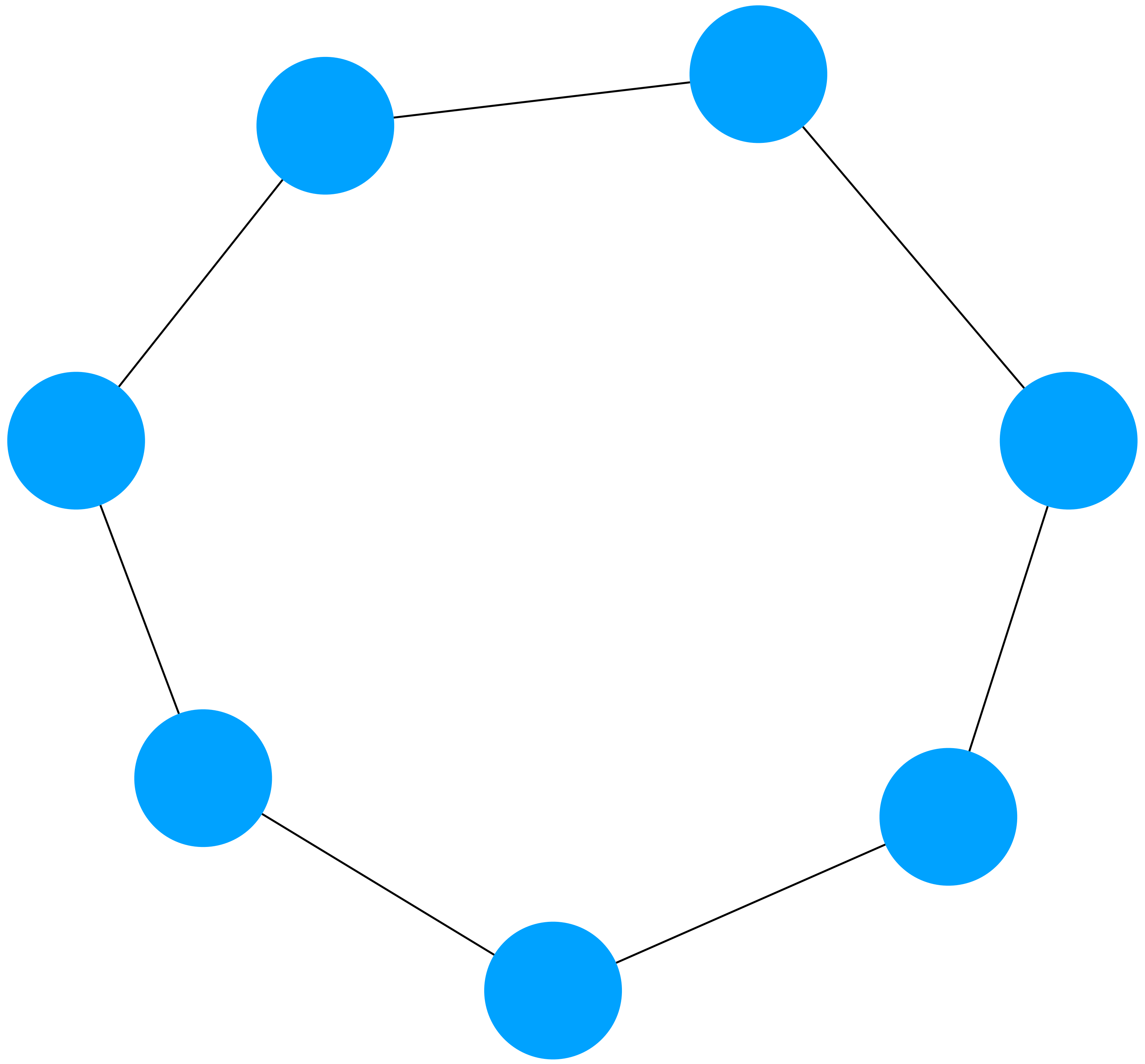

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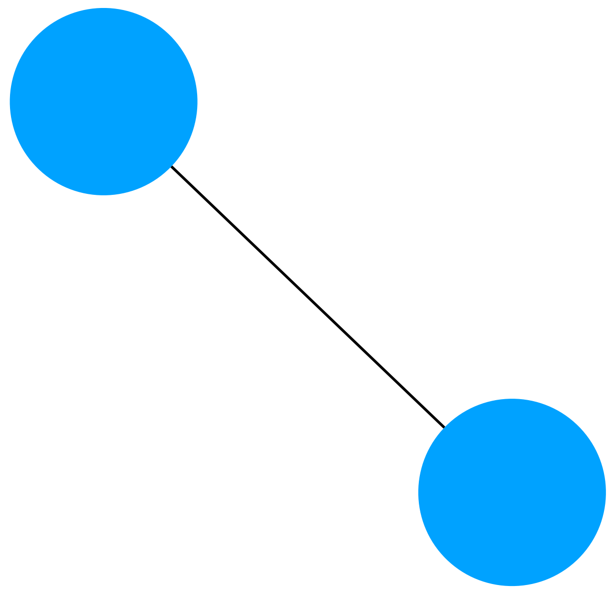

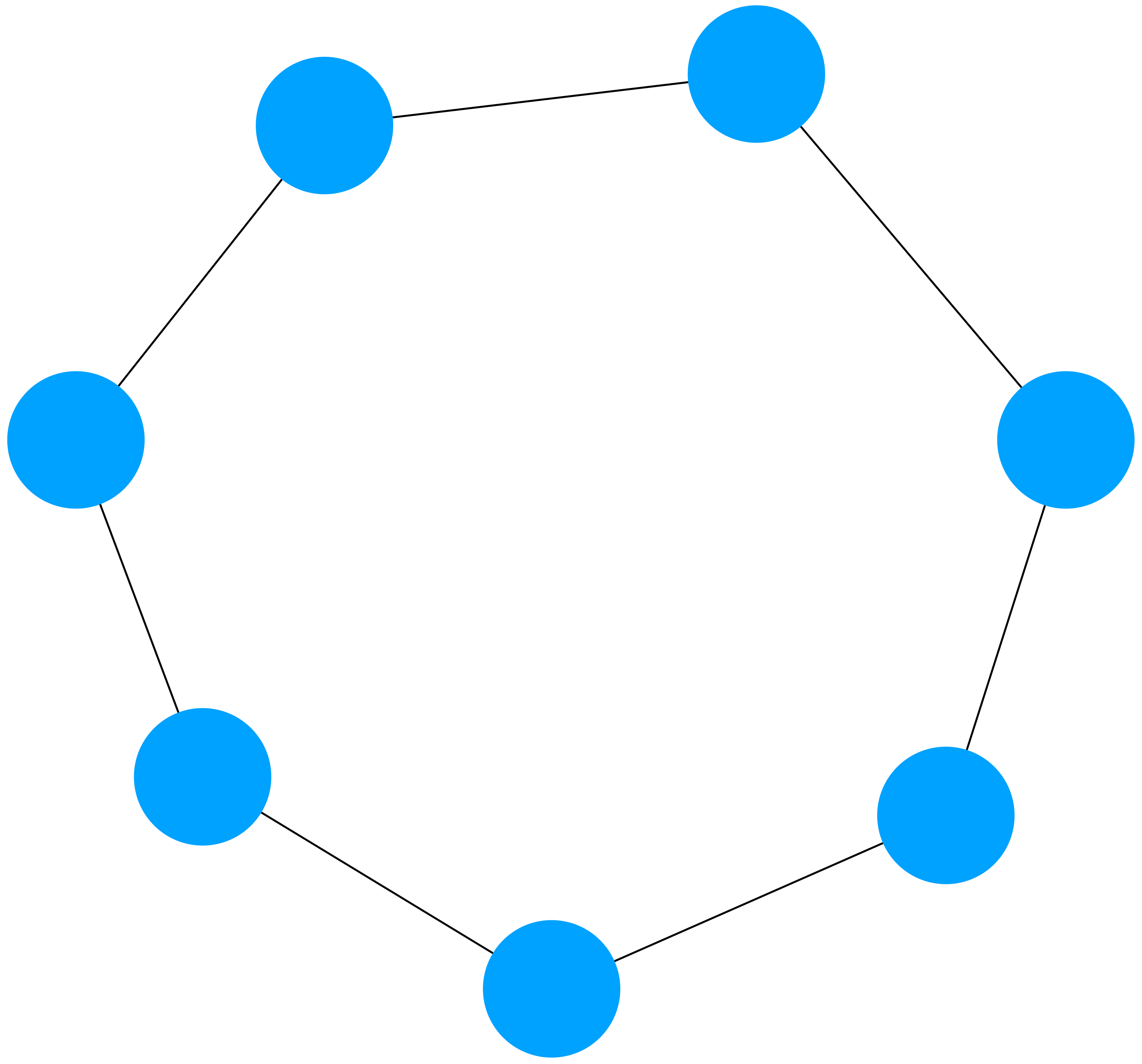

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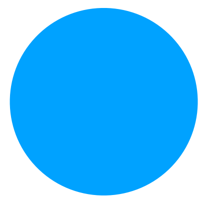

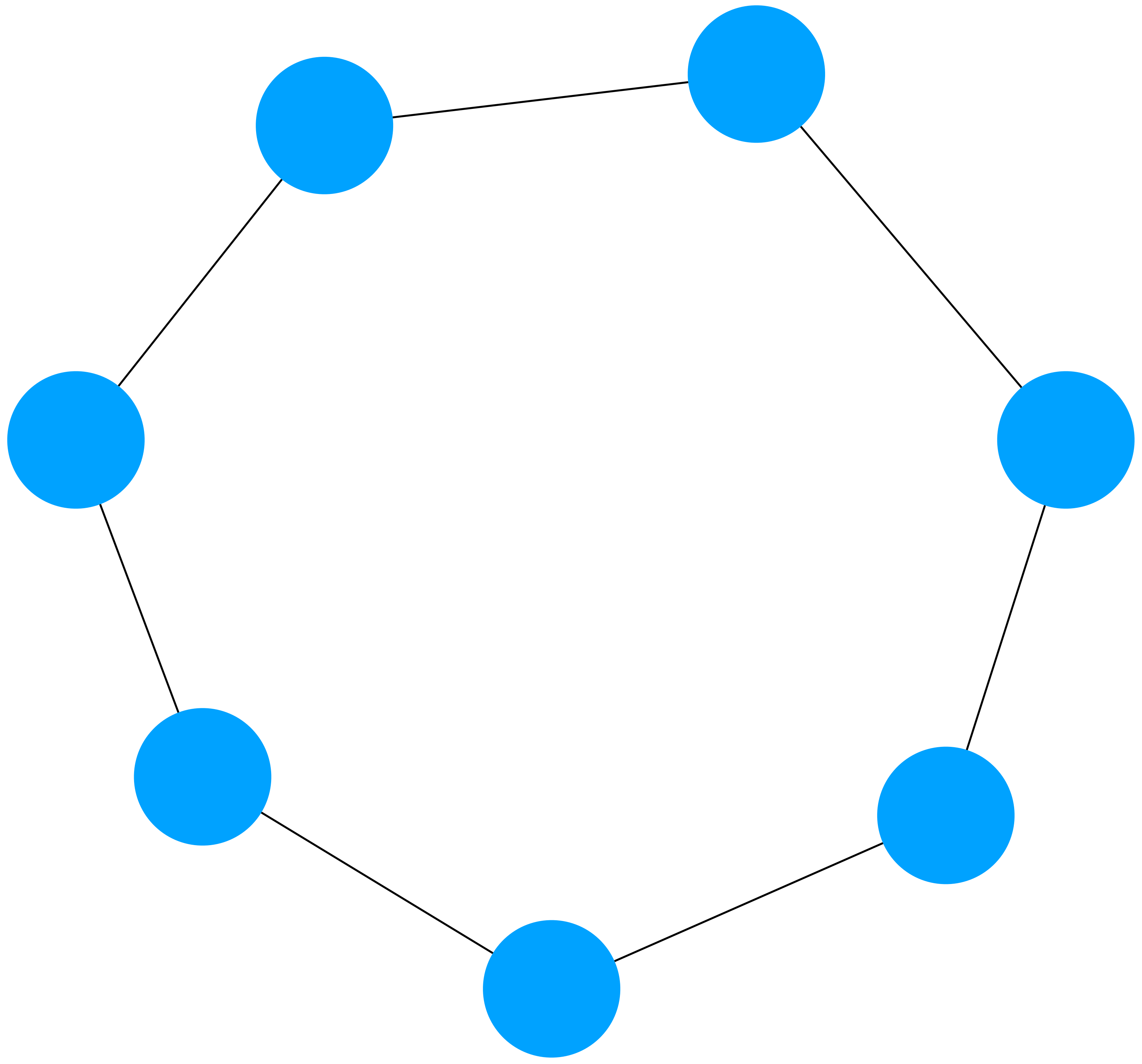

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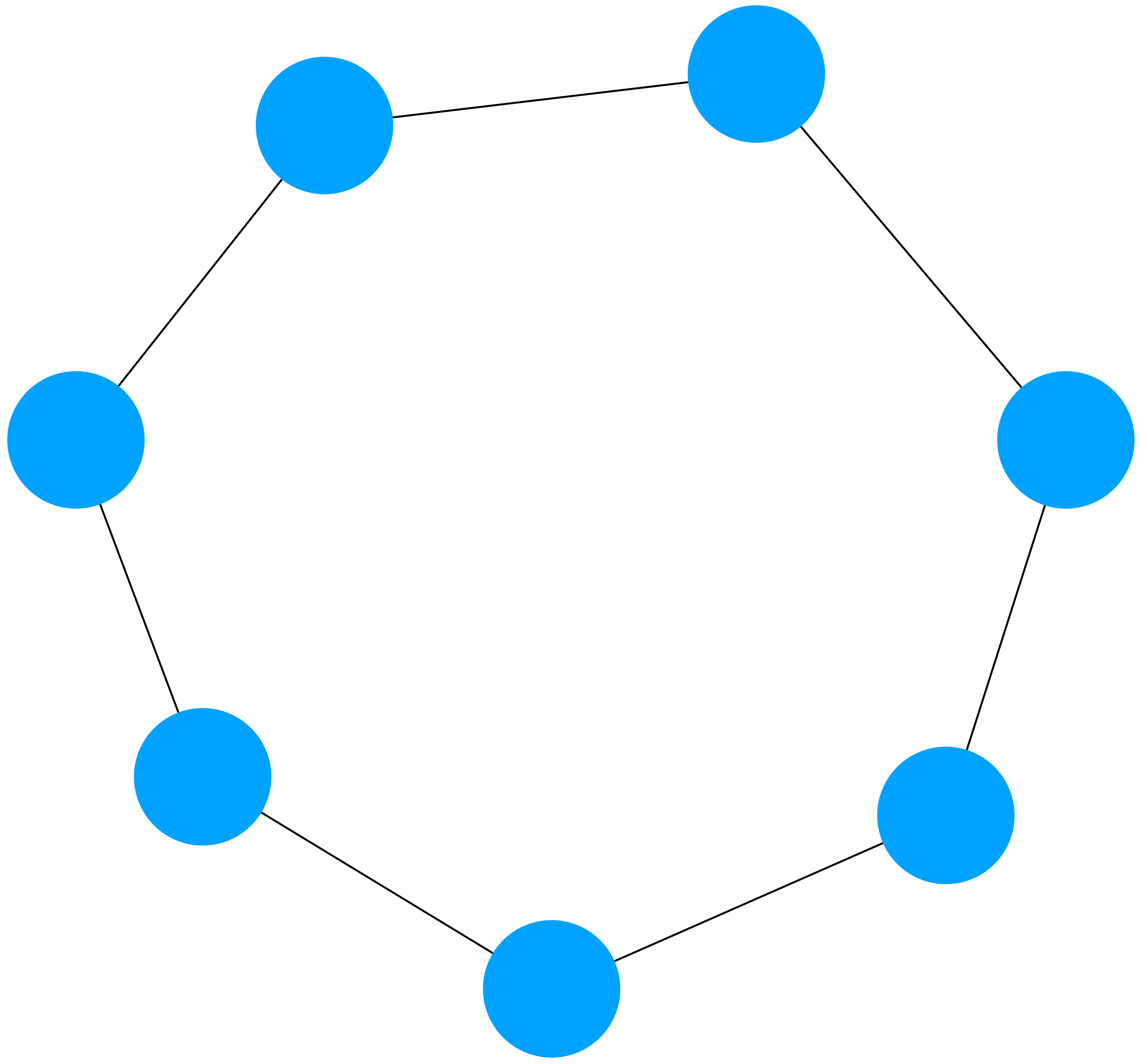

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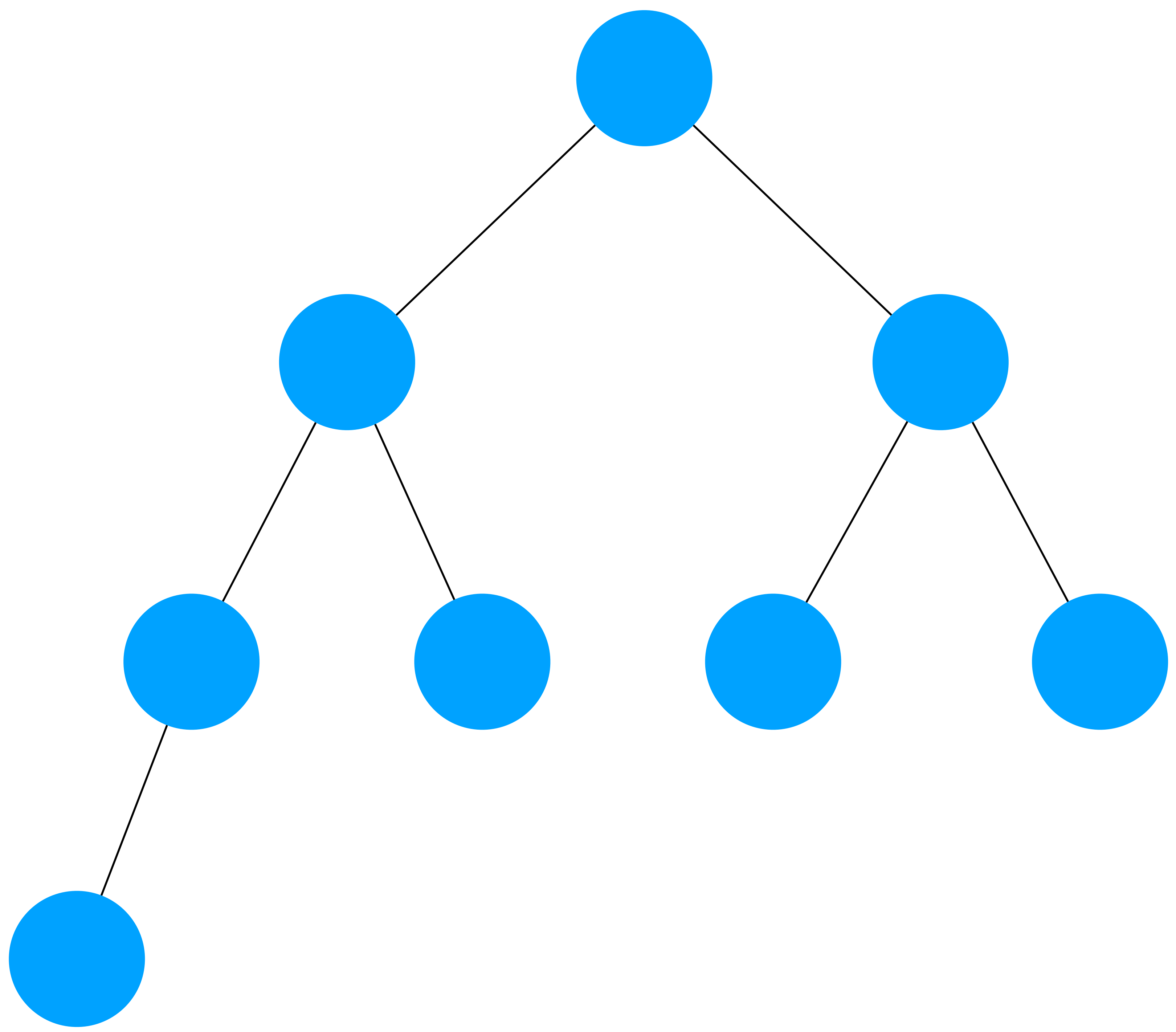
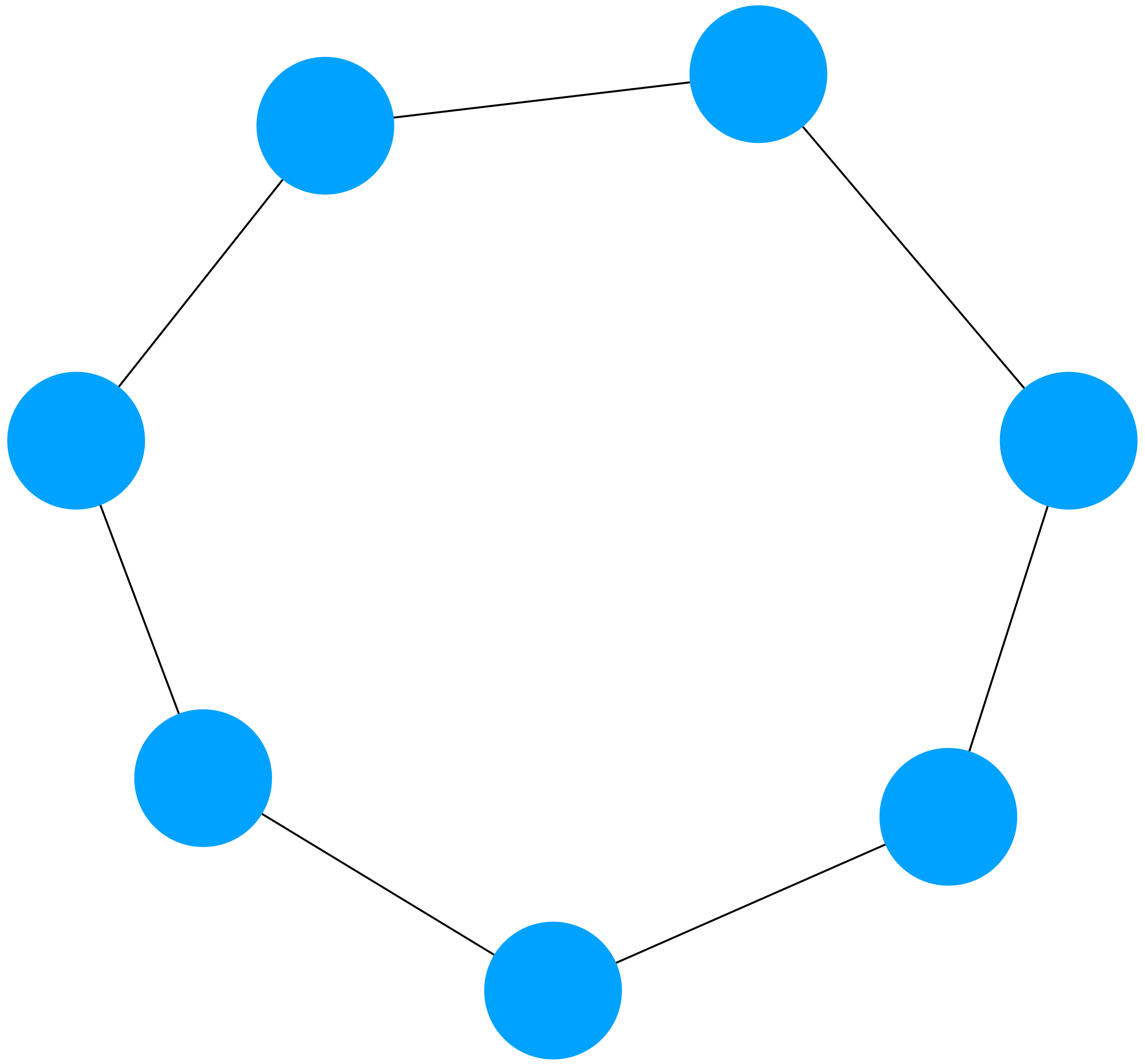
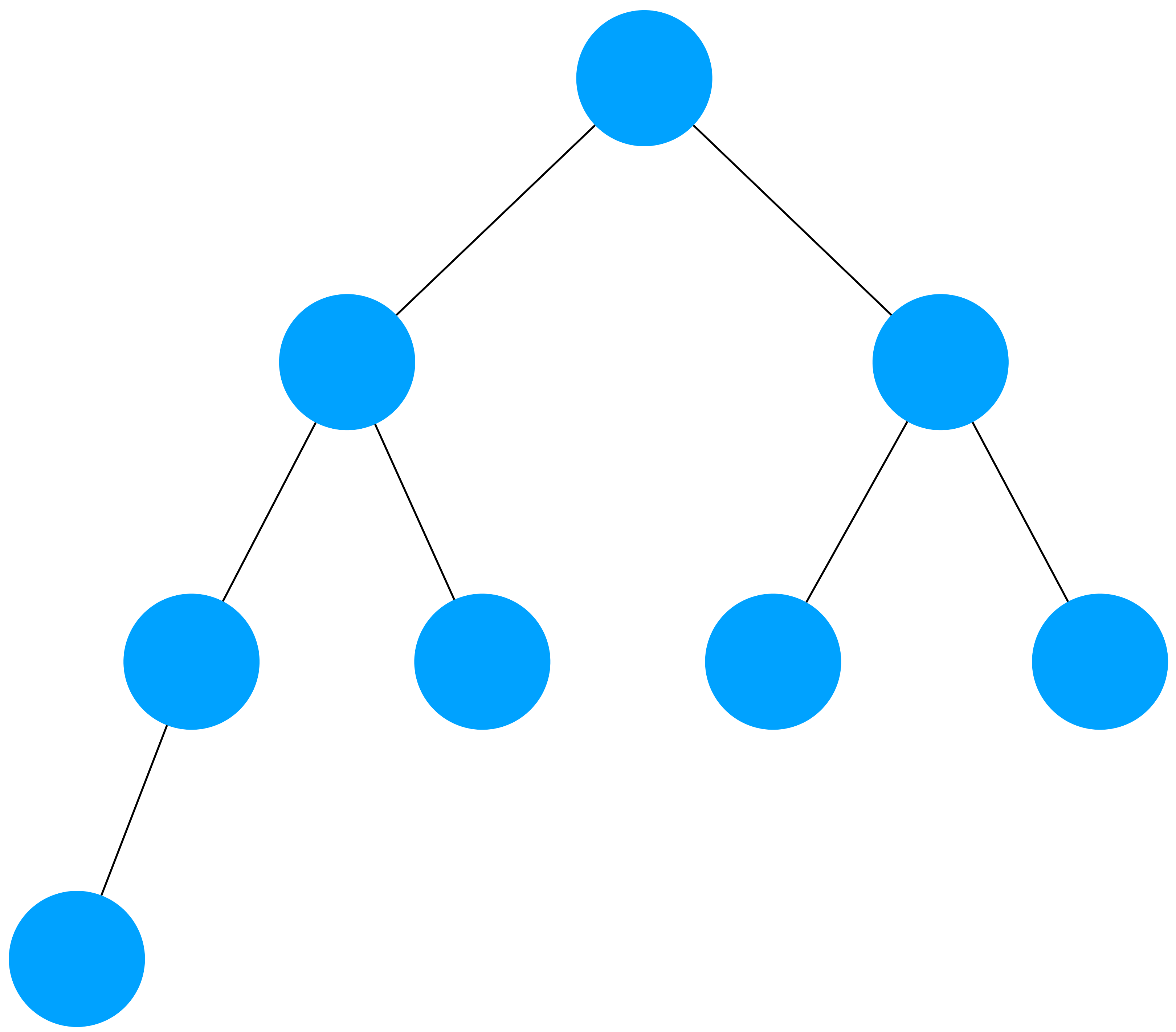


width 1



Examples



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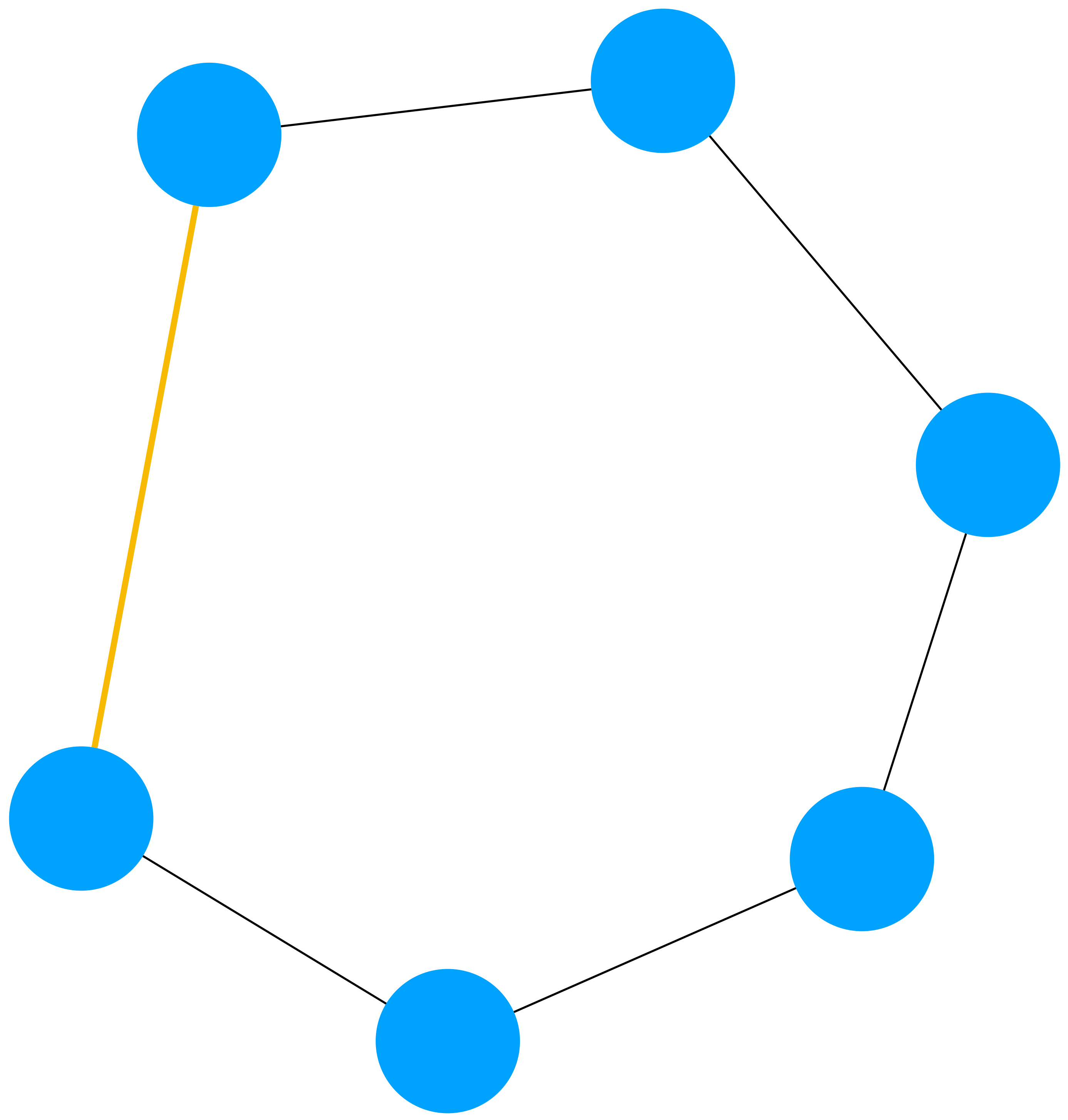
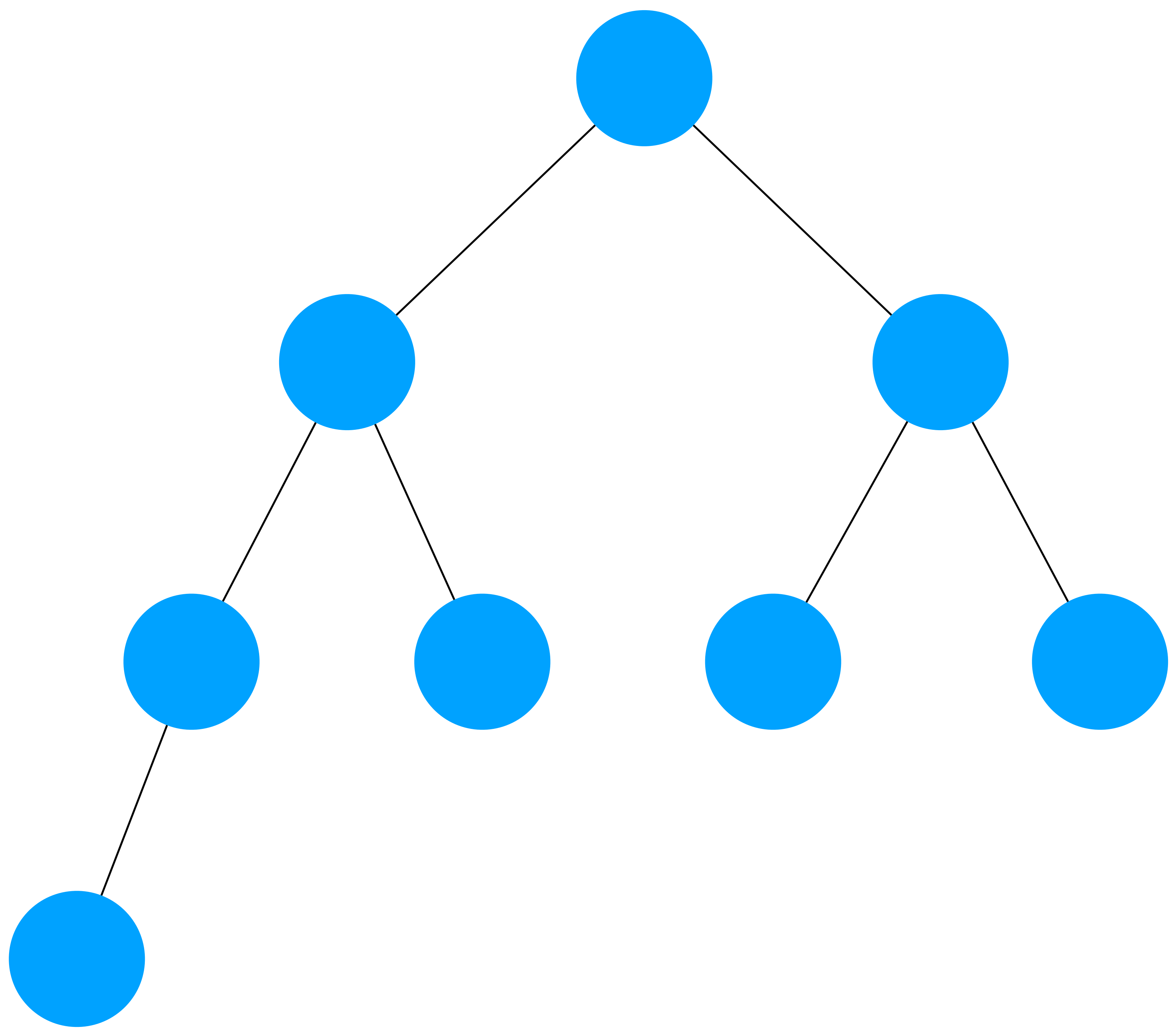


Examples



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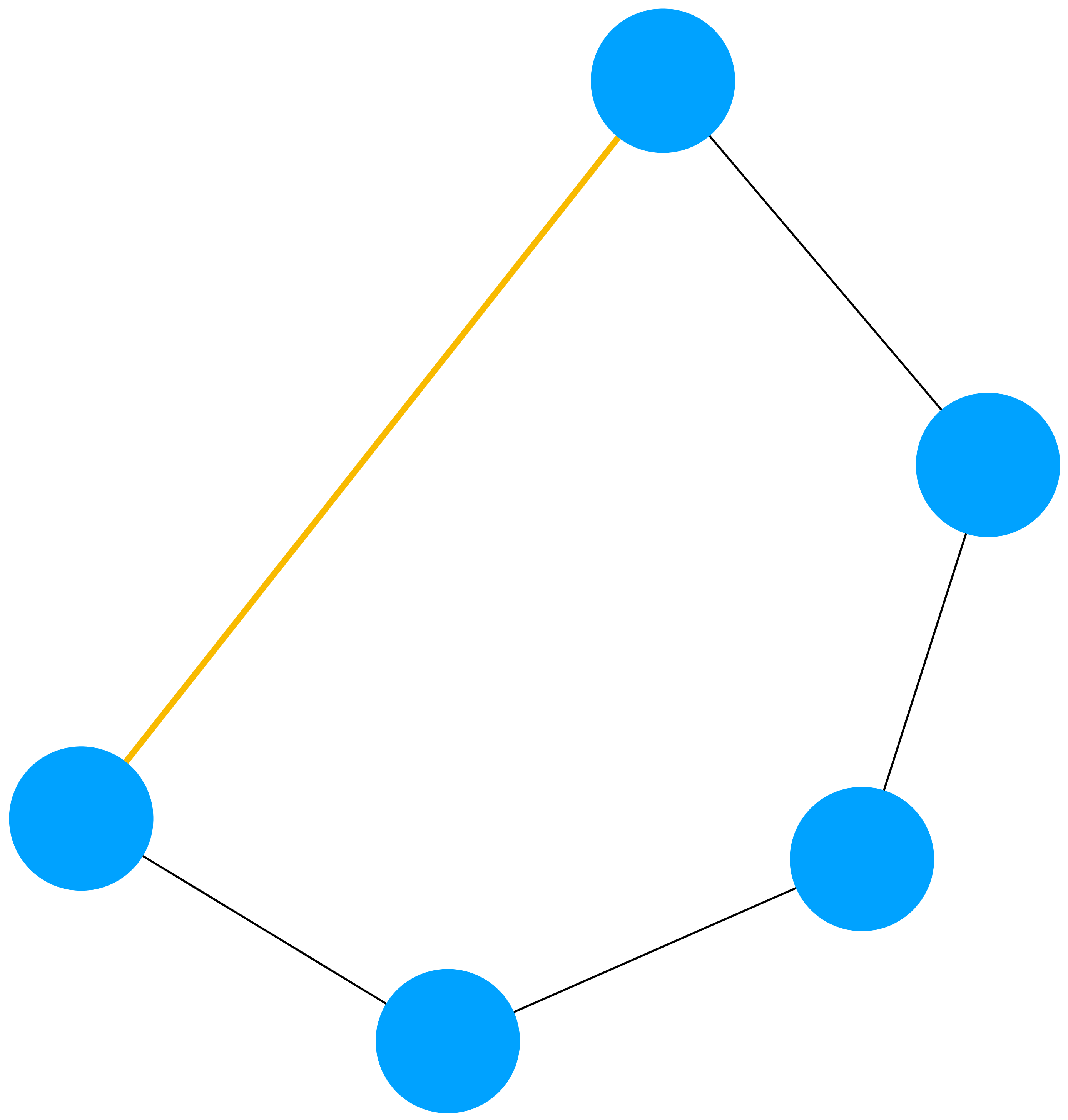
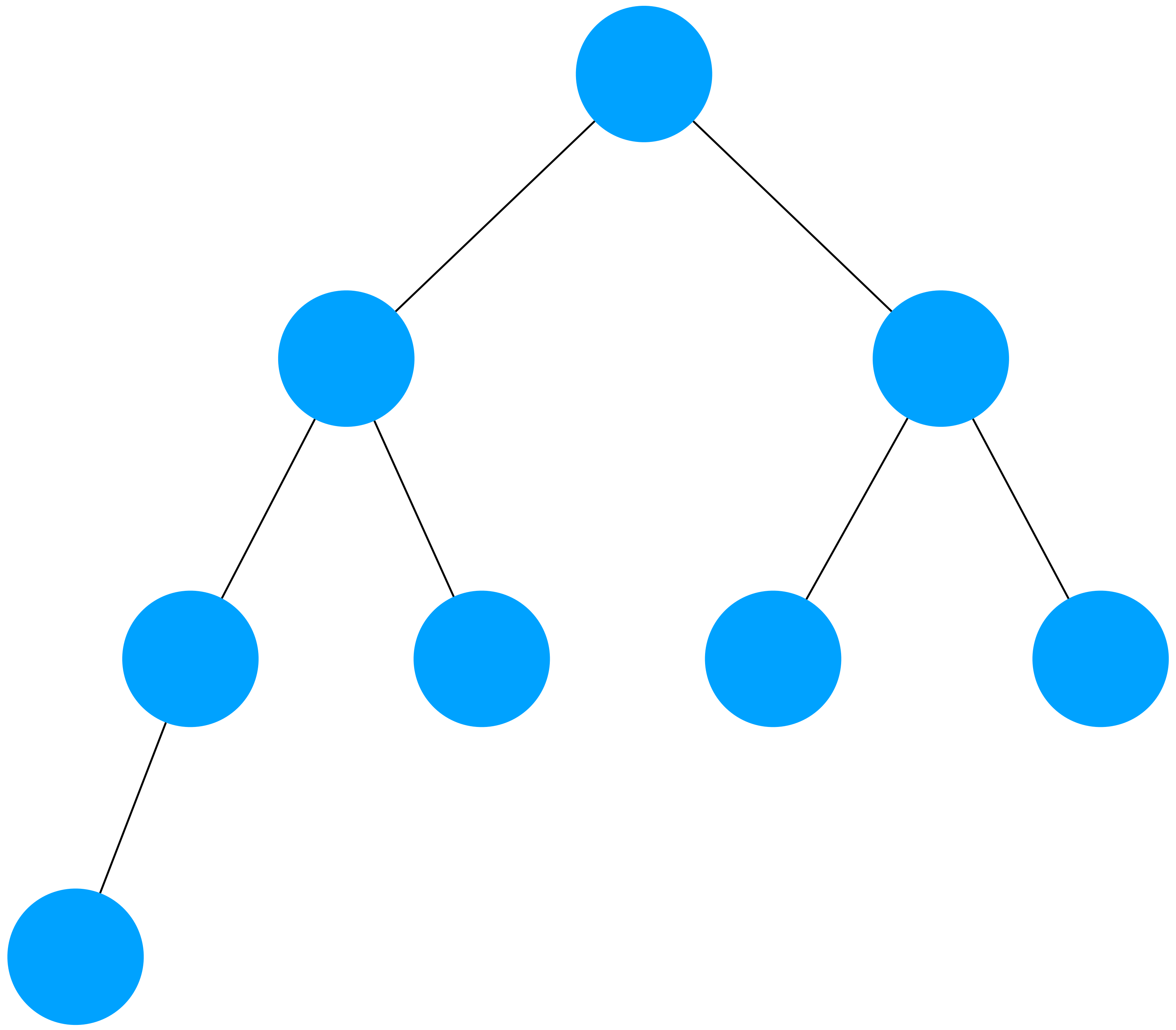


Examples



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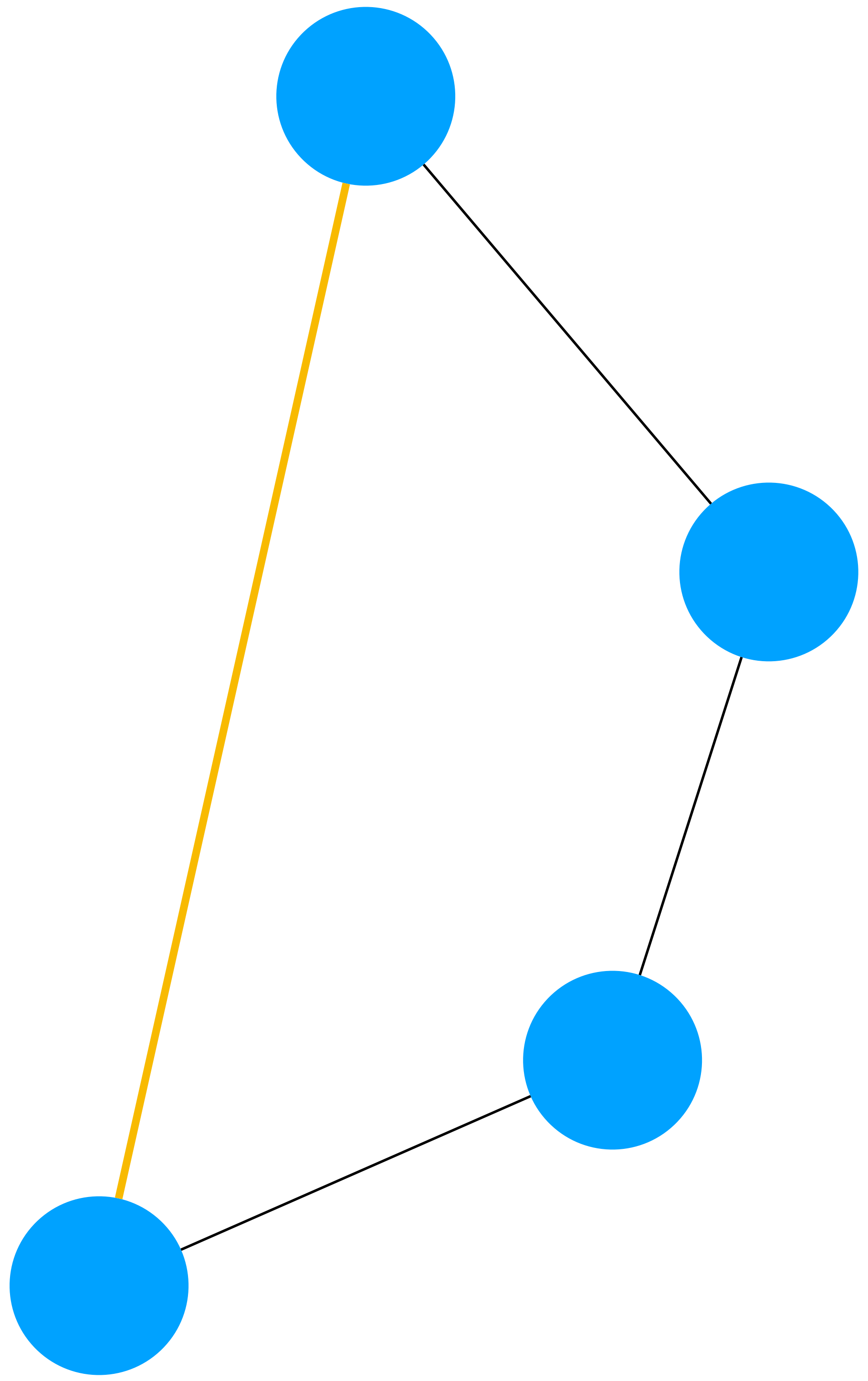

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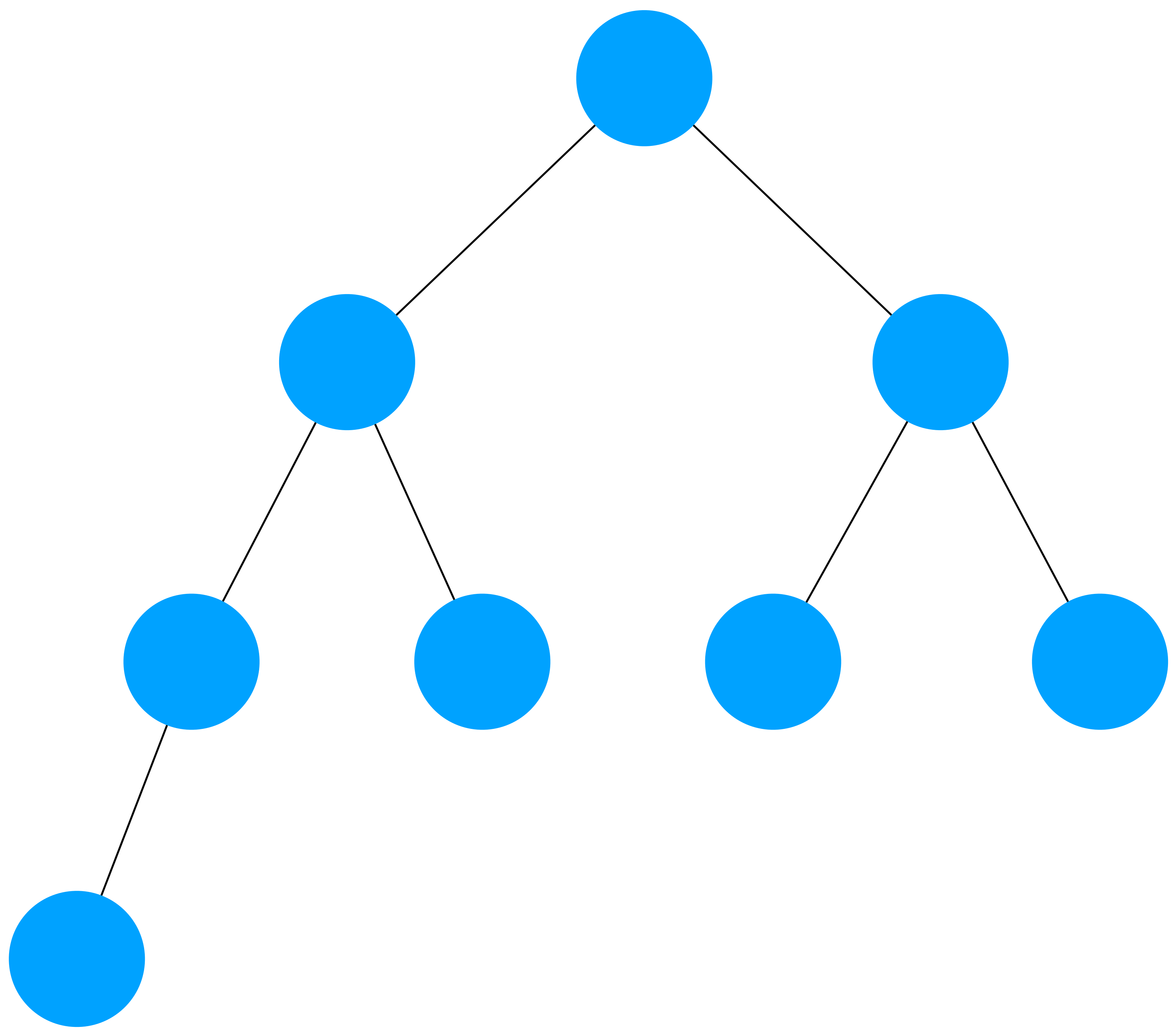

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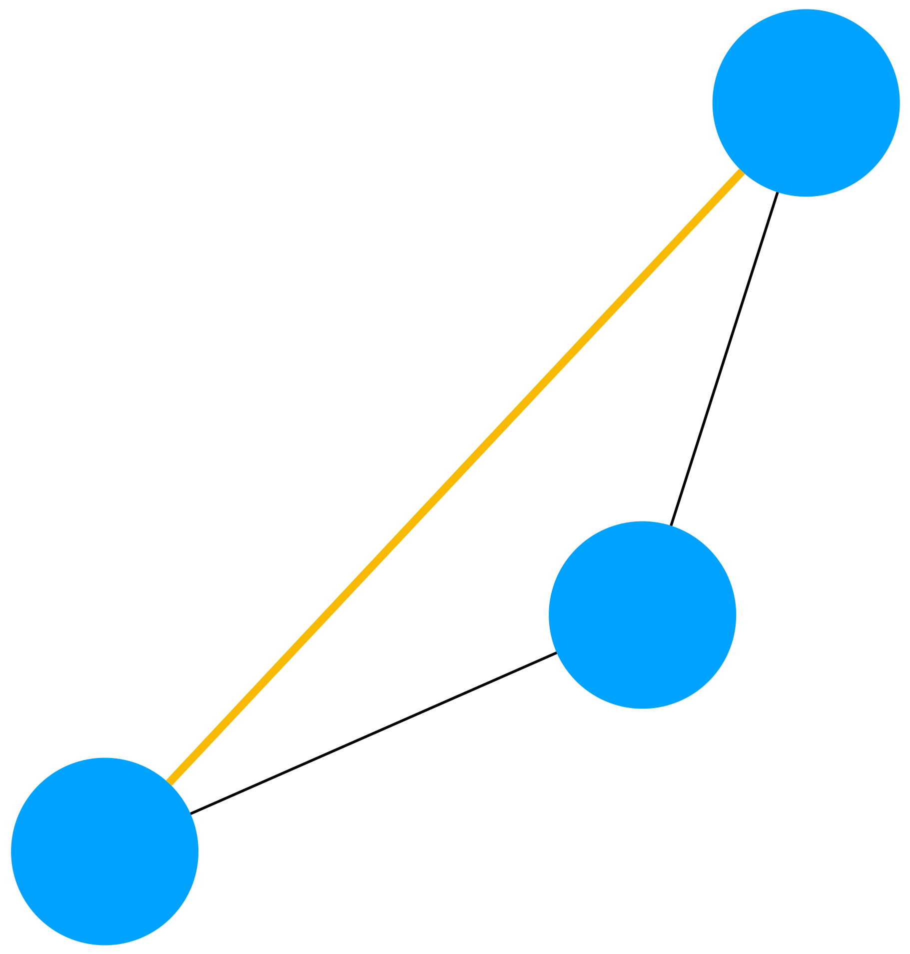

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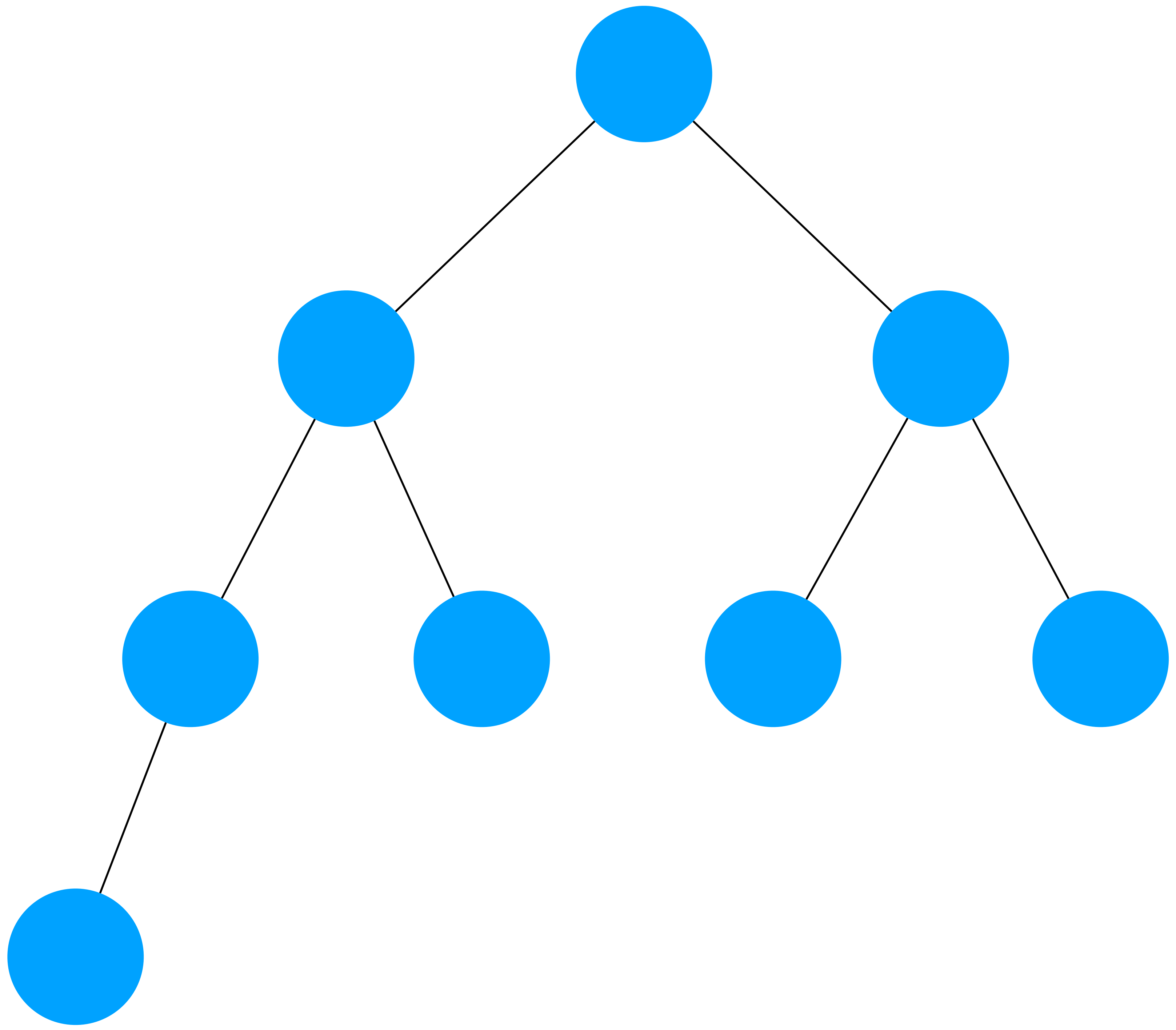

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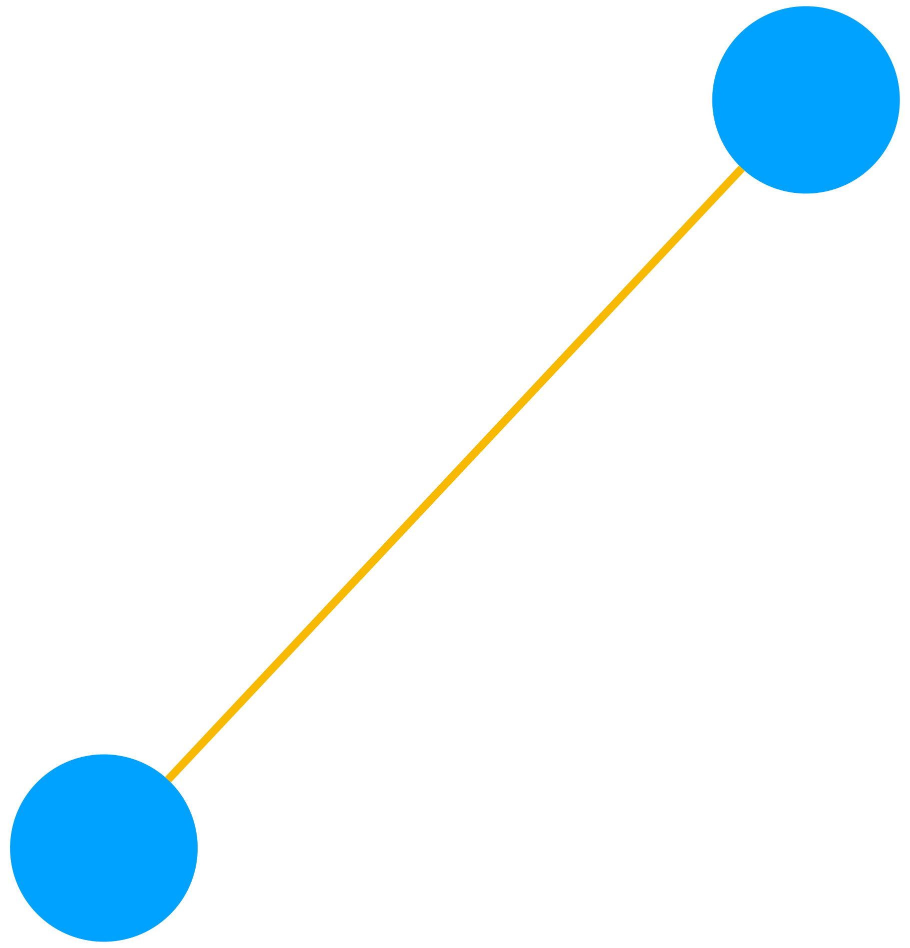

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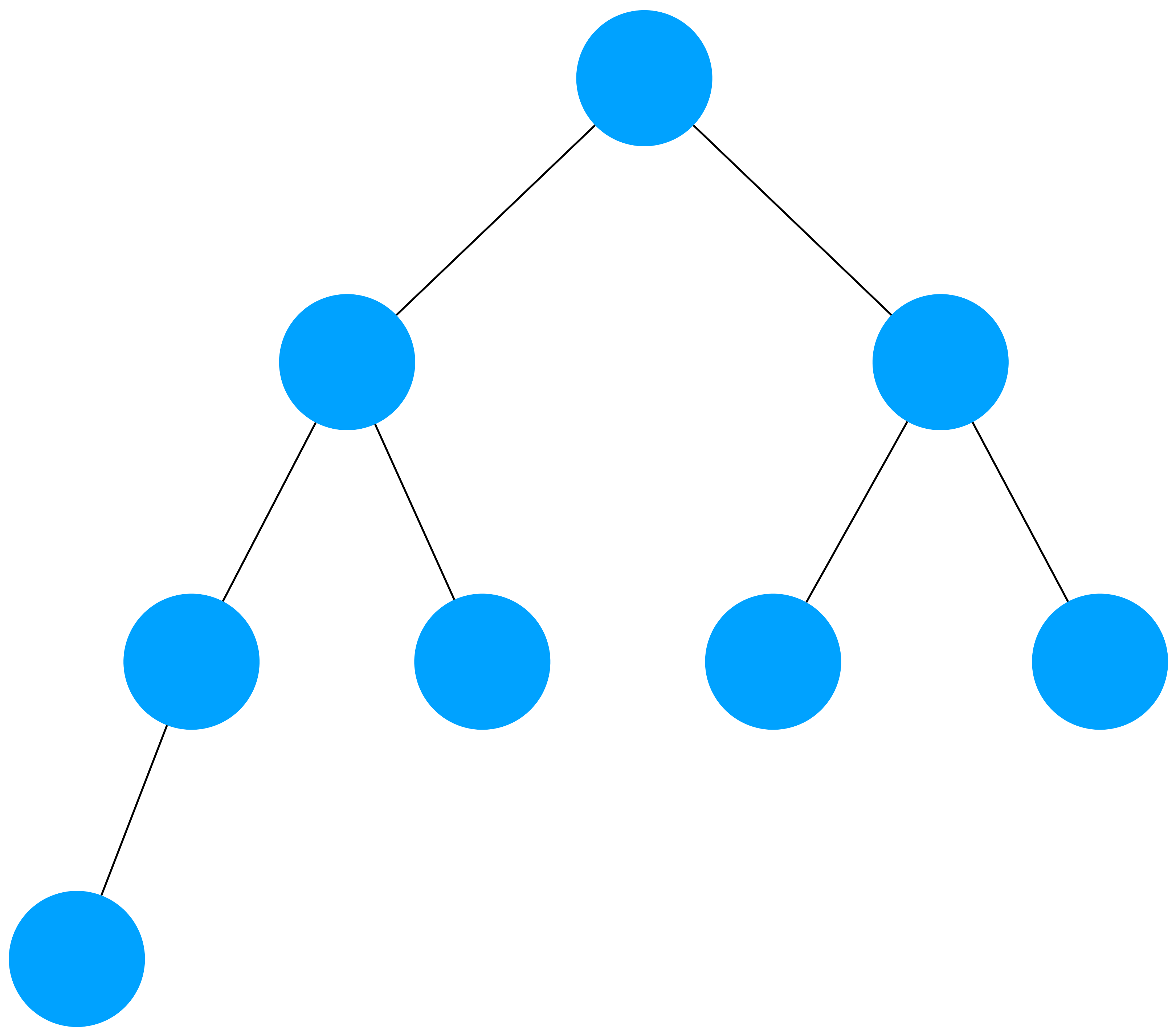

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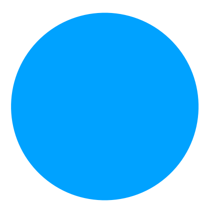

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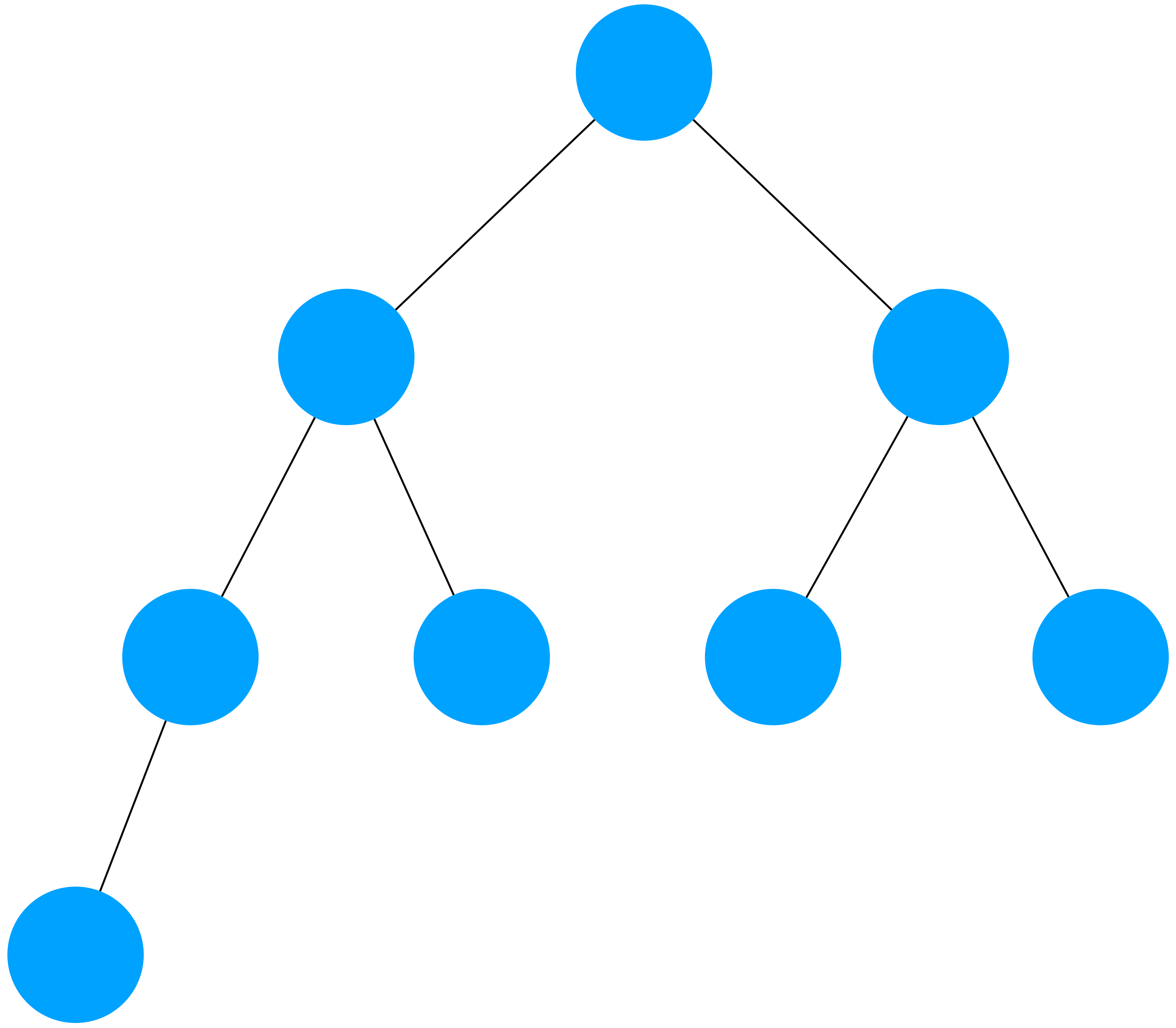

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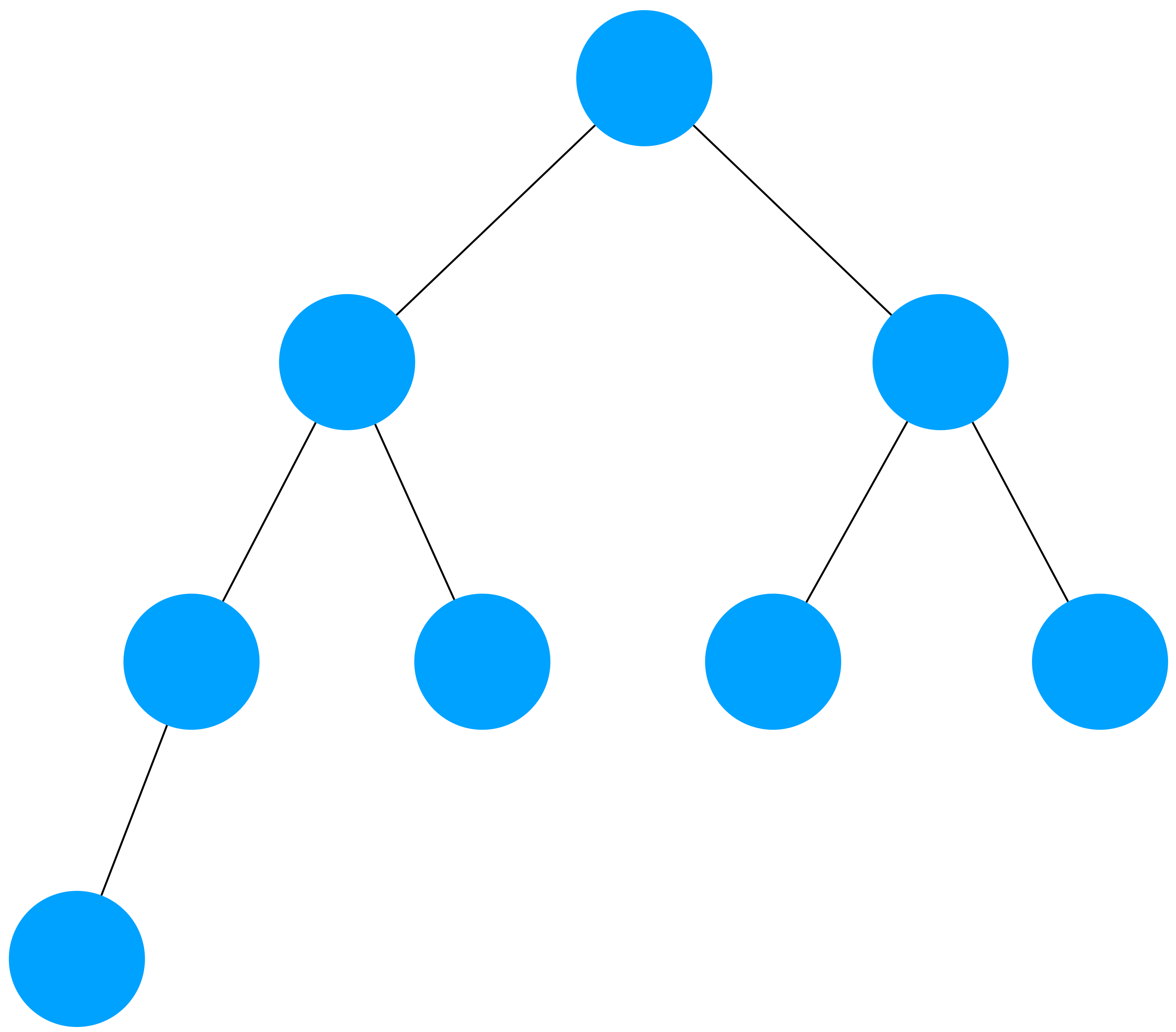

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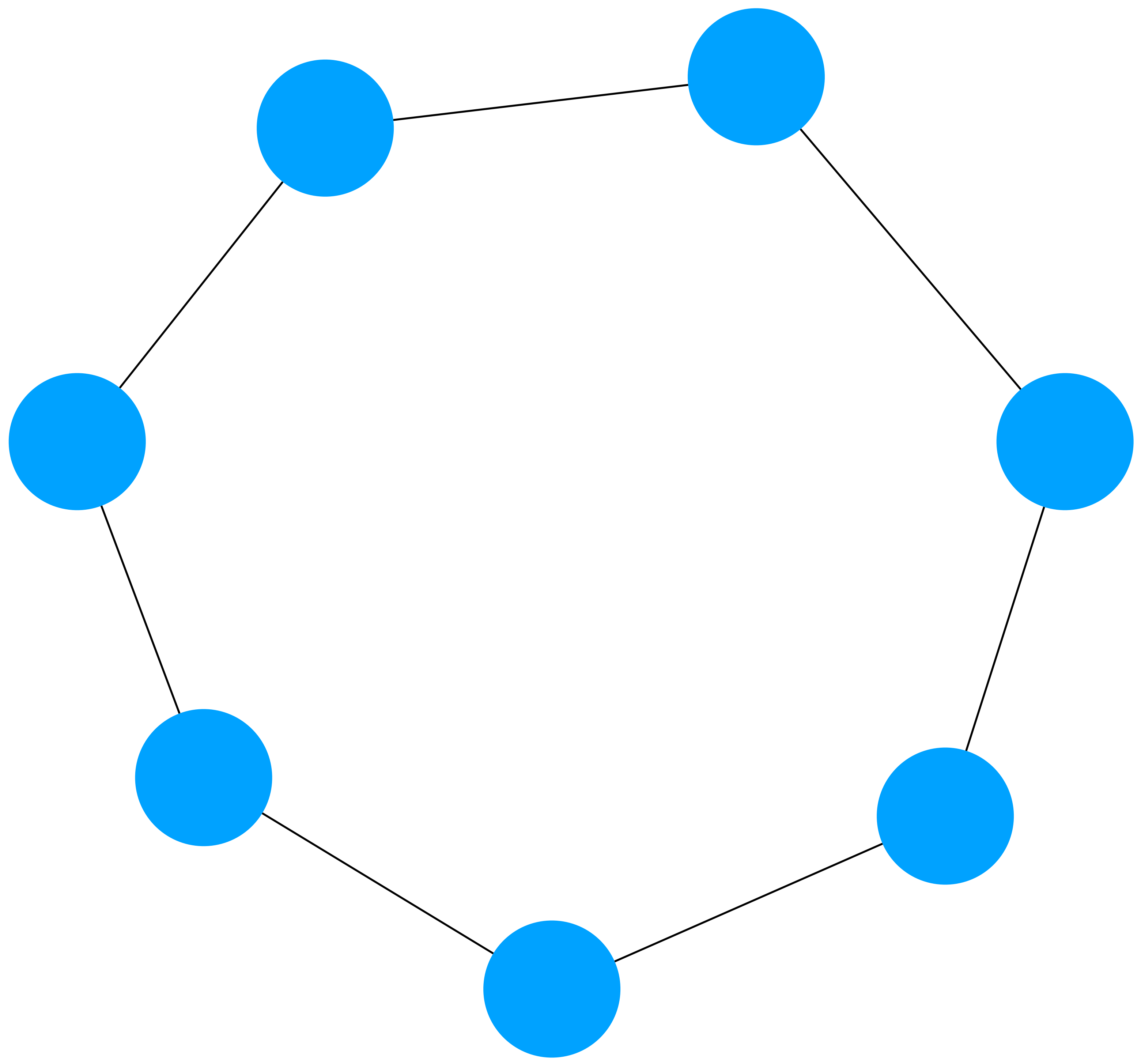

width 2

Treewidth

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Definition

The **treewidth** of a graph G is the minimum width of an elimination ordering of G.

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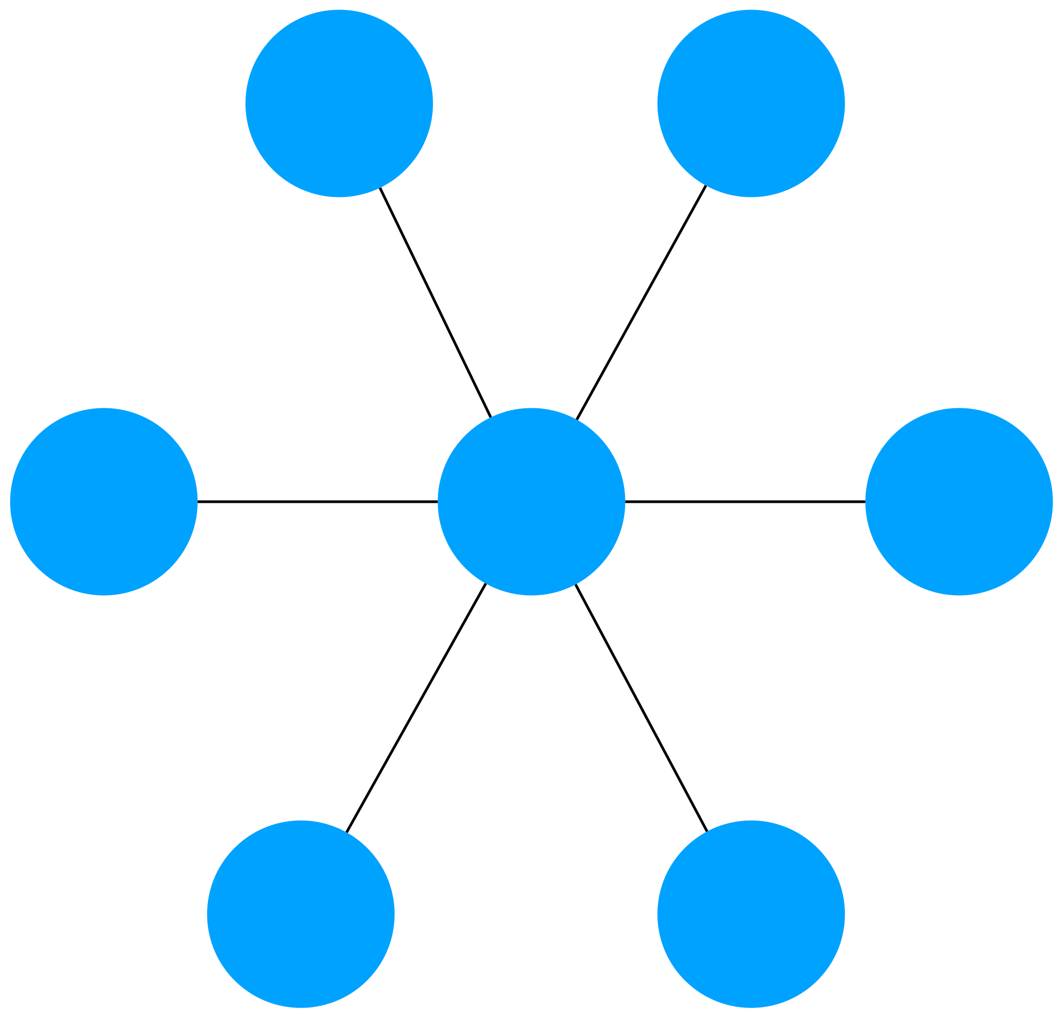

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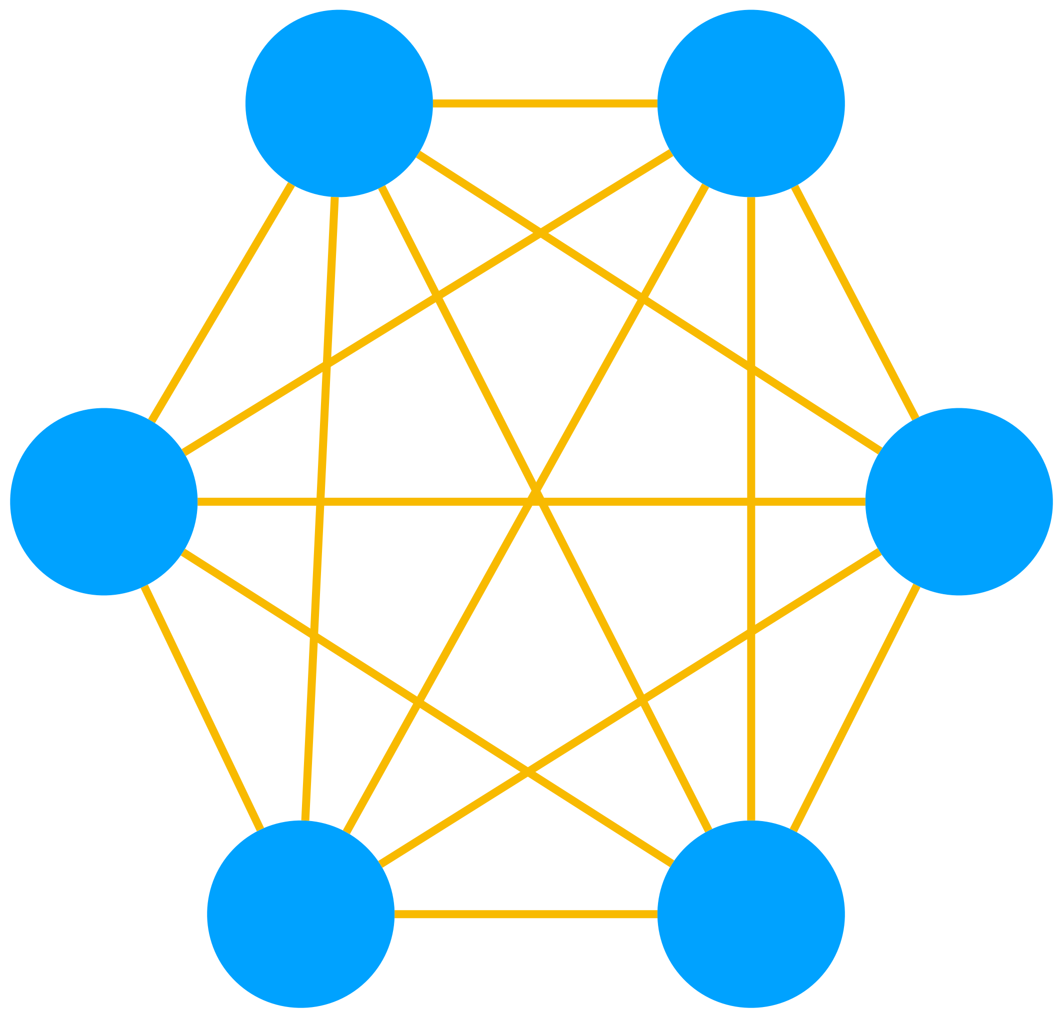

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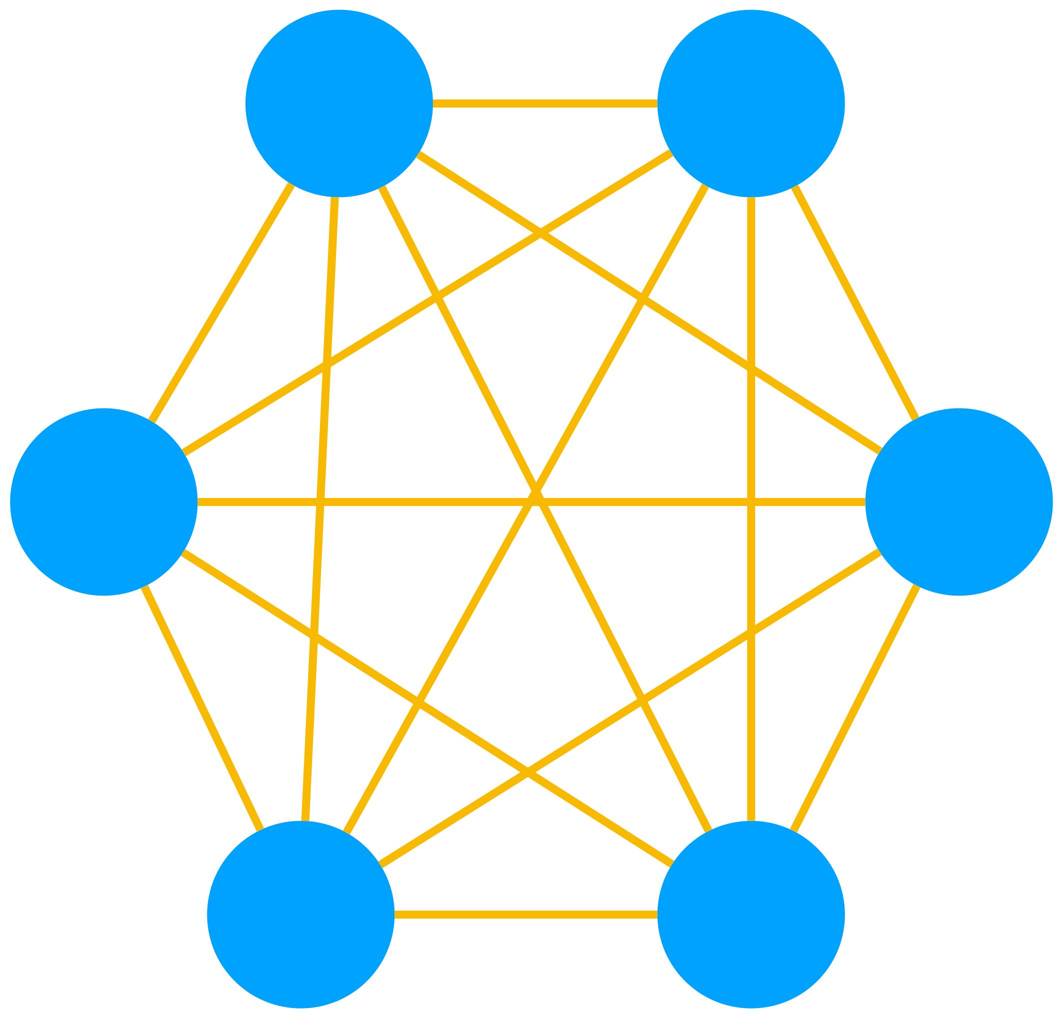
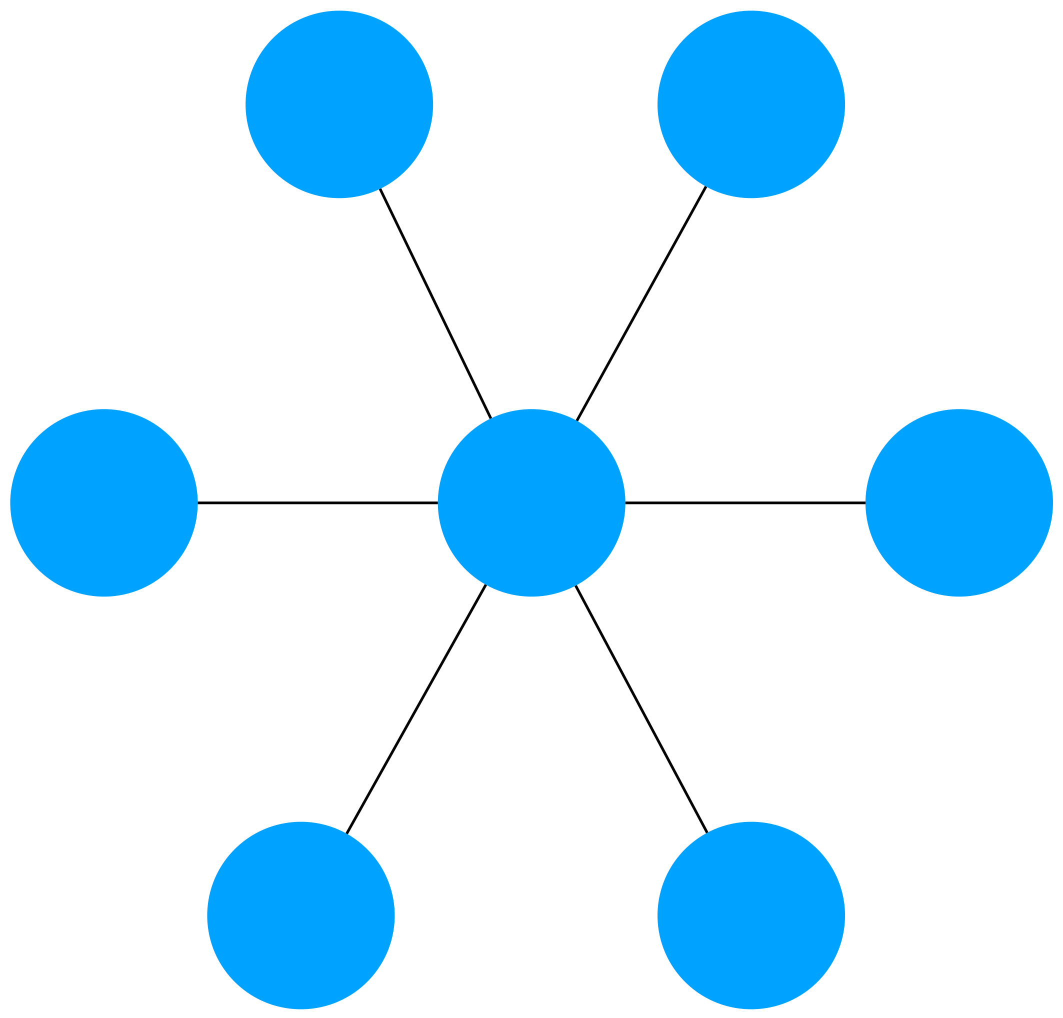


width 6



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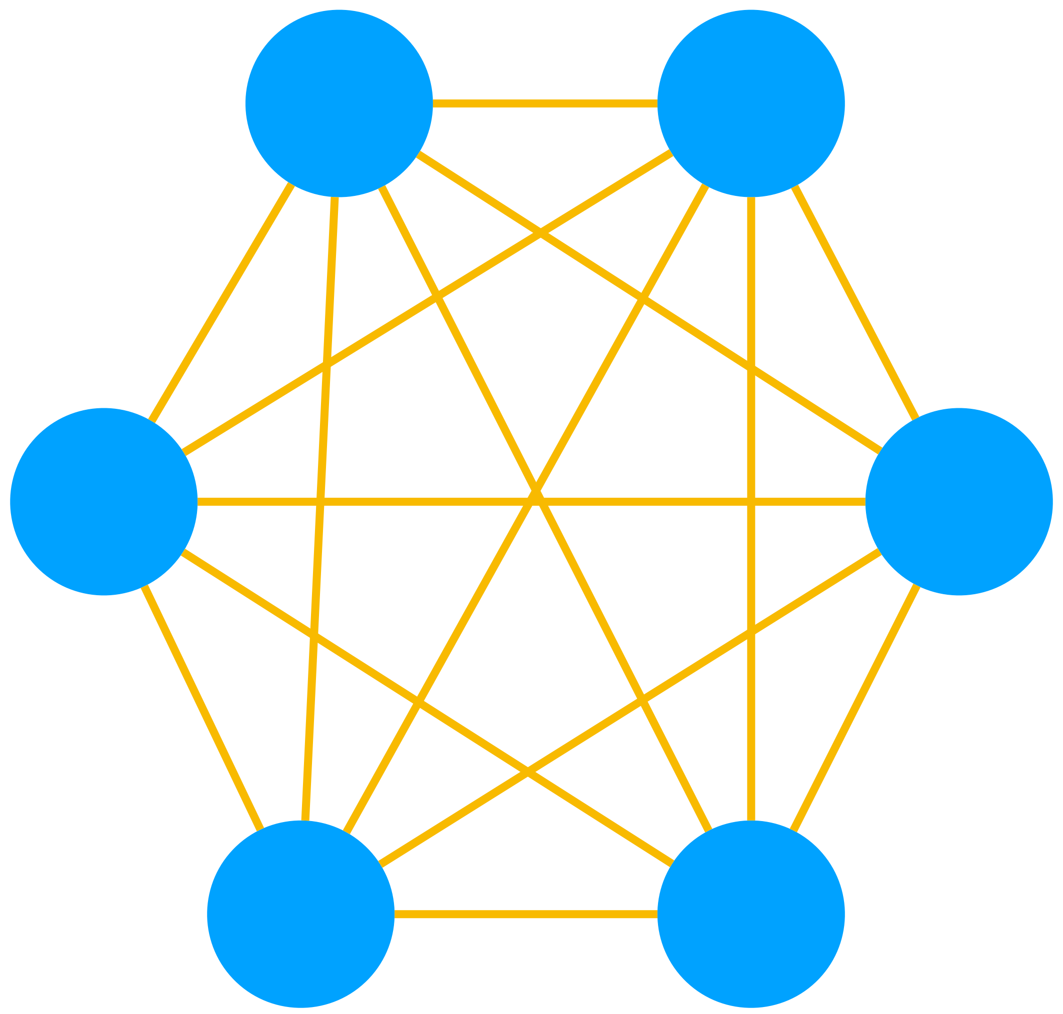

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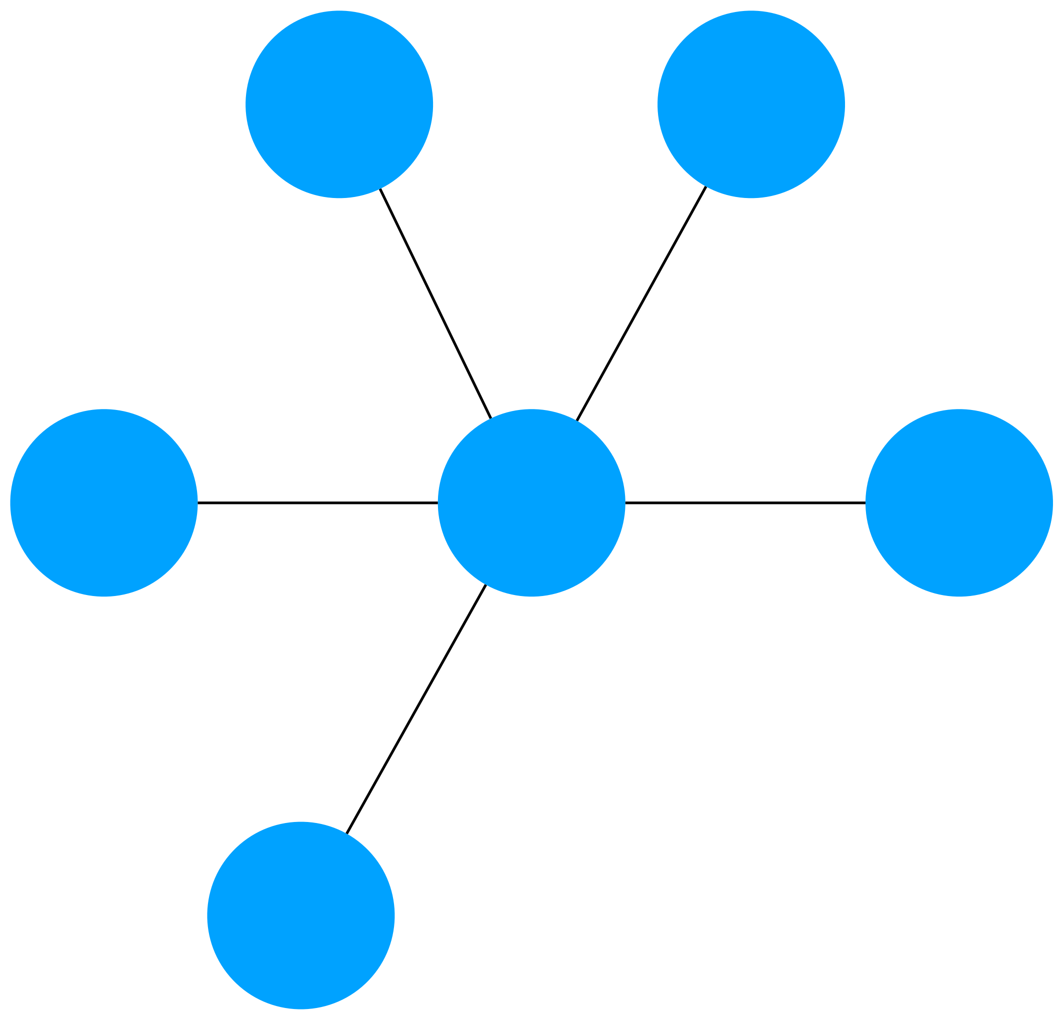

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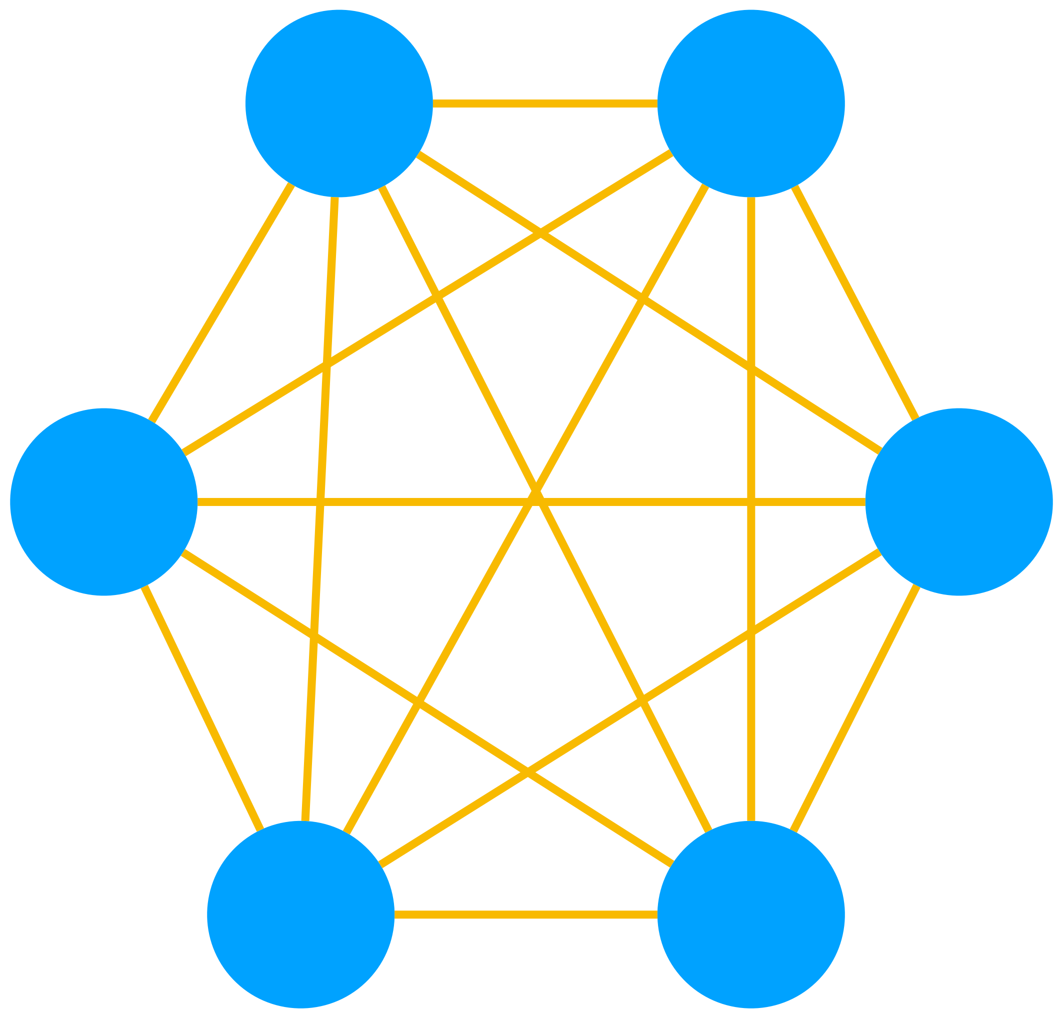

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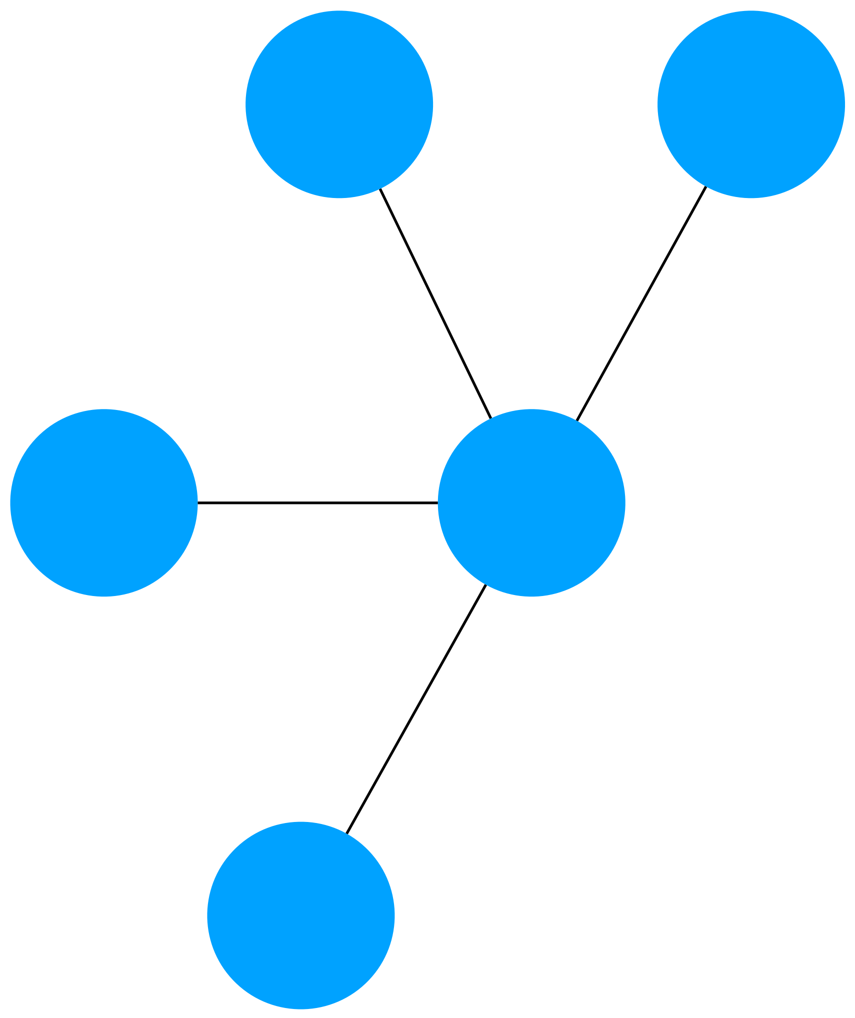

