Part 4: Structural Decompositions and Algorithms

Friedrich Slivovsky

Dynamic Programming on Tree Decompositions

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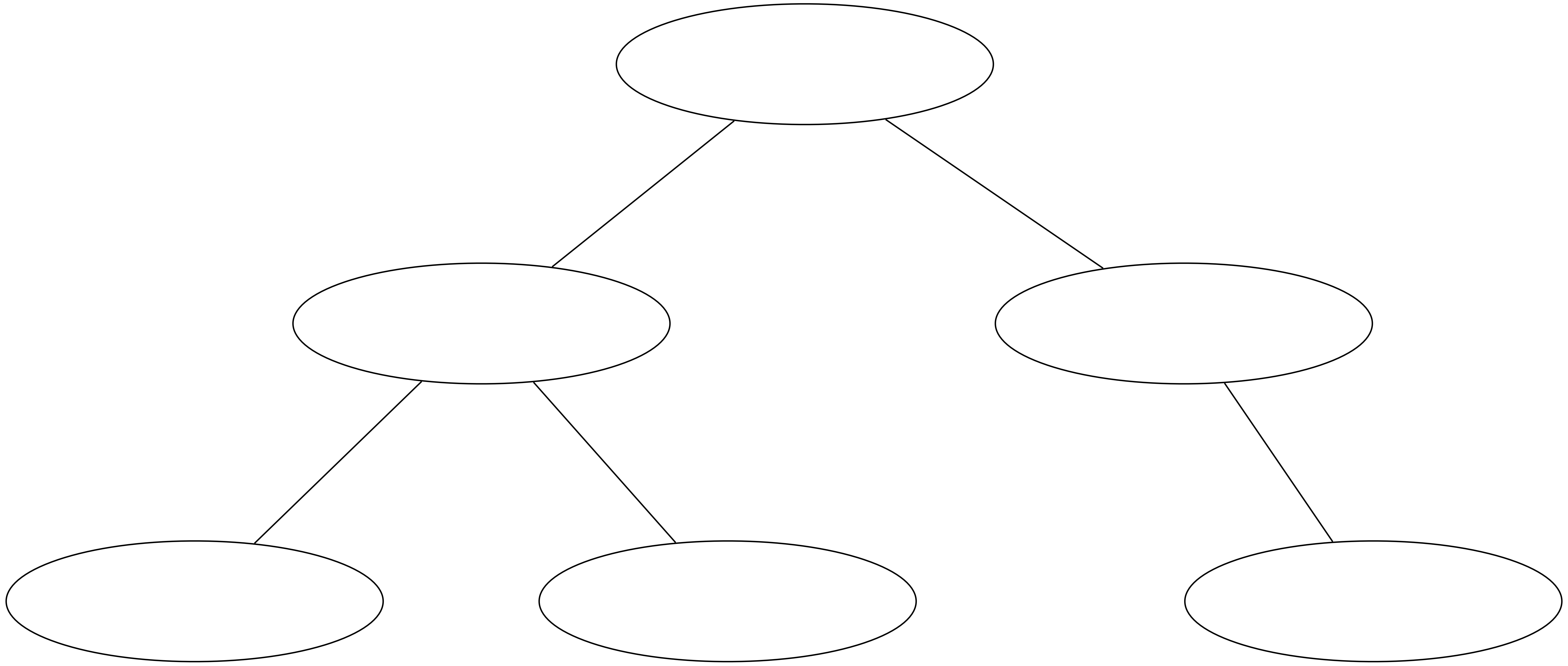
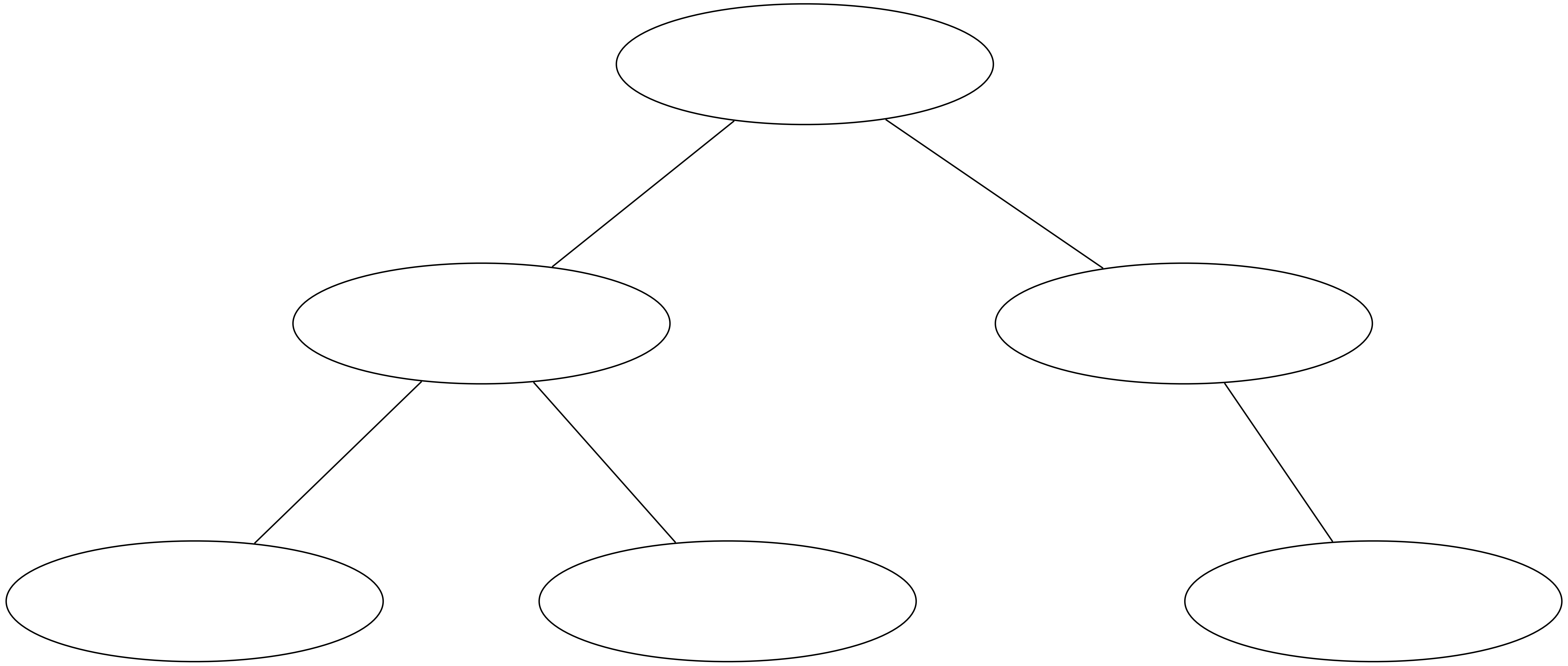


Dynamic Programming on Tree Decompositions

1. Compute a (nice) tree decomposition.

2. Go from leaves to root and compute table for each node.



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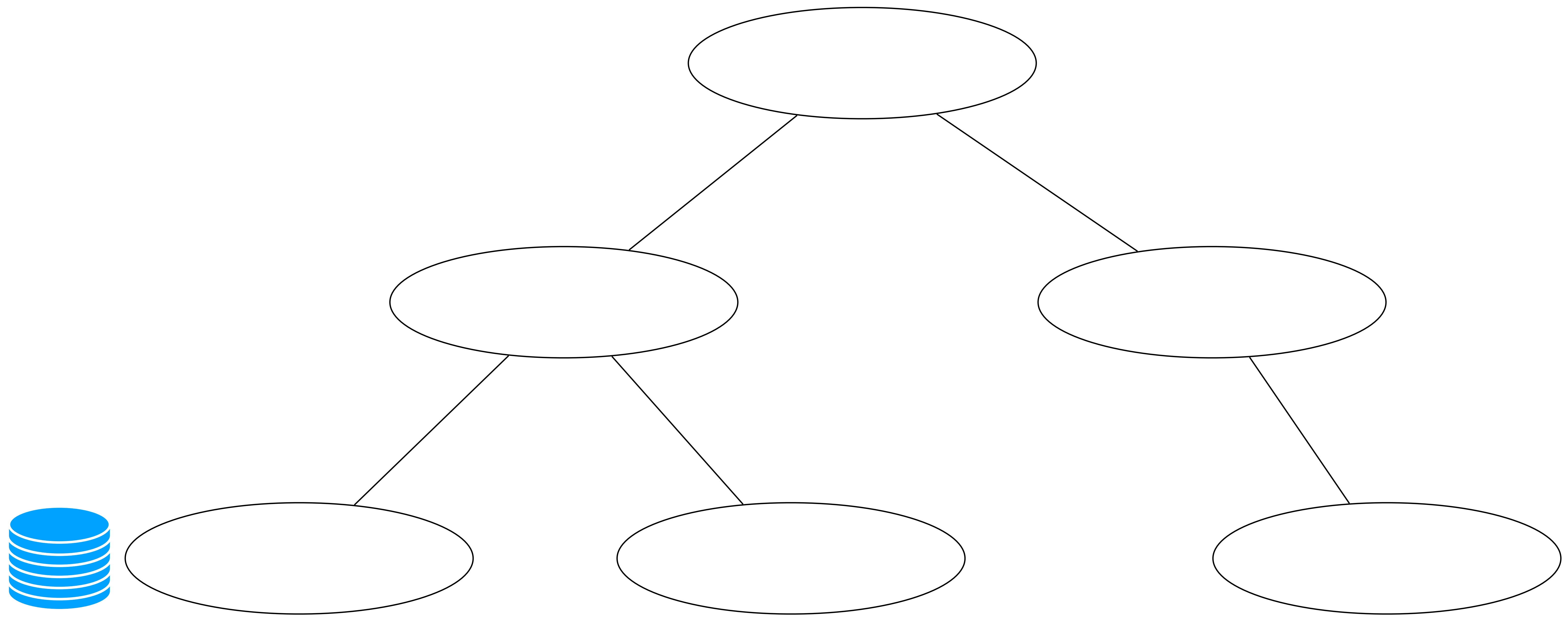

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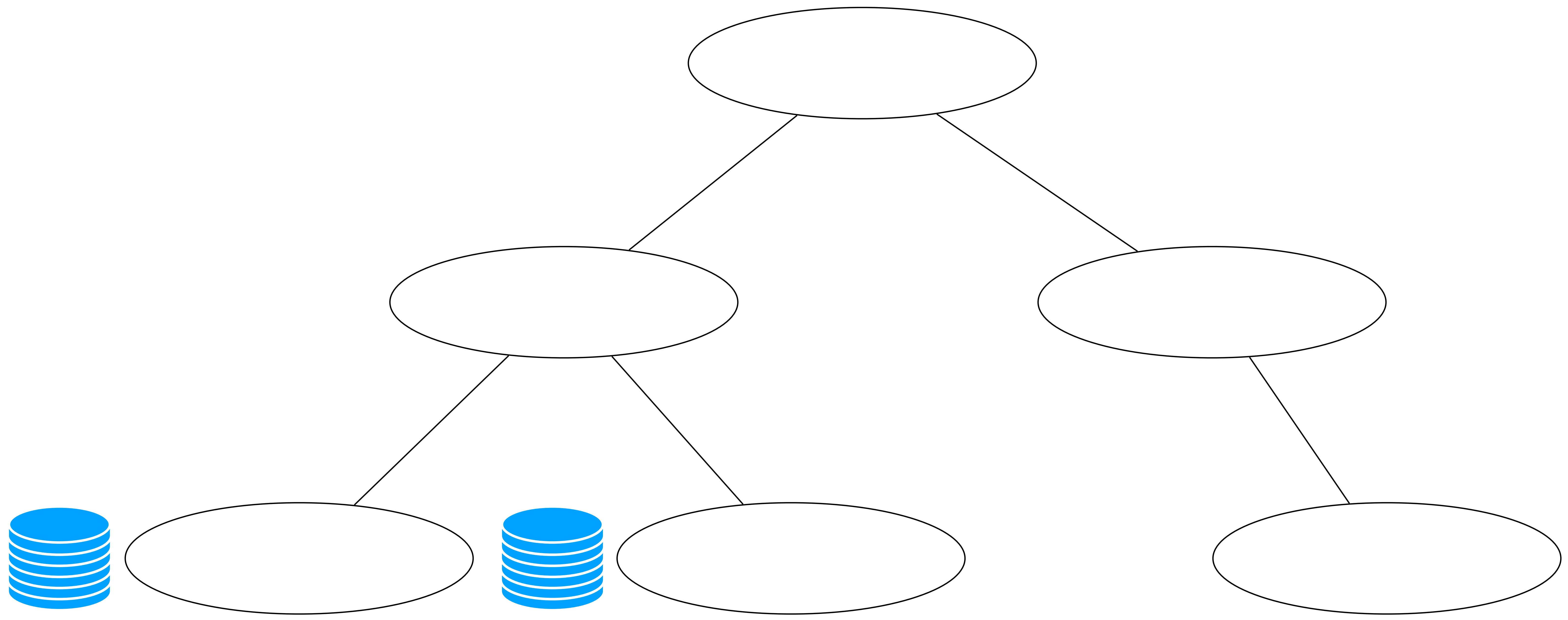

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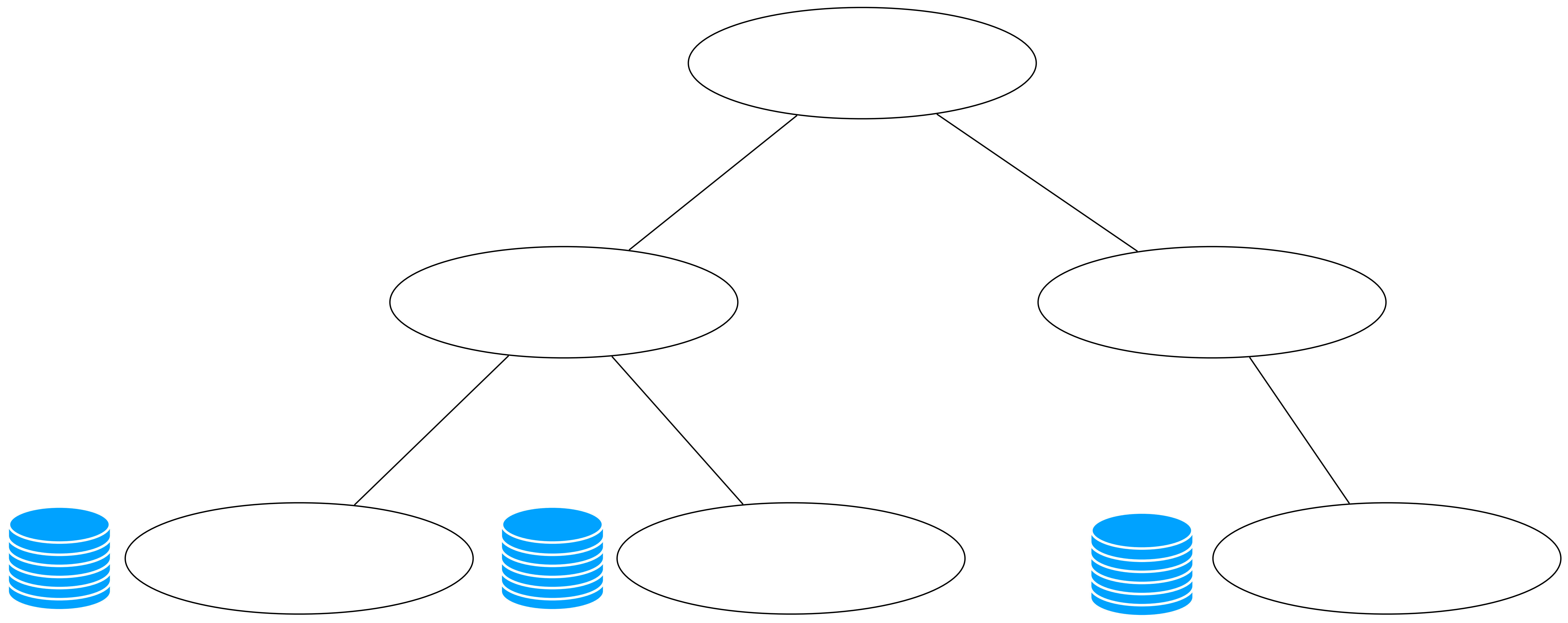

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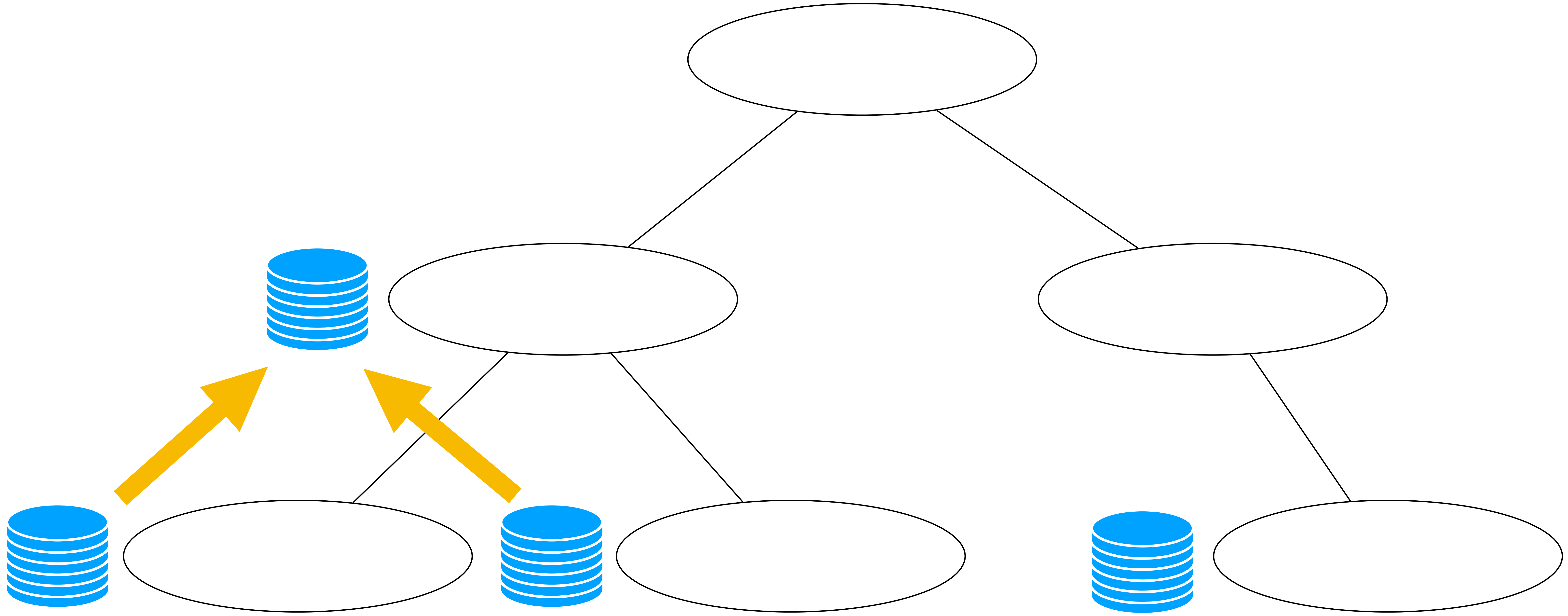