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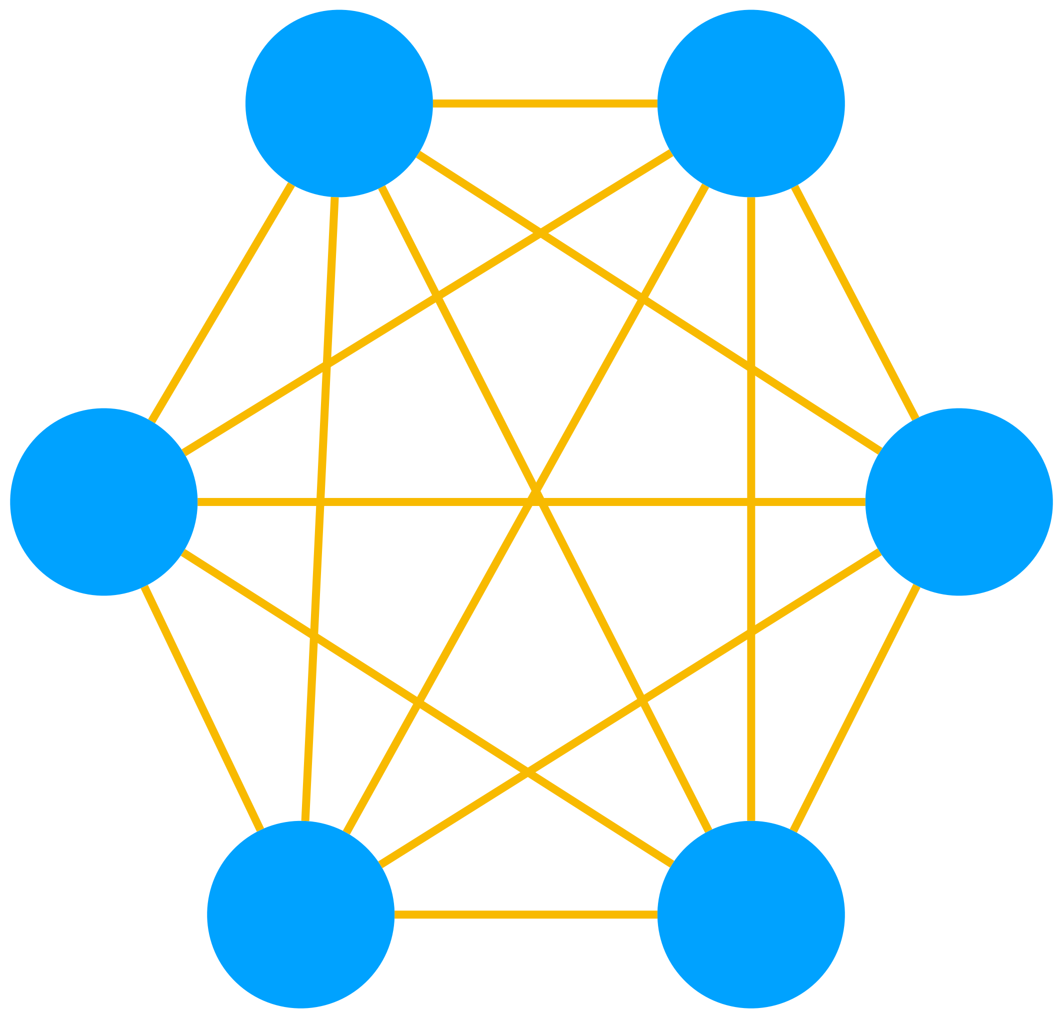

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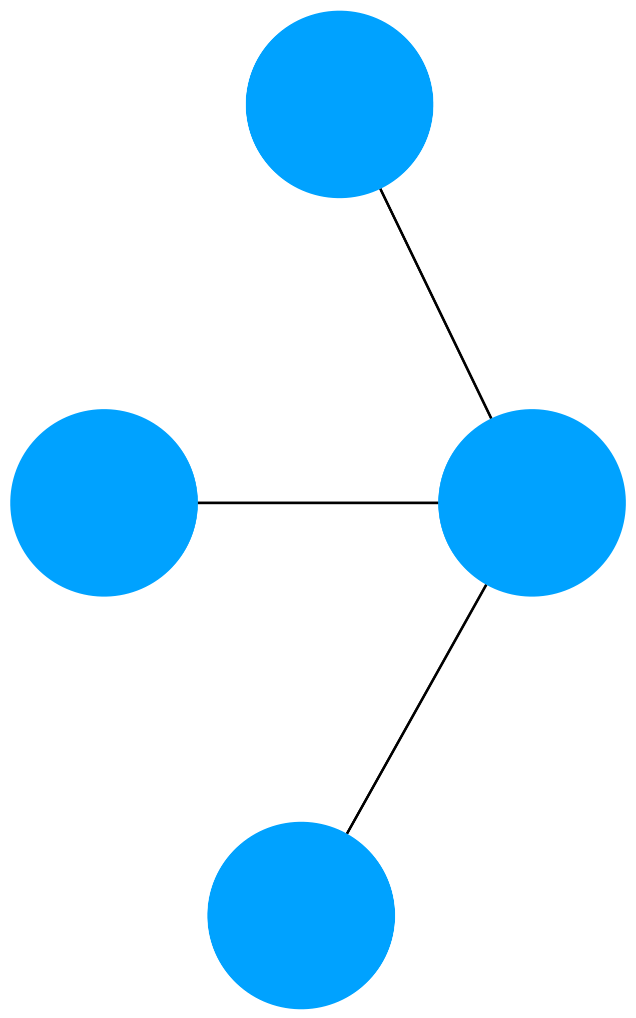

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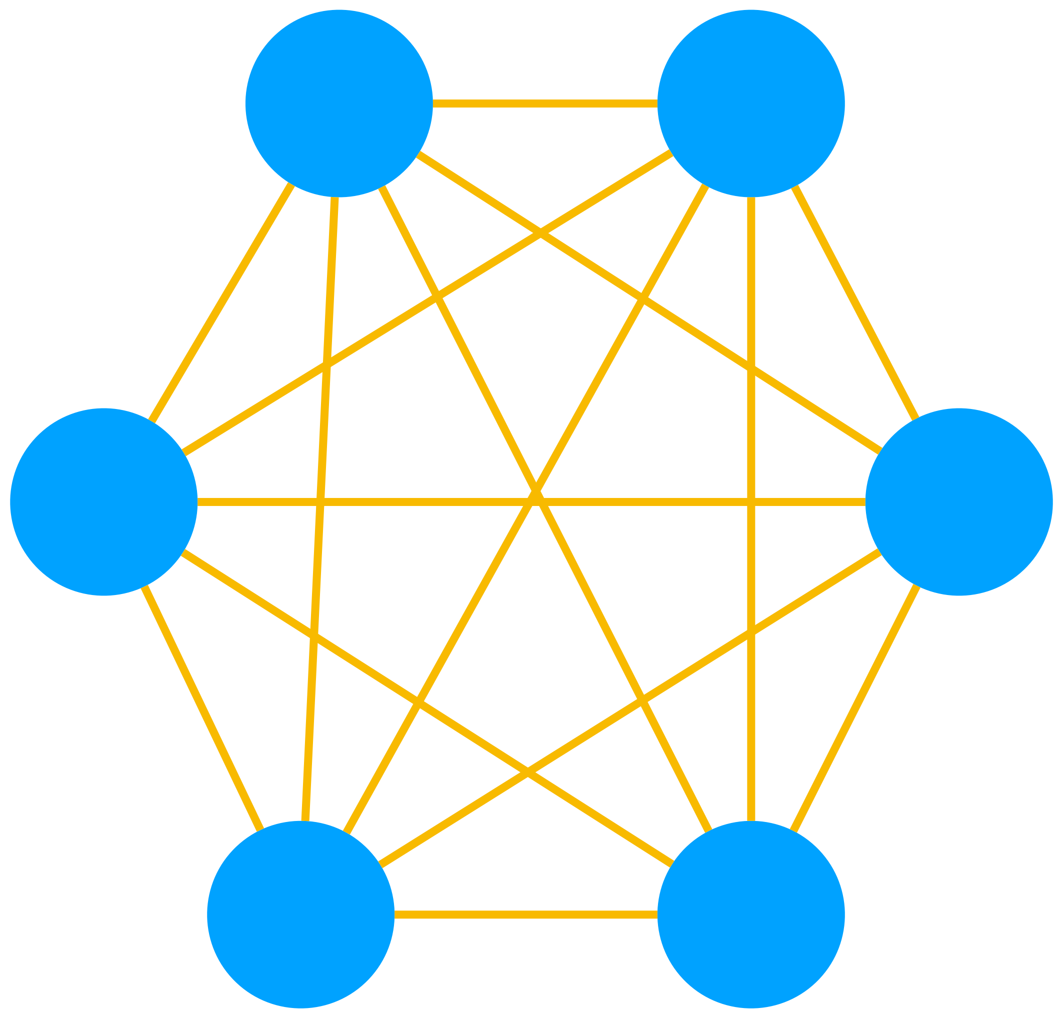

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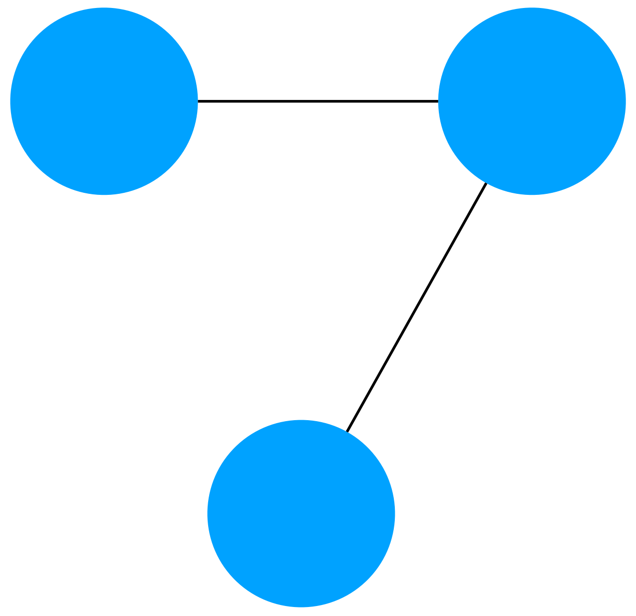

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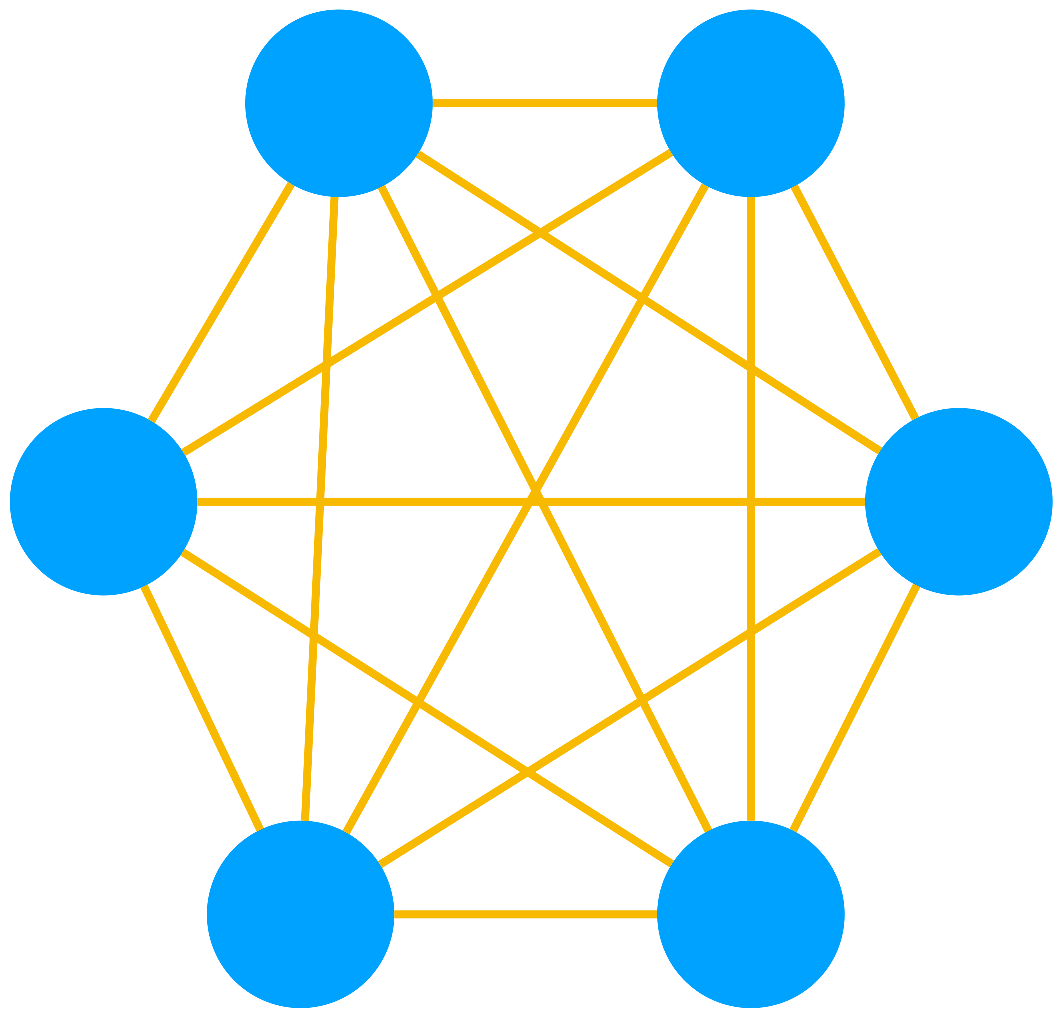

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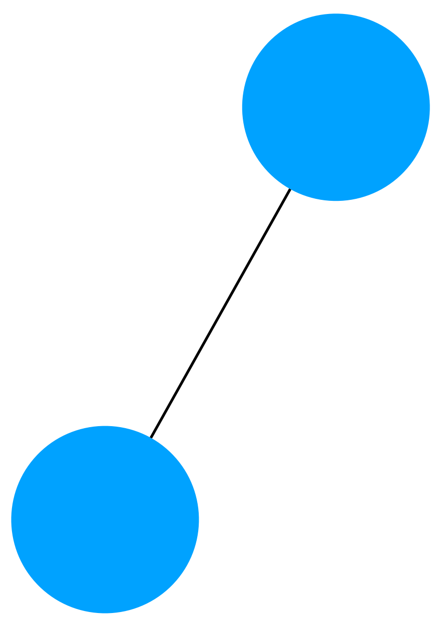

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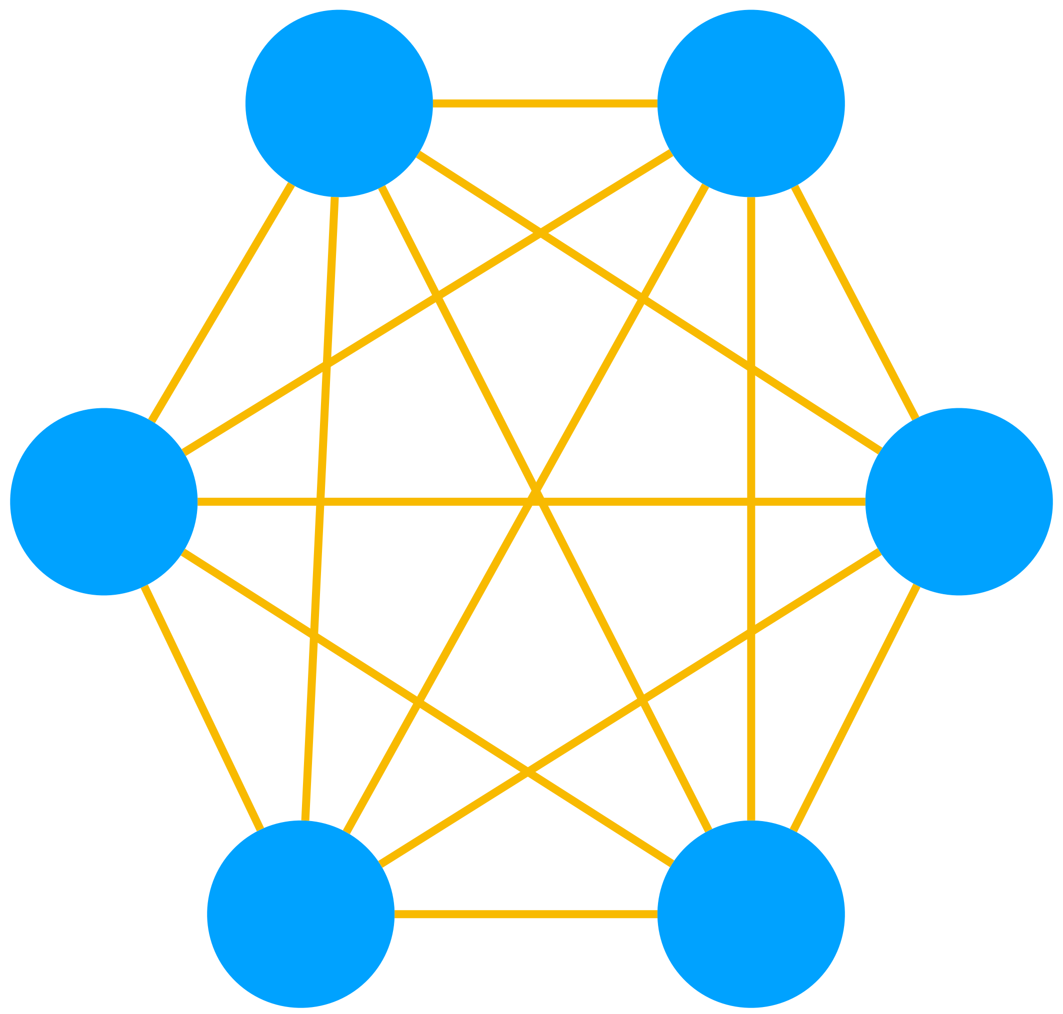

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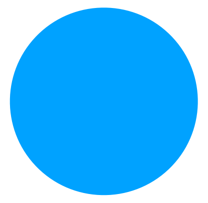

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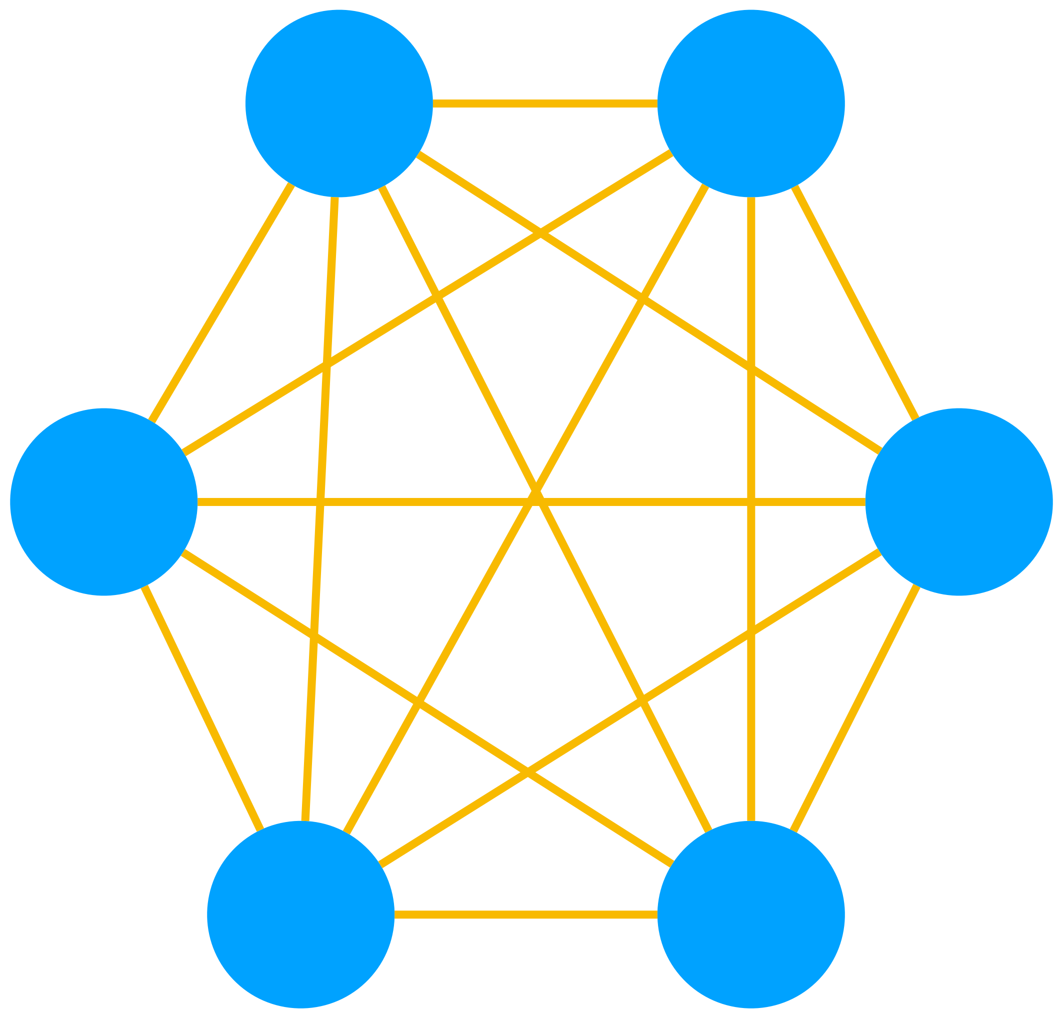

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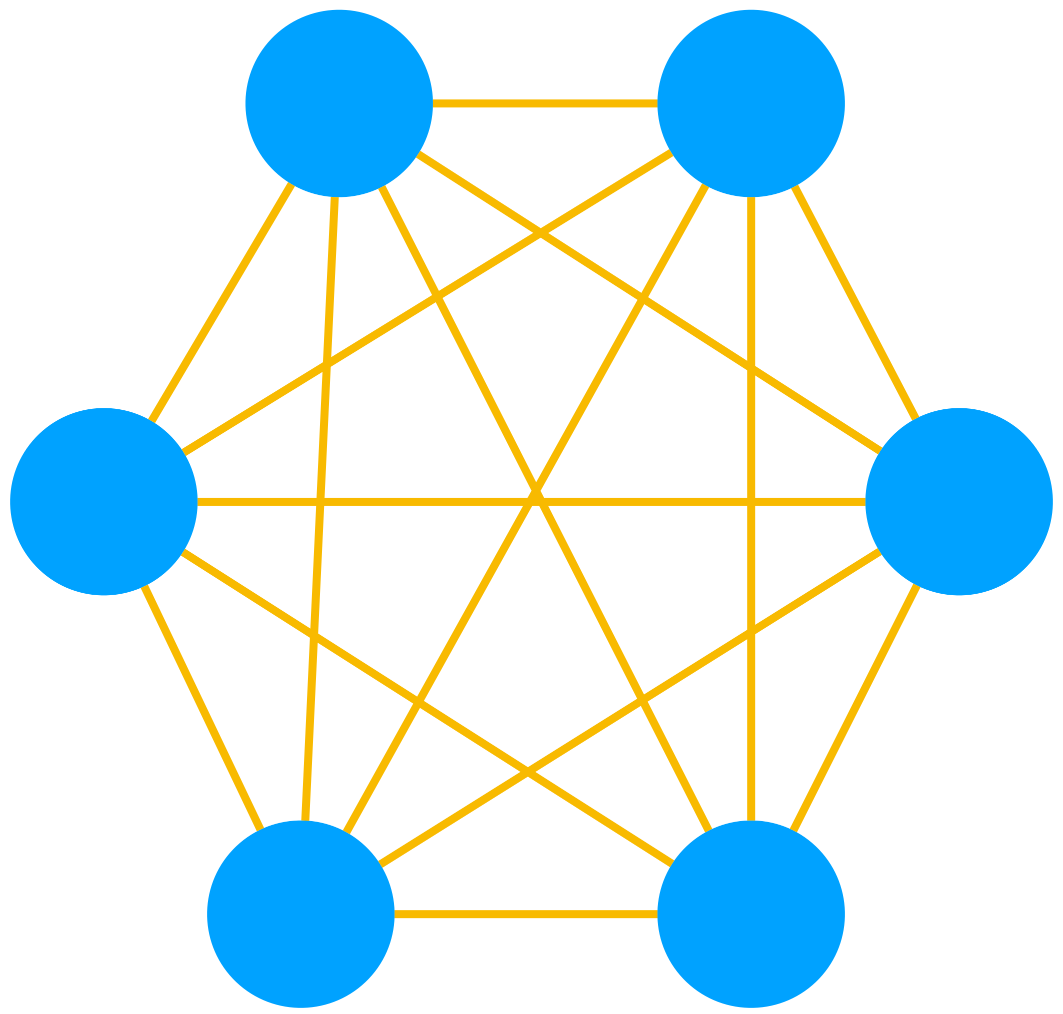

width 6

width 1

Degree and Treewidth

Proposition

If G has minimum degree d then $d \leq \mathbf{tw}(G)$.

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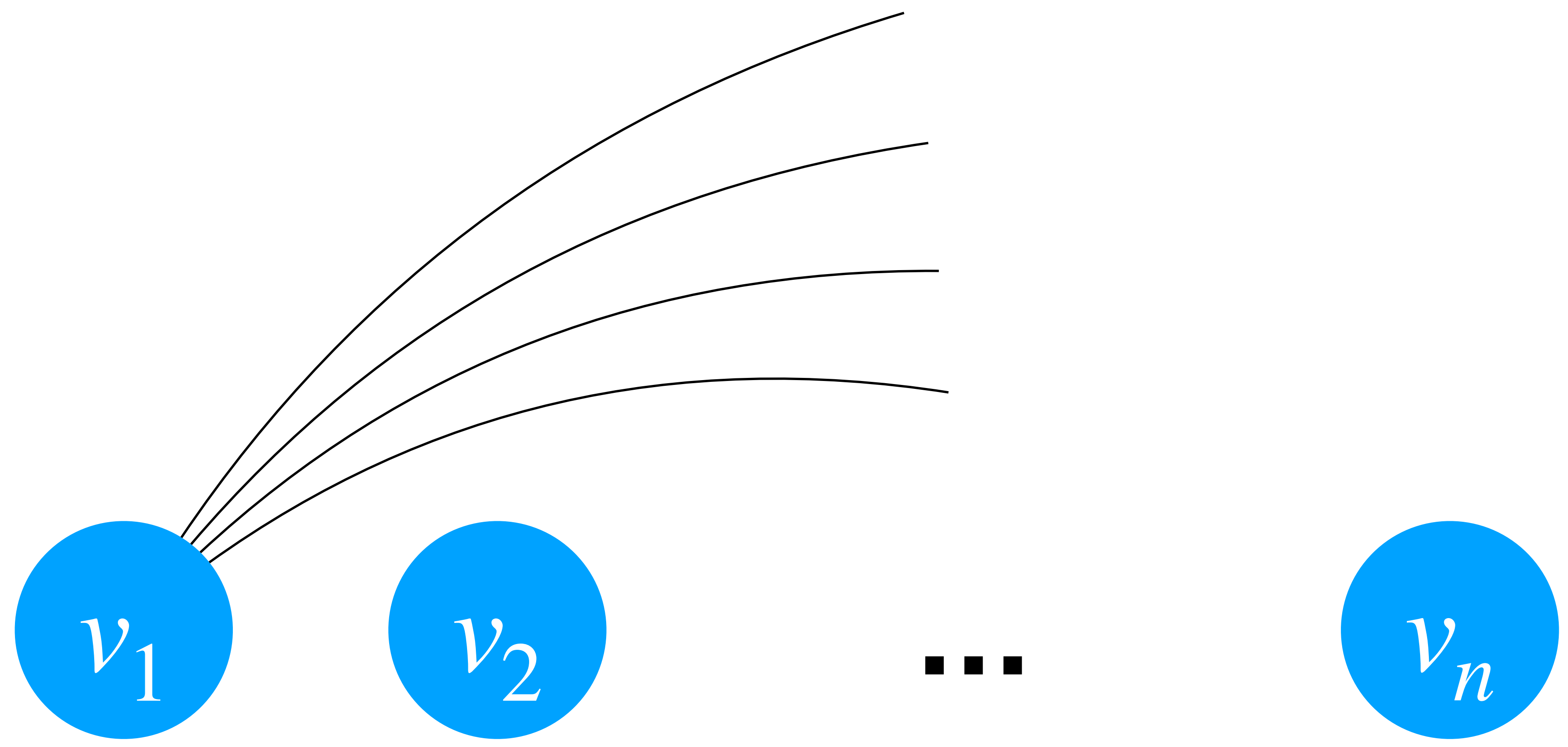

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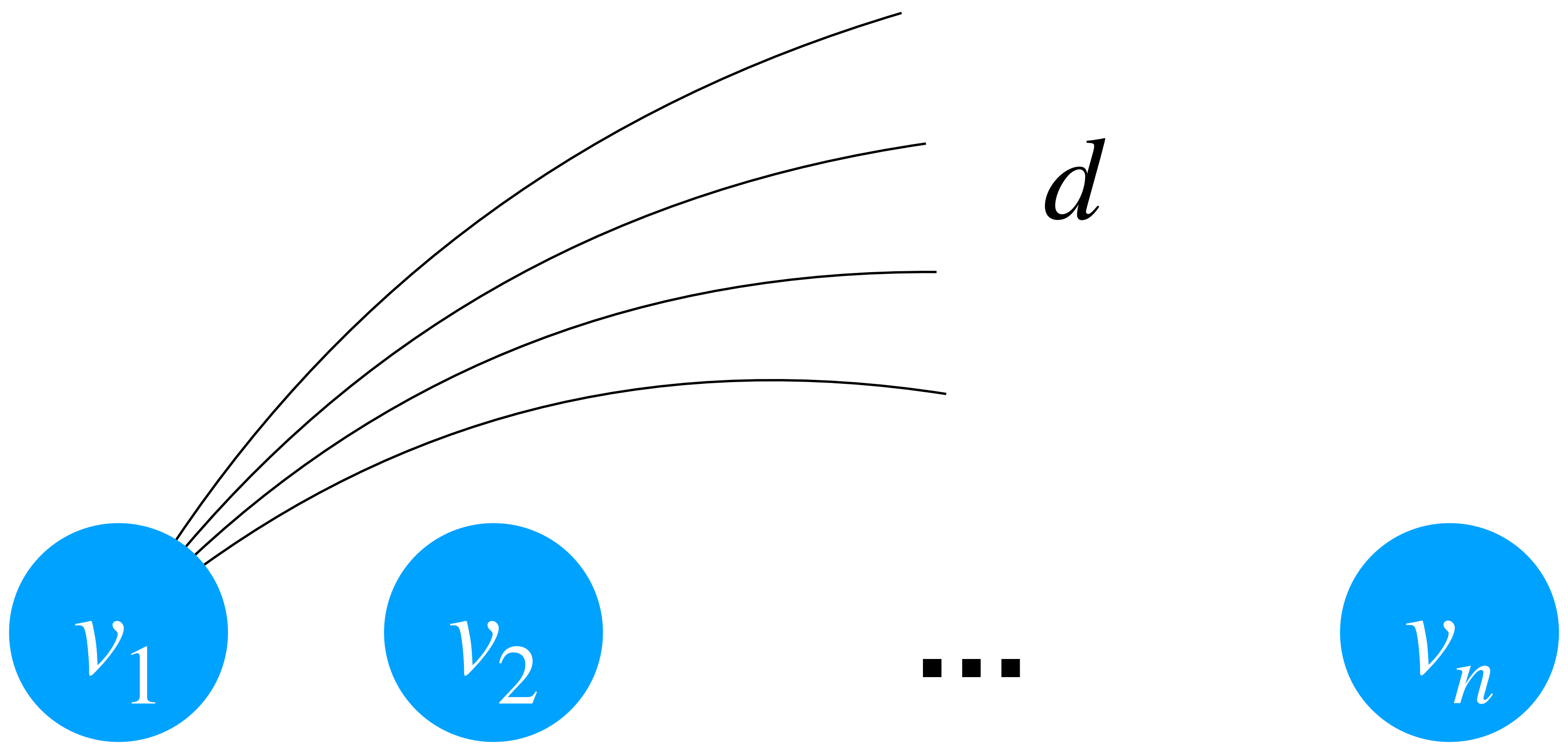

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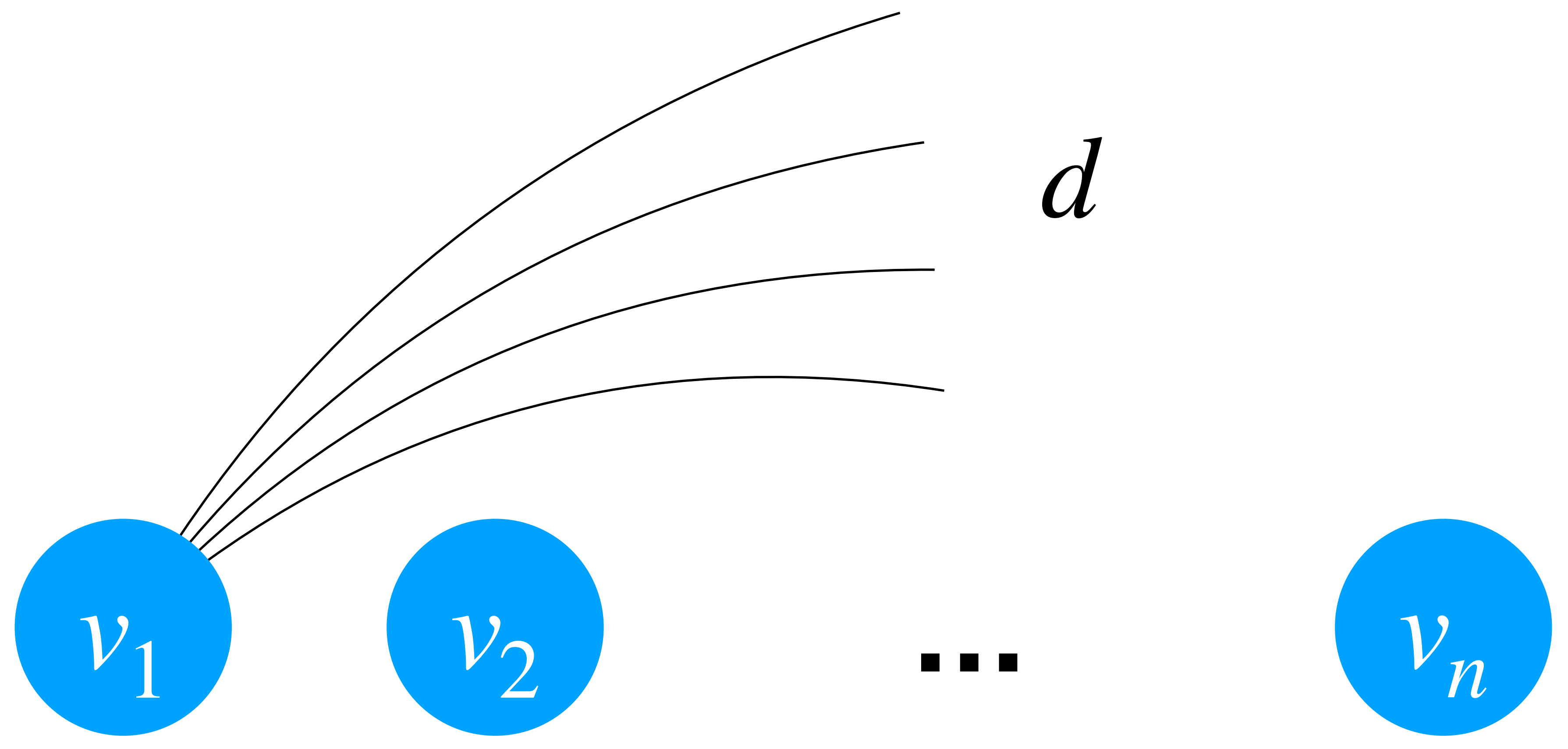
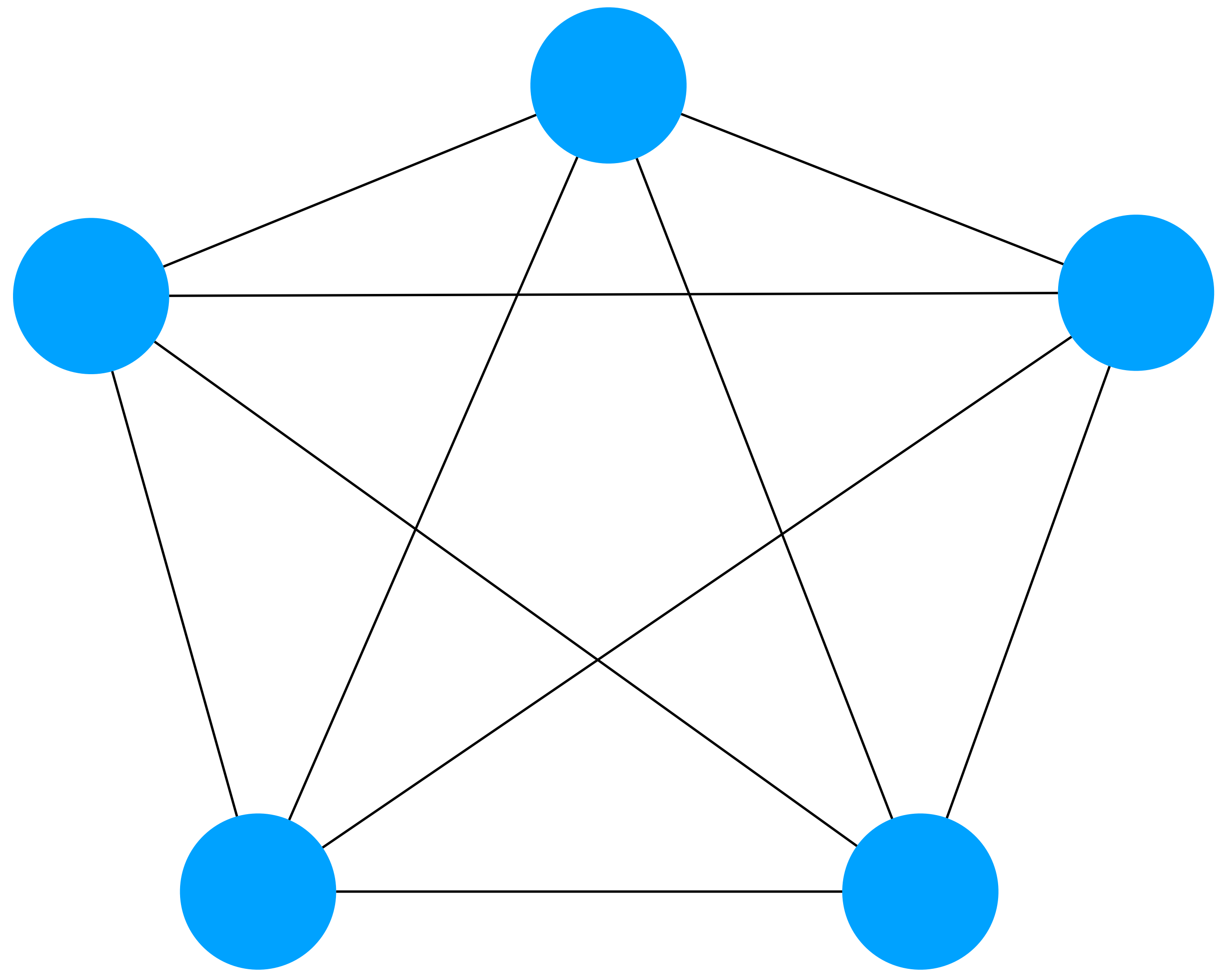


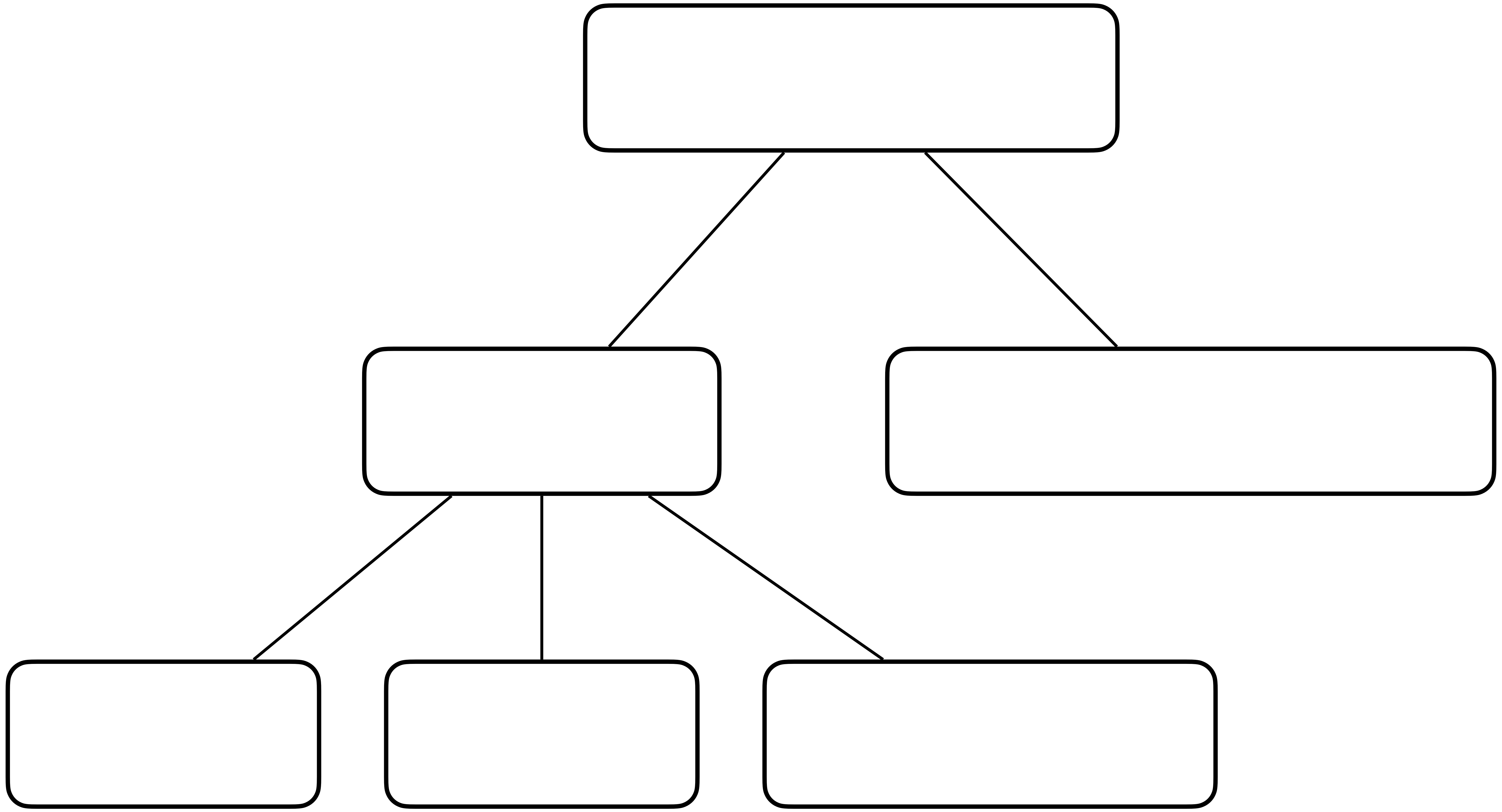


K_n treewidth $n - 1$

Tree Decompositions

Tree Decomposition

Tree Decomposition



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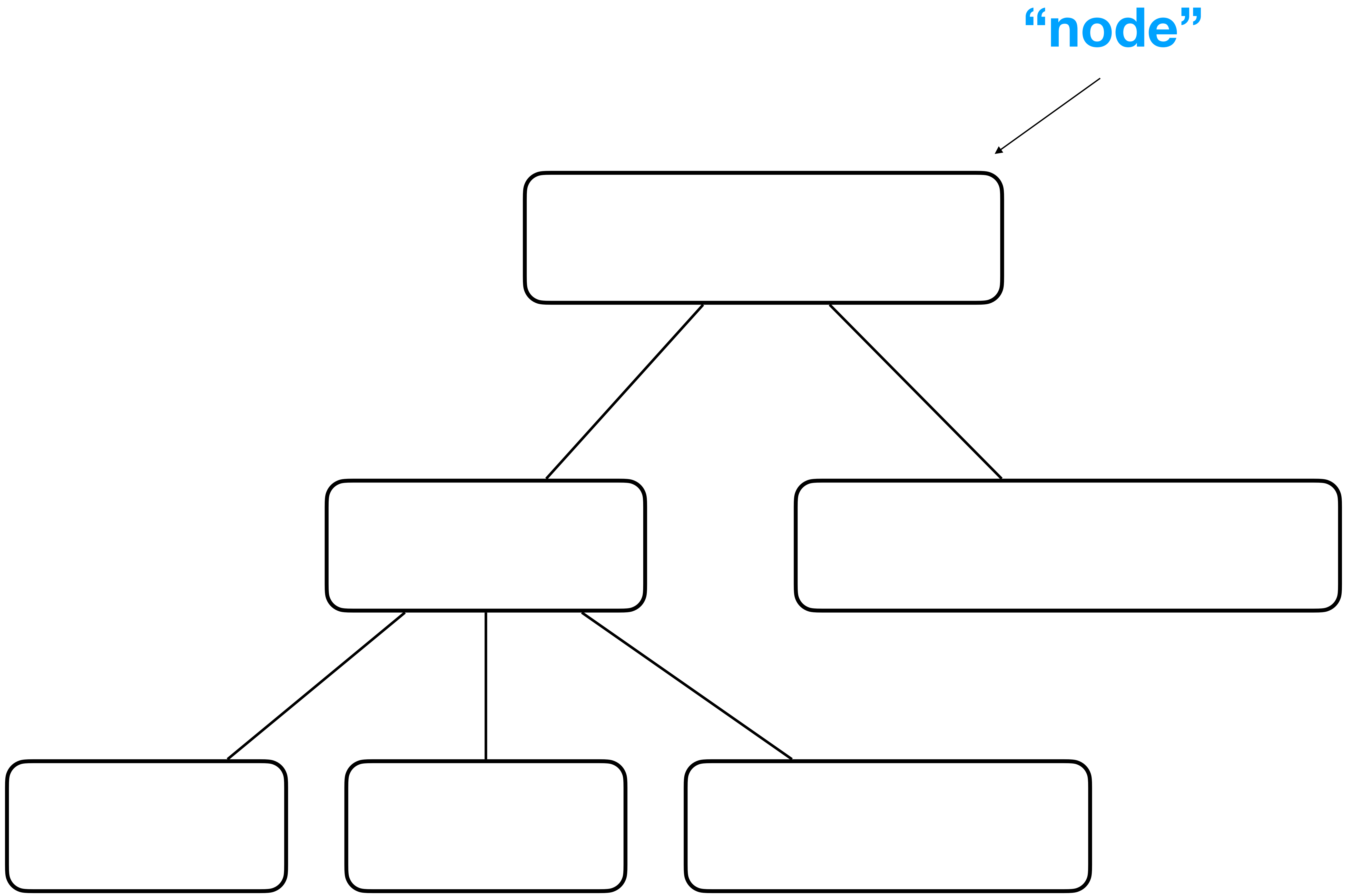

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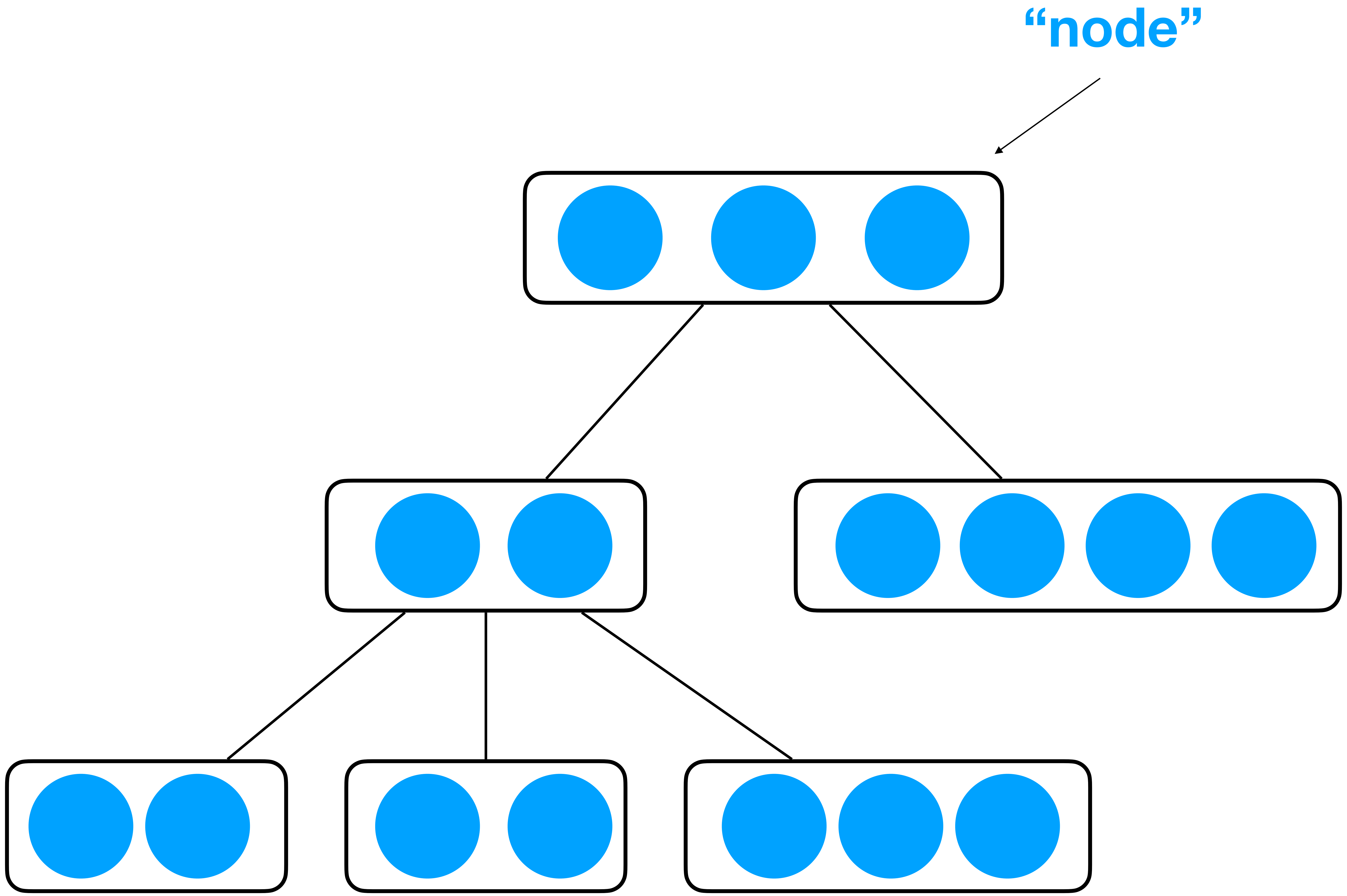

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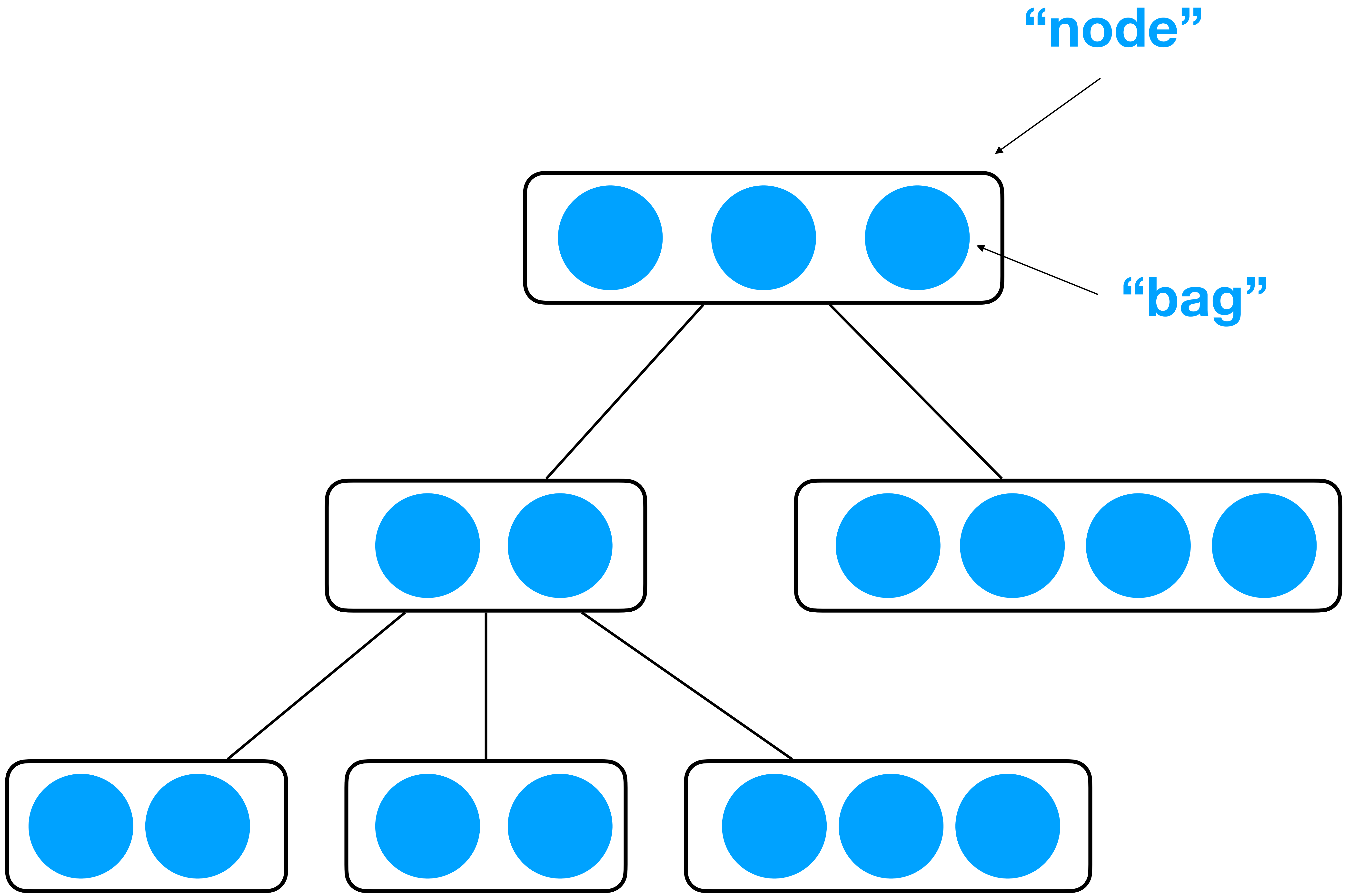
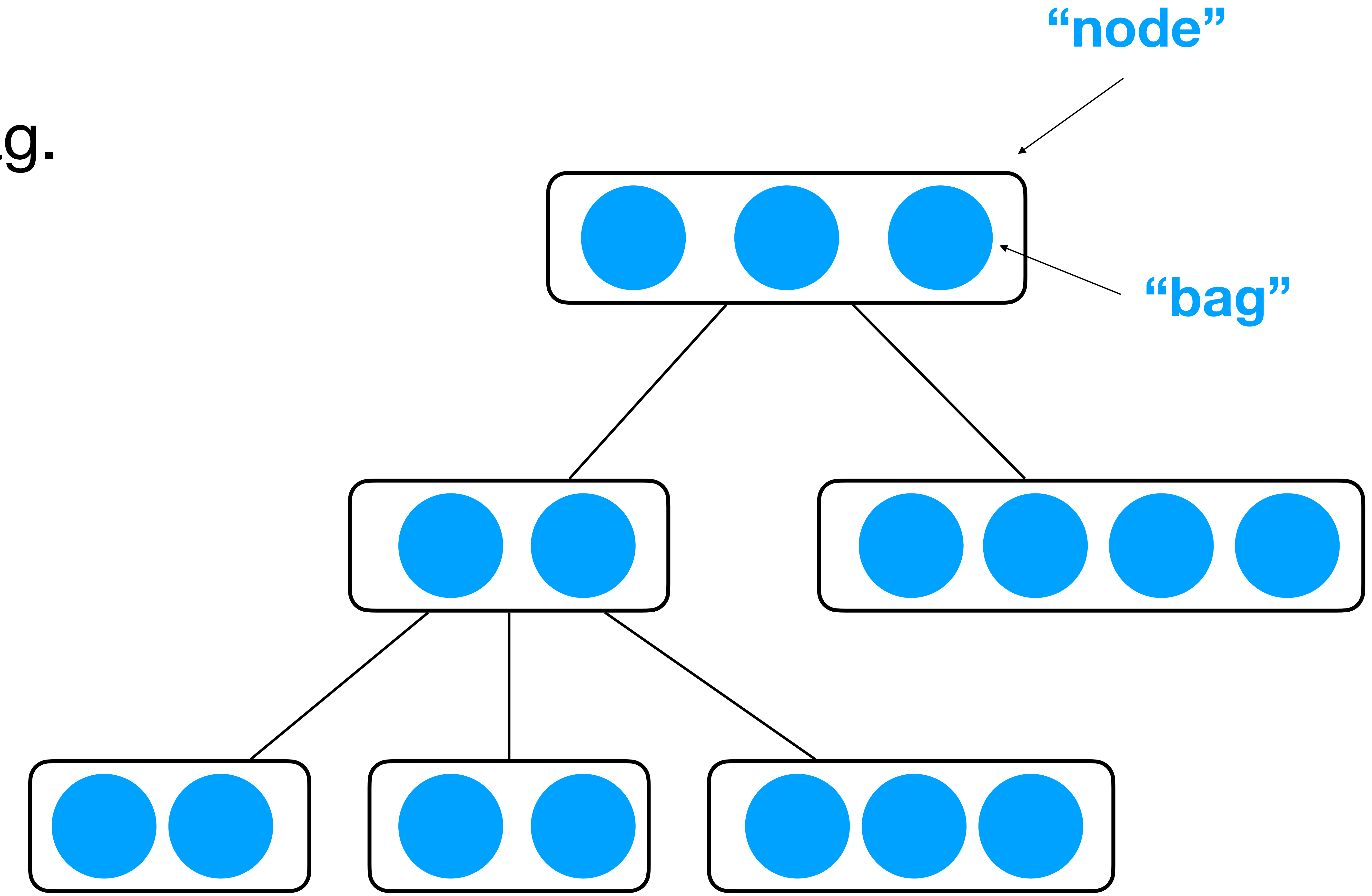


Tree Decomposition

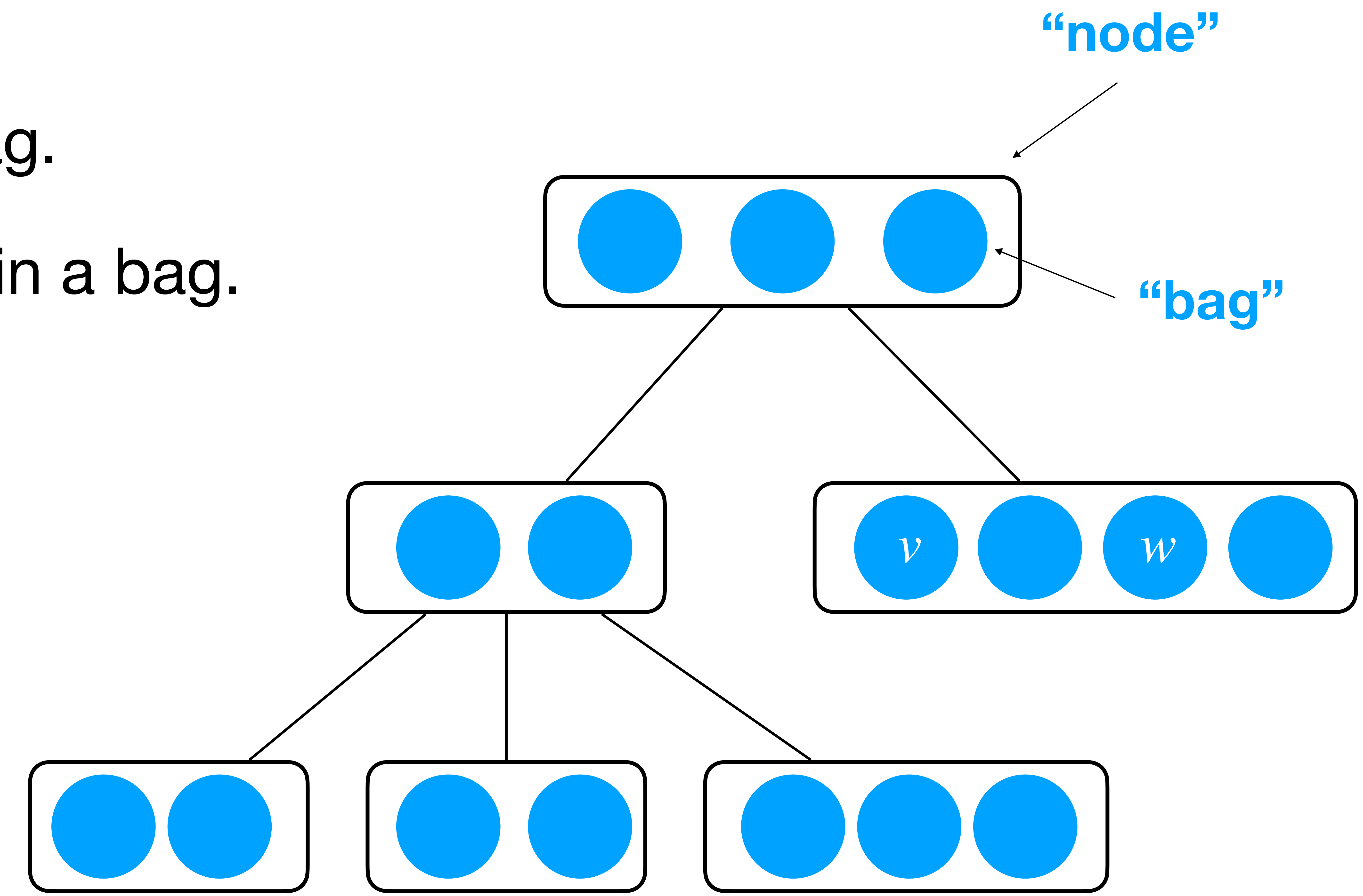
1. Each **vertex** appears in a bag.



Tree Decomposition

1. Each **vertex** appears in a bag.

2. Each **edge** vw is contained in a bag.



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1. Each **vertex** appears in a bag.
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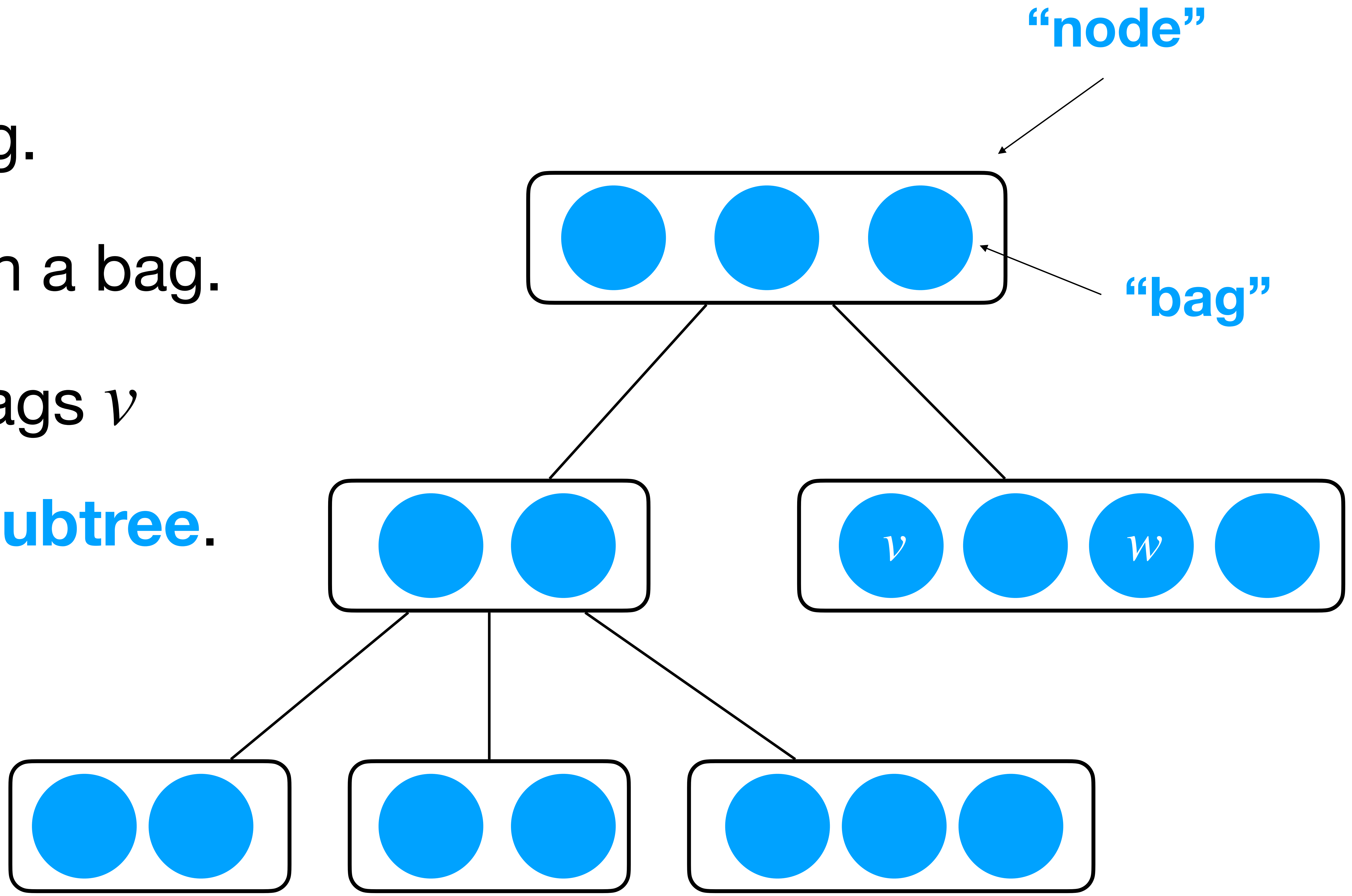

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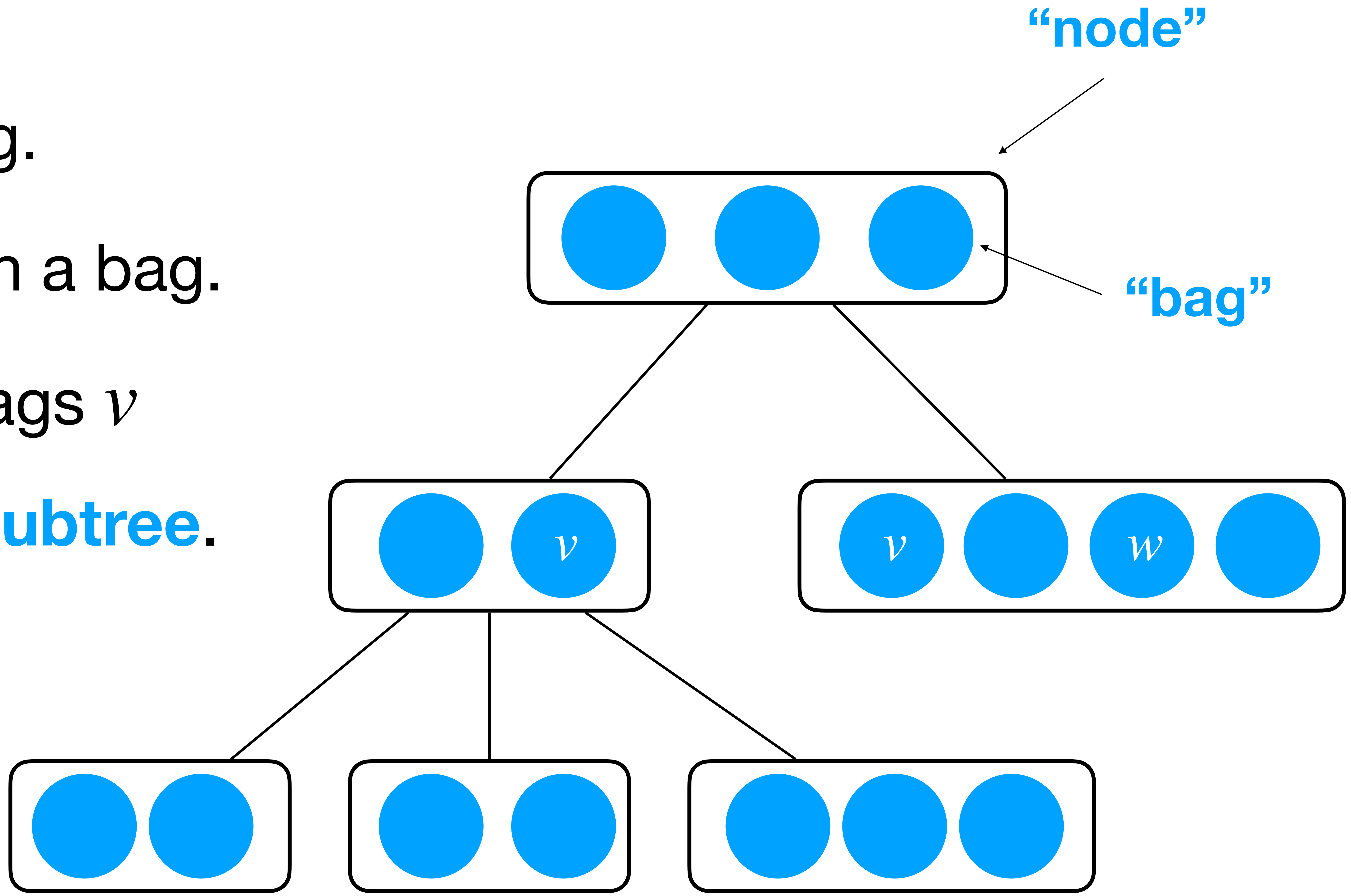

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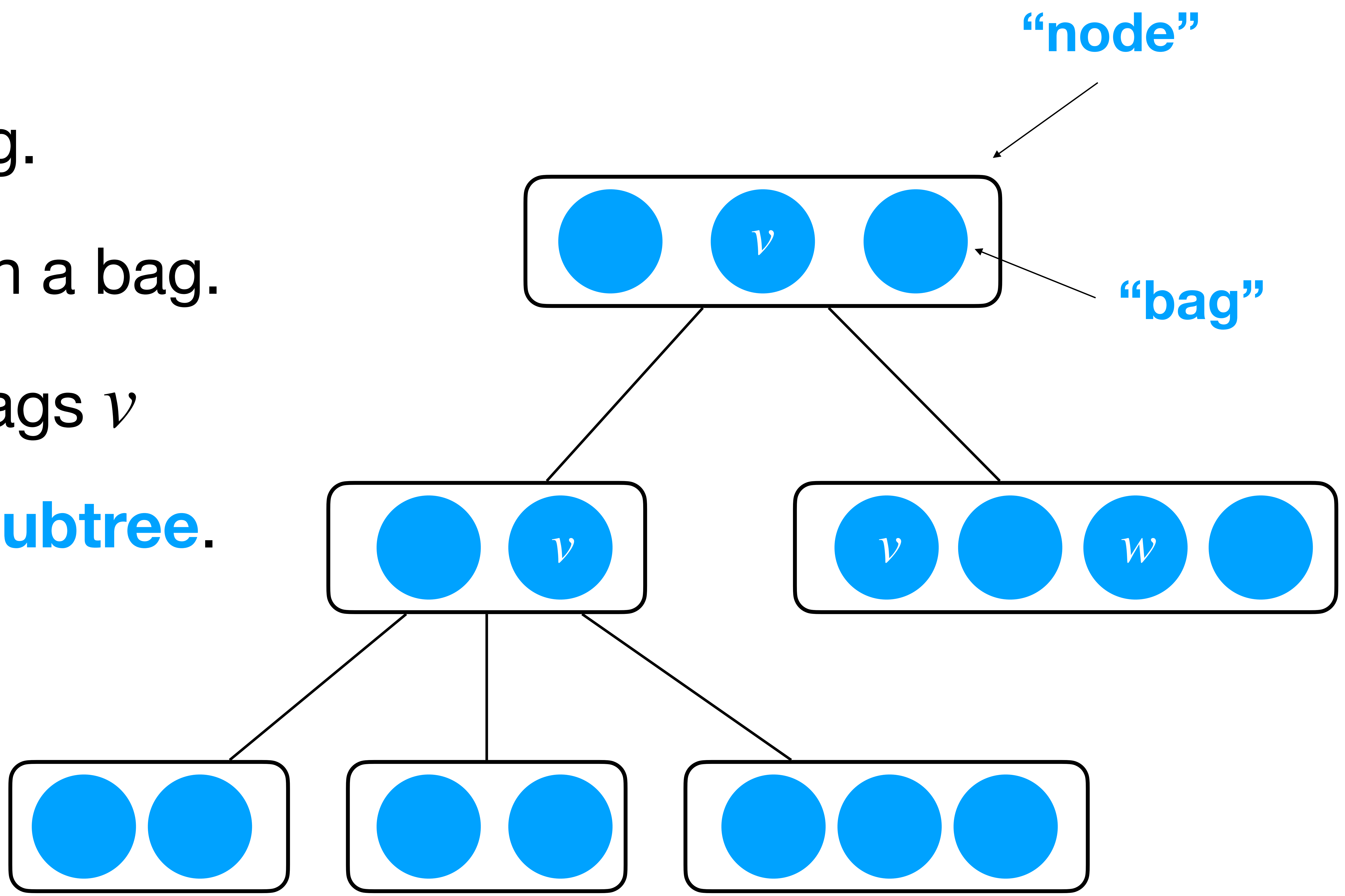

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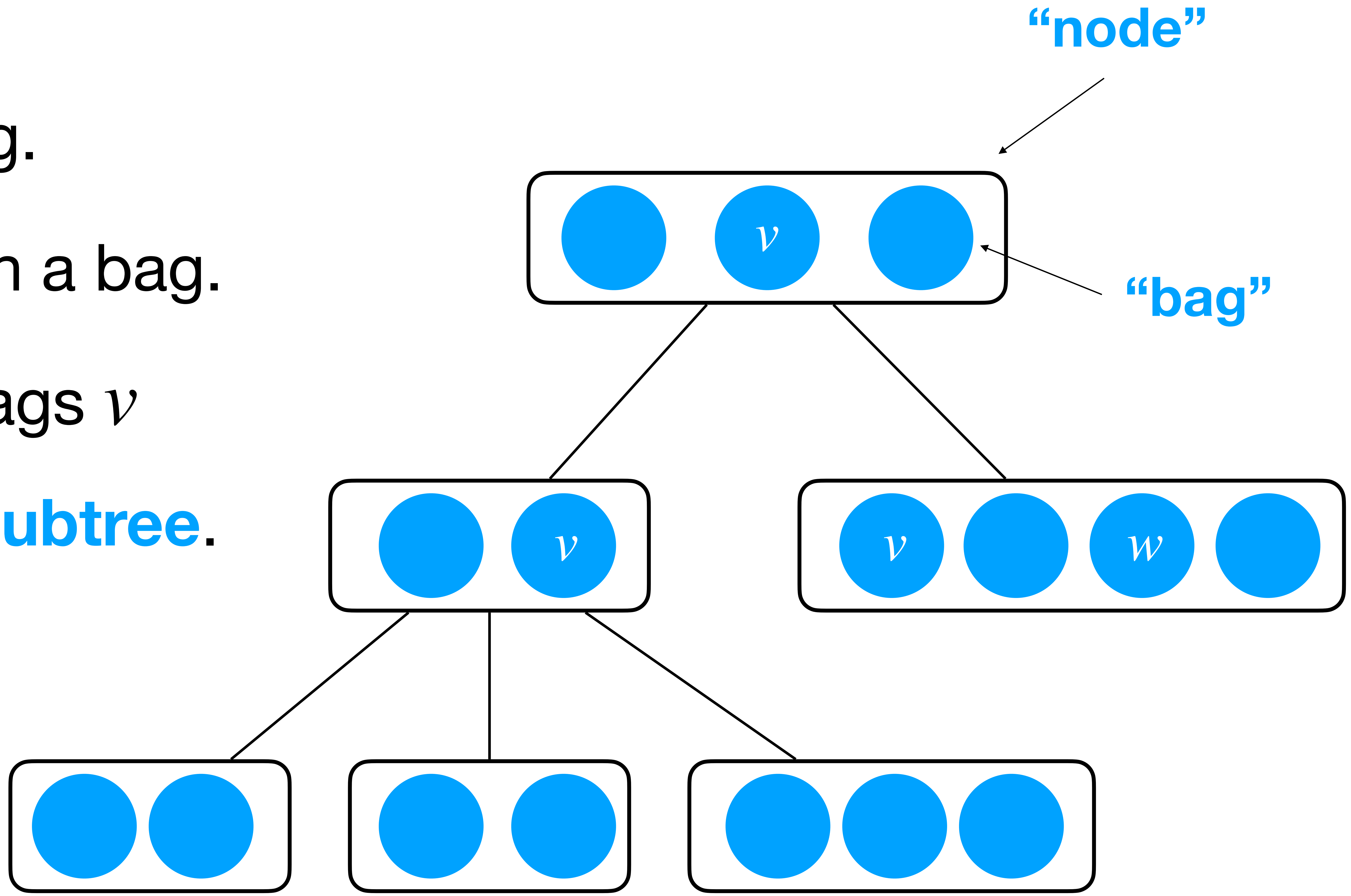
The **width** of a tree decomposition is the size of its largest bag - 1.



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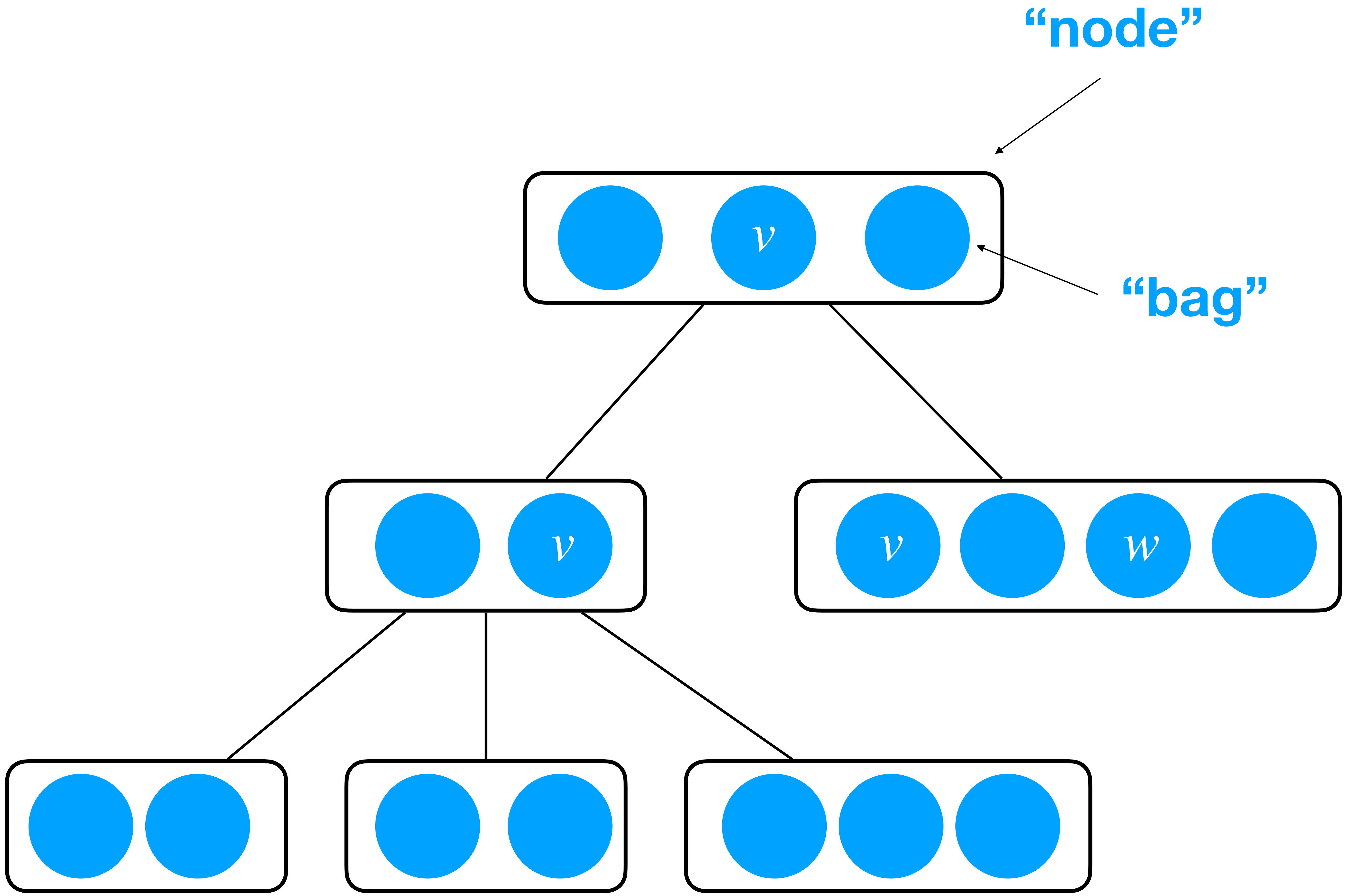
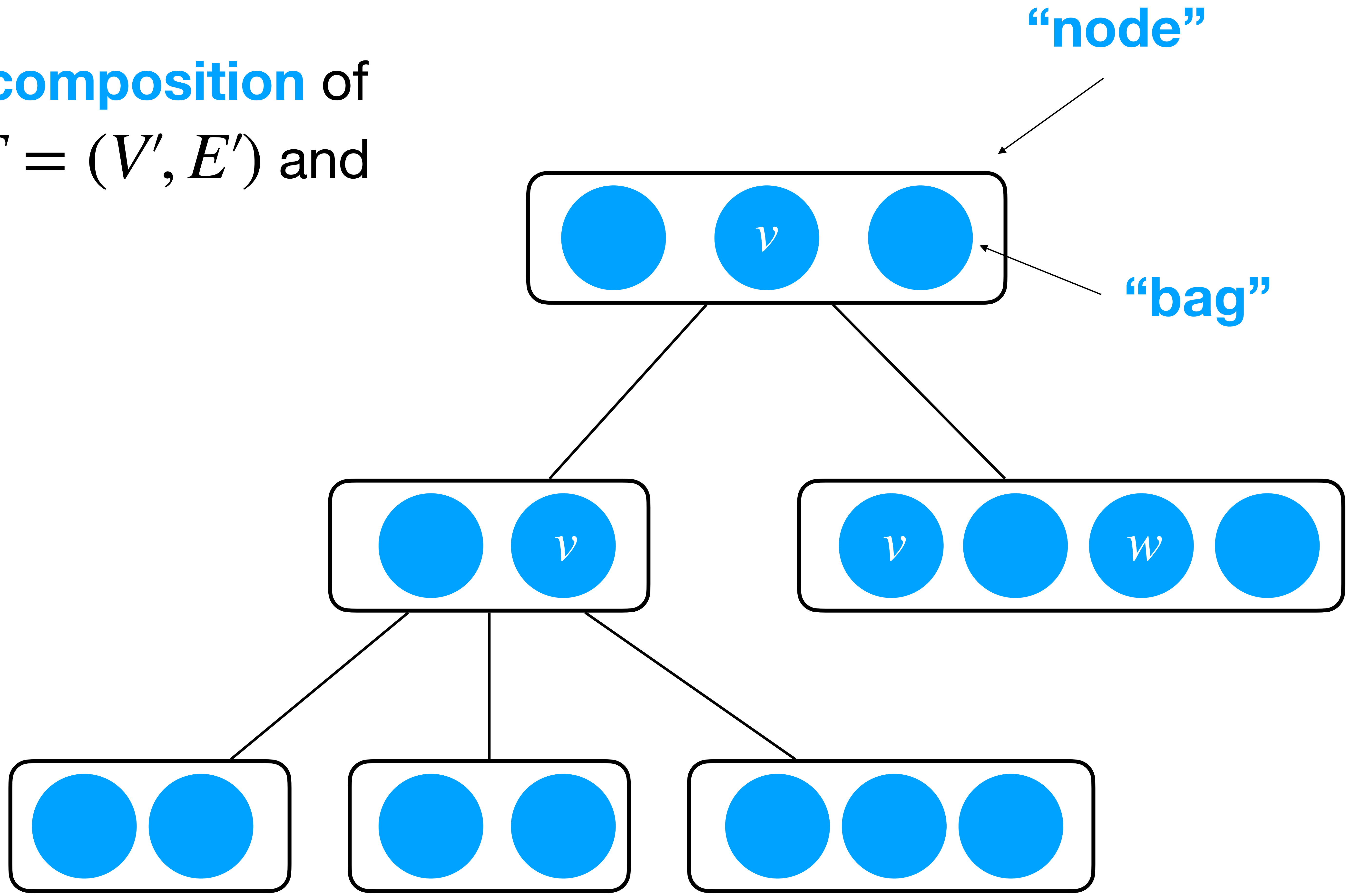


Tree Decomposition

Definition

Let $G = (V, E)$ be a graph. A **tree decomposition** of G is a pair (T, χ) consisting of a tree $T = (V', E')$ and a mapping $\chi : V' \to 2^V$ such that



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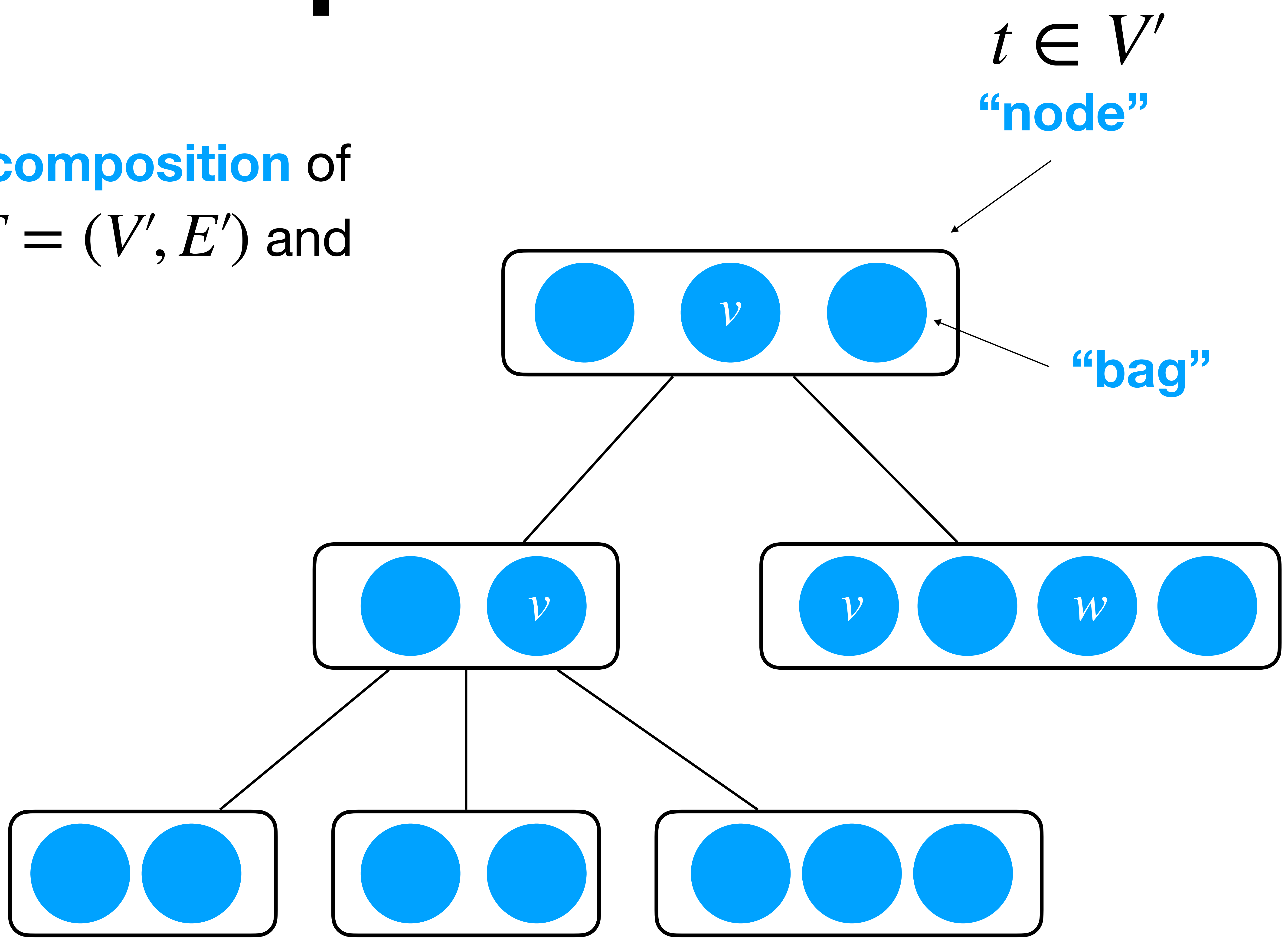

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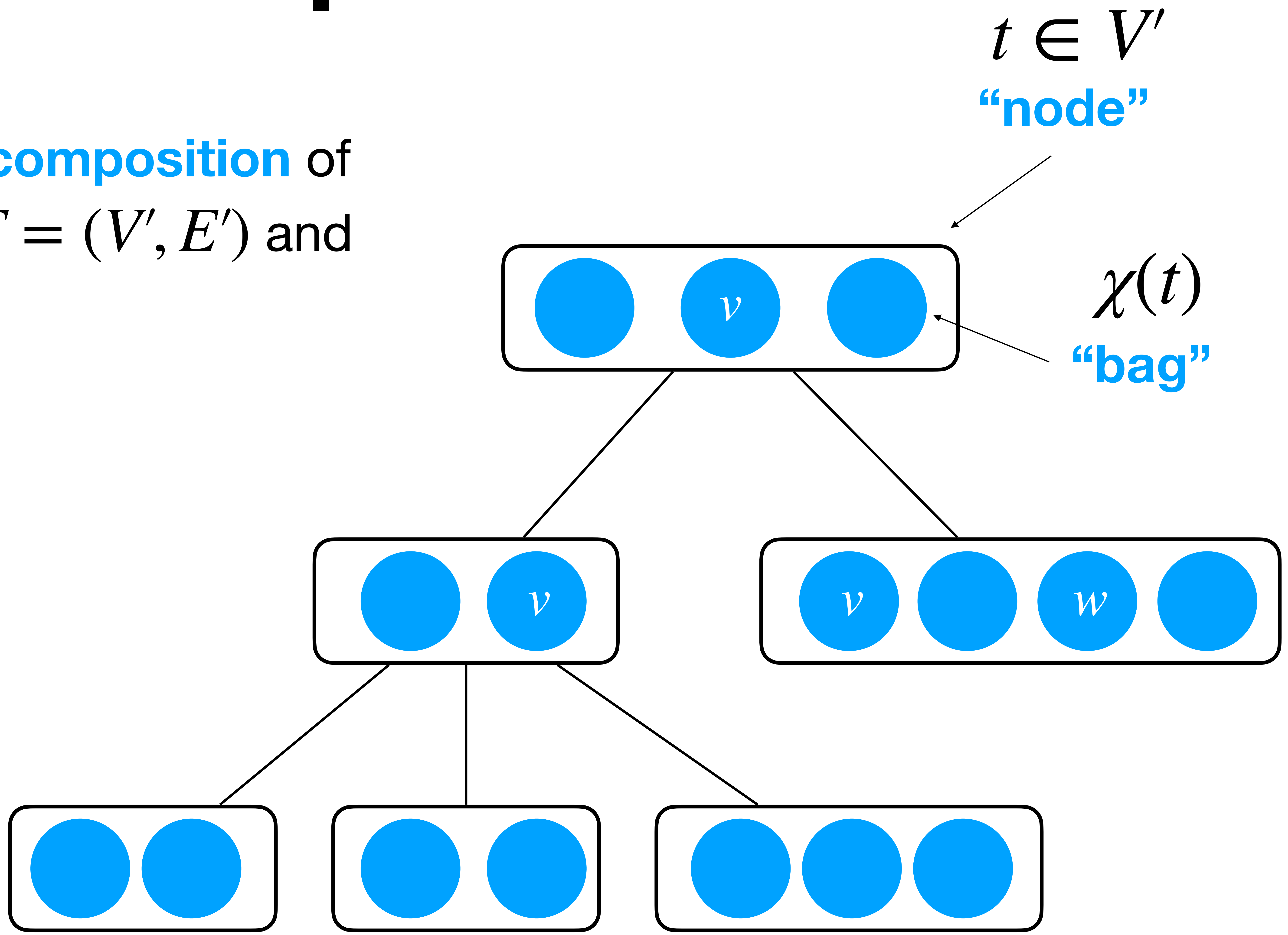

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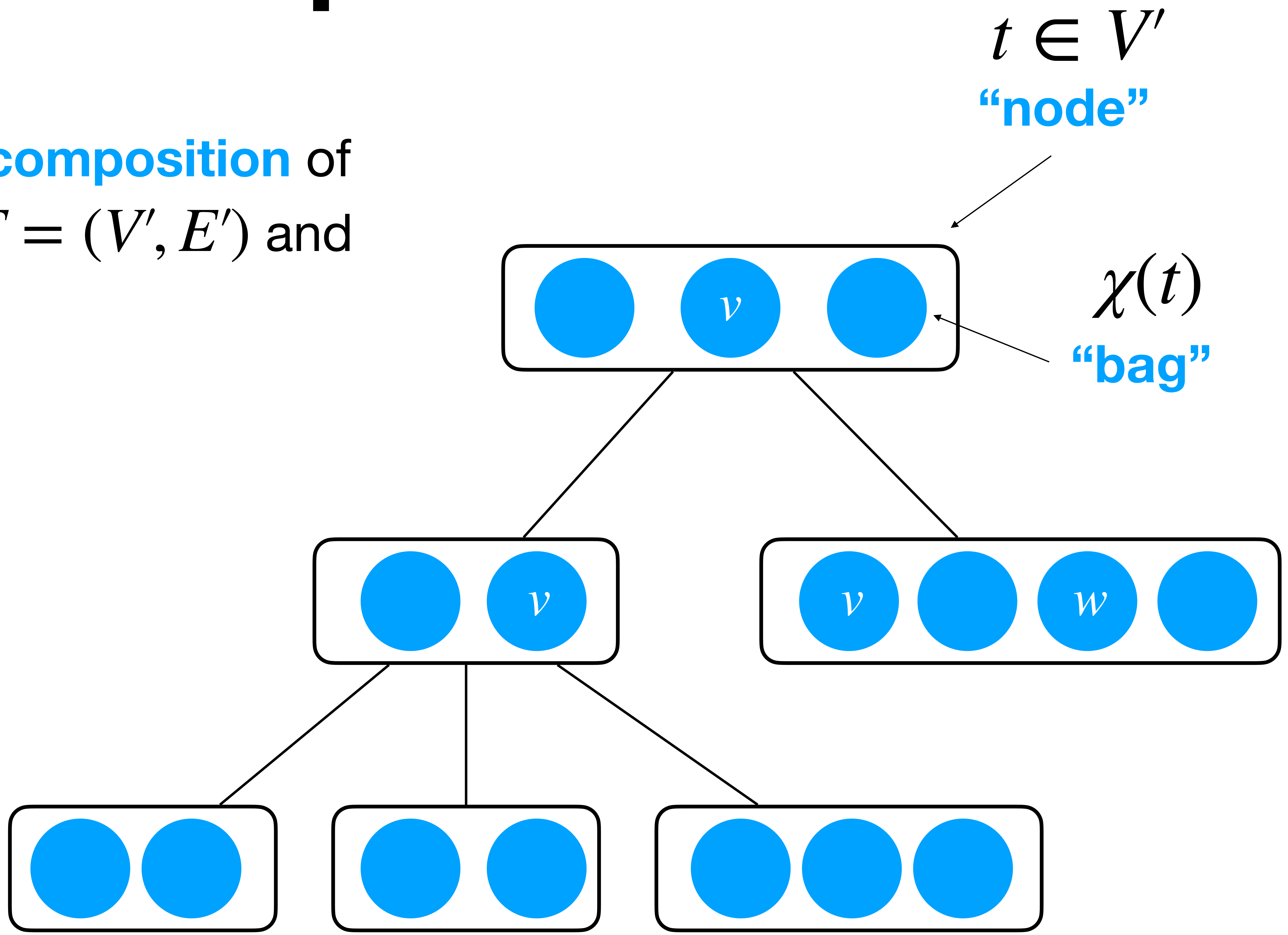

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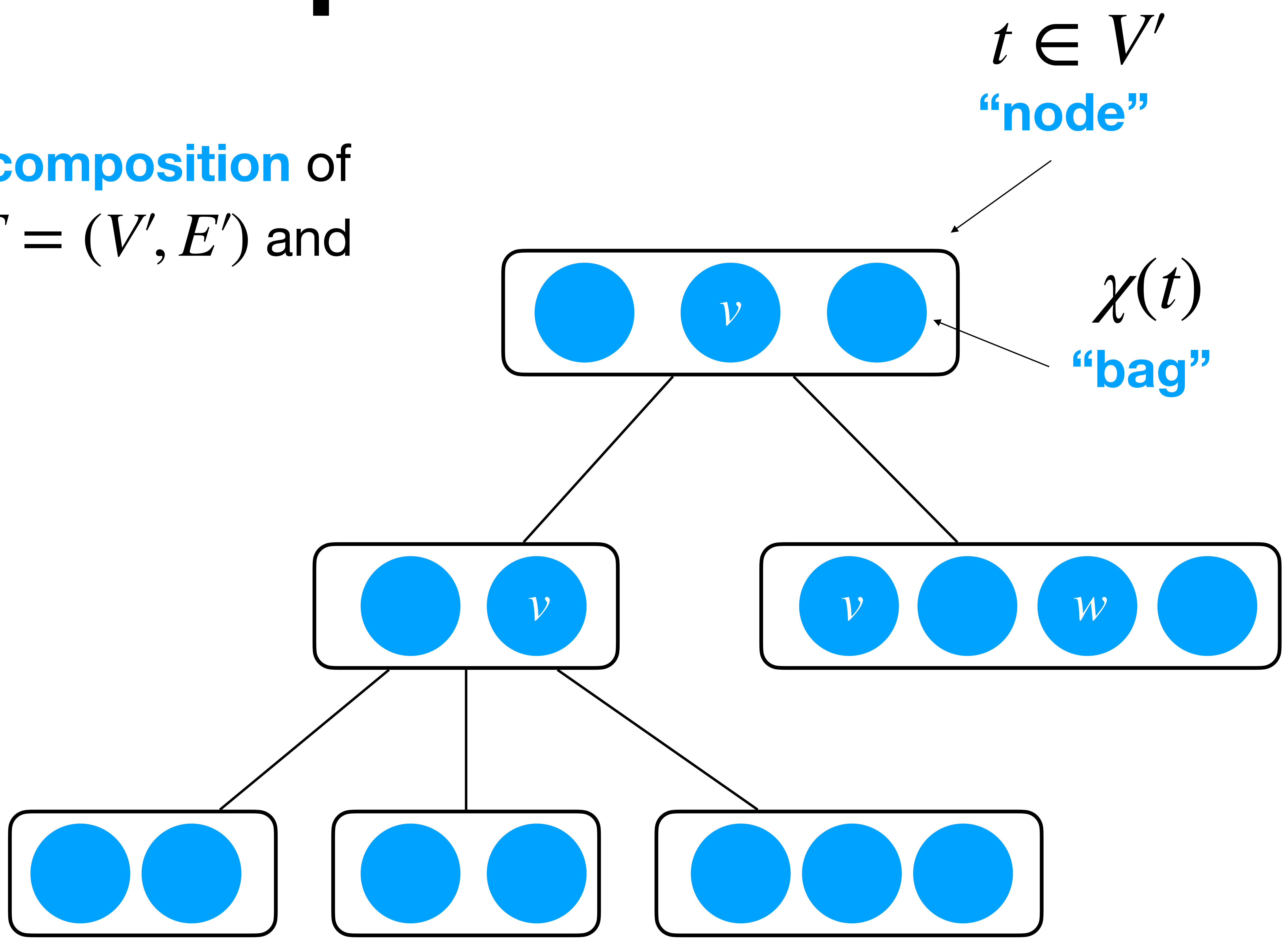

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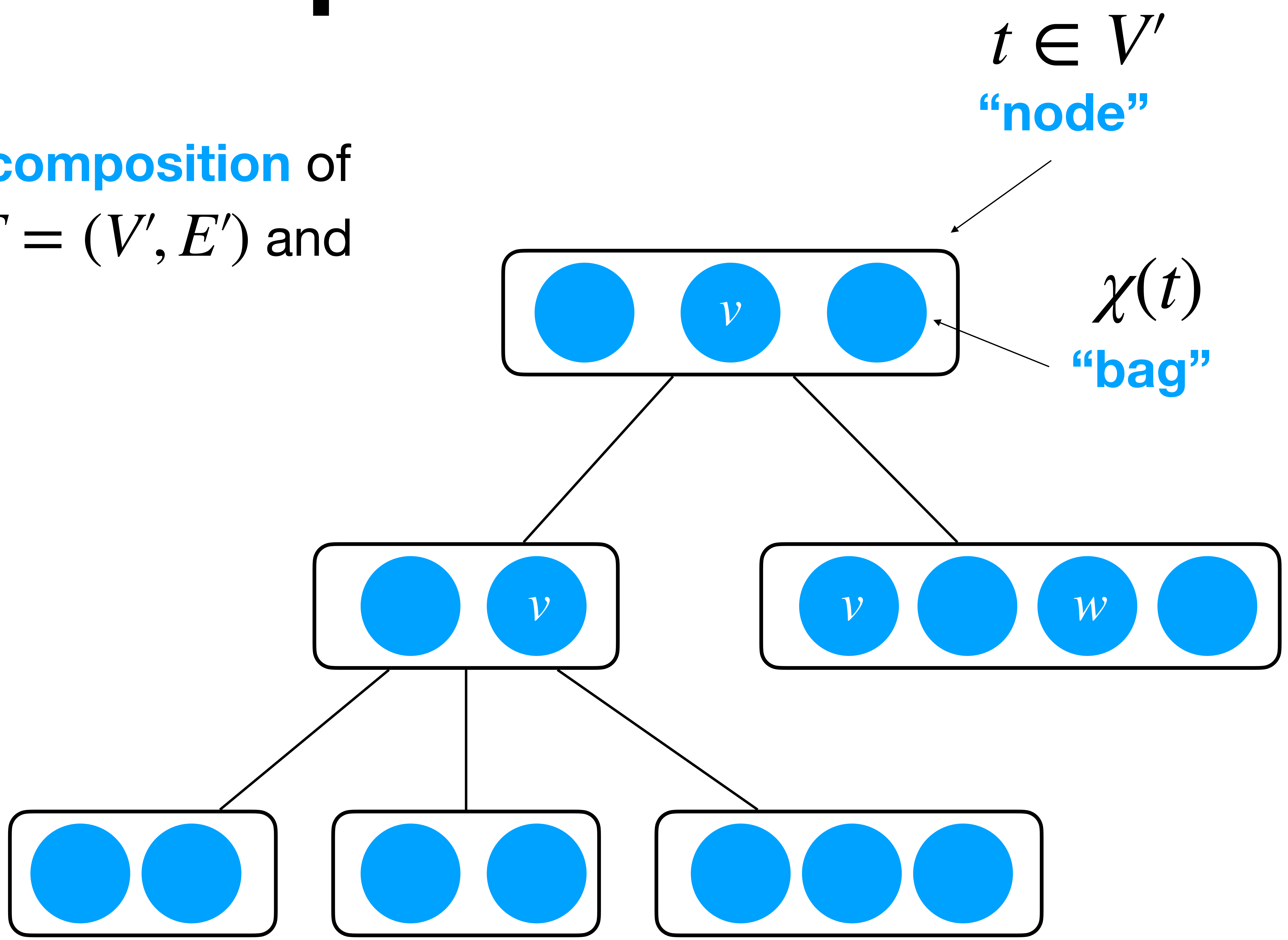

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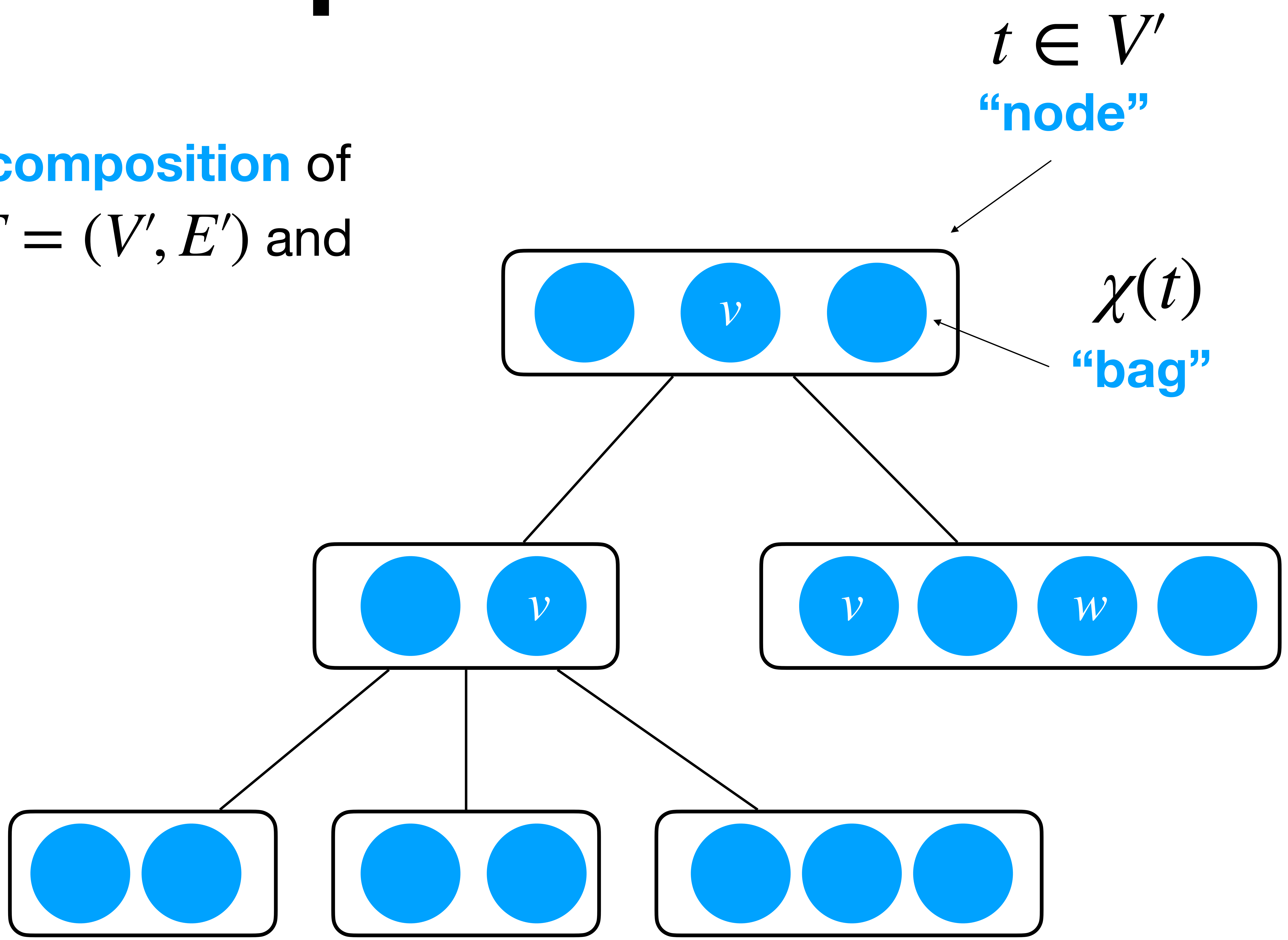
The **width** of (T, χ) is $\max_{t \in V'} |\chi(t)| - 1$.



Tree Decompositions and Treewidth

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Fact

A graph has a tree decomposition of width k if, and only if, it has an elimination ordering of width k.

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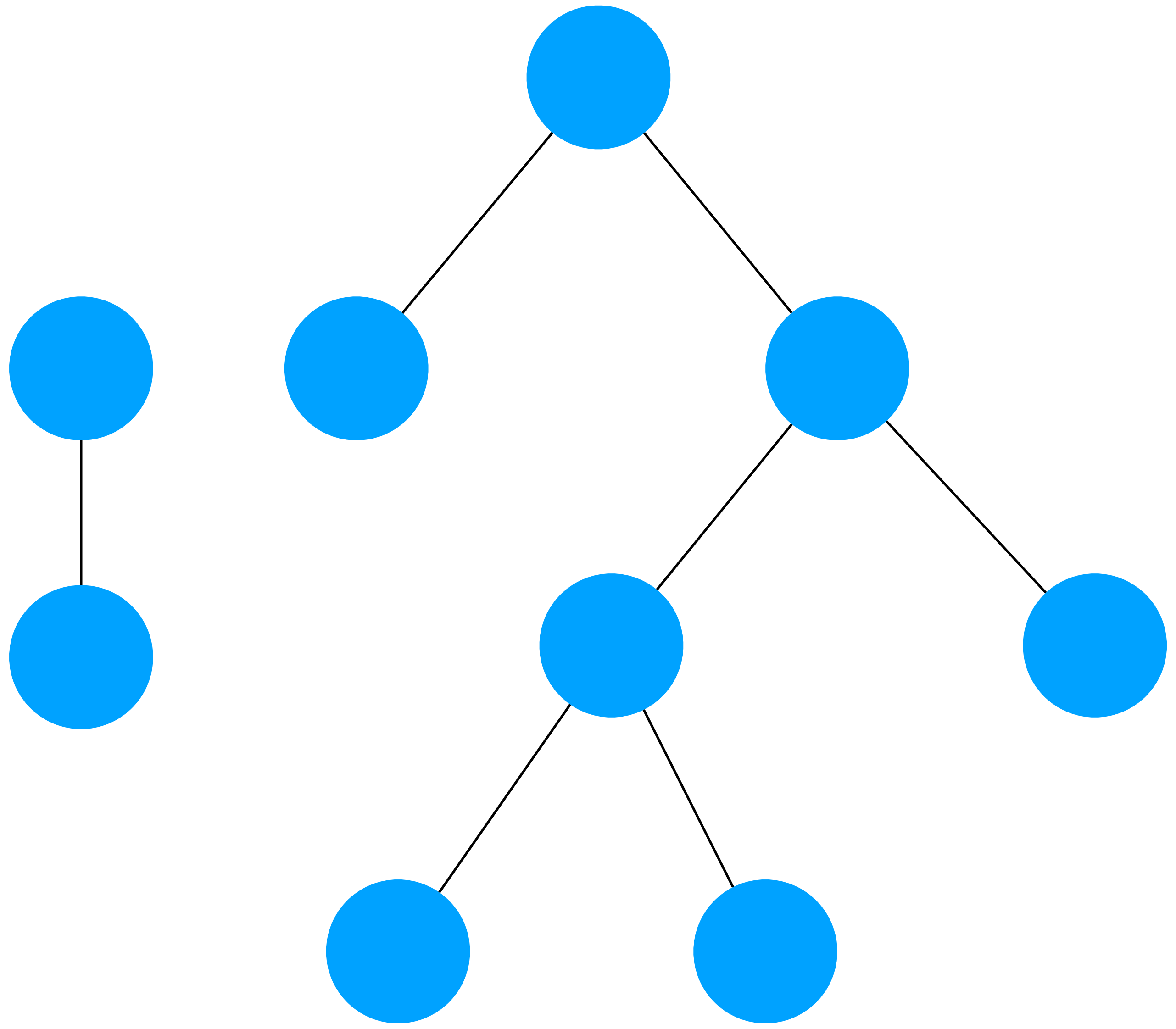
The treewidth of a graph G is the minimum width of a tree decomposition of G.

Examples

Examples

Forests



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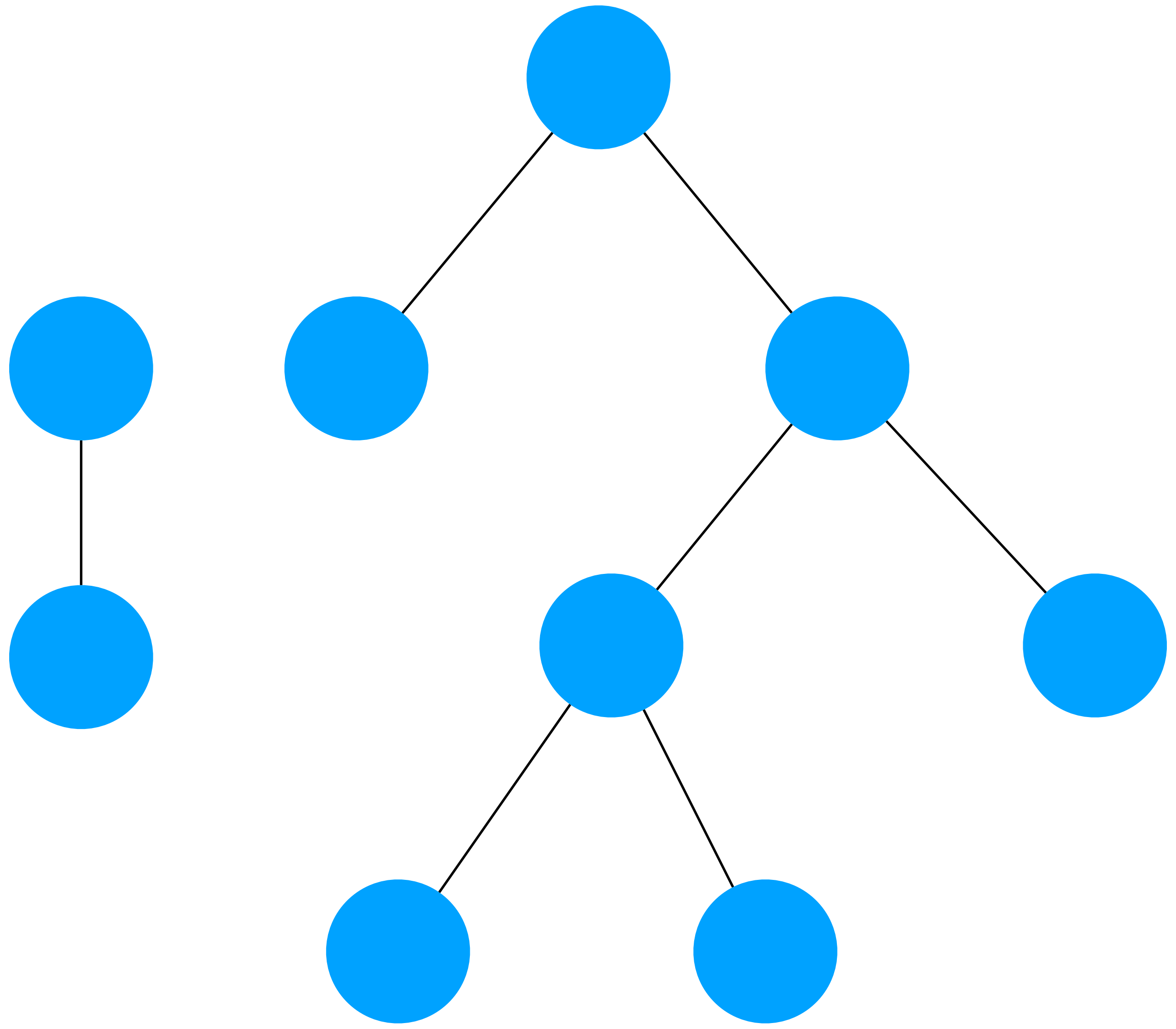

treewidth 1

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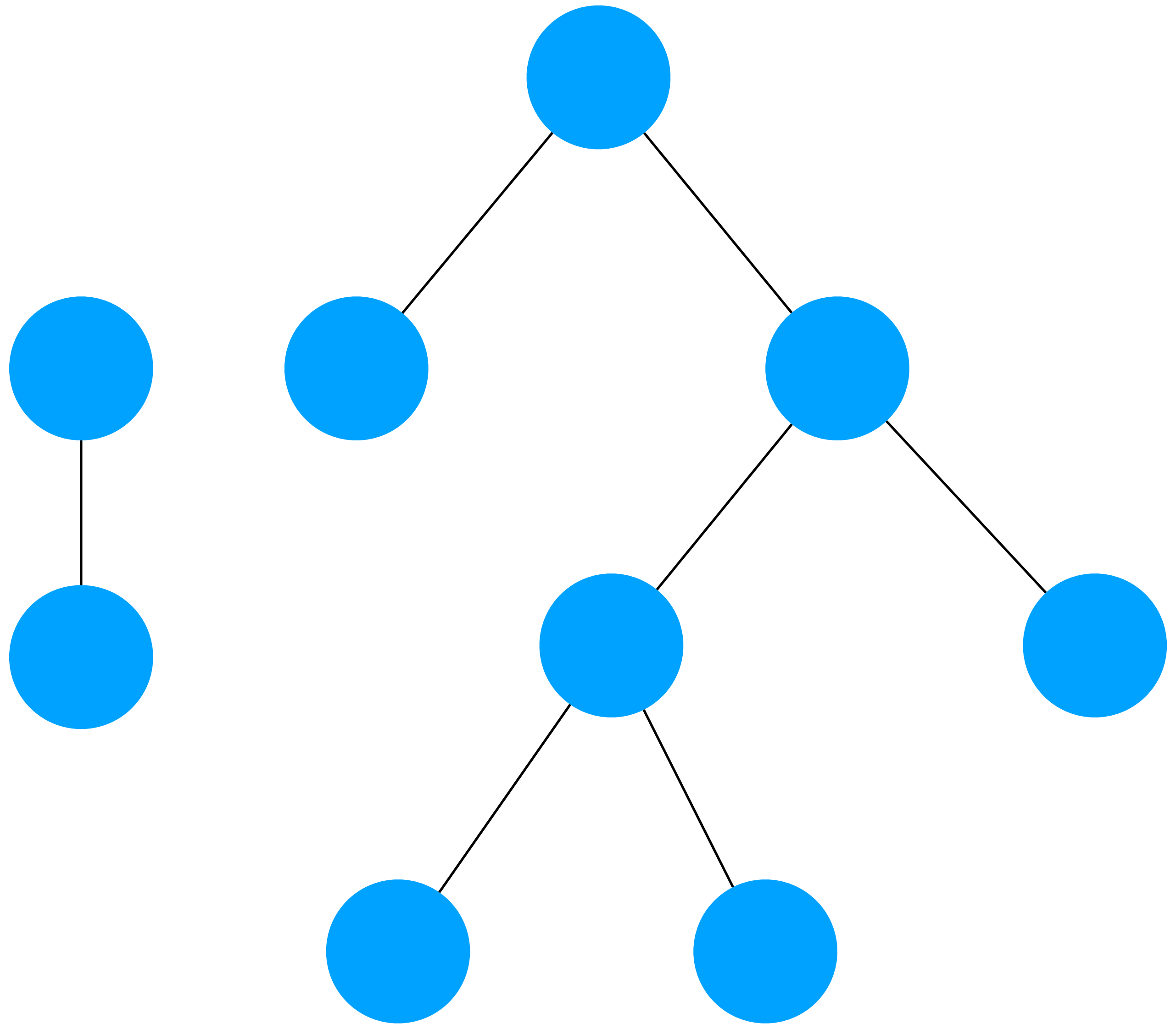
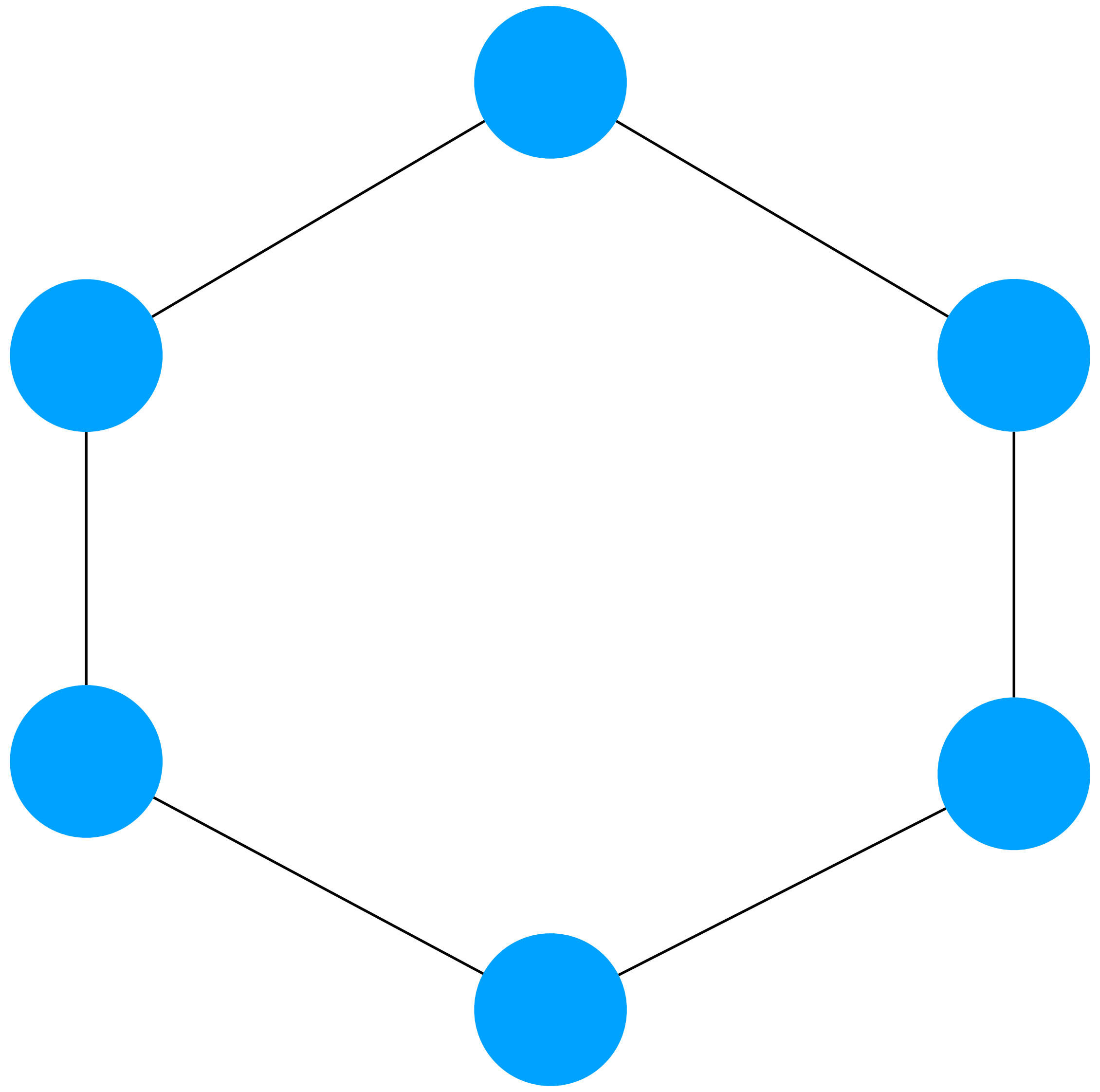


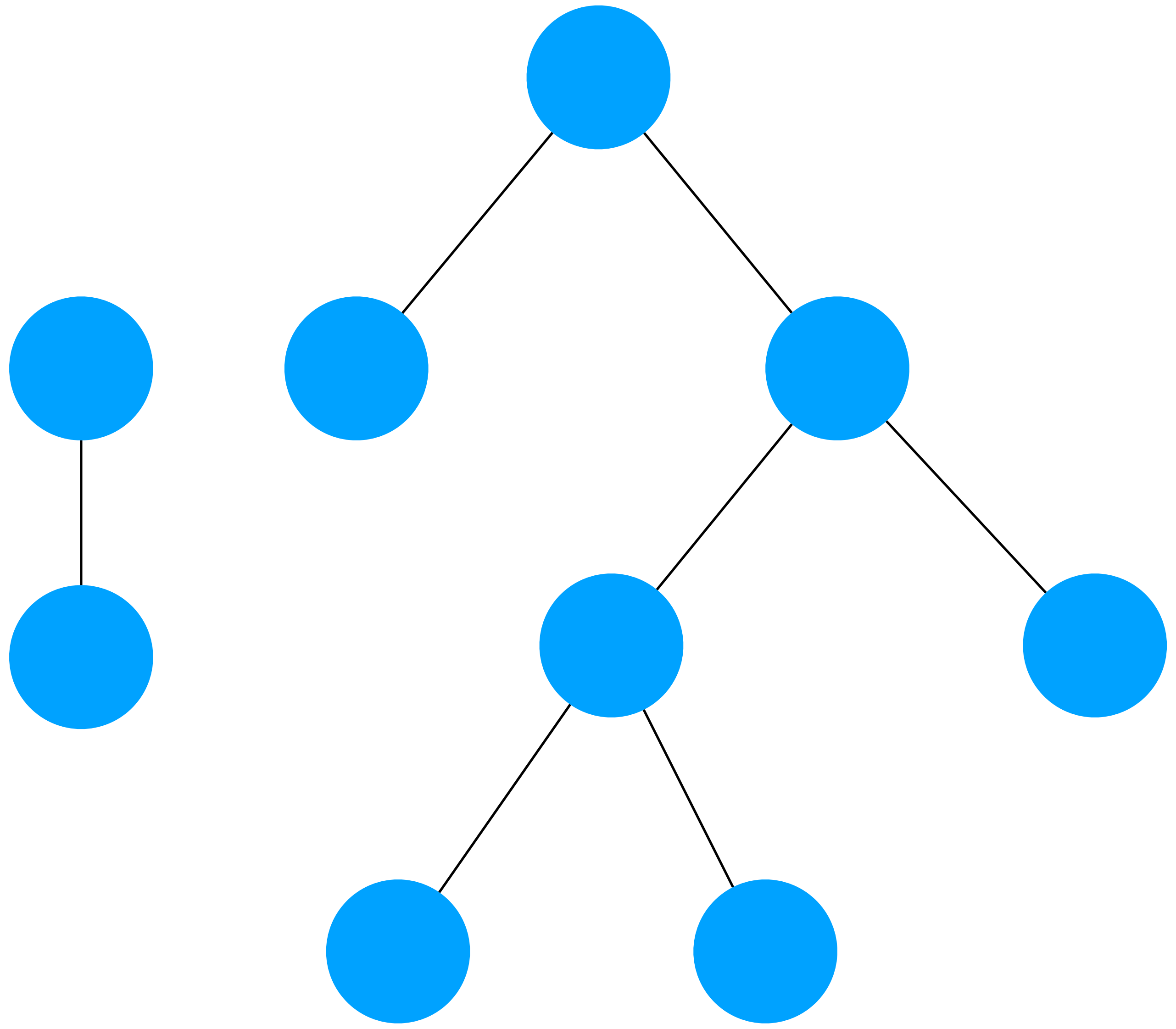
treewidth 1

Cycles



Examples

Forests



treewidth 1

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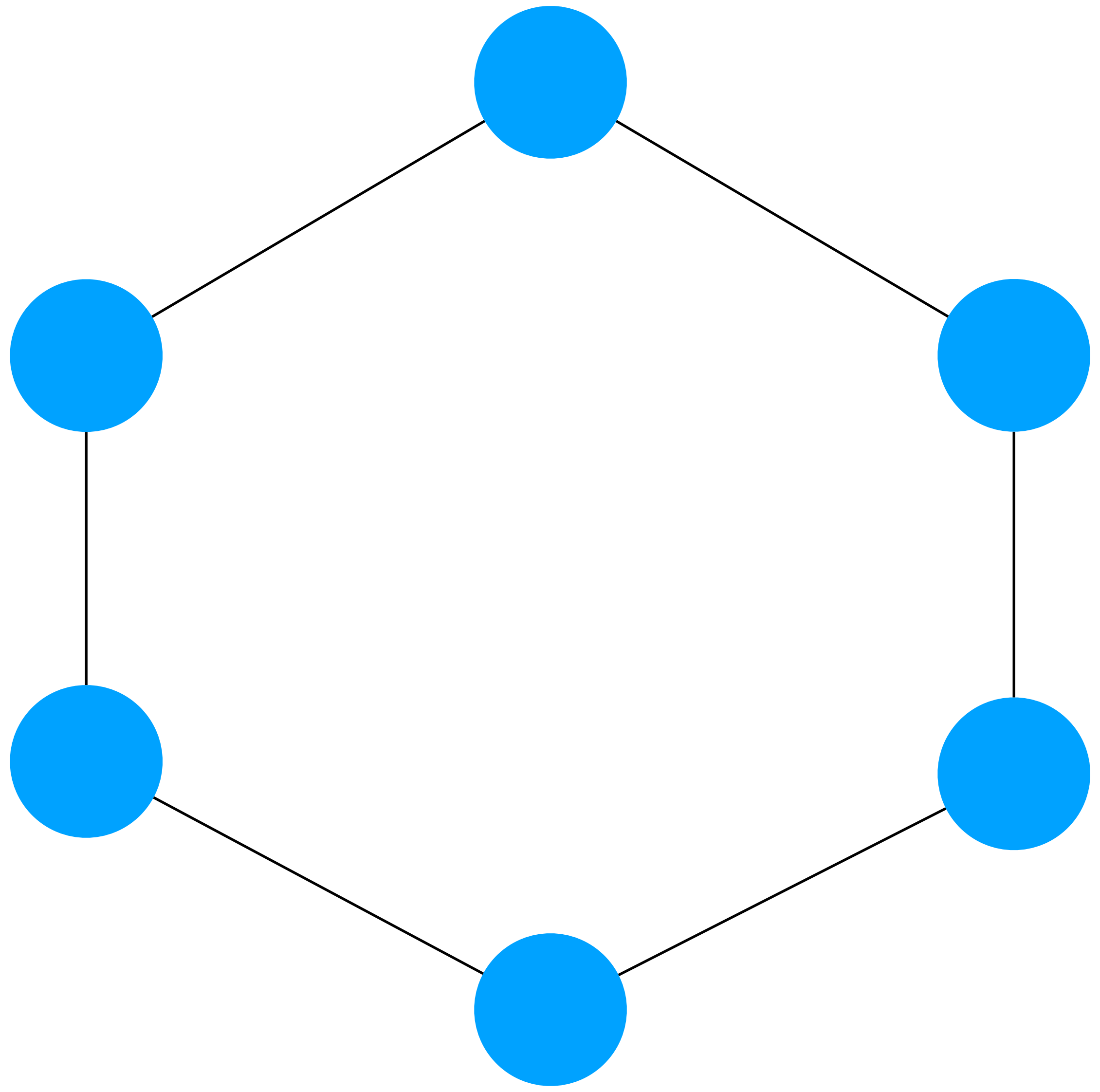
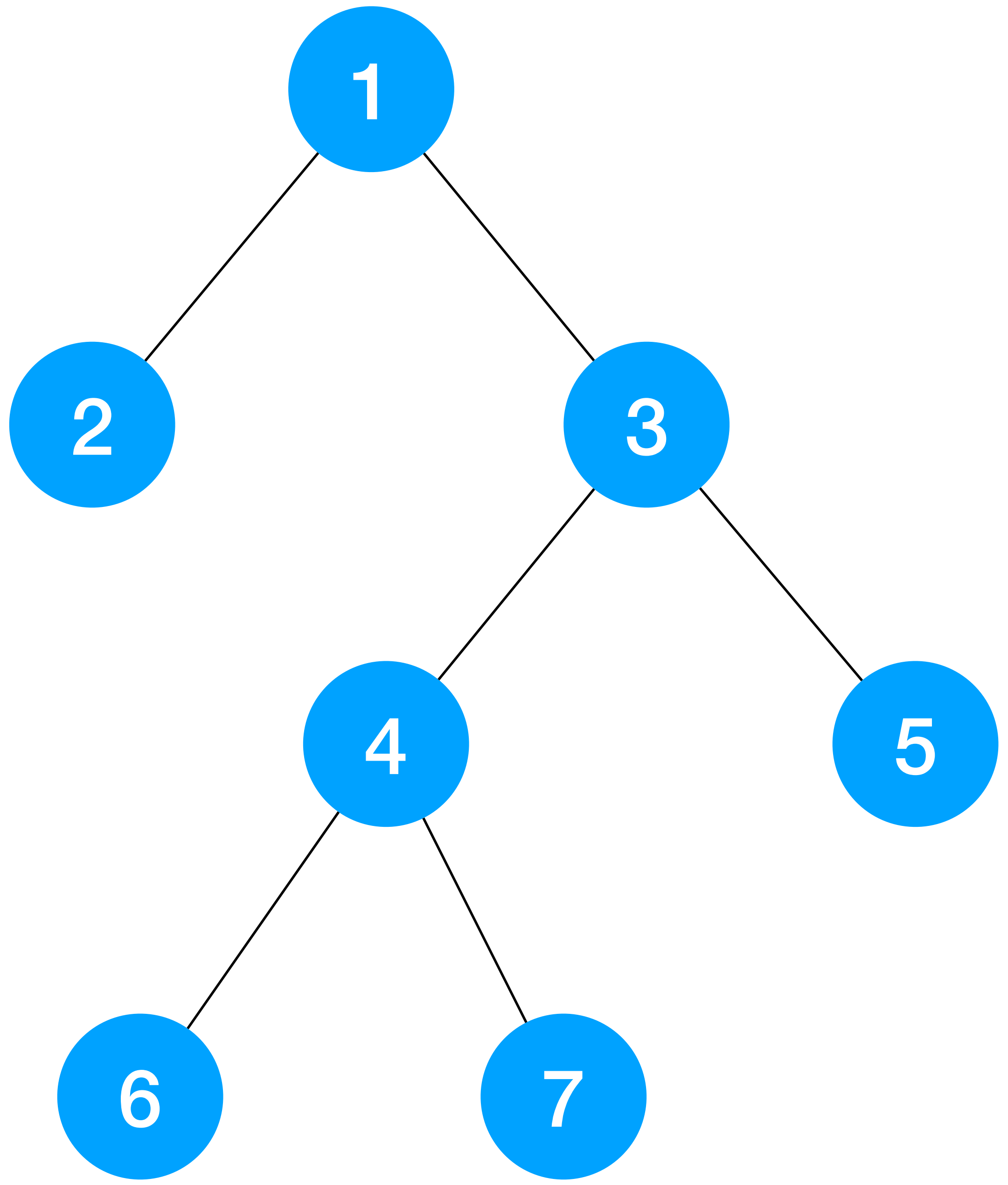
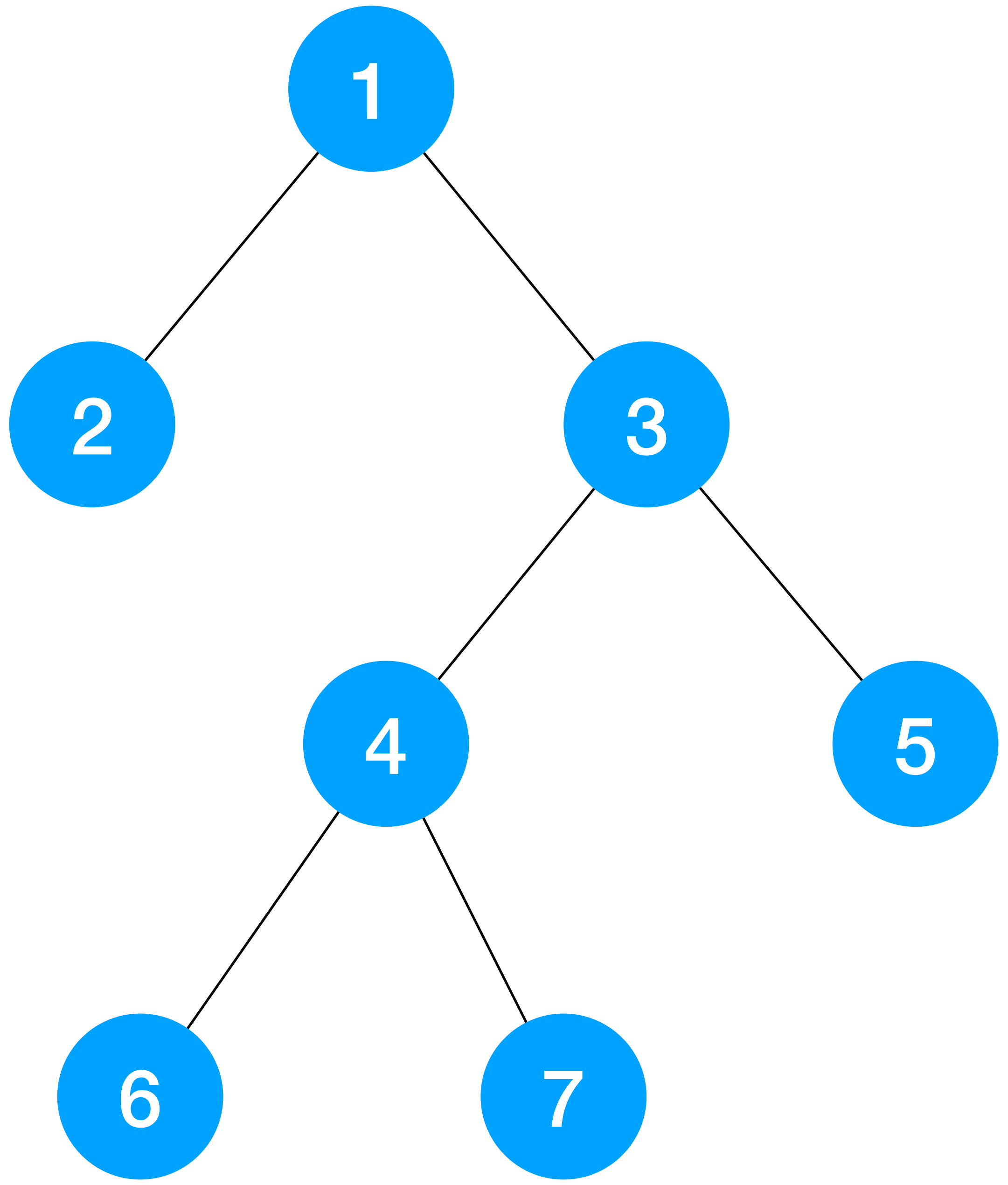


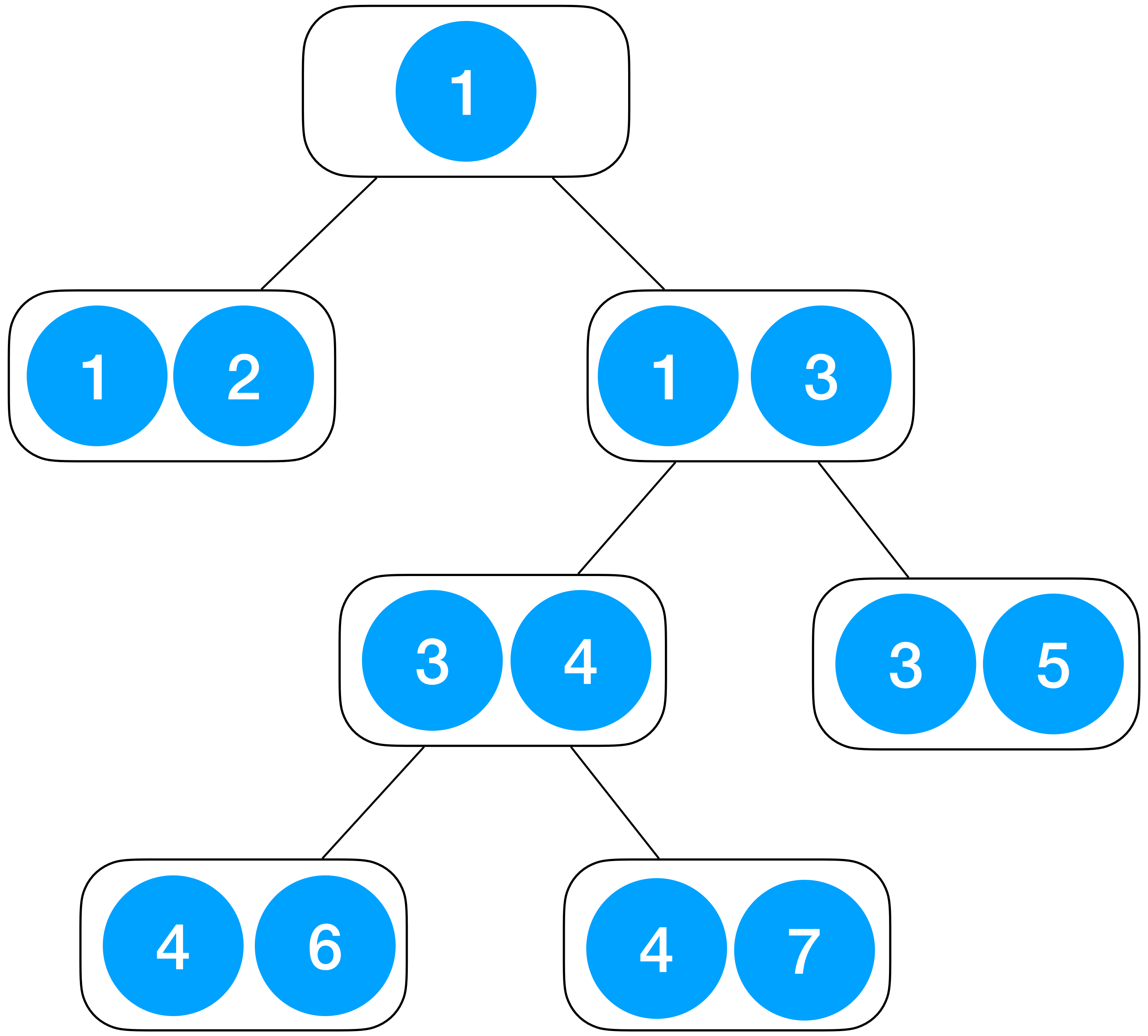
treewidth 2

Trees



Trees





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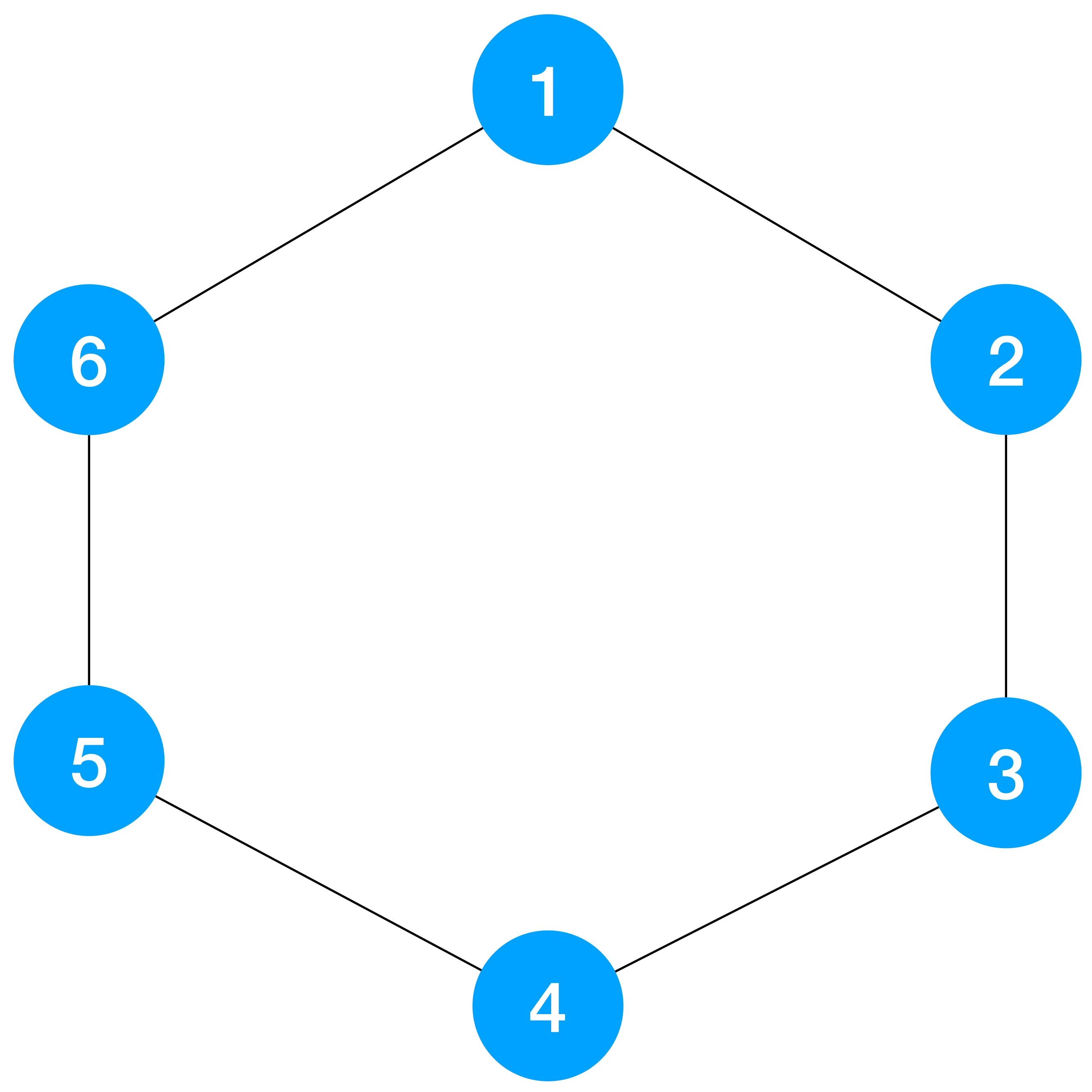

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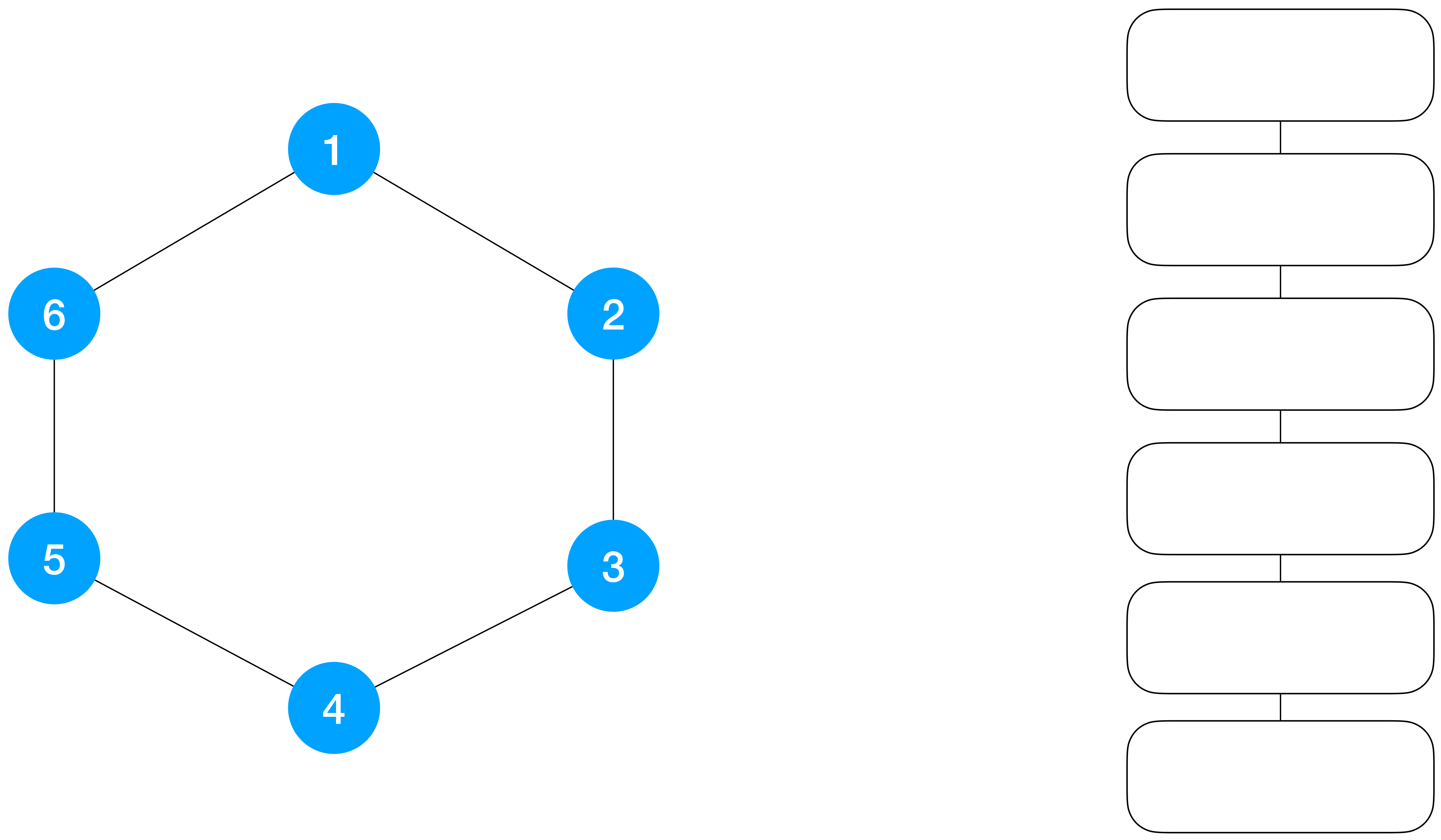

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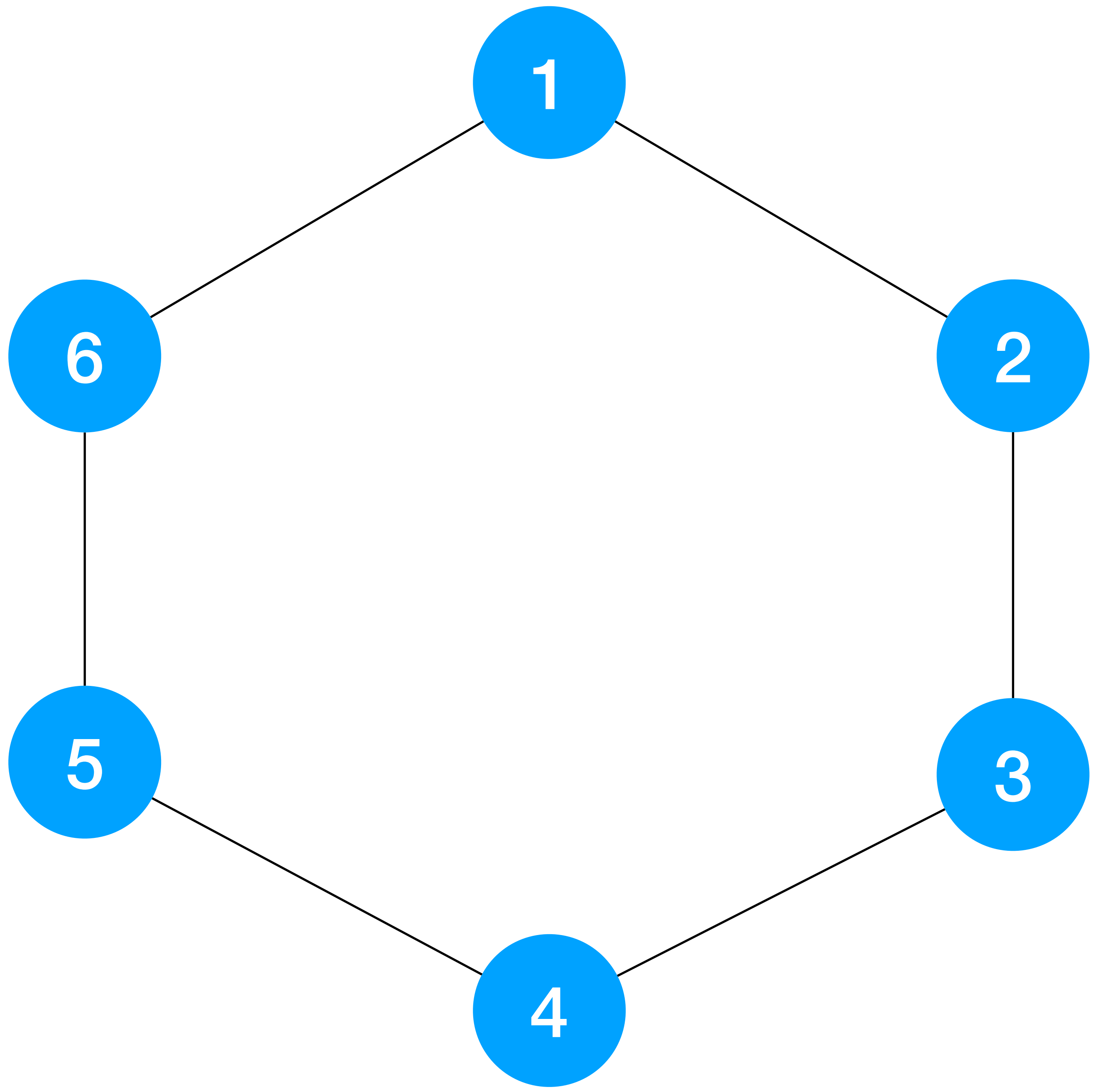

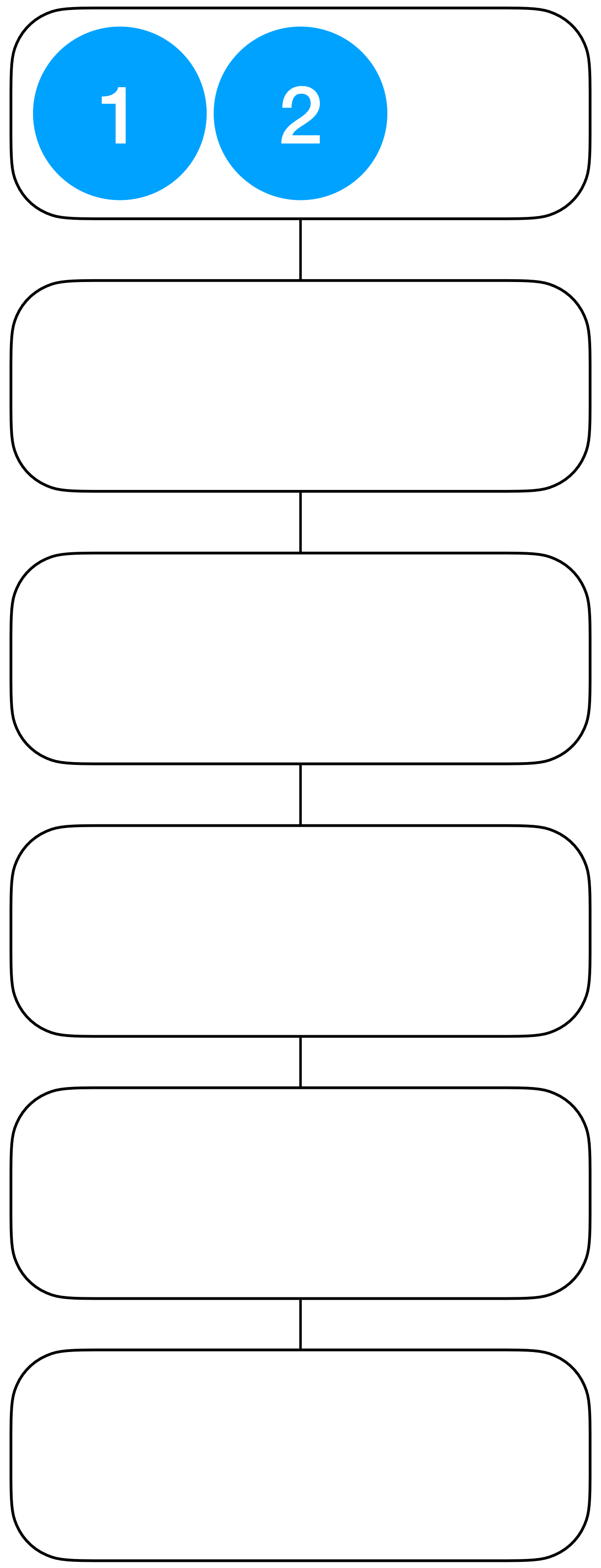

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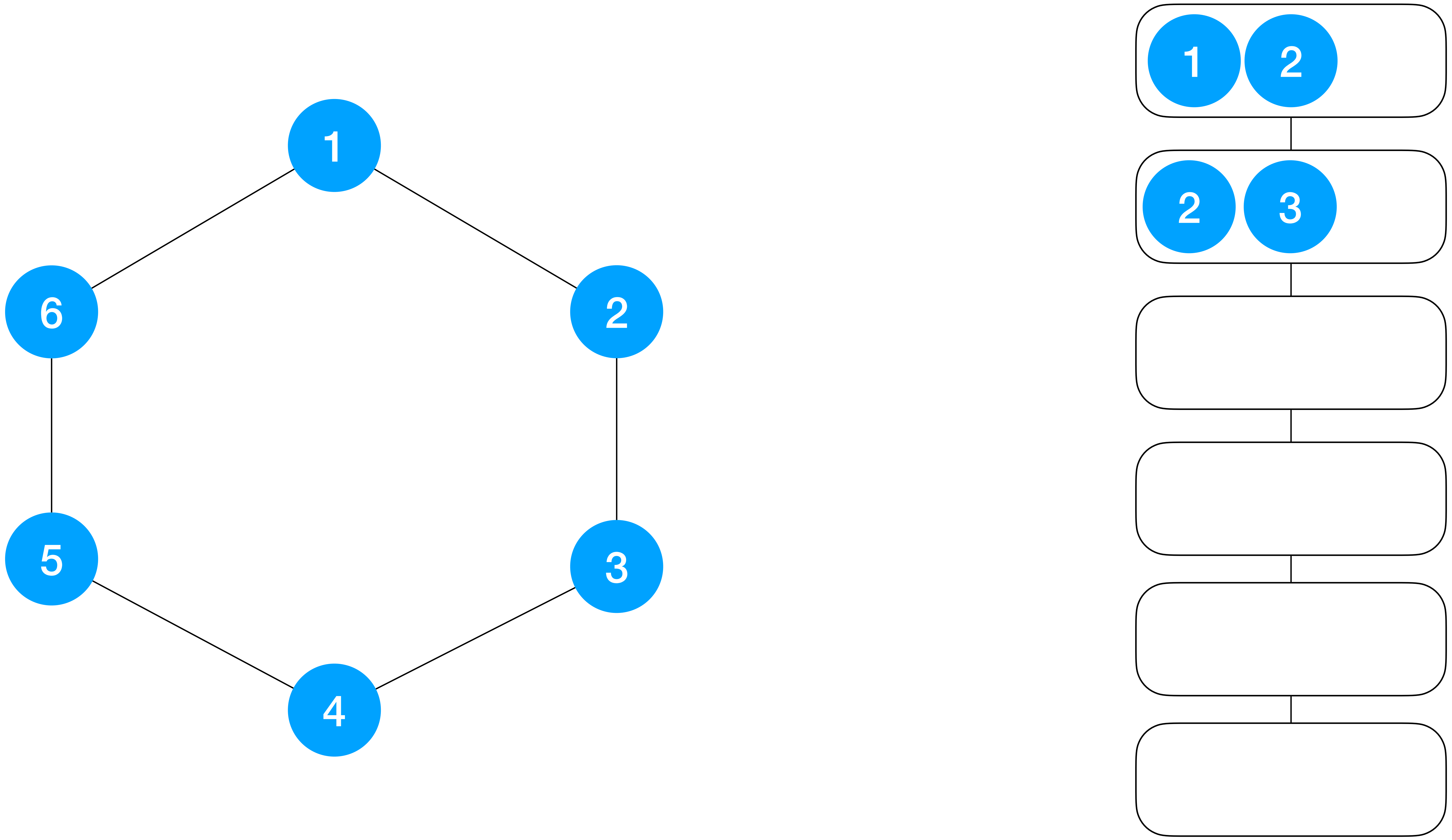

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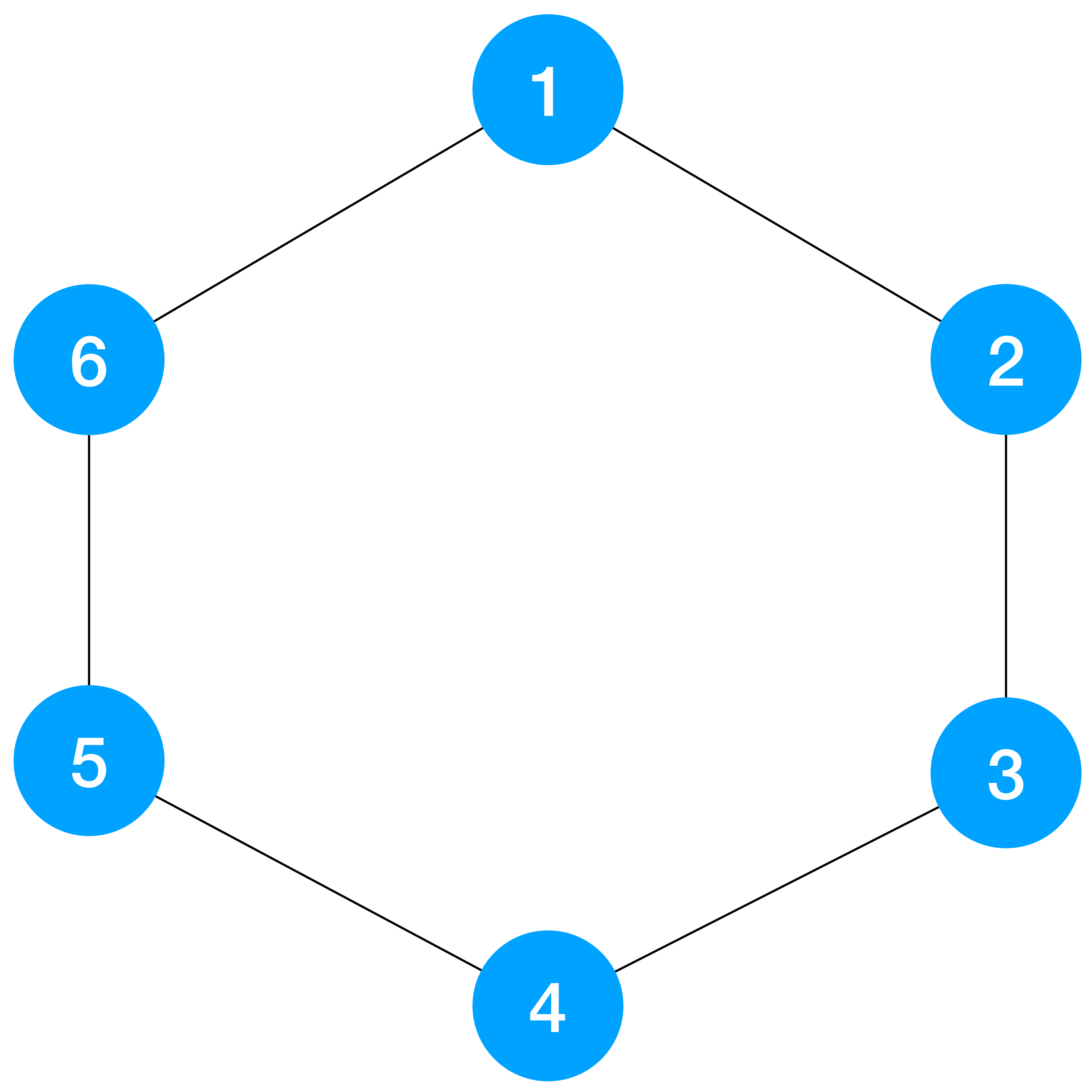

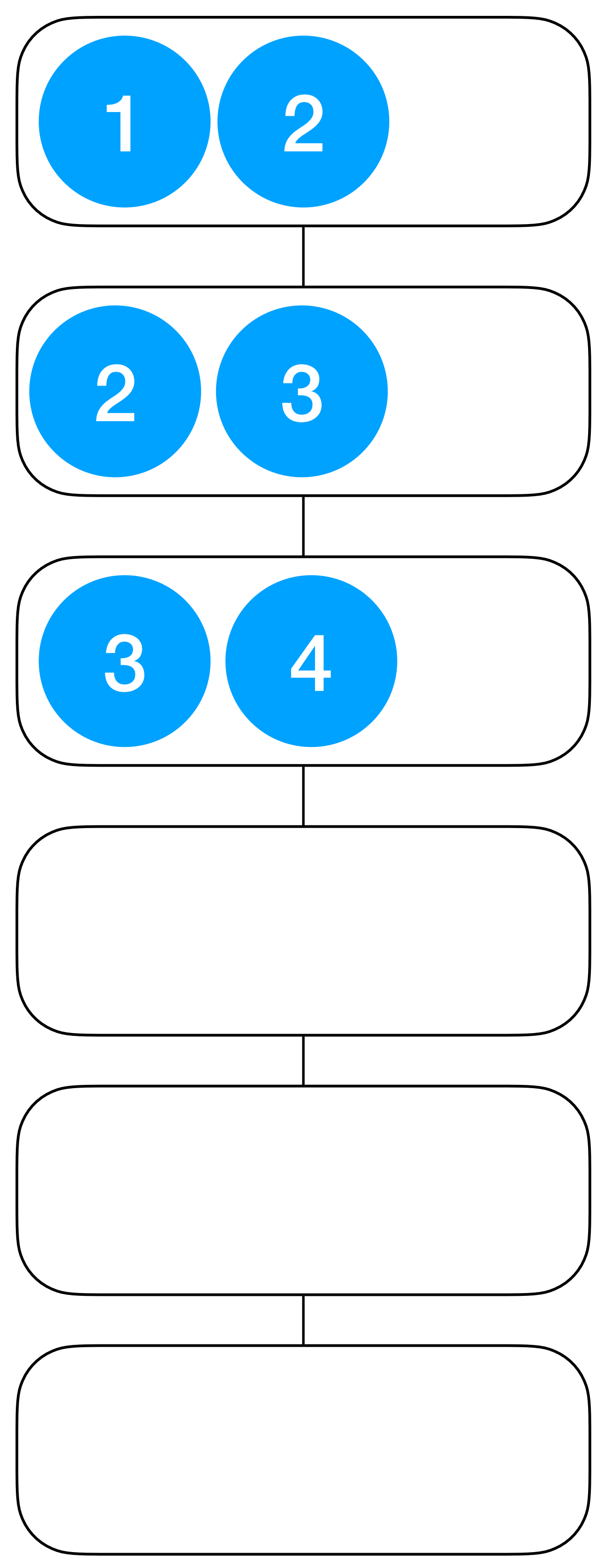

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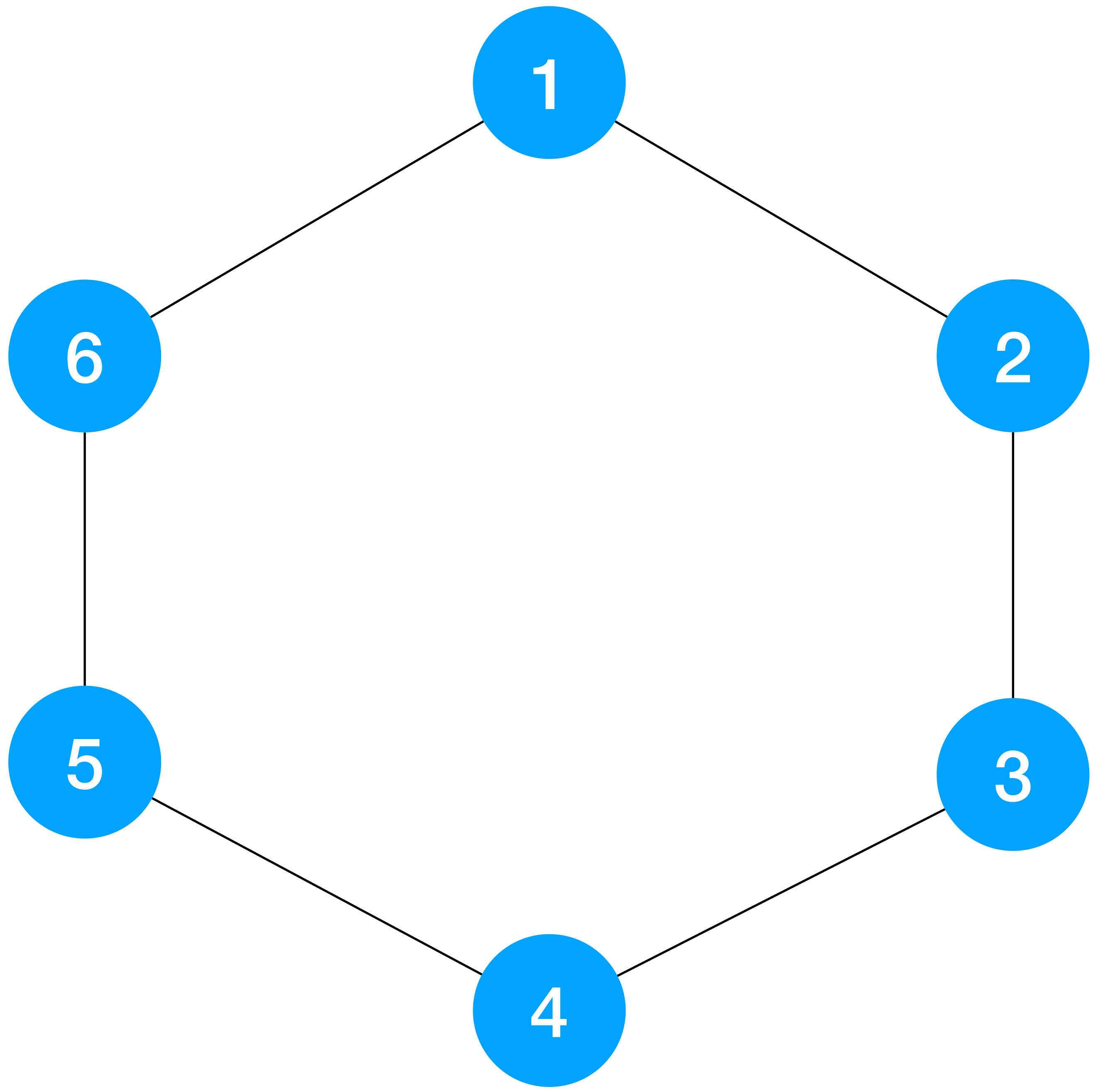

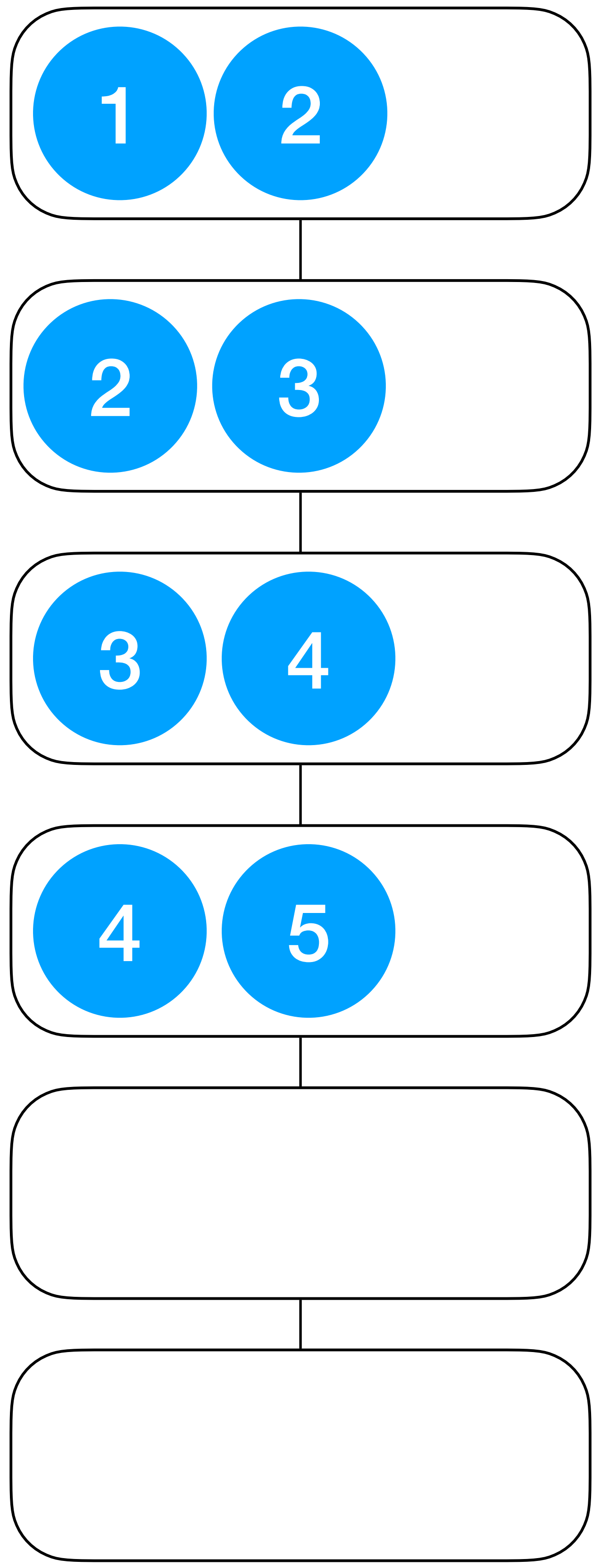

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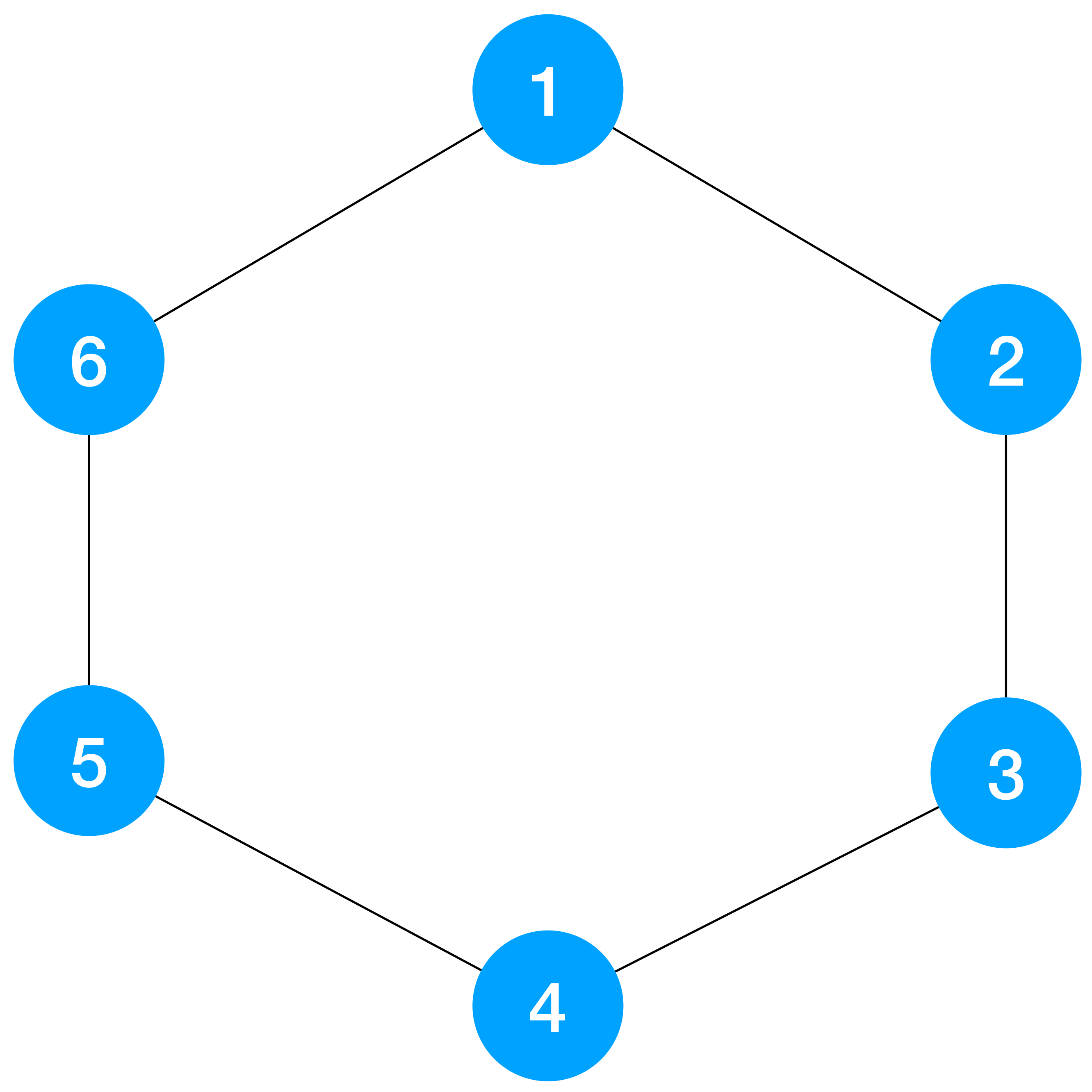

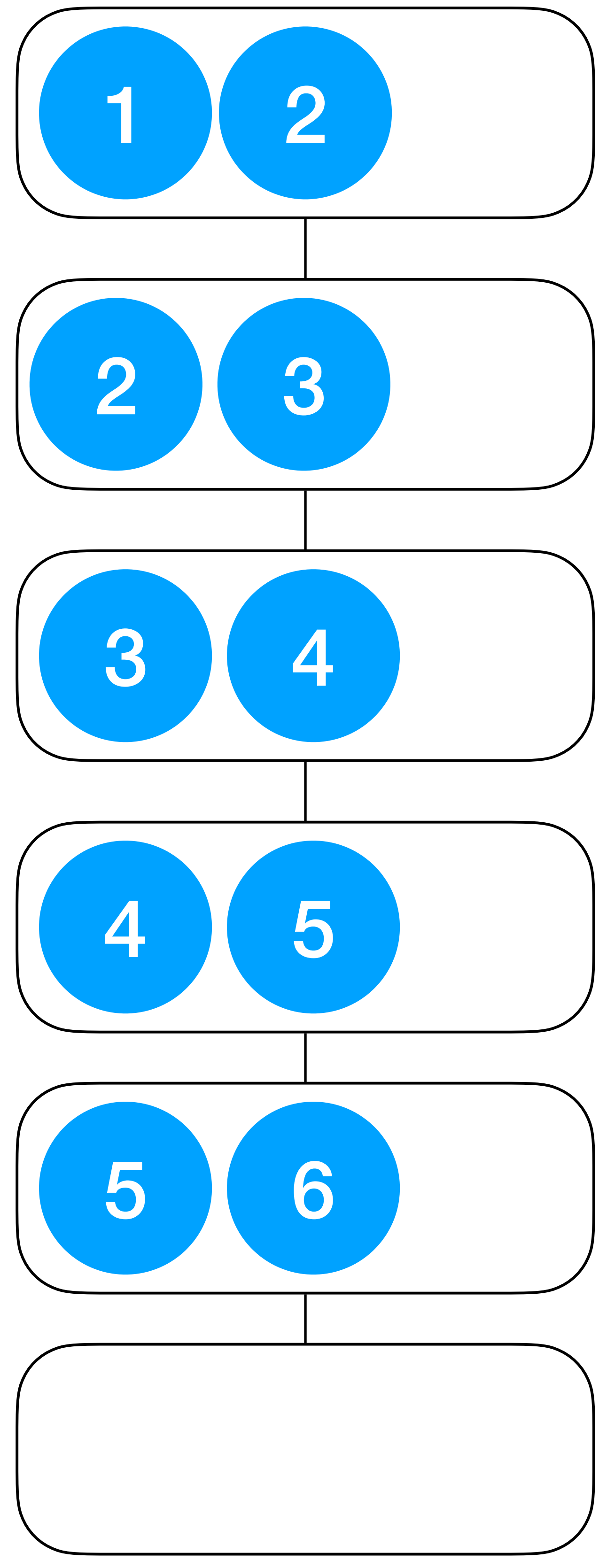

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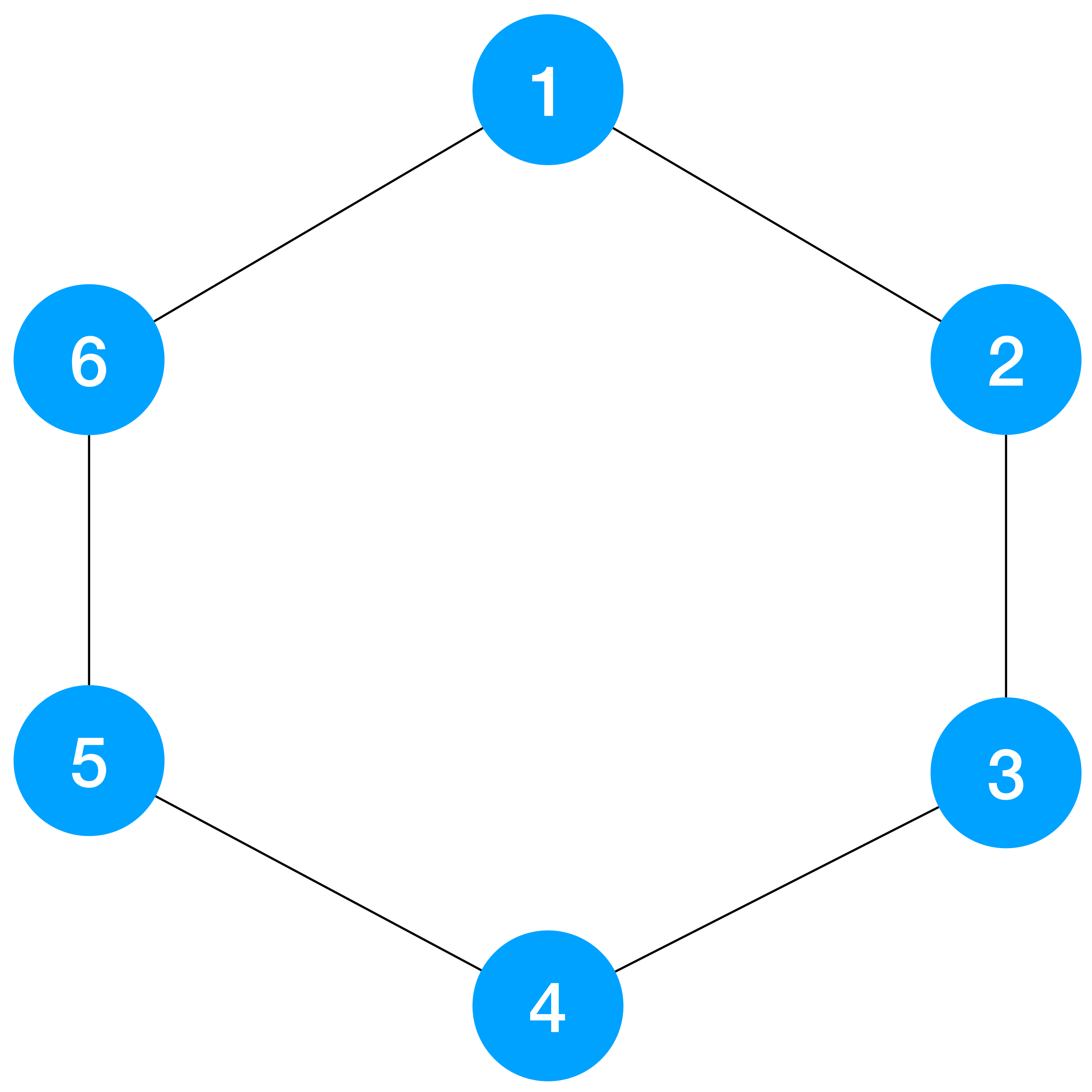

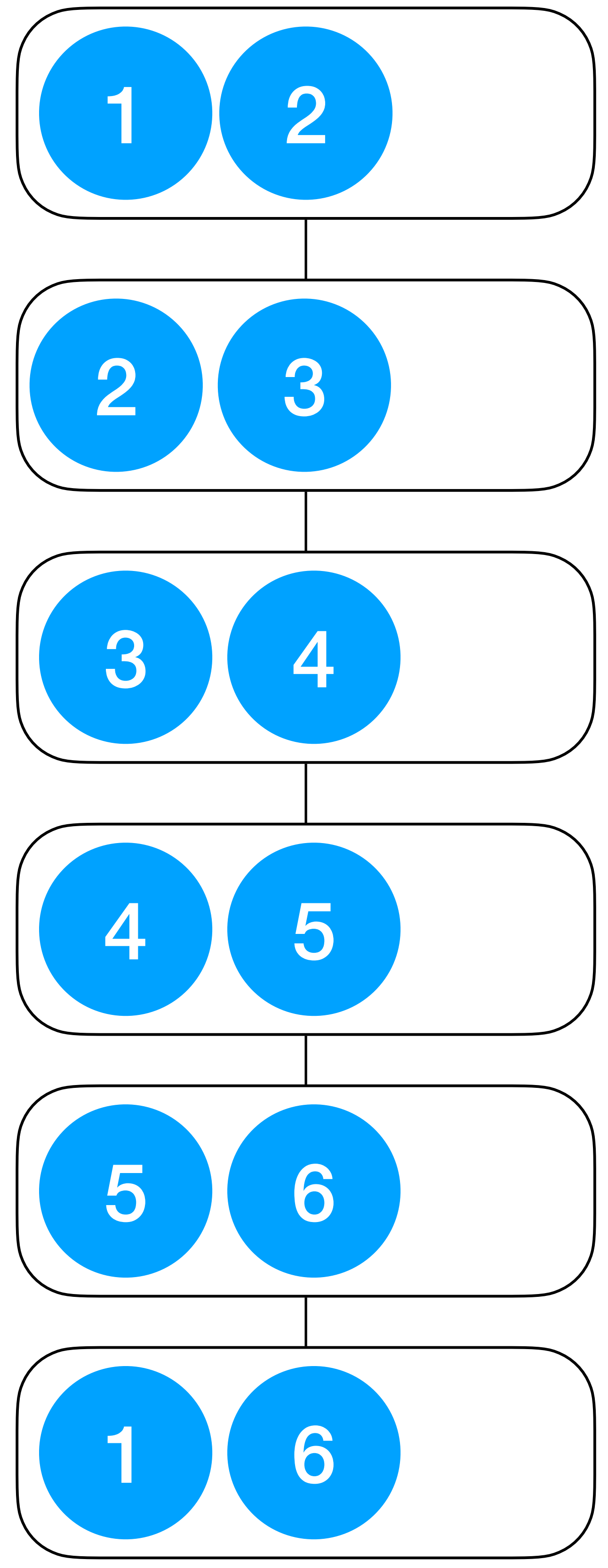

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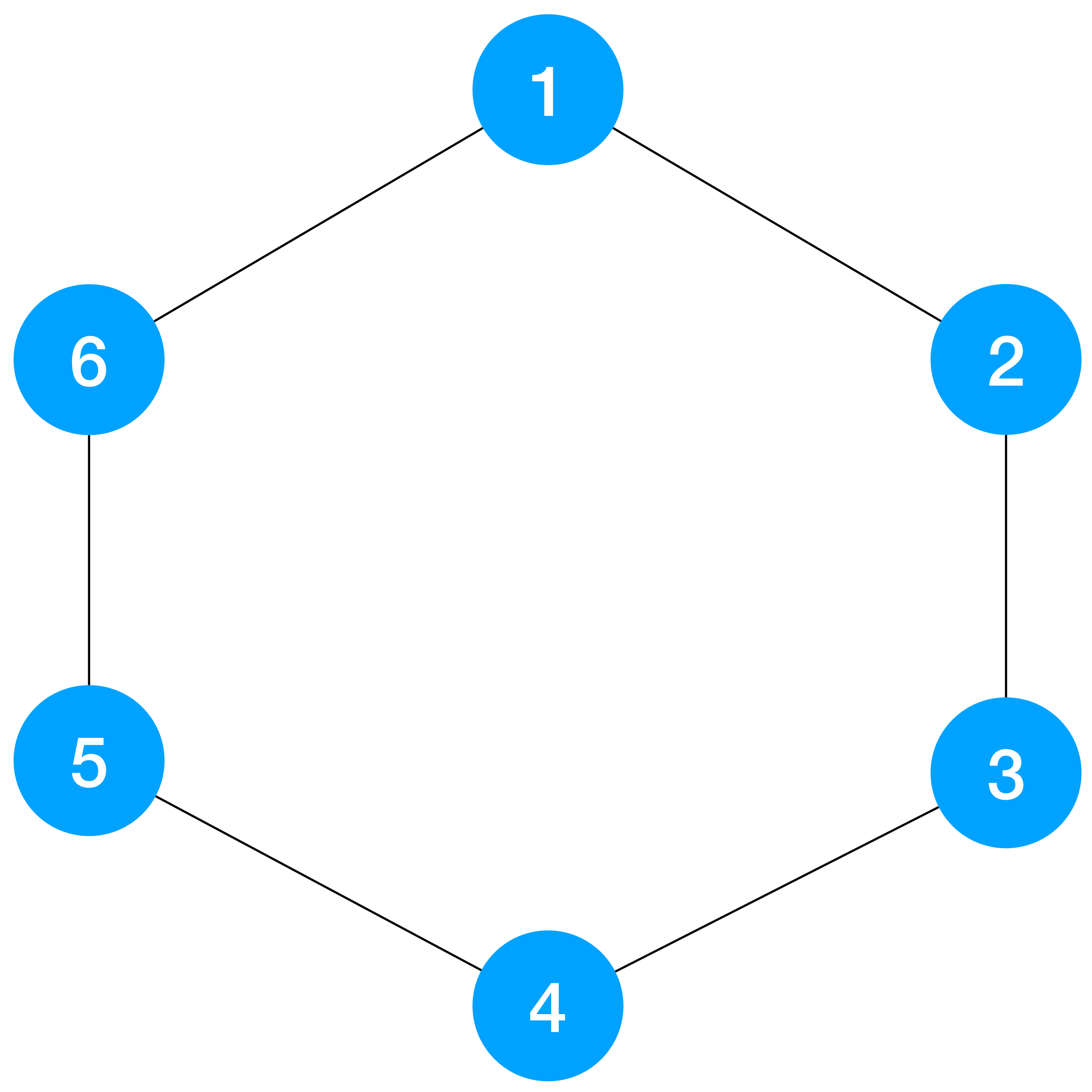

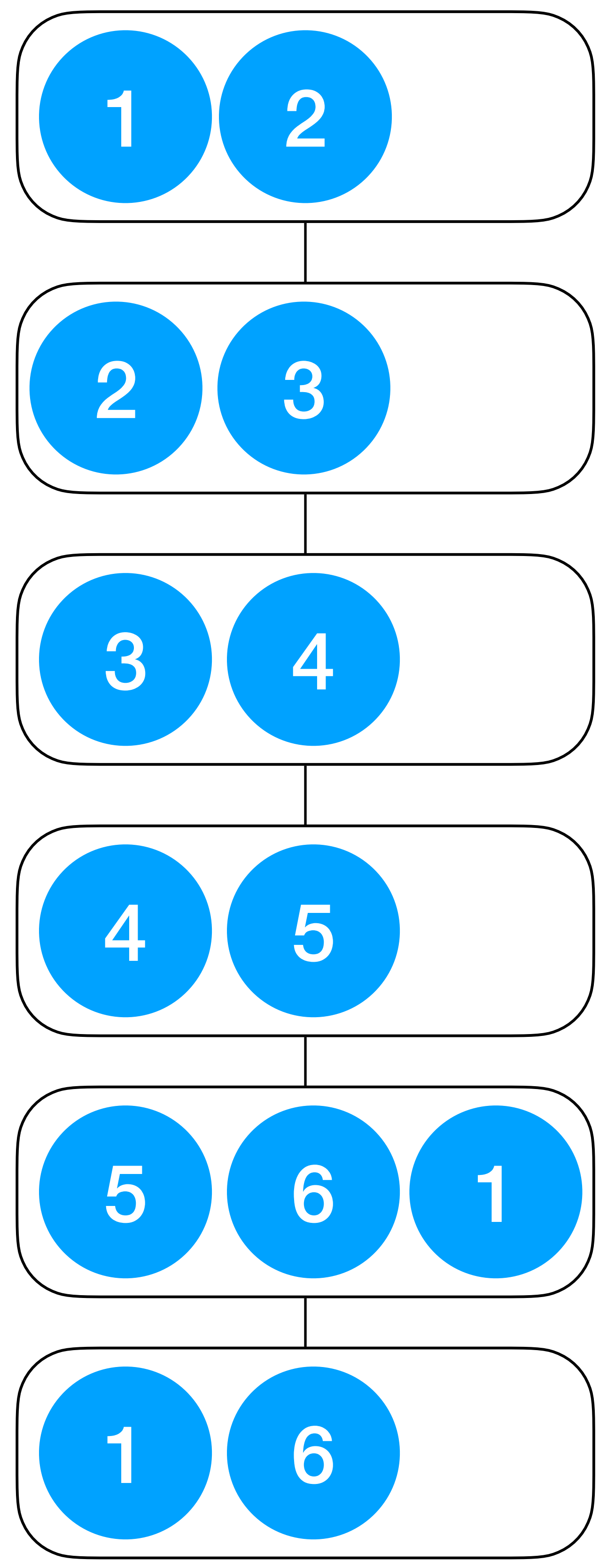

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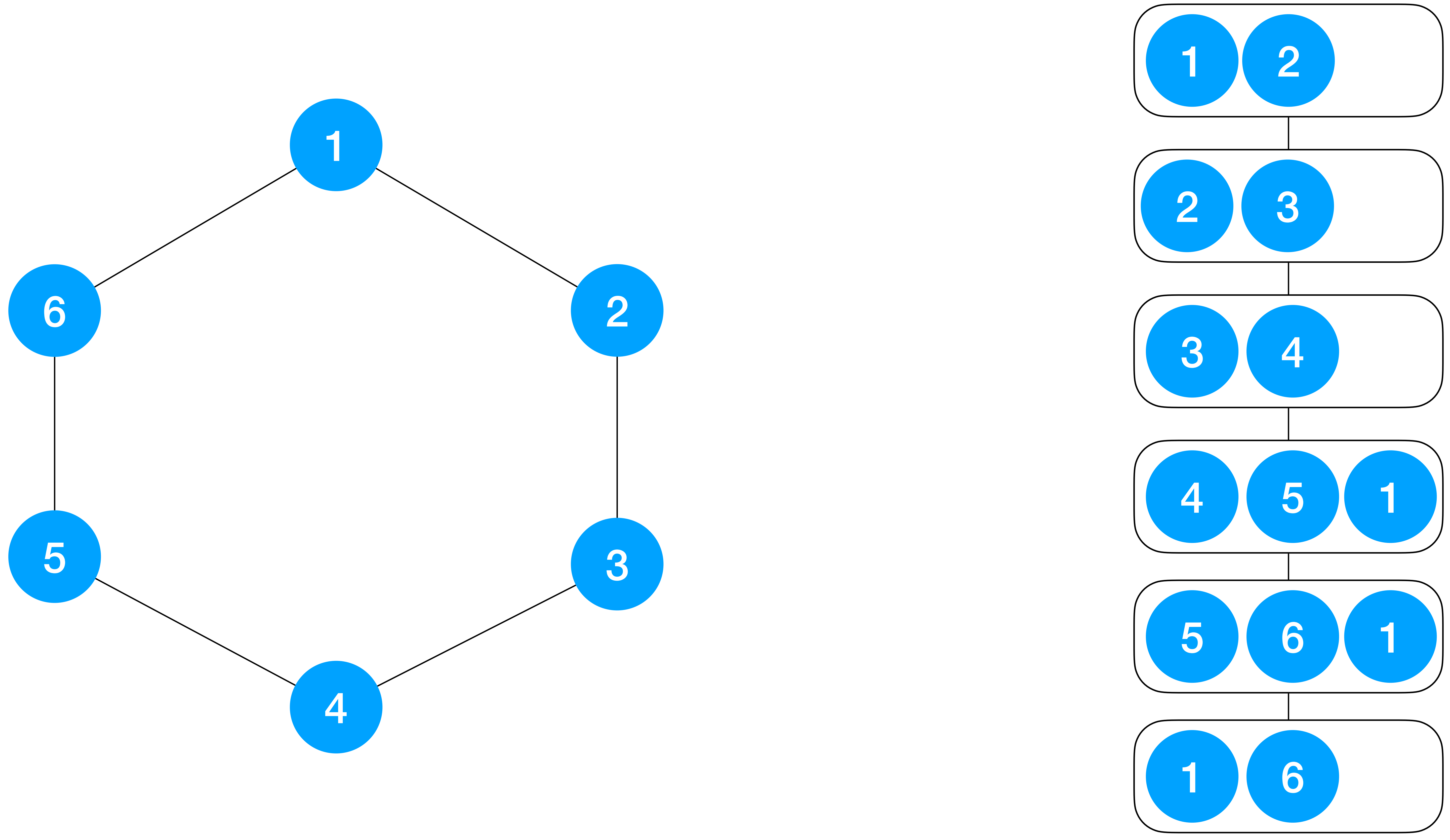

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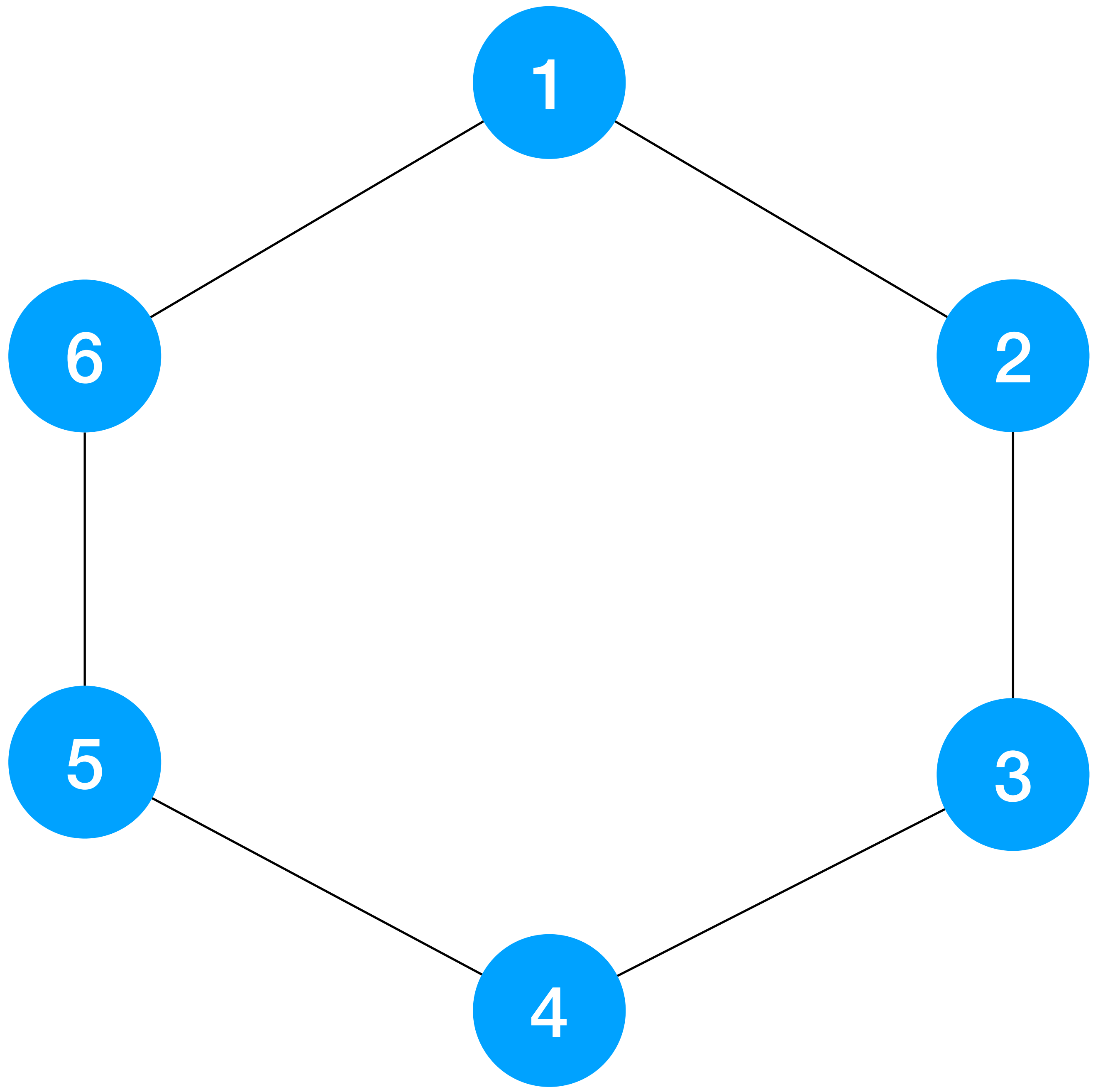