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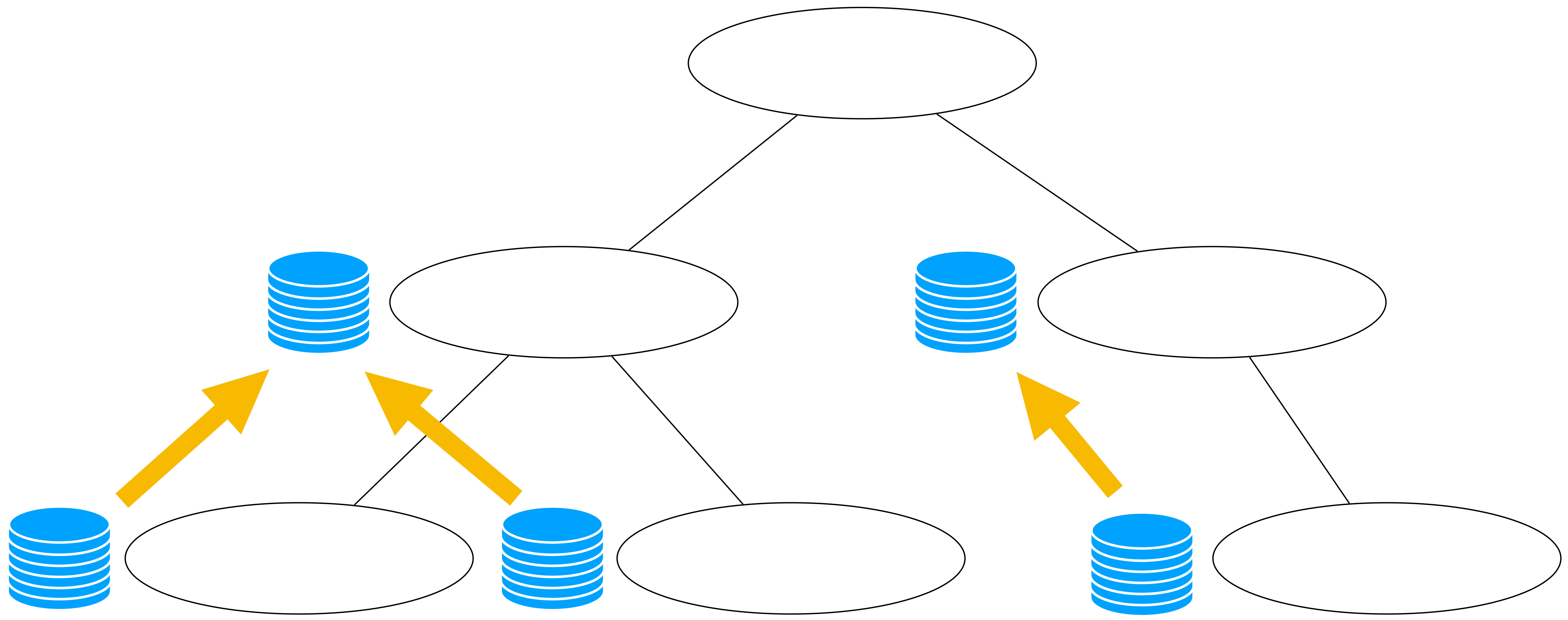

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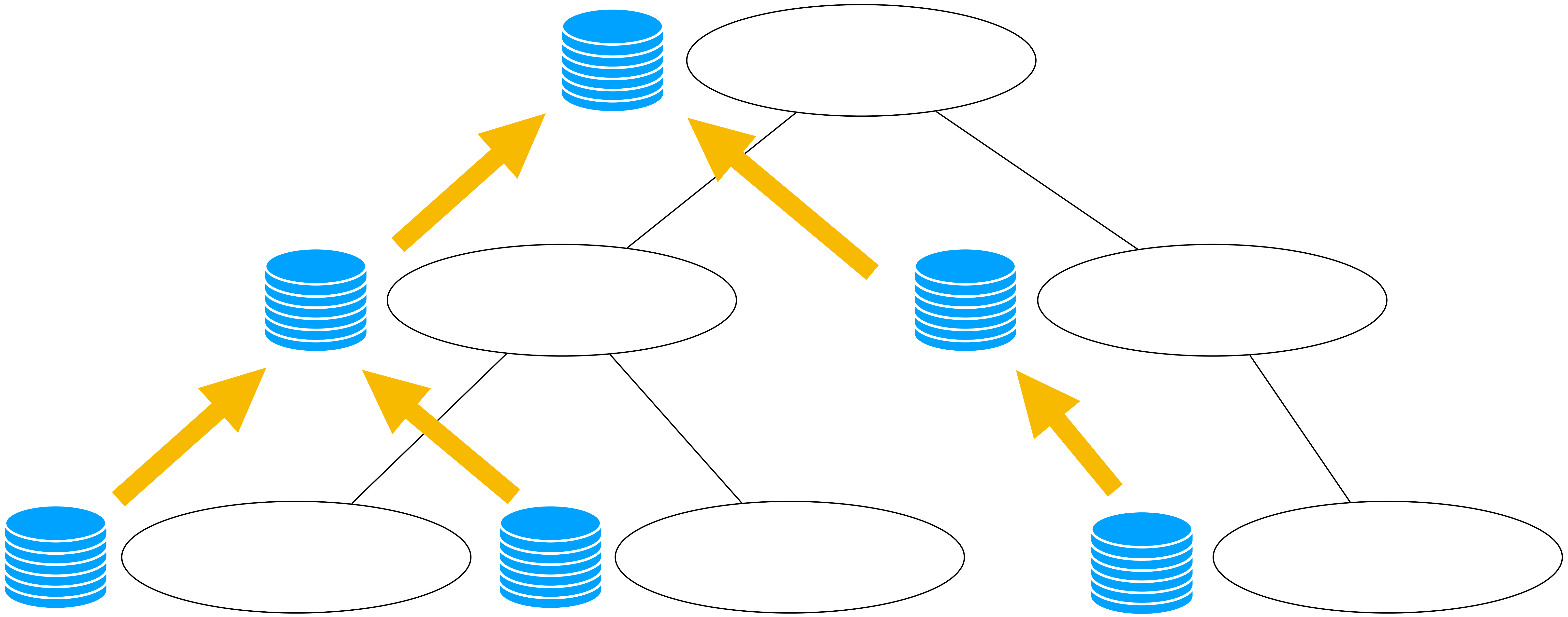

Dynamic Programming on Tree Decompositions

1. Compute a (nice) tree decomposition.
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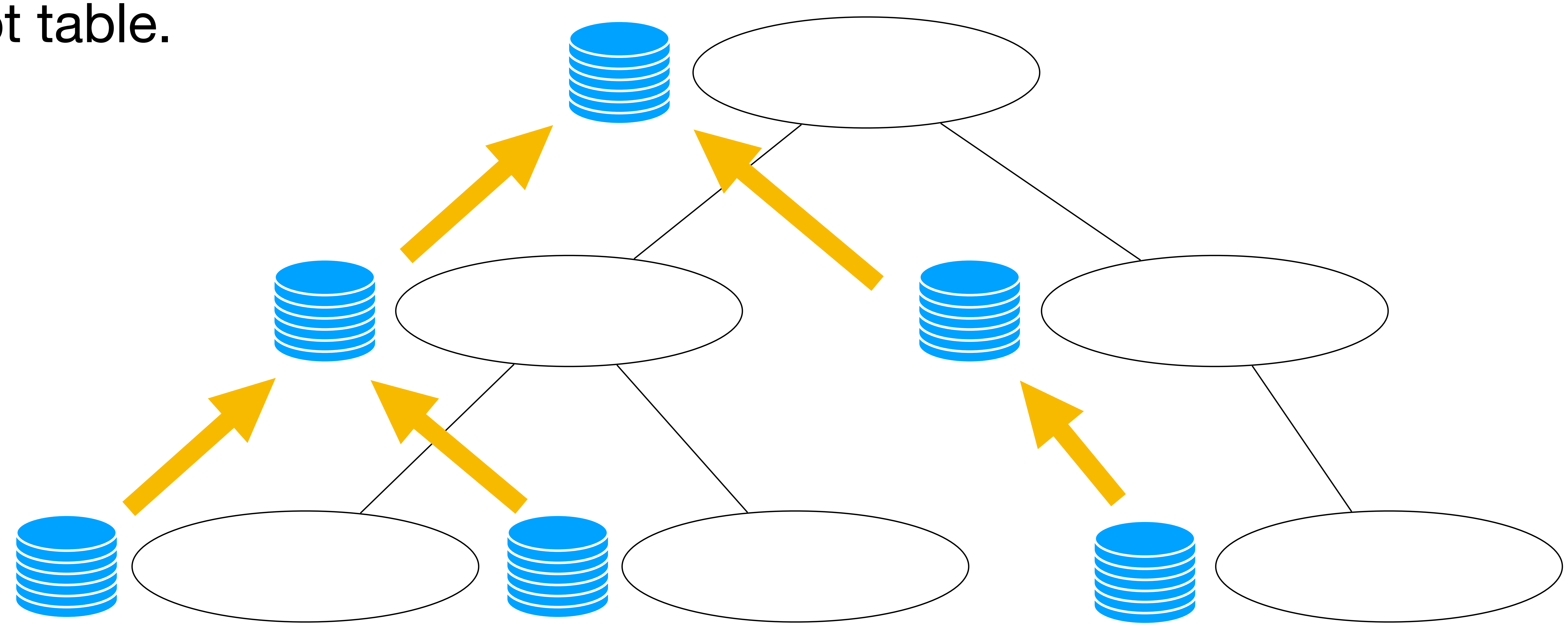
3. Retrieve solution from root table.



Dynamic Programming on Tree Decompositions

Treewidth k

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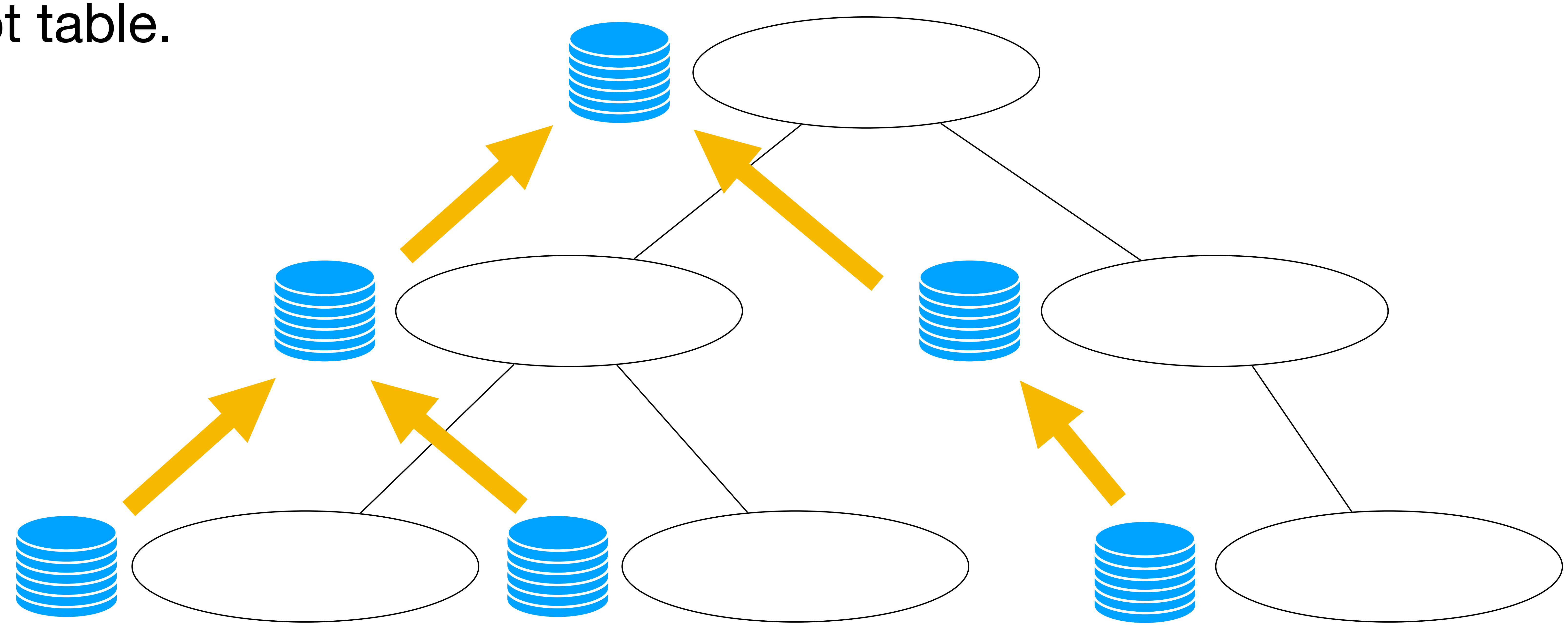

Dynamic Programming on Tree Decompositions

$f(k)\,p(n)$

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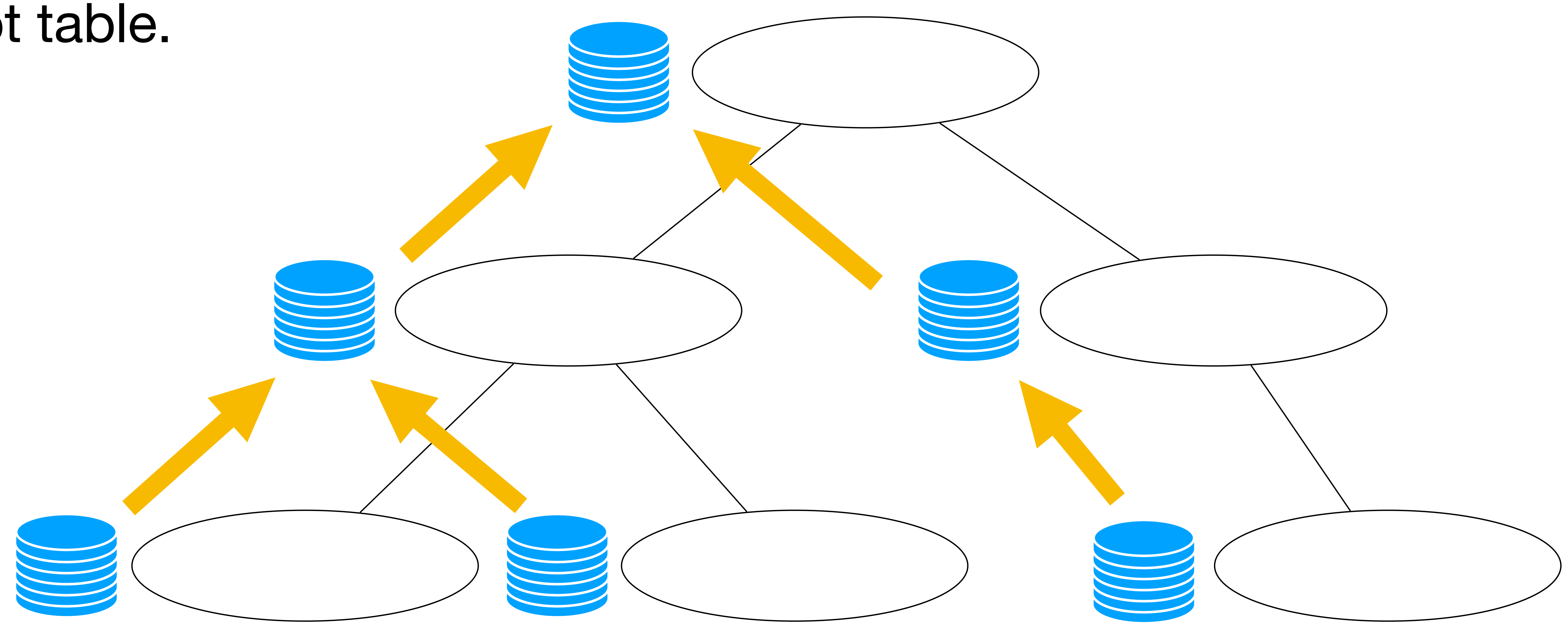

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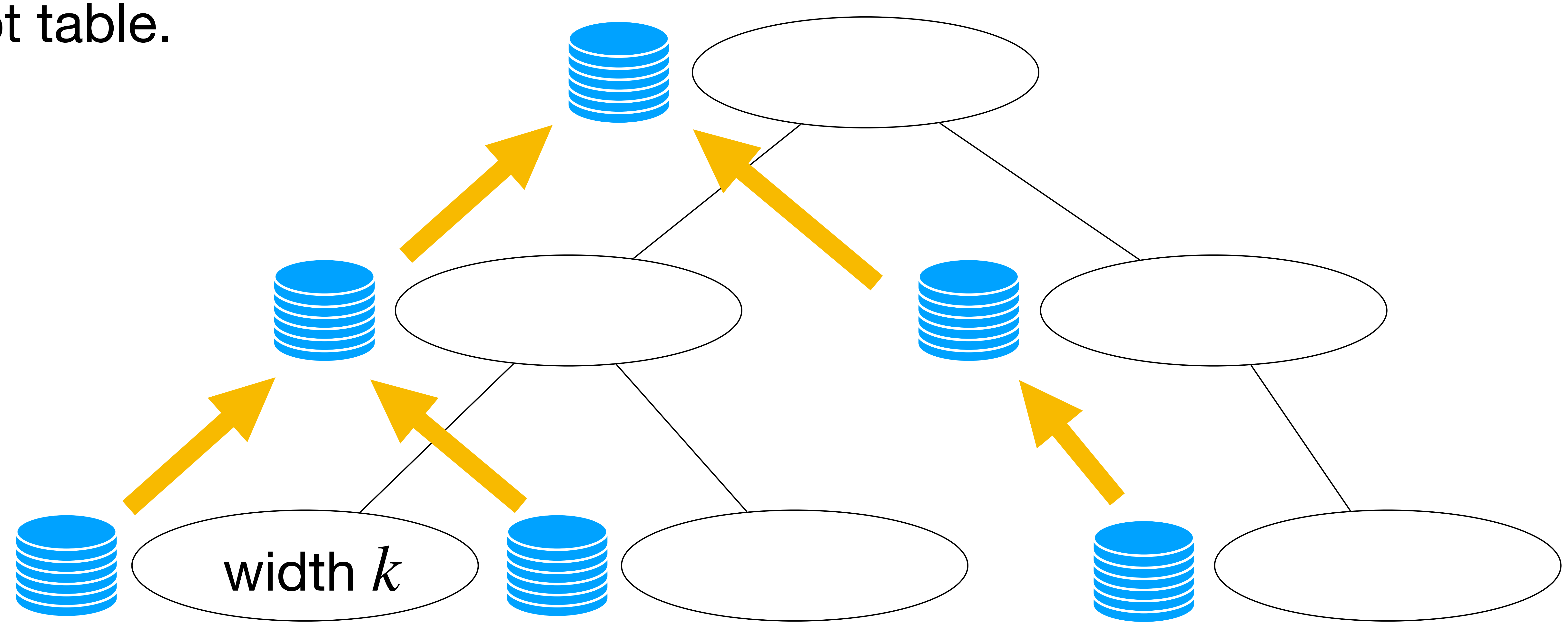