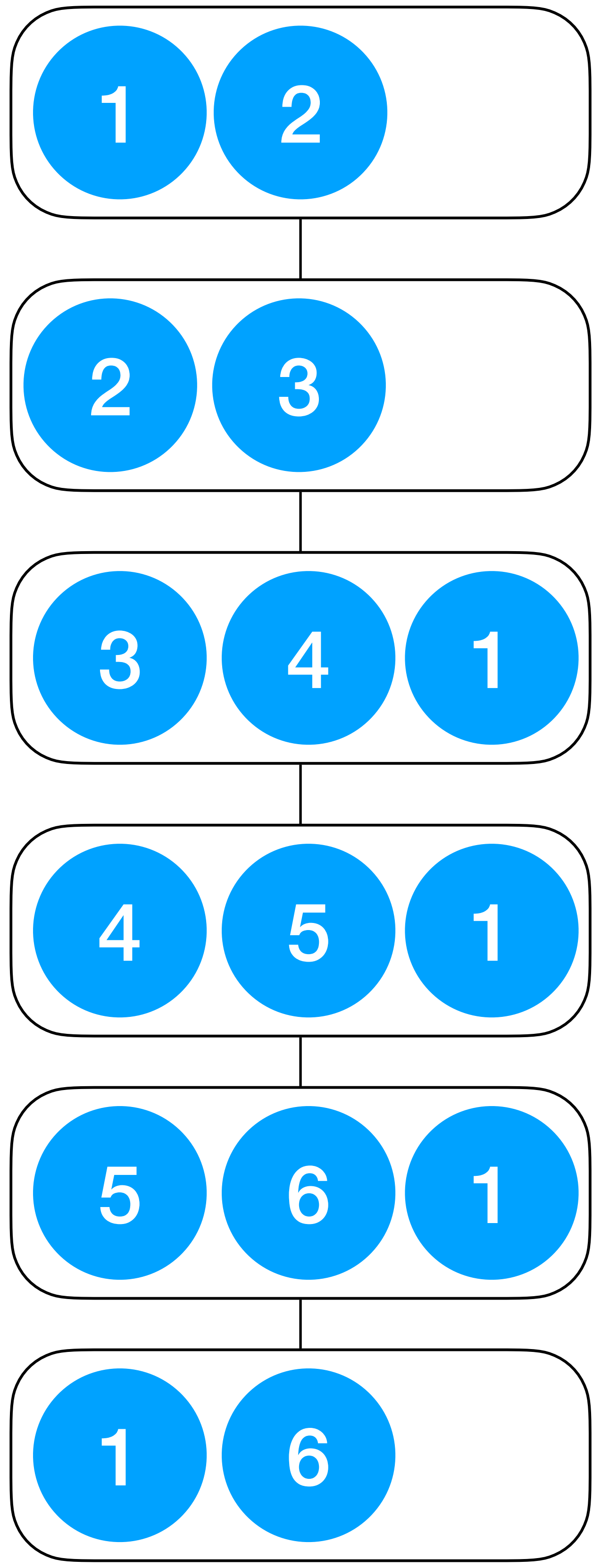

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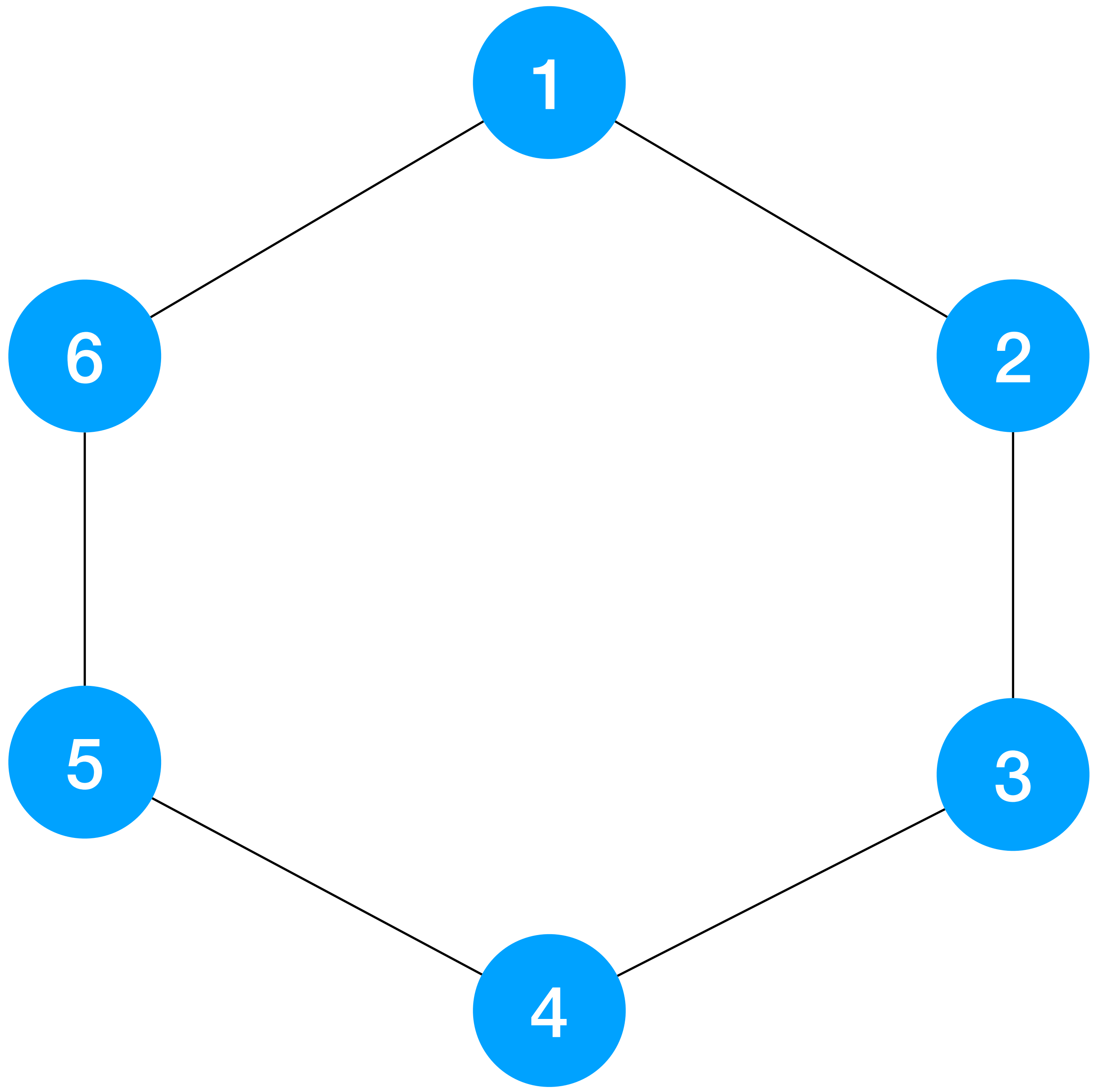
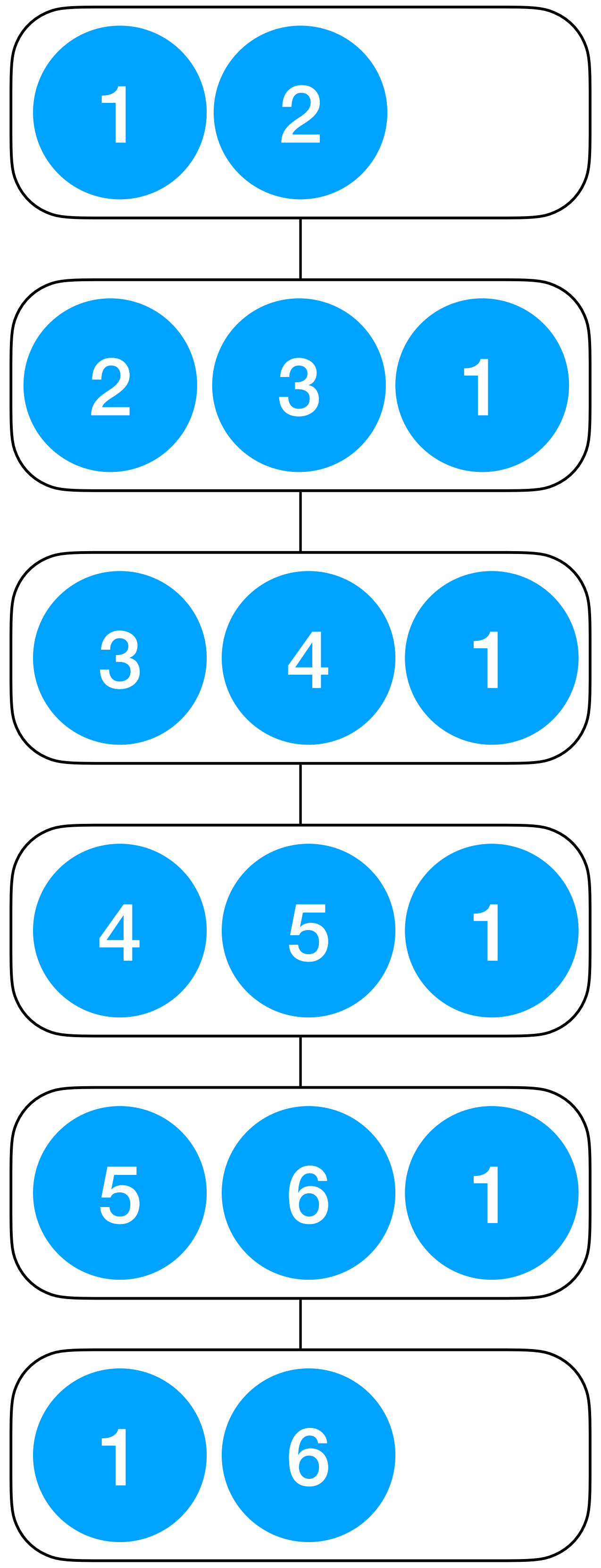
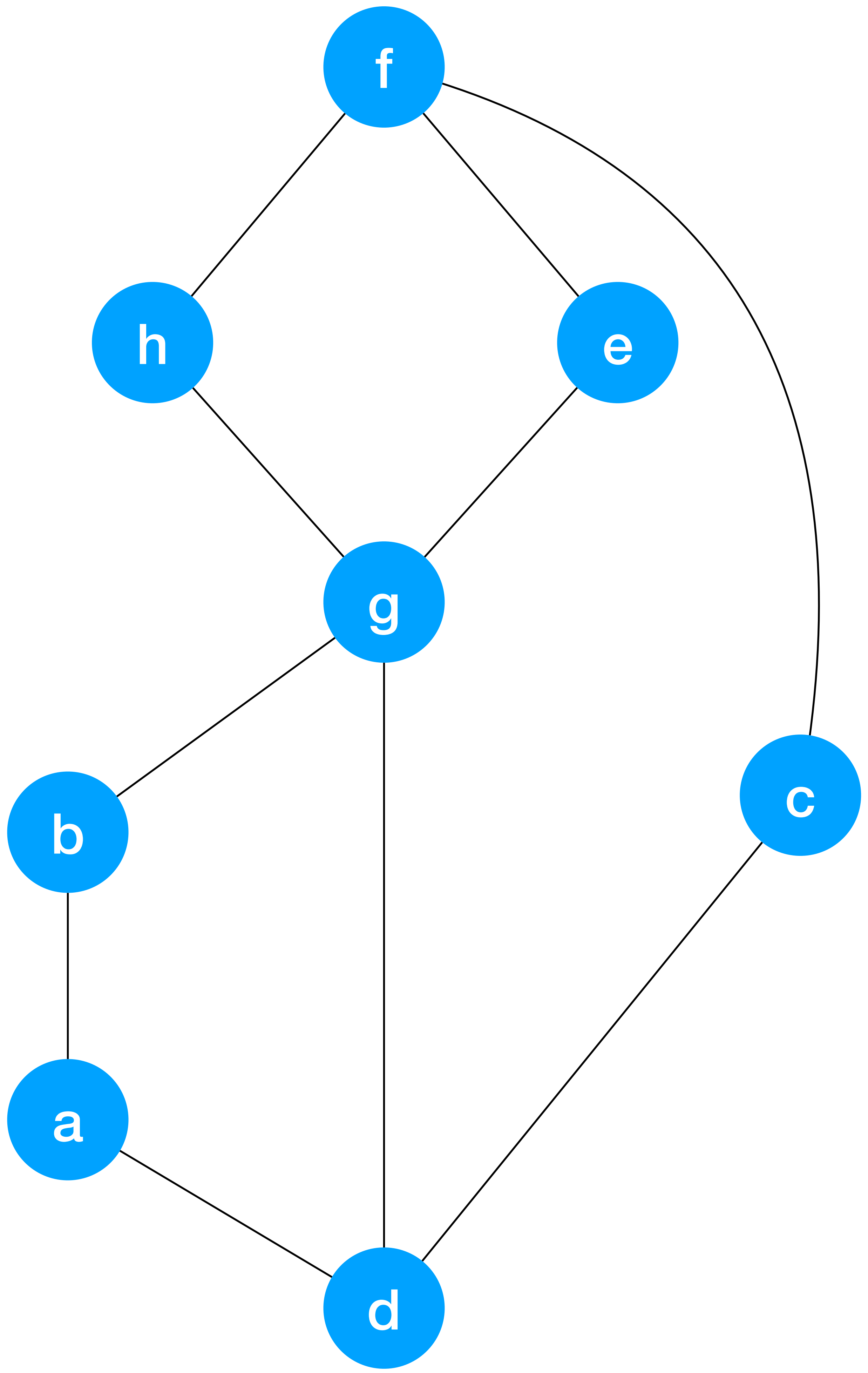




Another Example



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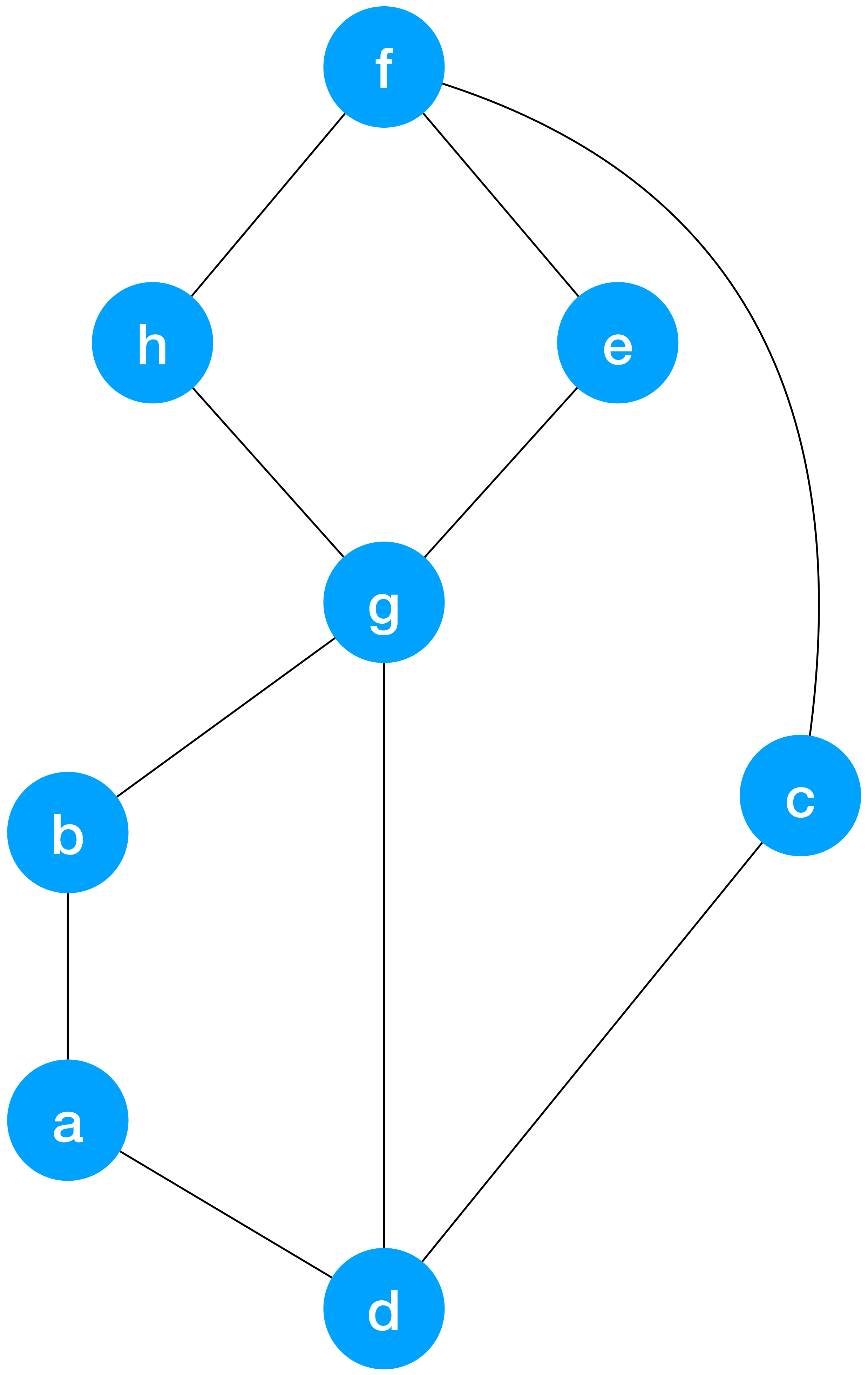

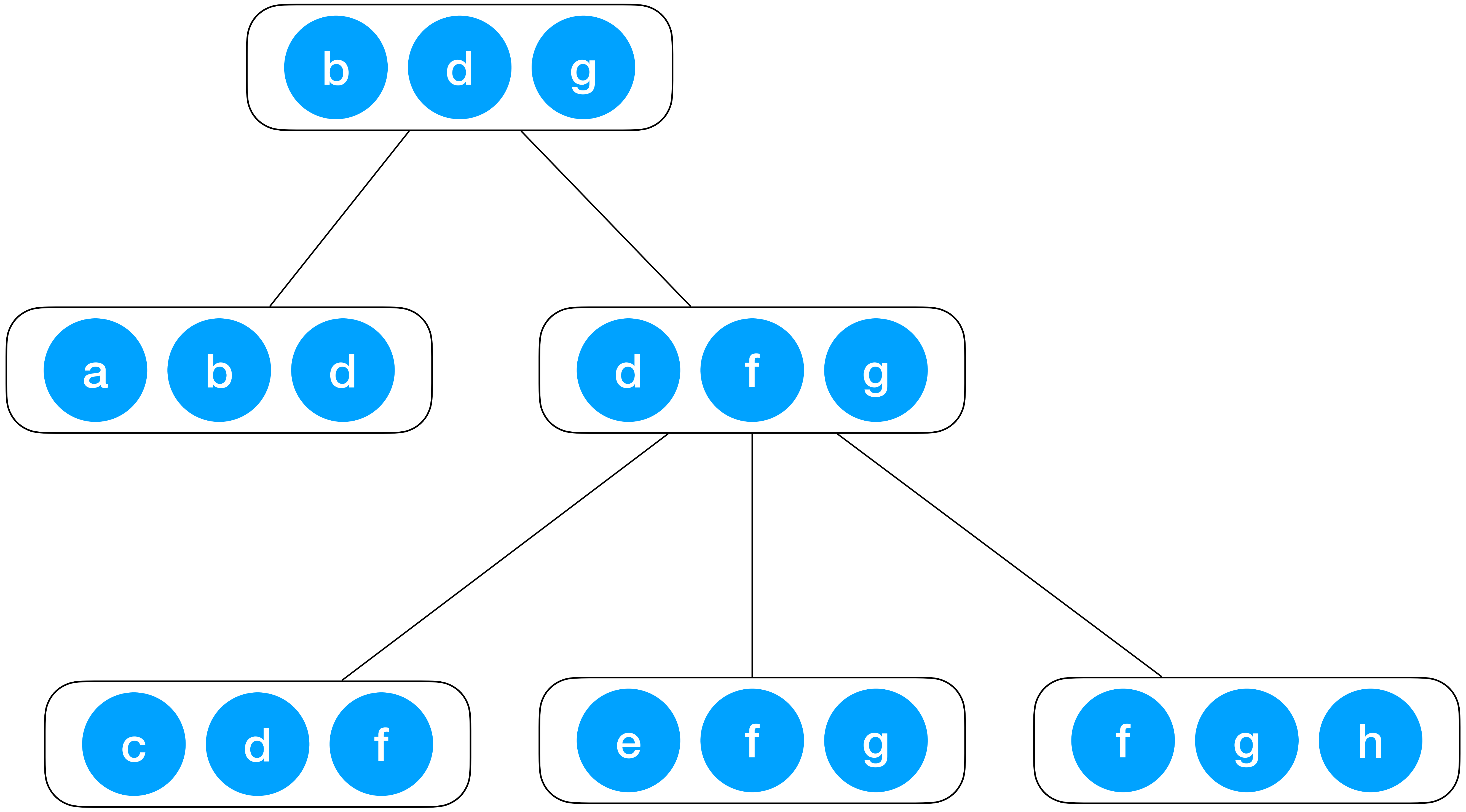

Properties of Treewidth

Subgraphs

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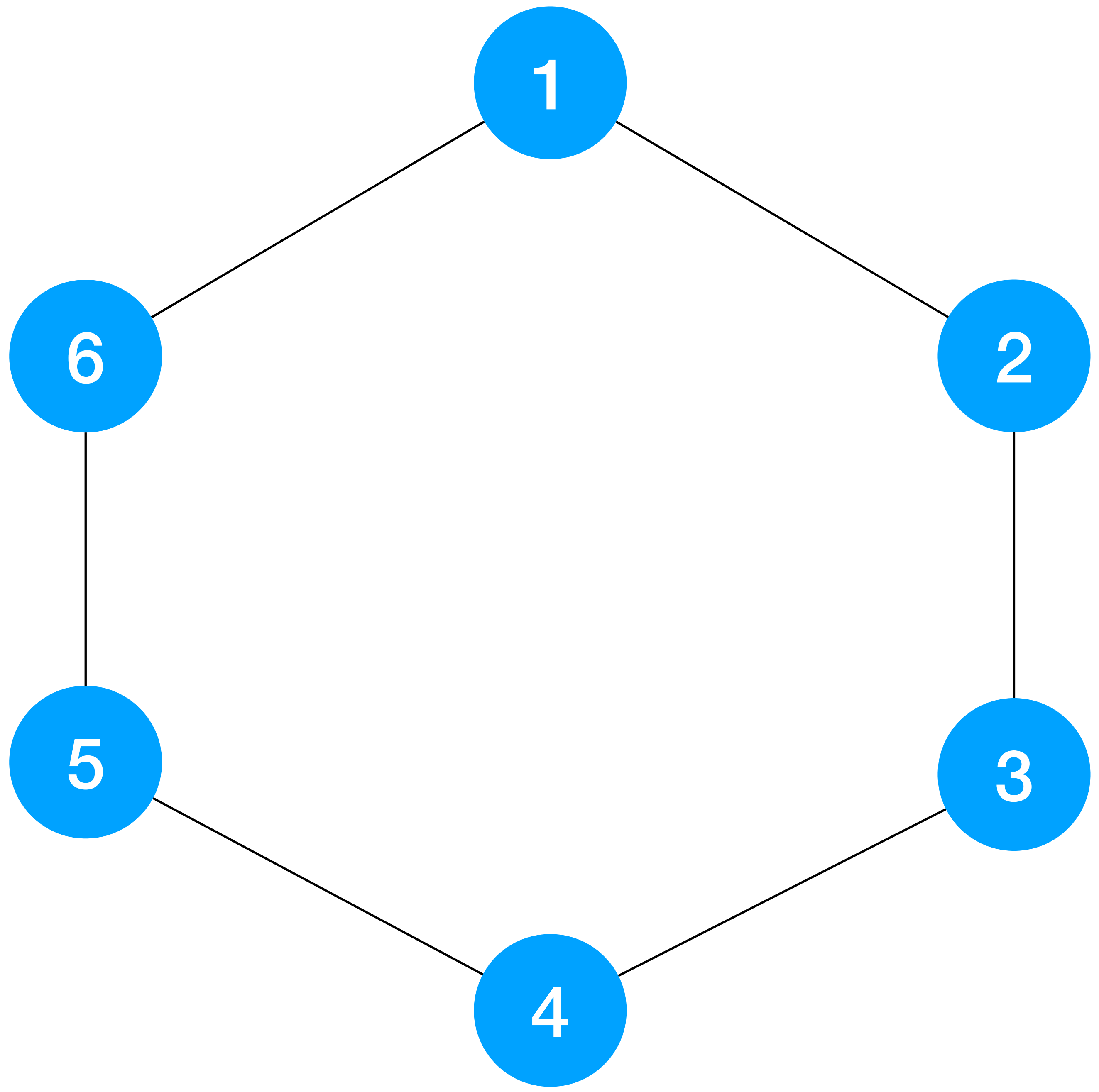

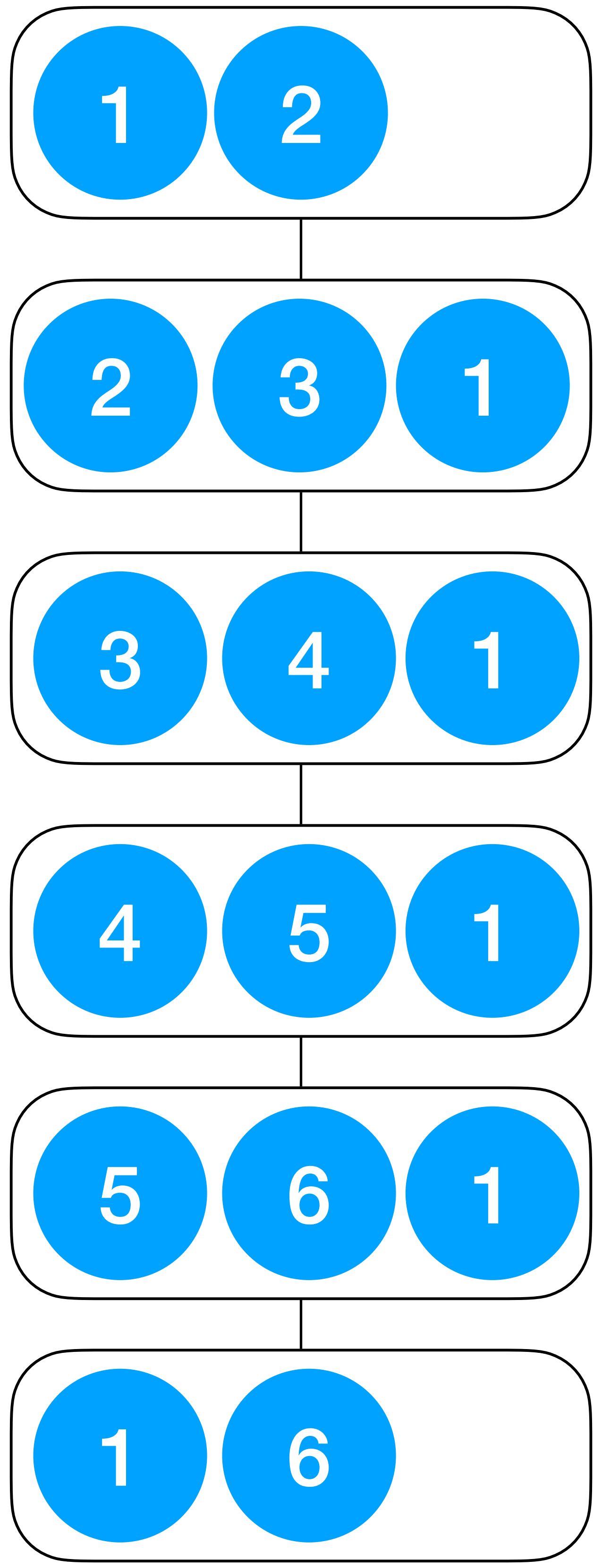

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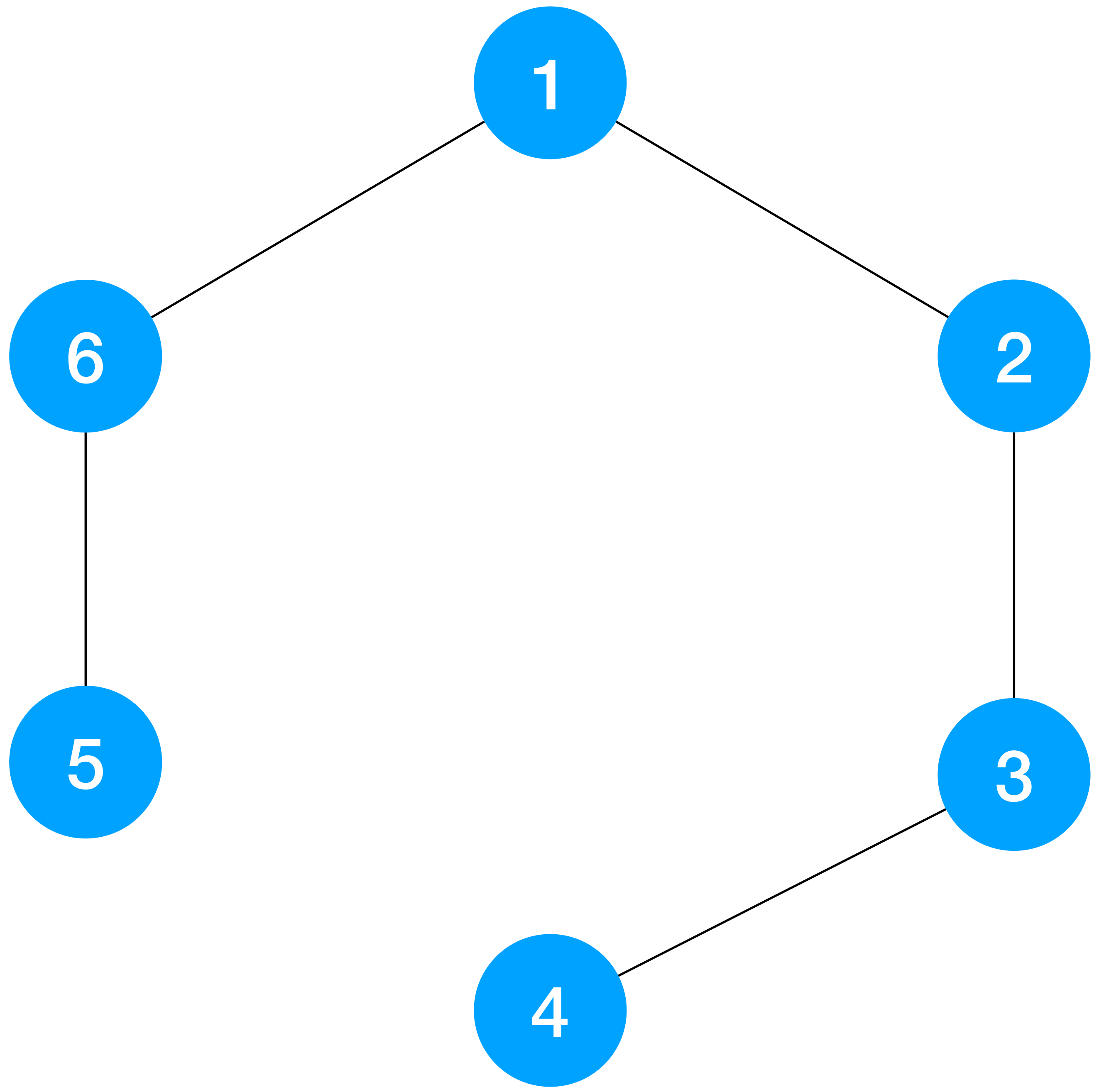

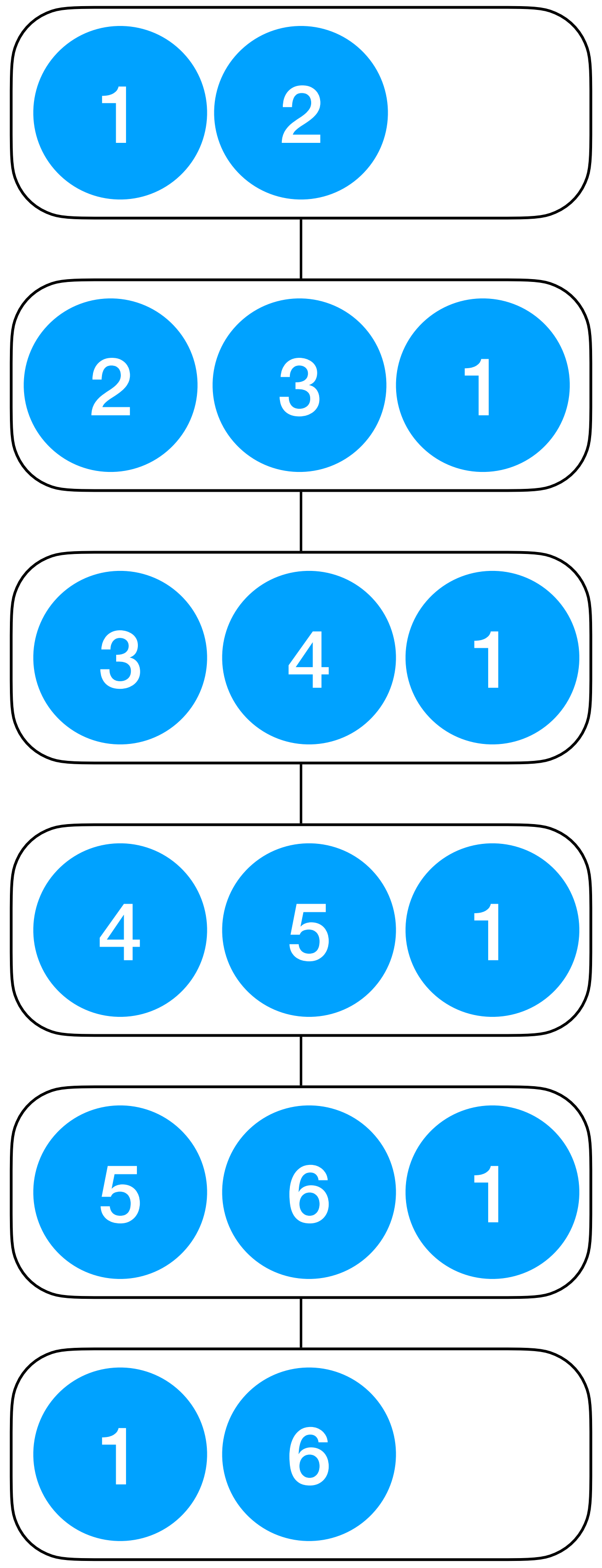

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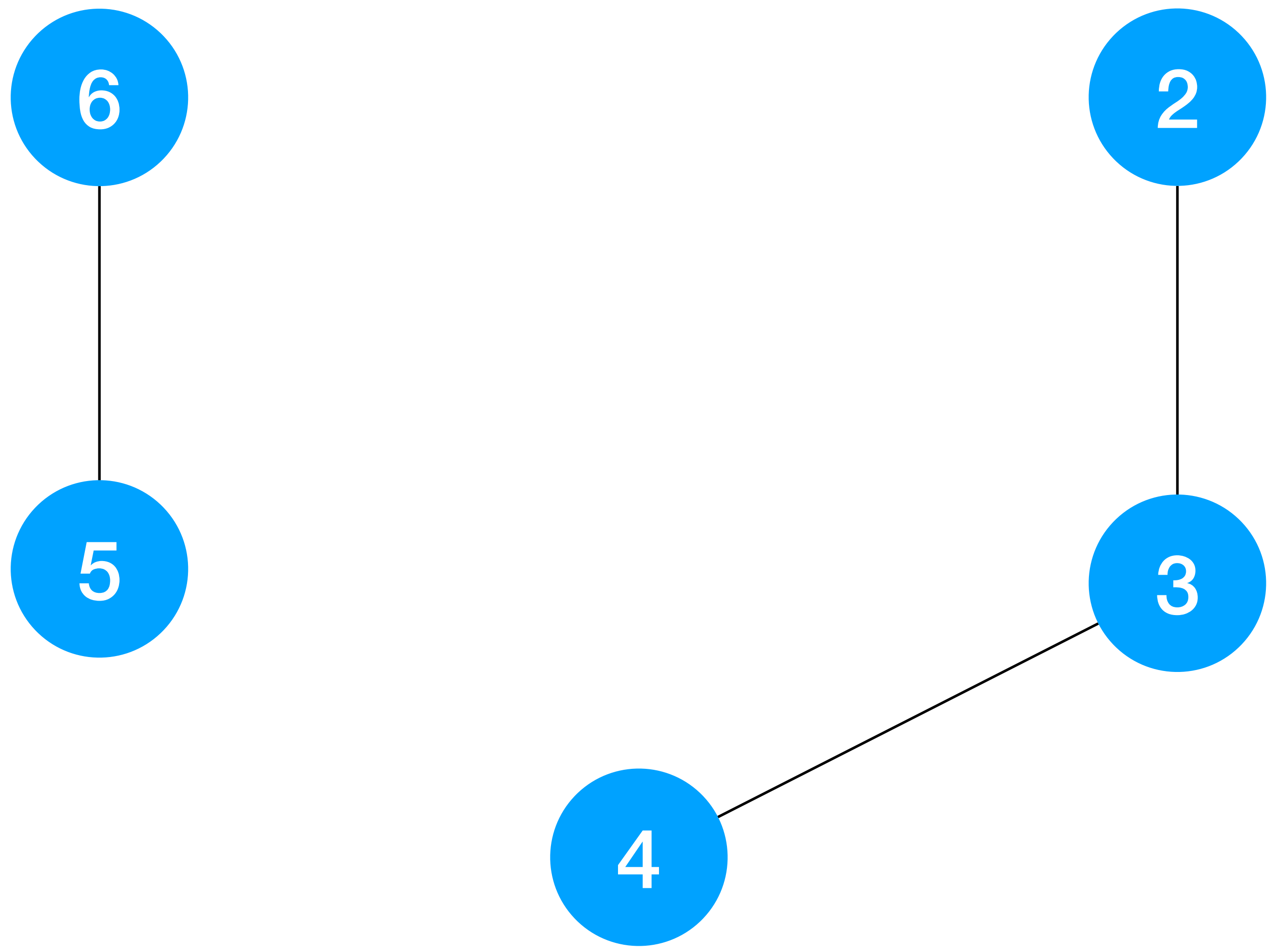

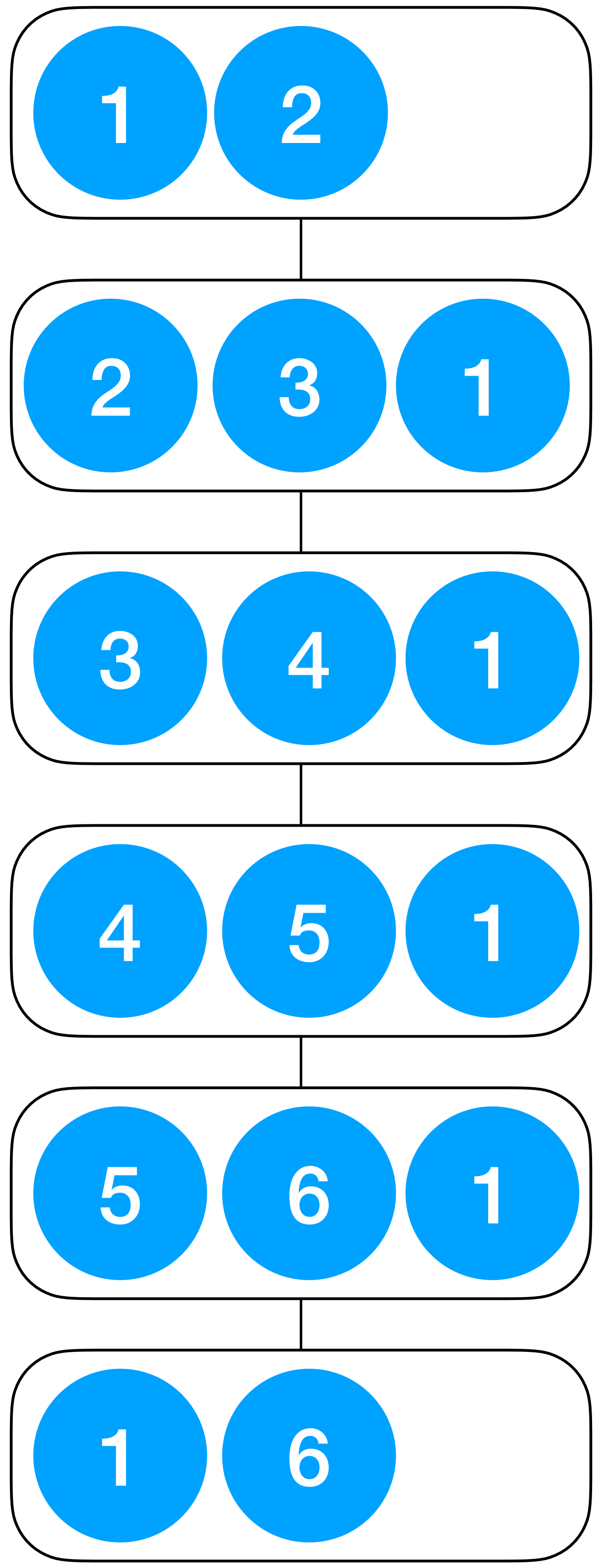

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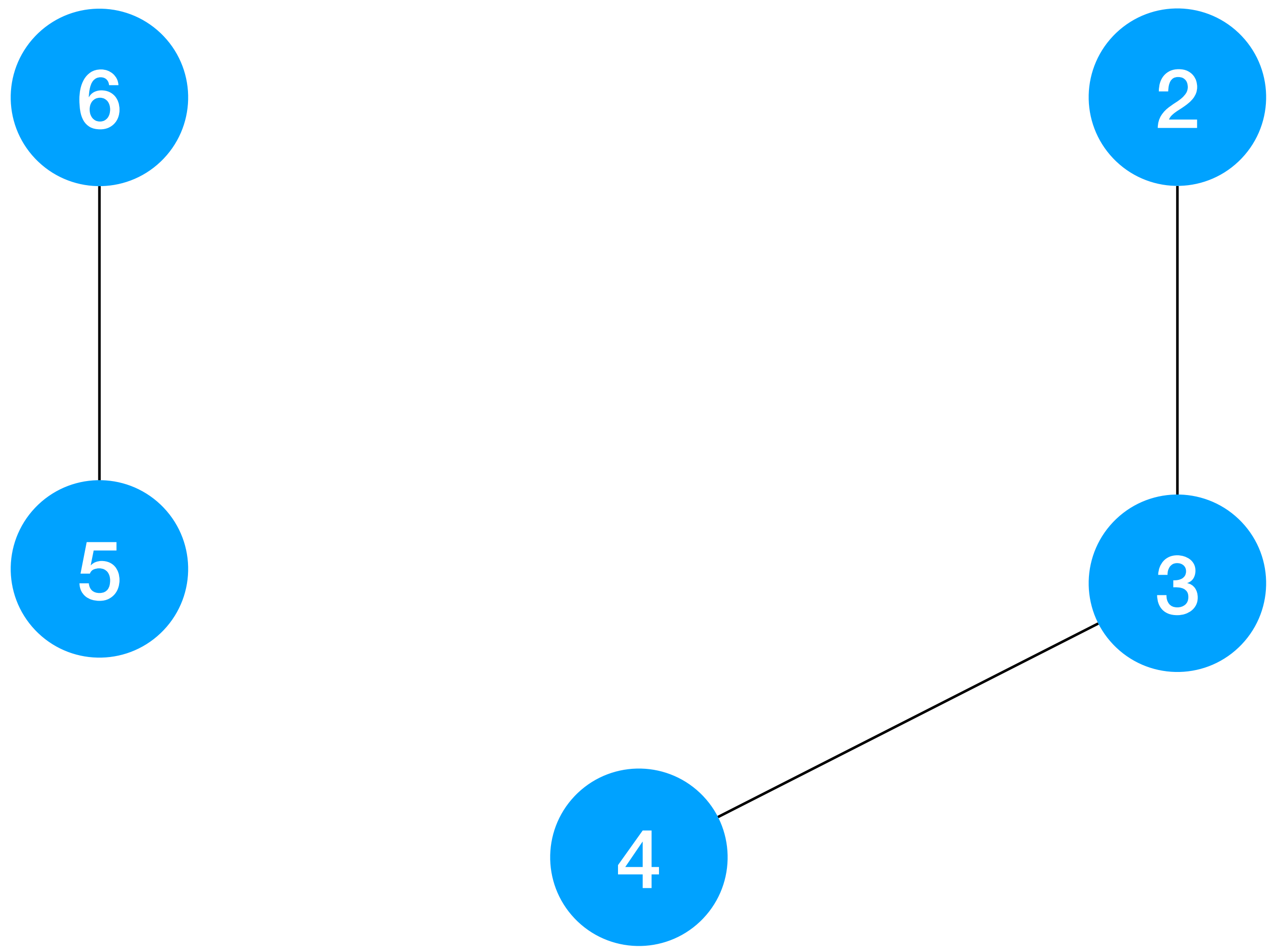

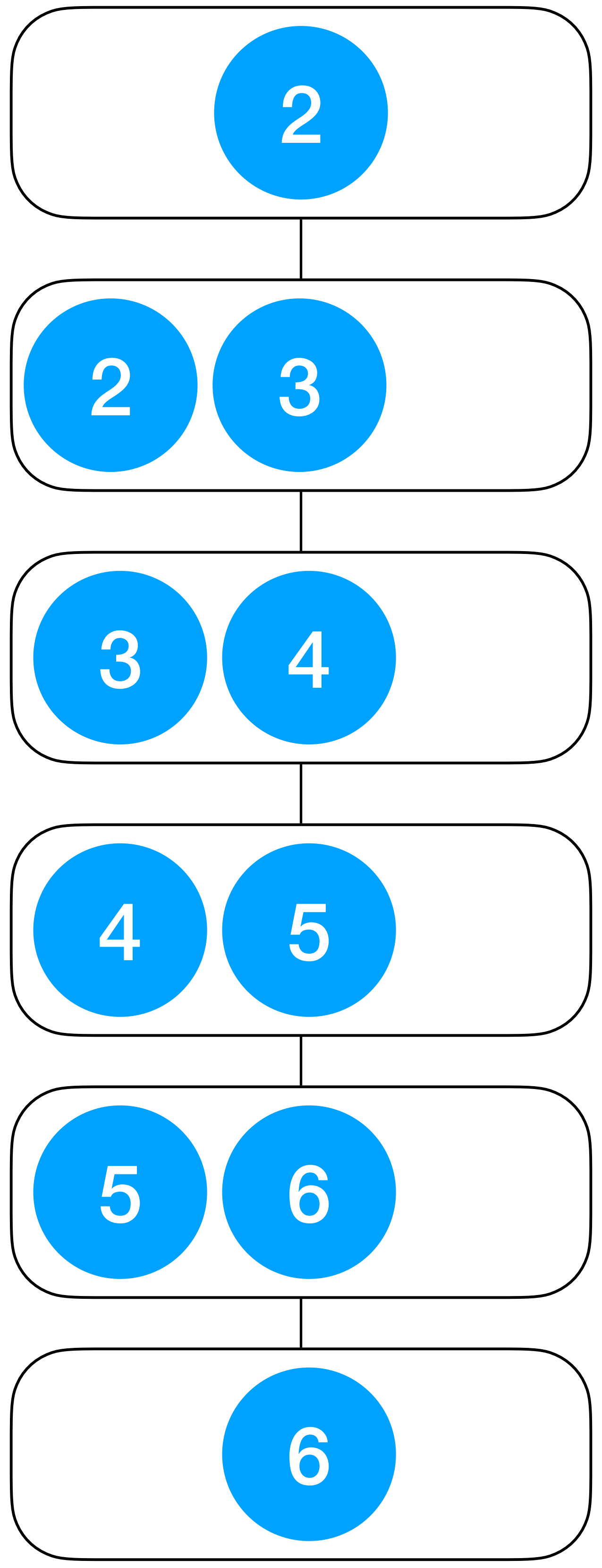

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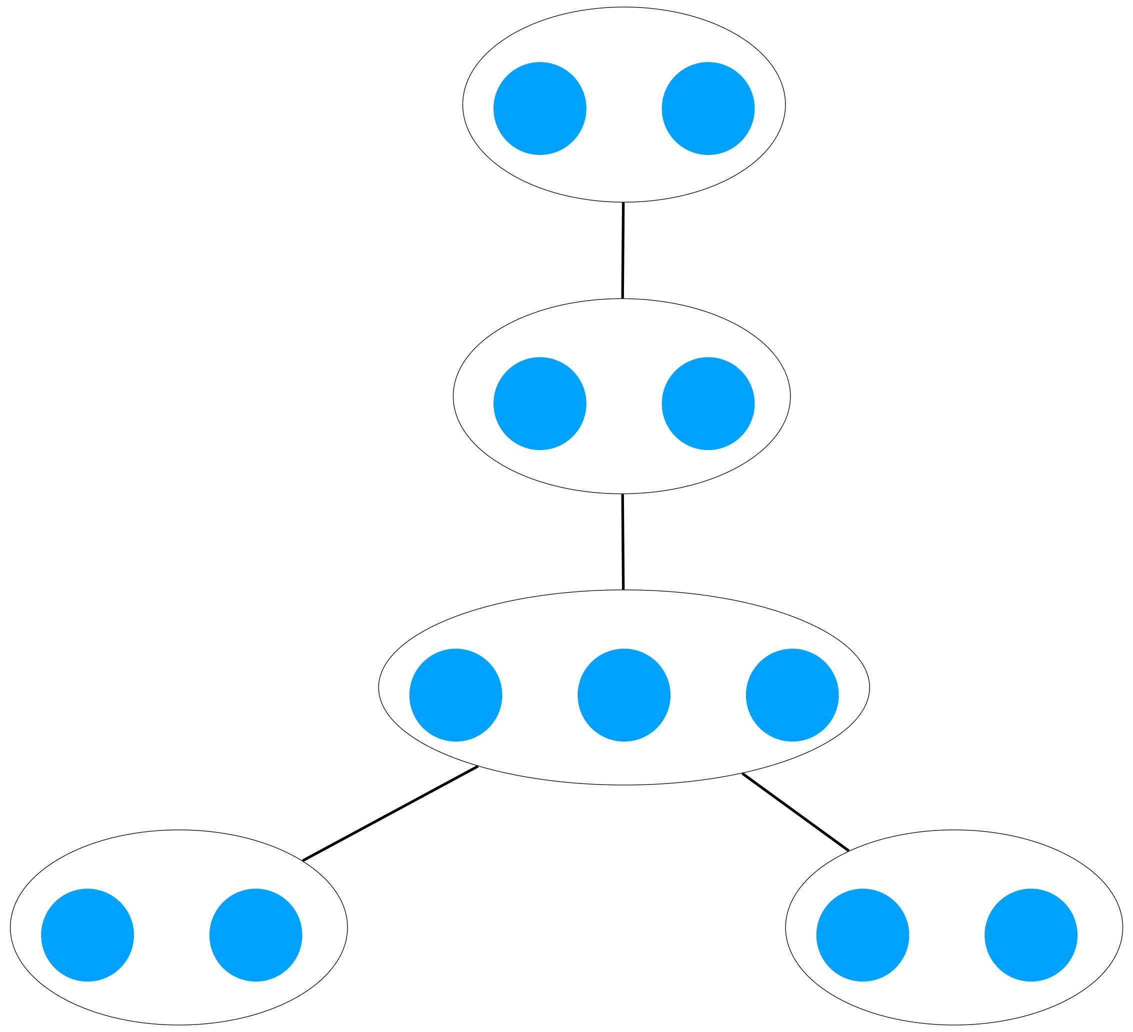

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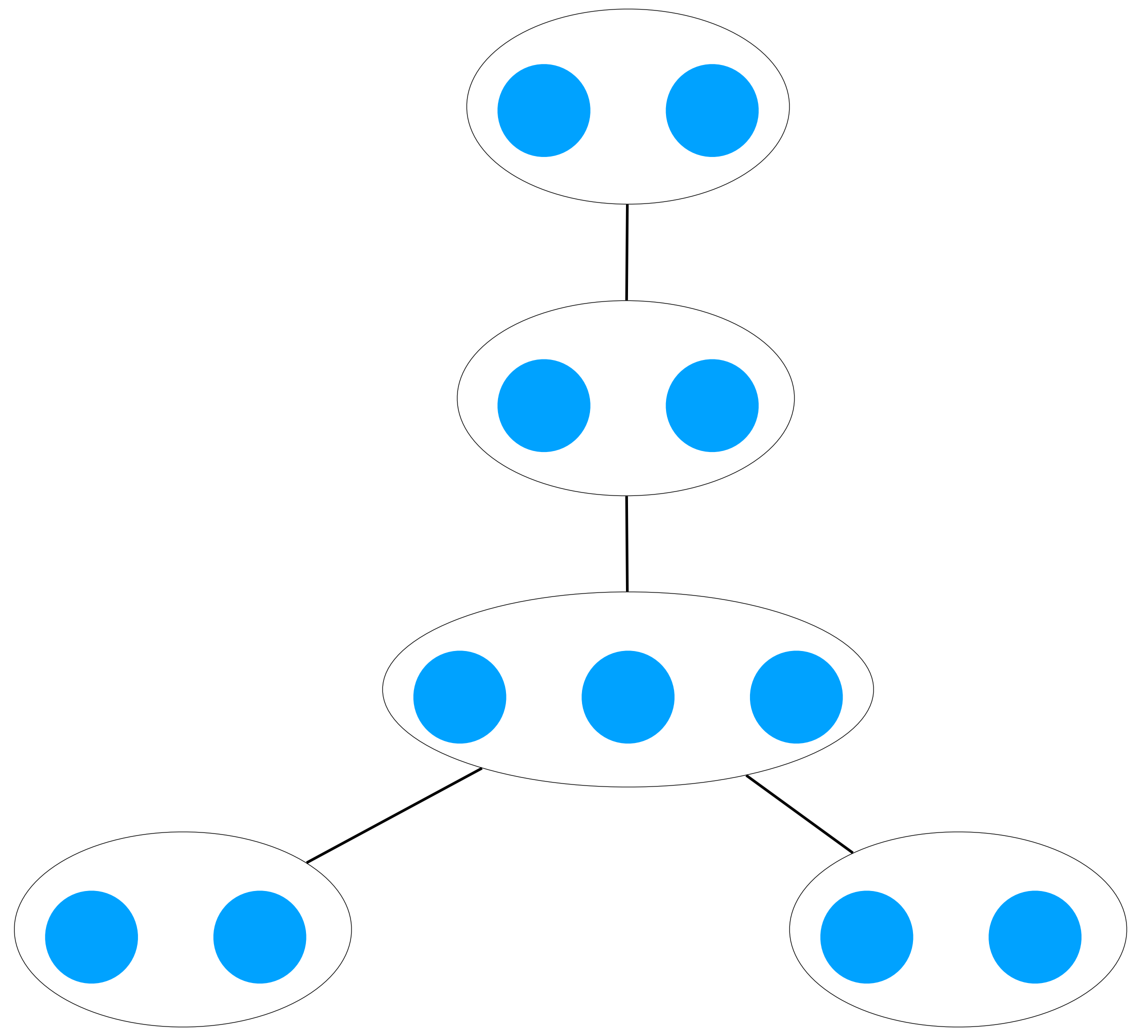

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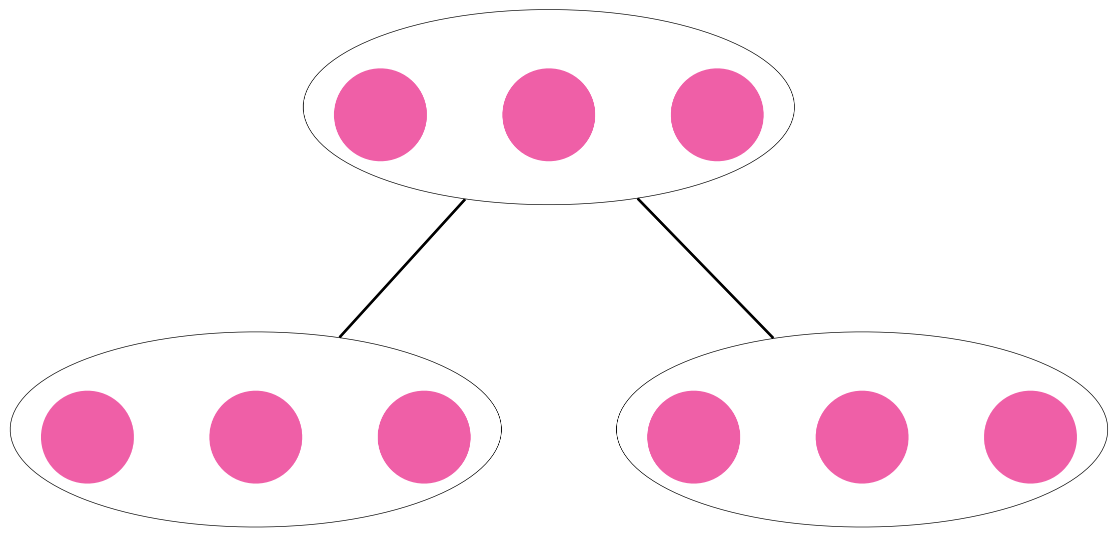

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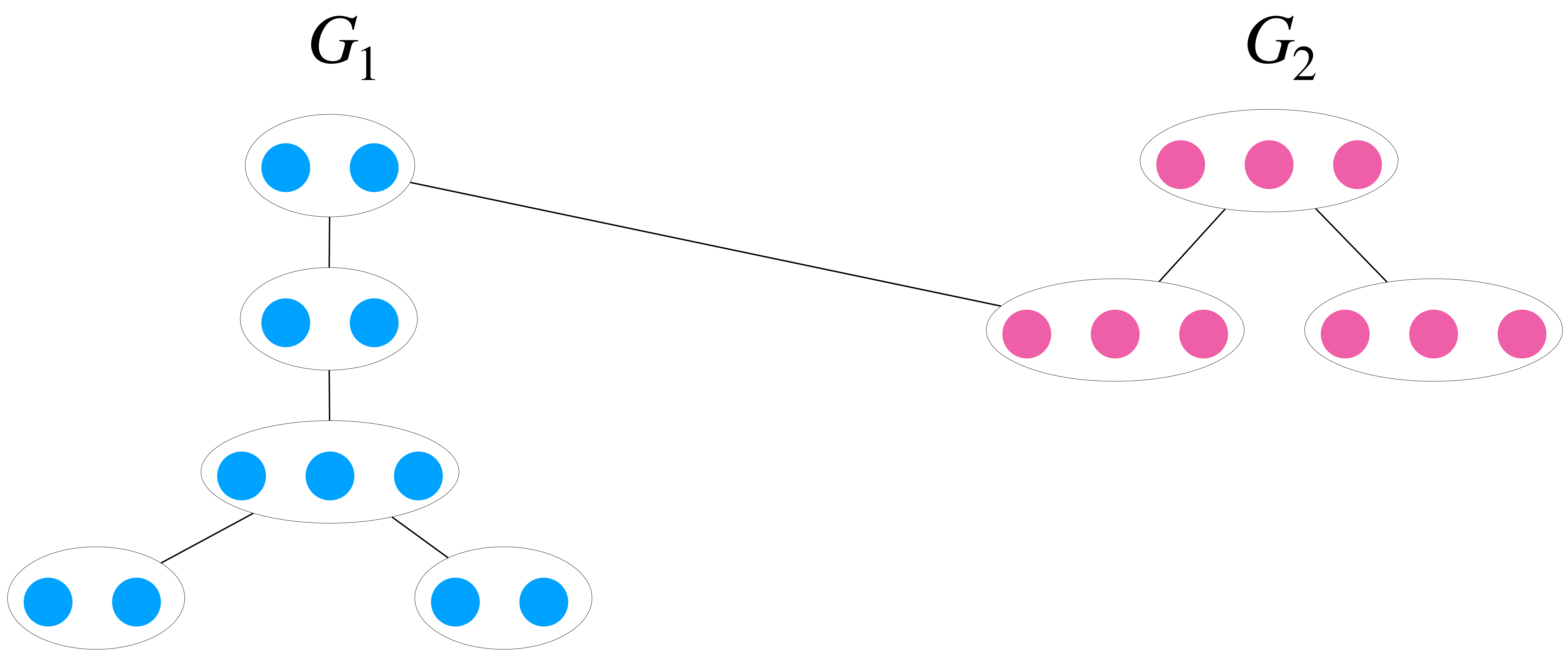

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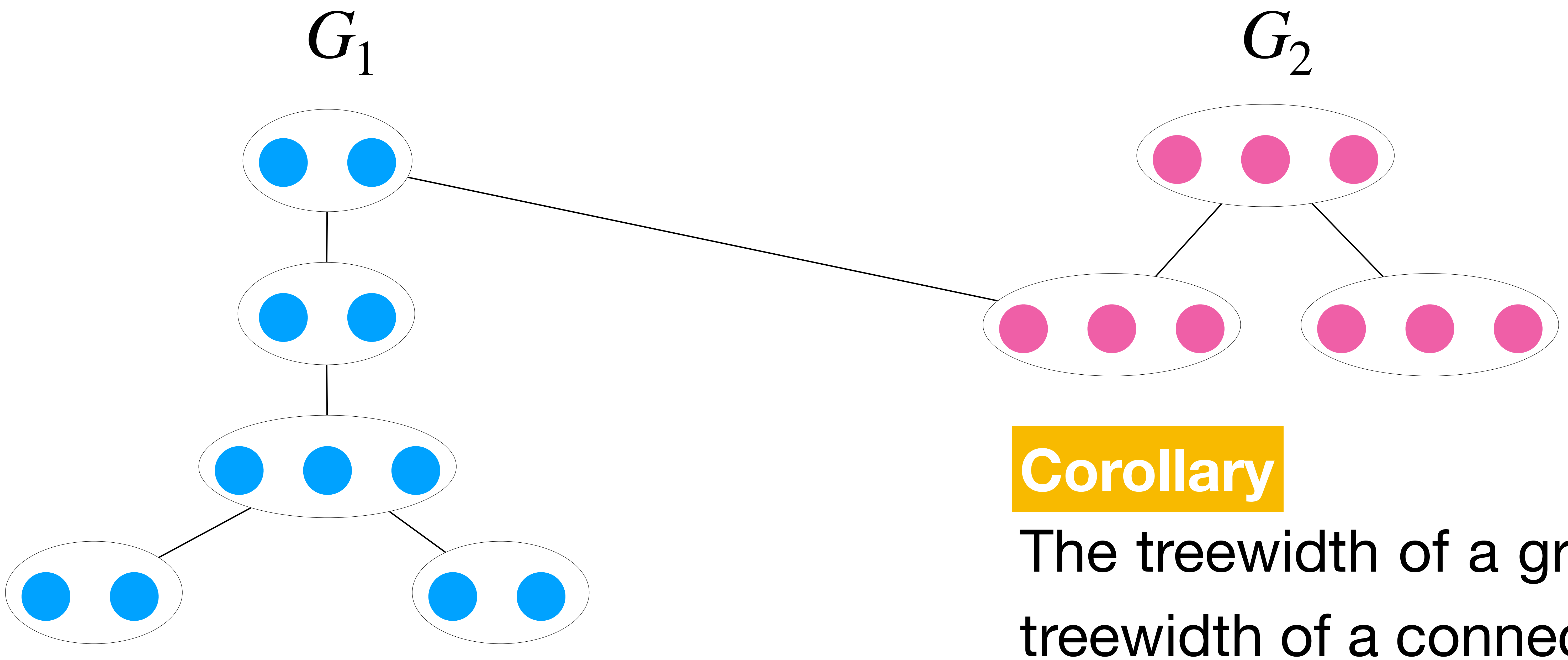

Corollary

The treewidth of a graph G is the maximum treewidth of a connected component of G.

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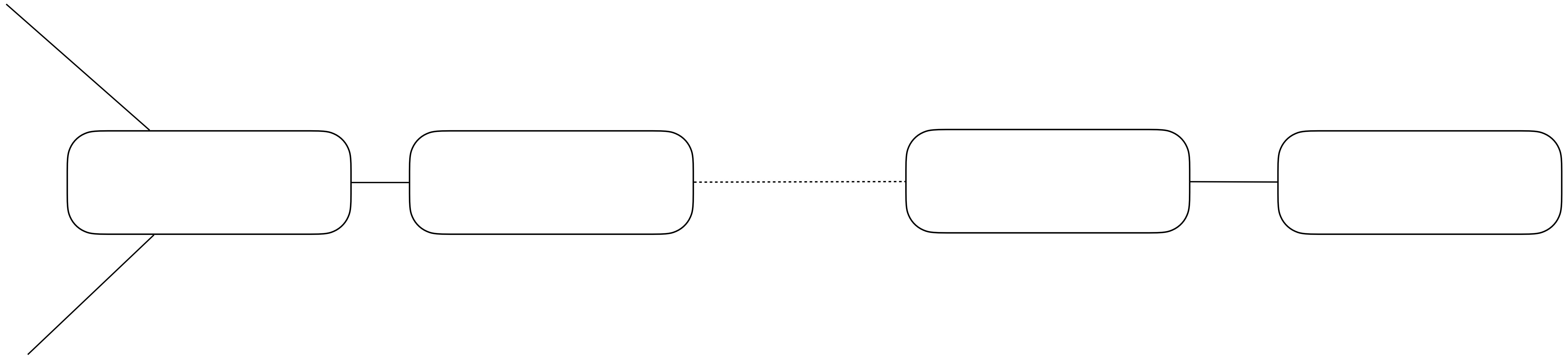

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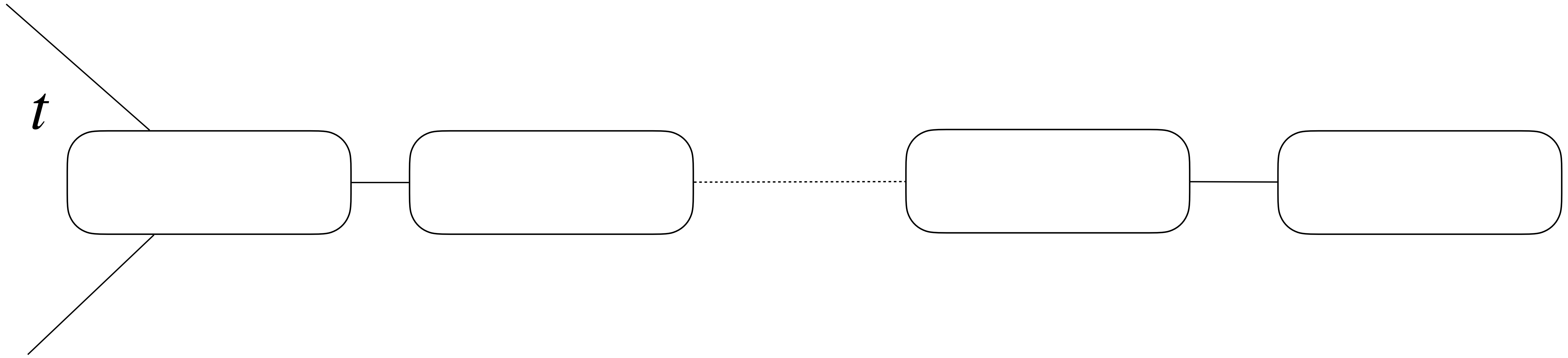

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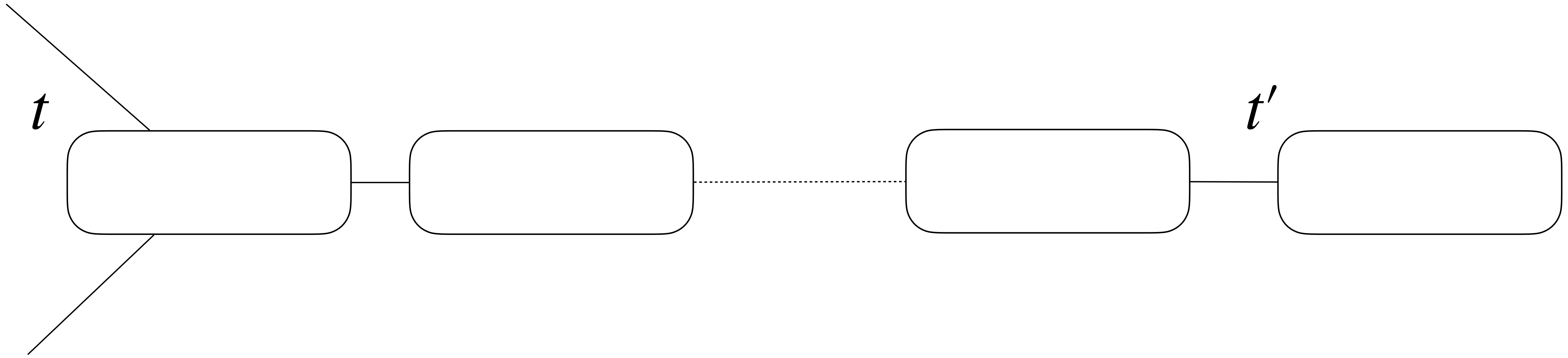

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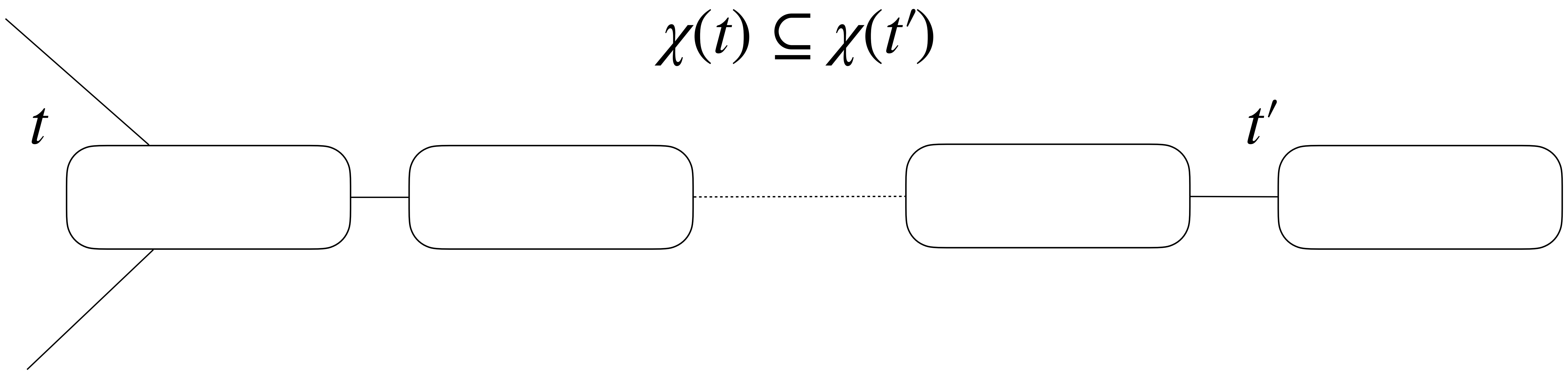

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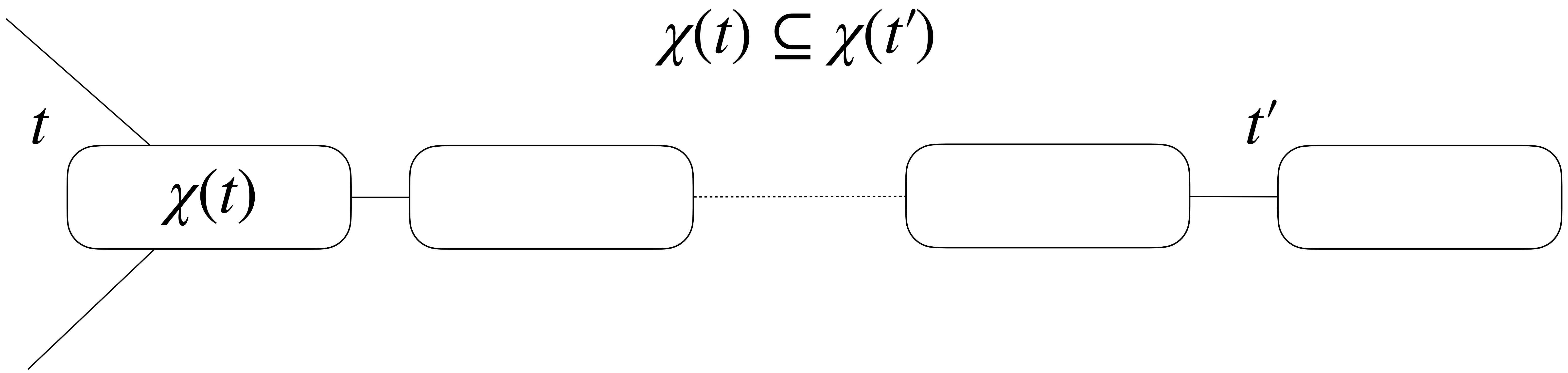

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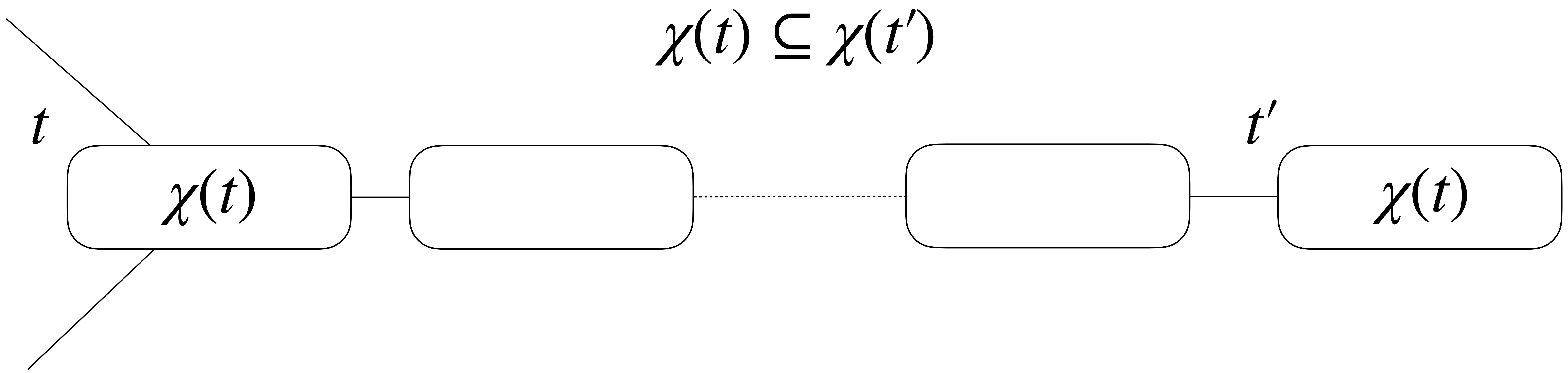

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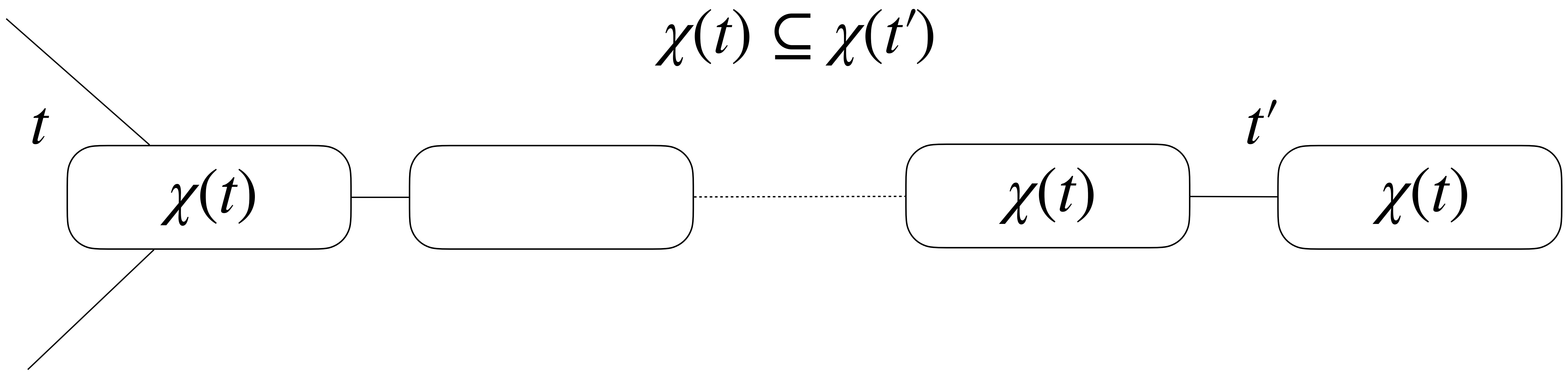

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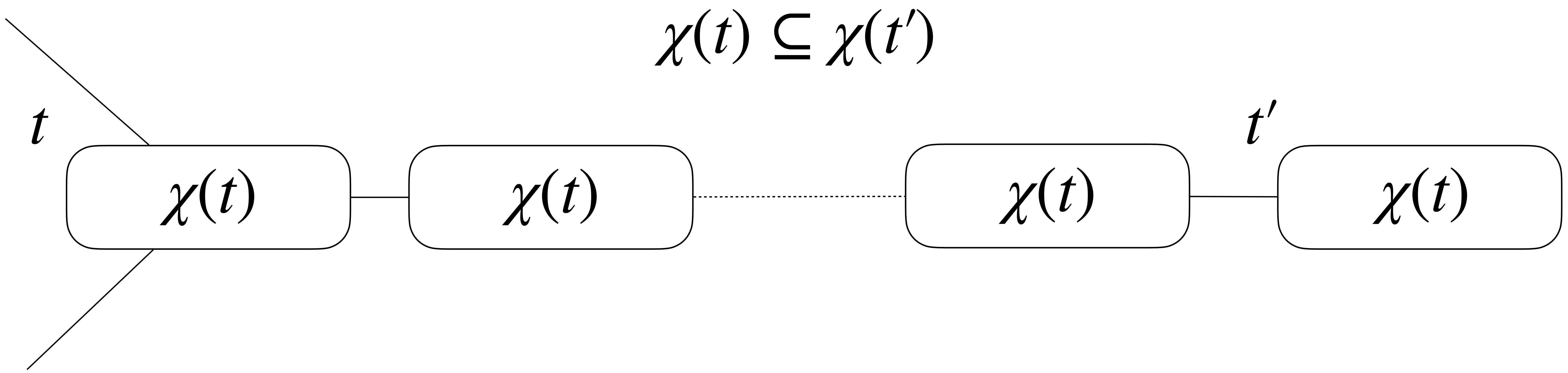

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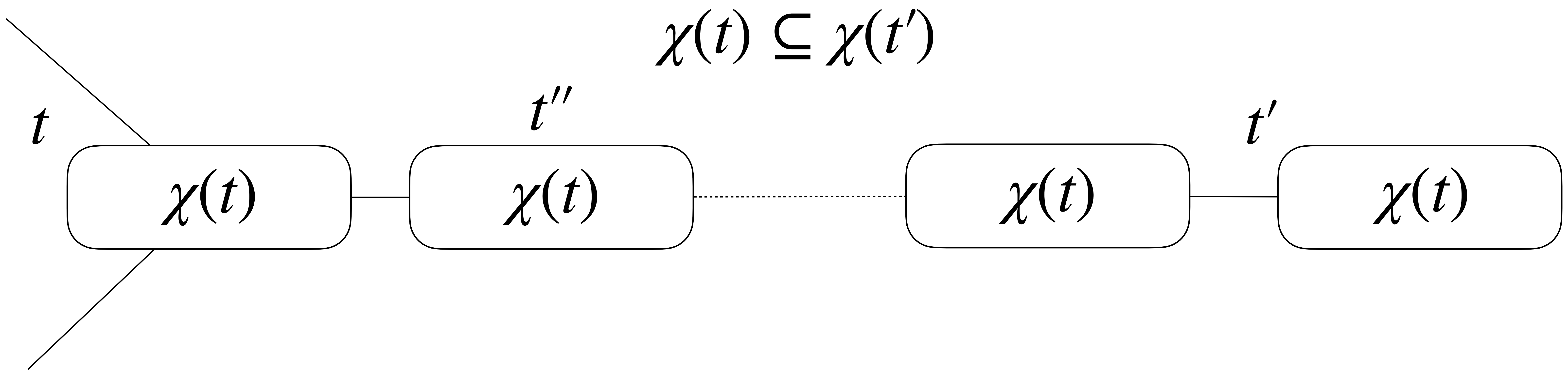

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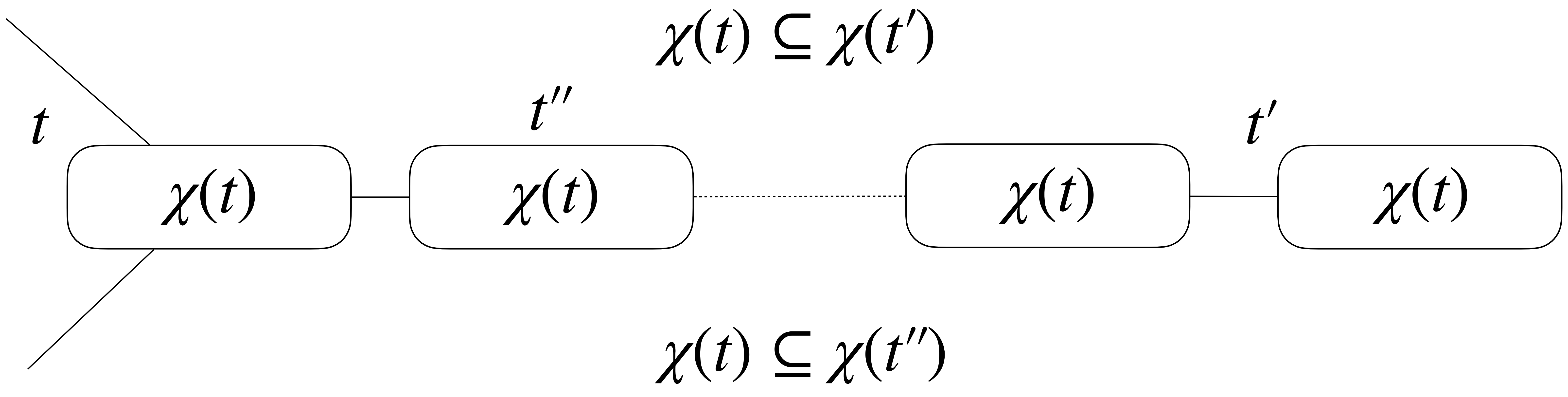

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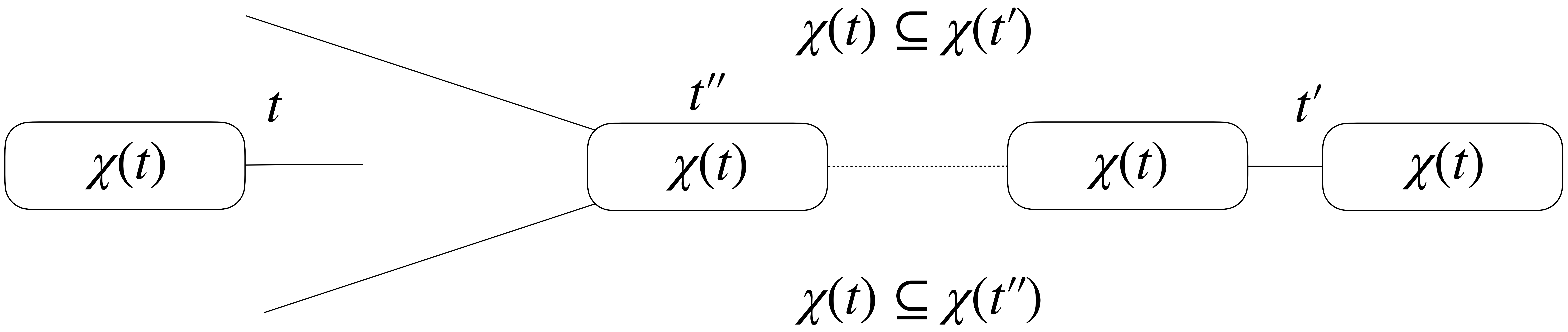

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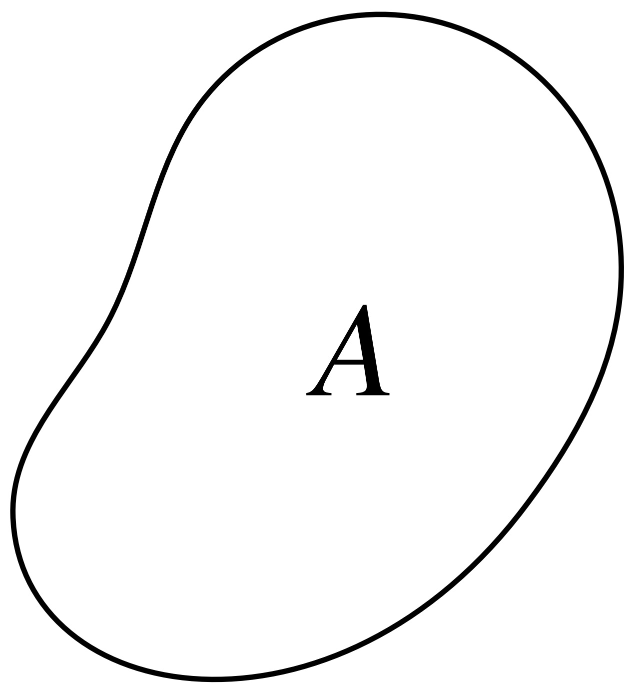

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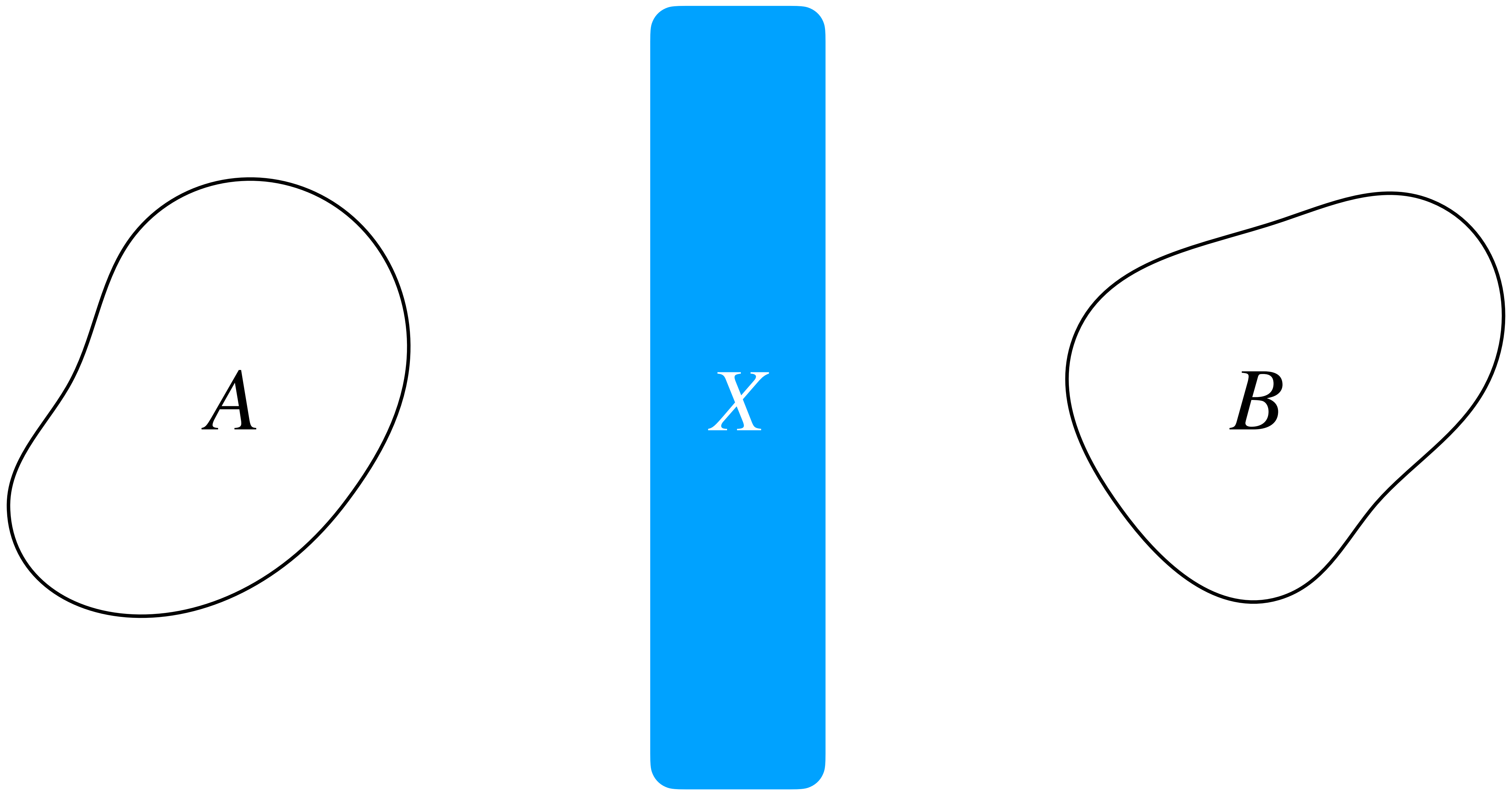

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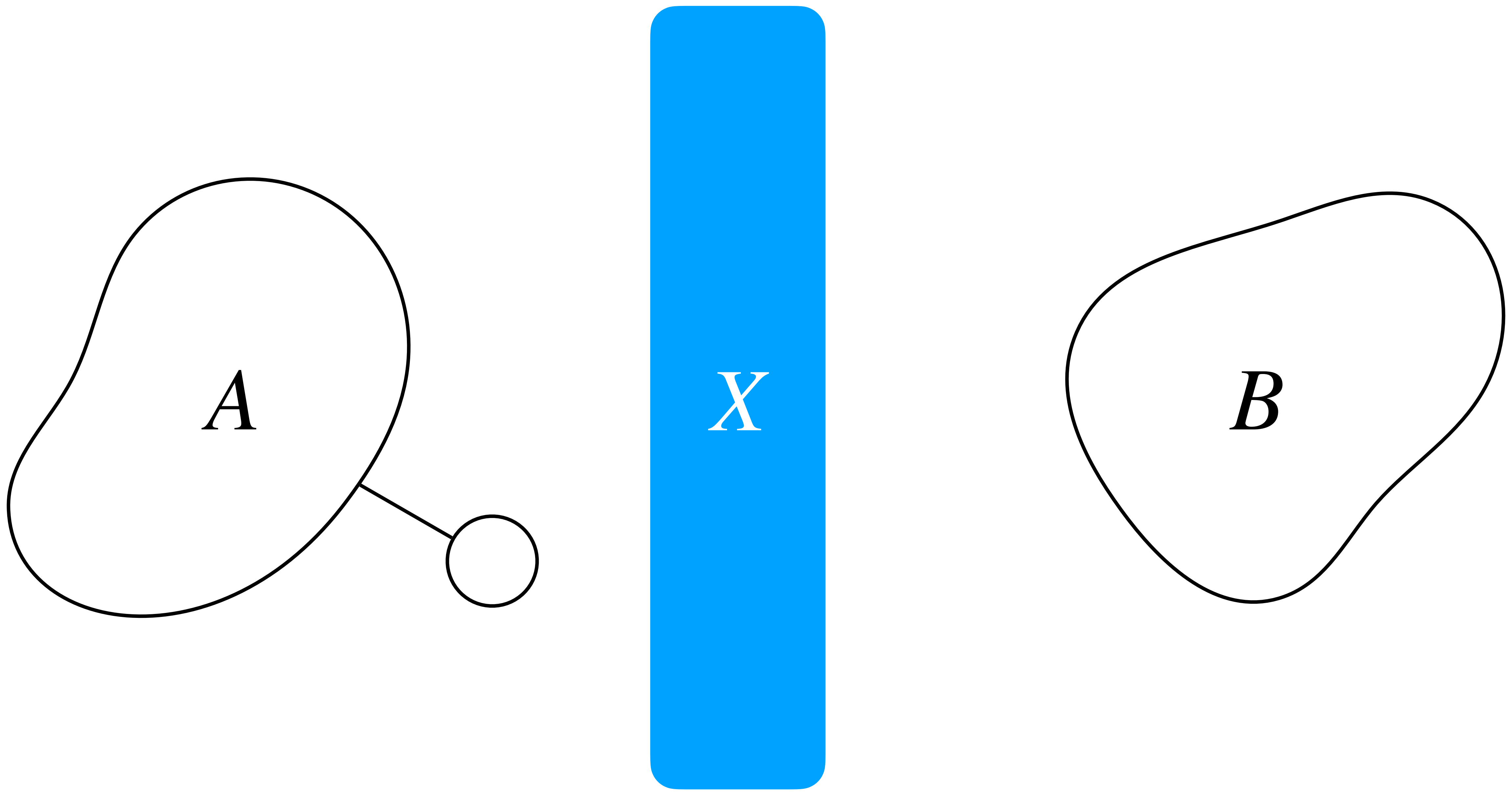

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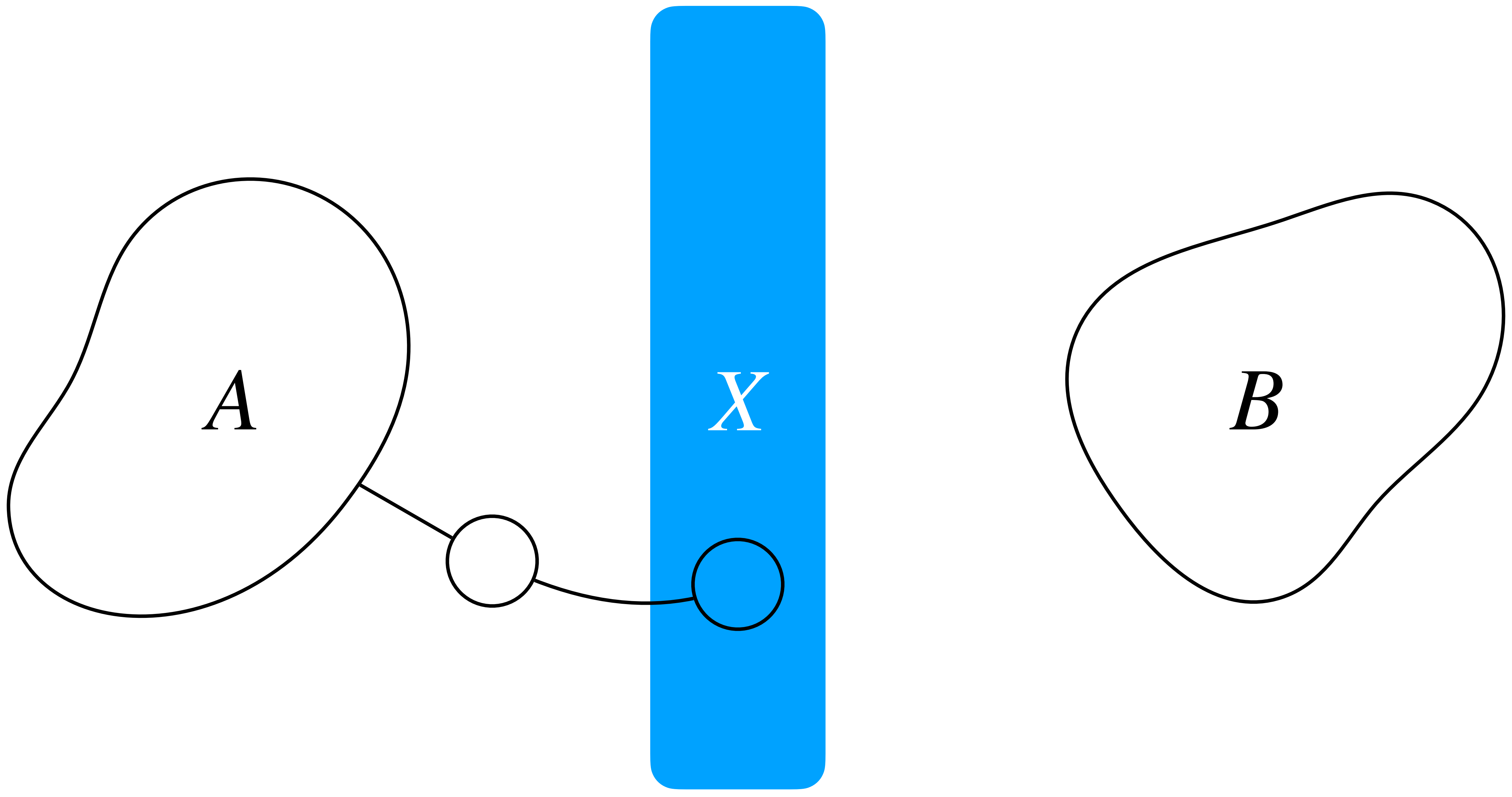

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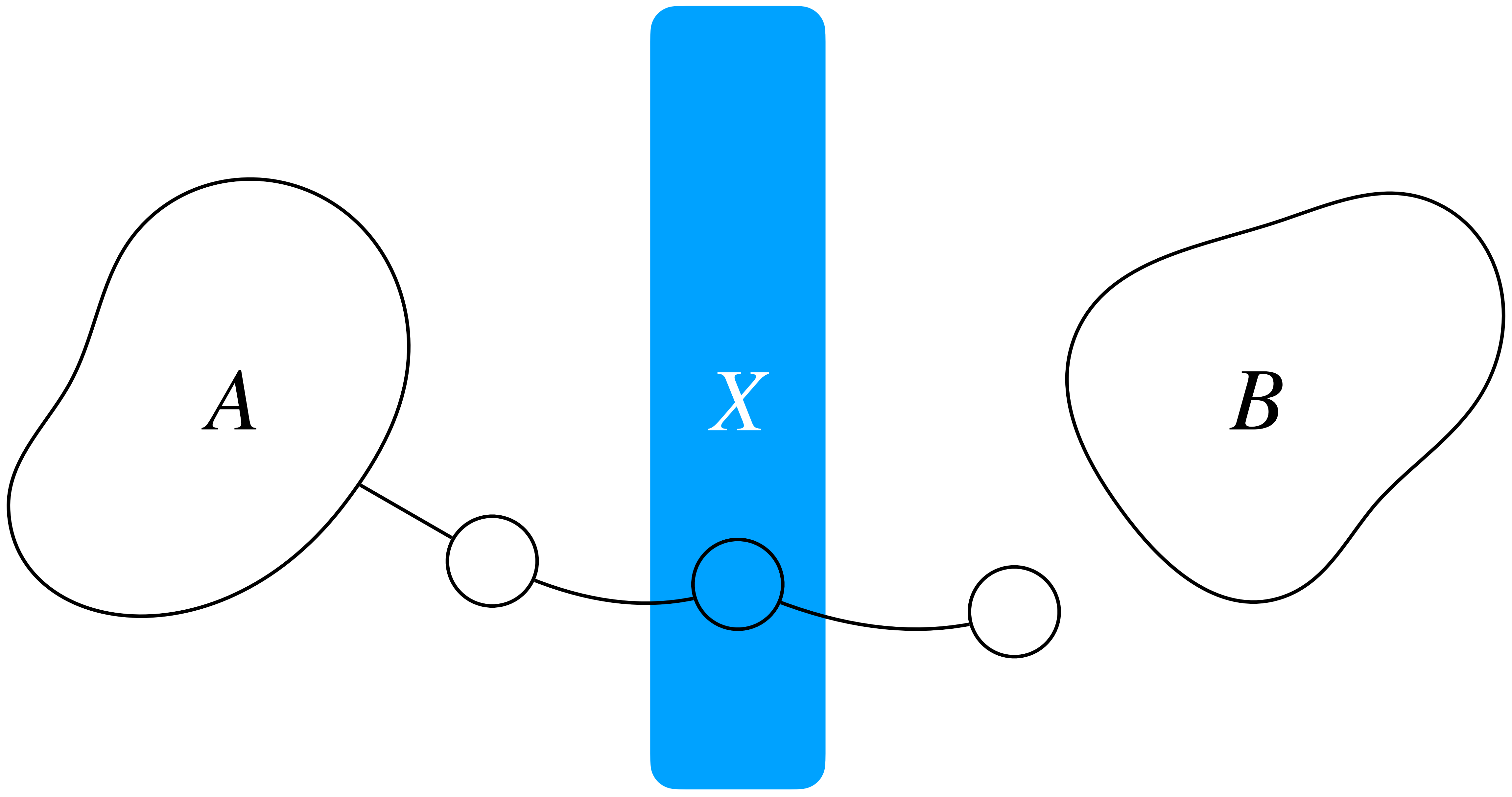

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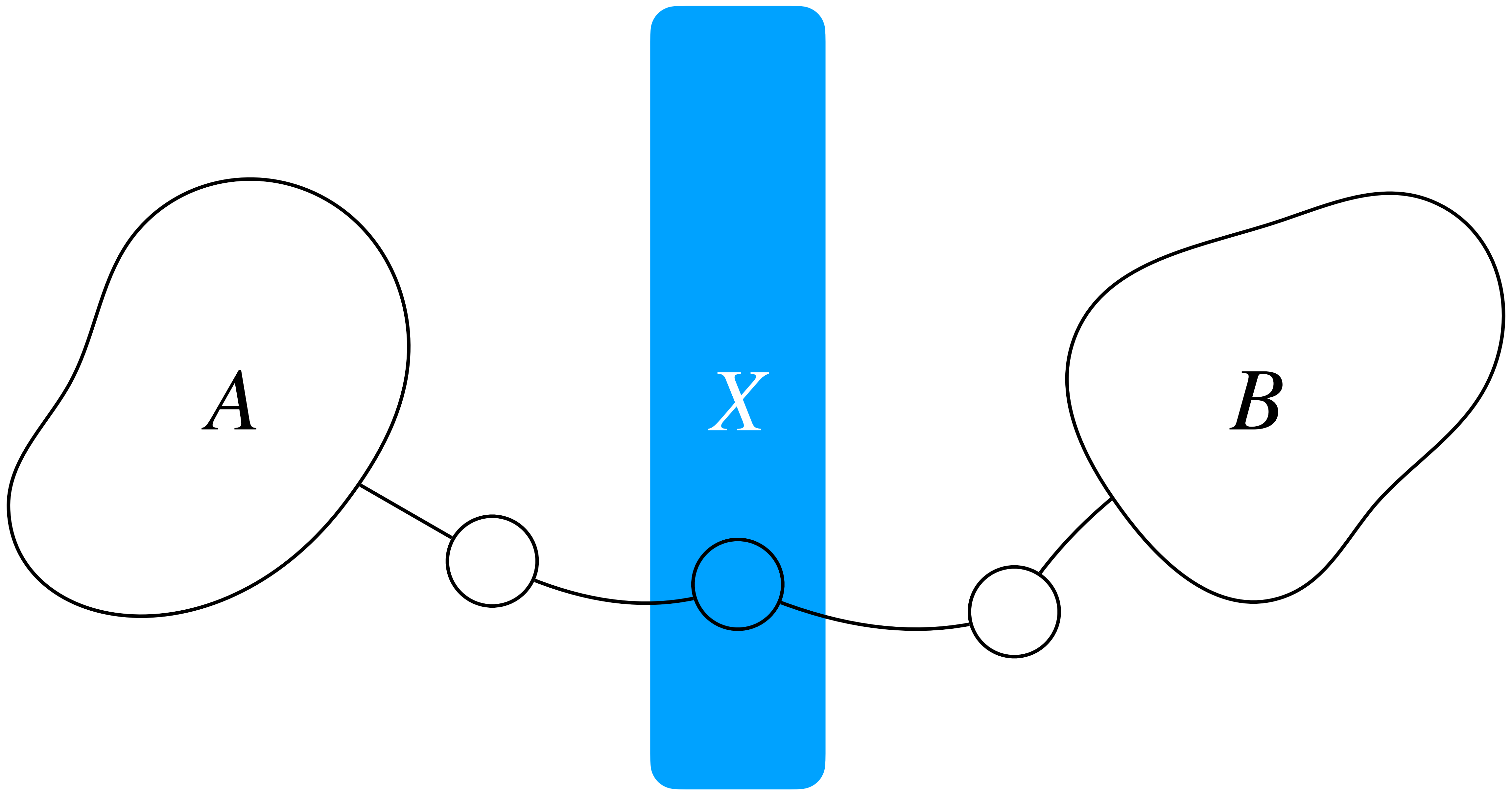

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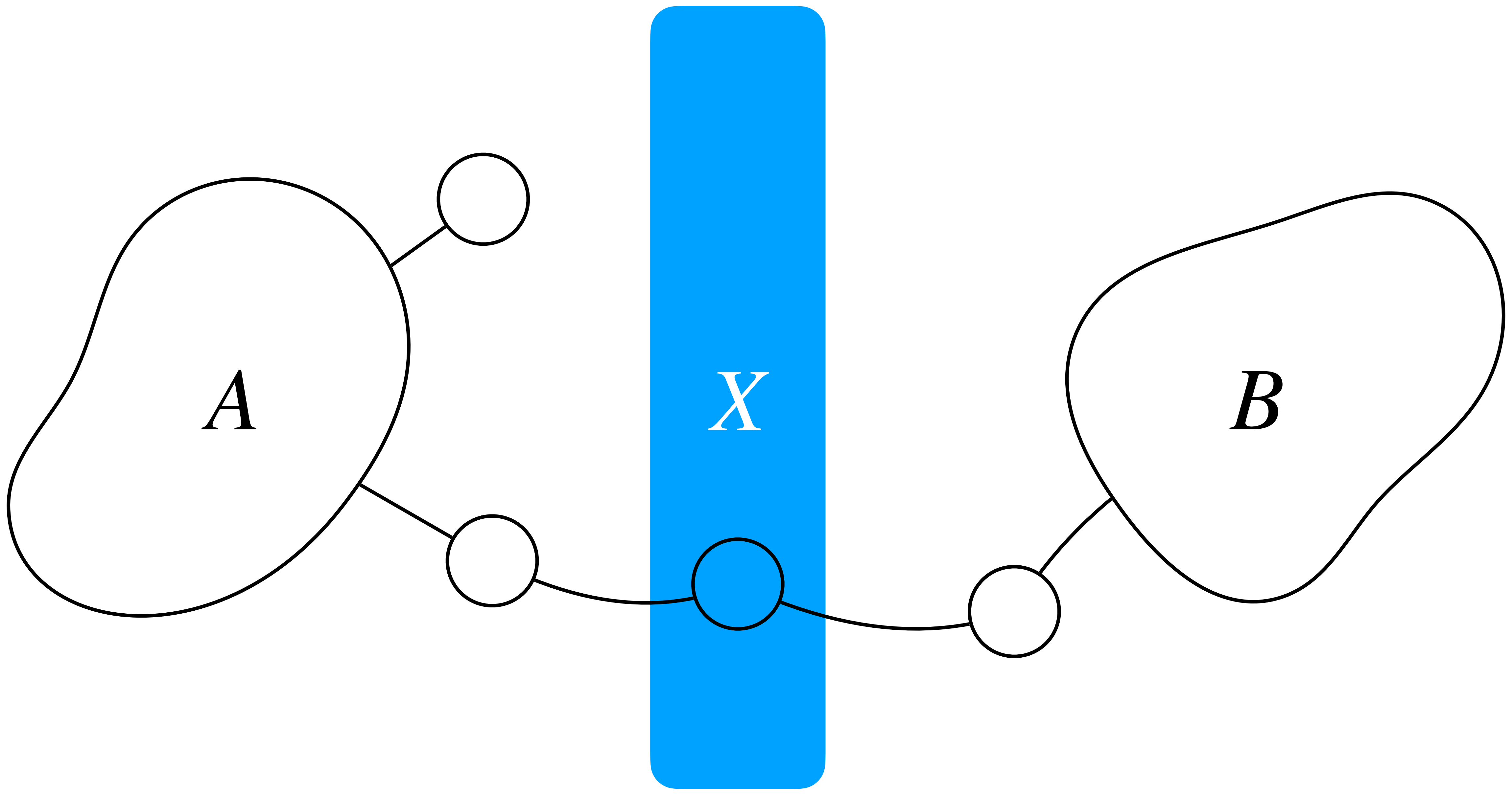