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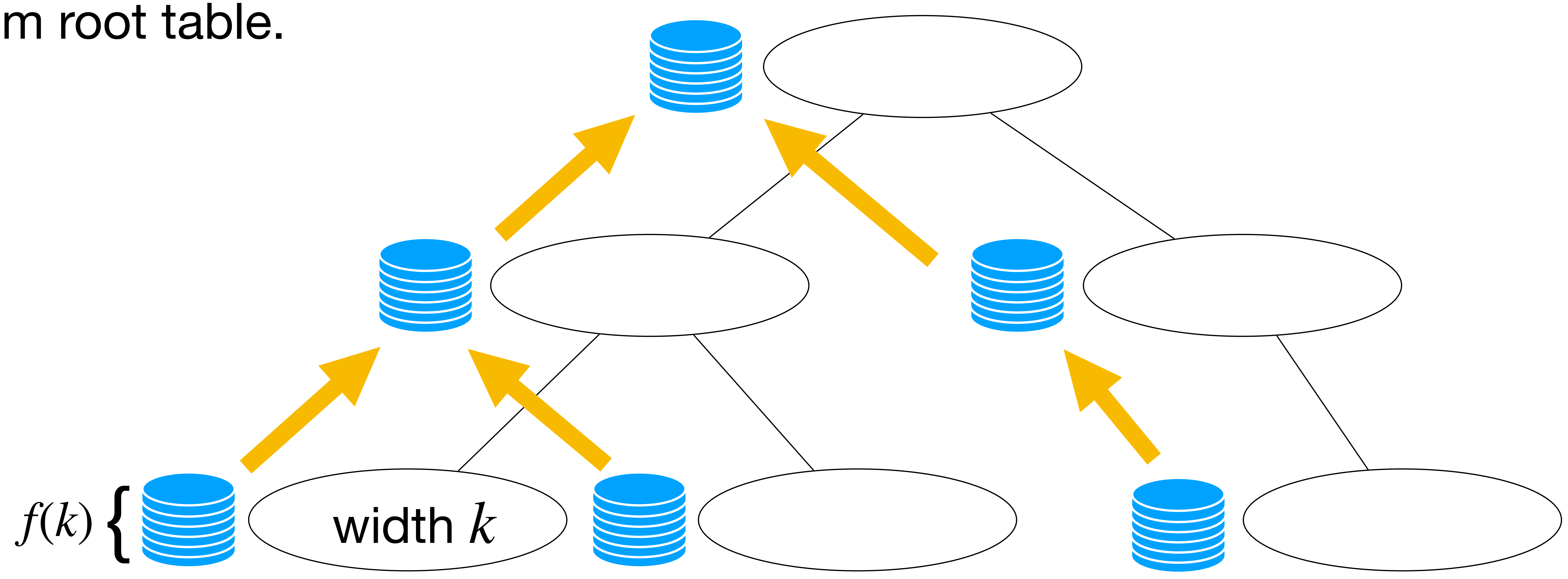

Dynamic Programming on Tree Decompositions

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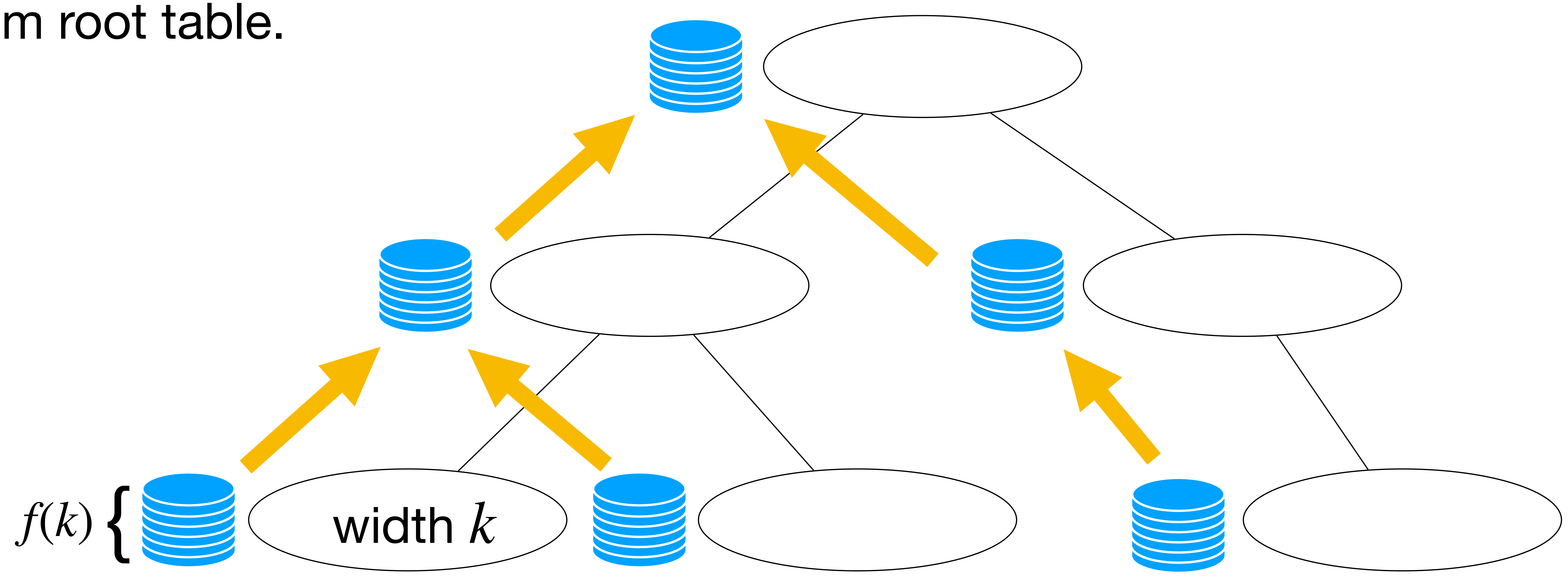

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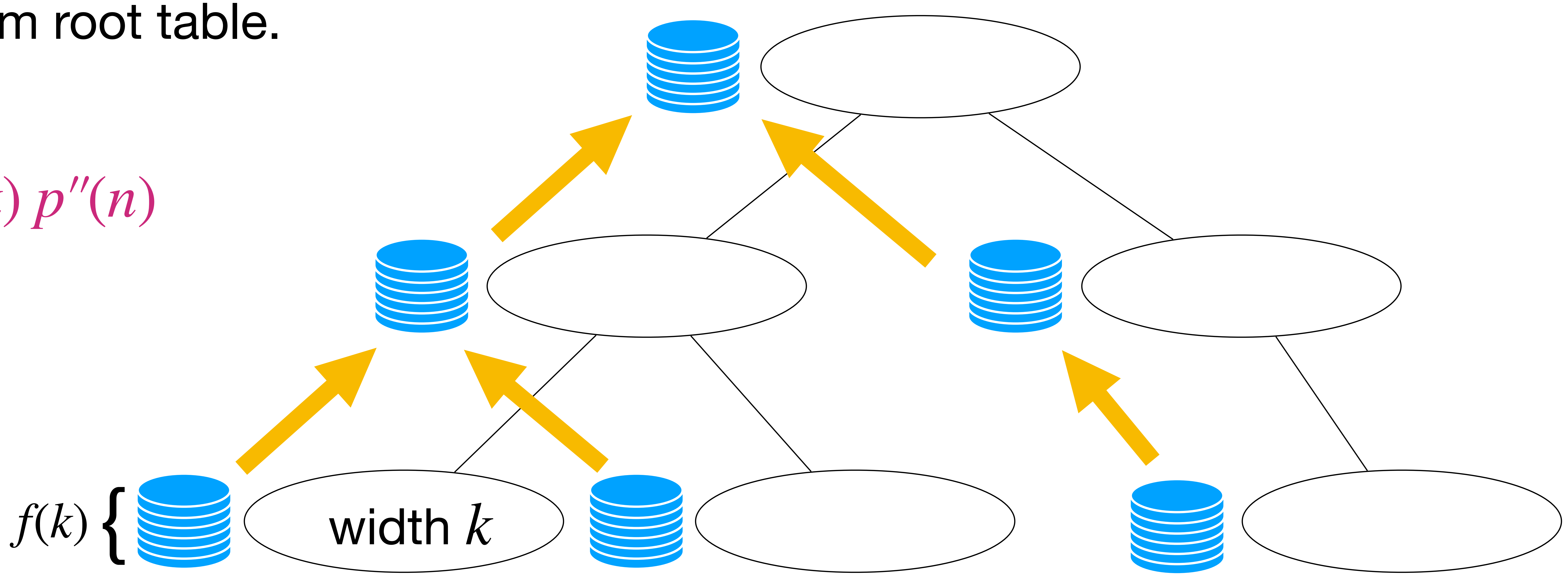
$h(k)\, p''(n)$



Independent Set

Definition

An **independent set** of a graph is a subset of vertices such that no two vertices are adjacent.

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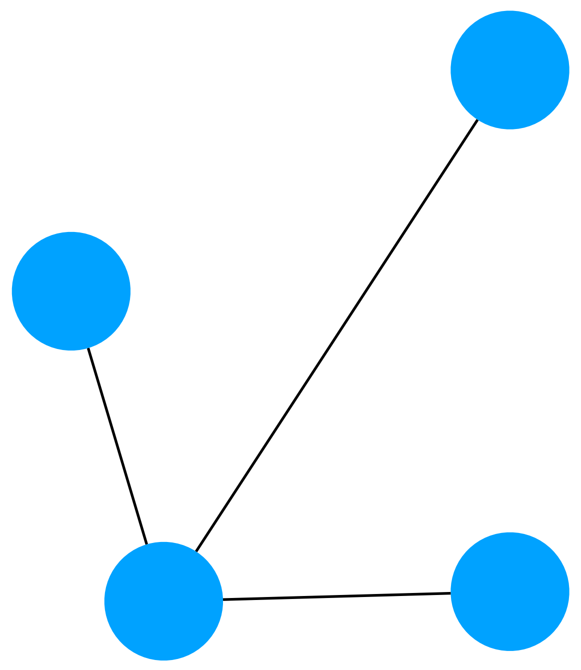

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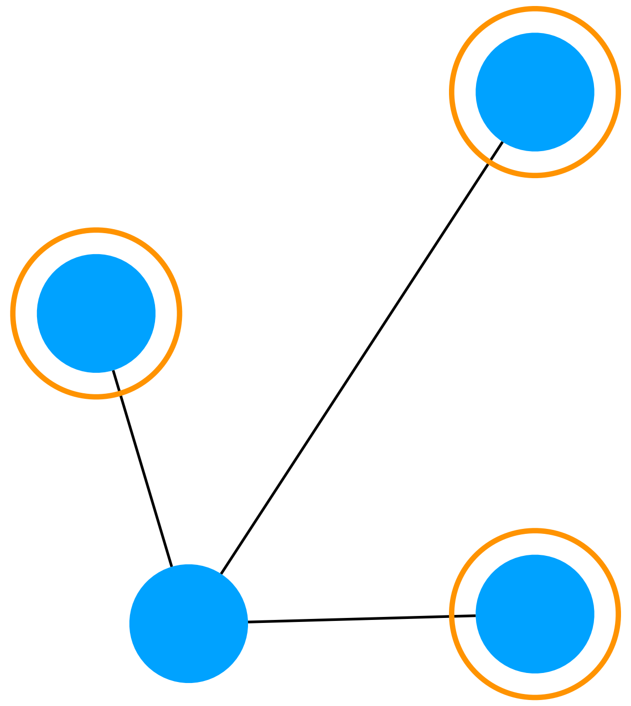

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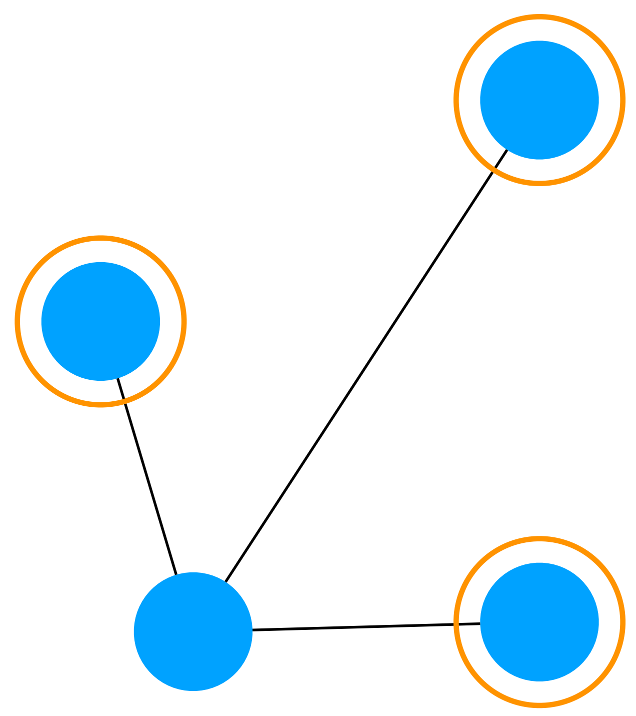

INDEPENDENT SET

Input: A graph G and an integer k.

Question: Does G have an **IS** of size k?

Independent Set on Trees

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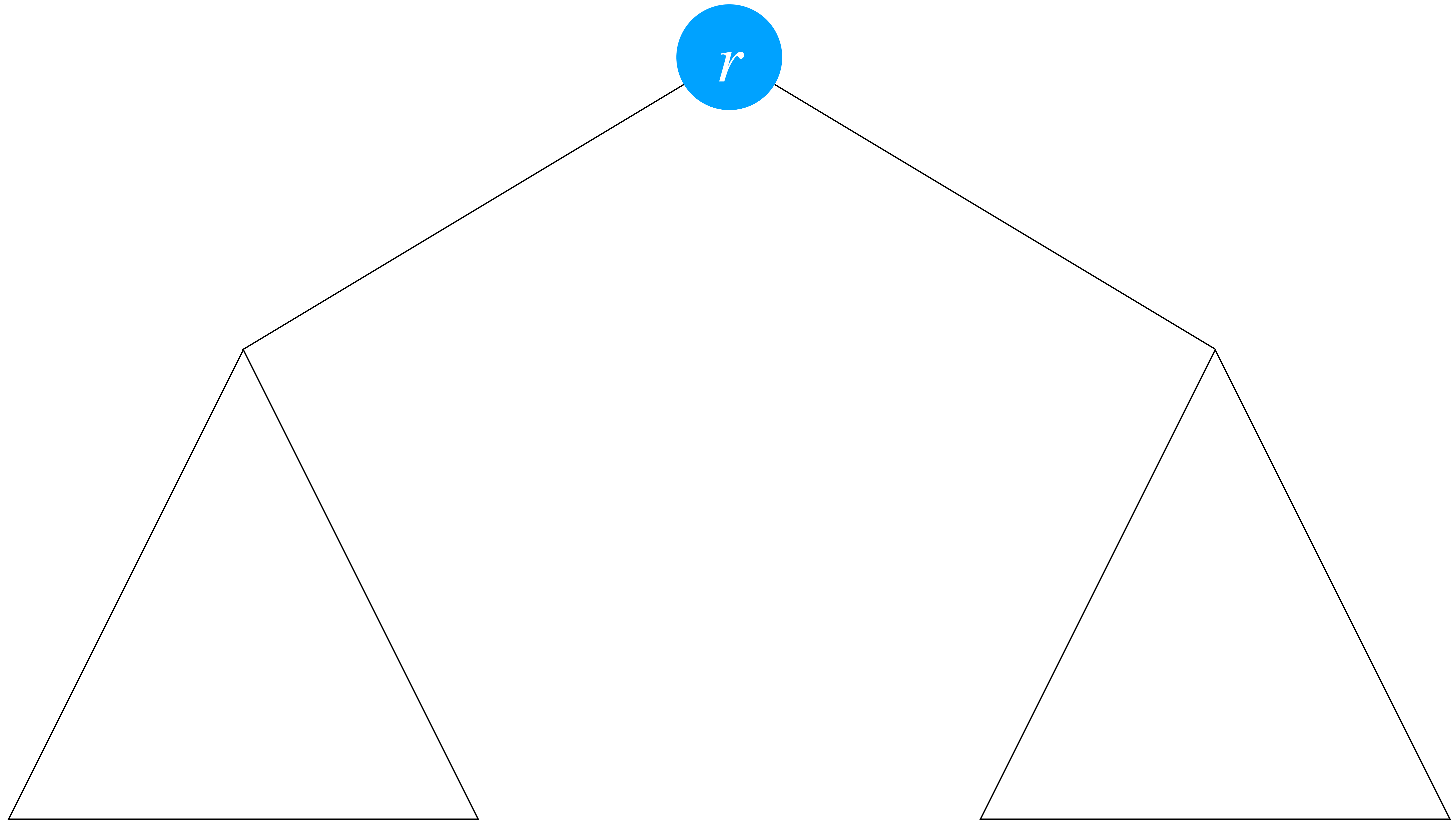