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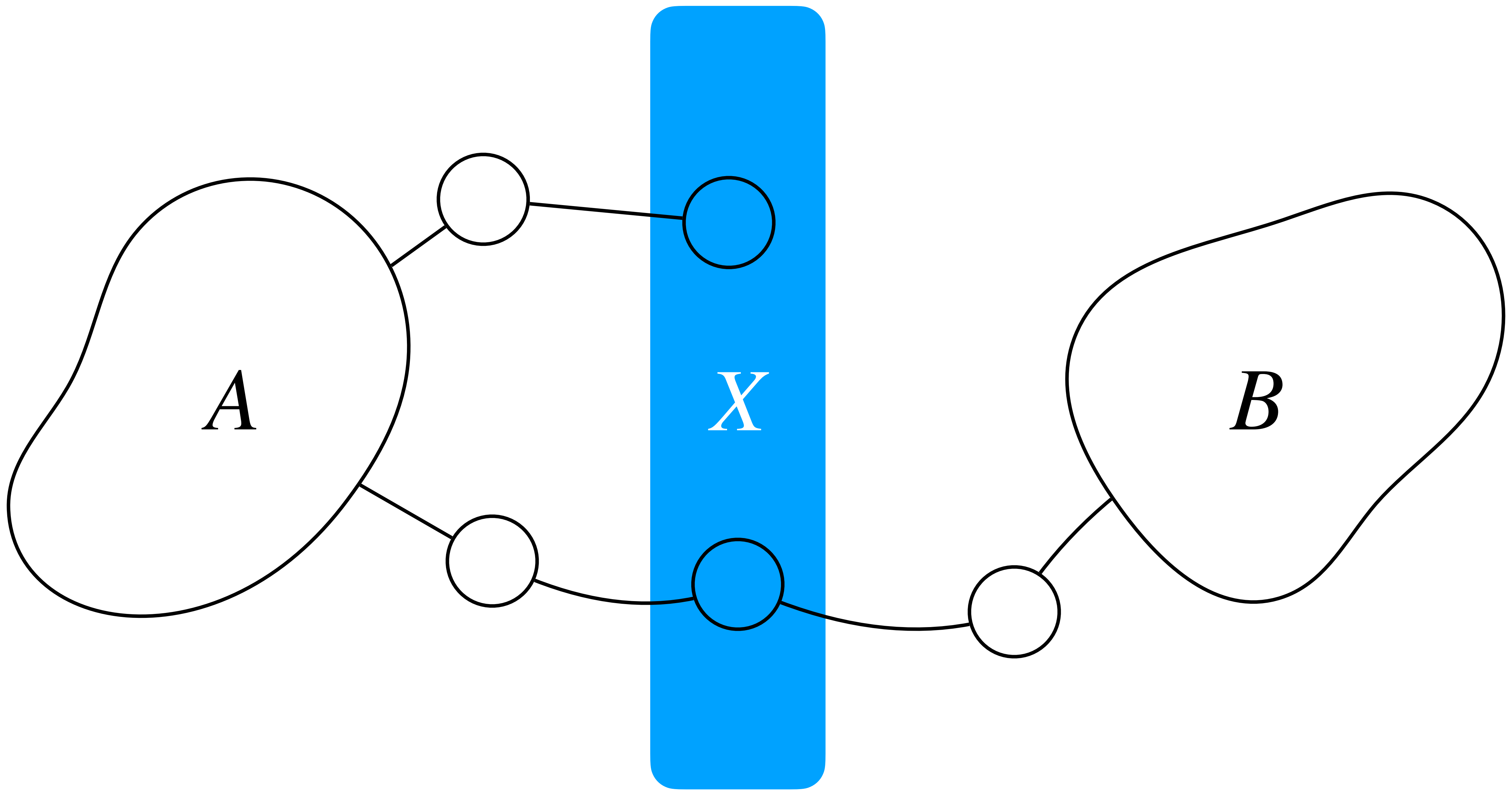

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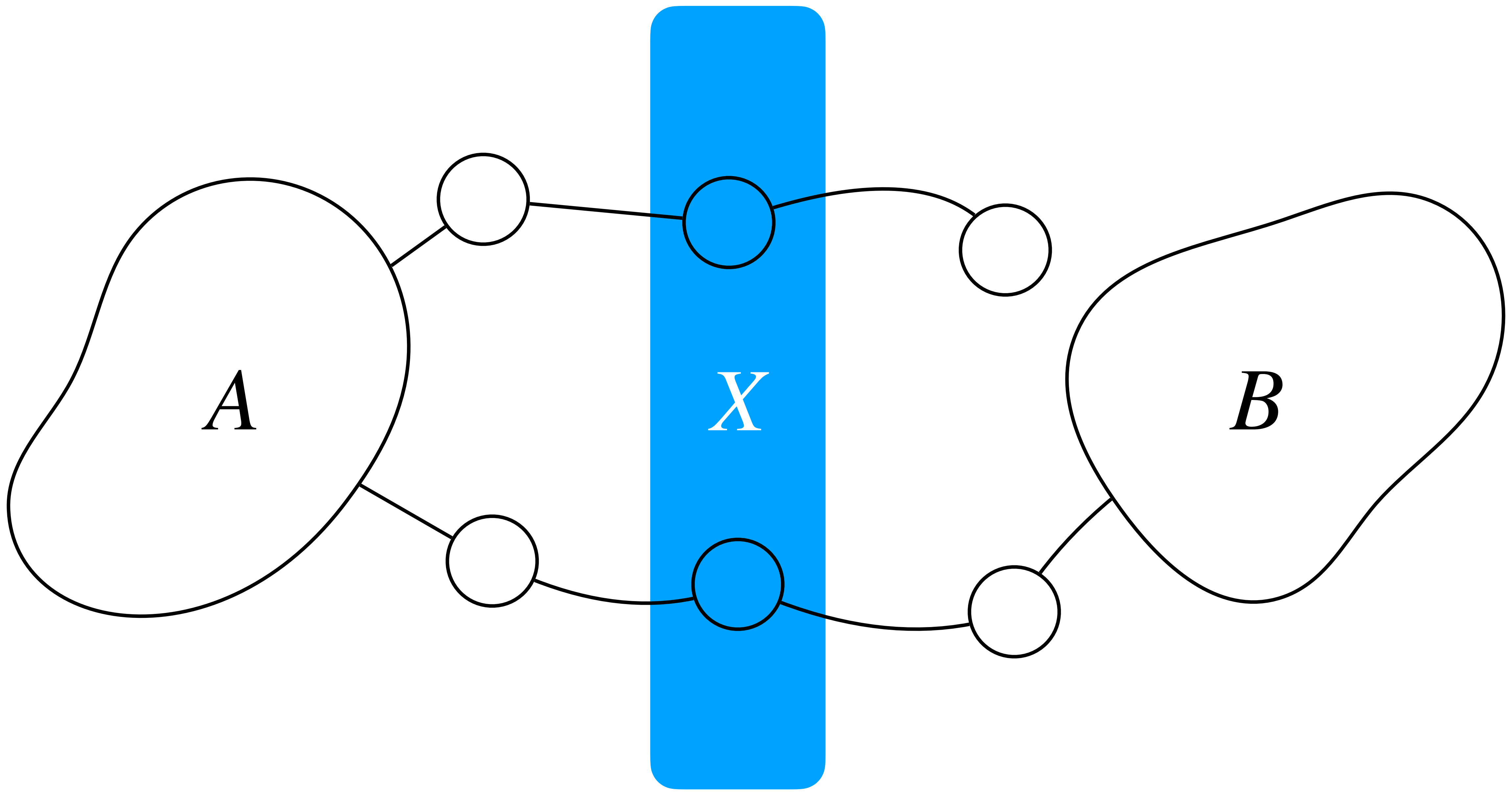

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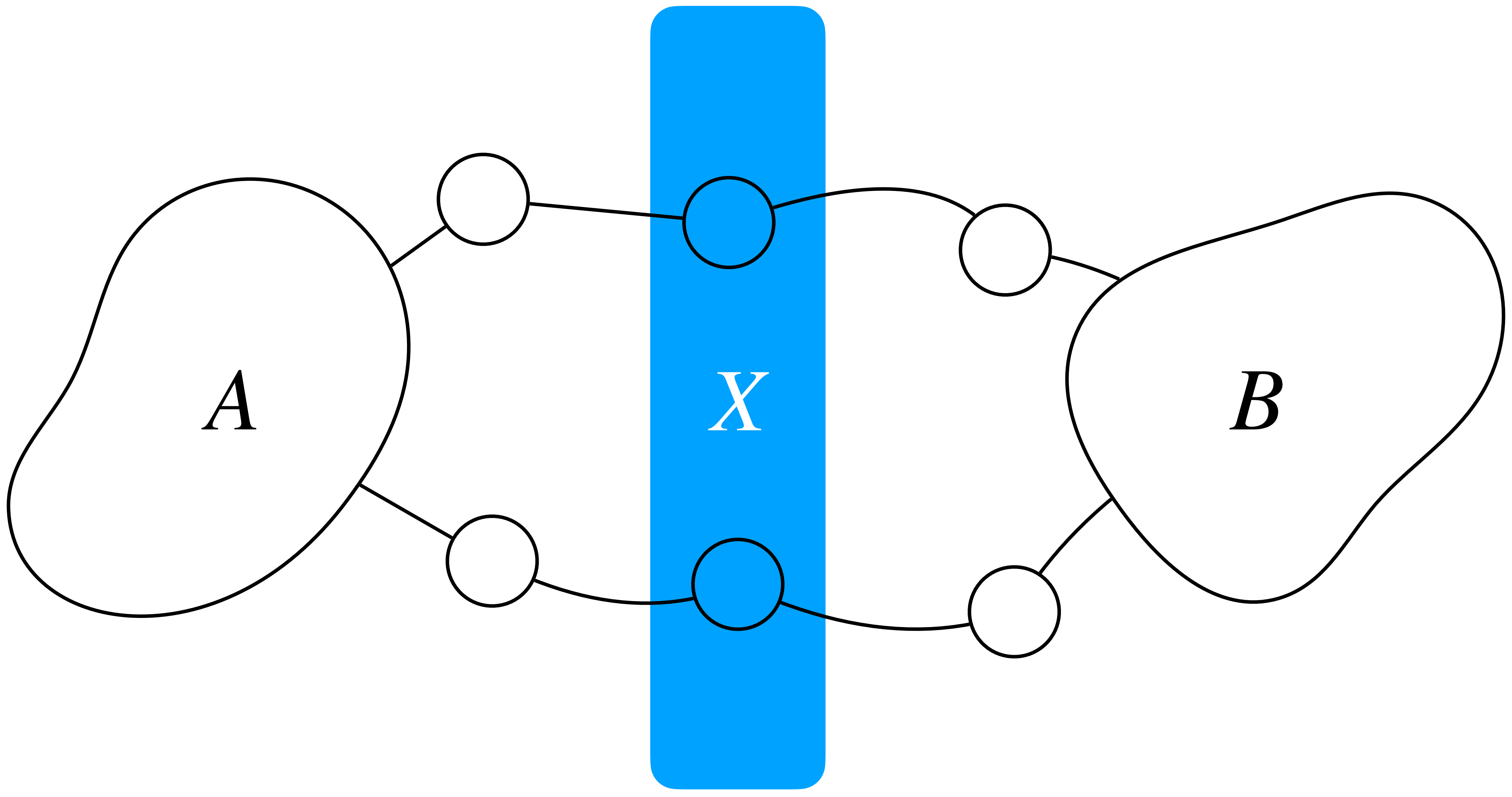

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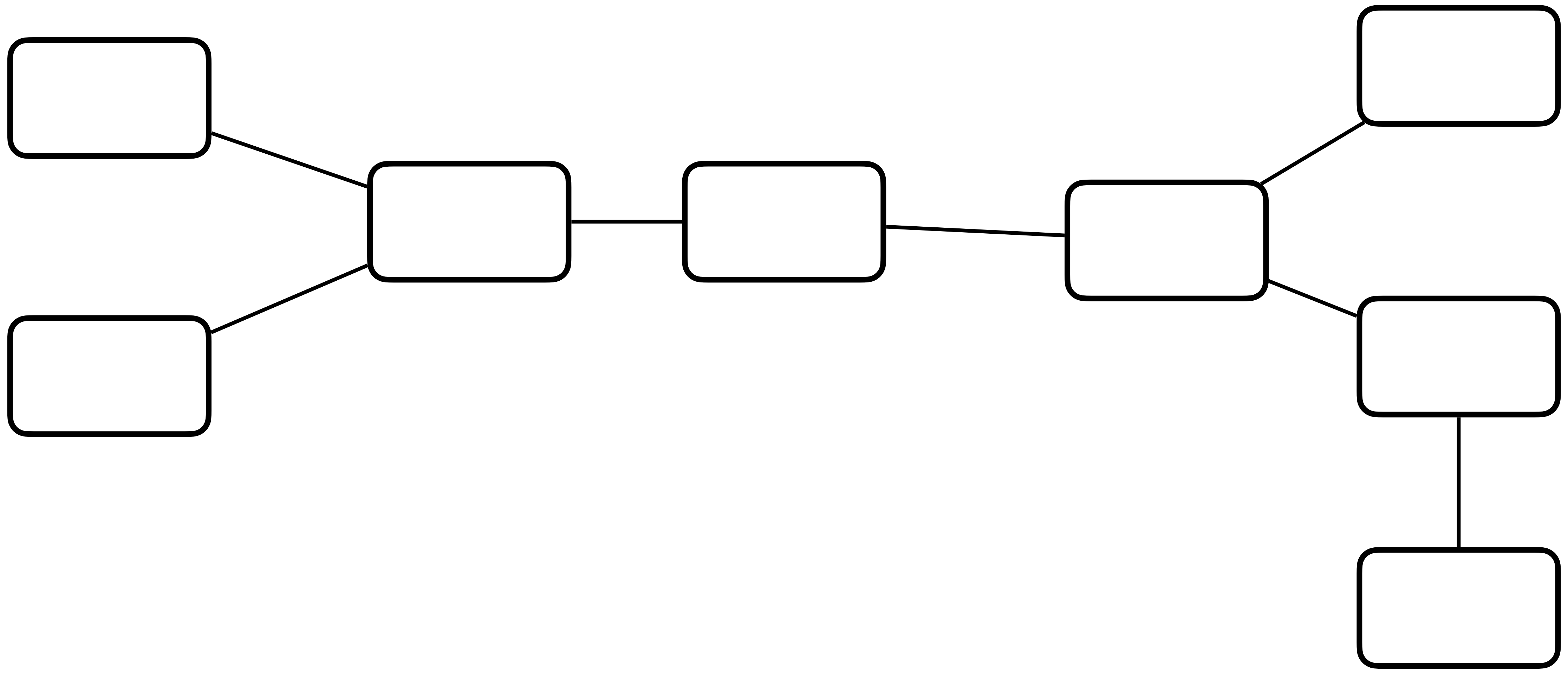

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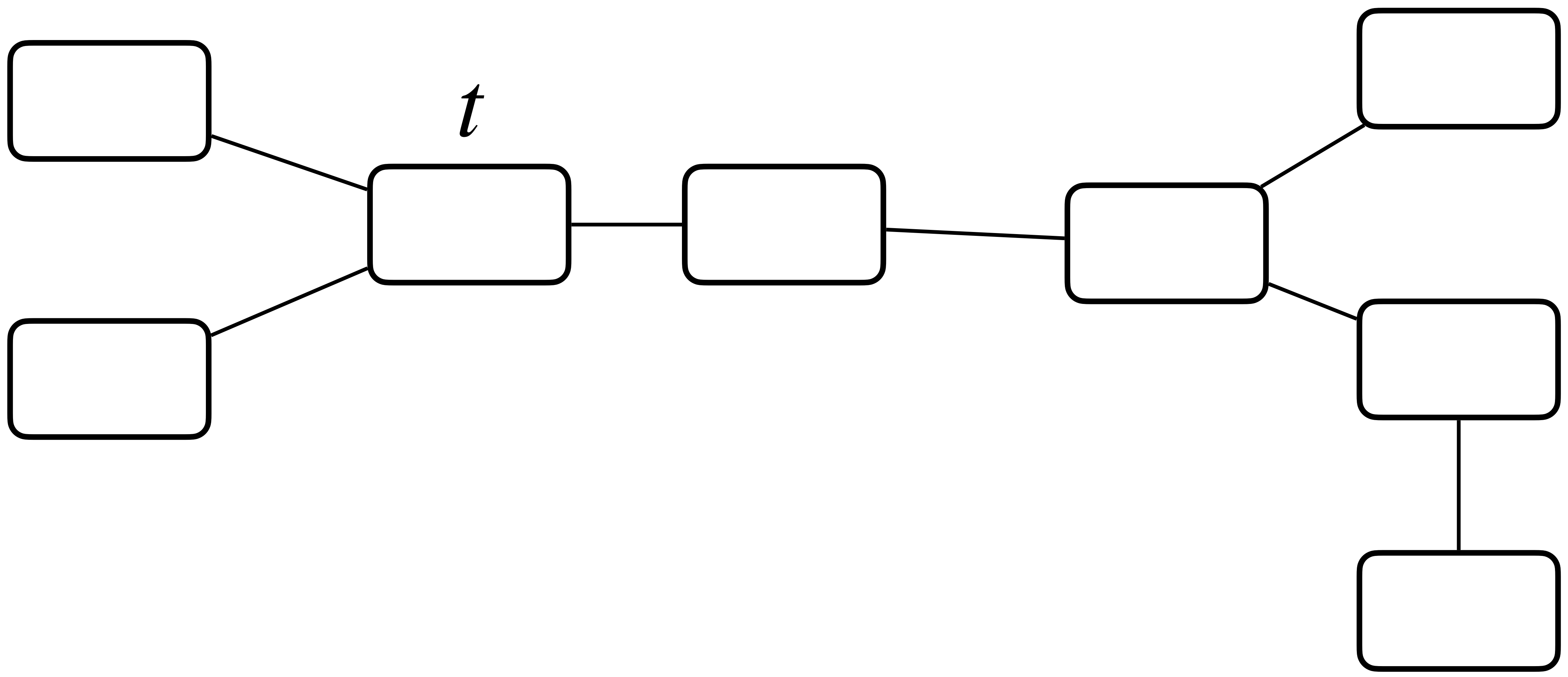

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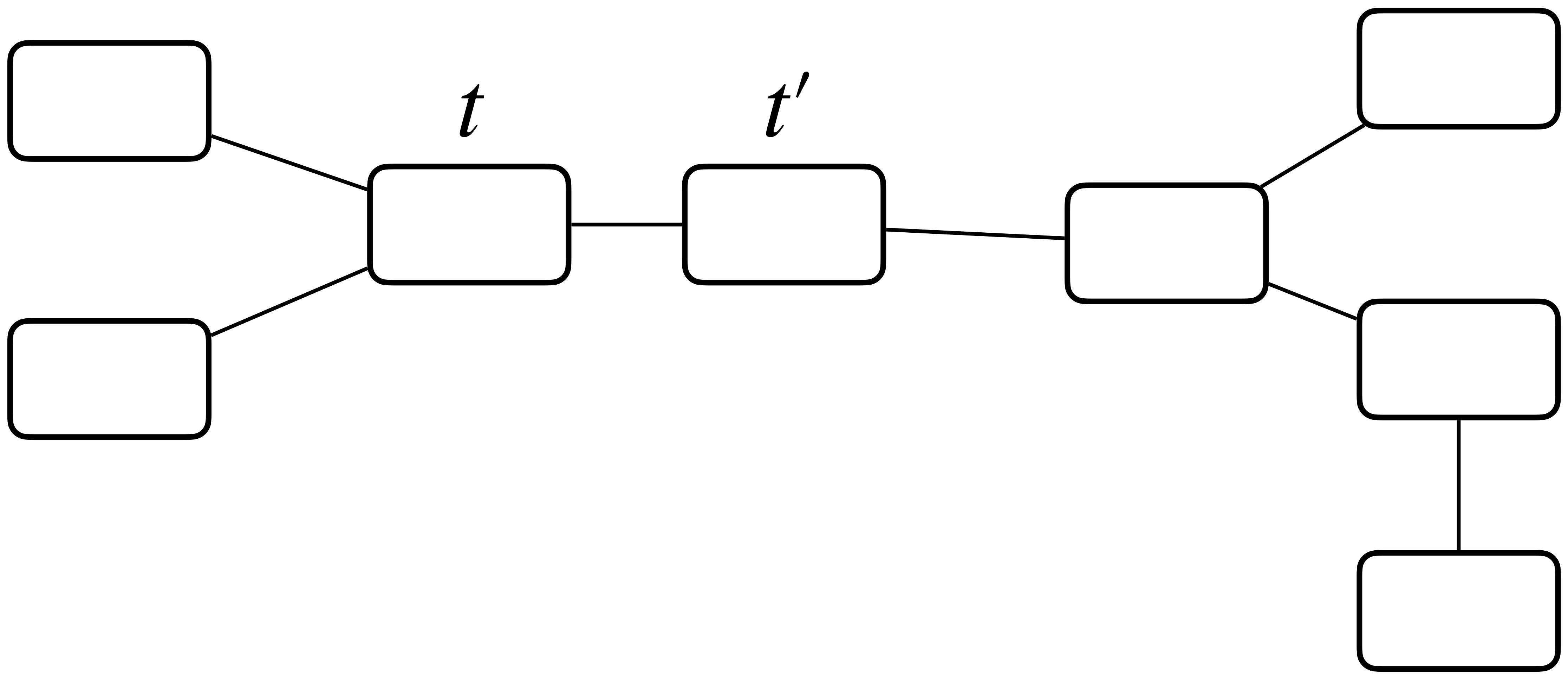

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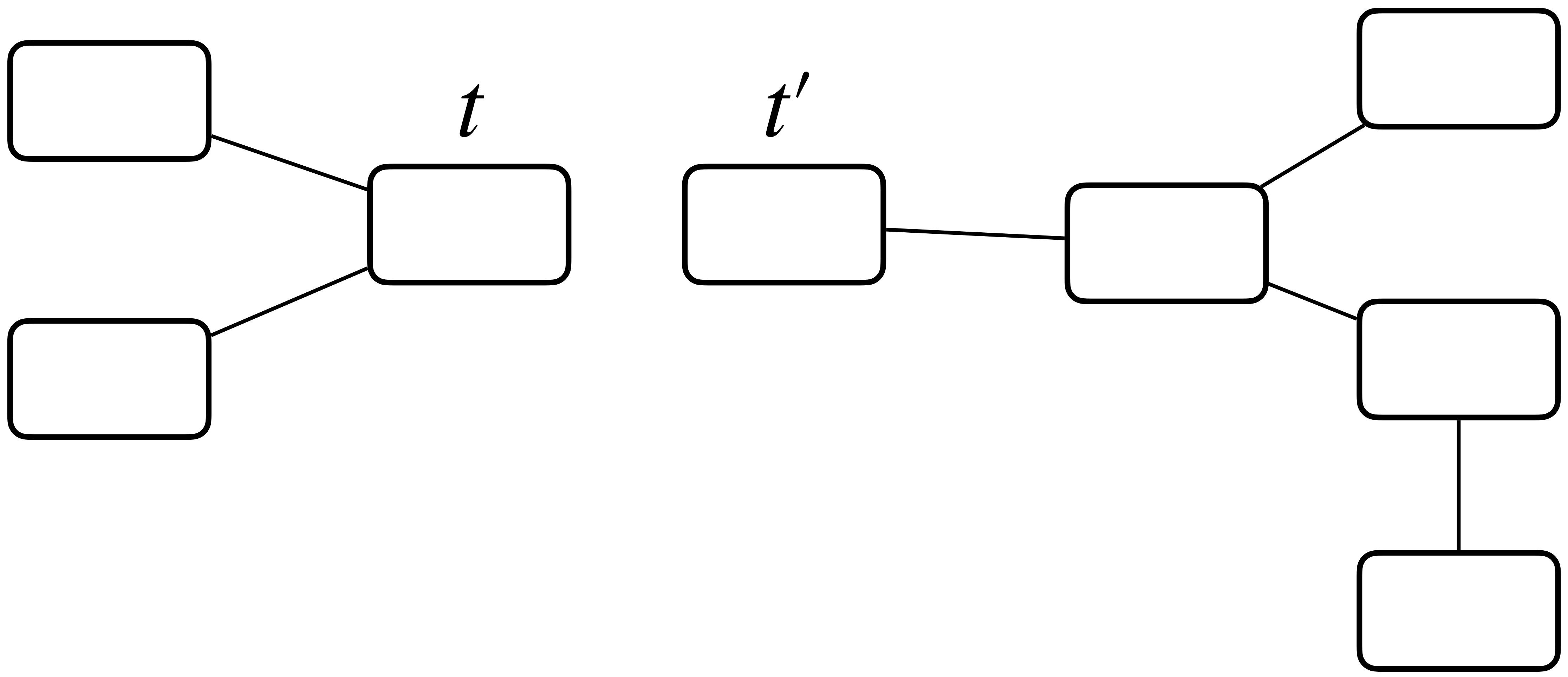

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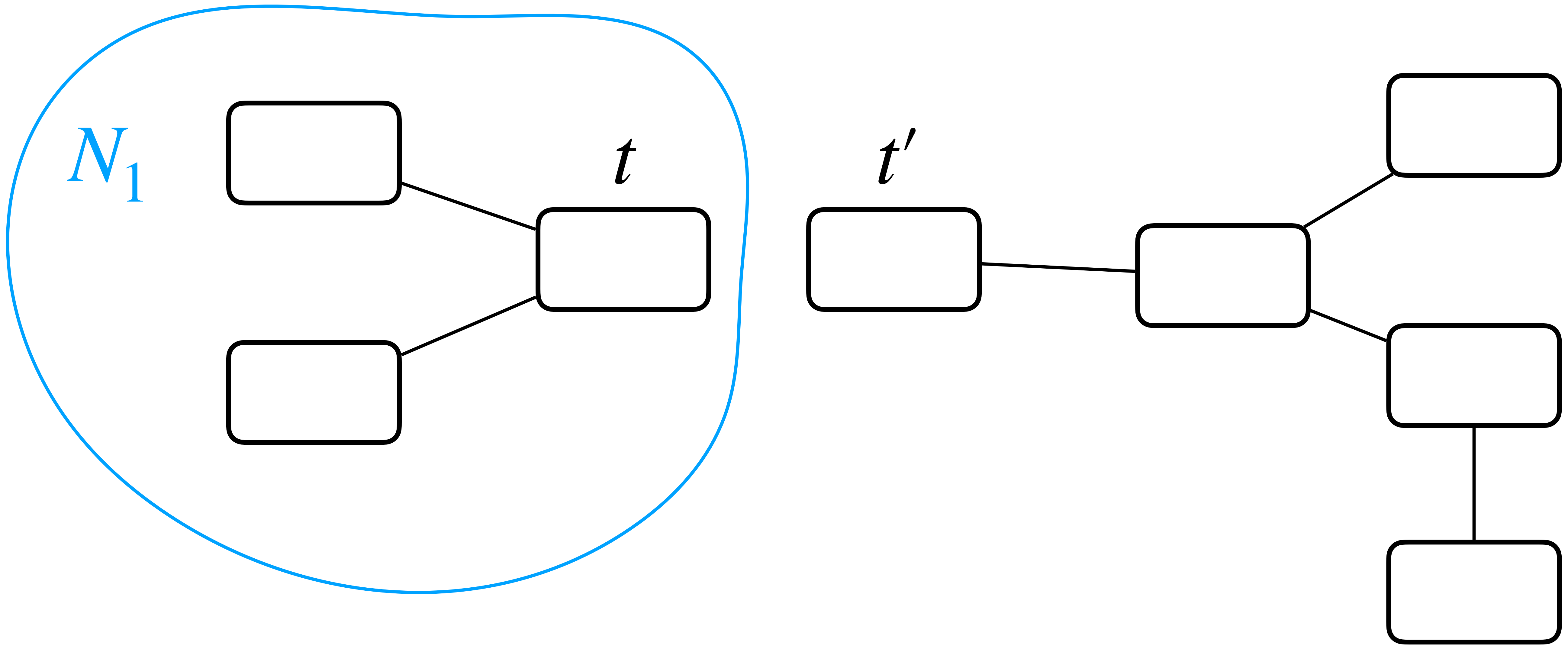

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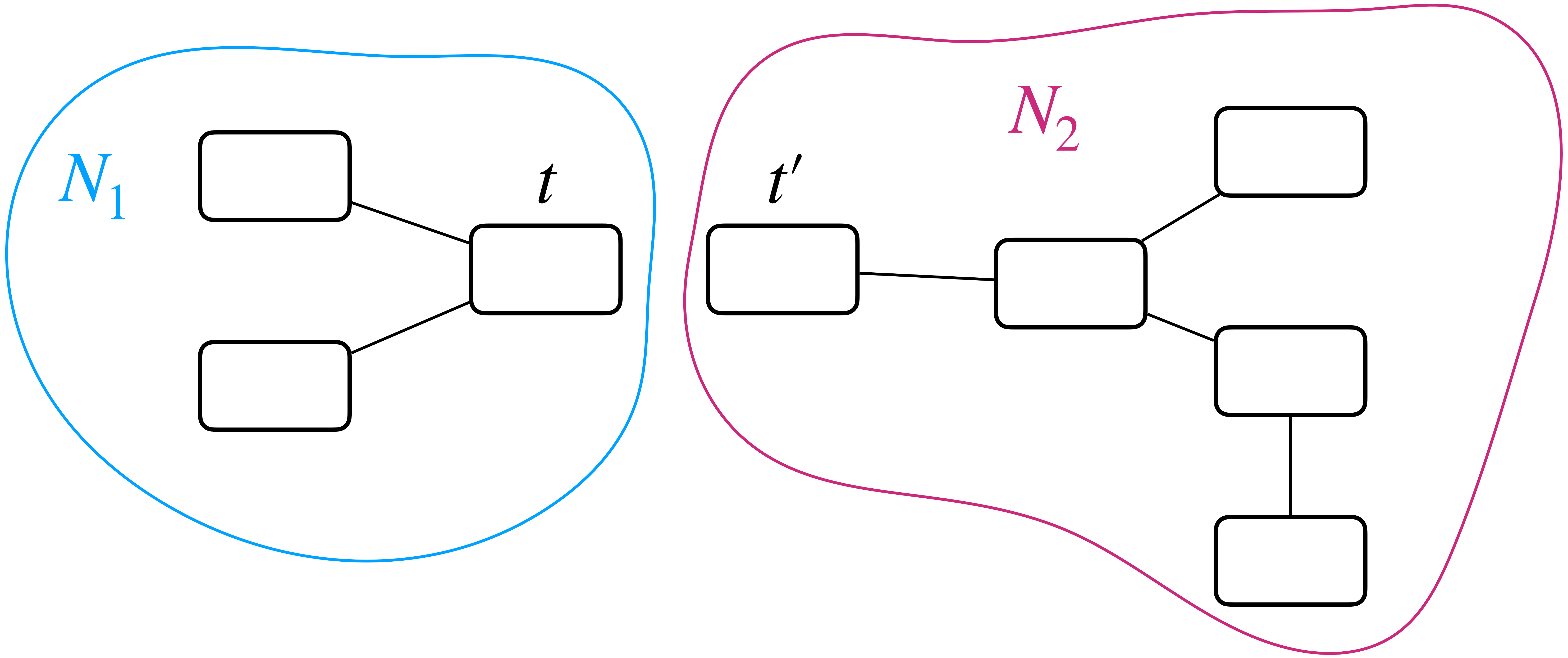

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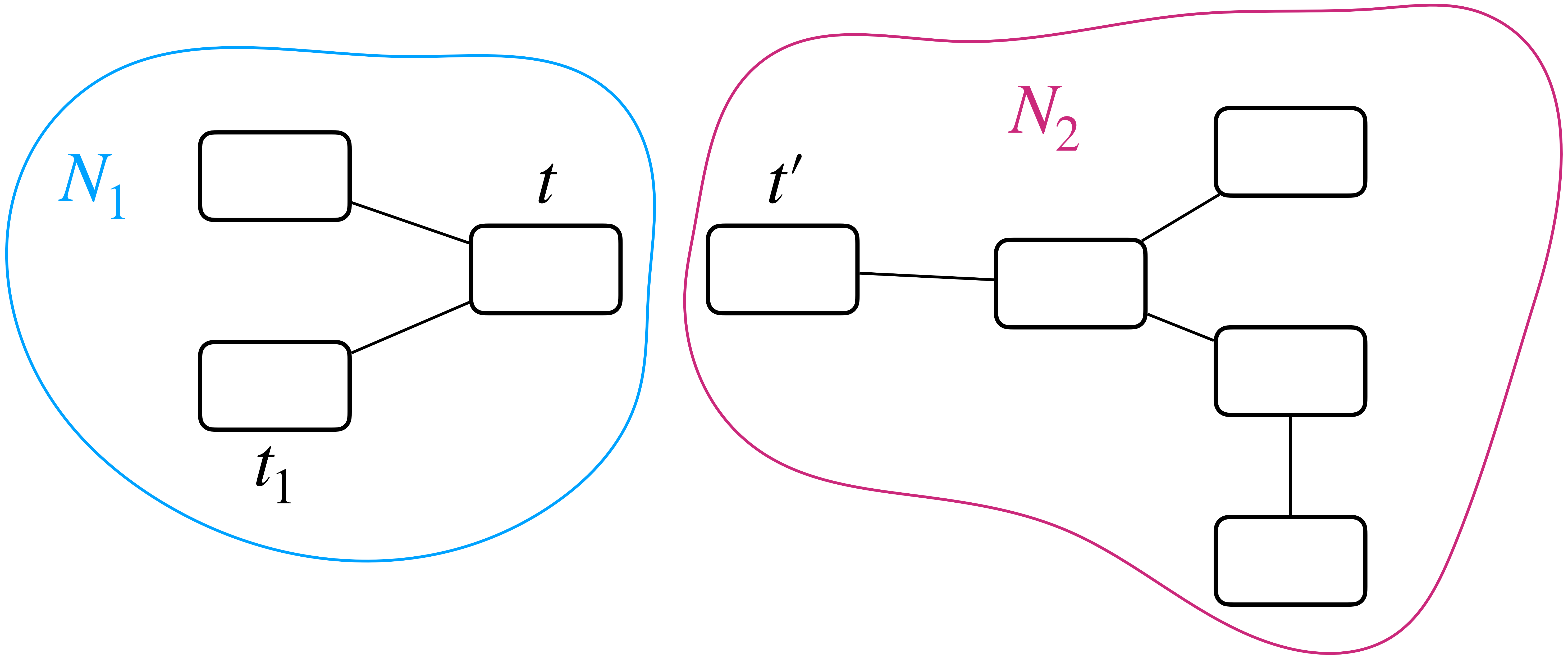

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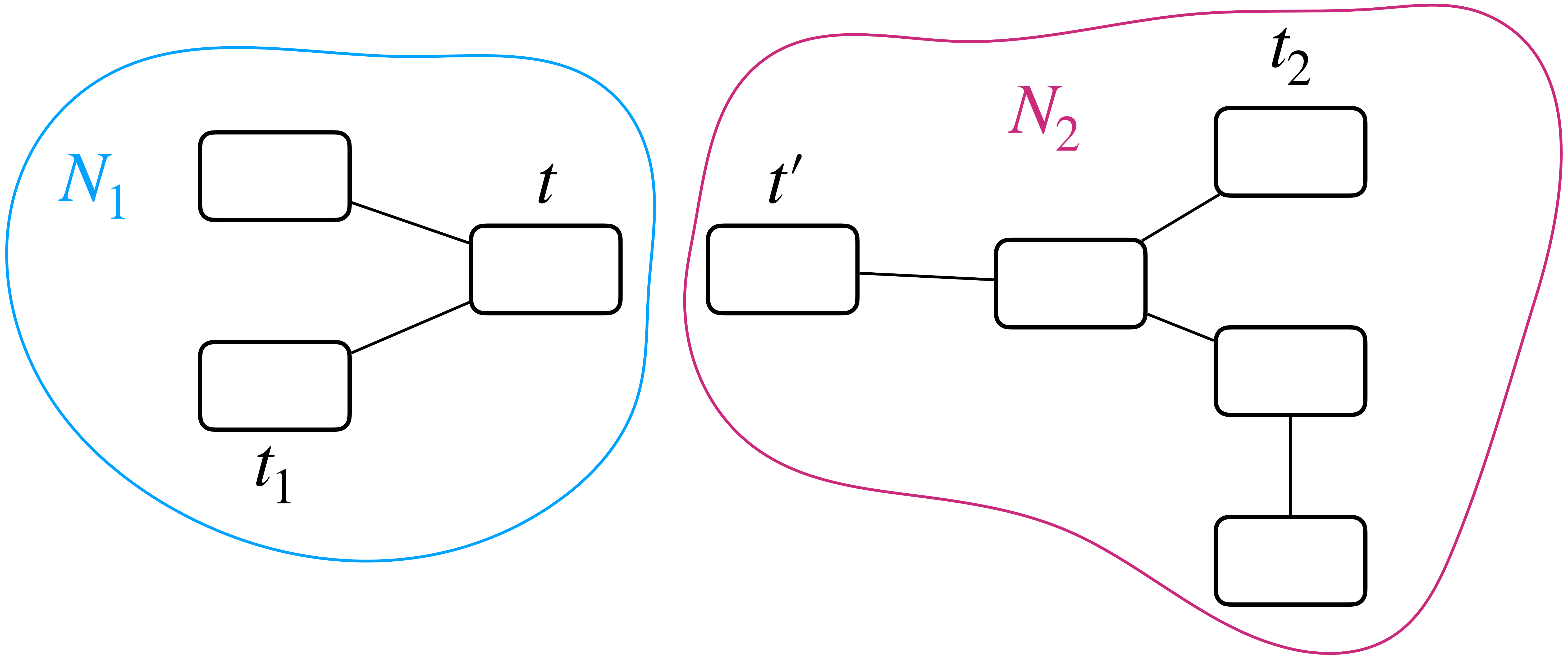

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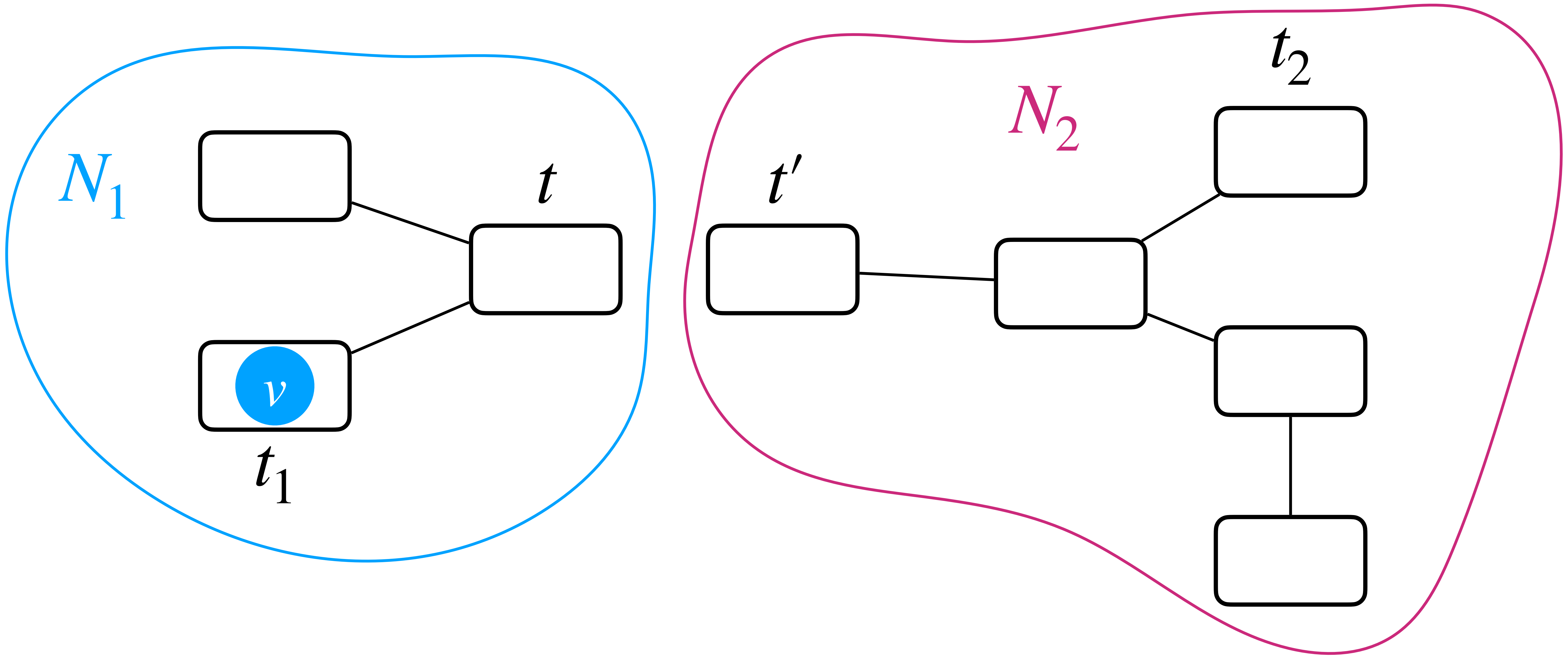

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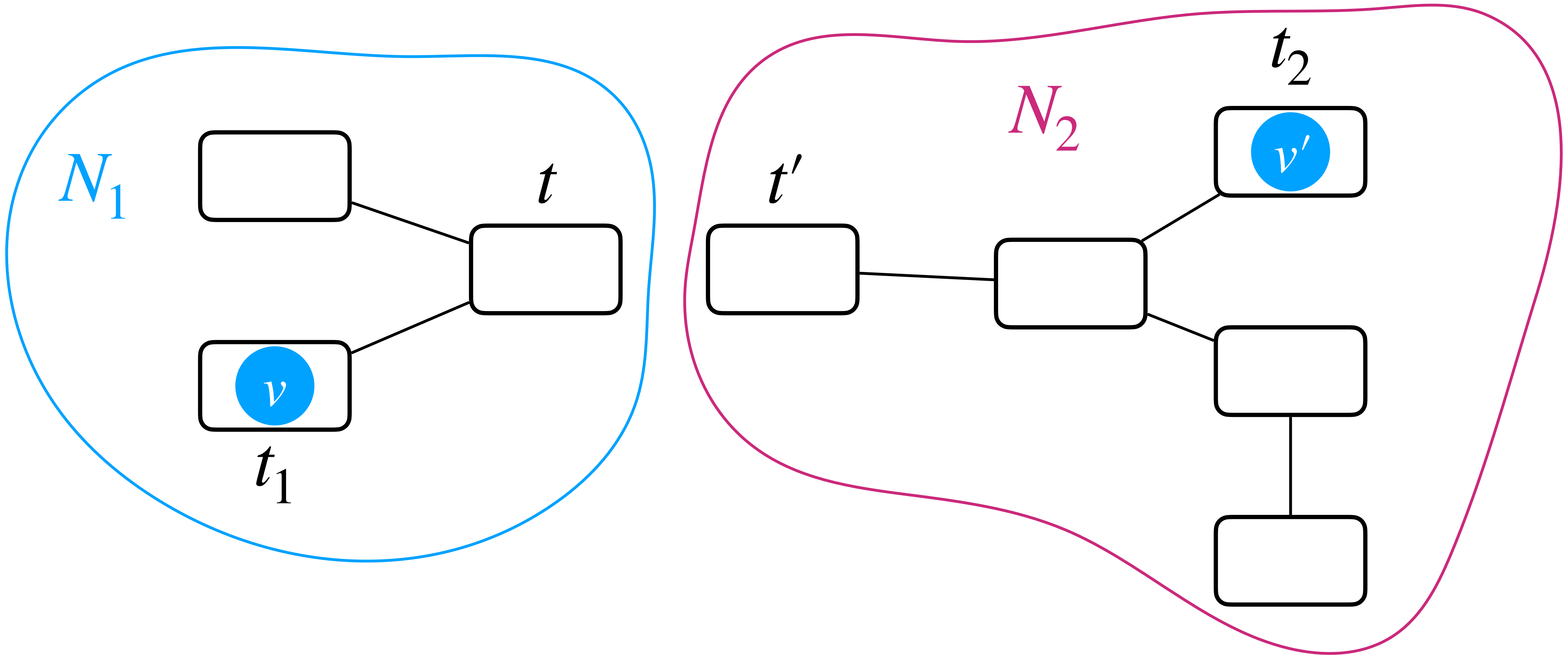

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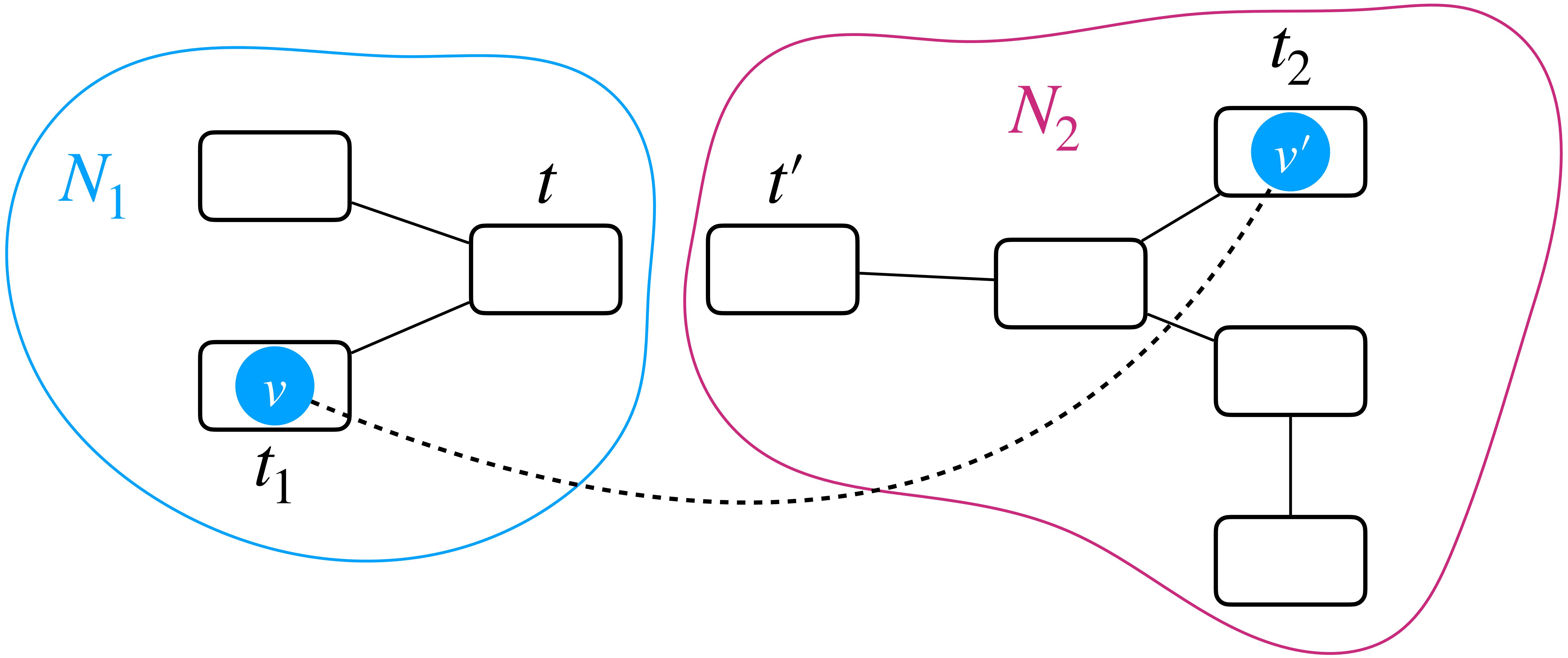

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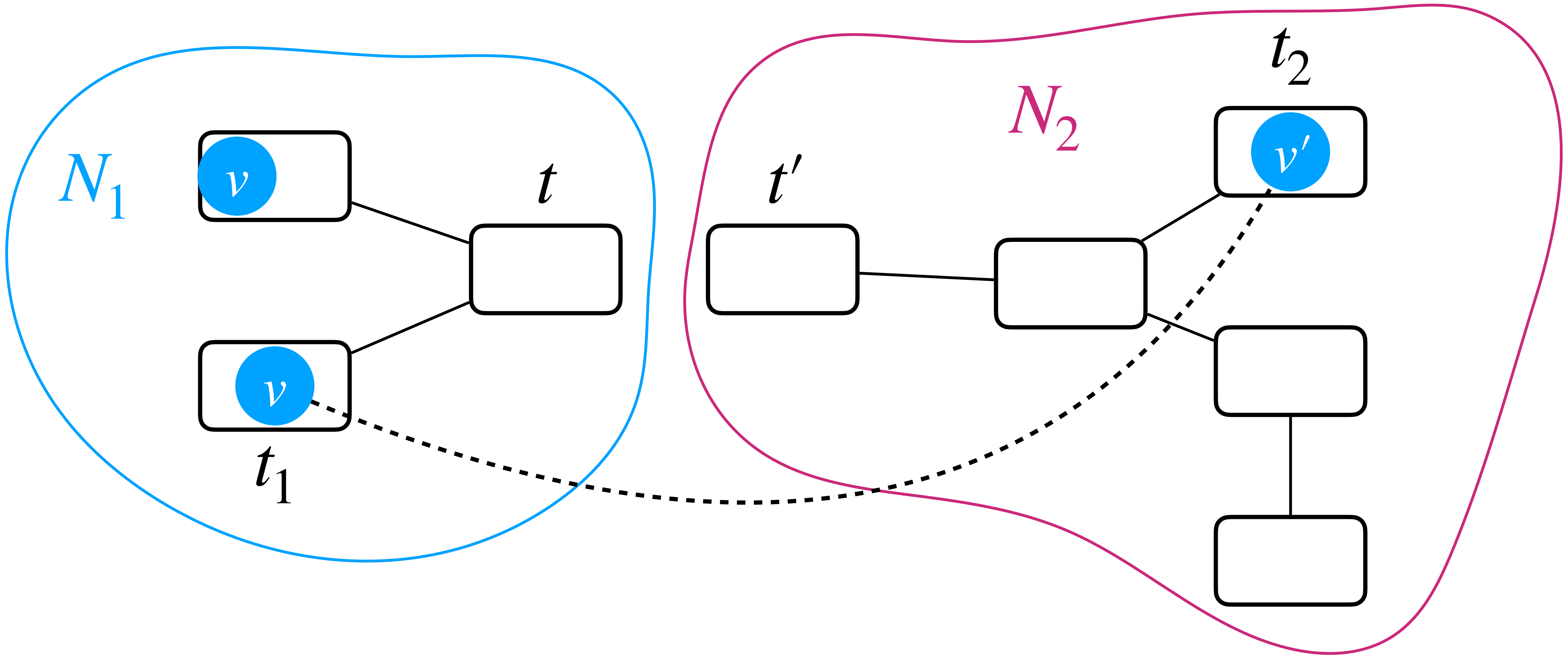

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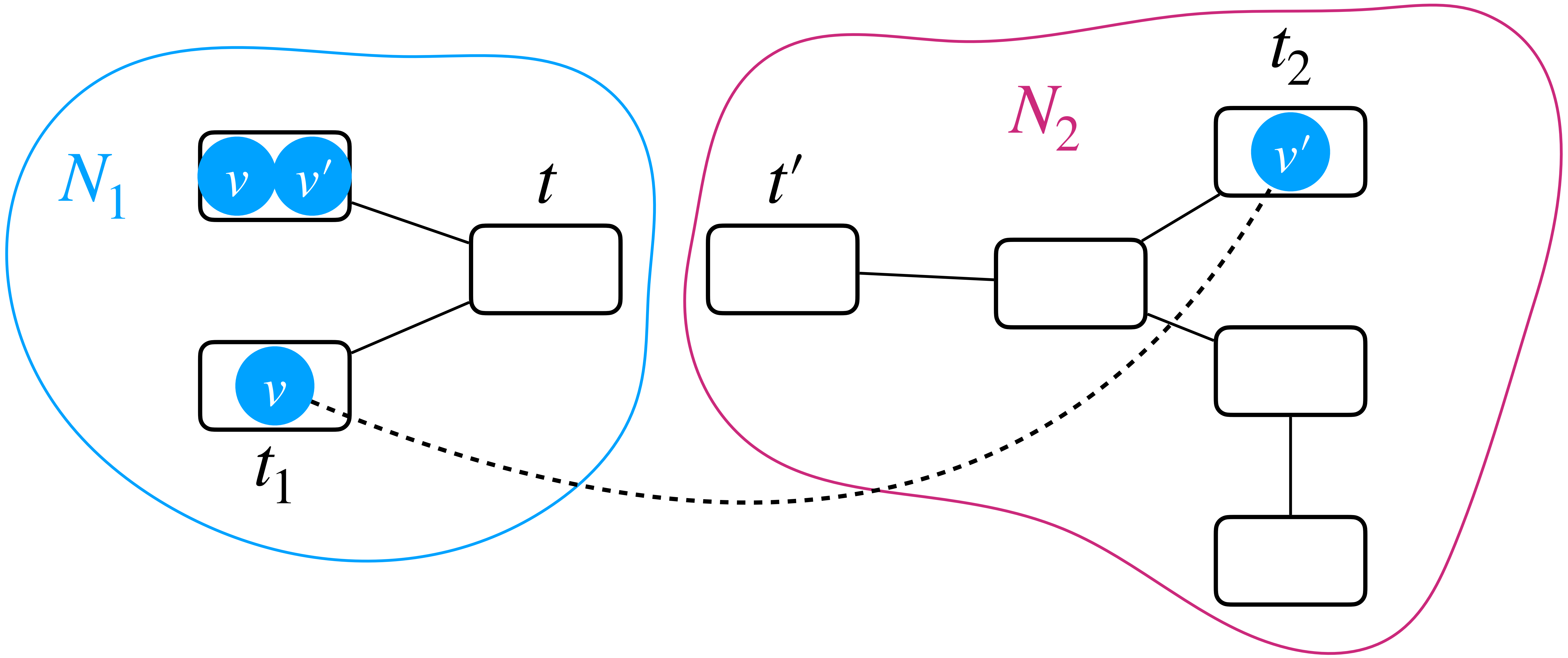

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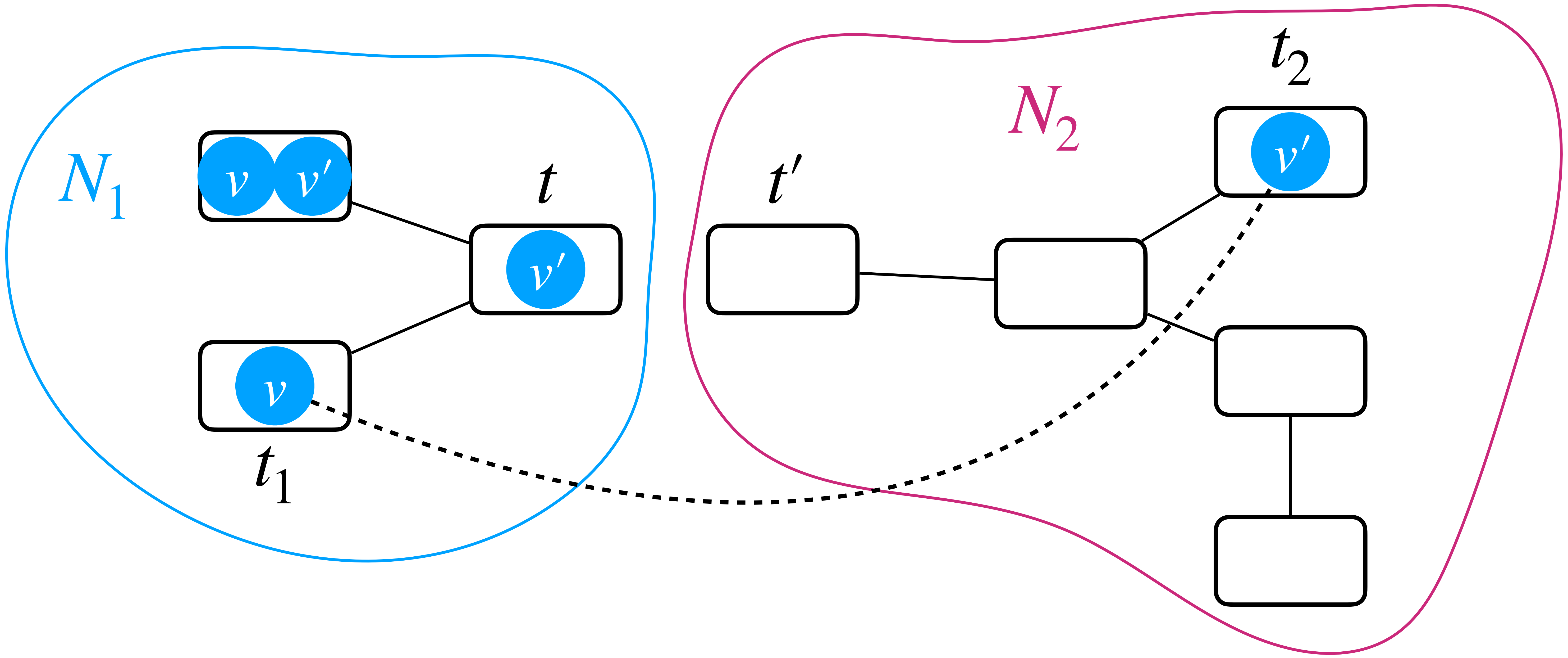

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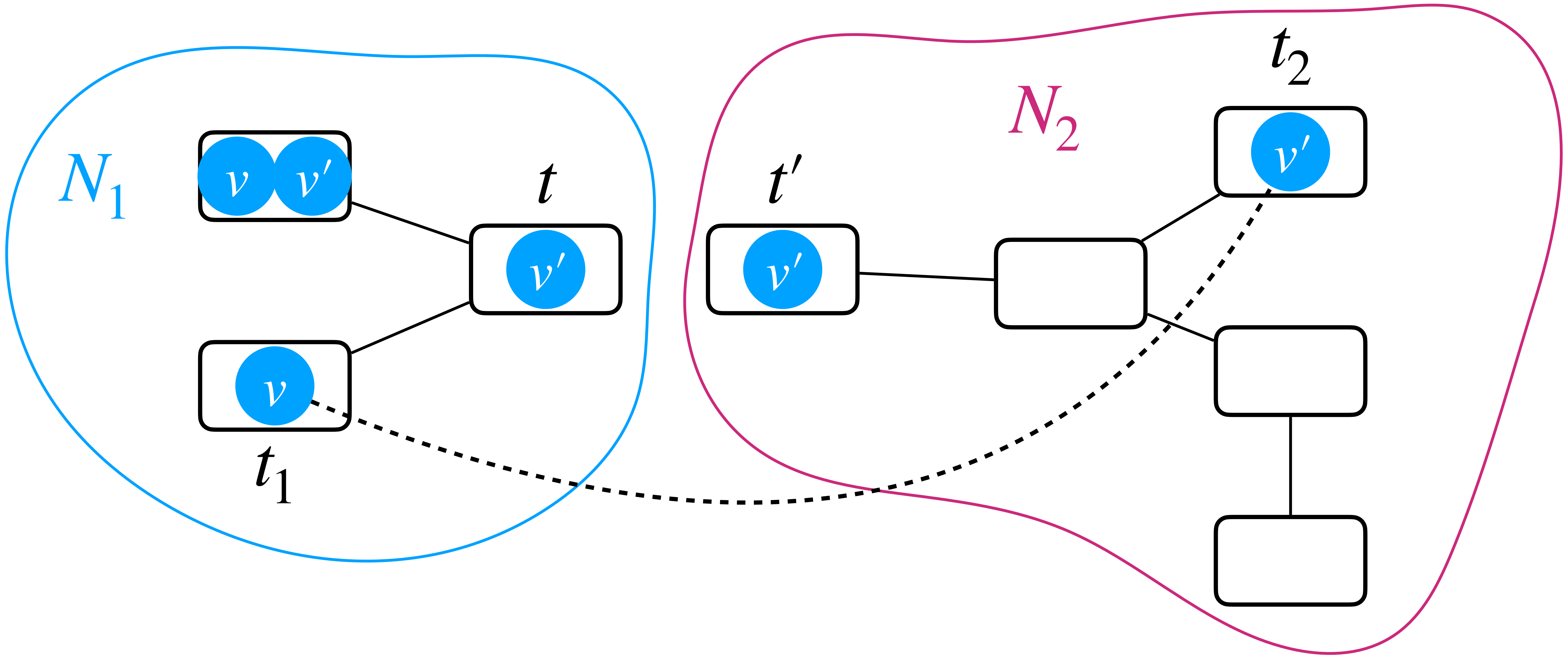

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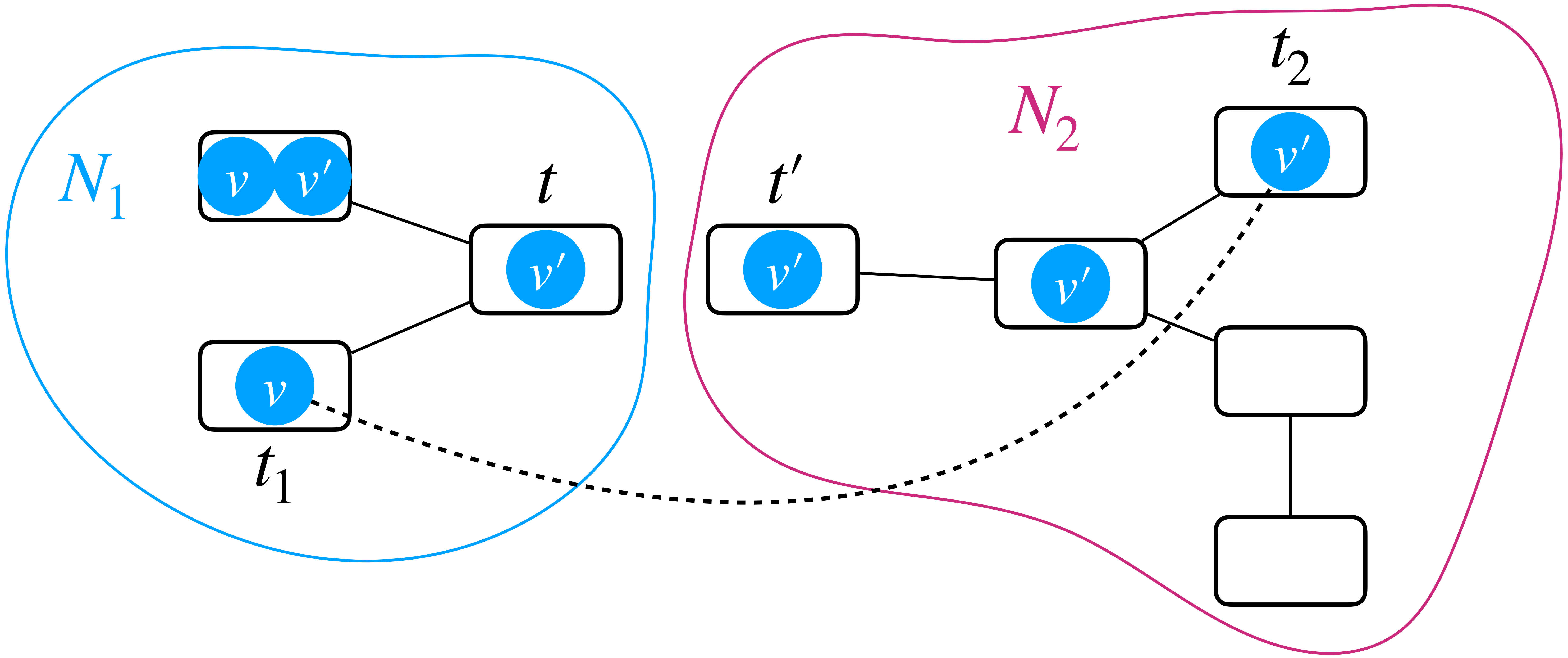

Graph Minors and Treewidth

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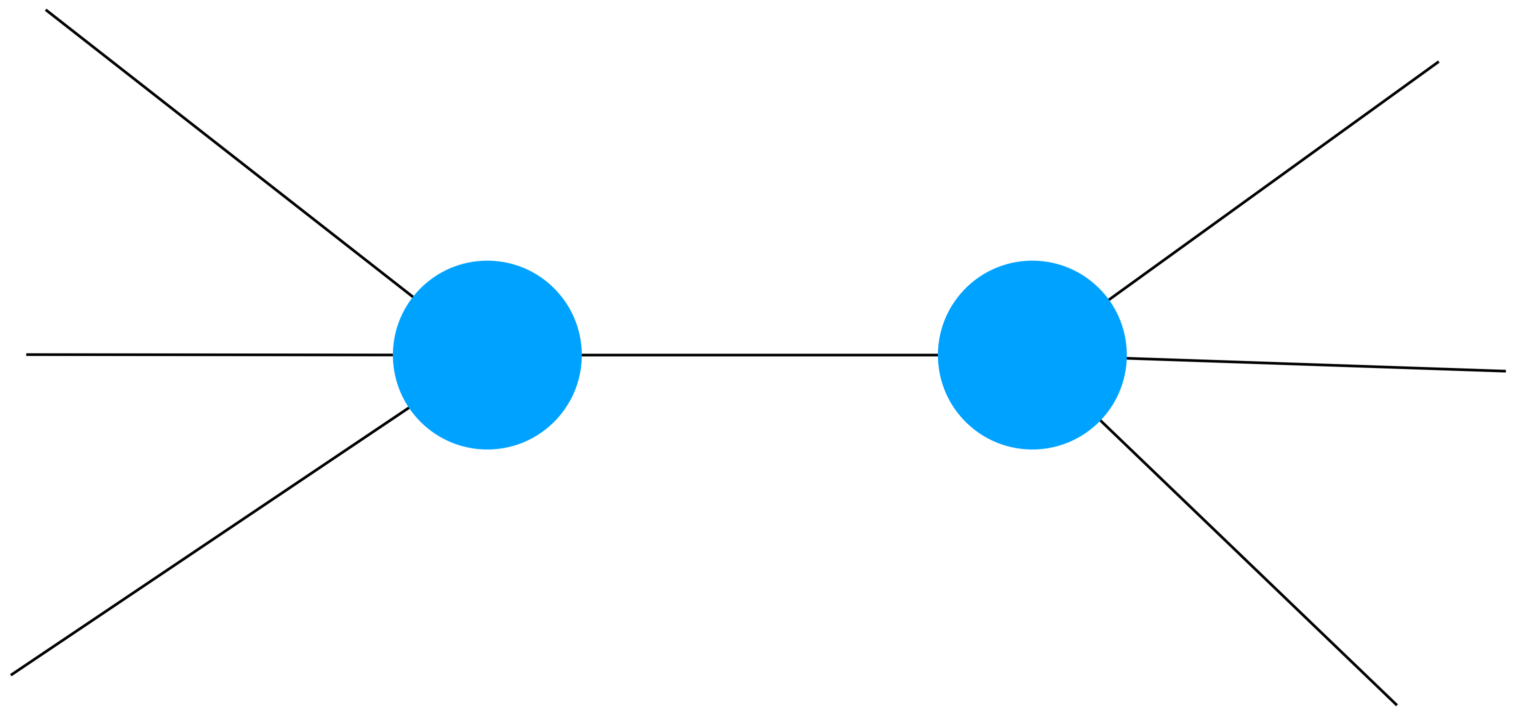

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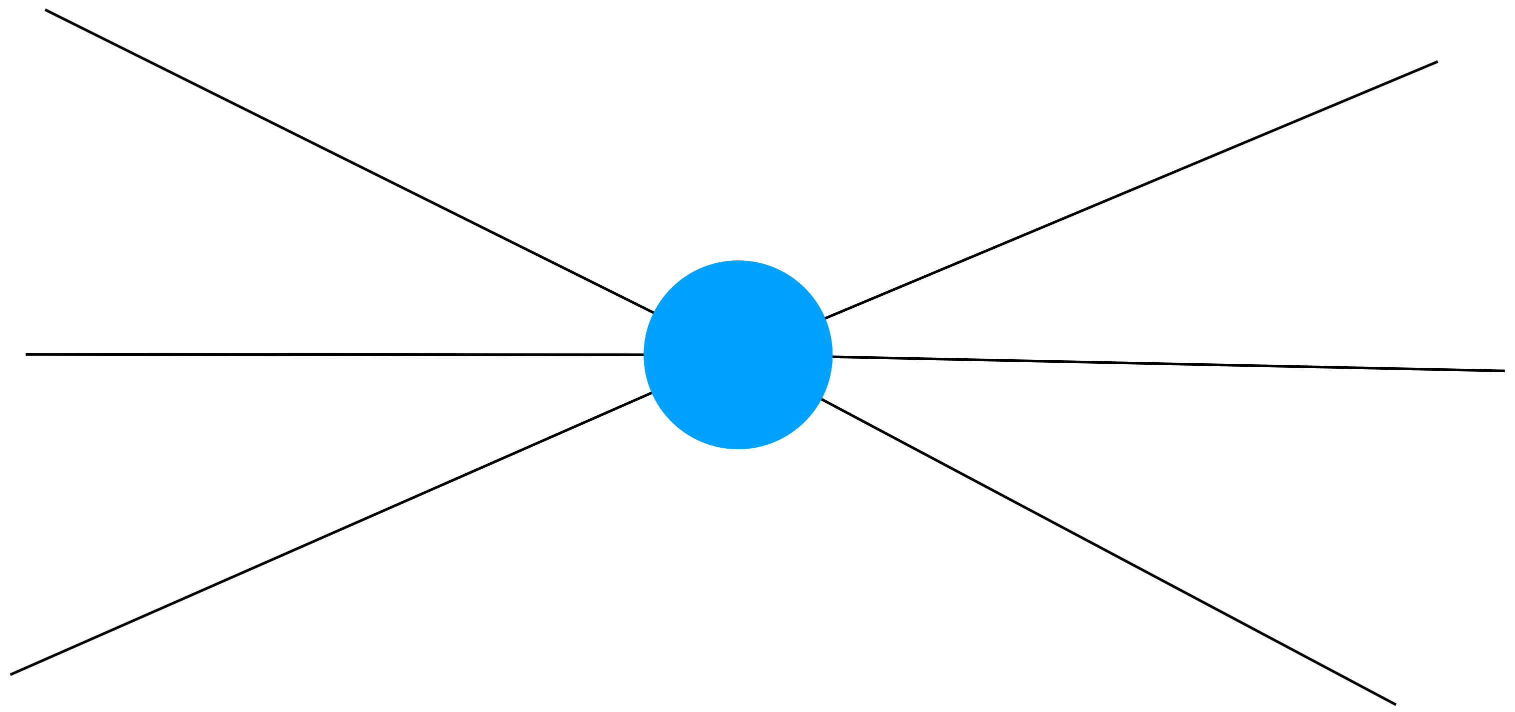

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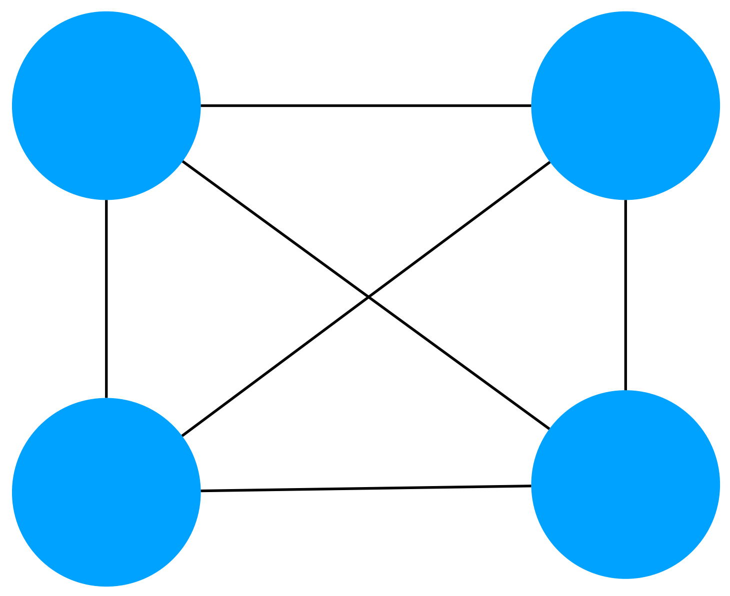

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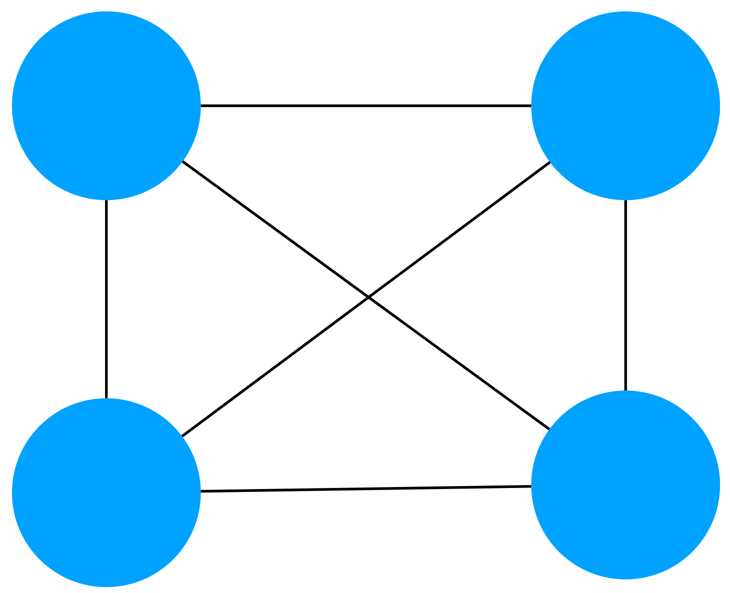

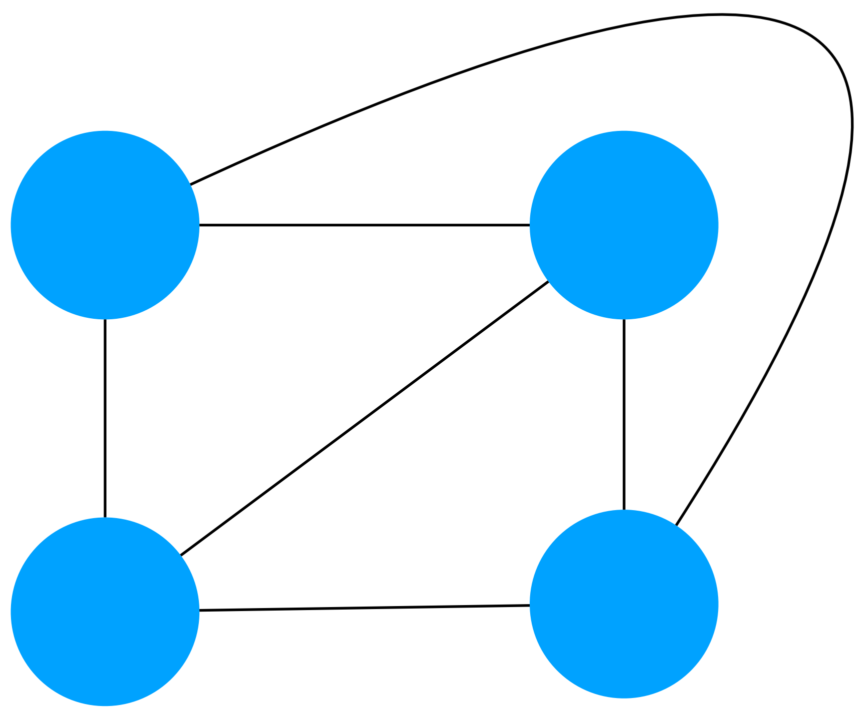

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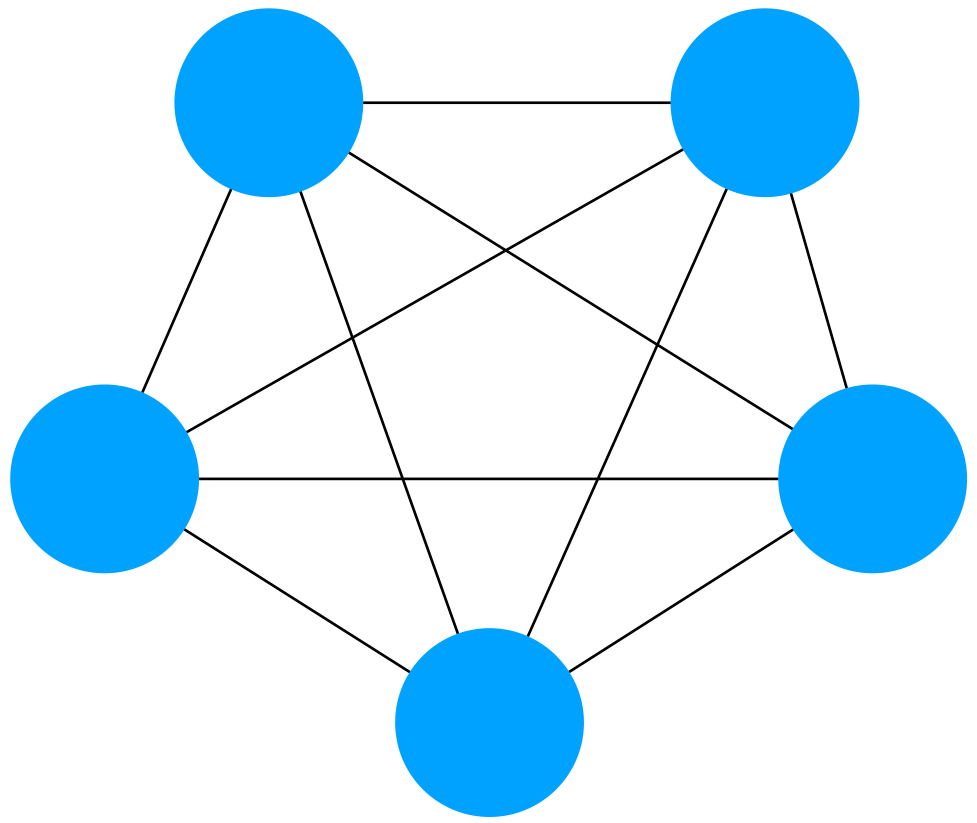

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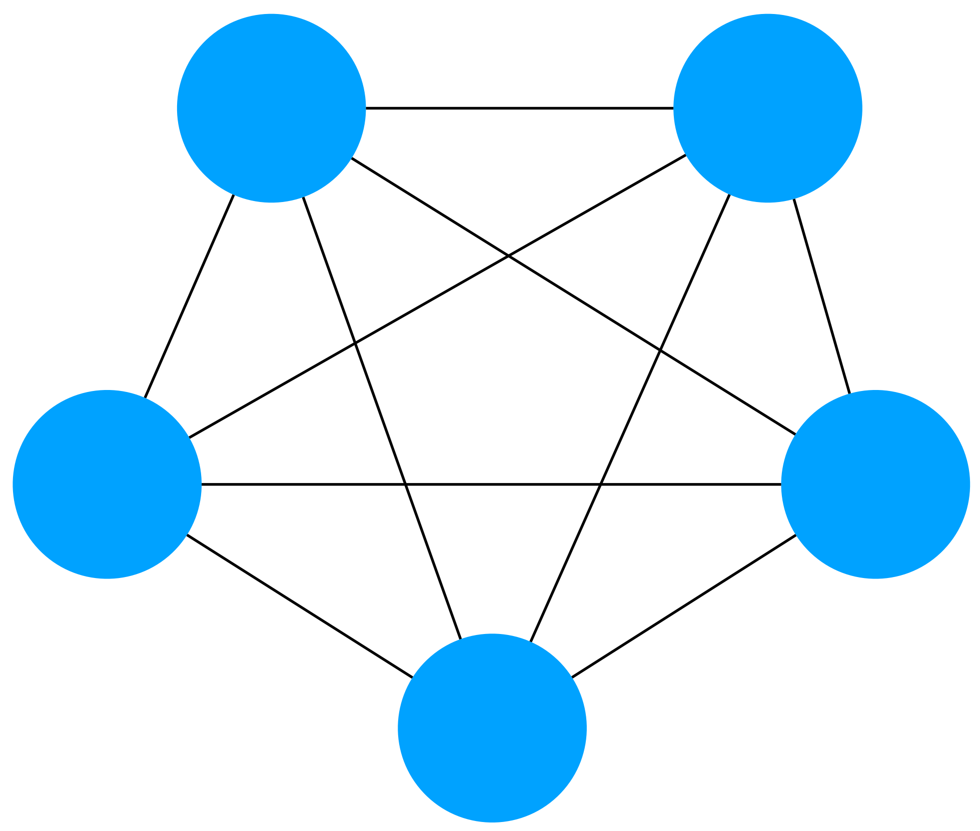

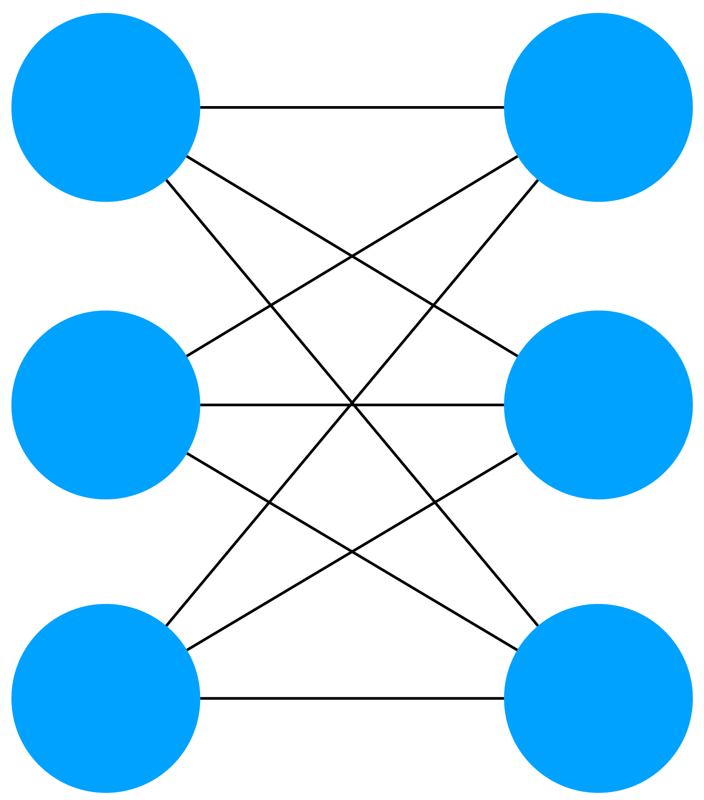

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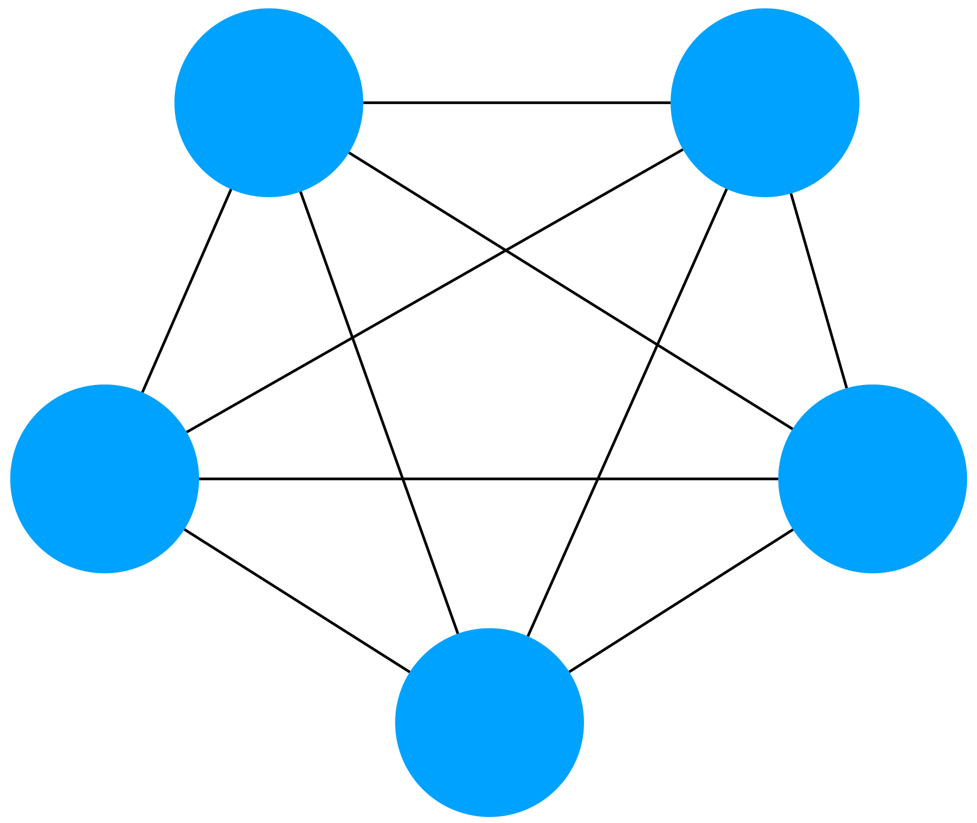

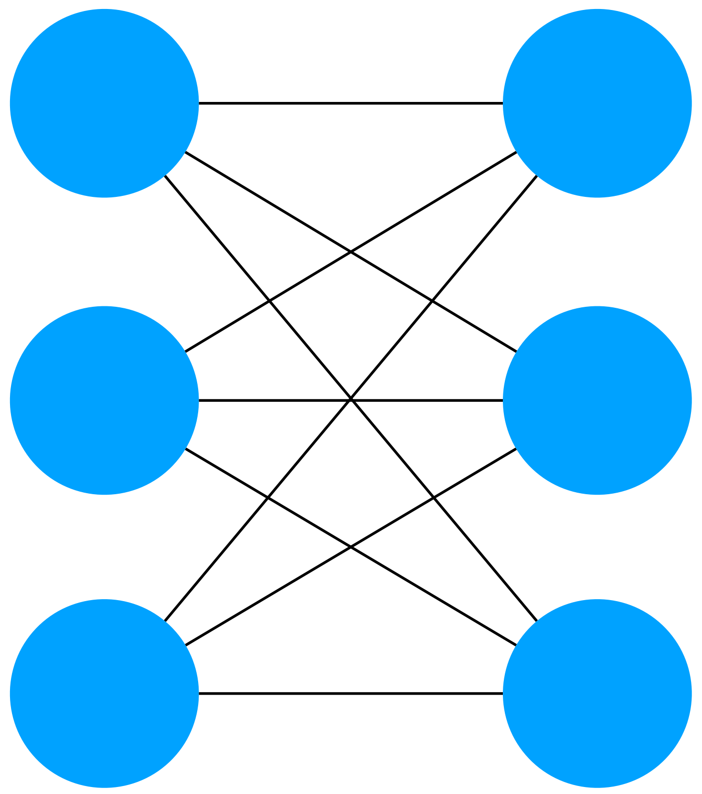

Theorem (Kuratowski's Theorem)

A graph is planar if, and only if, it does not contain K_5 or $K_{3,3}$ as a minor.

Treewidth and Minors

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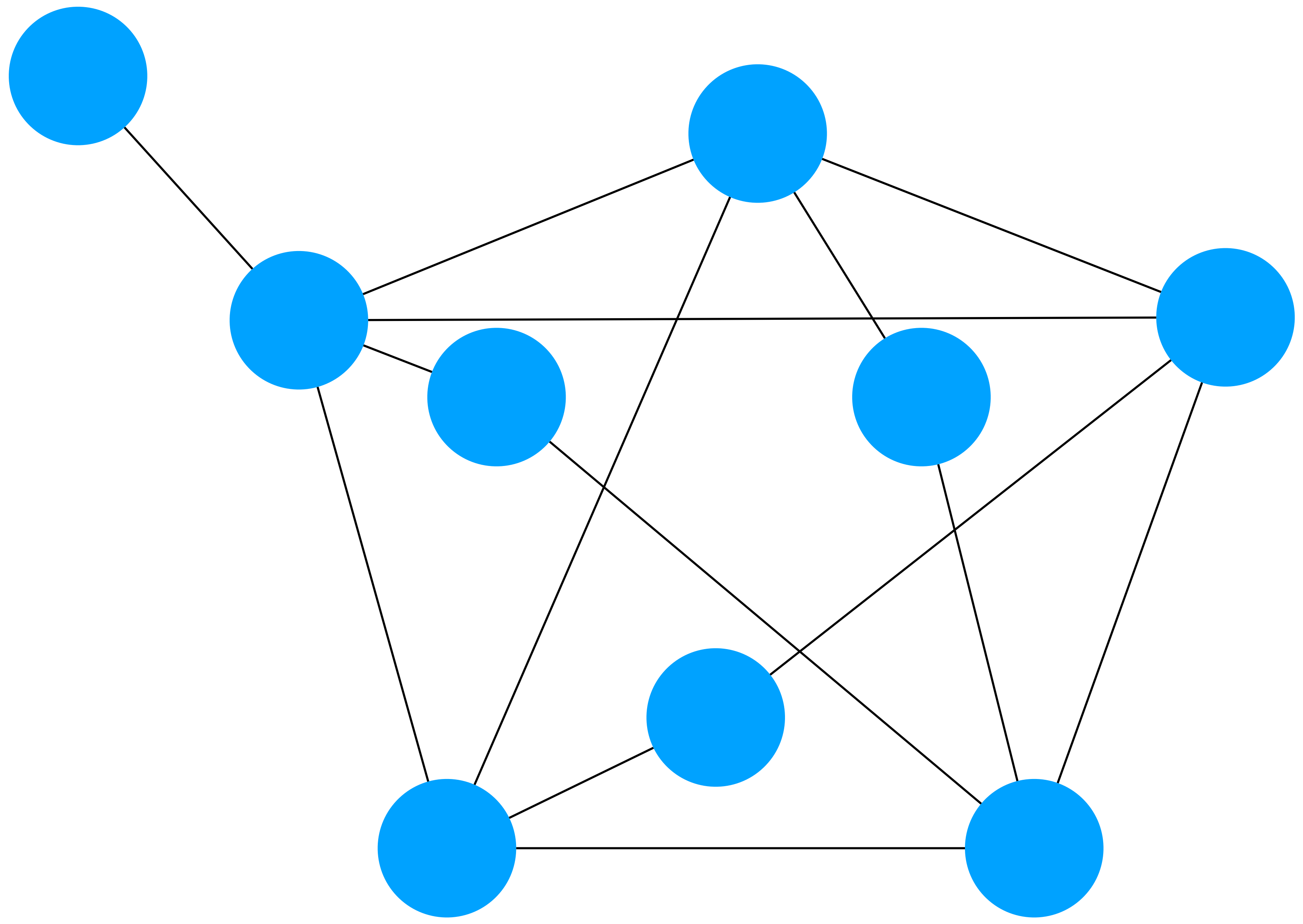

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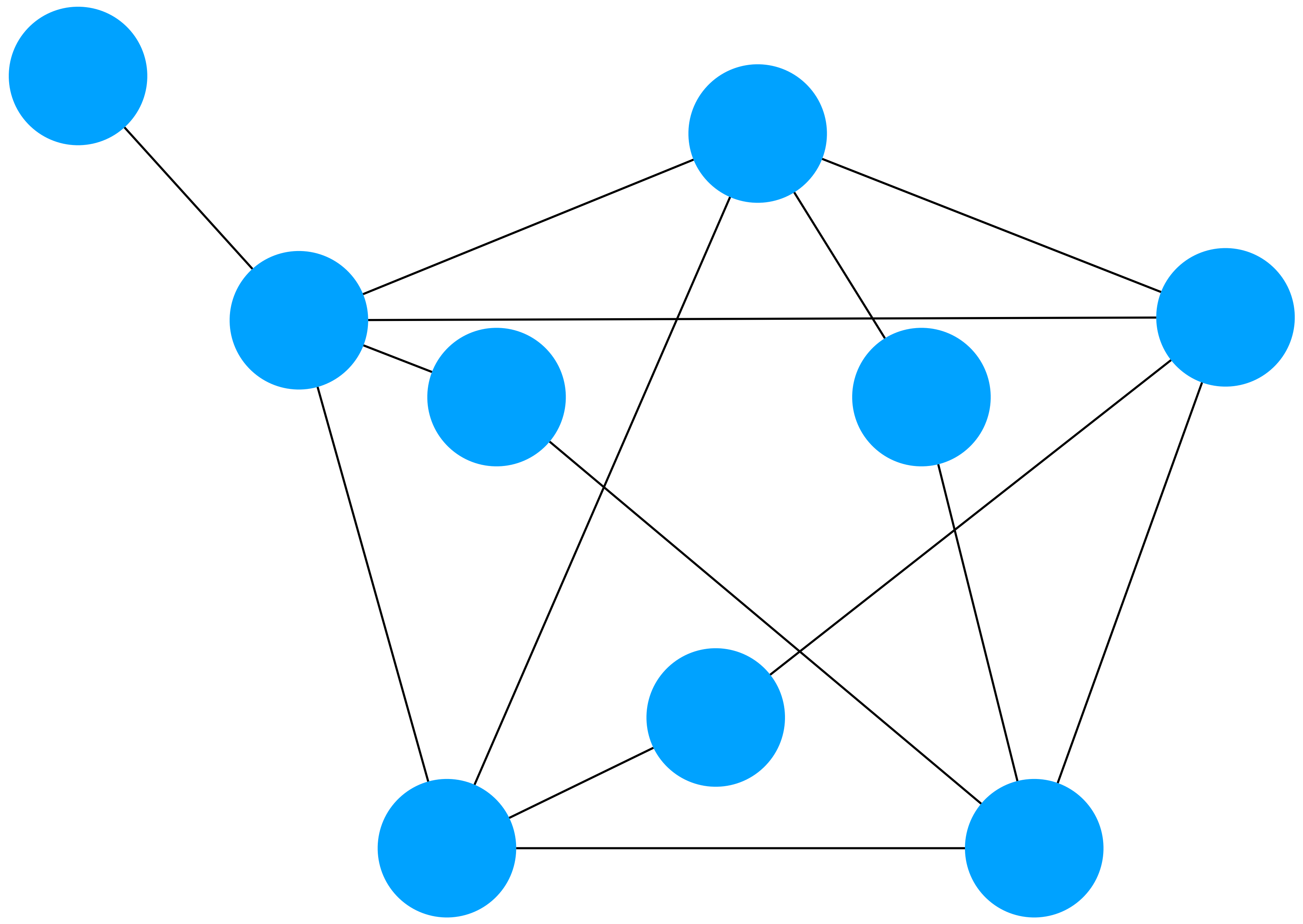

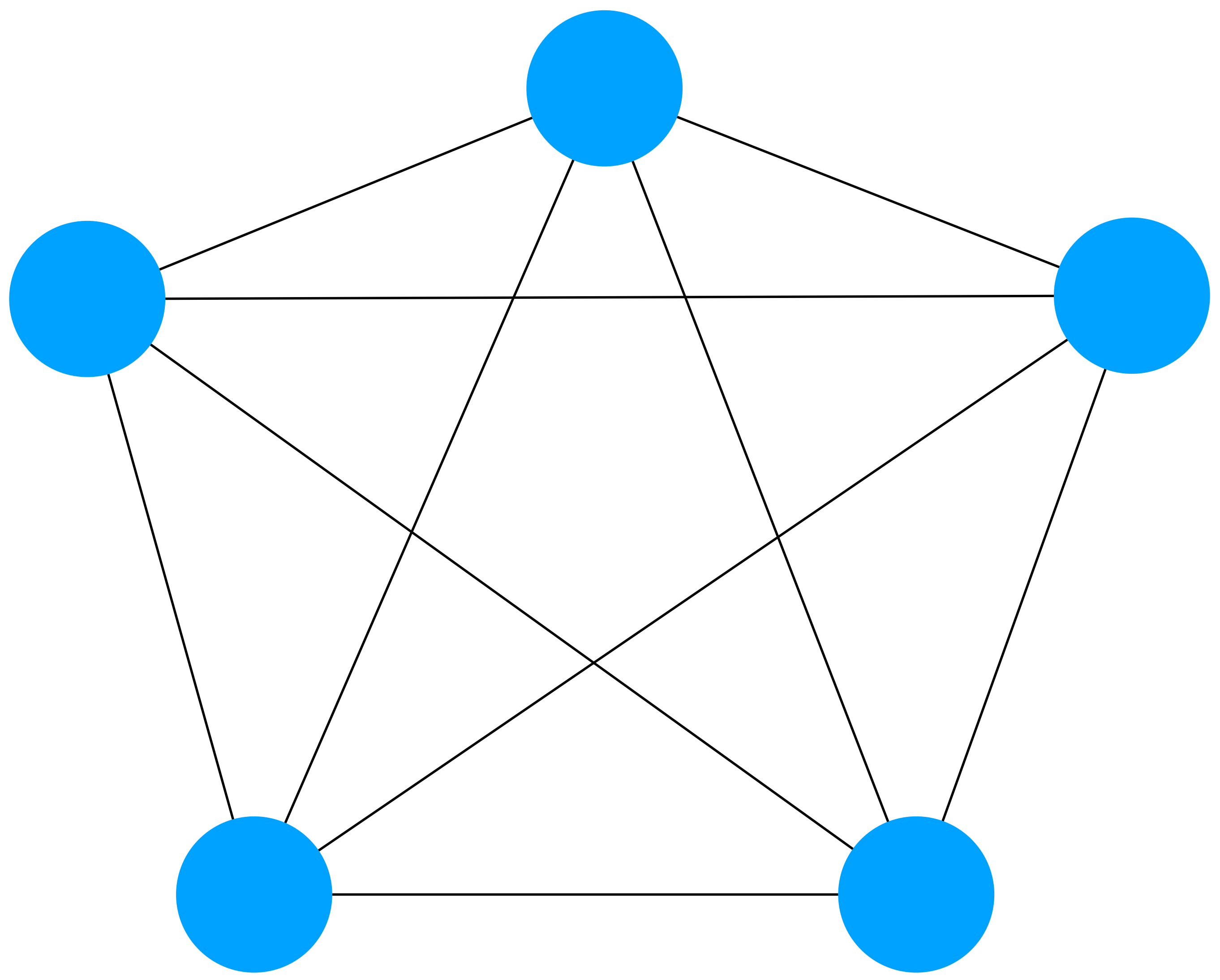

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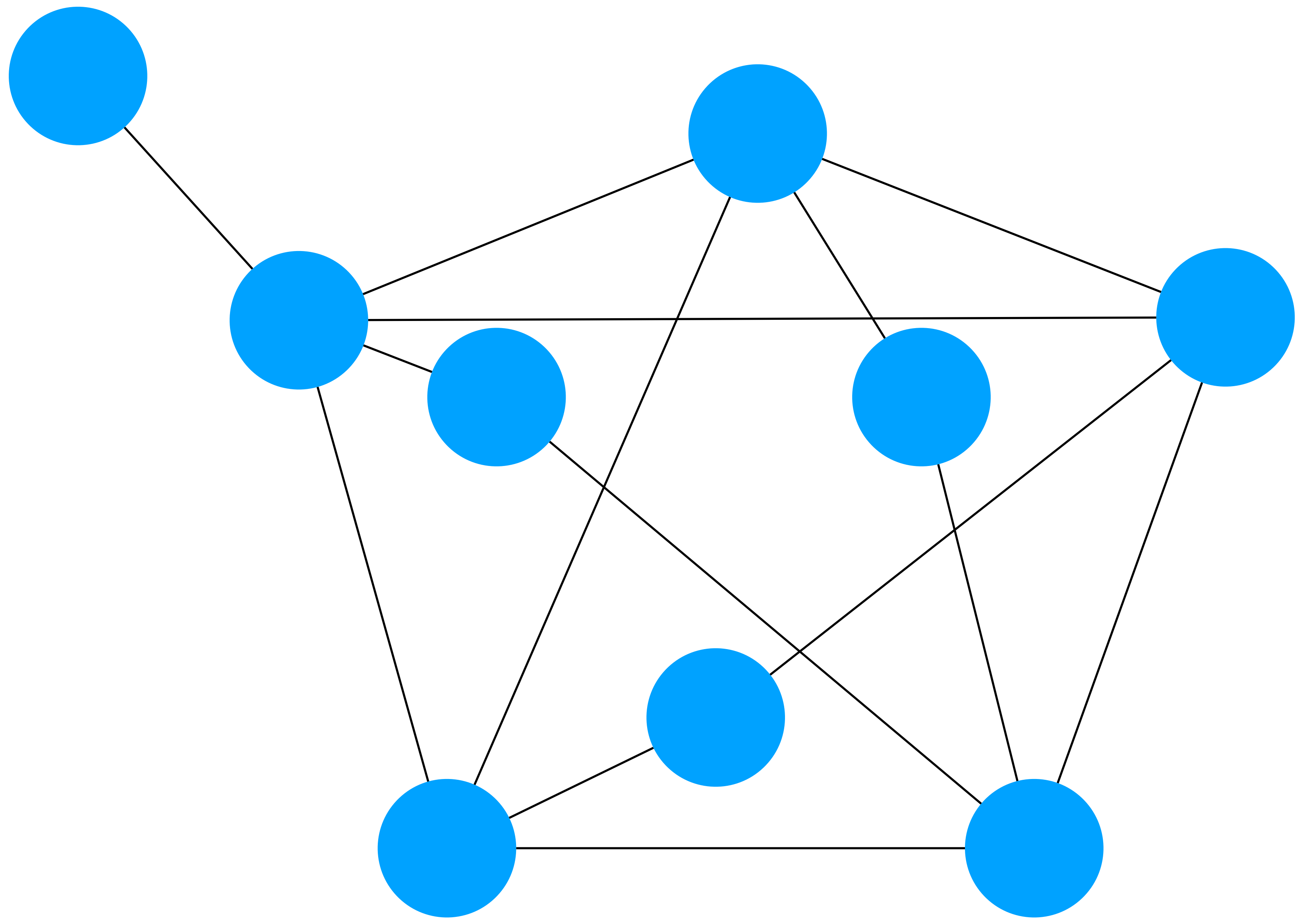

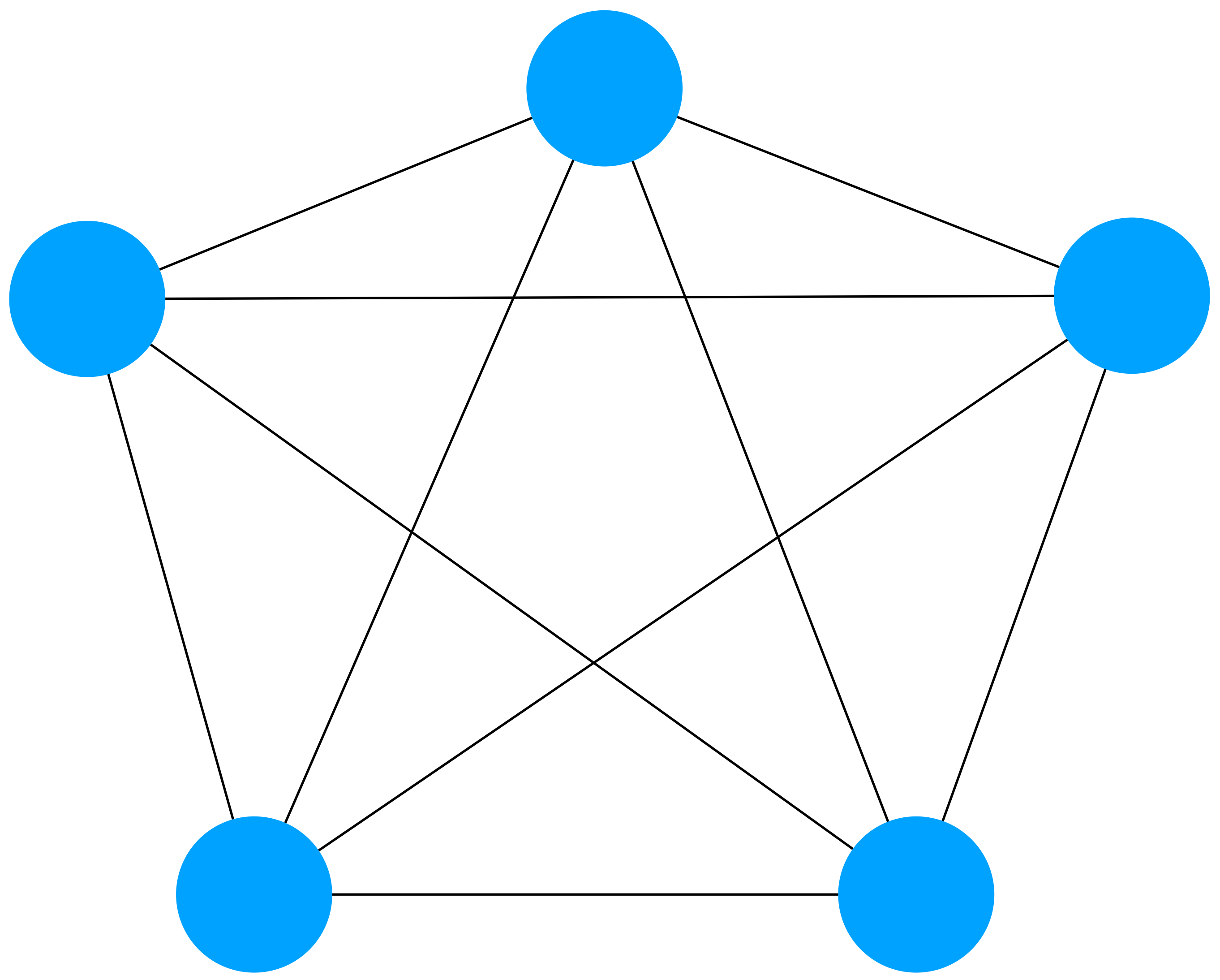

K_5 treewidth 4

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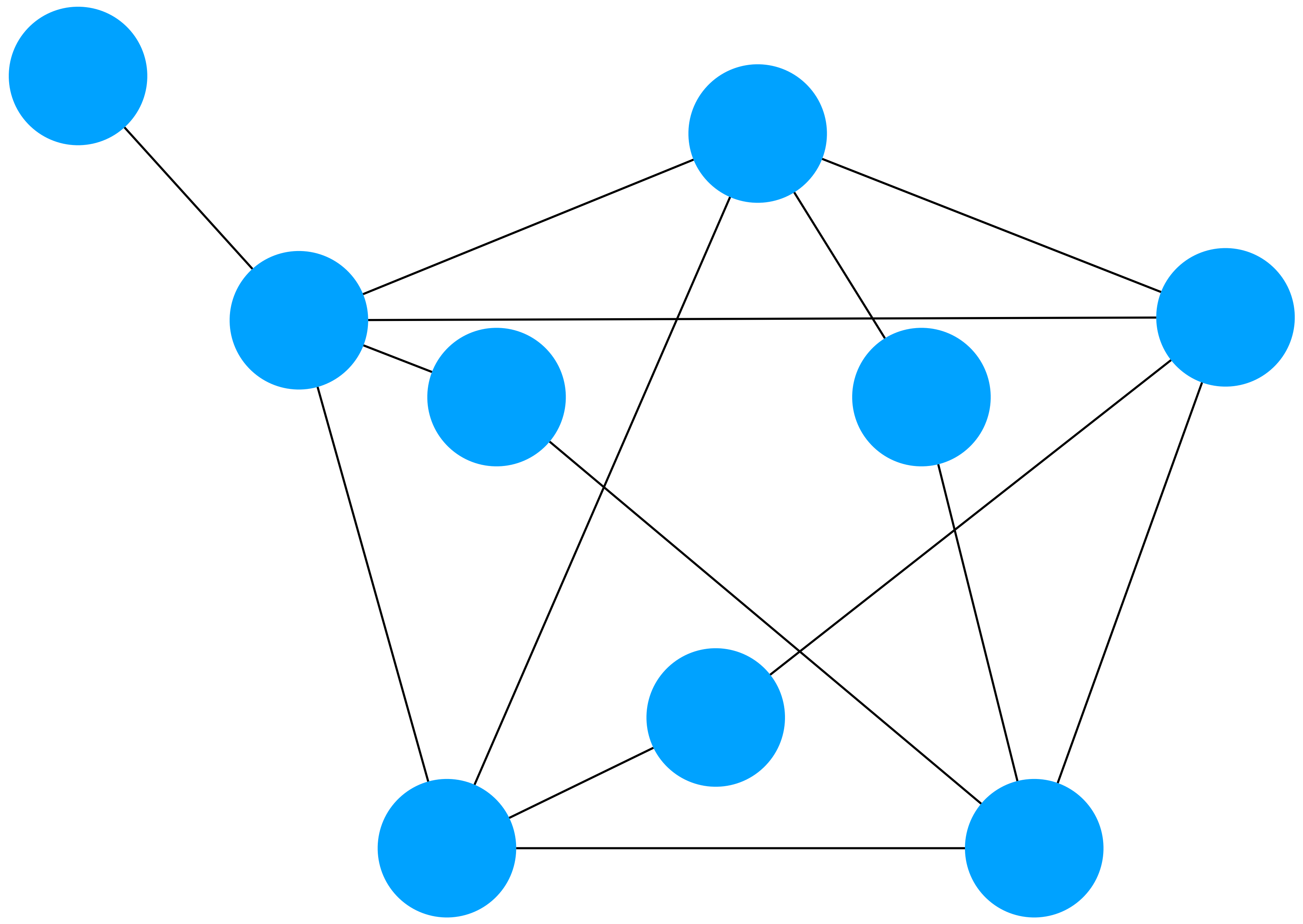
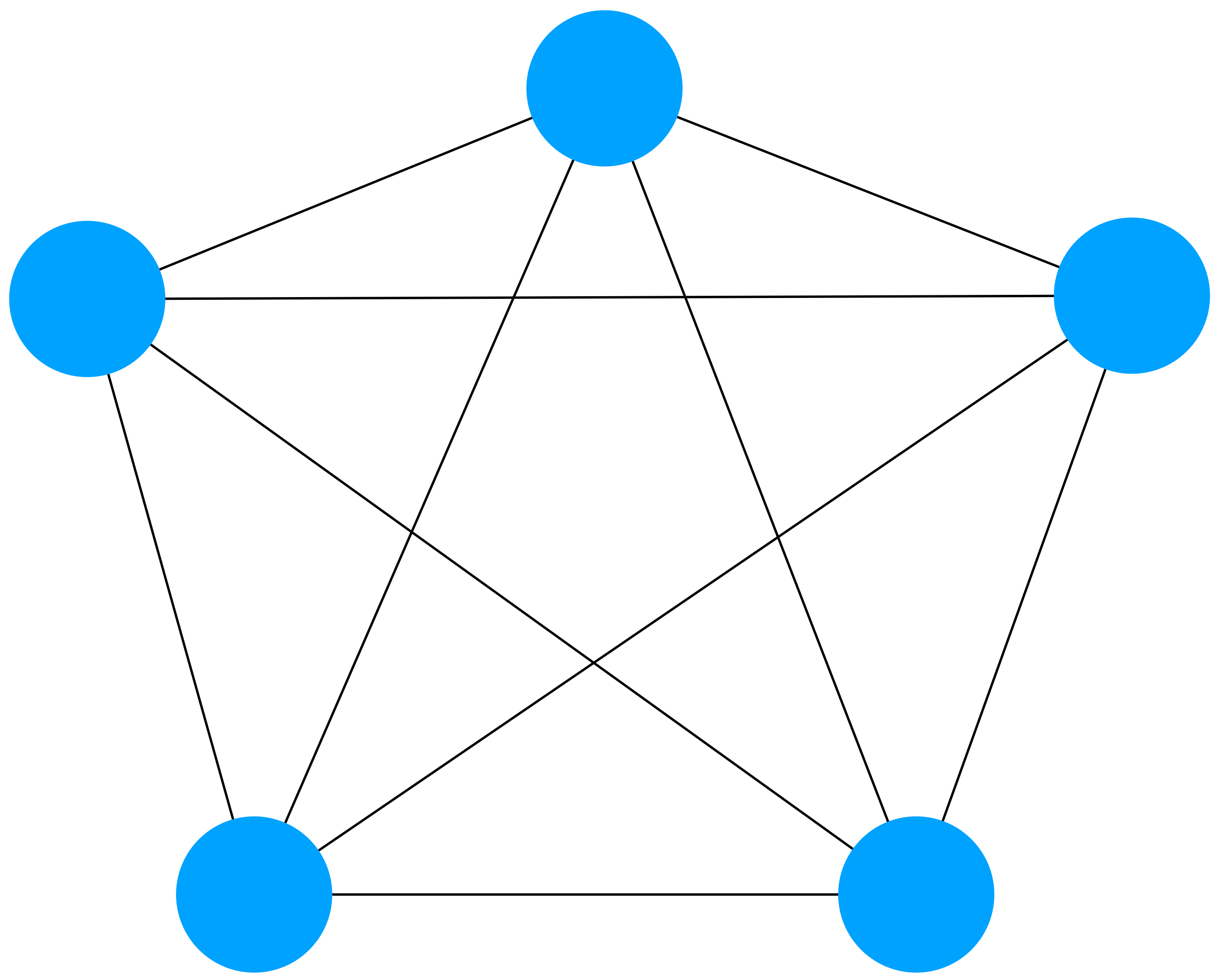


treewidth ≥ 4



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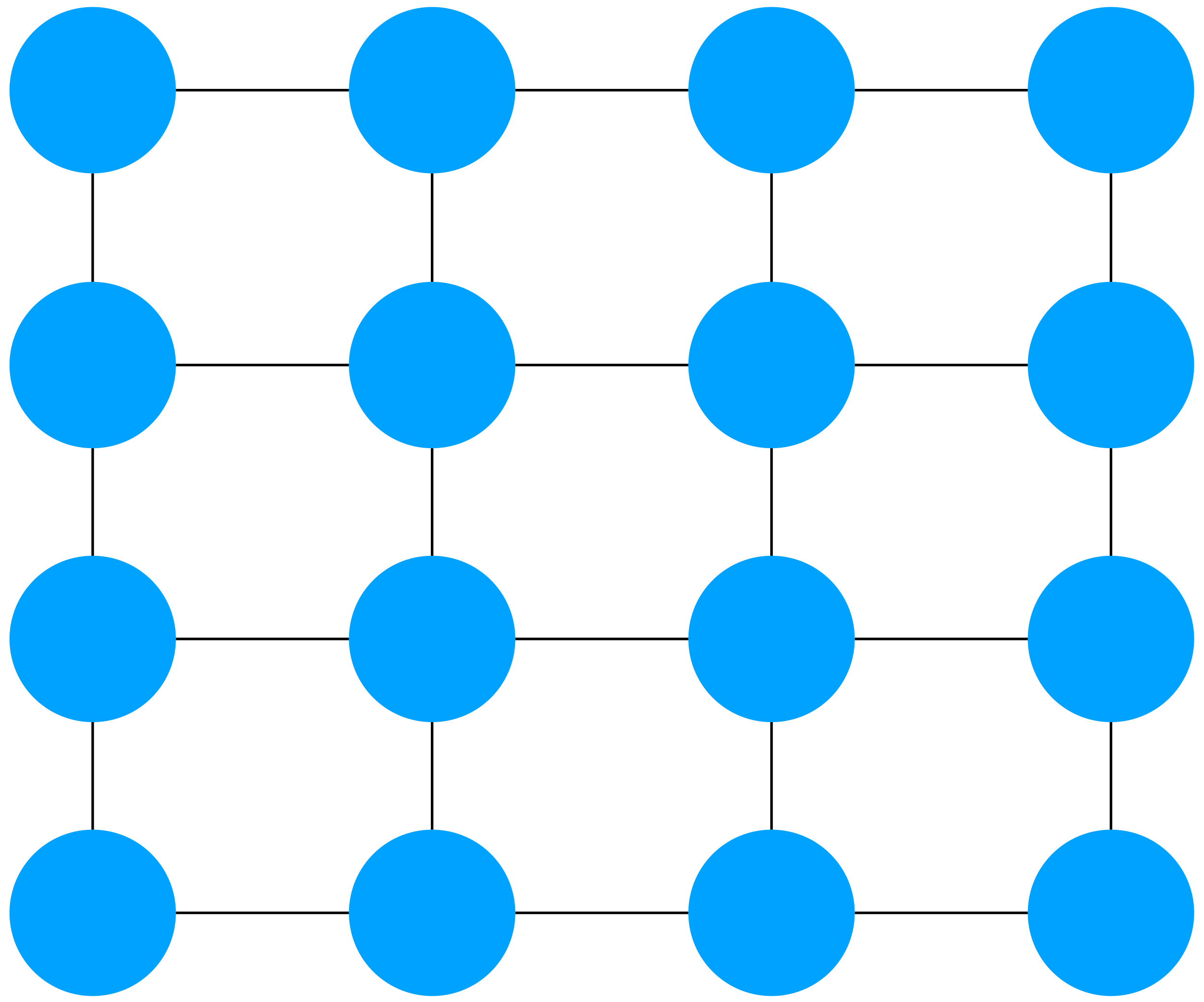
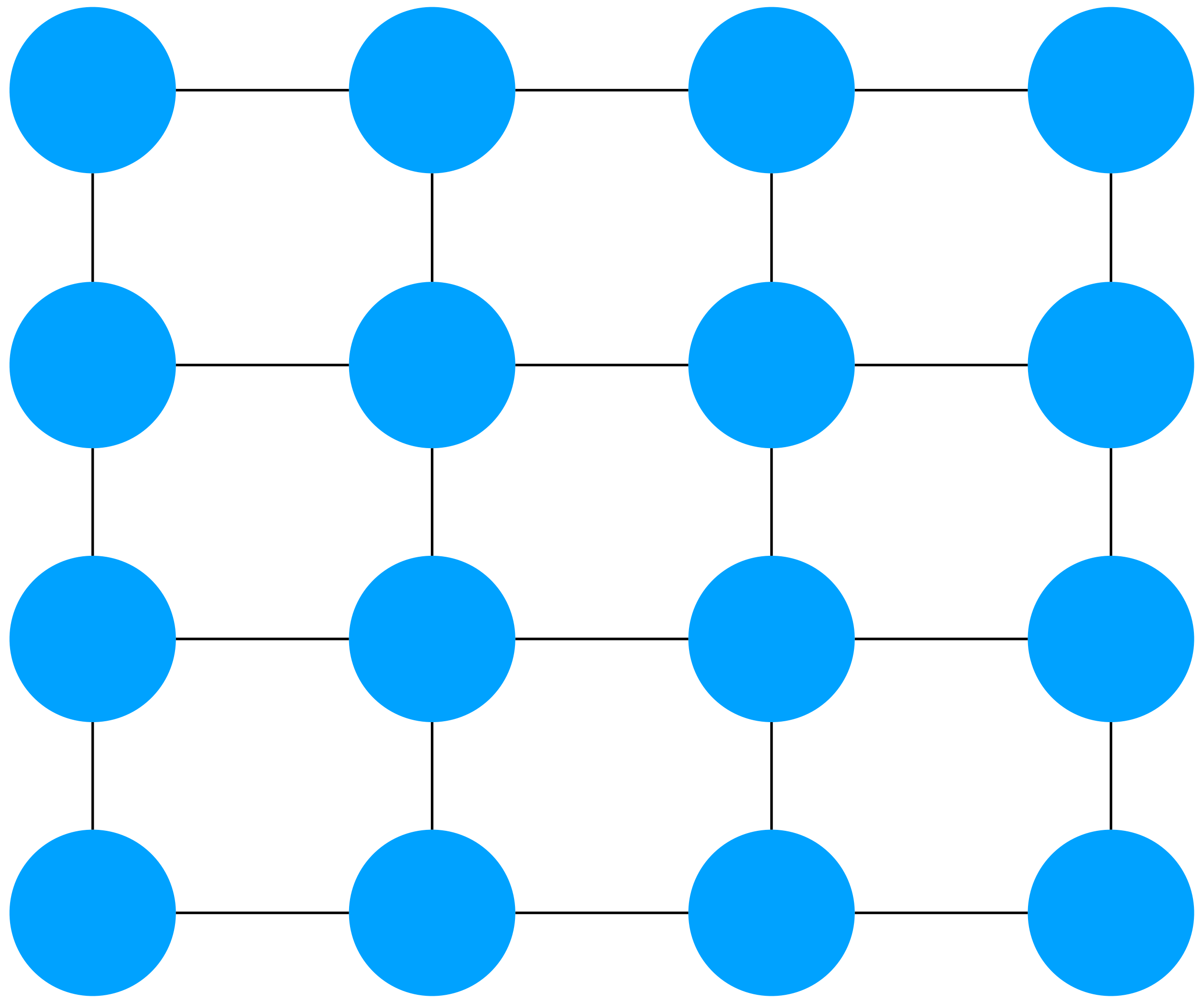
Treewidth and Grid Minors



Q_4

Treewidth and Grid Minors

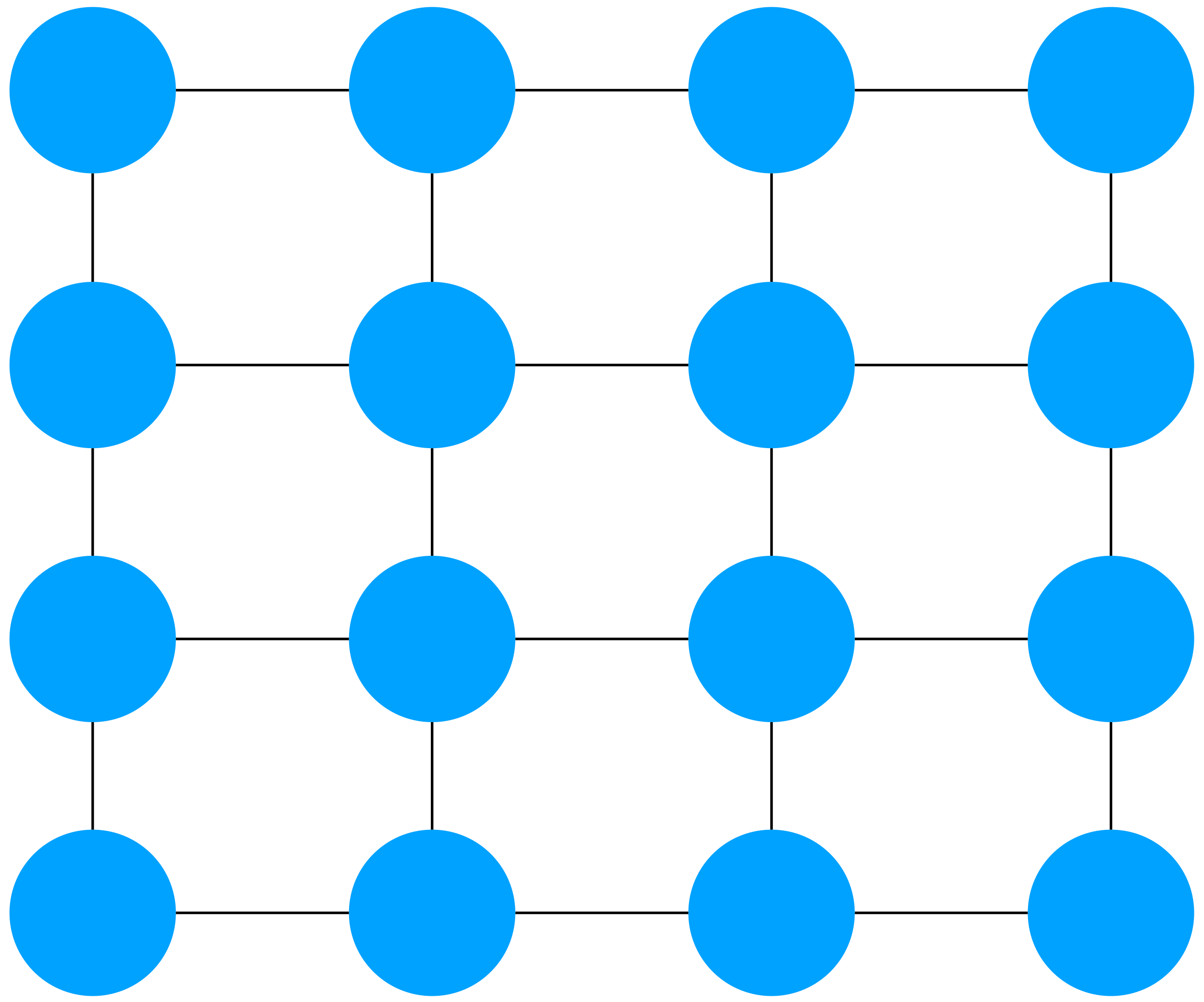


Q_4

Theorem (Robertson and Seymour)

A graph class $\mathscr{C}$ has bounded treewidth if, and only if, there is a k such that the grid Q_k is not a minor of any graph in $\mathscr{C}$.

Treewidth and Grid Minors



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Theorem (Chekuri and Chuzhoy)

There is a polynomial p such that every graph of treewidth larger than $p(k)$ contains Q_k as a minor.

Cops and Robbers

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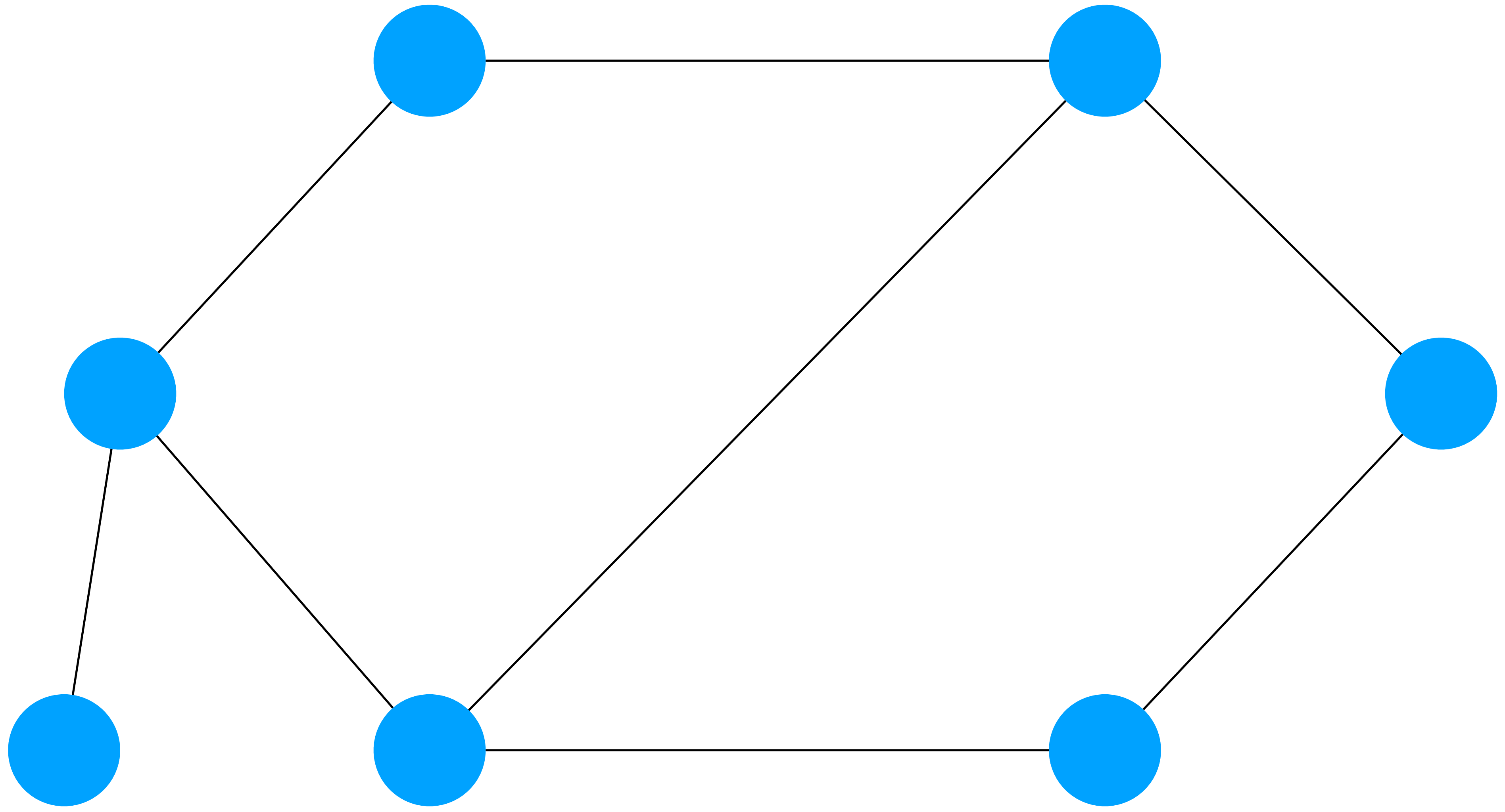

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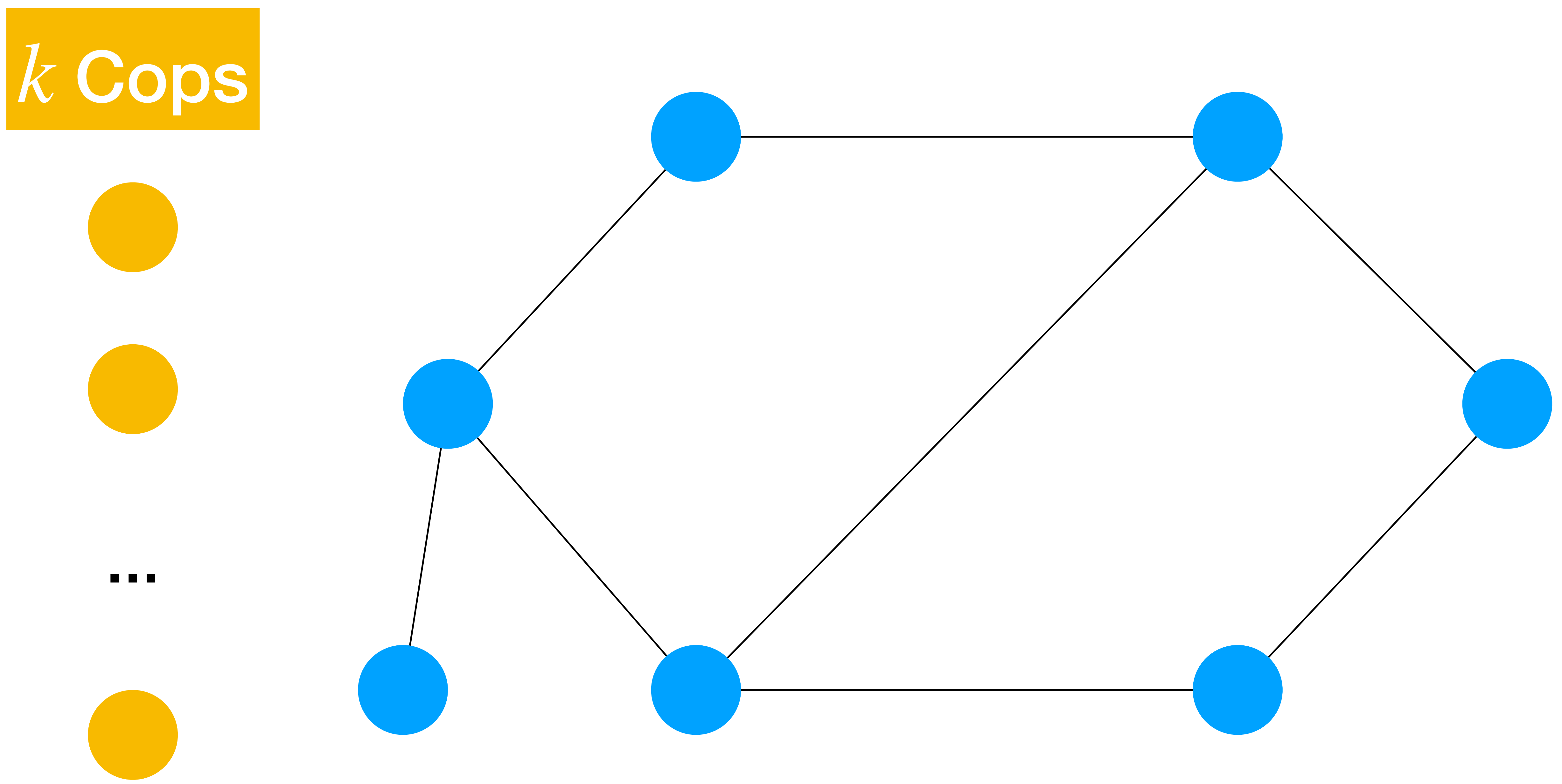
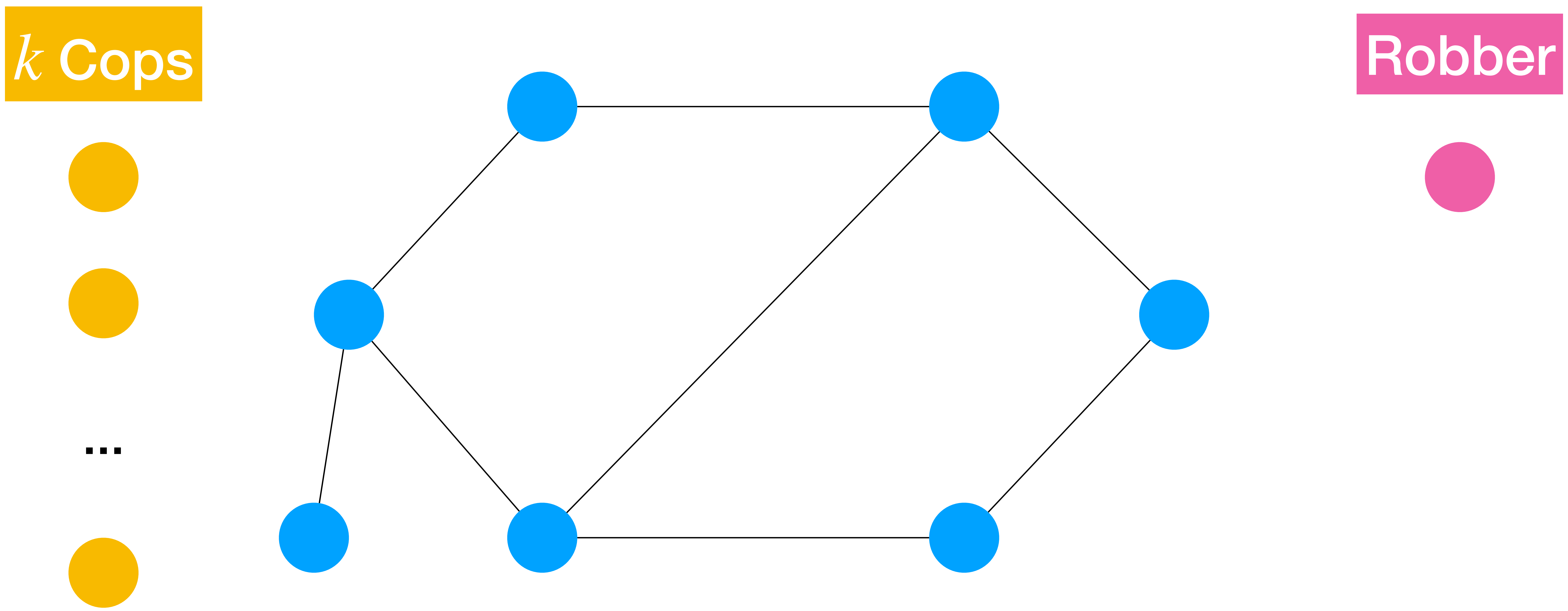
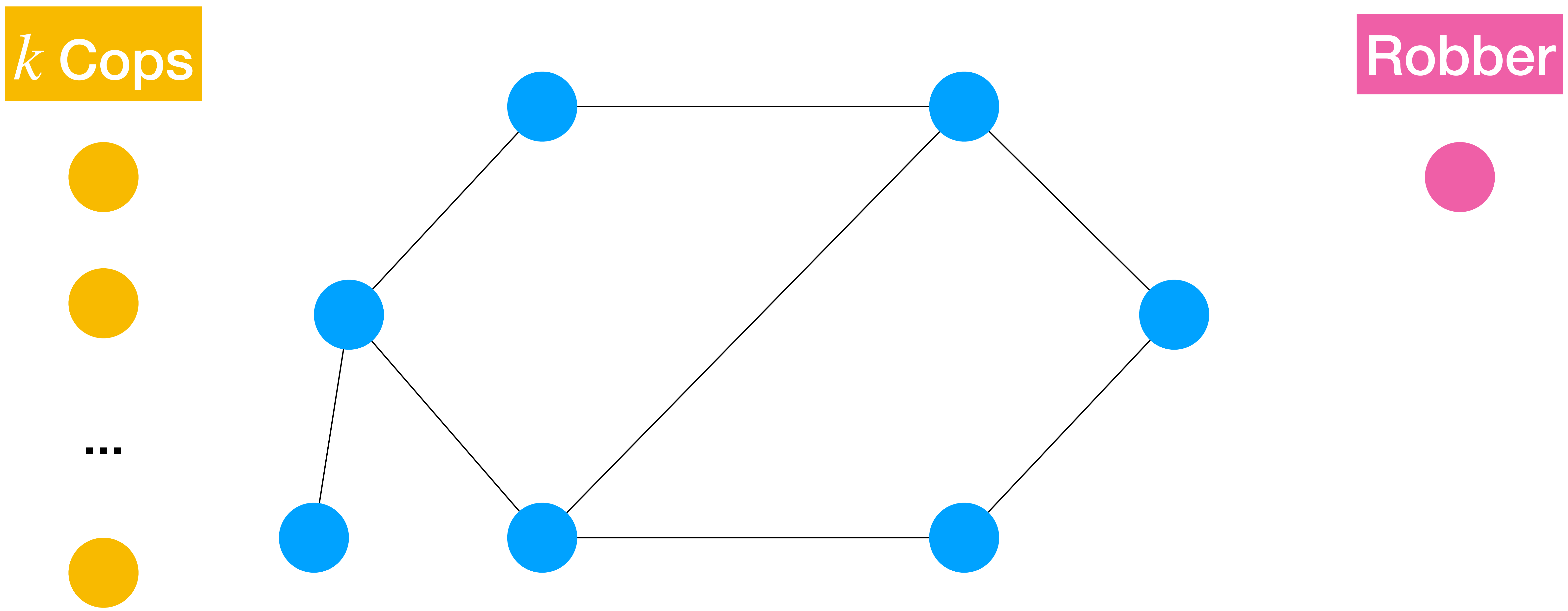


Cops and Robbers

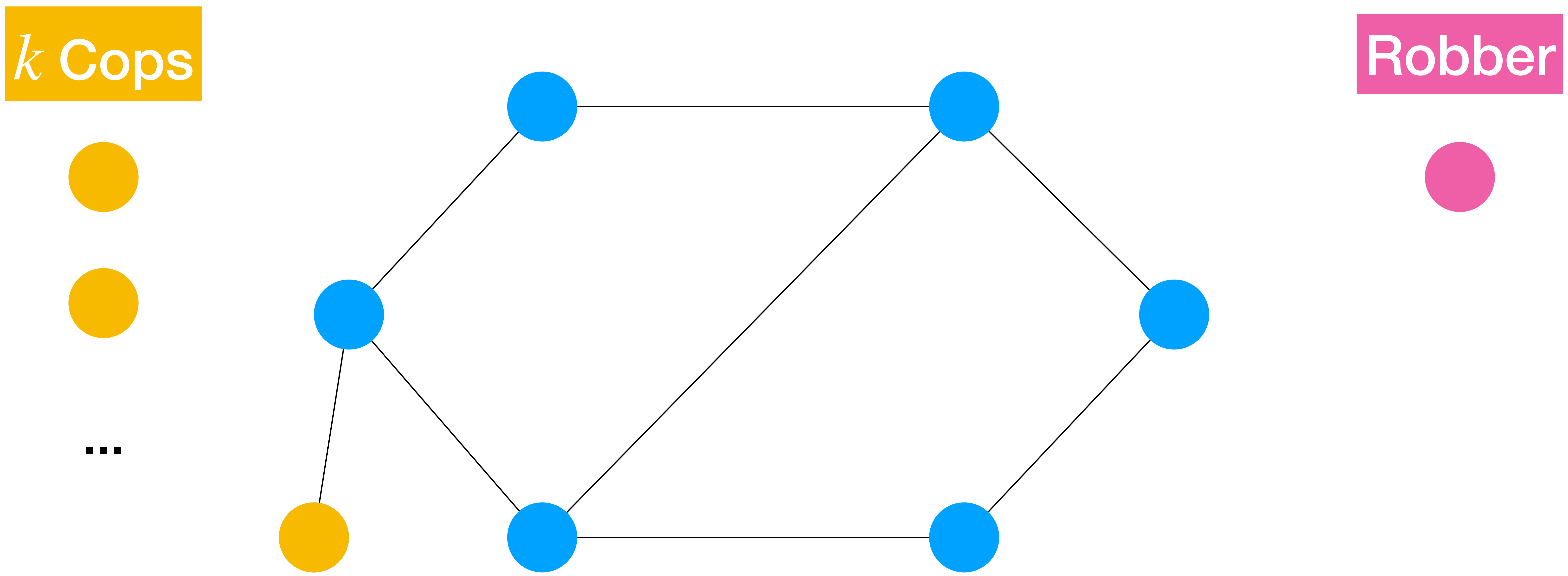


Cops and Robbers



1. First, cops position themselves on the graph.

Cops and Robbers



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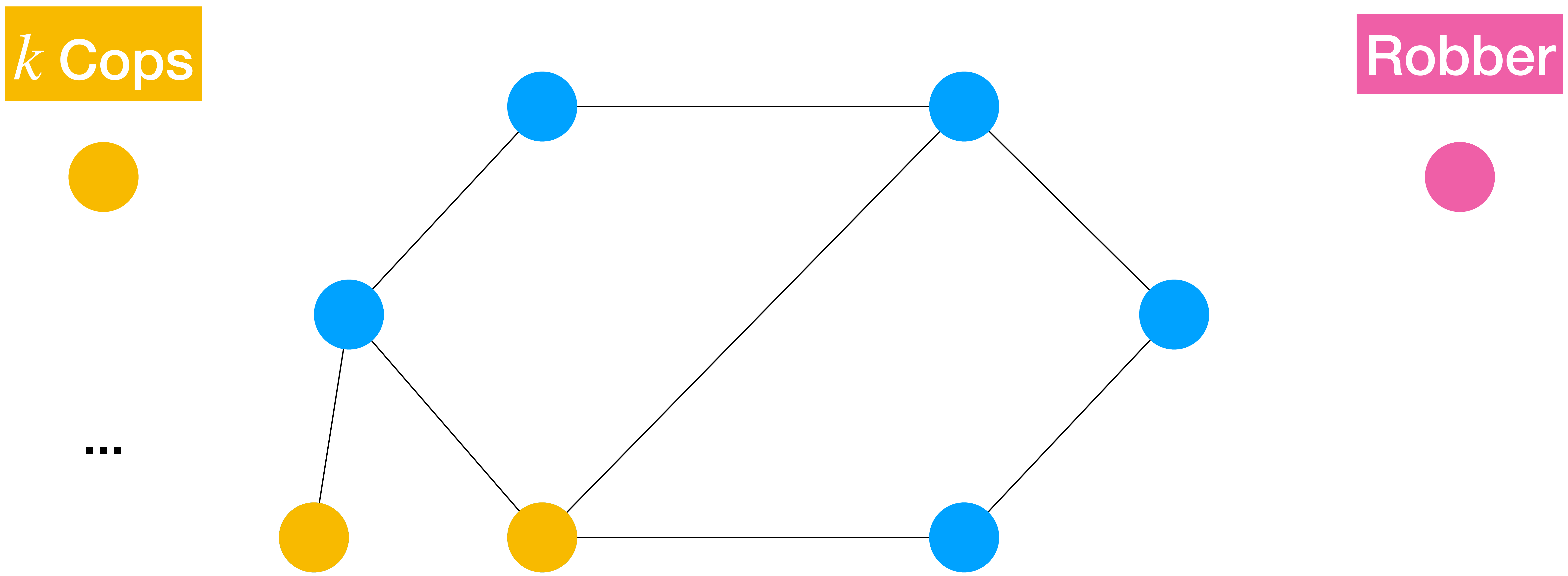

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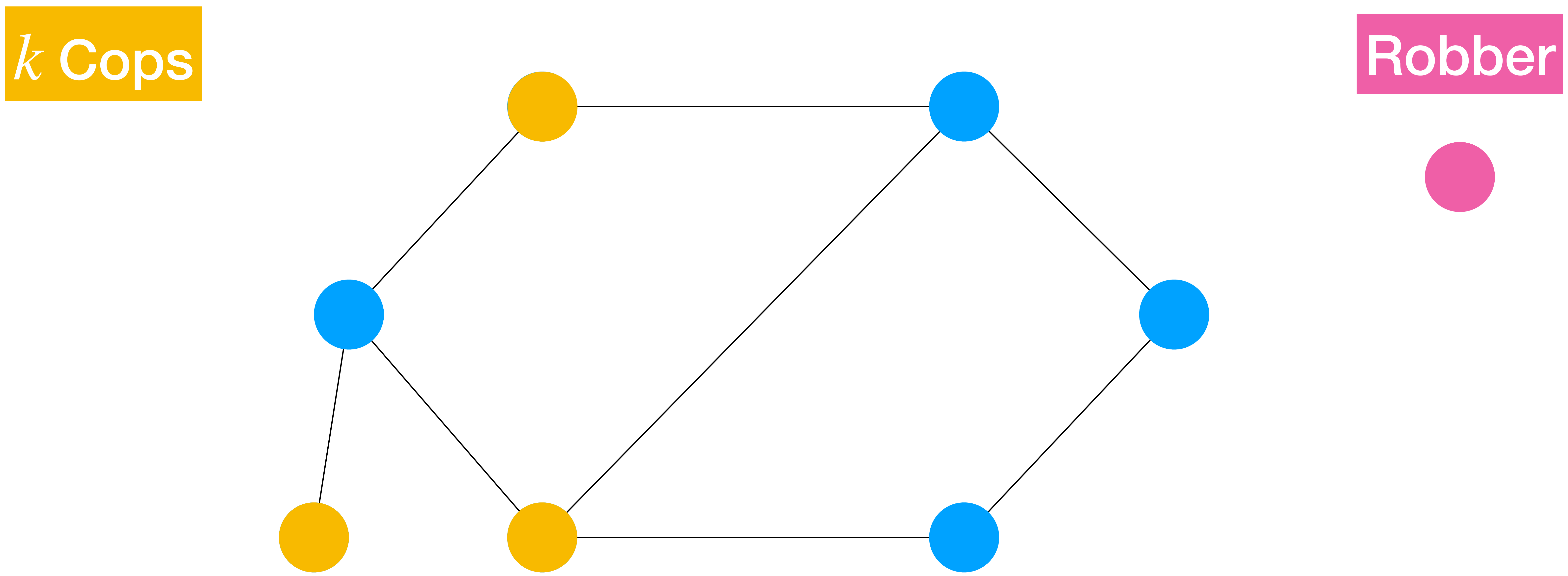

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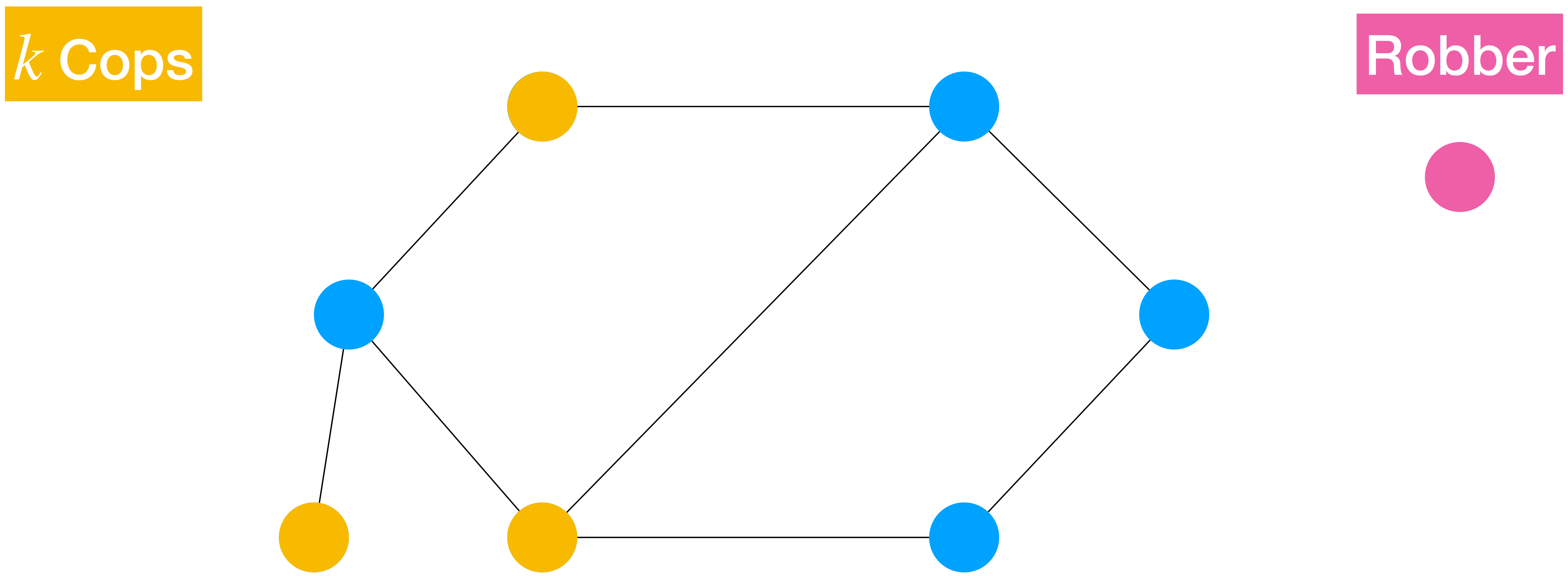

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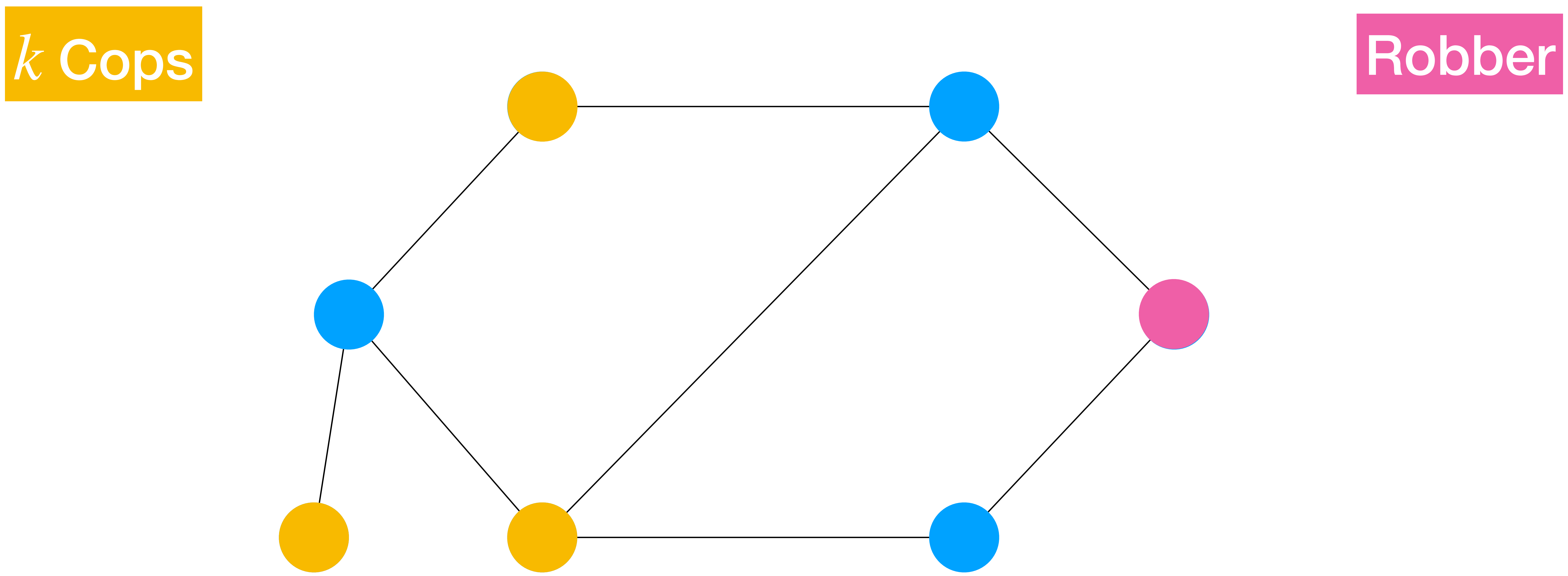
2. Then the robber chooses a vertex.

Cops and Robbers



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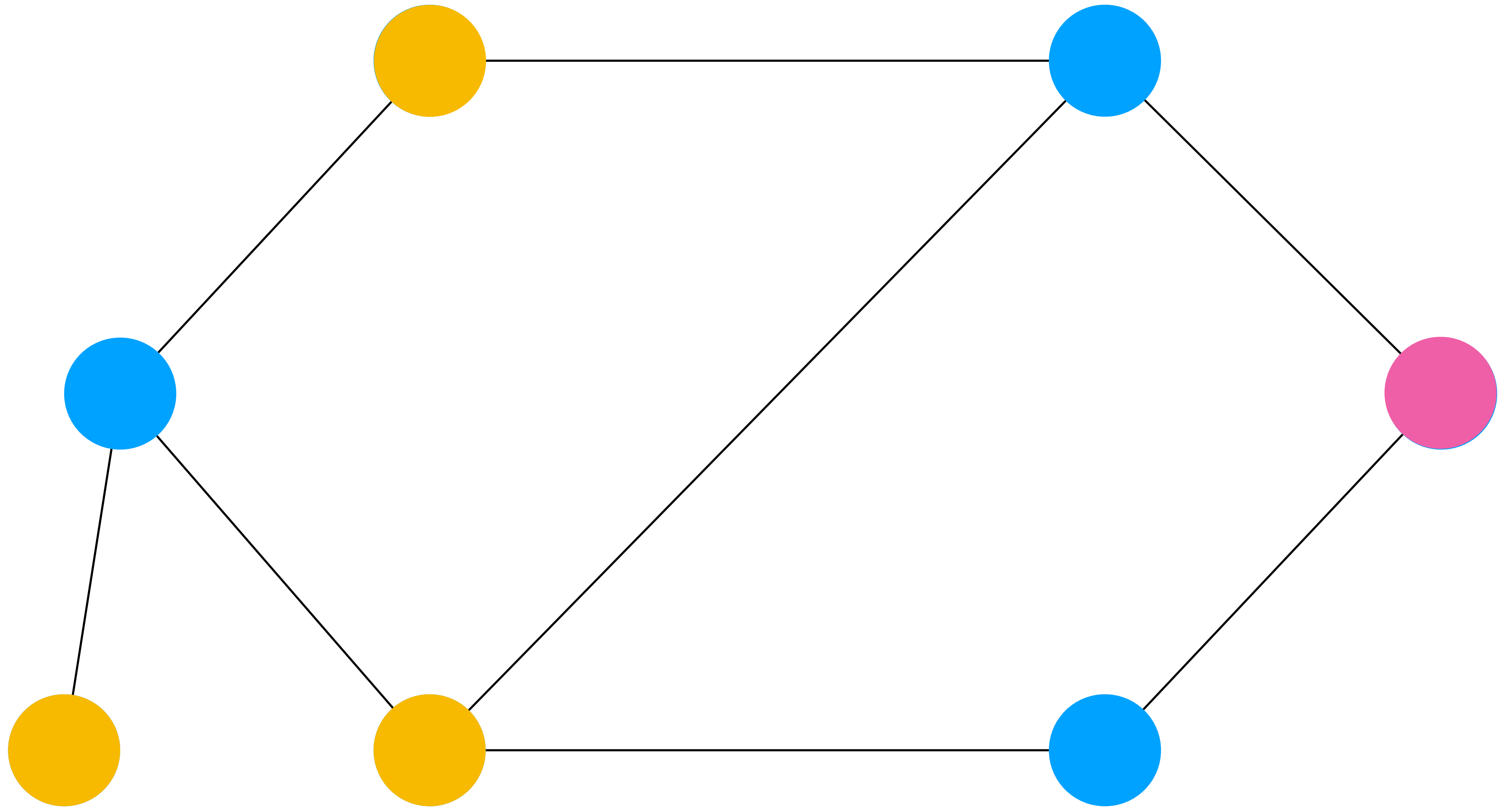
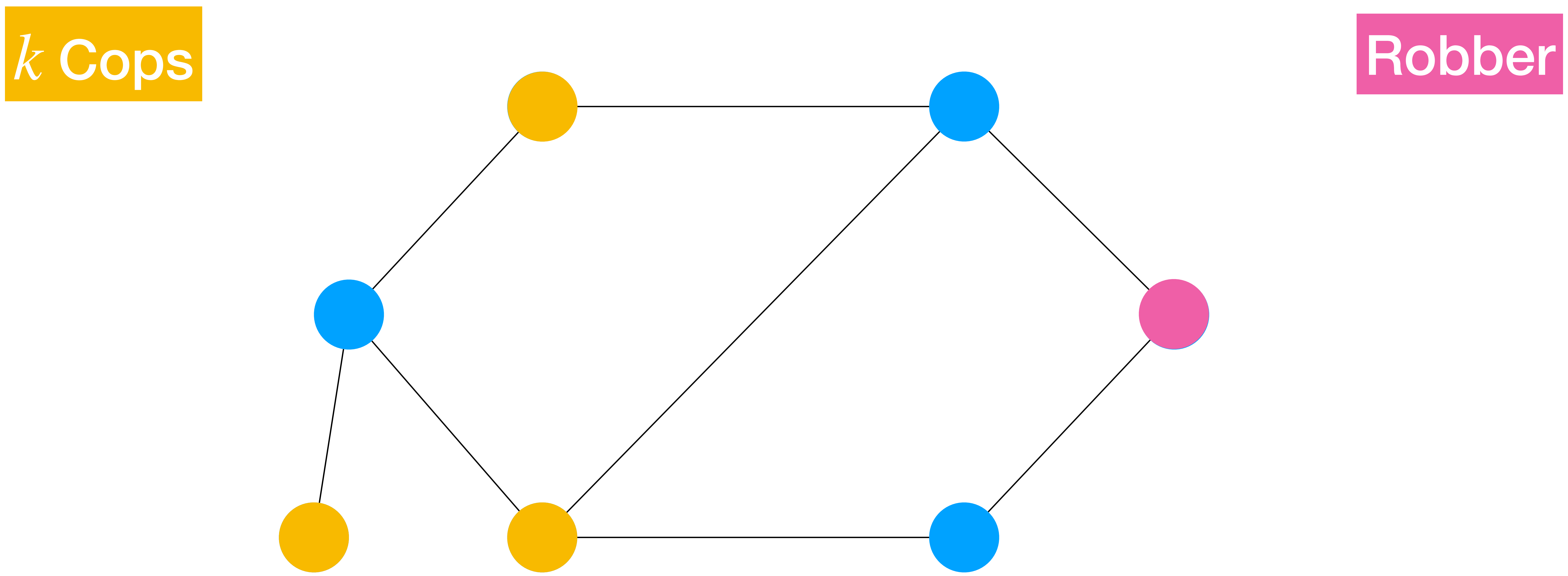
Cops and Robbers

k Cops

Robber

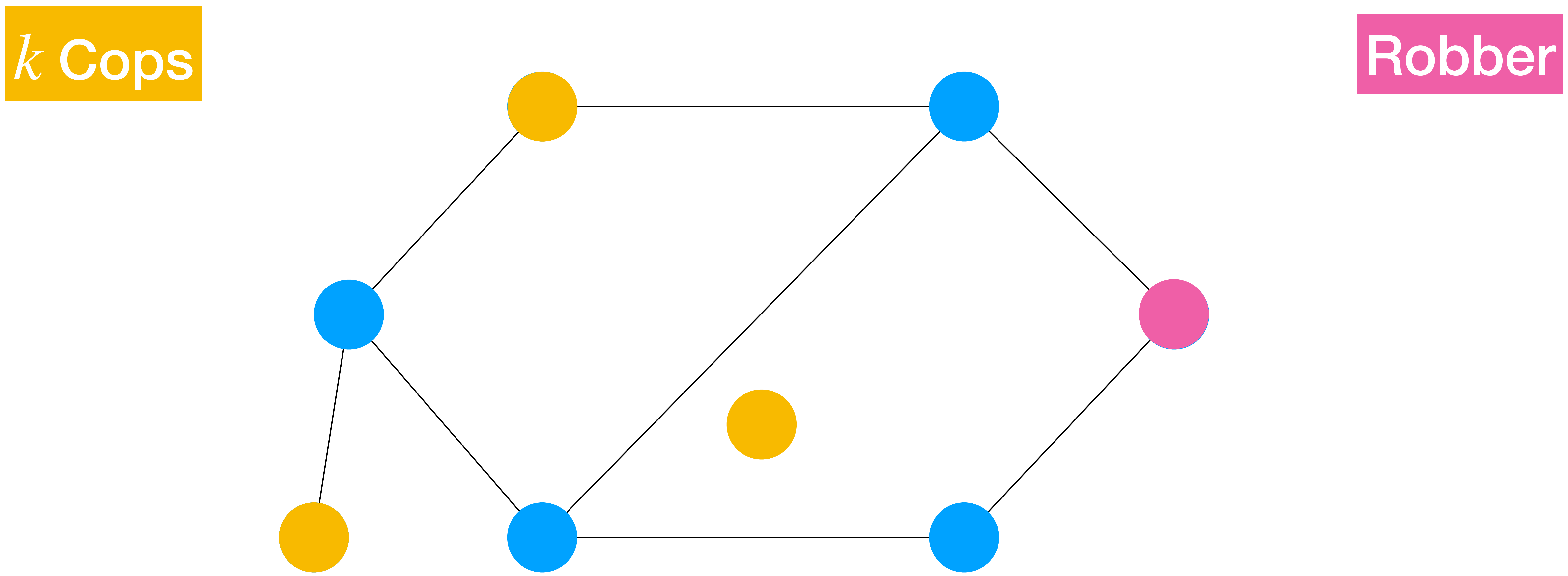


Cops and Robbers



- Cops may move anywhere, but have to **"leave the graph"** to do so.

Cops and Robbers



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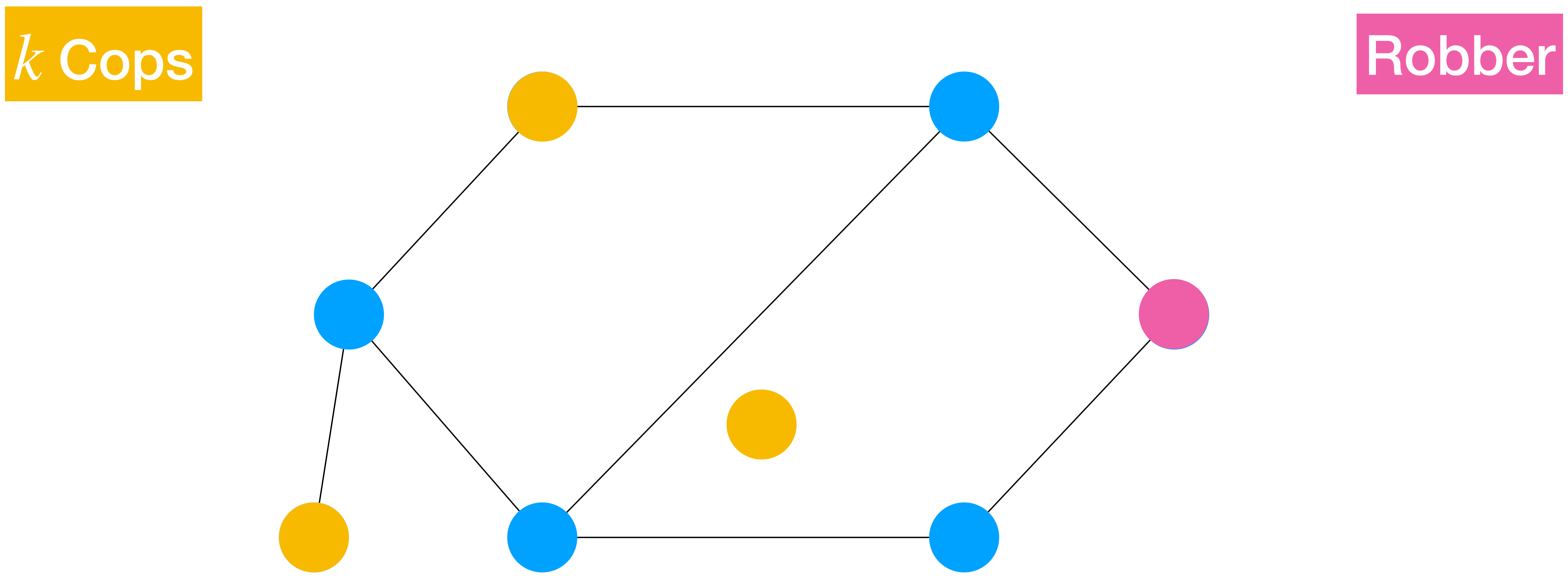

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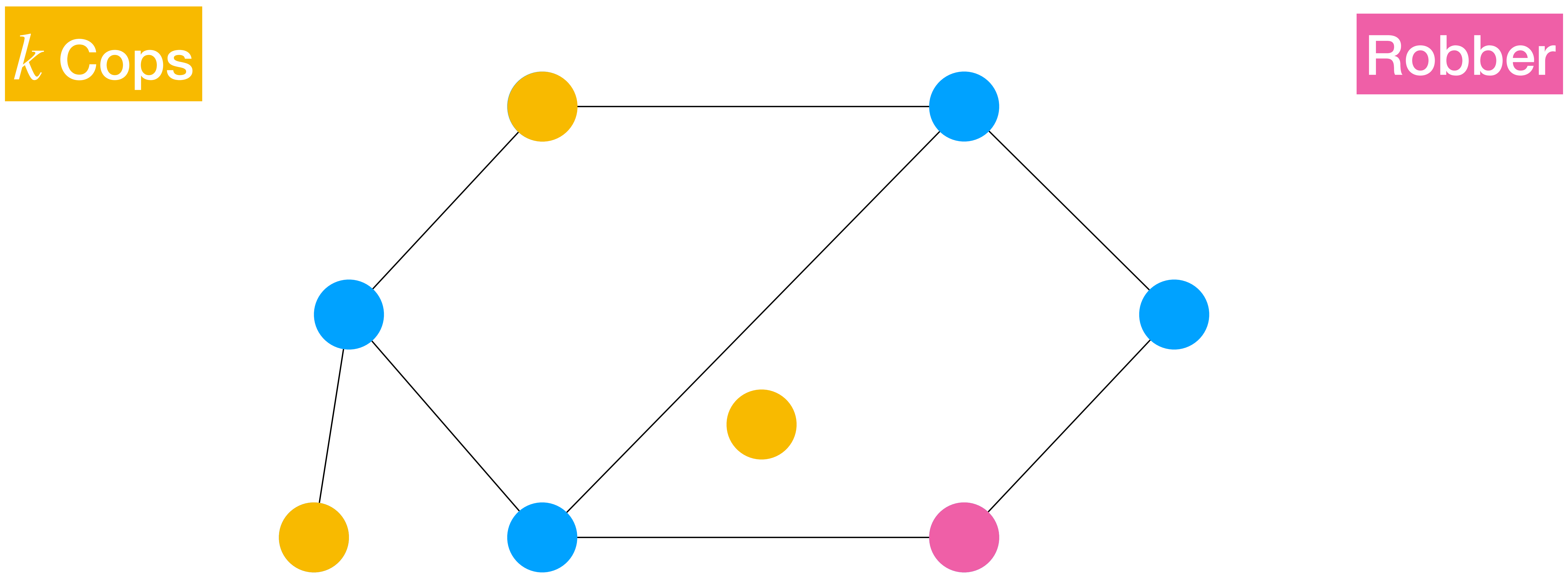
- Robber moves along the edges at **"infinite speed"**, must avoid cops.

Cops and Robbers



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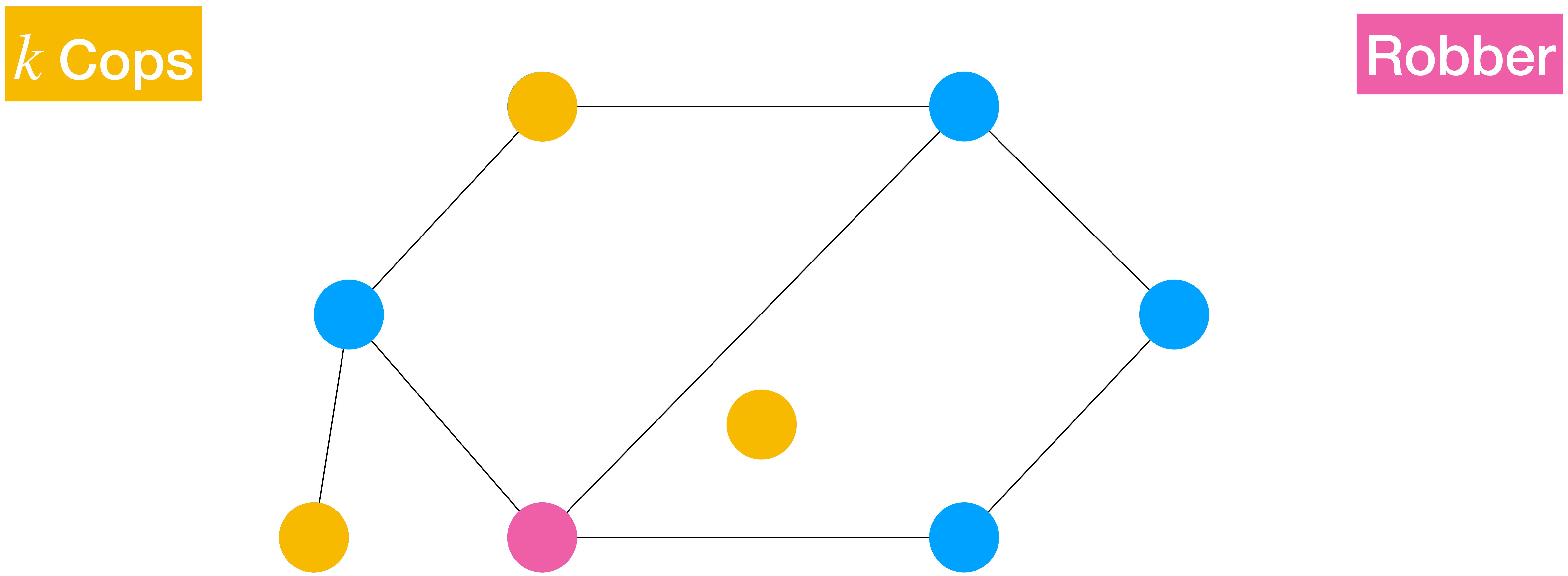

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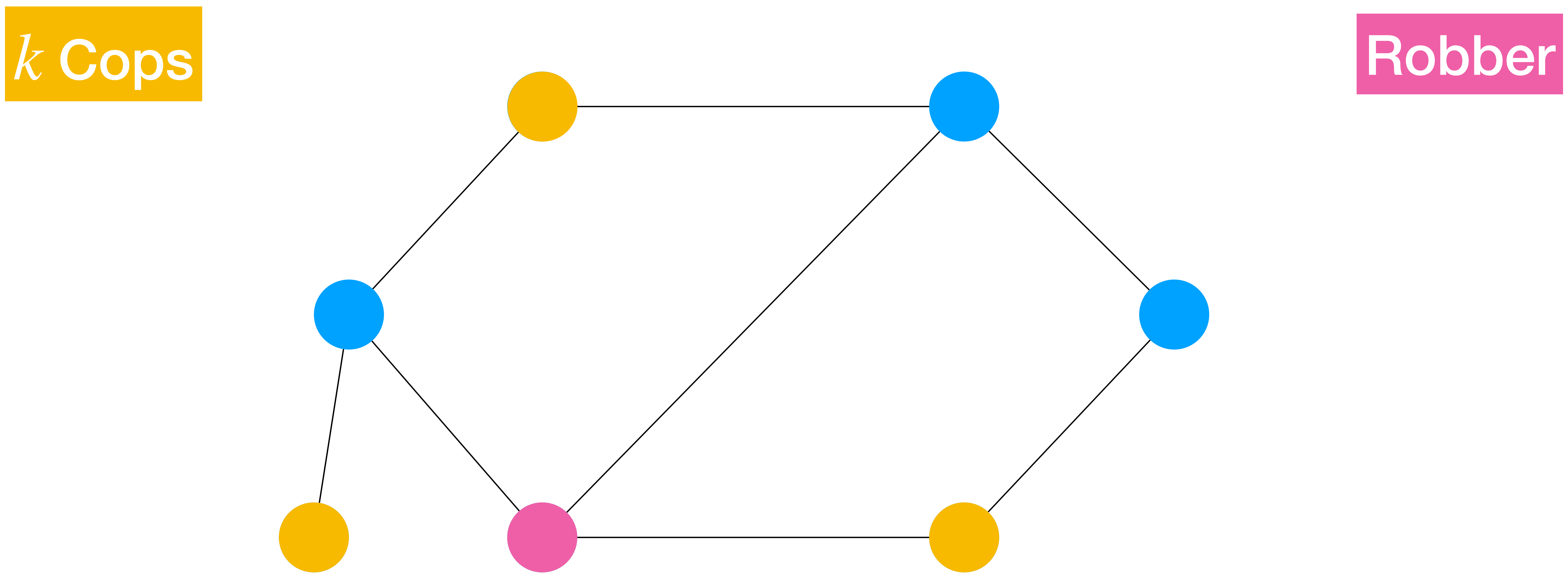

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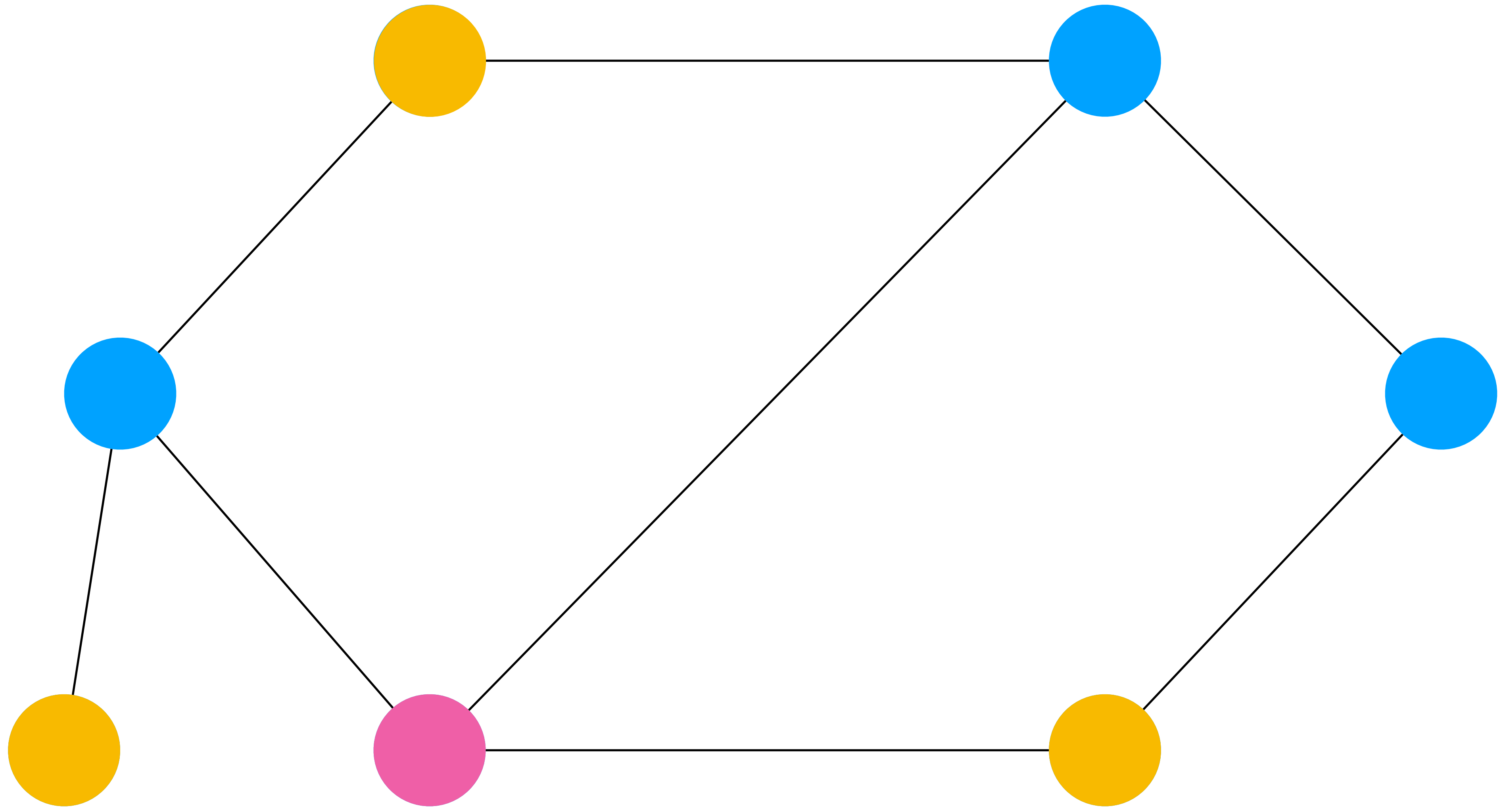
Cops and Robbers

k Cops

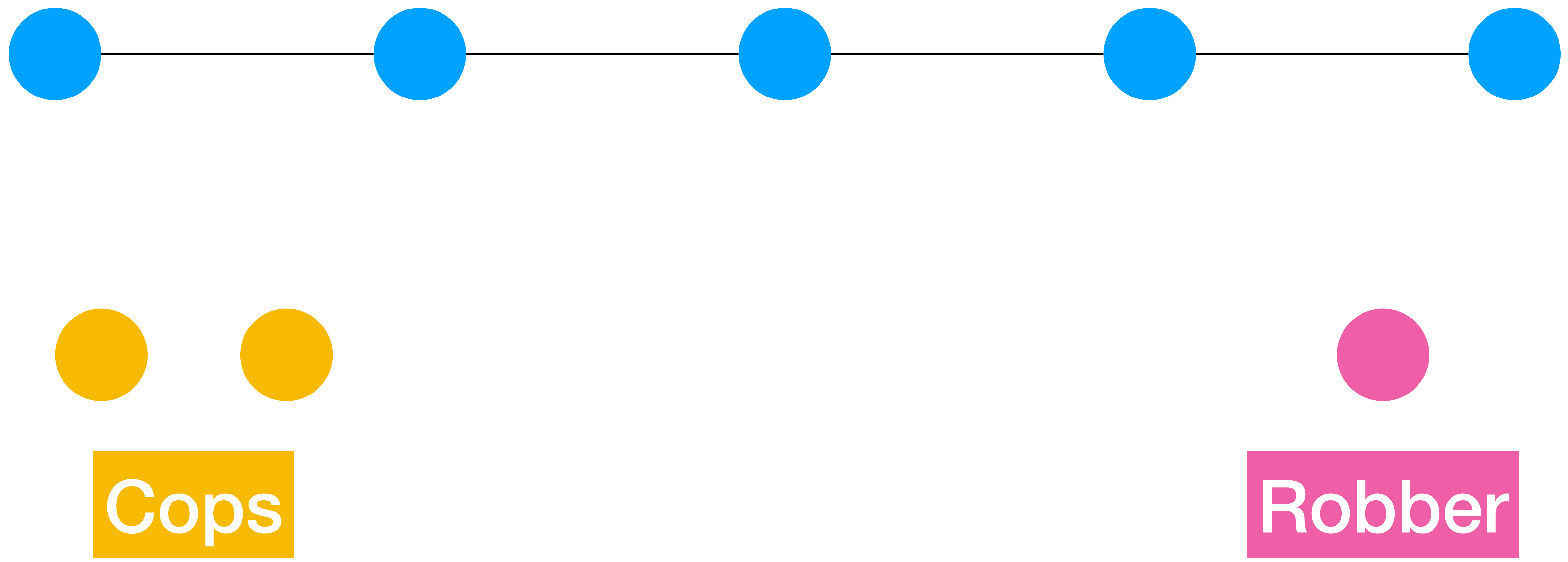
Robber



Cops win the game if the robber is caught.

- Cops may move anywhere, but have to **"leave the graph"** to do so.
- Robber moves along the edges at **"infinite speed"**, must avoid cops.

Cops and Robbers



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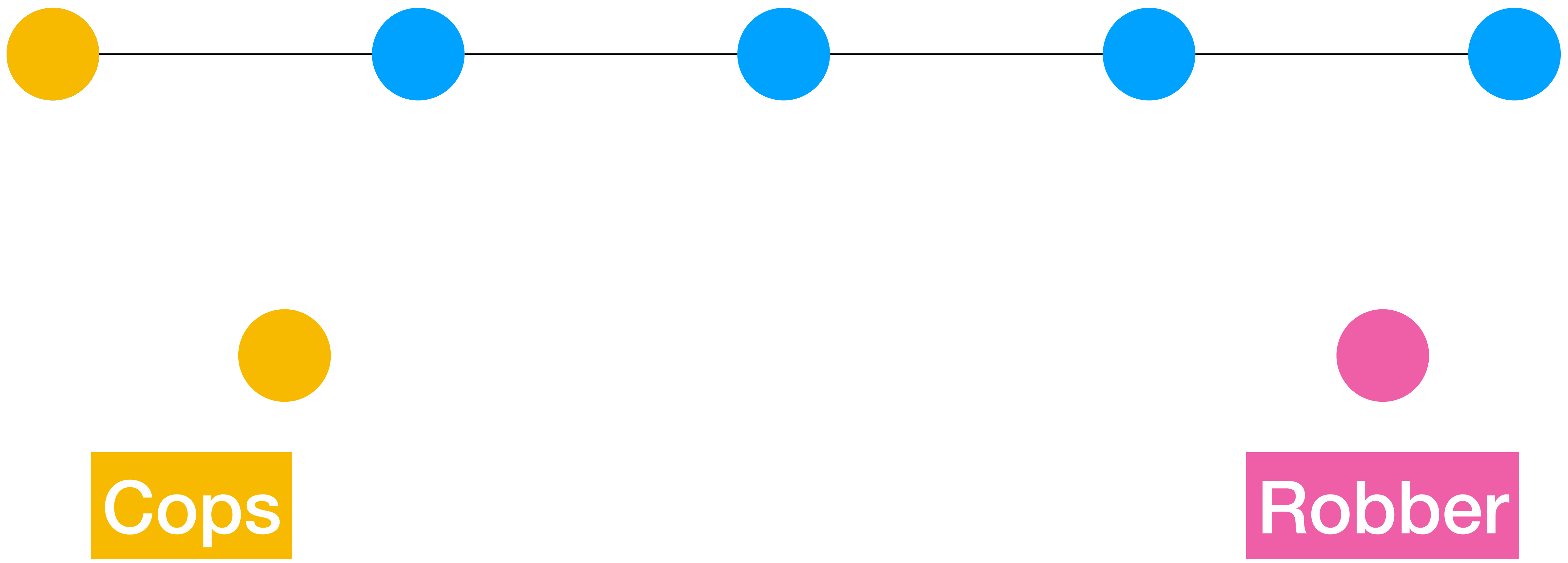

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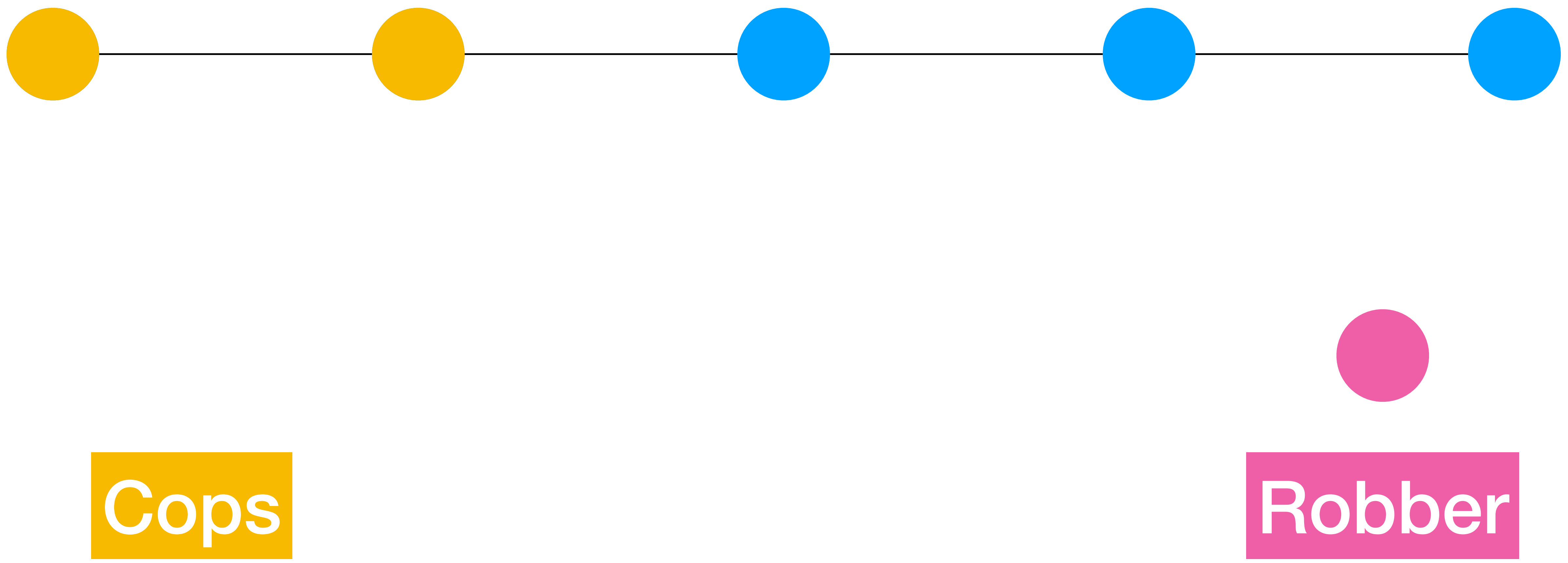

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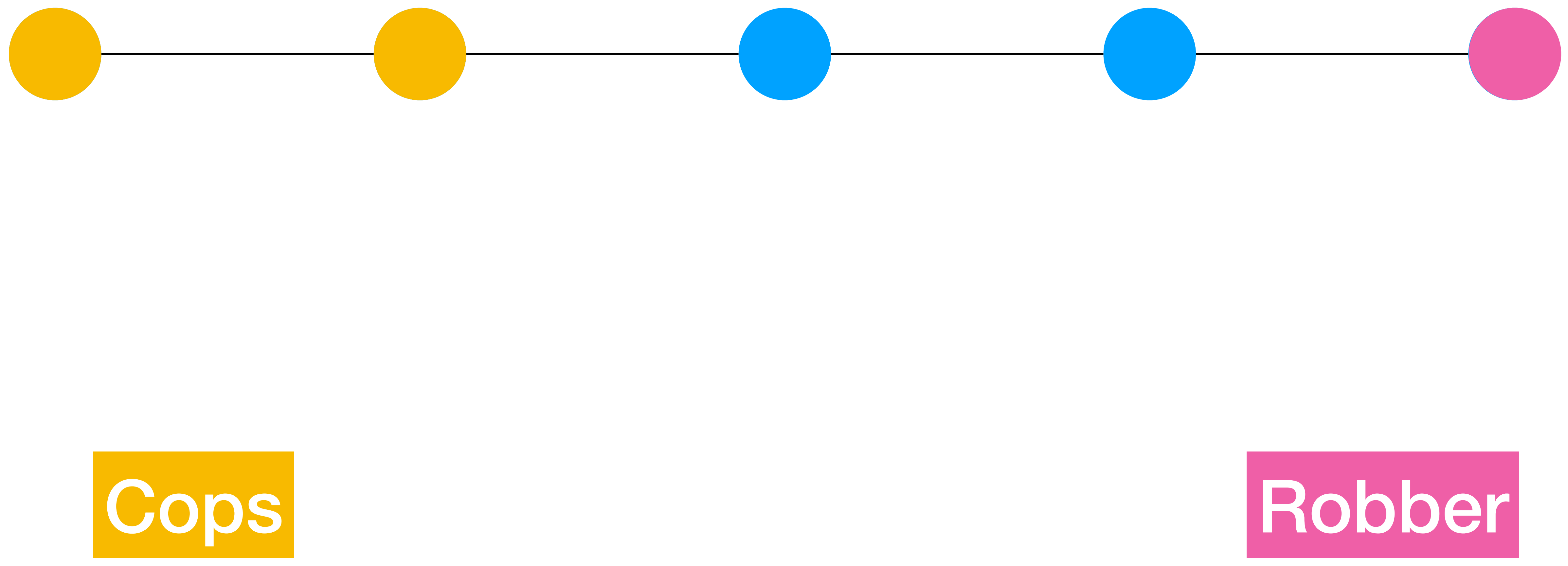

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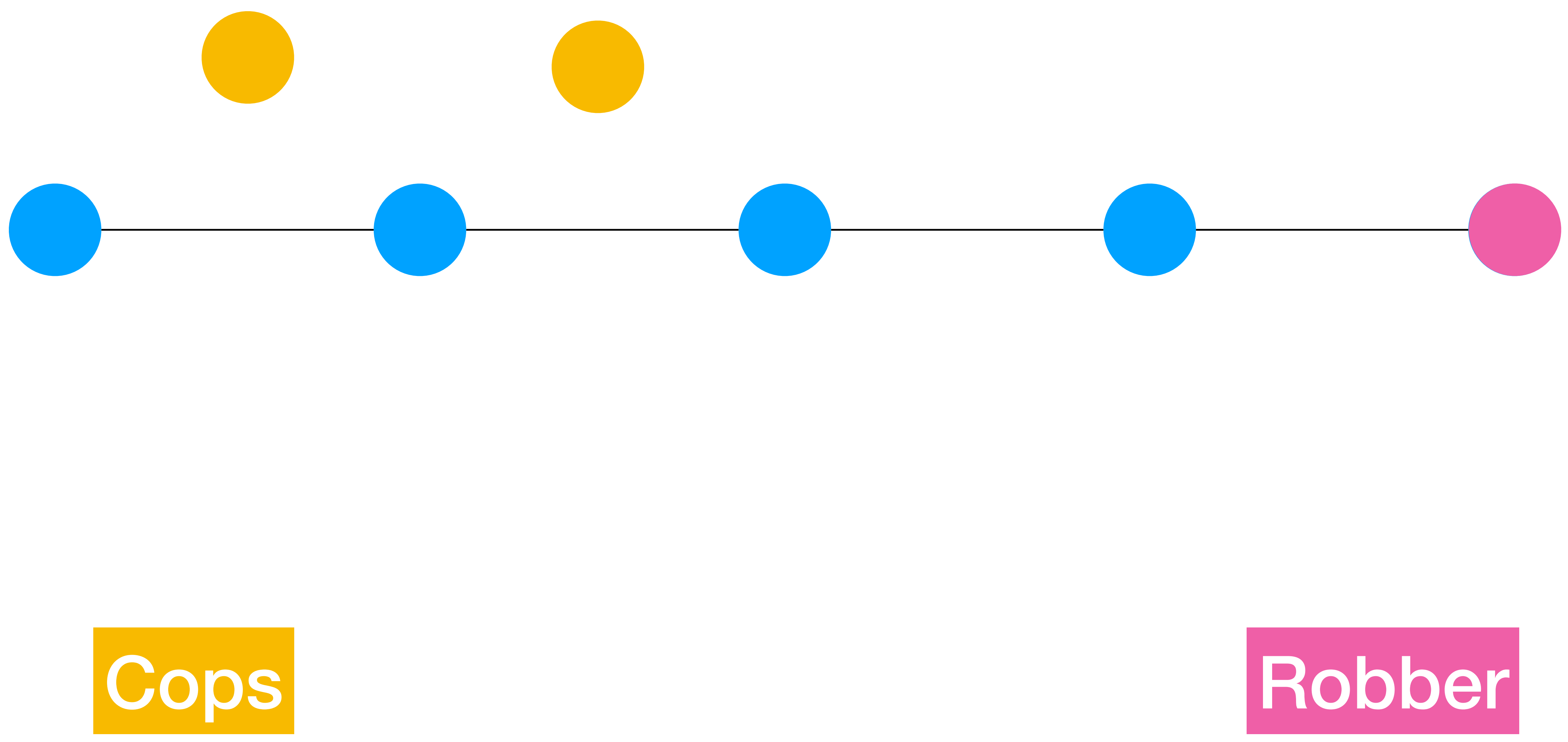

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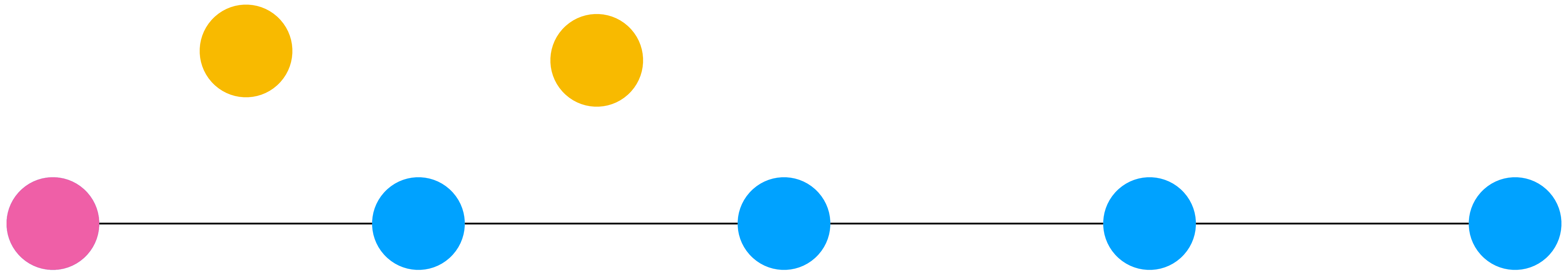
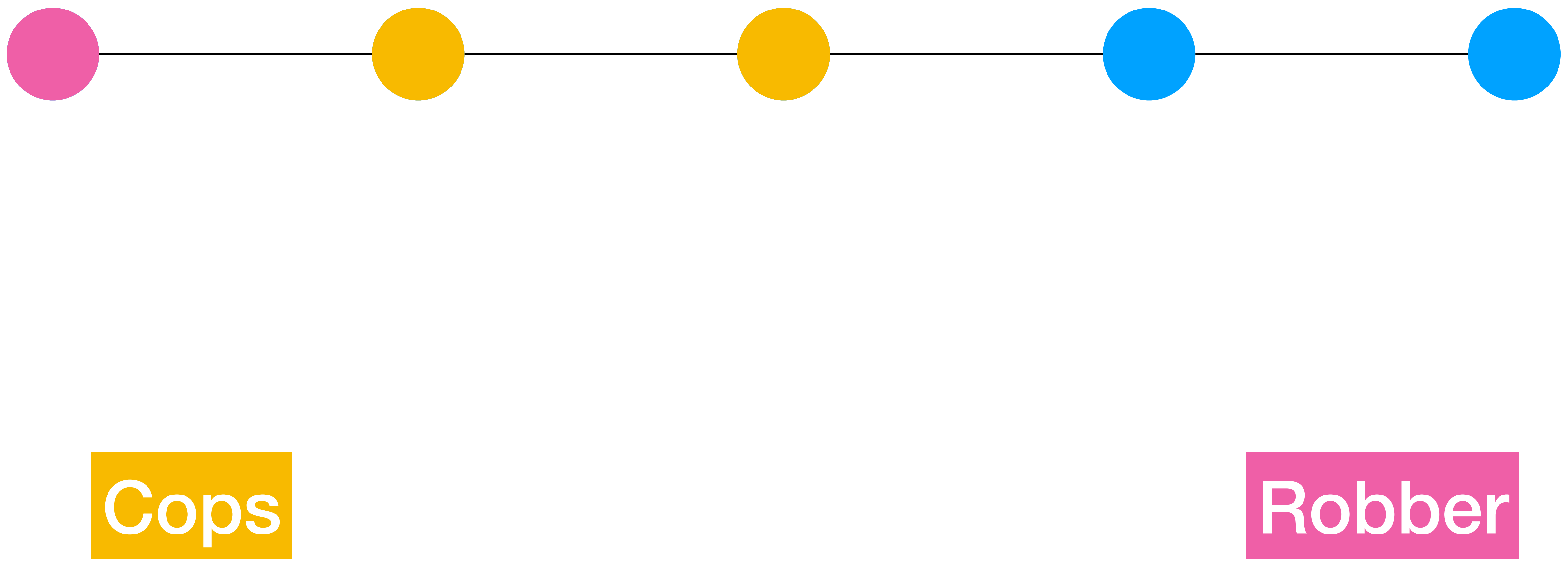


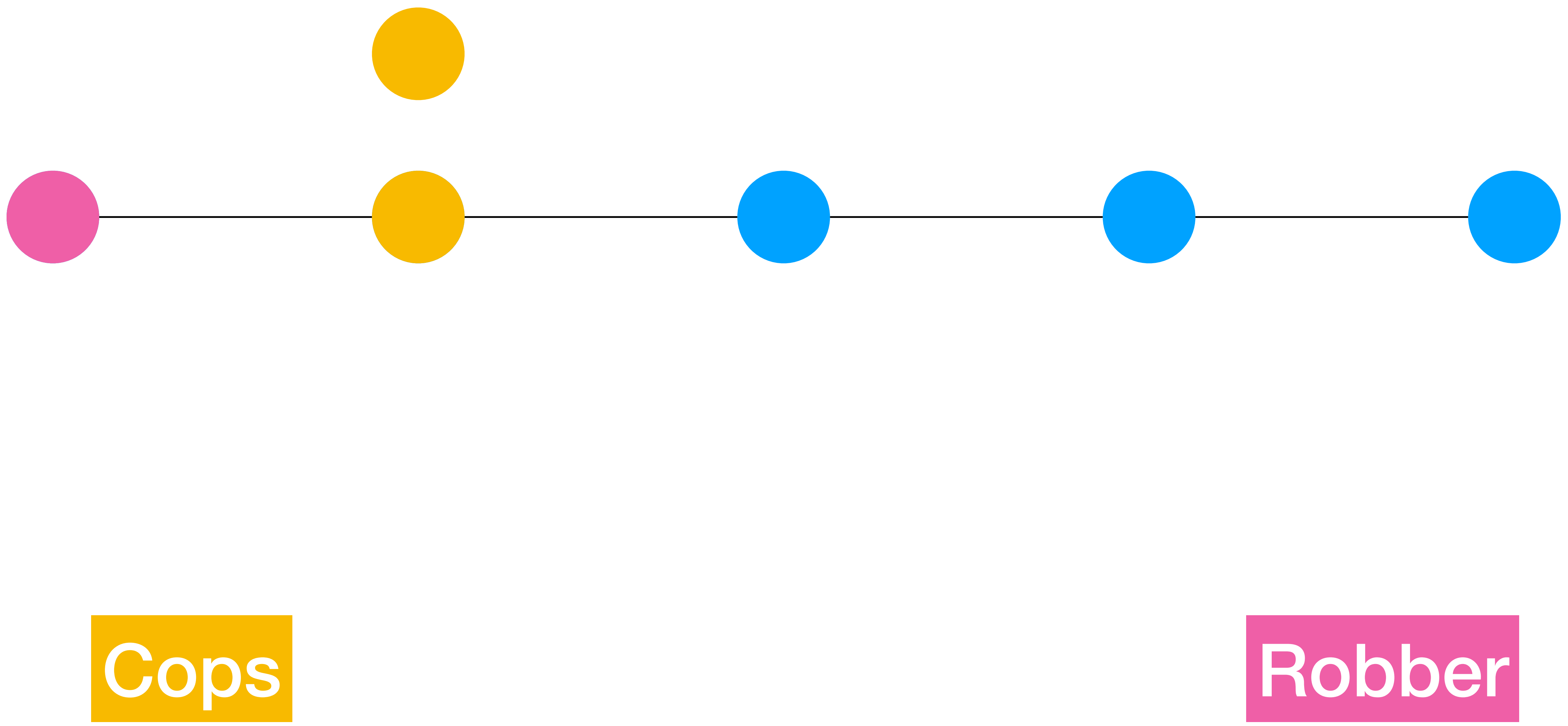
Cops

Robber

Cops and Robbers



Cops and Robbers



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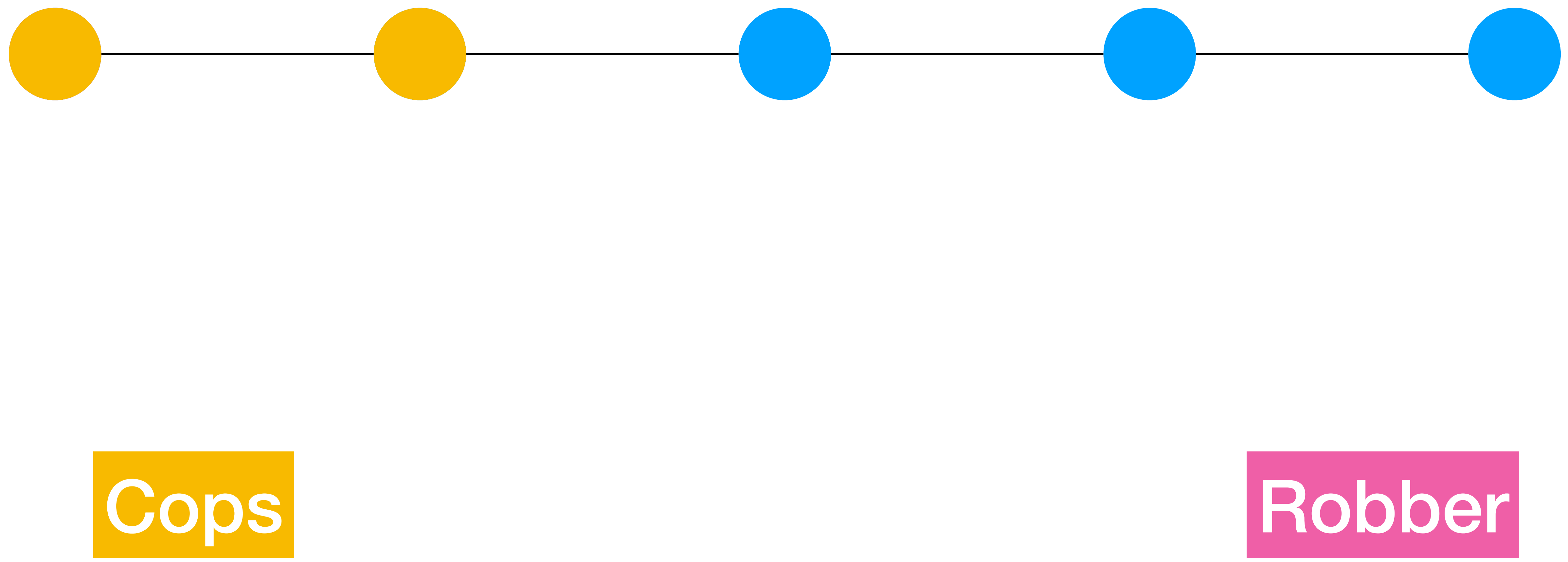

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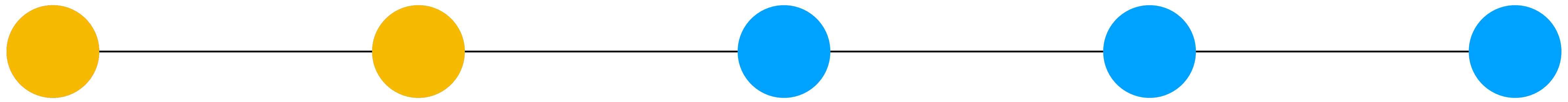


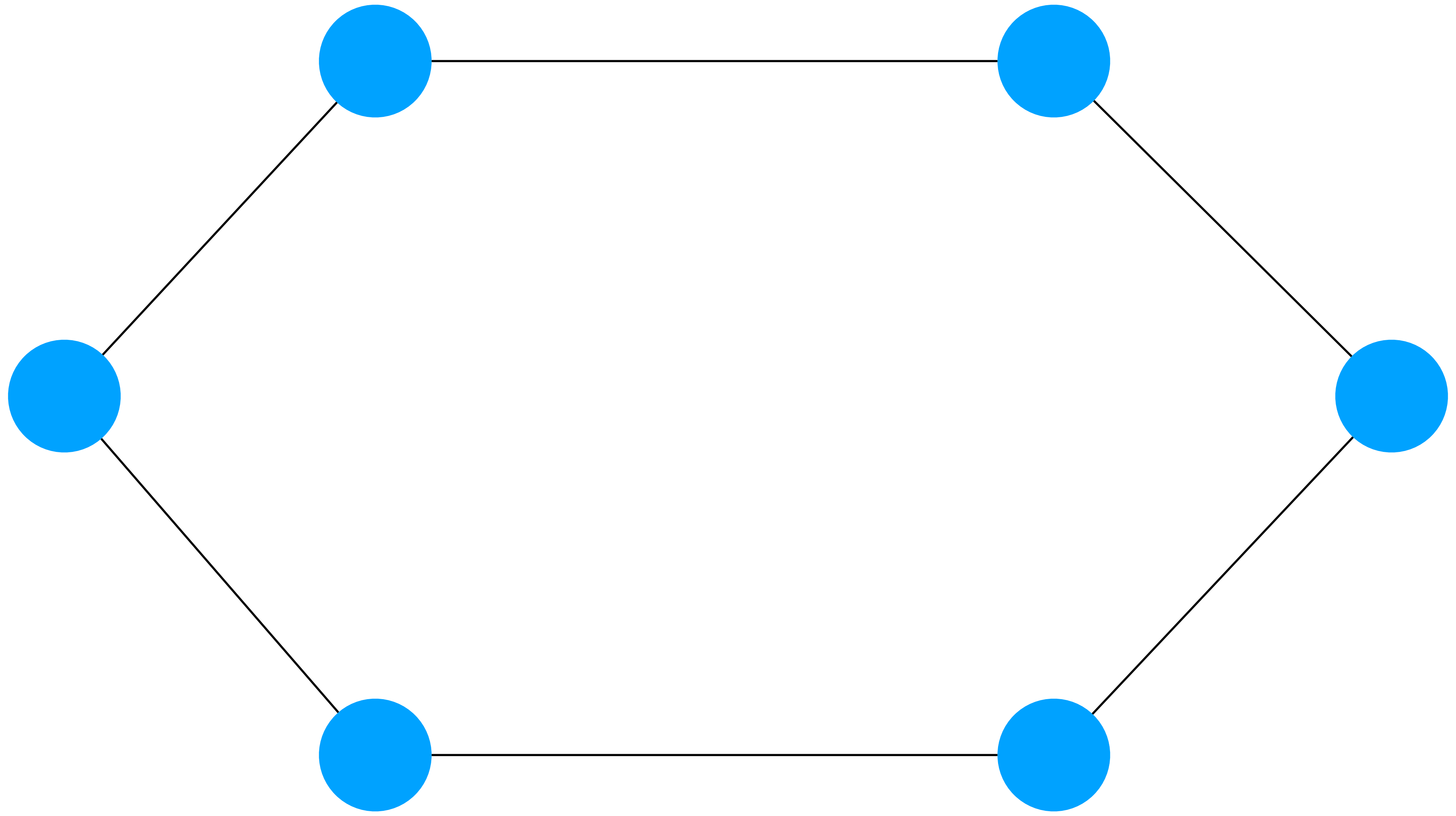
Robber caught.