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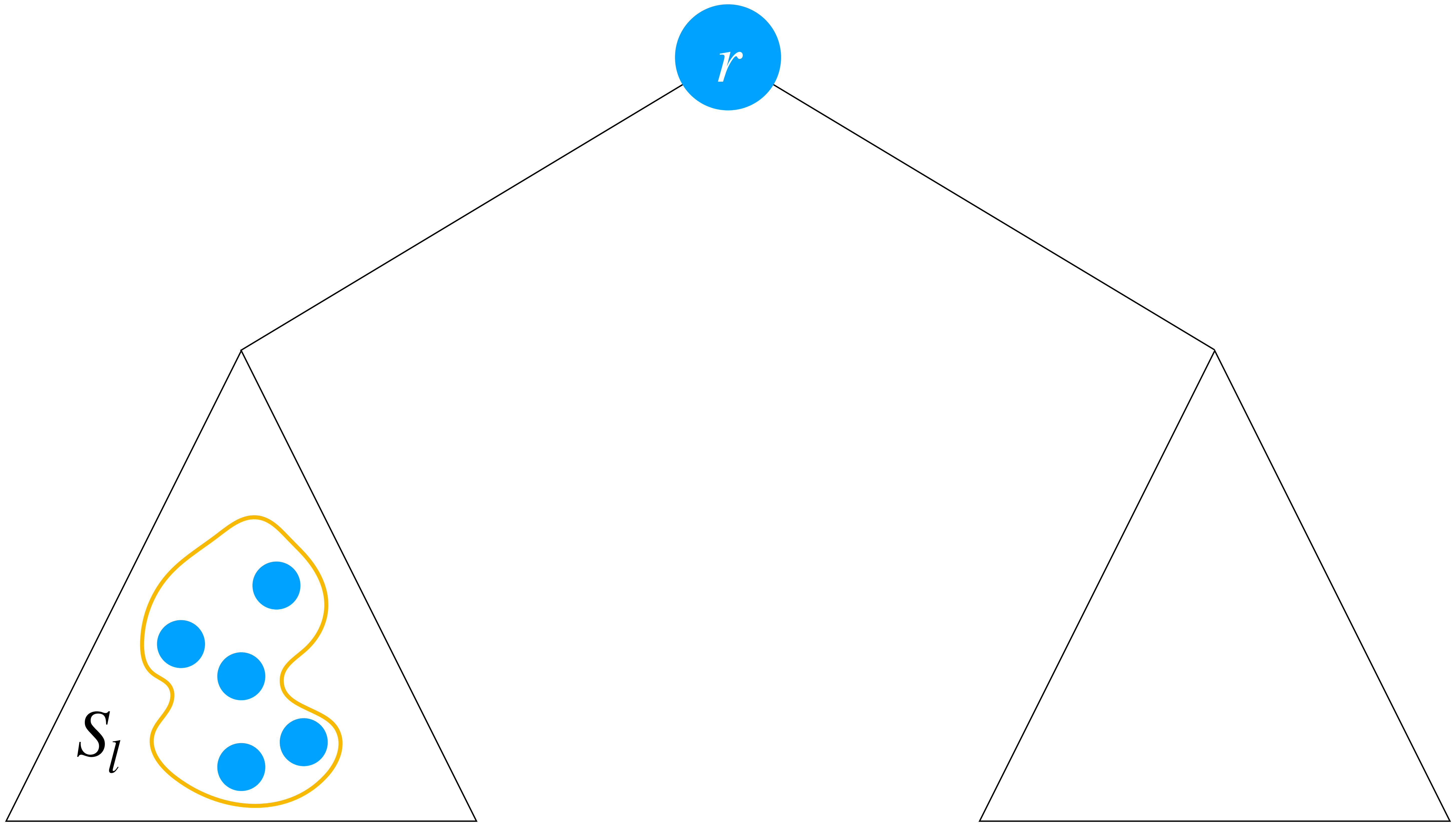

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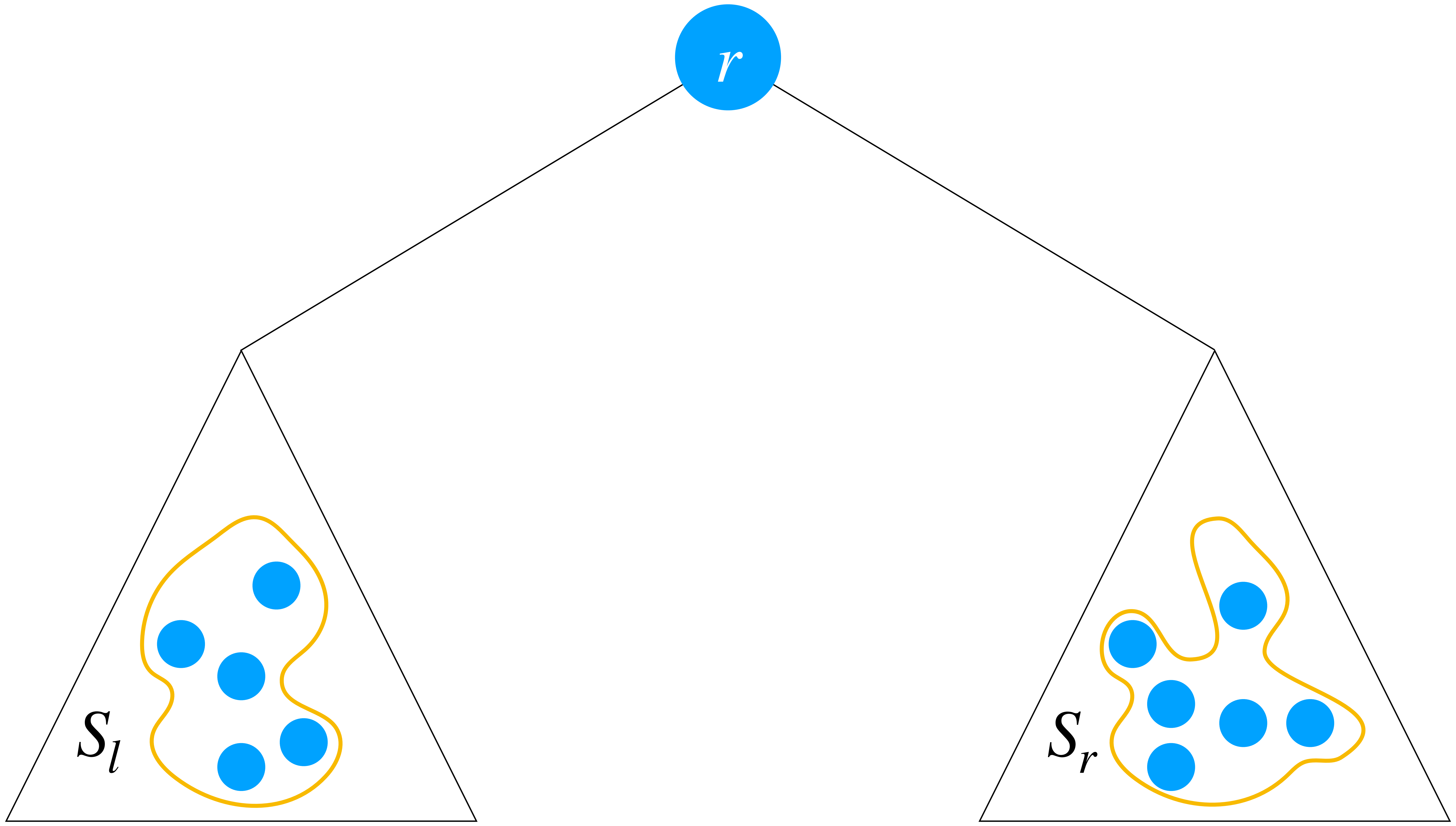
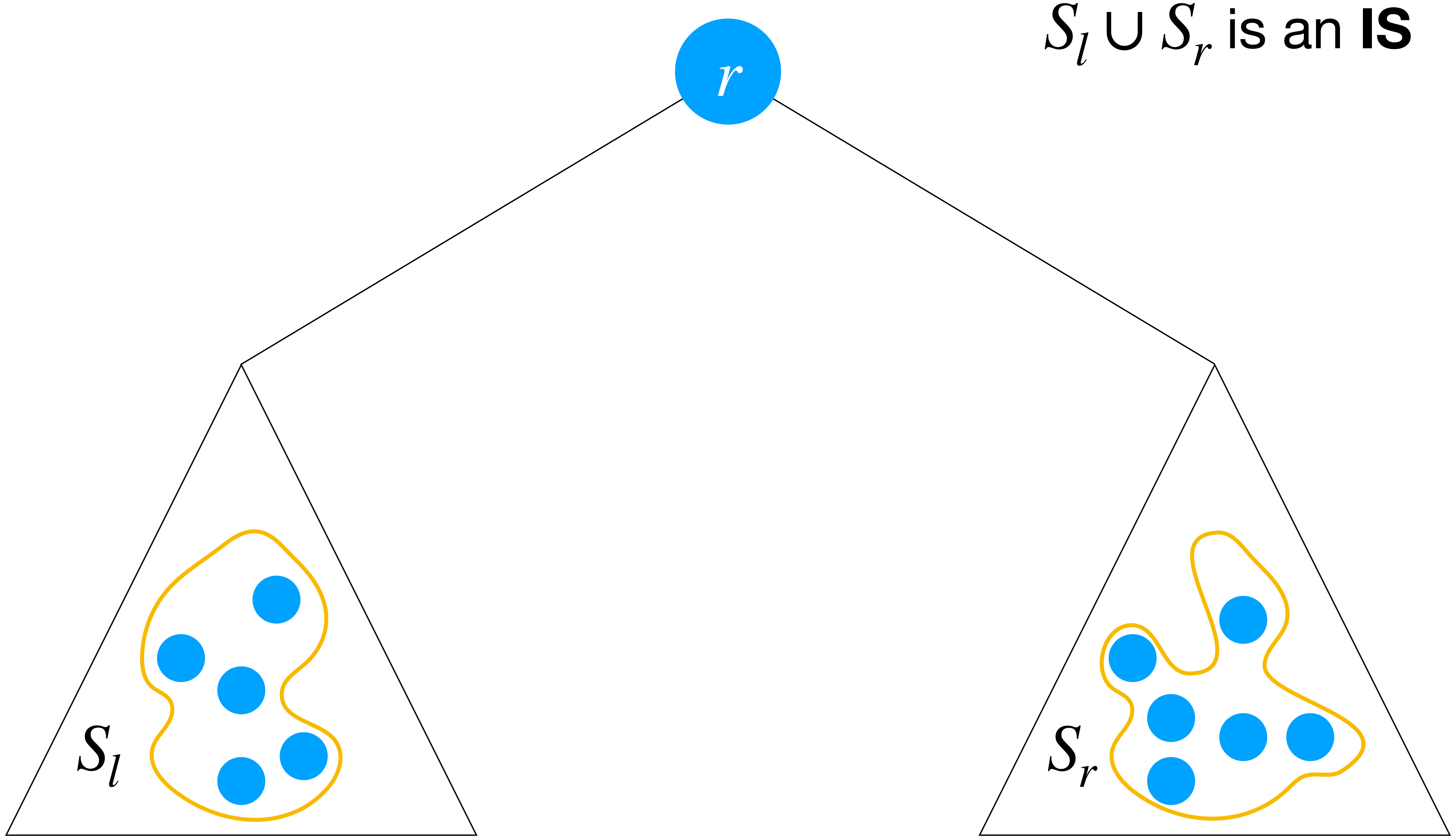