Another Definition of Treewidth

Fact

The **treewidth** of a graph G is the minimum k such that $k+1$ cops have a strategy to catch a robber in G.

Example



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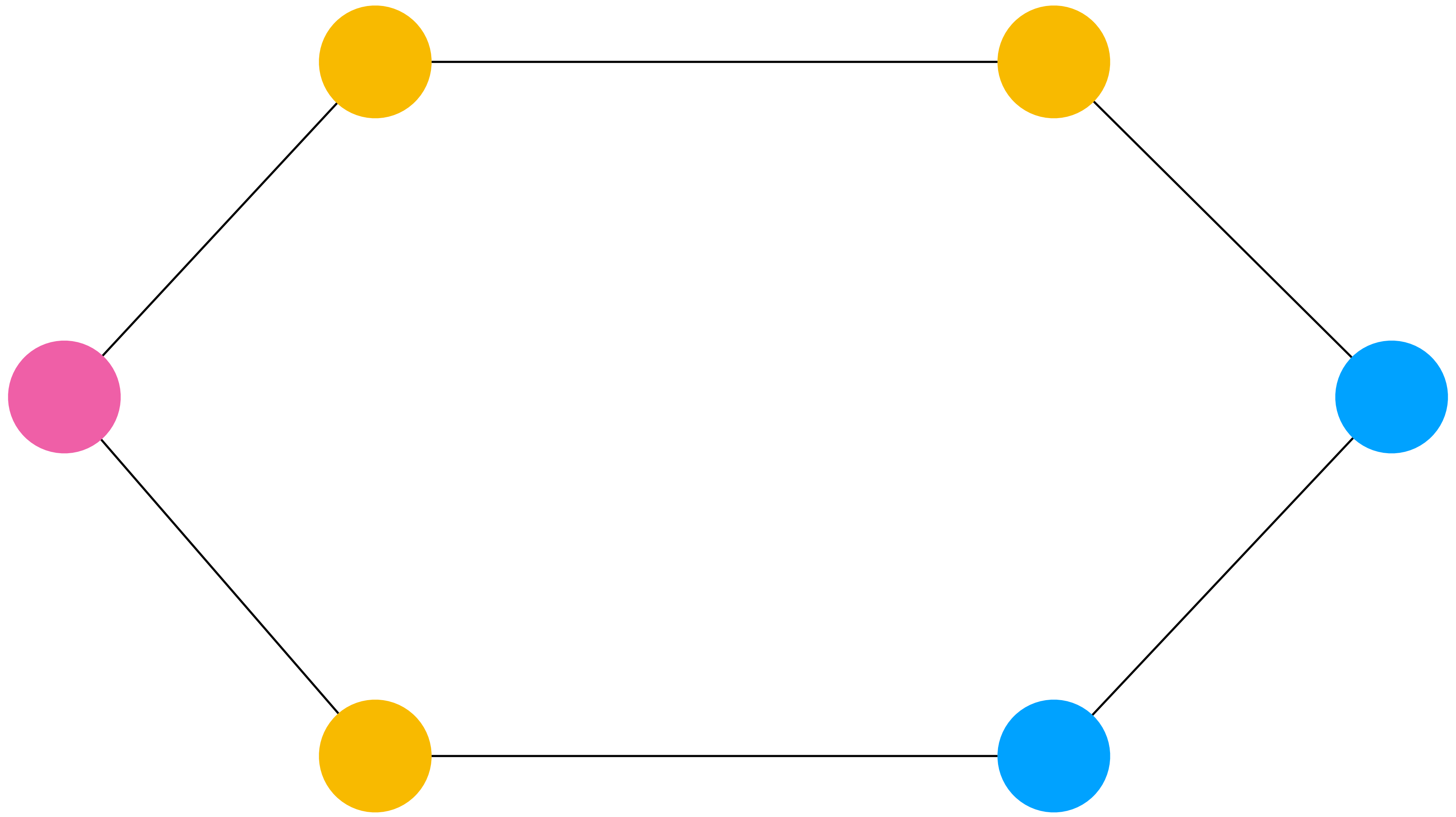

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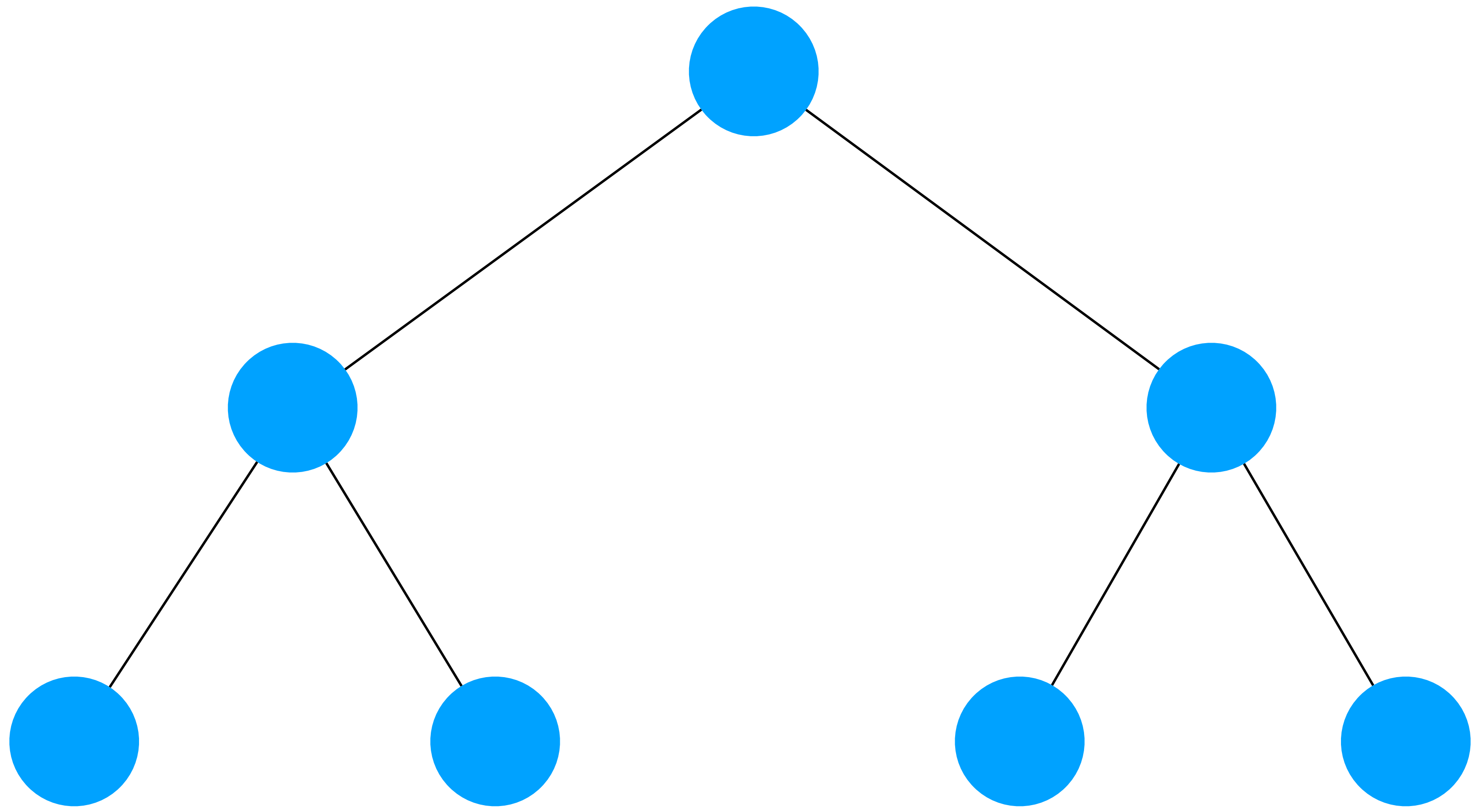

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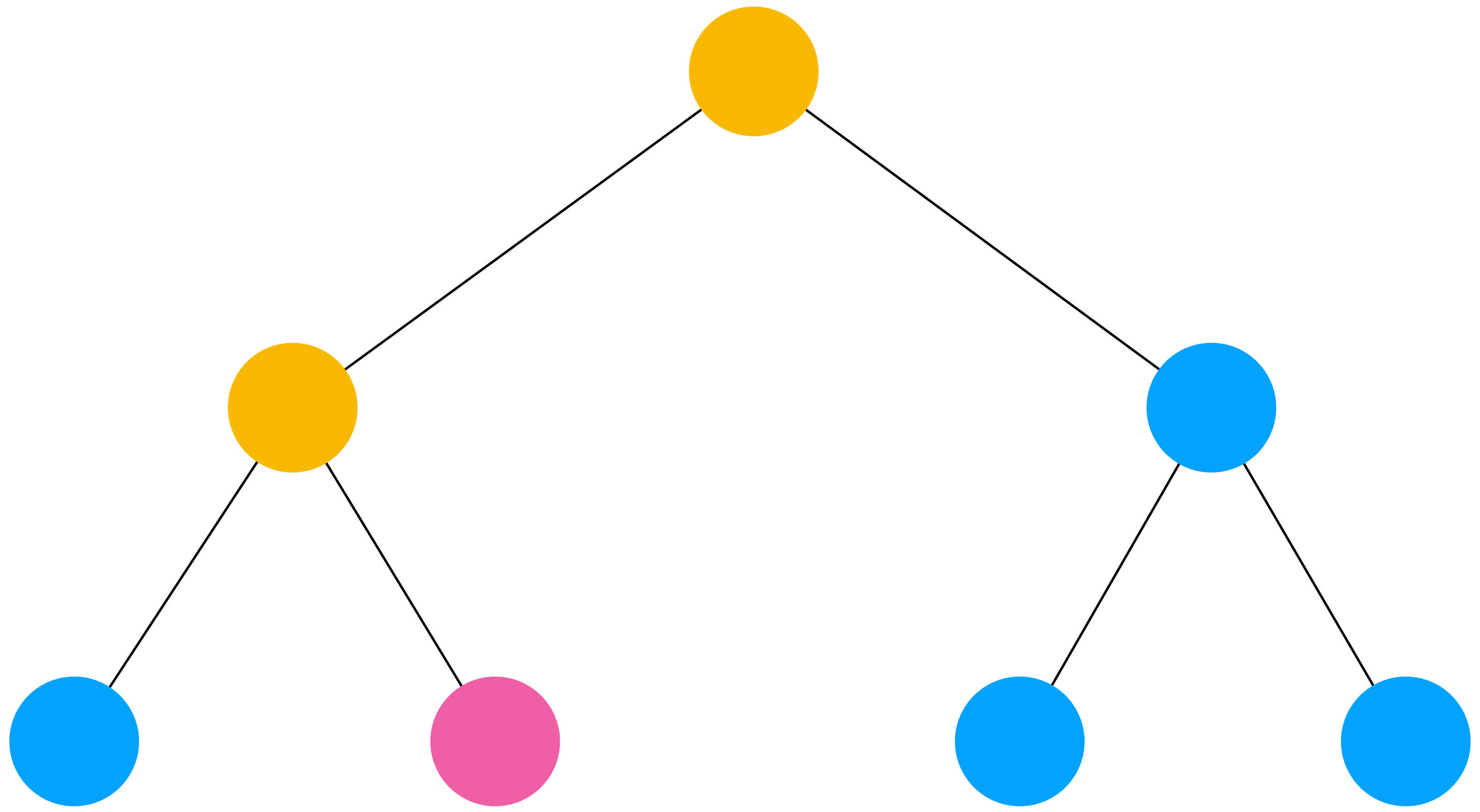

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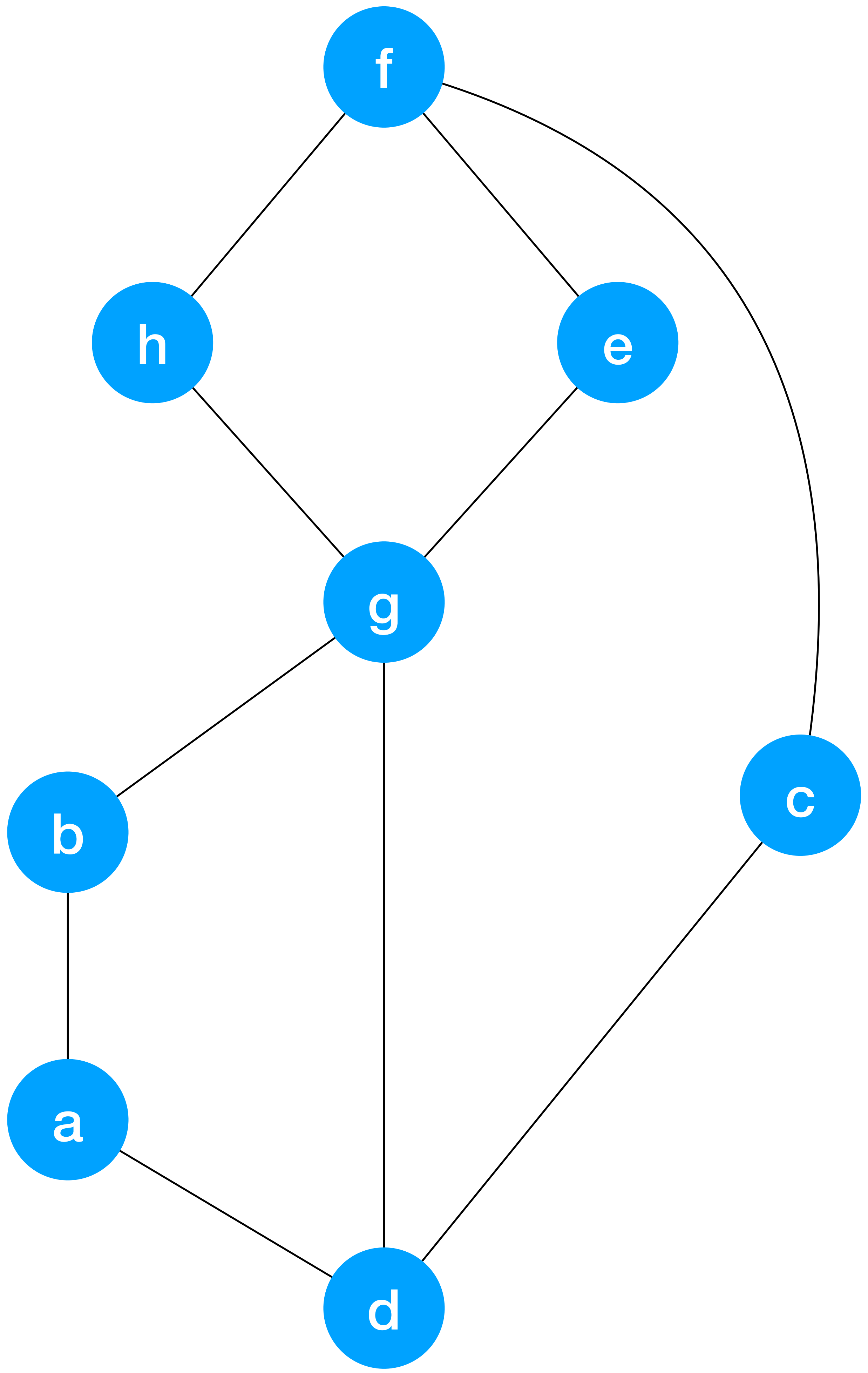
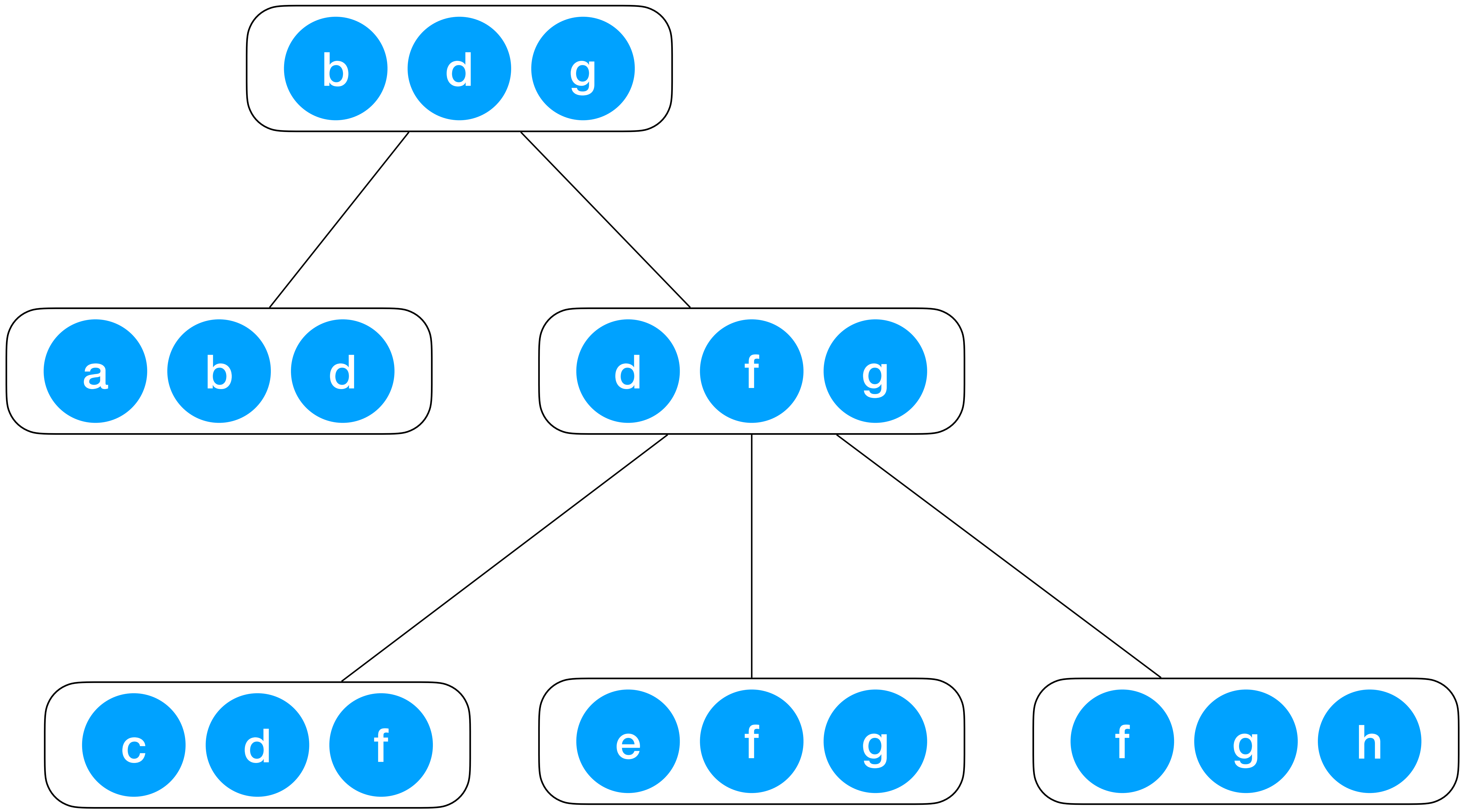
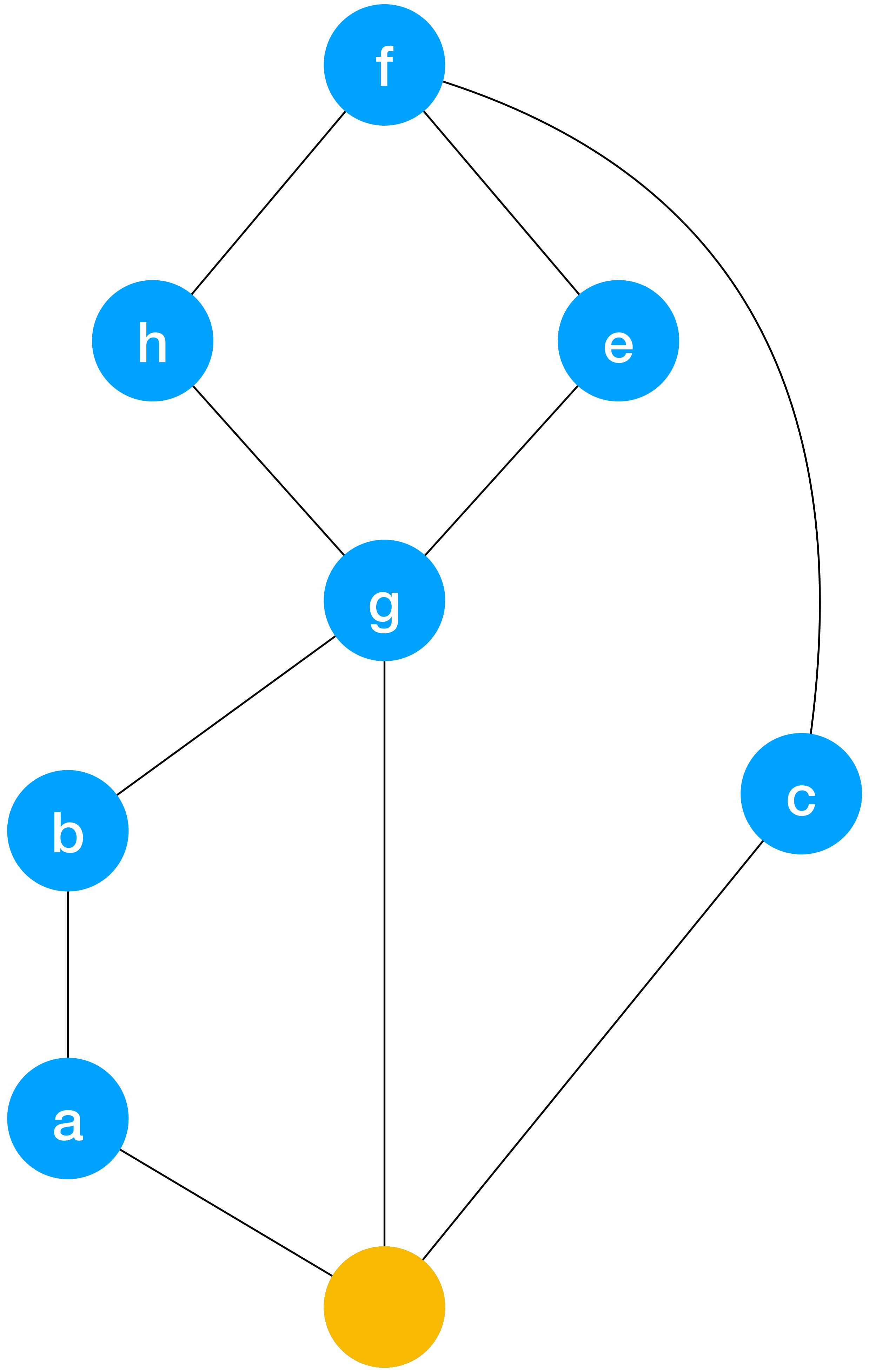
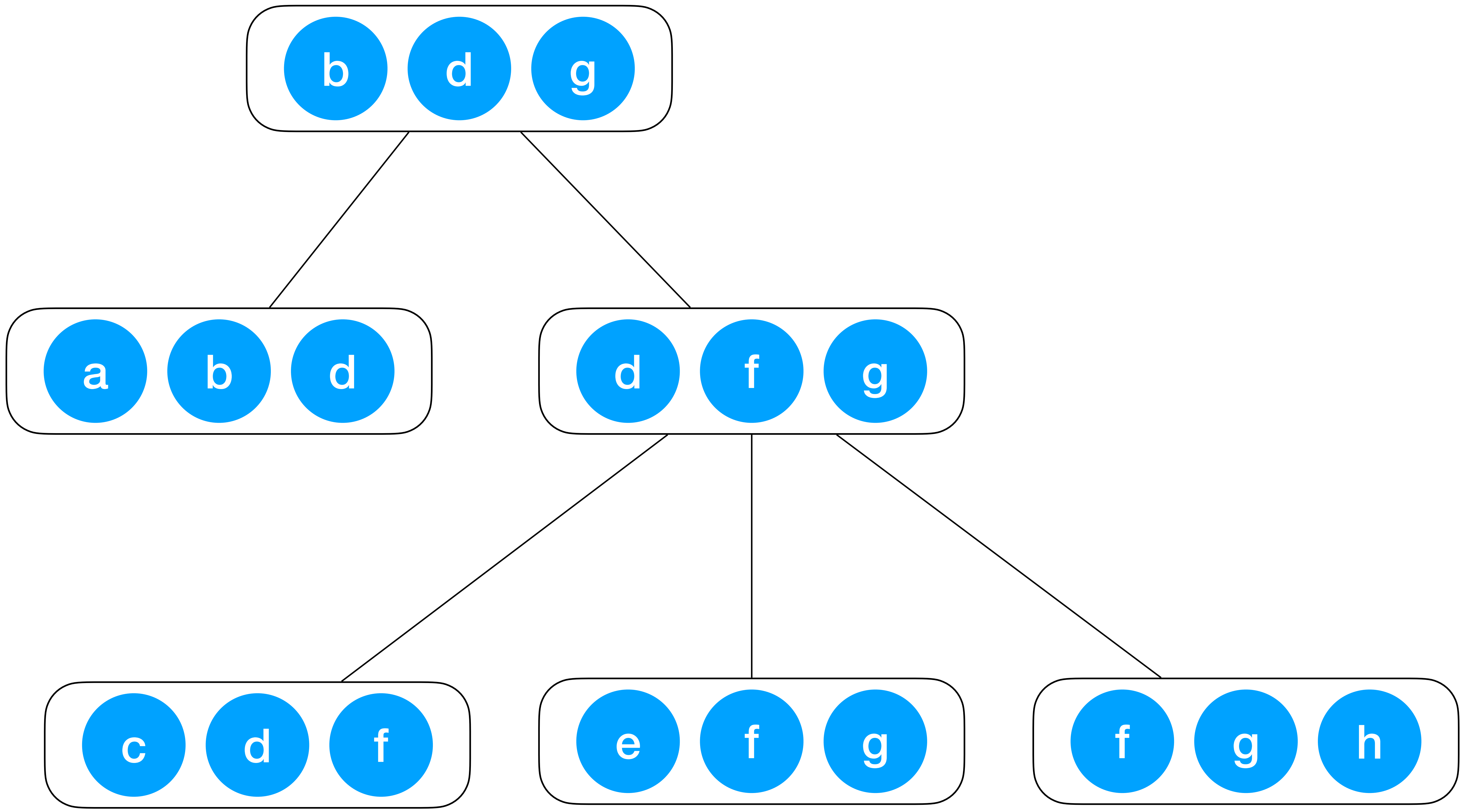




Example





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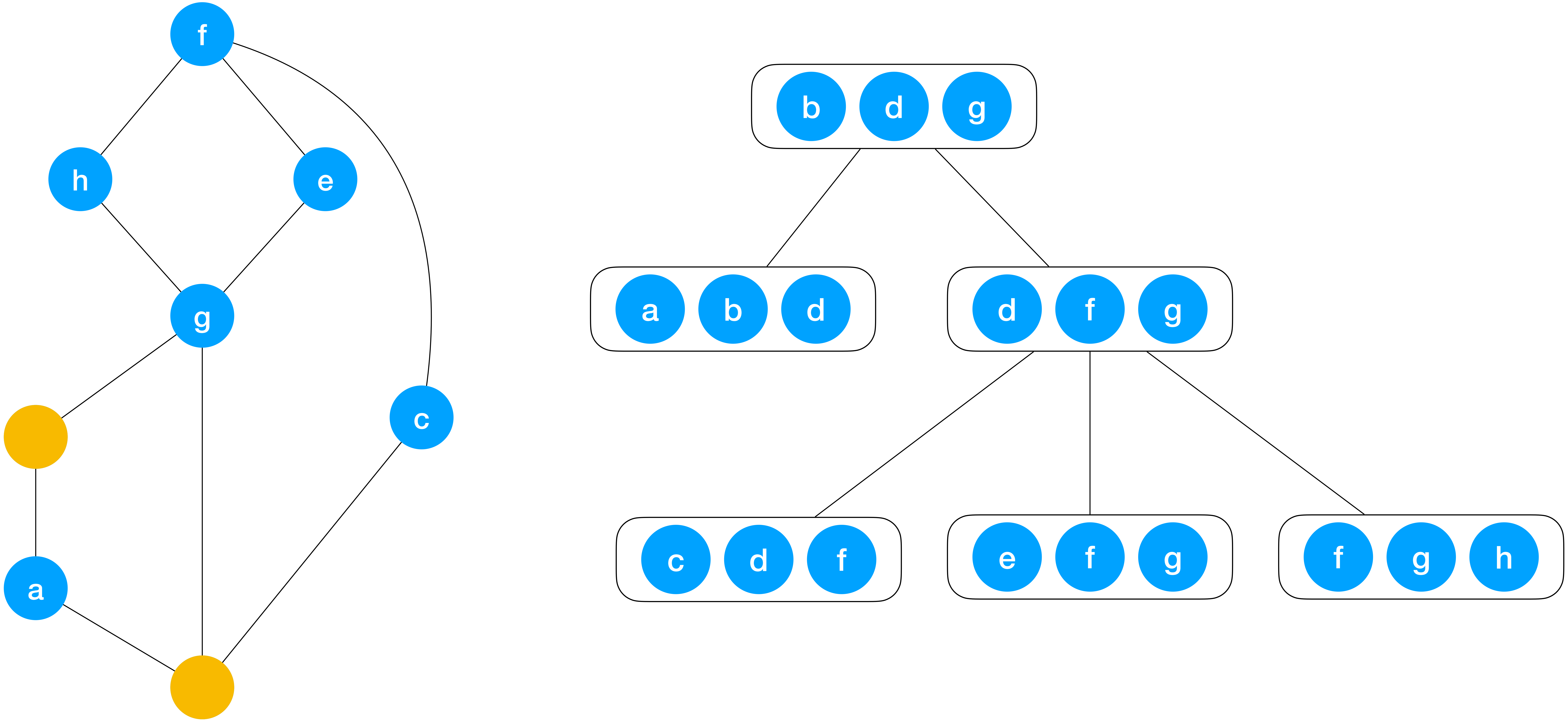

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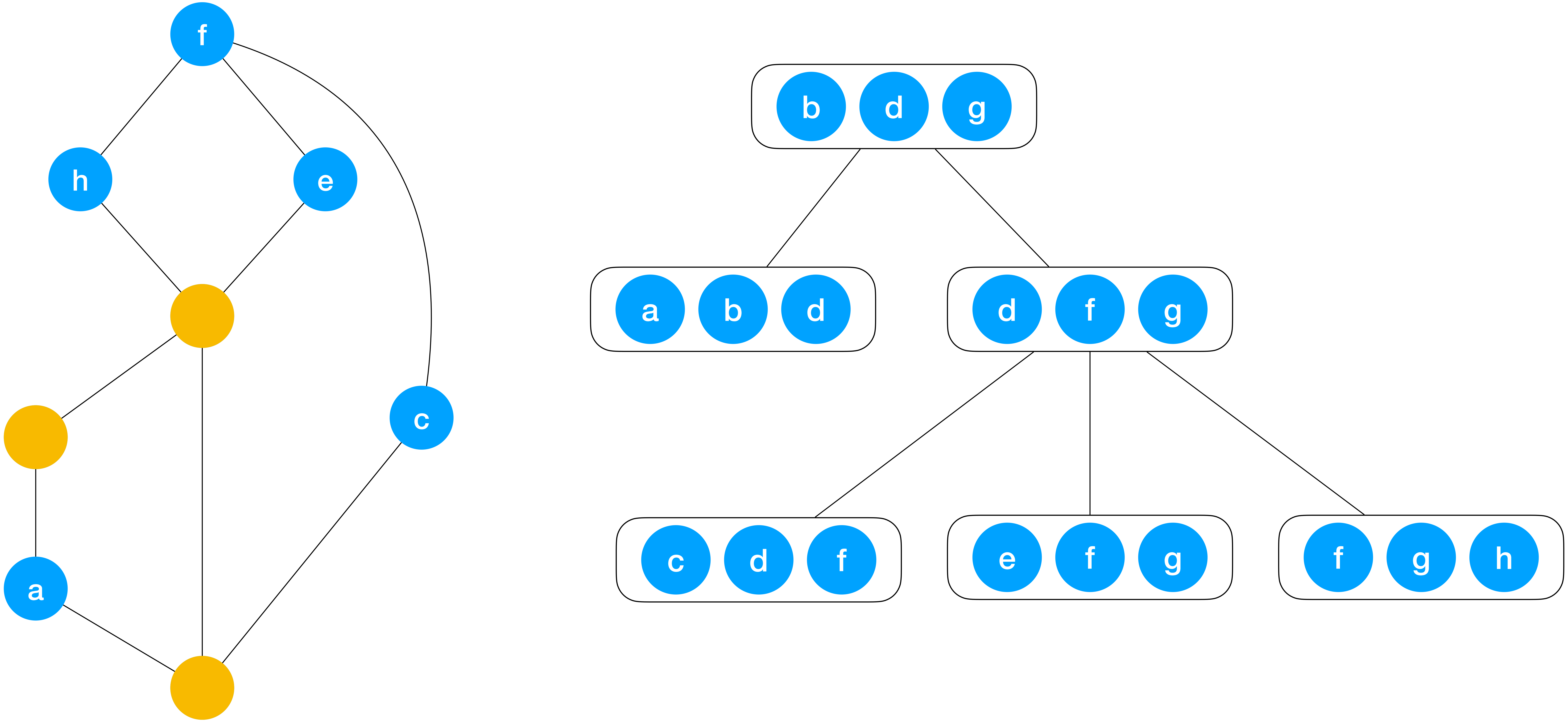

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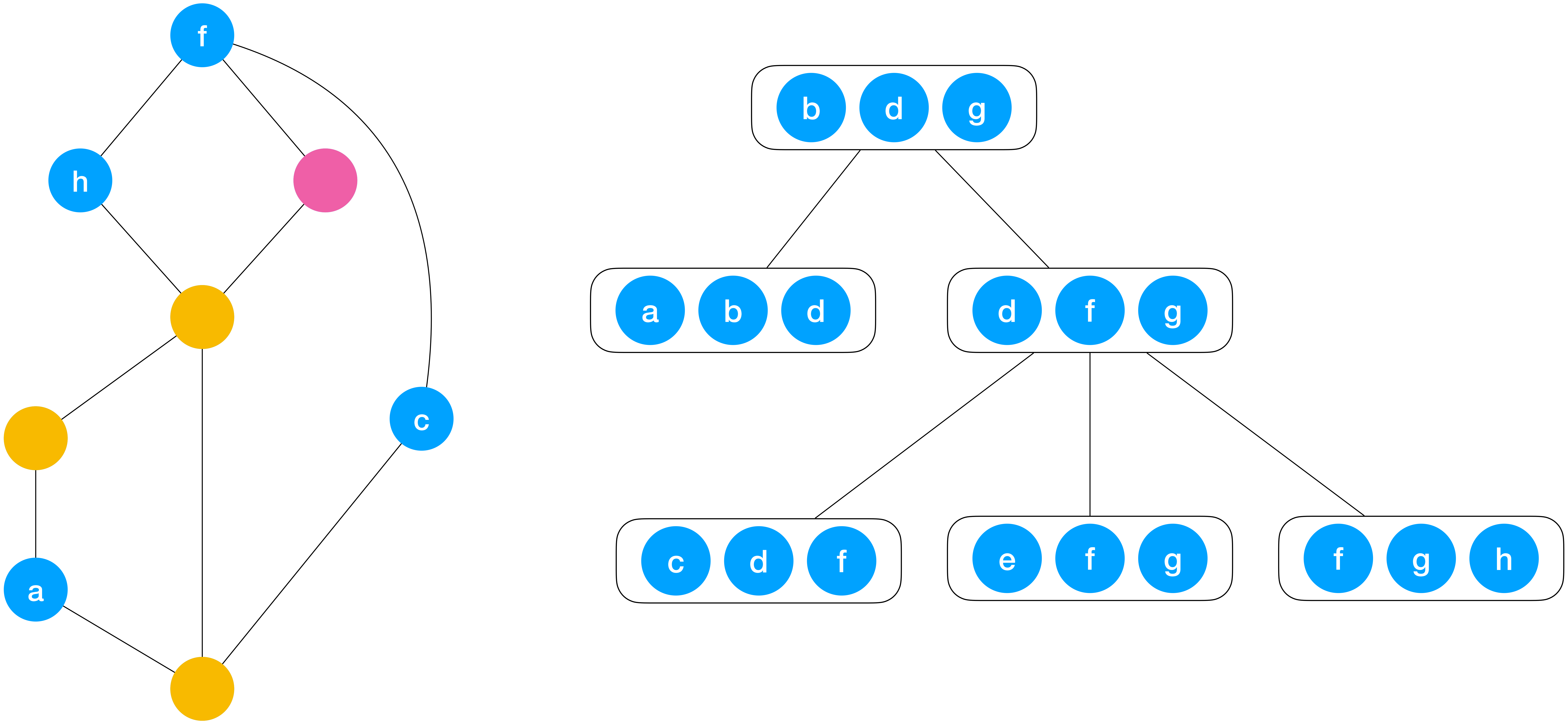

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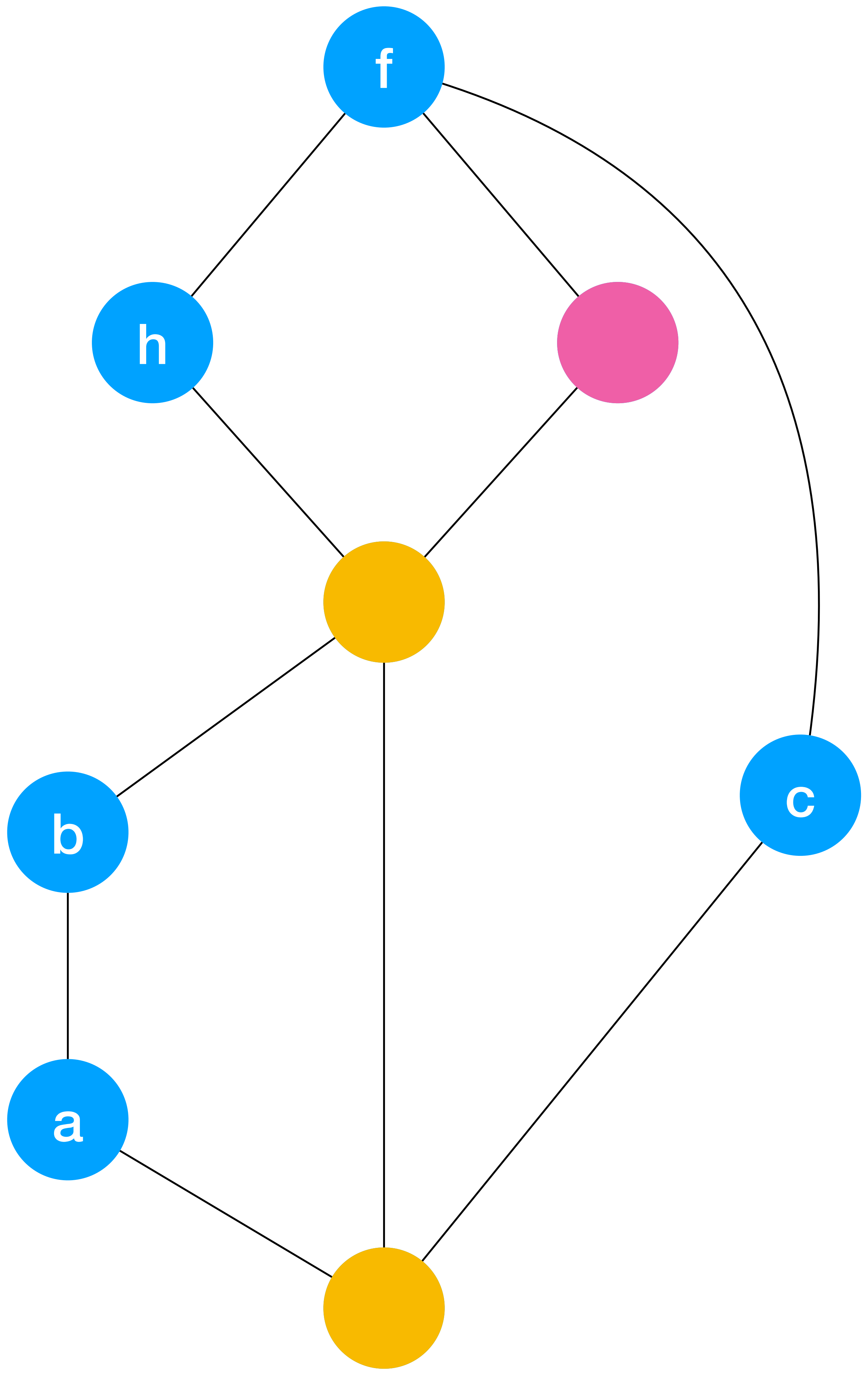

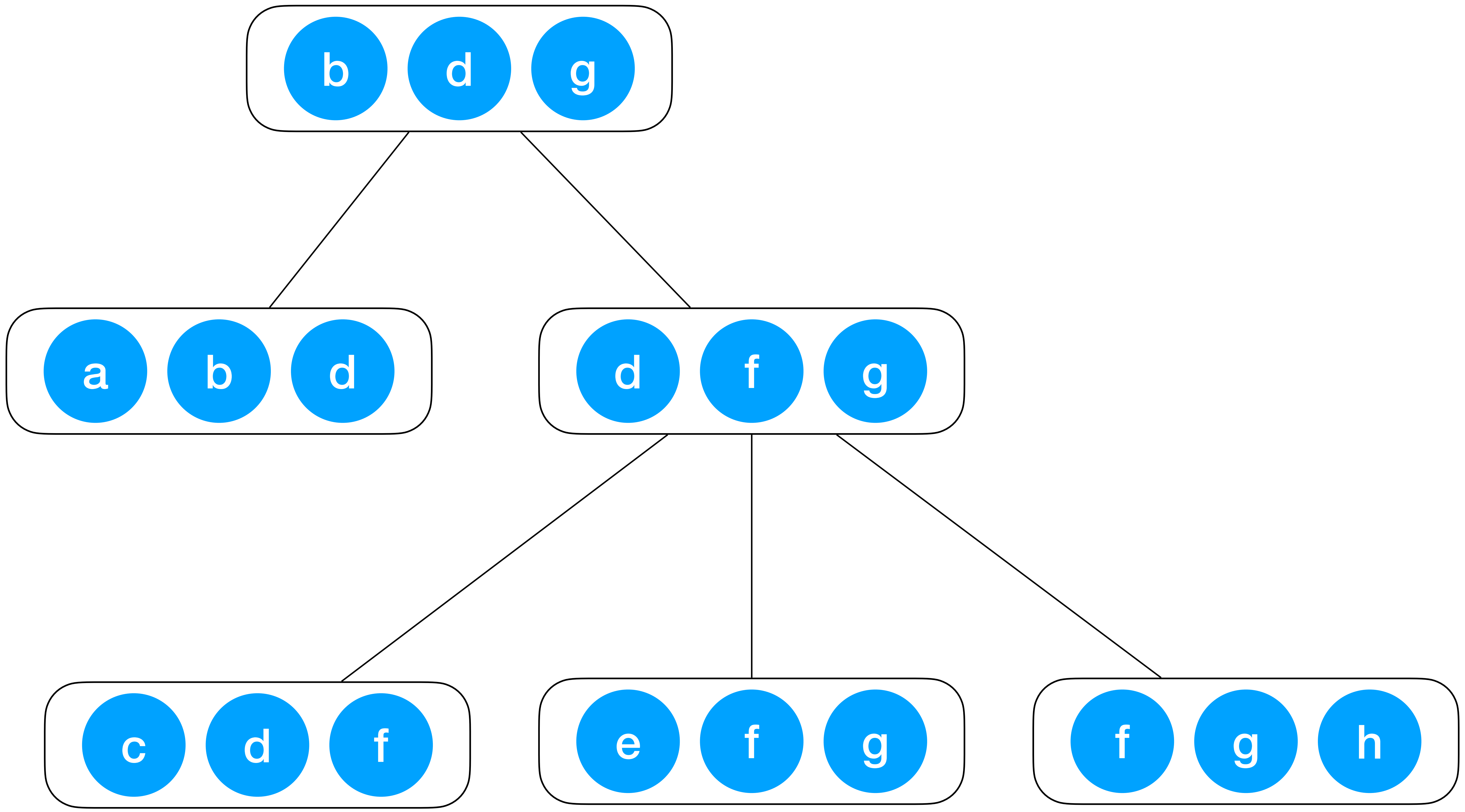

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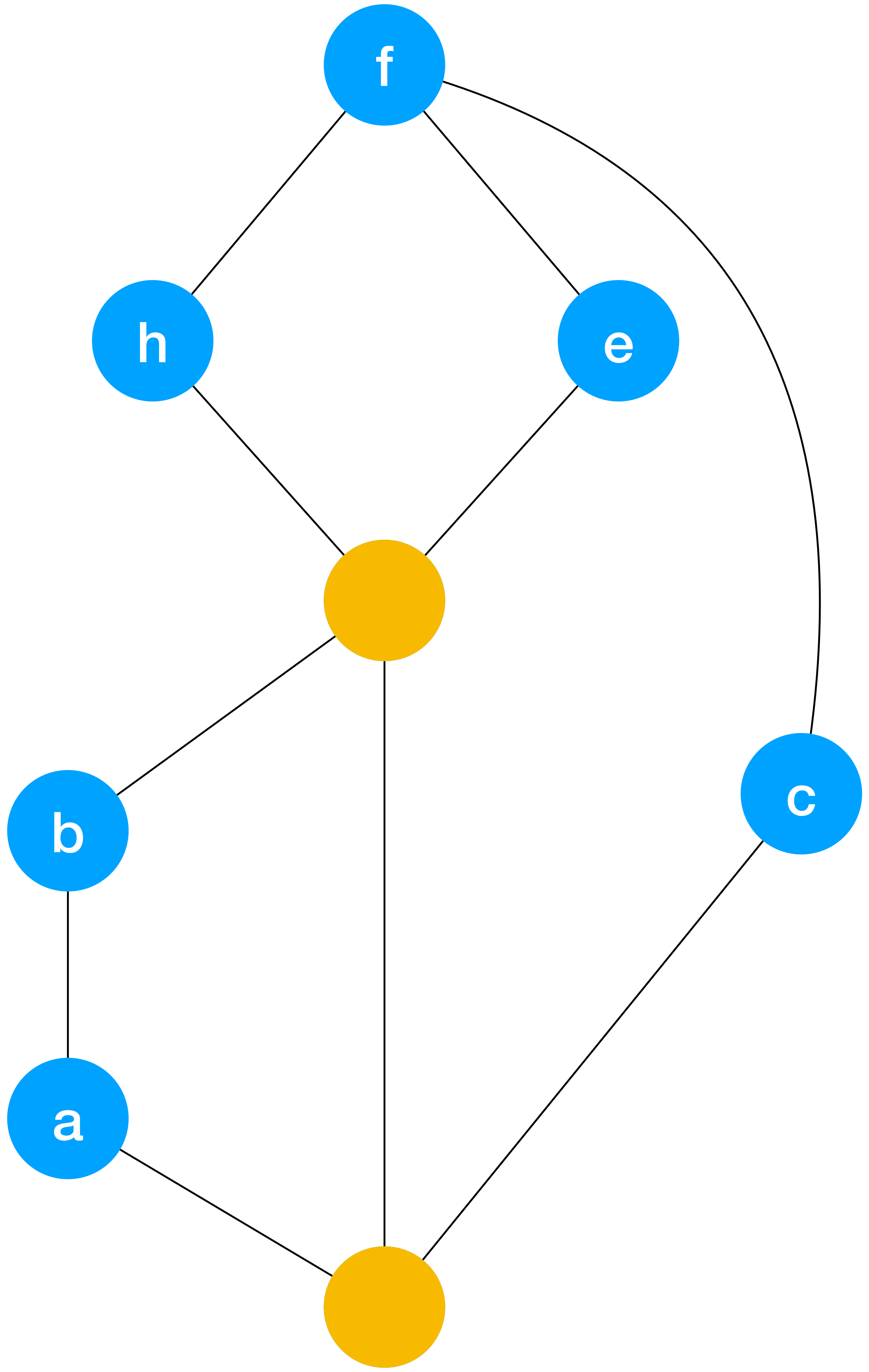

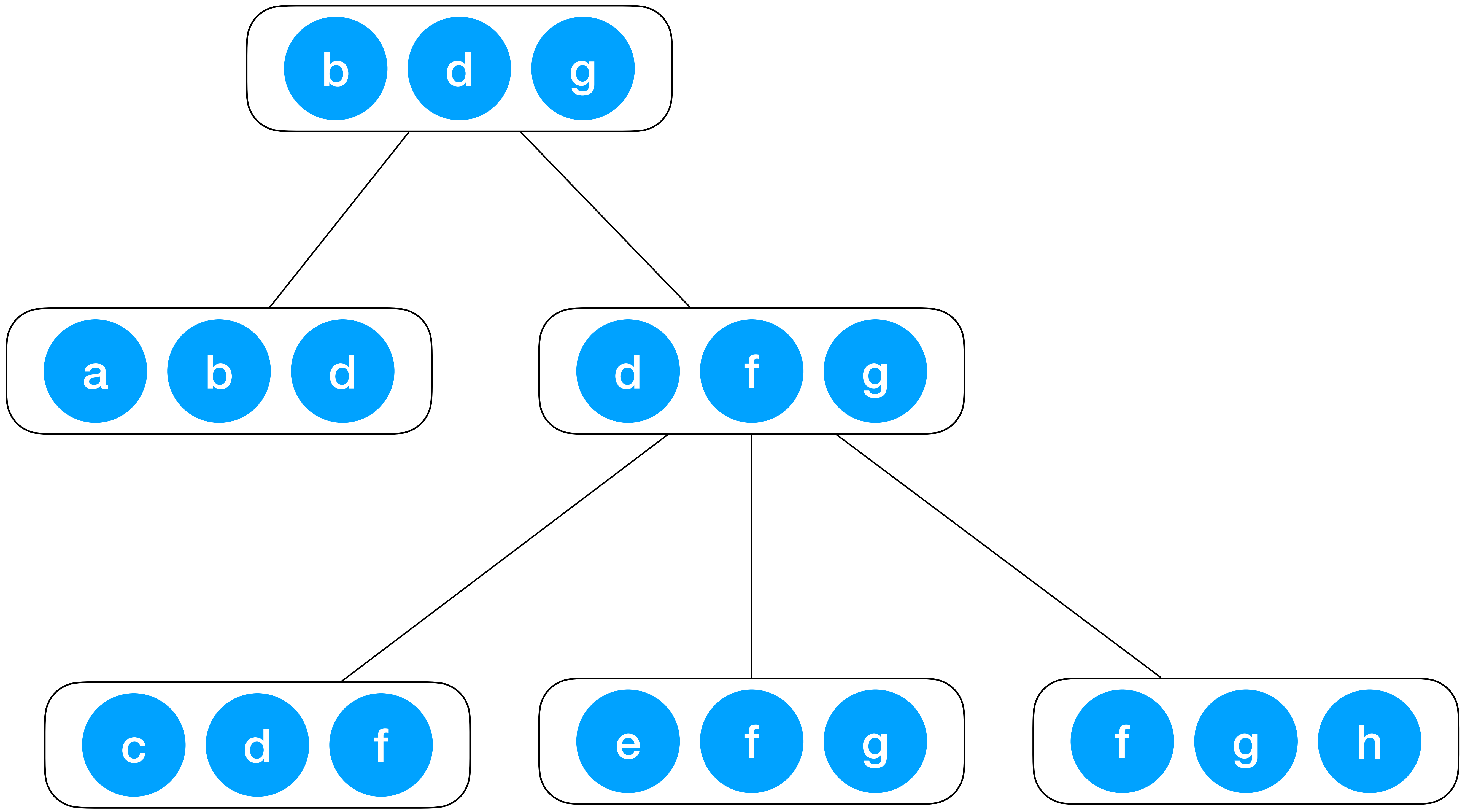

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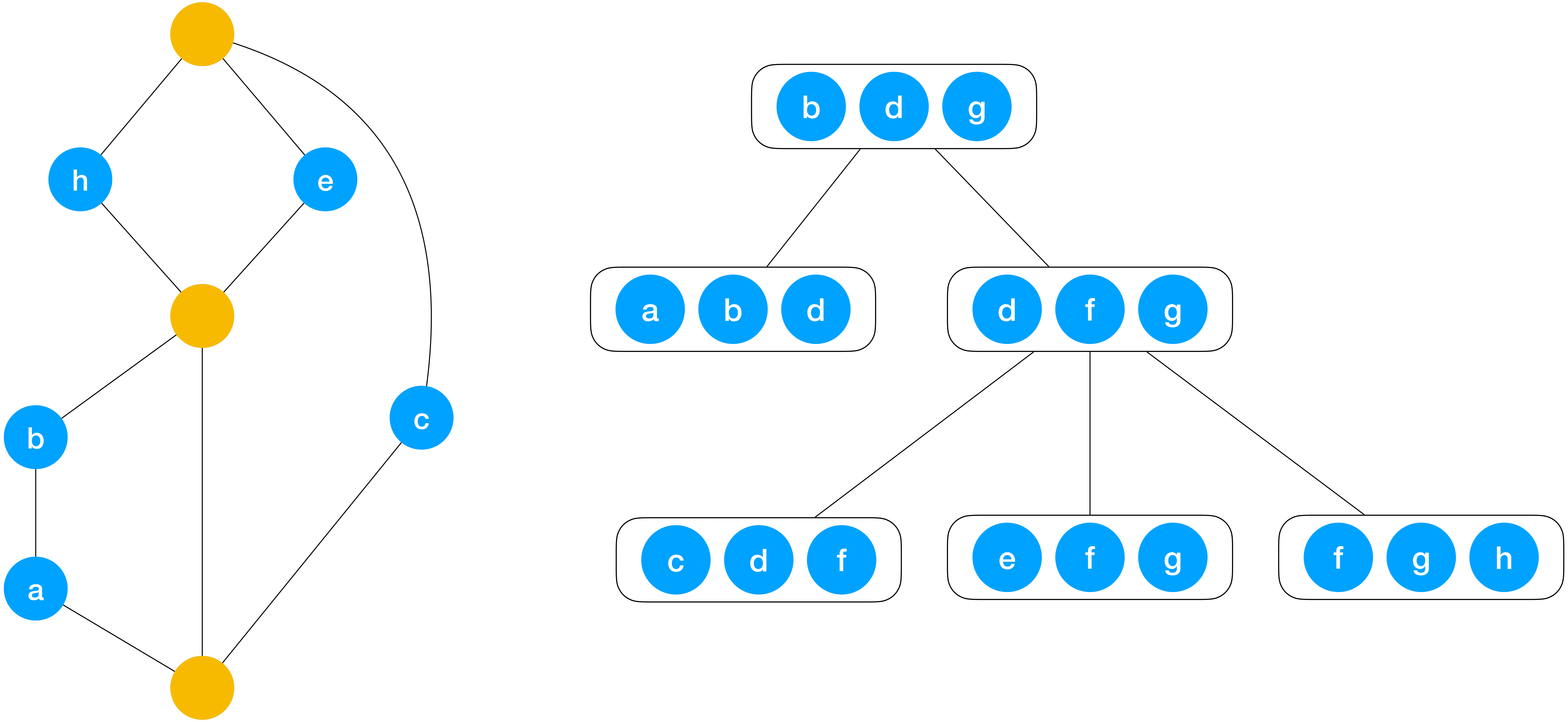

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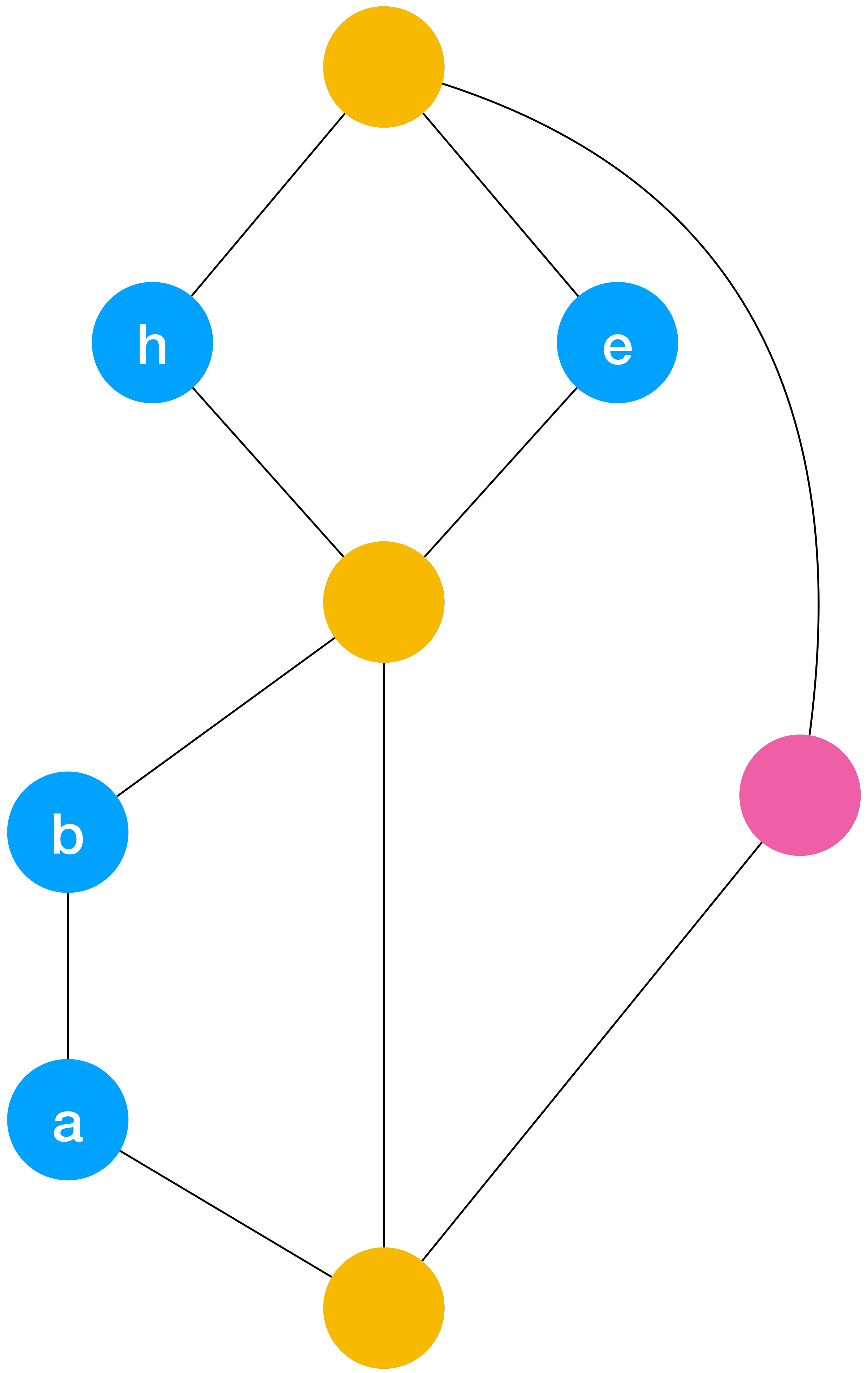

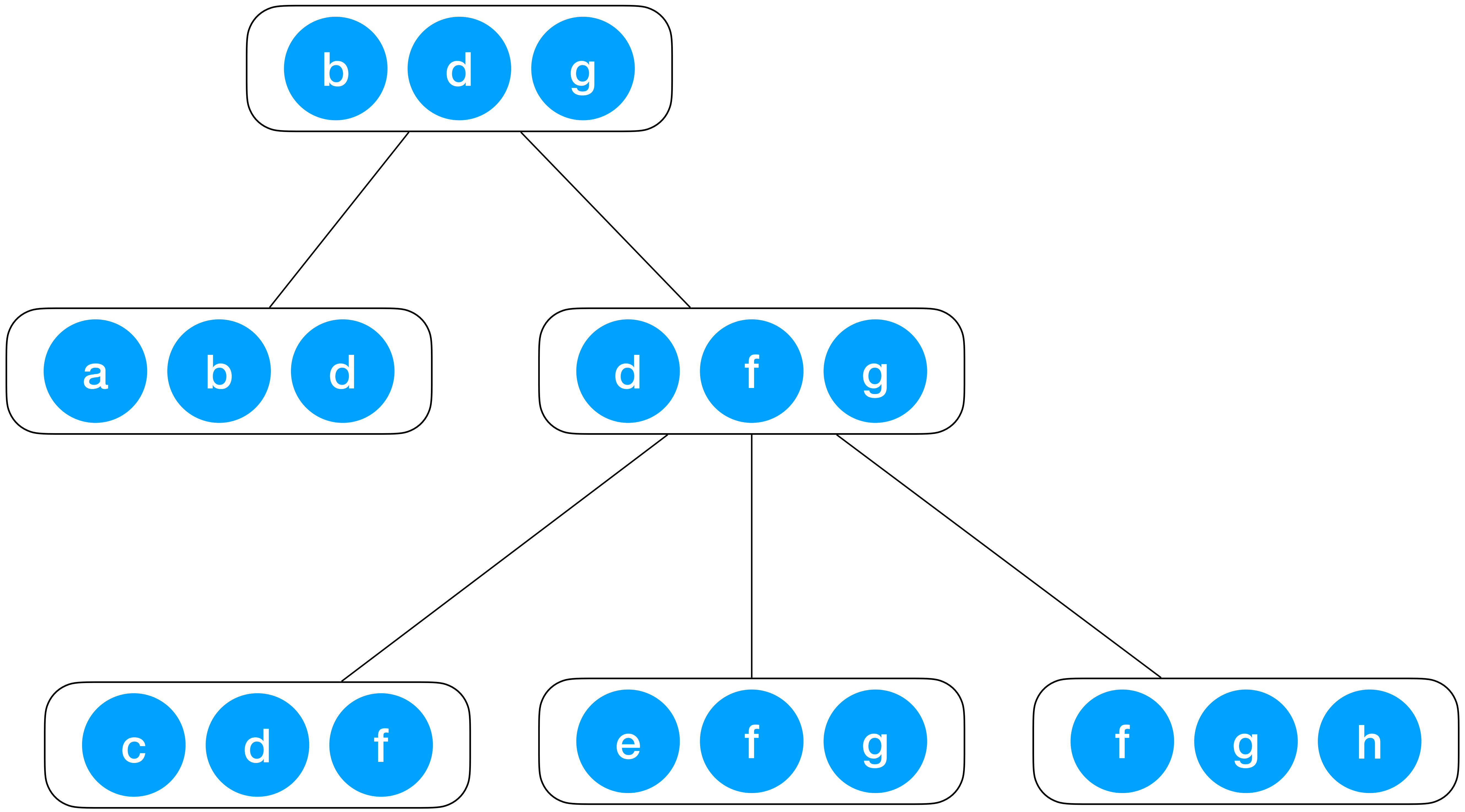

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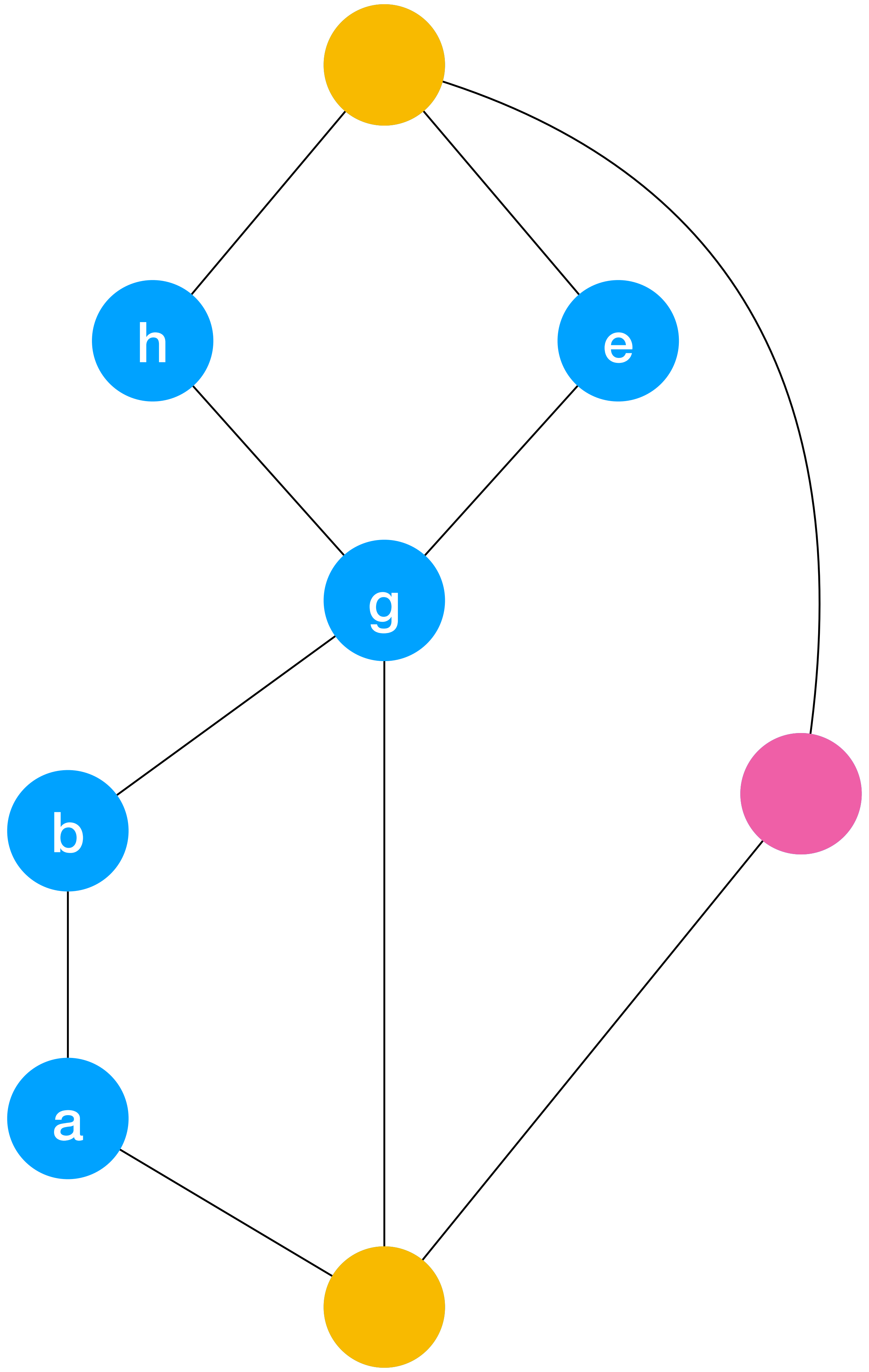

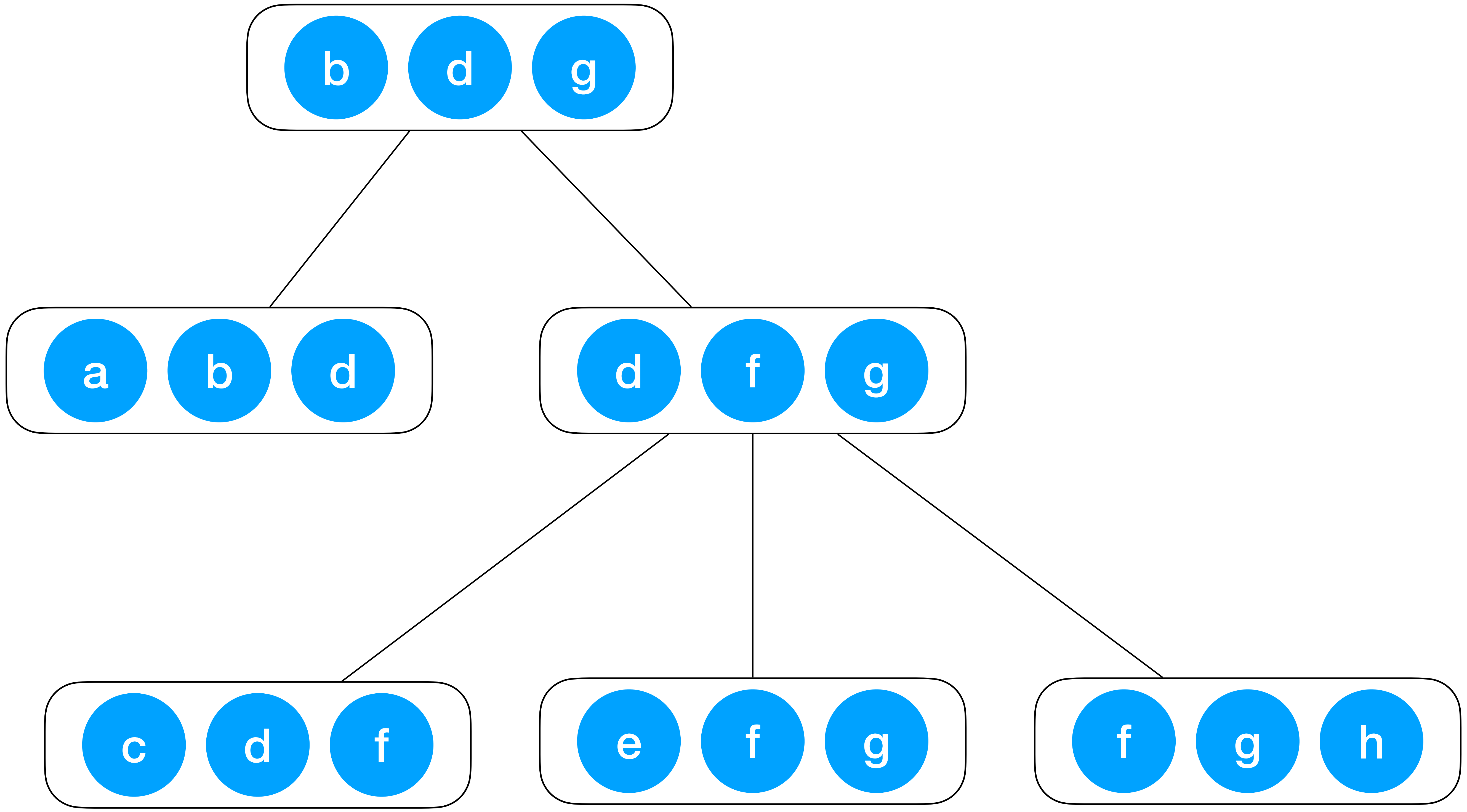

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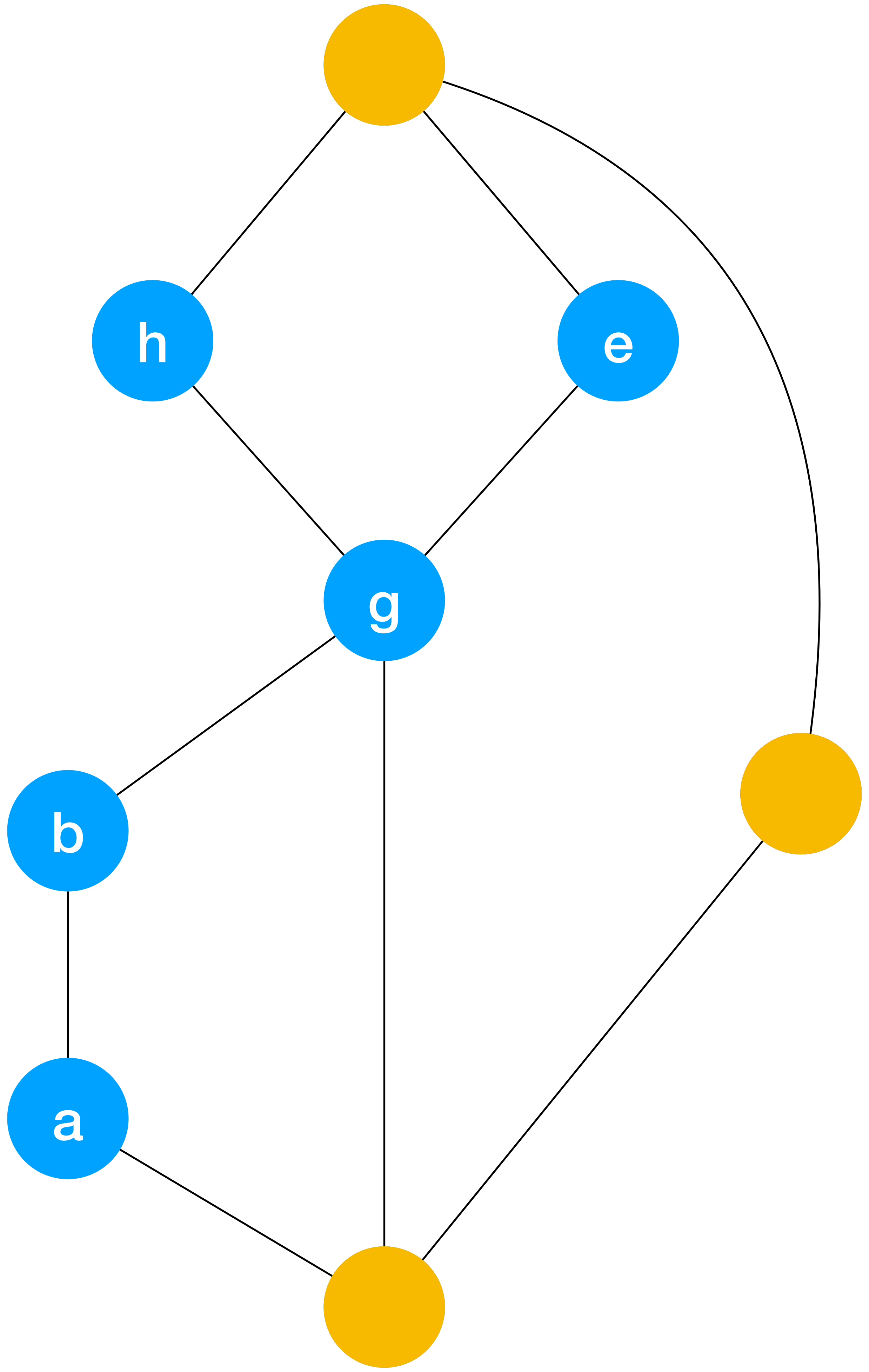
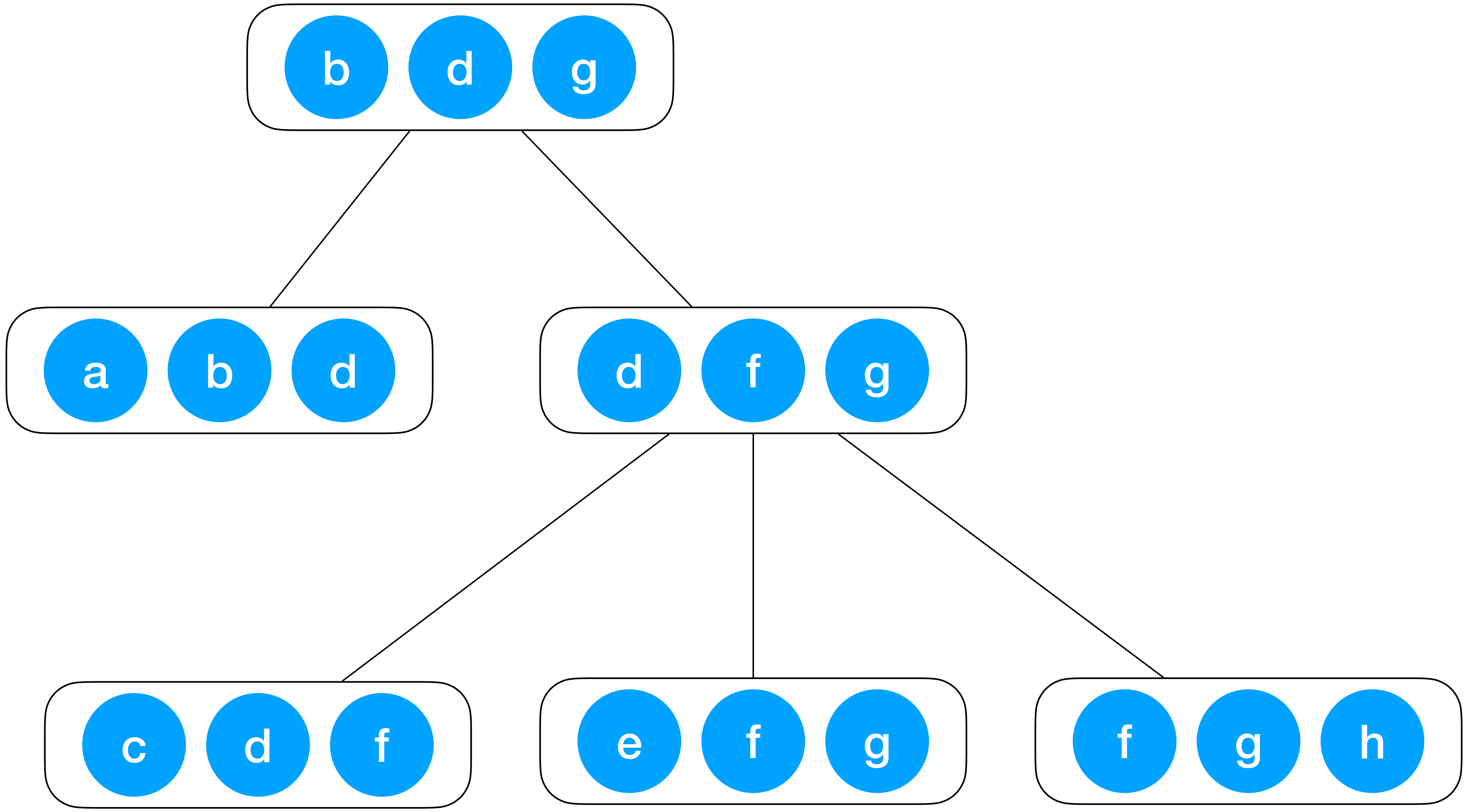
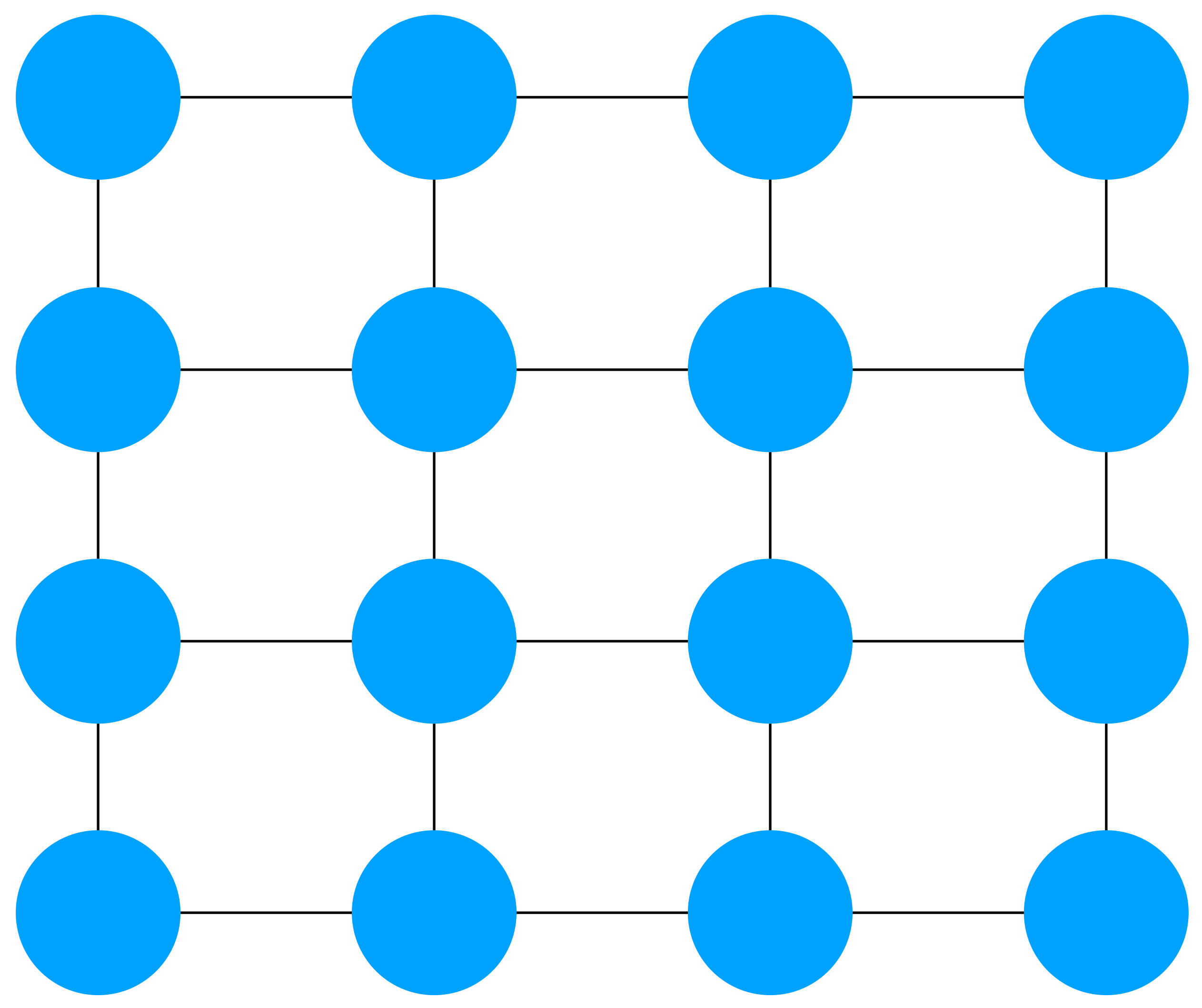




Grids have large Treewidth



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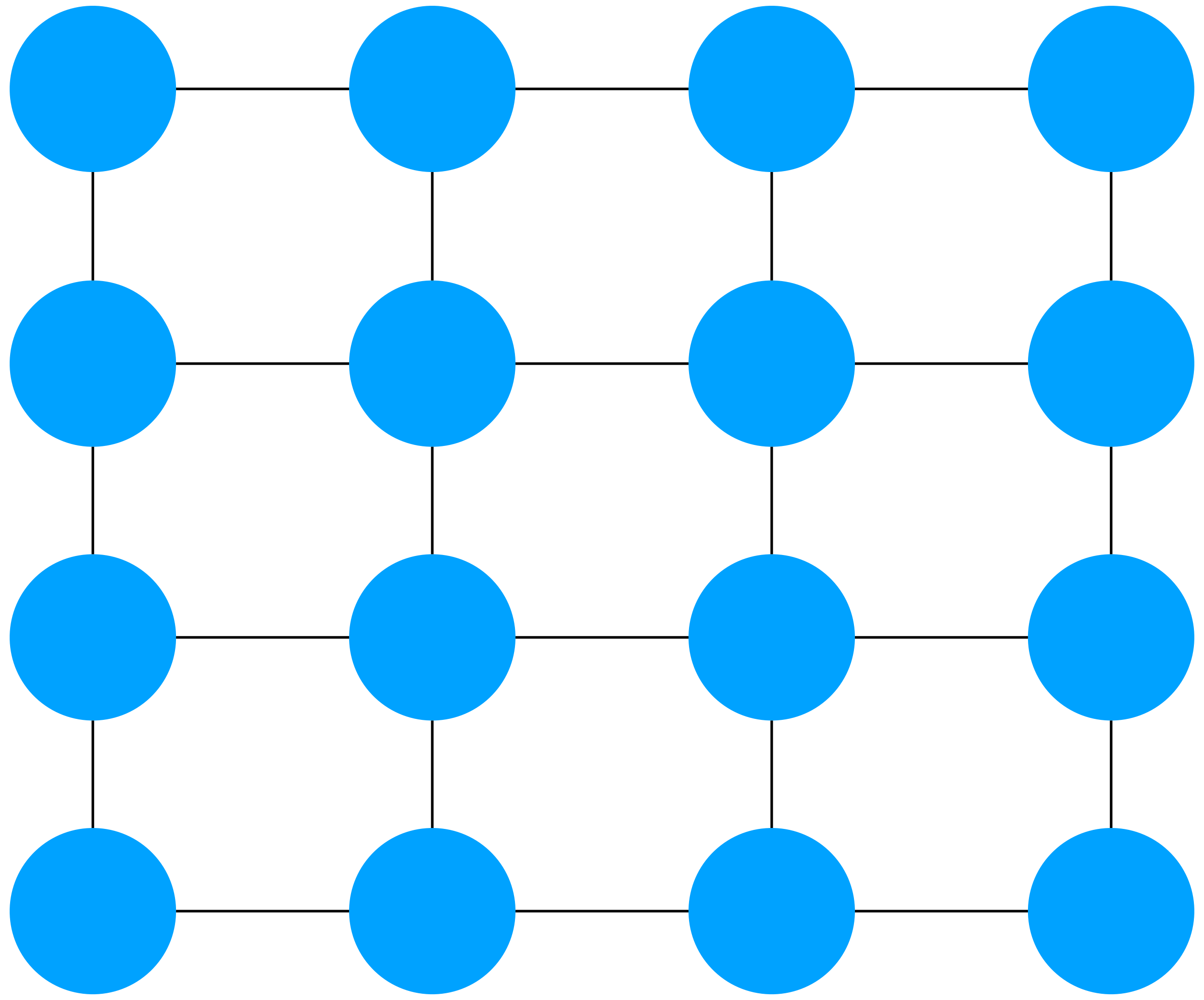
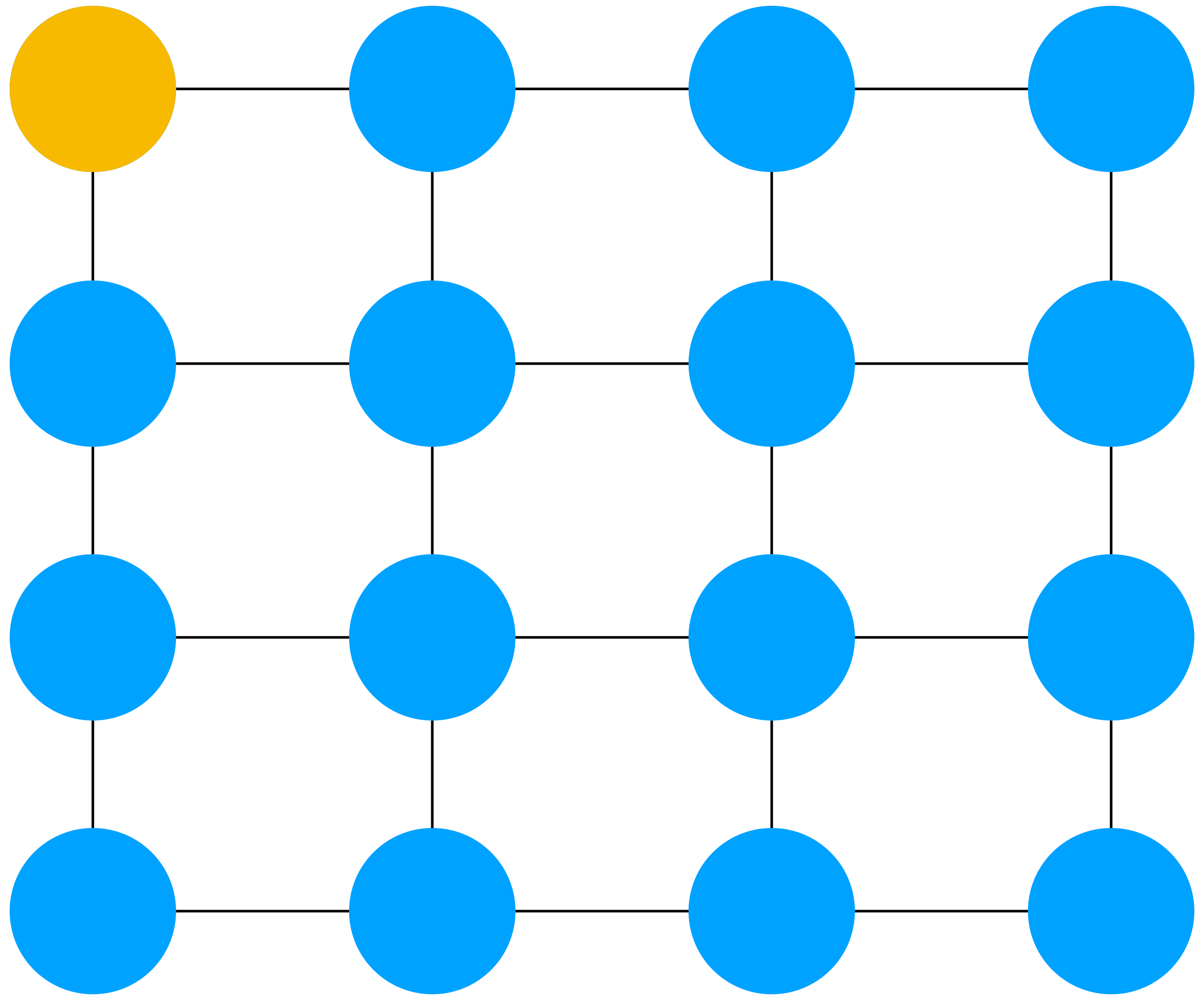


$n-1$ cops cannot win on Q_n

Grids have large Treewidth



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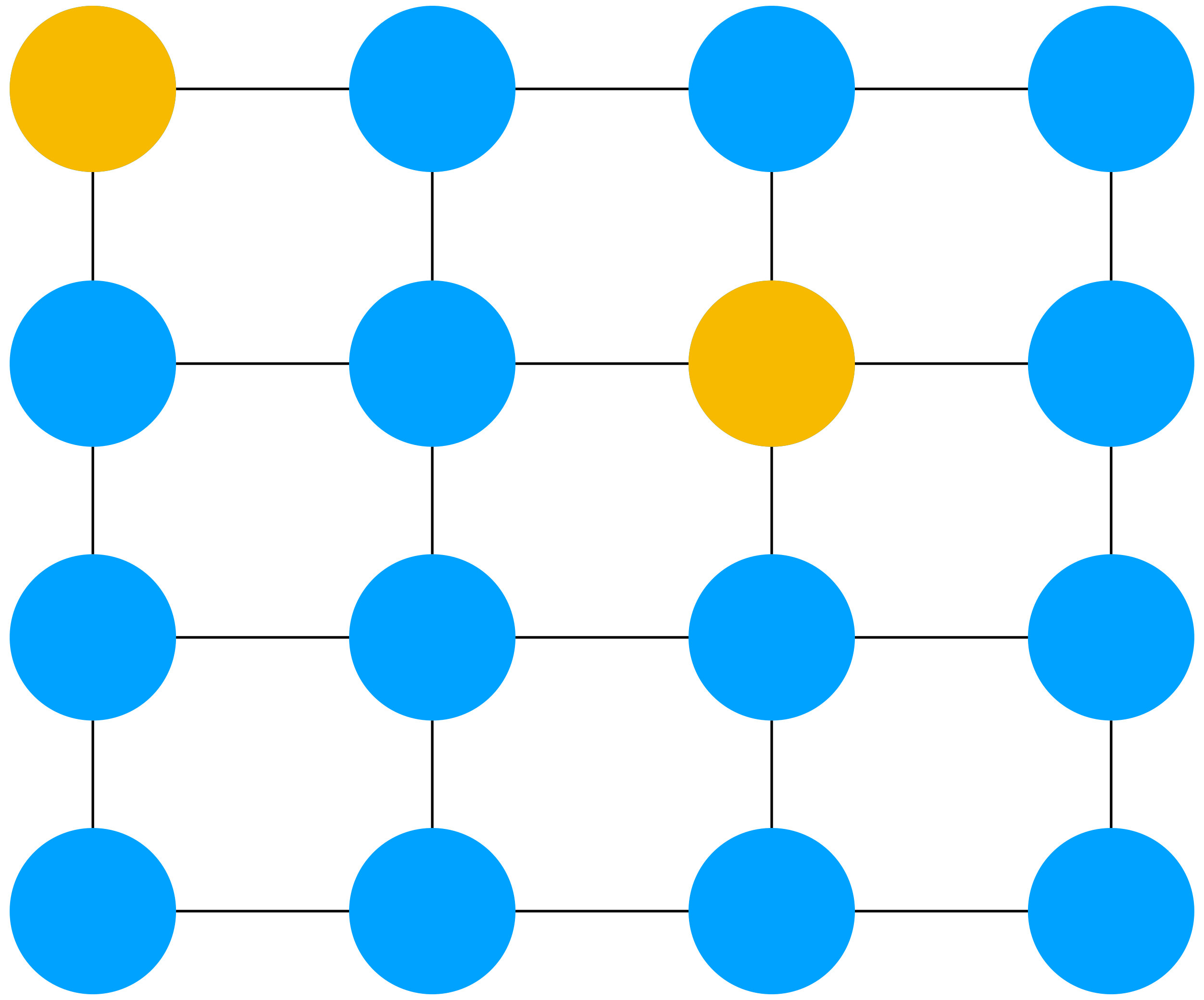

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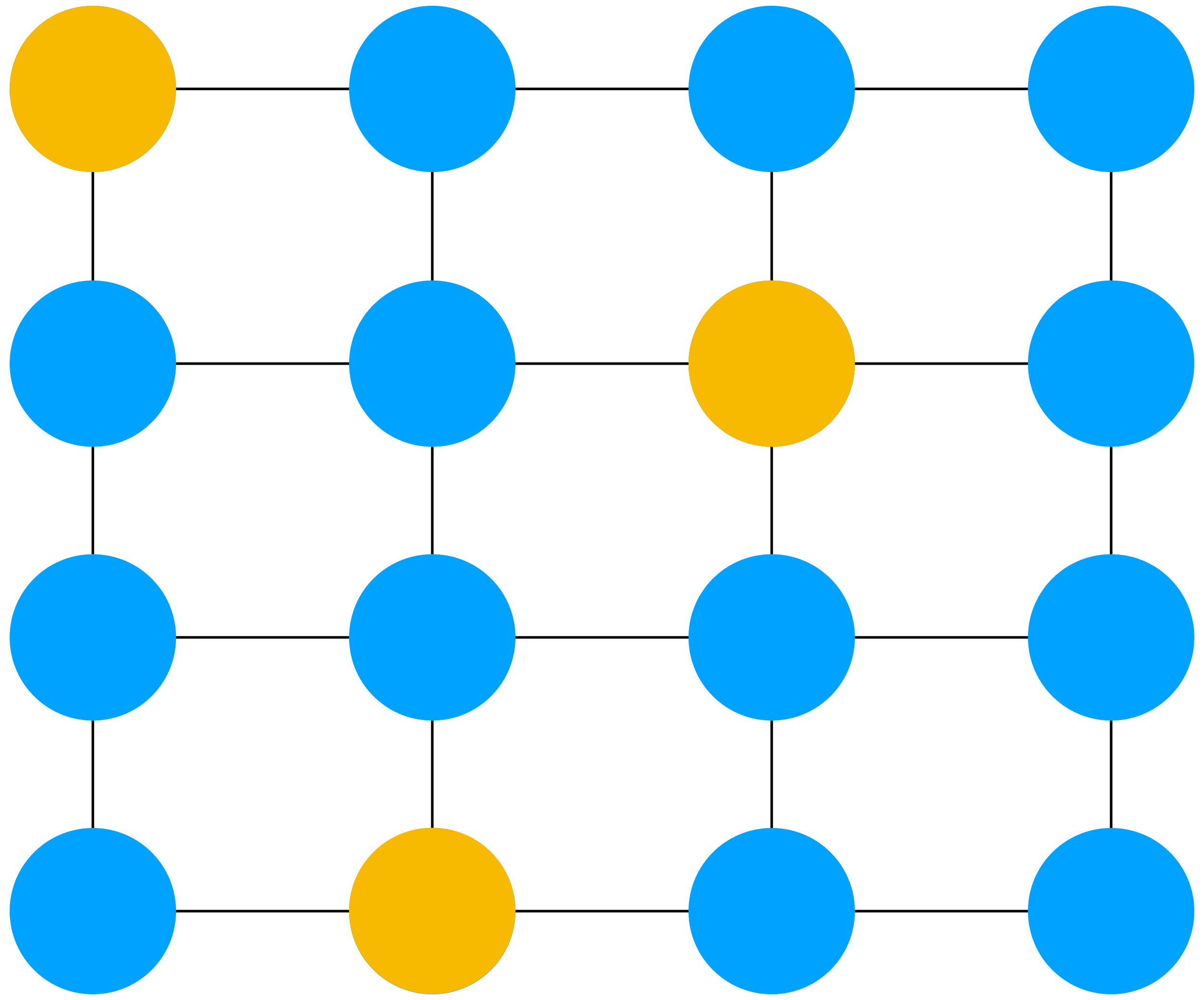

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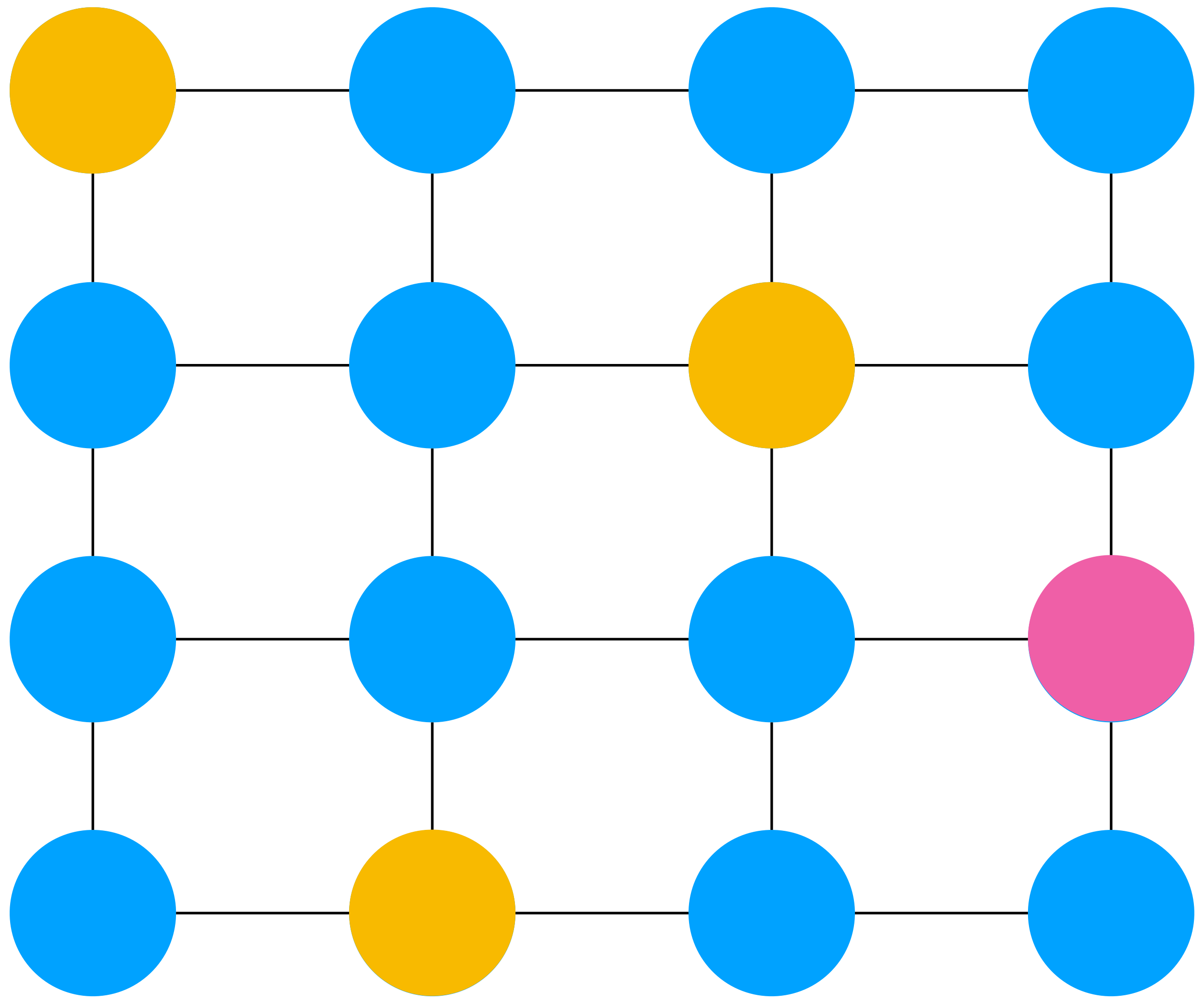

$n - 1$ cops cannot win on Q_n

In fact, the treewidth of Q_n is n.

Application: Variable Elimination for SAT

Propositional Satisfiability

Propositional Satisfiability

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2)$$

Propositional Satisfiability

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2)$$

SAT

Input: A CNF formula F.

Question: Does F have a satisfying assignment?

Propositional Satisfiability

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2)$$

$$x_1 \qquad x_2 \qquad x_3$$

SAT

Input: A CNF formula F.

Question: Does F have a satisfying assignment?

Propositional Satisfiability

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2)$$

x_1 x_2 x_3

SAT

Input: A CNF formula F.

Question: Does F have a satisfying assignment?

The Resolution Rule

$$\frac{C_1 \vee x \qquad\qquad \neg x \vee C_2}{C_1 \vee C_2}$$

The Resolution Rule

“pivot”

$$\frac{C_1 \lor x \qquad \neg x \lor C_2}{C_1 \lor C_2}$$

The Resolution Rule

“pivot”

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

“resolvent”

The Resolution Rule

"pivot"

$$\frac{C_1 \lor x \qquad \neg x \lor C_2}{C_1 \lor C_2}$$

"resolvent"

$C_1 \lor x$ $\quad \neg x \lor C_2$ $\quad$ both satisfied

The Resolution Rule

"pivot"

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

"resolvent"

$C_1 \vee x$ $\neg x \vee C_2$ both satisfied

x **false**

The Resolution Rule

“pivot”

$$\frac{C_1 \lor x \qquad \neg x \lor C_2}{C_1 \lor C_2}$$

“resolvent”

$C_1 \lor x$ $\qquad \neg x \lor C_2$ $\qquad$ both satisfied

x **false** ➡

The Resolution Rule

"pivot"

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

"resolvent"

$C_1 \vee x$ $\quad$ $\neg x \vee C_2$ $\quad$ both satisfied

x **false** ➡ C_1 satisfied

The Resolution Rule

“pivot”

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

“resolvent”

$C_1 \vee x$ $\neg x \vee C_2$ both satisfied

x **false** → C_1 satisfied →

The Resolution Rule

"pivot"

$$\frac{C_1 \lor x \qquad \neg x \lor C_2}{C_1 \lor C_2}$$

"resolvent"

$C_1 \lor x$ $\neg x \lor C_2$ both satisfied

x **false** → C_1 satisfied → $C_1 \lor C_2$ satisfied

The Resolution Rule

"pivot"

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

"resolvent"

$C_1 \vee x$ $\neg x \vee C_2$ both satisfied

x **false** ➡ C_1 satisfied ➡ $C_1 \vee C_2$ satisfied

x **true**

The Resolution Rule

"pivot"

$$\frac{C_1 \lor x \qquad \neg x \lor C_2}{C_1 \lor C_2}$$

"resolvent"

$C_1 \lor x$ $\quad$ $\neg x \lor C_2$ $\quad$ both satisfied

x **false** → C_1 satisfied → $C_1 \lor C_2$ satisfied

x **true** →

The Resolution Rule

"pivot"

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

"resolvent"

$C_1 \vee x$ $\quad$ $\neg x \vee C_2$ $\quad$ both satisfied

x **false** → C_1 satisfied → $C_1 \vee C_2$ satisfied

x **true** → C_2 satisfied

The Resolution Rule

"pivot"

$$\frac{C_1 \lor x \qquad \neg x \lor C_2}{C_1 \lor C_2}$$

"resolvent"

$C_1 \lor x$ $\quad$ $\neg x \lor C_2$ $\quad$ both satisfied

x **false** → C_1 satisfied → $C_1 \lor C_2$ satisfied

x **true** → C_2 satisfied →

The Resolution Rule

"pivot"

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

"resolvent"

$C_1 \vee x$ $\quad \neg x \vee C_2$ $\quad$ both satisfied

x **false** → C_1 satisfied → $C_1 \vee C_2$ satisfied

x **true** → C_2 satisfied → $C_1 \vee C_2$ satisfied

The Resolution Rule

"pivot"

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

"resolvent"

Theorem

Resolution is sound.

$C_1 \vee x$ $\quad$ $\neg x \vee C_2$ $\quad$ both satisfied

x **false** → C_1 satisfied → $C_1 \vee C_2$ satisfied

x **true** → C_2 satisfied → $C_1 \vee C_2$ satisfied

The Resolution Rule

"pivot"

$$\frac{C_1 \vee x \qquad \neg x \vee C_2}{C_1 \vee C_2}$$

"resolvent"

Theorem

Resolution is sound.

Adding resolvents does not make a formula unsatisfiable.

$C_1 \vee x$ $\quad$ $\neg x \vee C_2$ $\quad$ both satisfied

x **false** → C_1 satisfied → $C_1 \vee C_2$ satisfied

x **true** → C_2 satisfied → $C_1 \vee C_2$ satisfied

Examples

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$(x_1 \vee x_2 \vee \neg x_3) \quad (\neg x_1 \vee x_2)$

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$$(x_1 \vee x_2 \vee \neg x_3) \quad (\neg x_1 \vee x_2)$$

Examples

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tautology (always satisfied)

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$$(x) \qquad (\neg x)$$

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Examples

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$$\frac{(x) \quad (\neg x)}{()}$$

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tautology (always satisfied)

$$\frac{(x) \quad (\neg x)}{()}$$

"empty clause"

Examples

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tautology (always satisfied)

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"empty clause"

Observation

The empty clause cannot be satisfied.

Refutations

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The empty clause cannot be satisfied.

Theorem

Resolution is sound.

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"refutation"

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"refutation"

Corollary

If a formula has a refutation it is unsatisfiable.

Completeness

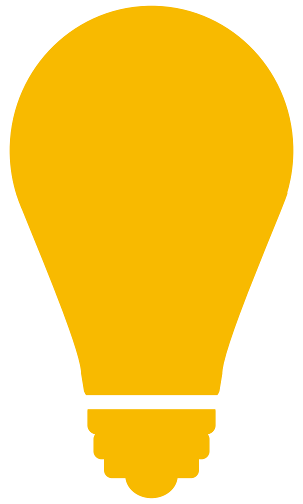
Theorem

Every unsatisfiable formula has a refutation.

Completeness

Theorem

Every unsatisfiable formula has a refutation.



Completeness

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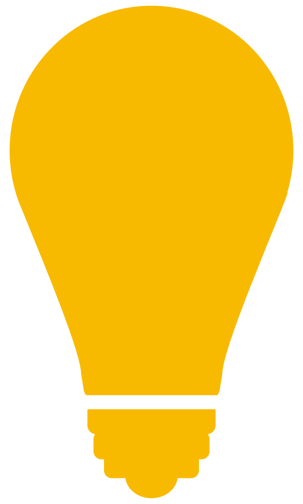

Algorithm for SAT: decide if there is a refutation.

Davis-Putnam Resolution

Davis-Putnam Resolution

Davis & Putnam 1960

Davis-Putnam Resolution

Davis & Putnam 1960

Input: A CNF formula F with m clauses

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"variable elimination"

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Davis & Putnam 1960

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remove clauses containing x_i from F

remove tautologies from F

return false $\Leftrightarrow$ F contains the empty clause

Worst case: m^2 resolvents in each iteration.

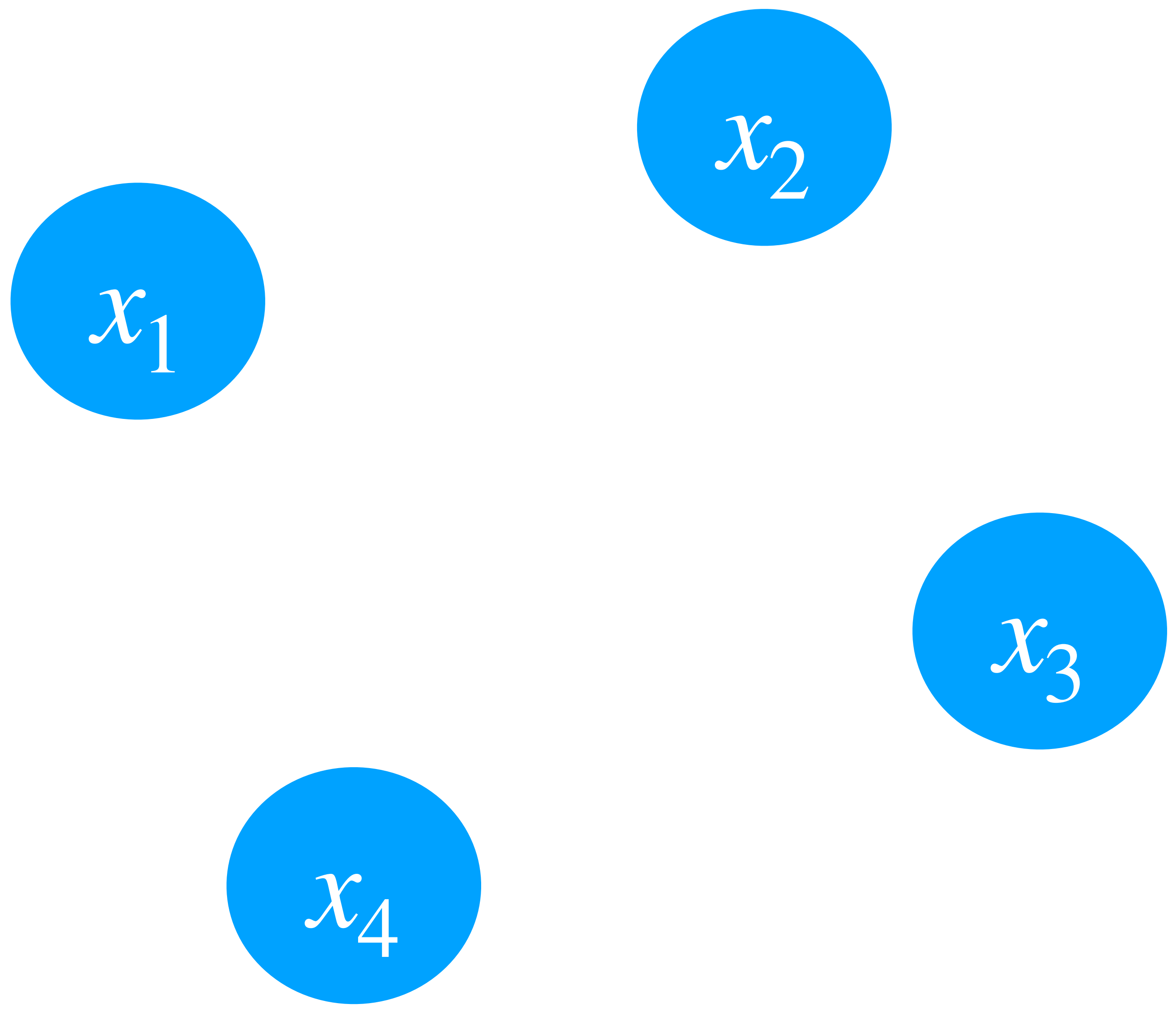
"variable elimination"

Primal Graph

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_3 \lor \neg x_2) \land (x_1 \lor x_2 \lor x_4)$$

Primal Graph

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee x_4)$$



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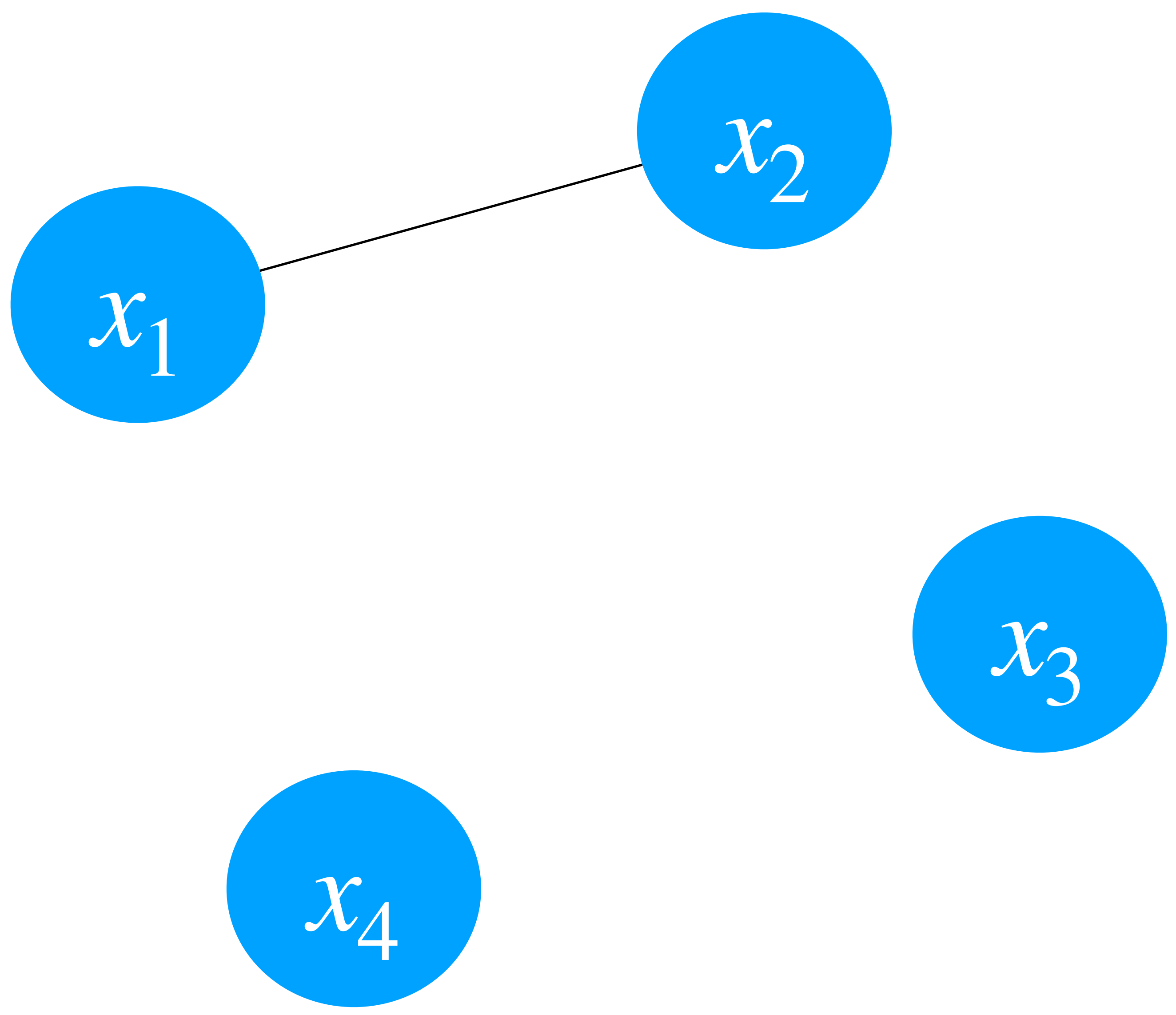

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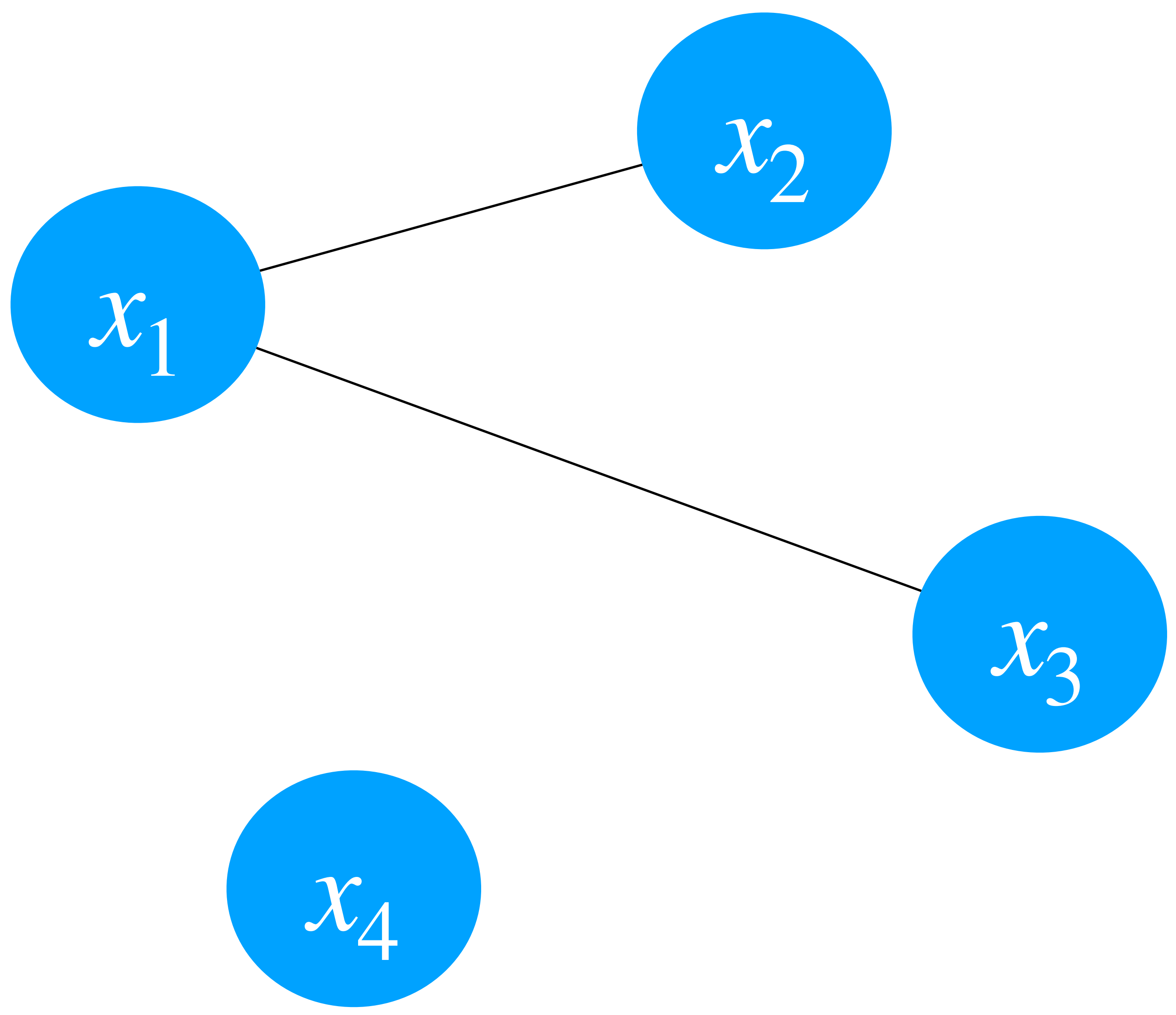

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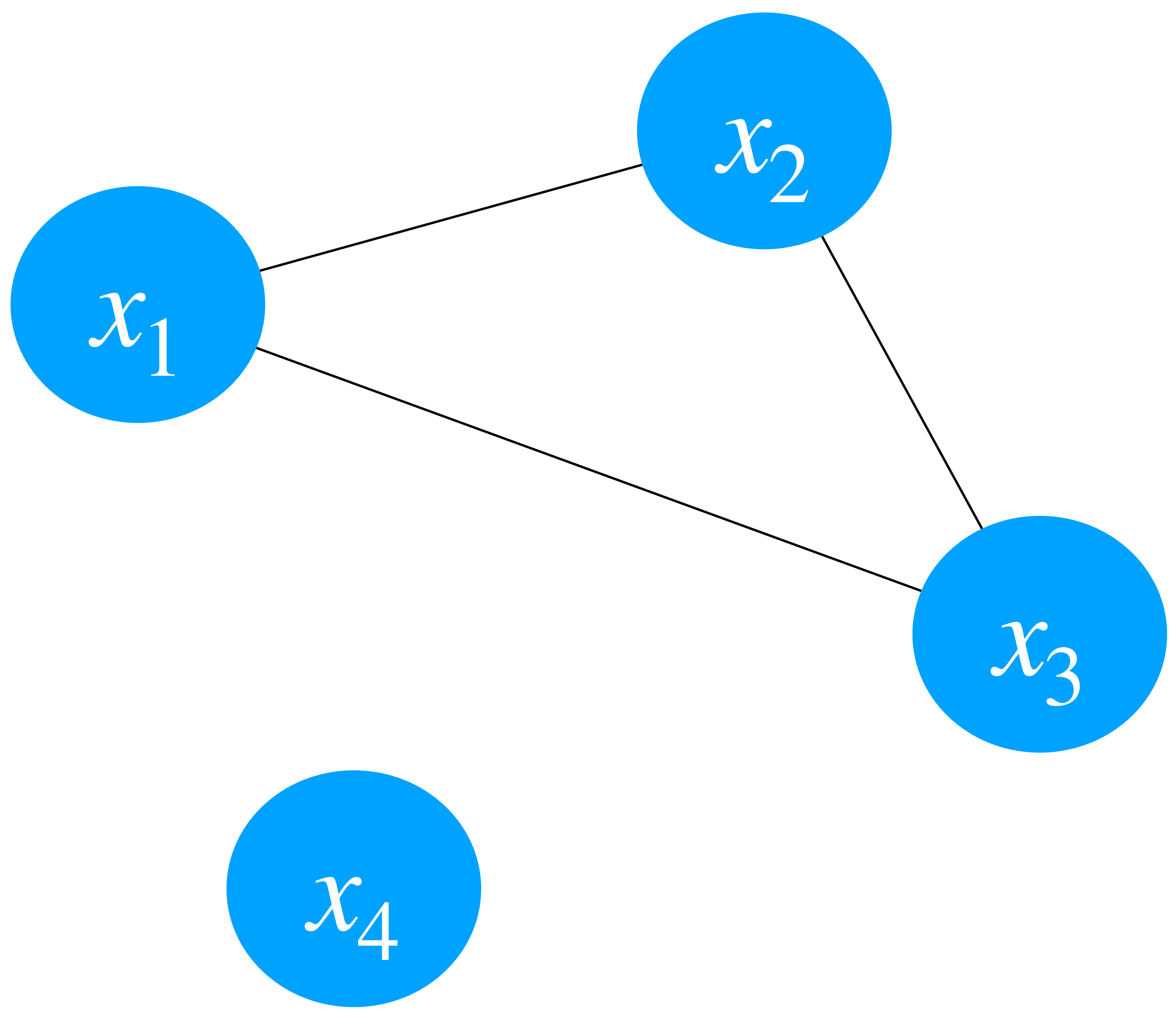

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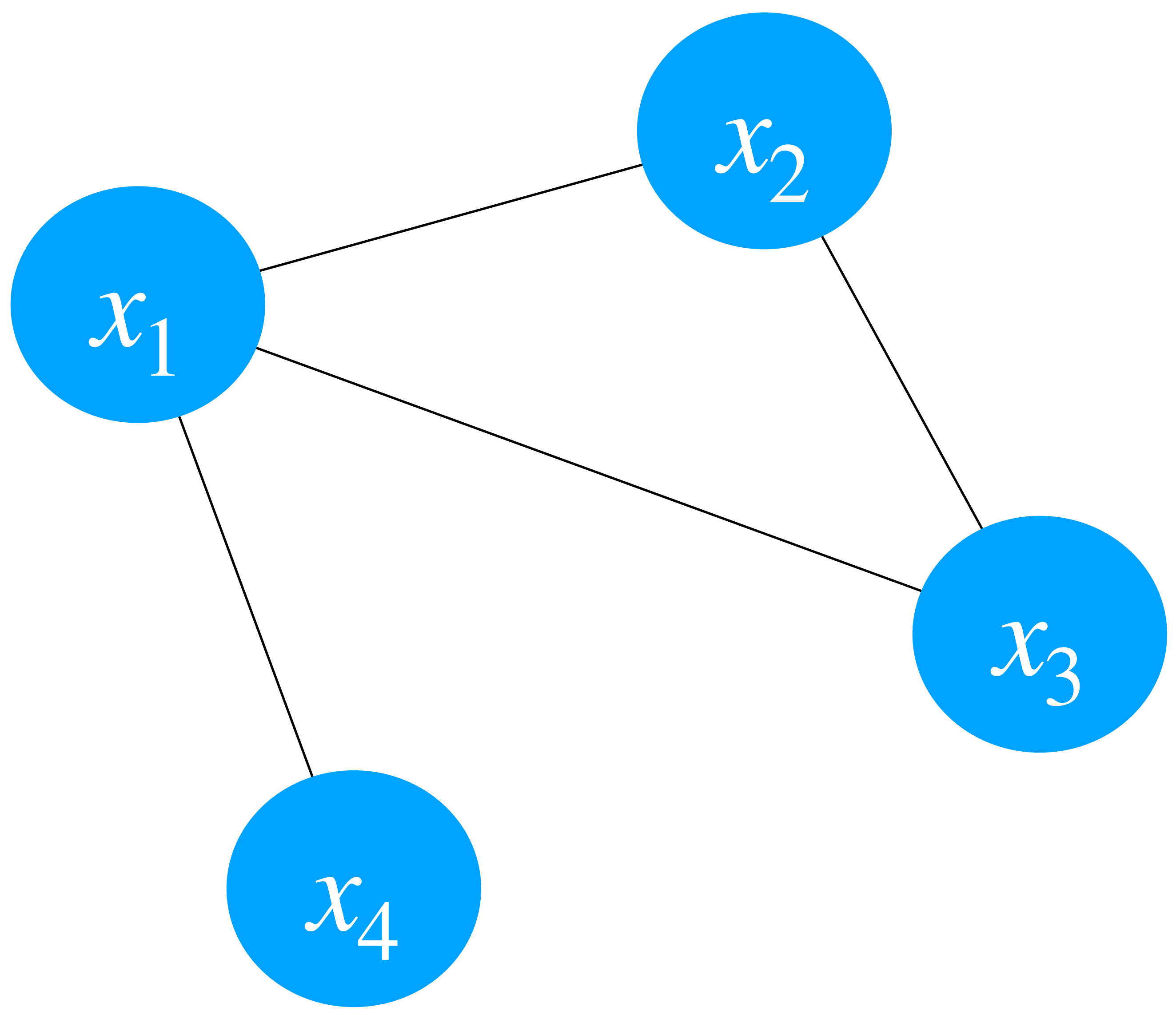

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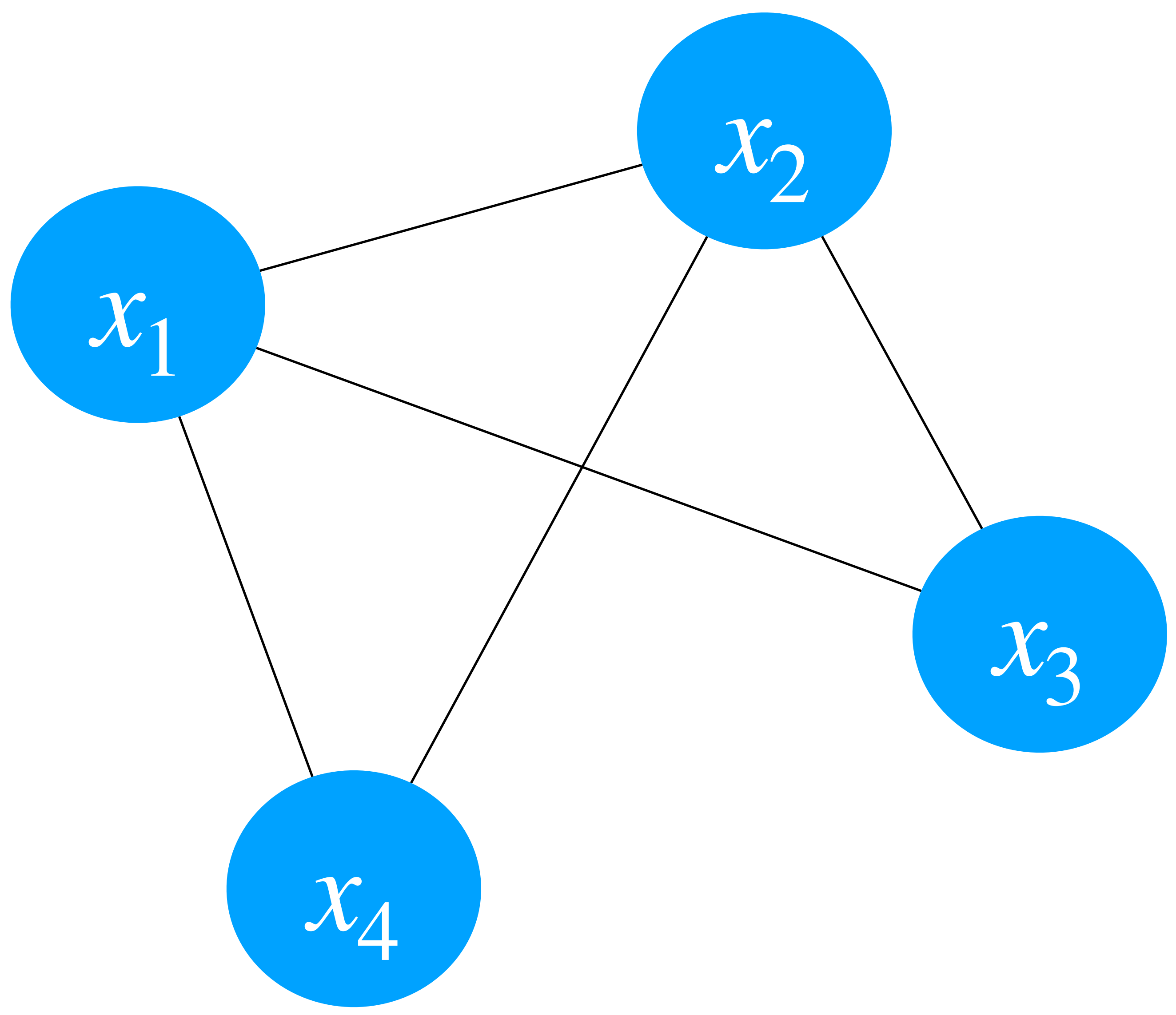
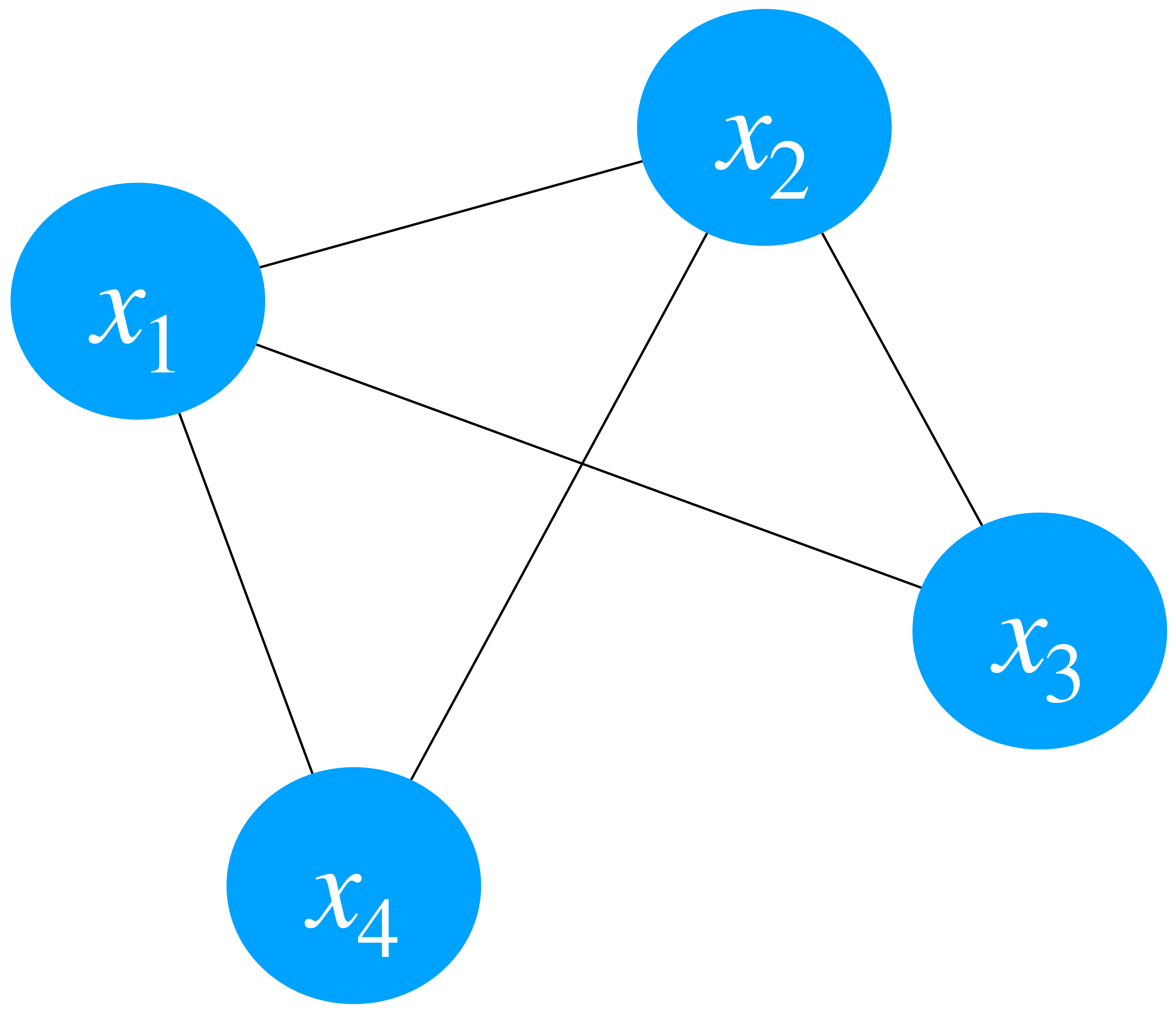


Primal Graph

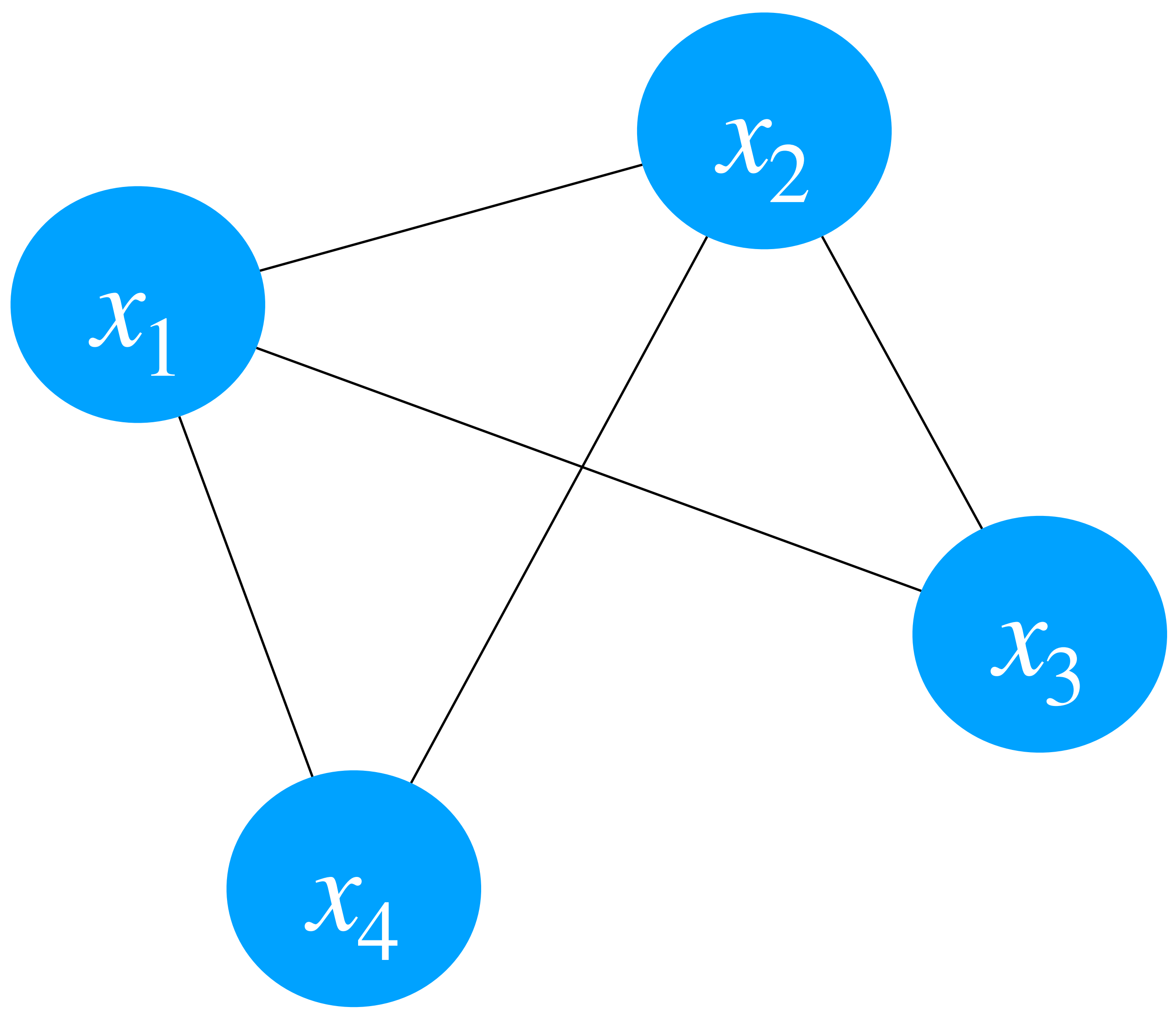
$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_3 \lor \neg x_2) \land (x_1 \lor x_2 \lor x_4)$$



$$(x_3 \lor \neg x_2)$$

Primal Graph

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee x_4)$$



$(x_3 \vee \neg x_2)$ $\quad$ $(x_1 \vee x_2 \vee x_4)$

Primal Graph

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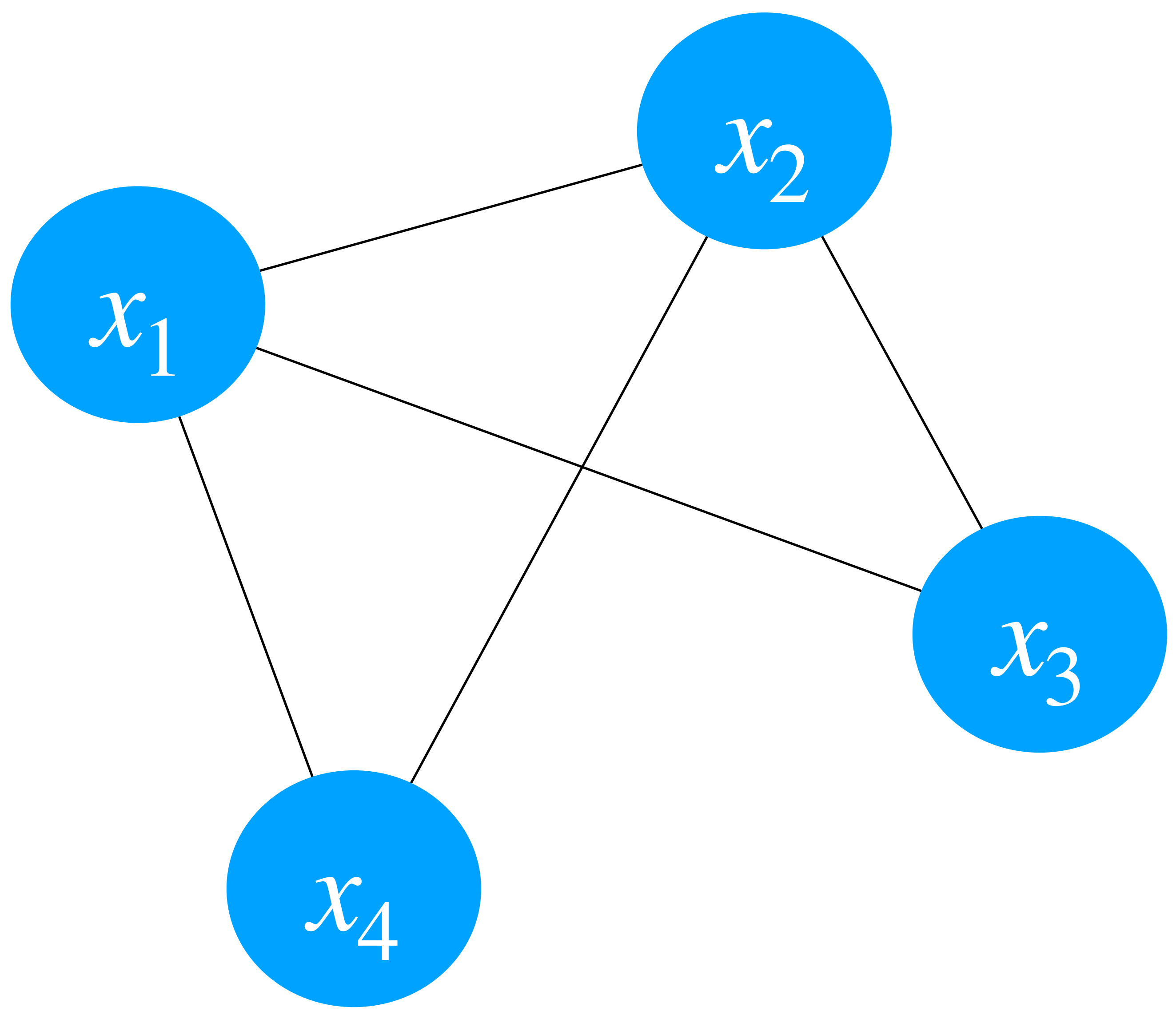

$$\frac{(x_3 \vee \neg x_2) \qquad (x_1 \vee x_2 \vee x_4)}{(x_1 \vee x_3 \vee x_4)}$$

Primal Graph

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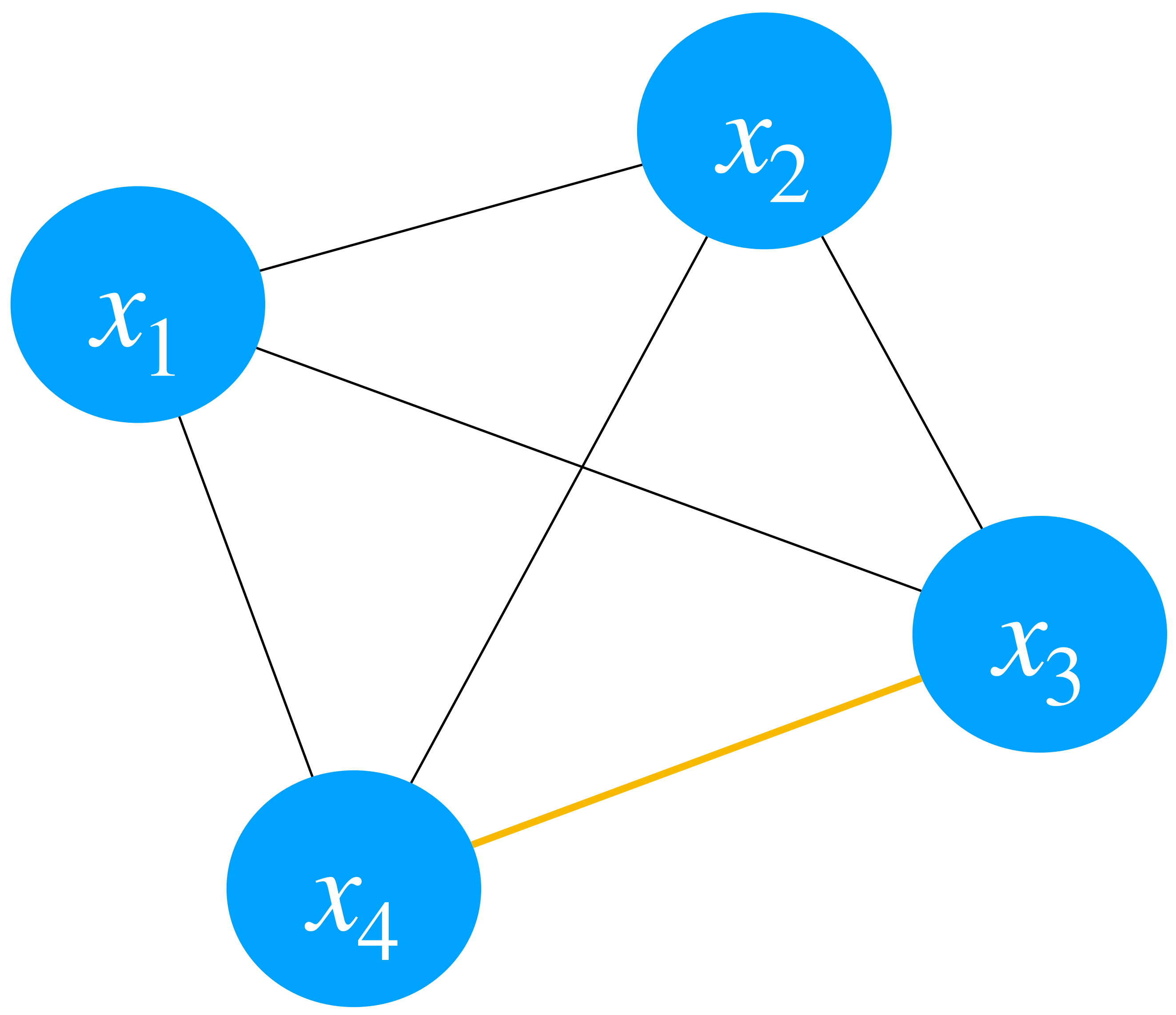

$$\frac{(x_3 \vee \neg x_2) \qquad (x_1 \vee x_2 \vee x_4)}{(x_1 \vee x_3 \vee x_4)}$$

Resolution in the Primal Graph

$$\frac{C_1 \lor x \qquad\qquad \neg x \lor C_2}{C_1 \lor C_2}$$

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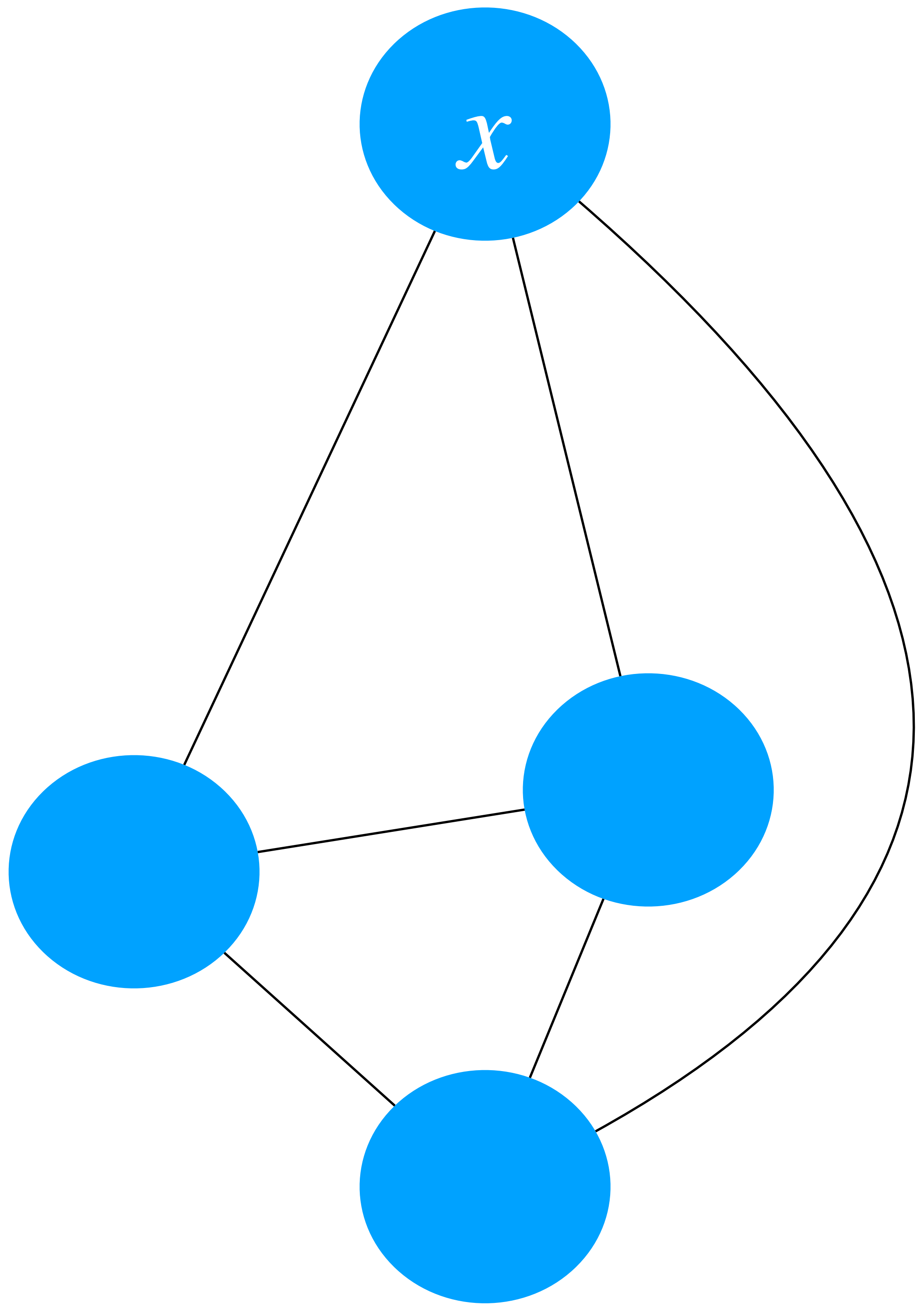

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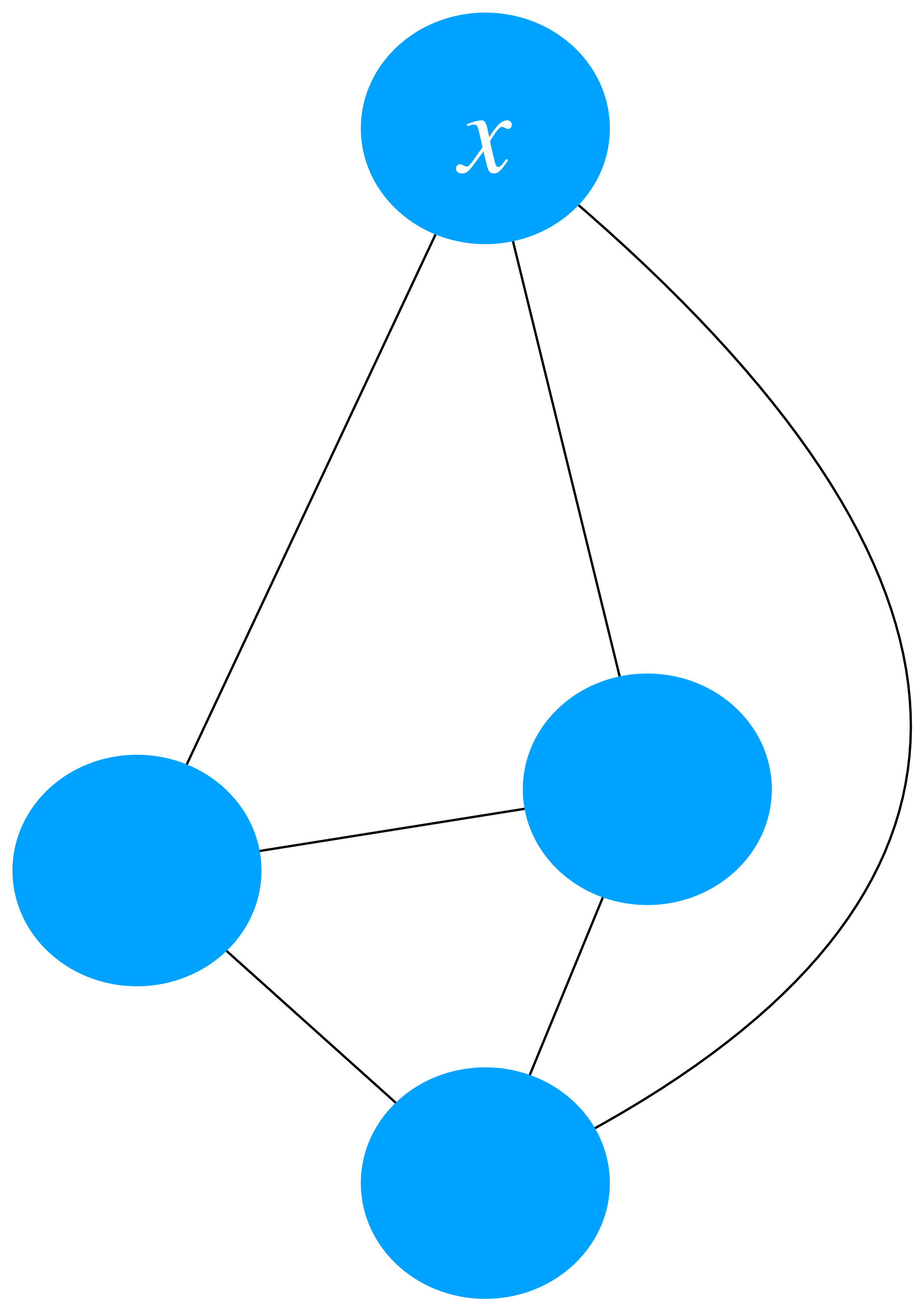

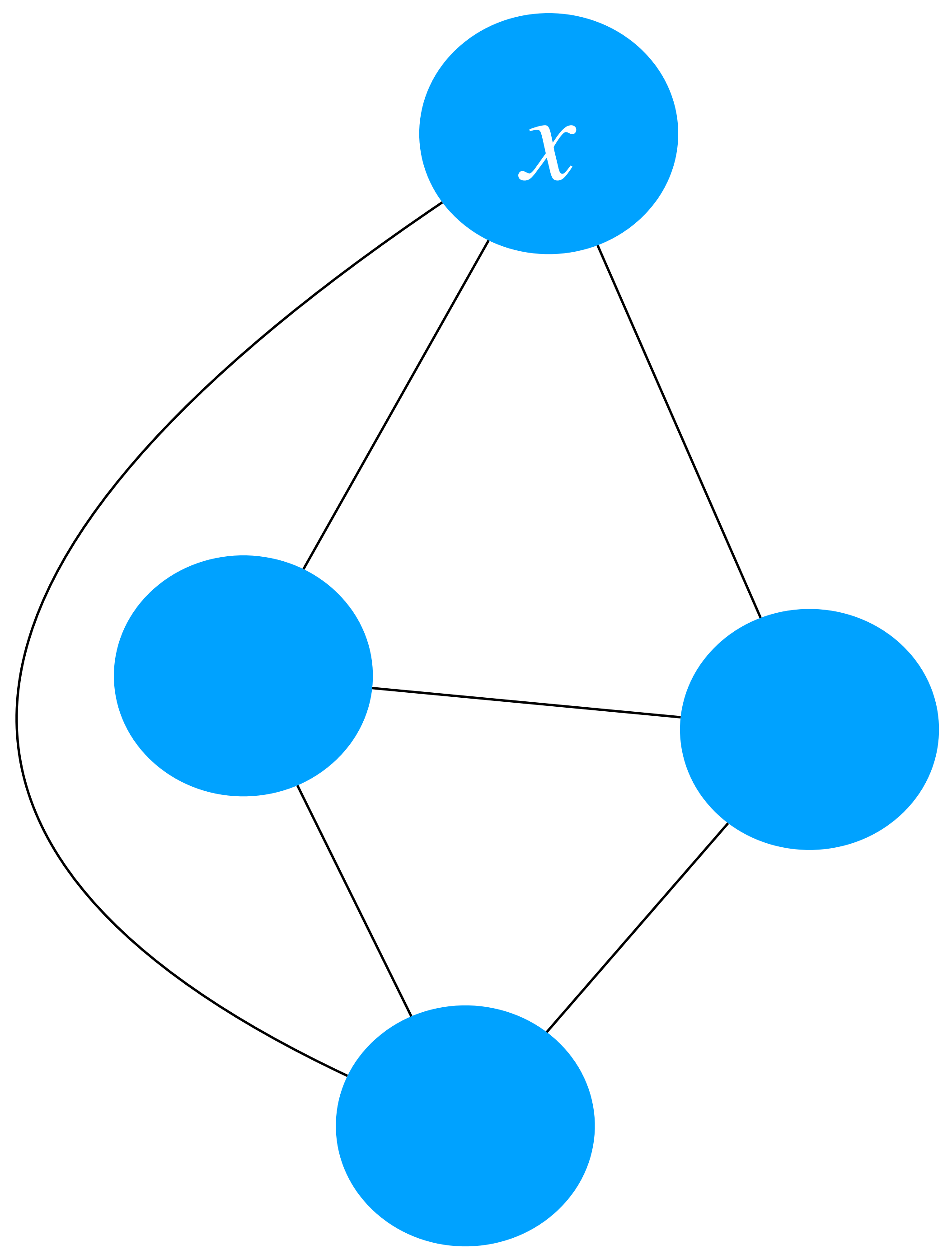

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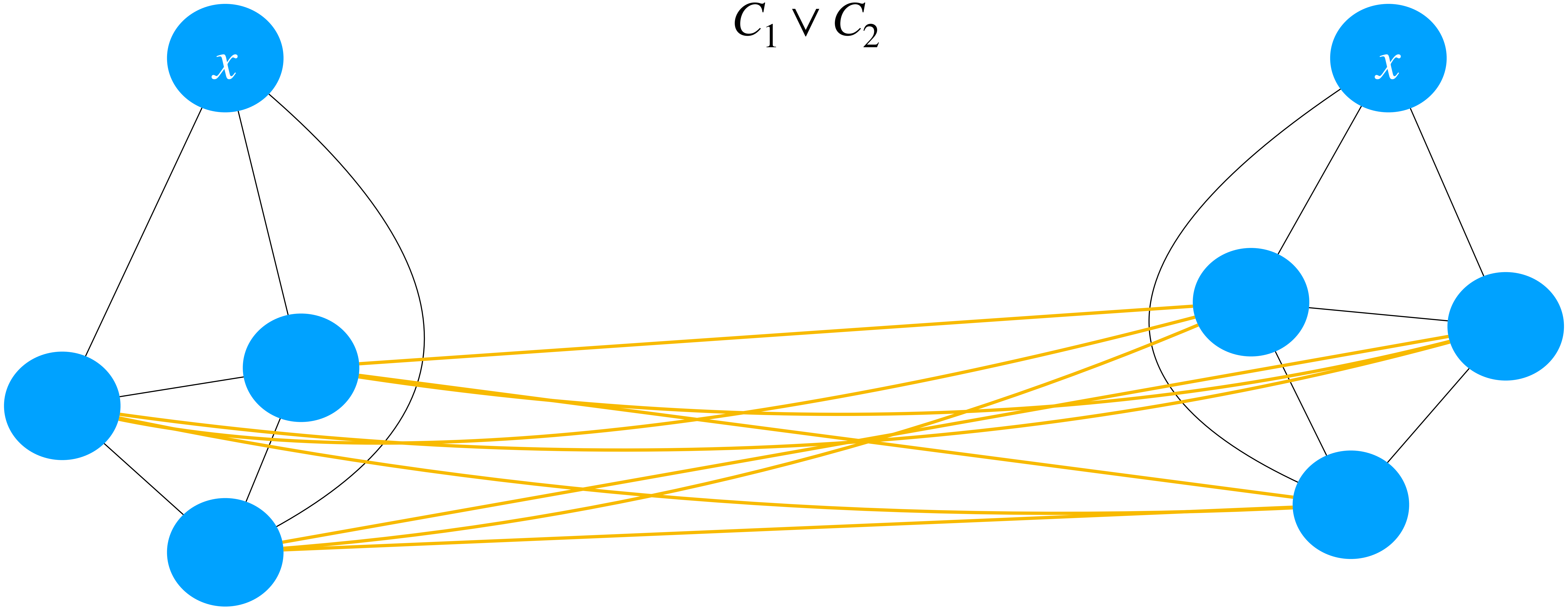

Variable Elimination in the Primal Graph

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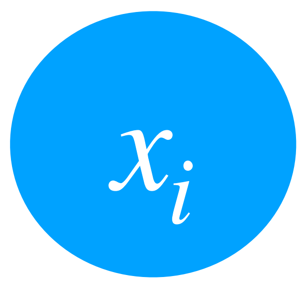

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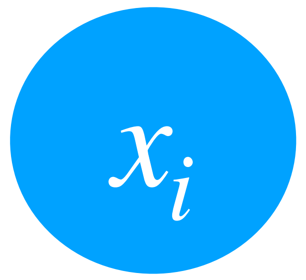
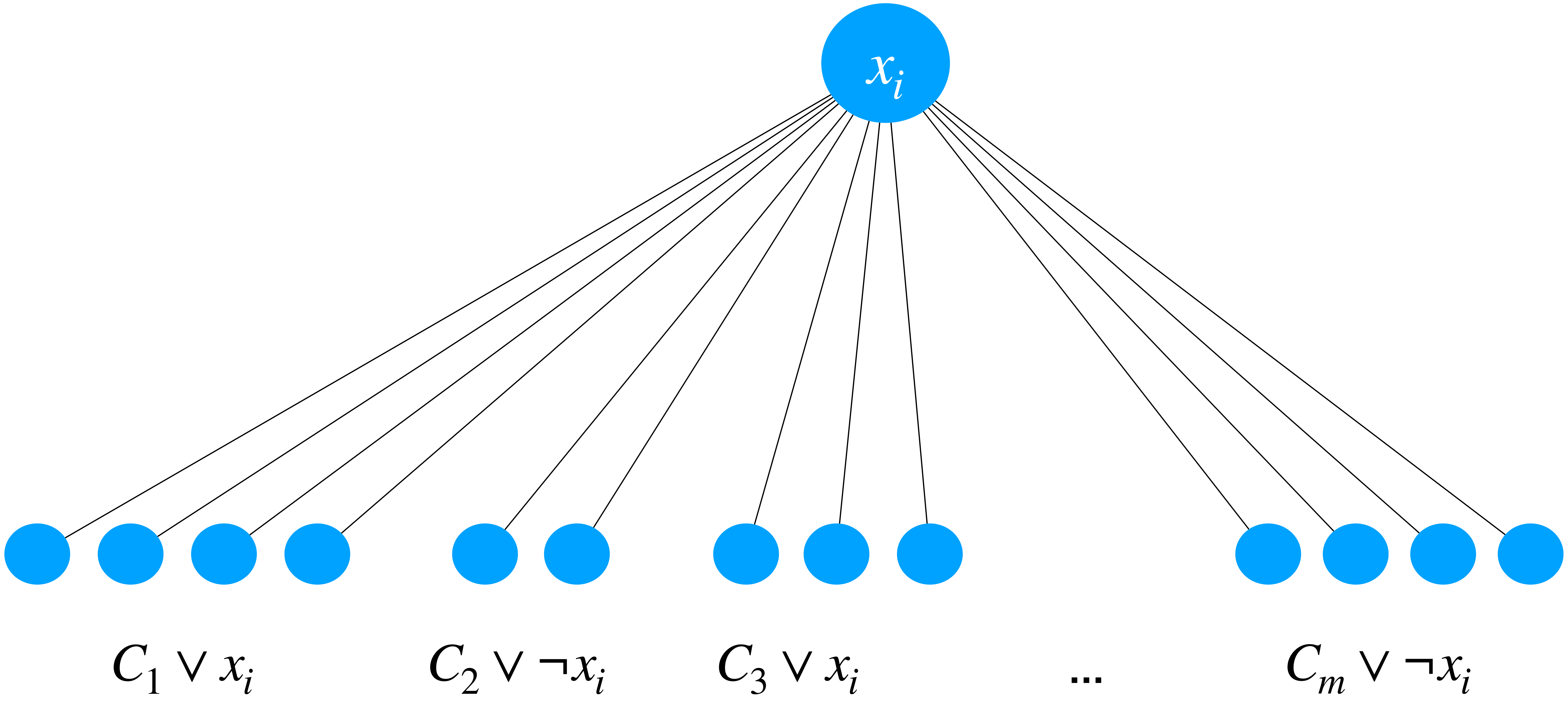


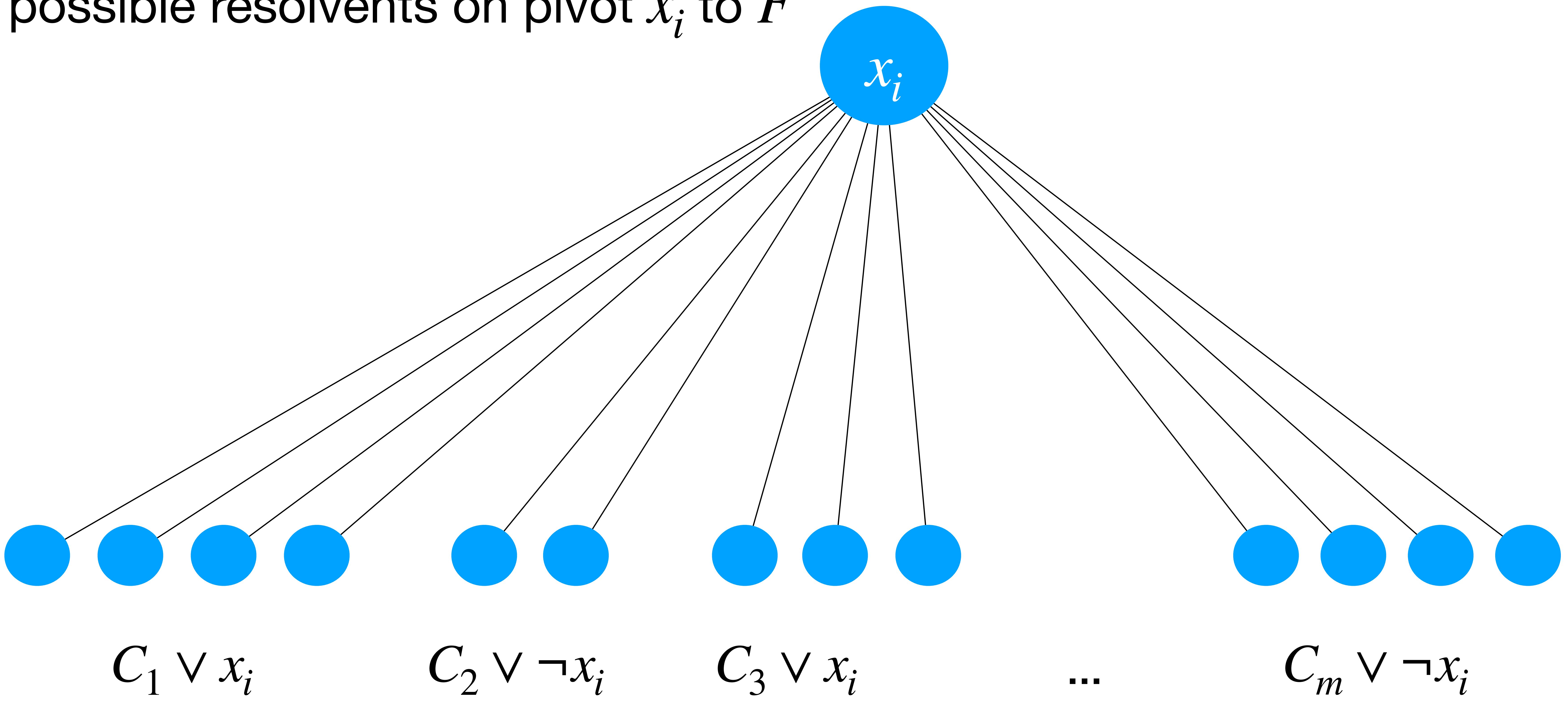
$C_1 \vee x_i$ $C_2 \vee \neg x_i$ $C_3 \vee x_i$ … $C_m \vee \neg x_i$

Variable Elimination in the Primal Graph



Variable Elimination in the Primal Graph

add all possible resolvents on pivot x_i to F



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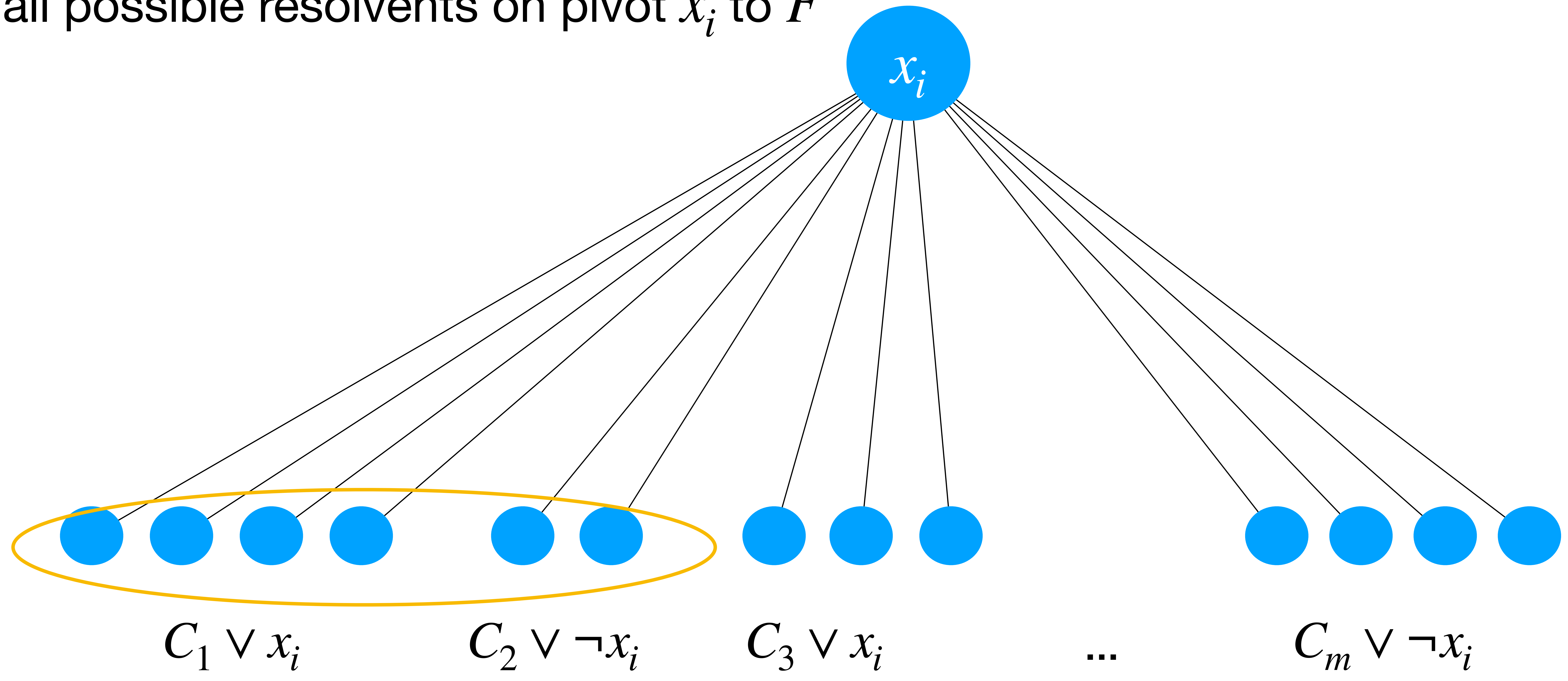

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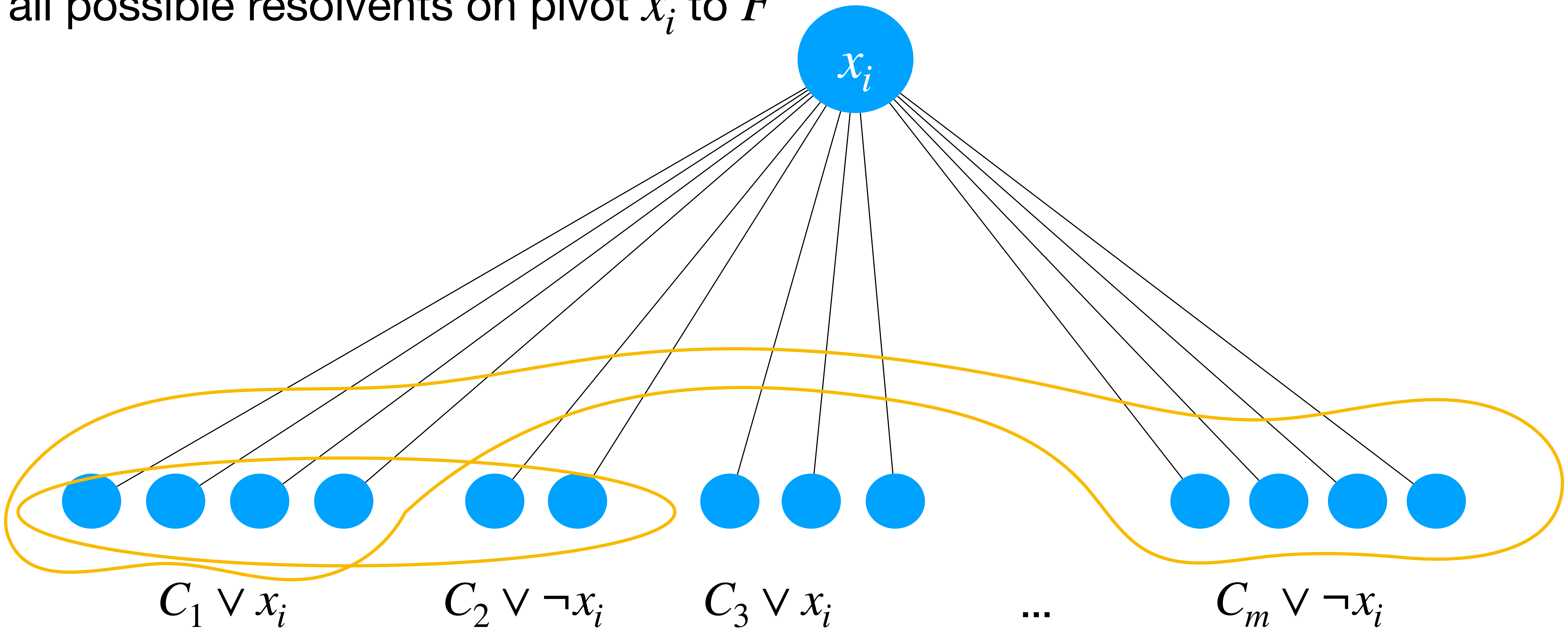

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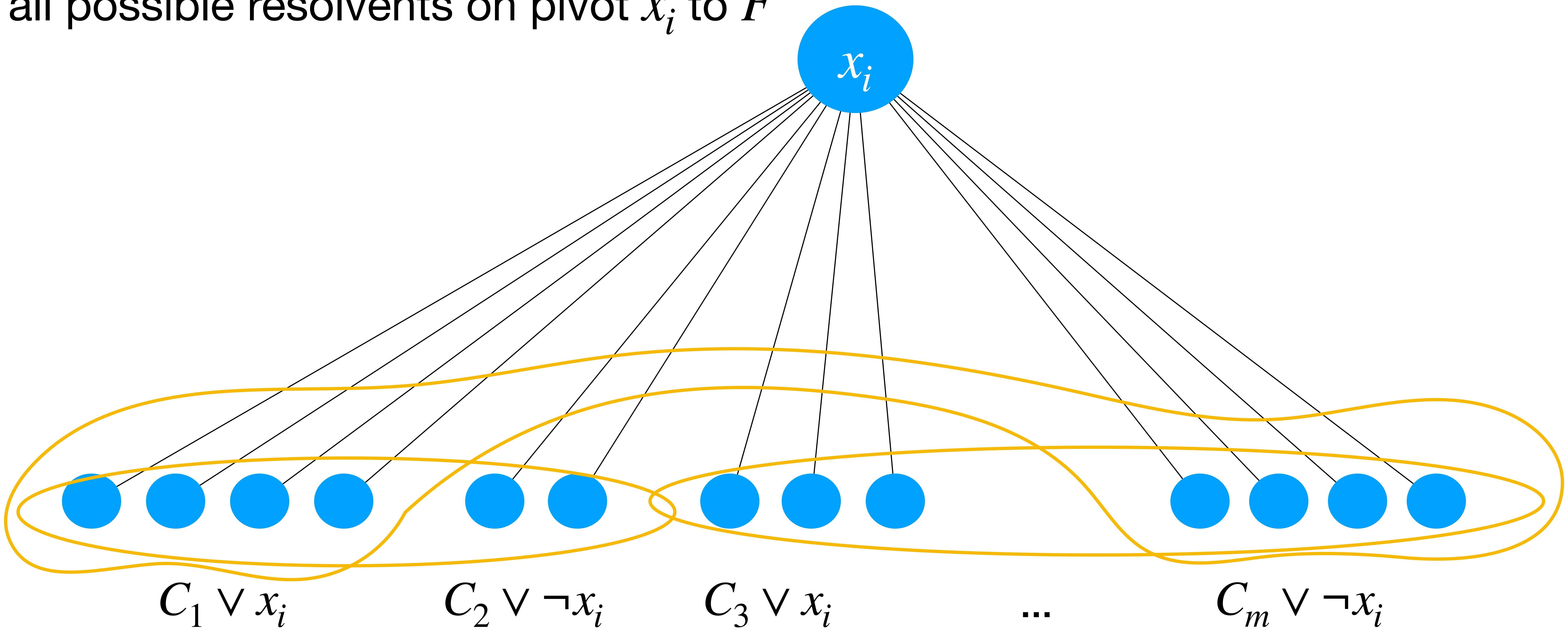

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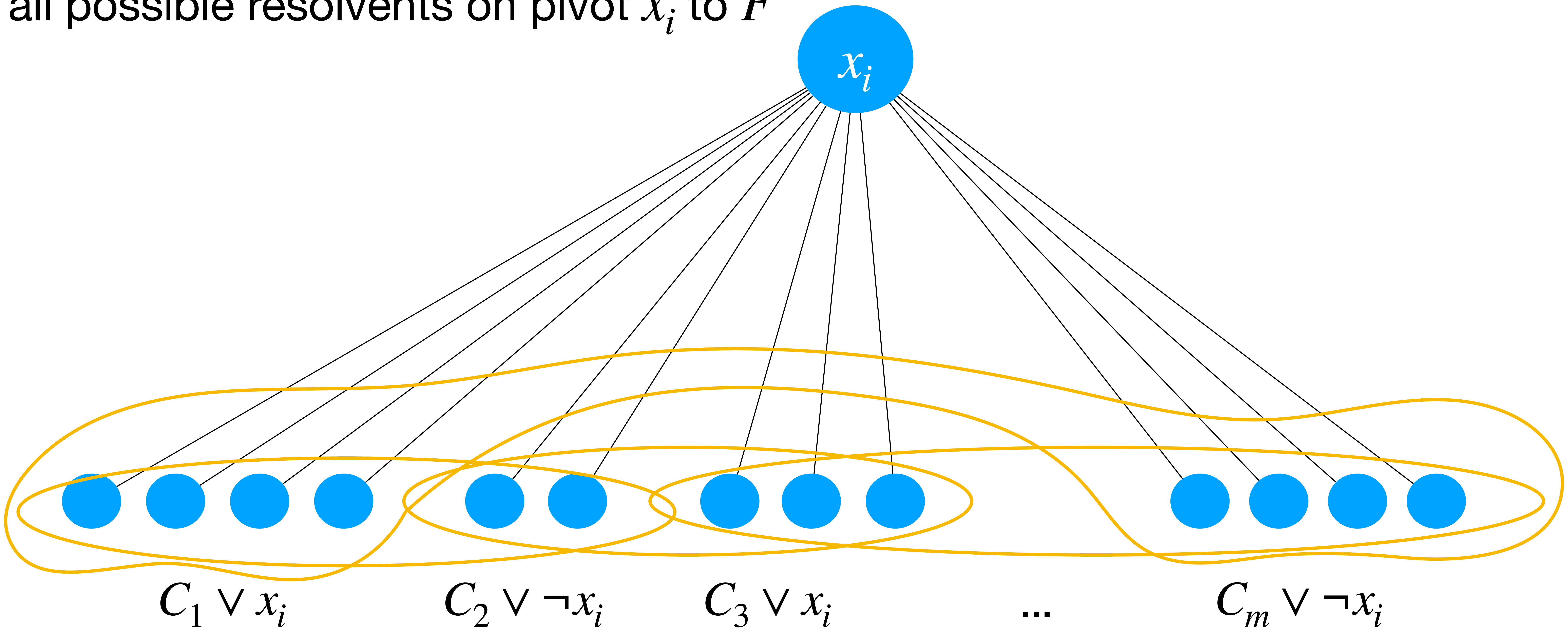

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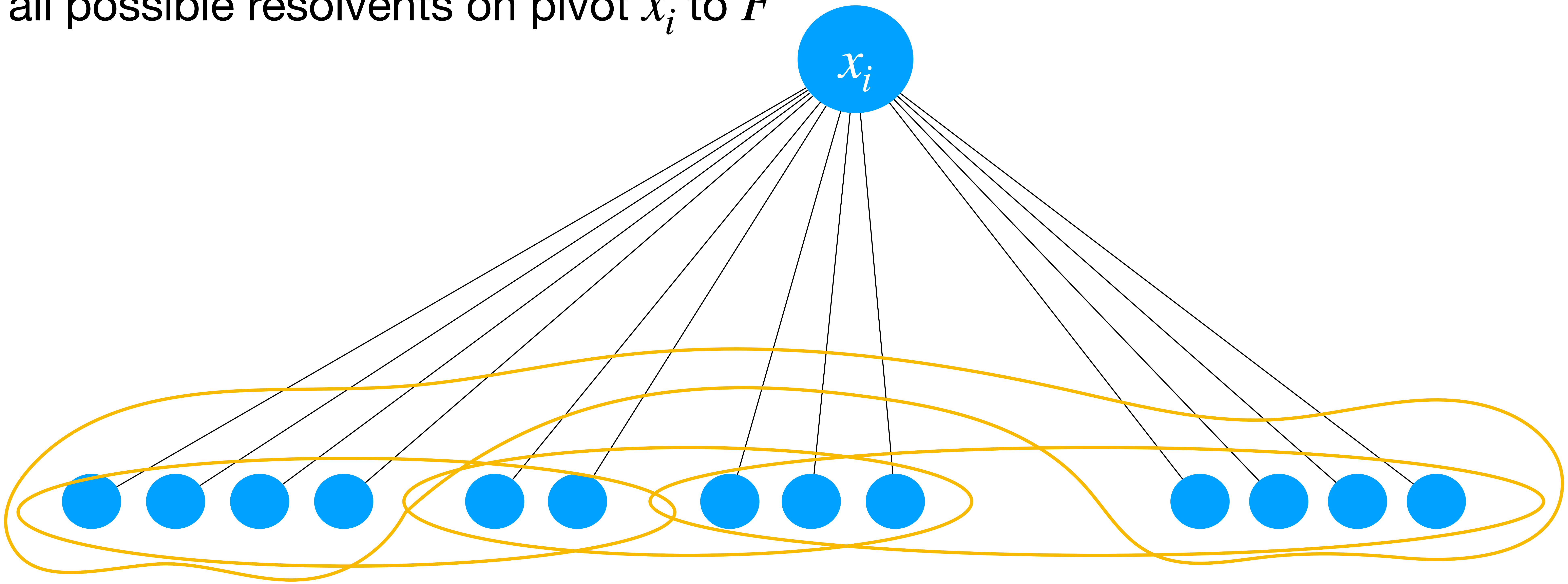

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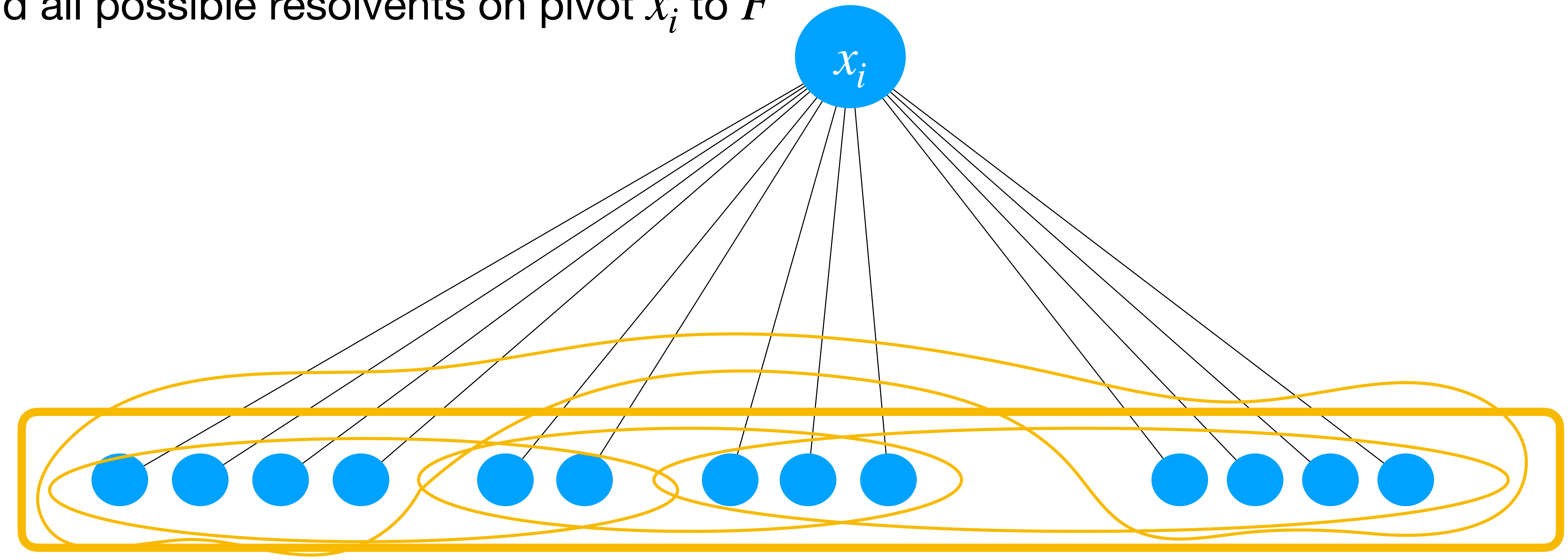
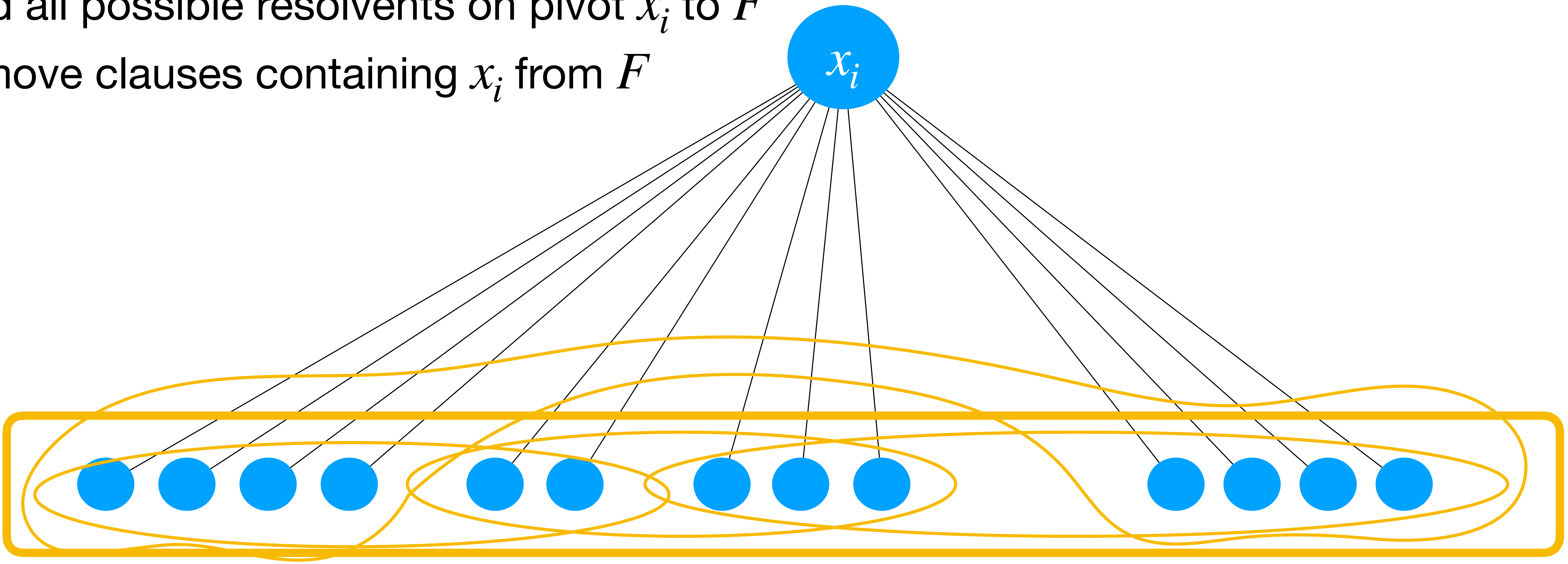


Variable Elimination in the Primal Graph

add all possible resolvents on pivot x_i to F

remove clauses containing x_i from F



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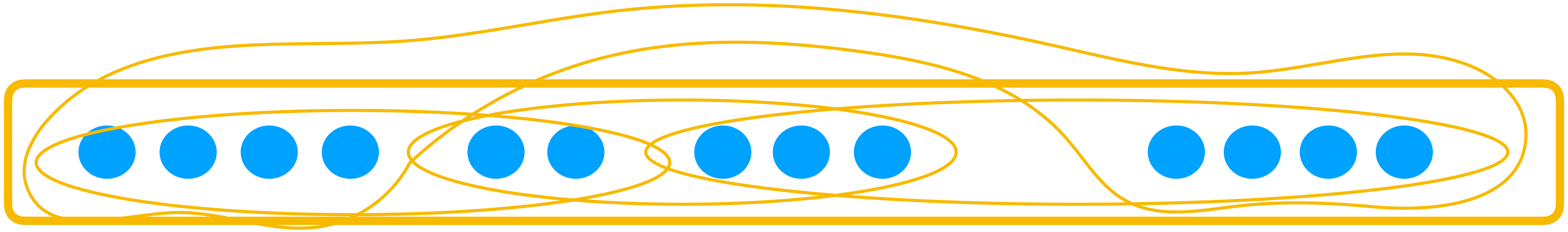

DP-Resolution and Elimination Orderings

Input: A CNF formula F with m clauses

Pick an ordering $\sigma := x_1, \ldots, x_n$ of variables

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  remove tautologies from F

return **false** $\Leftrightarrow$ F contains the empty clause

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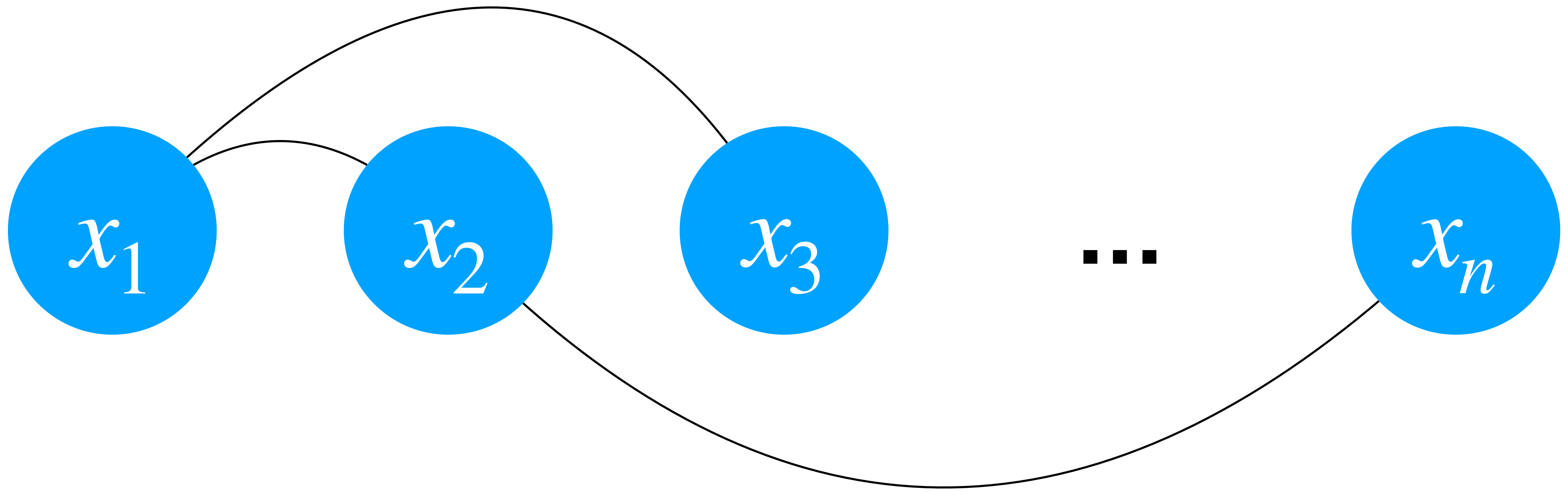

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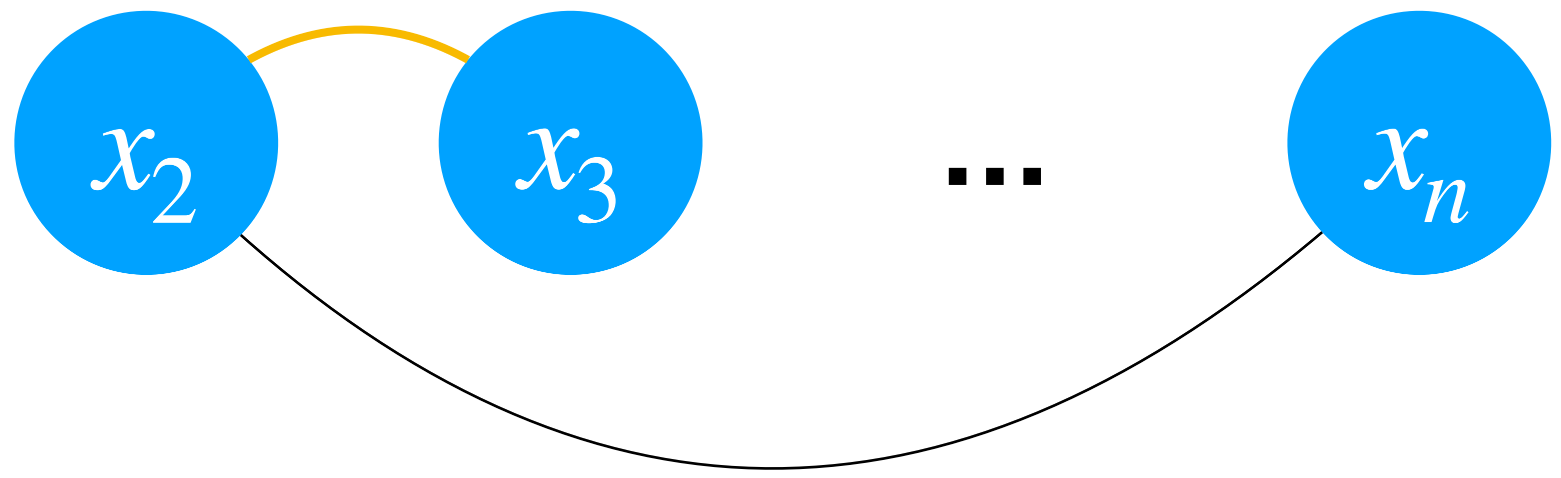

DP-Resolution and Elimination Orderings

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Pick an ordering $\sigma := x_1, \ldots, x_n$ of variables

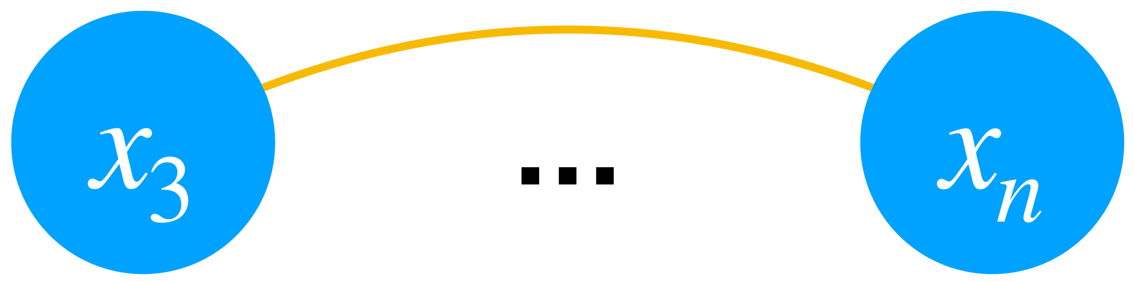
for x_i **in** σ **:**

add all possible resolvents on pivot x_i to F

remove clauses containing x_i from F

remove tautologies from F

return false $\Leftrightarrow$ F contains the empty clause



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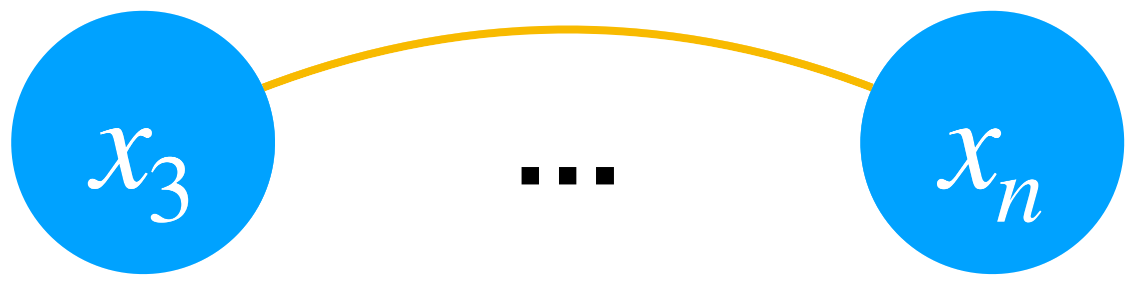

Observation

The size of any clause generated by Davis-Putnam Resolution is at most the width k of the elimination ordering in the primal graph.

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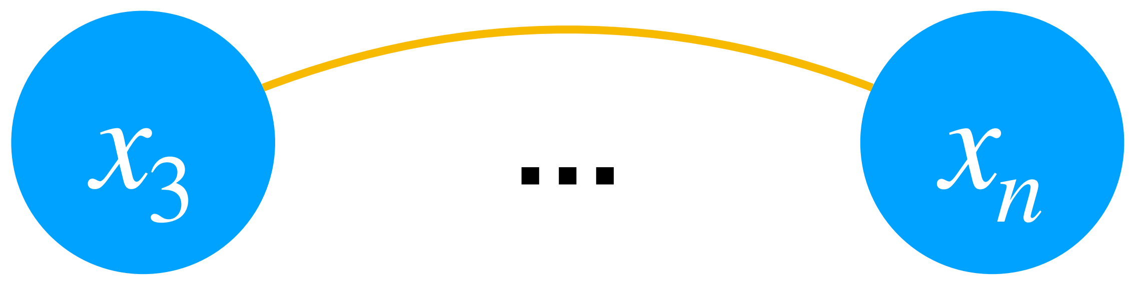

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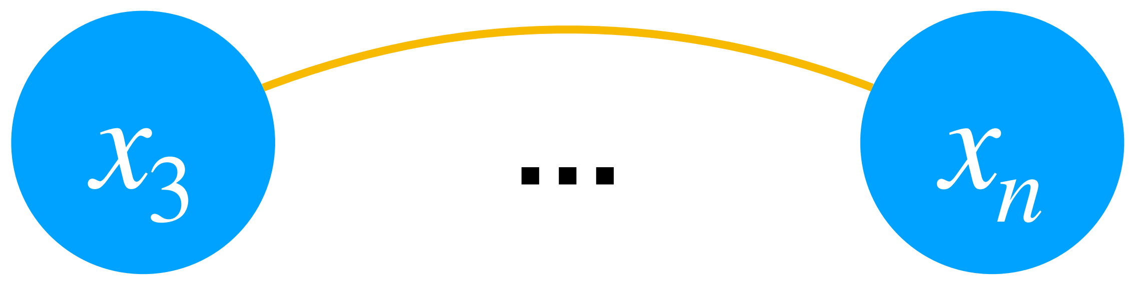

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Theorem

SAT is FPT parameterized by the treewidth of the primal graph.