Independent Set on Trees

$S_l \cup S_r$ is an **IS**



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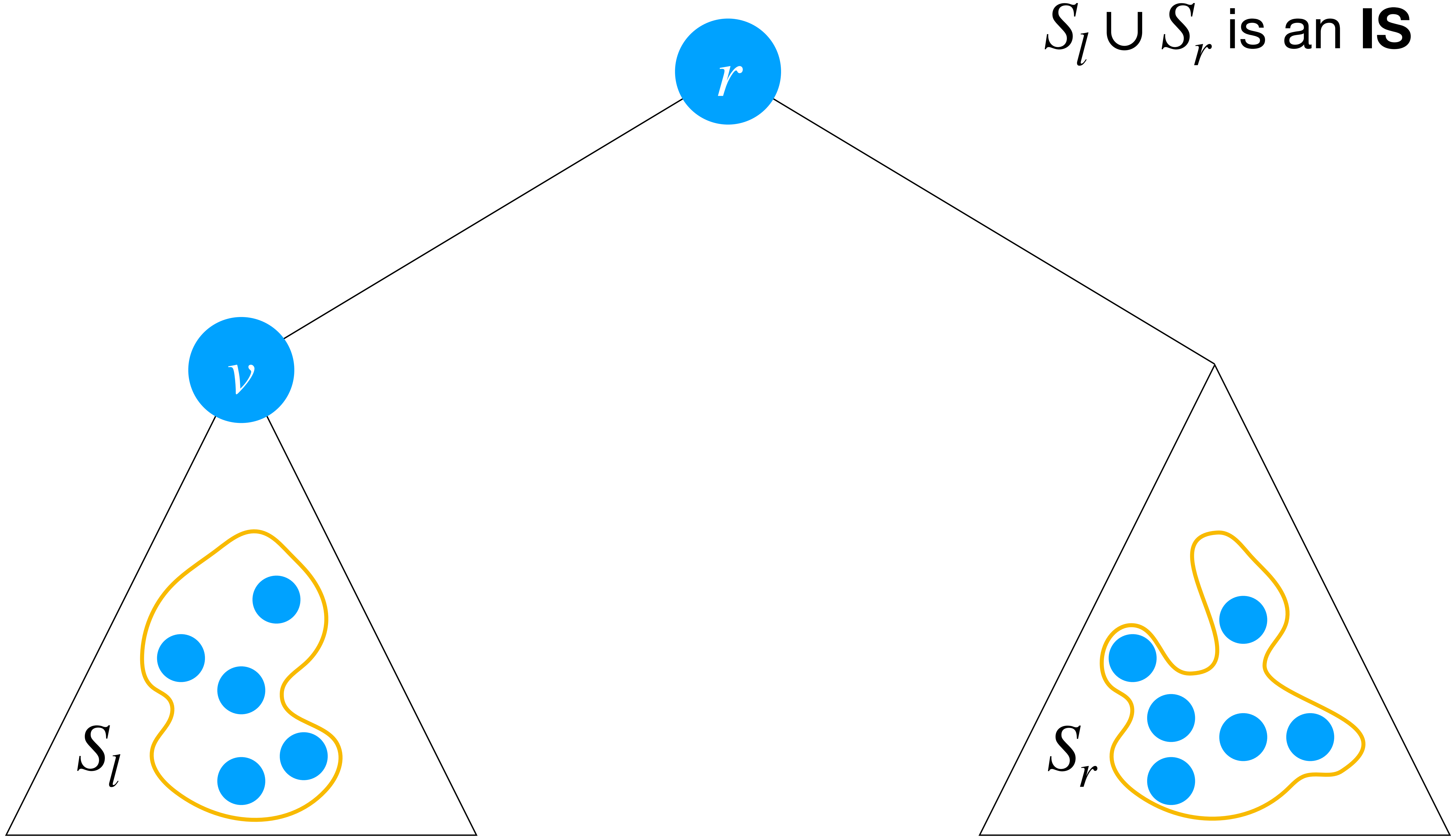

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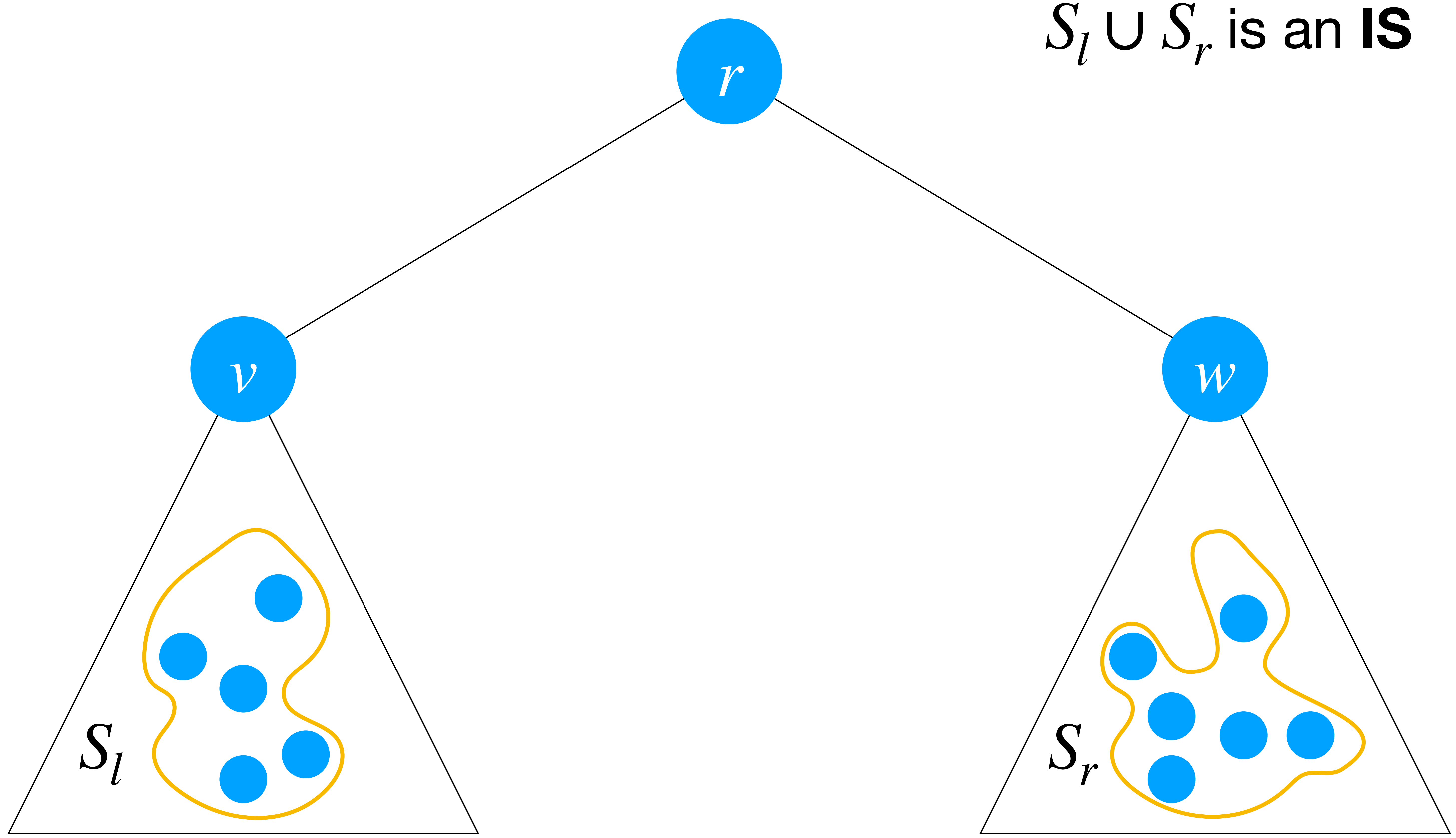

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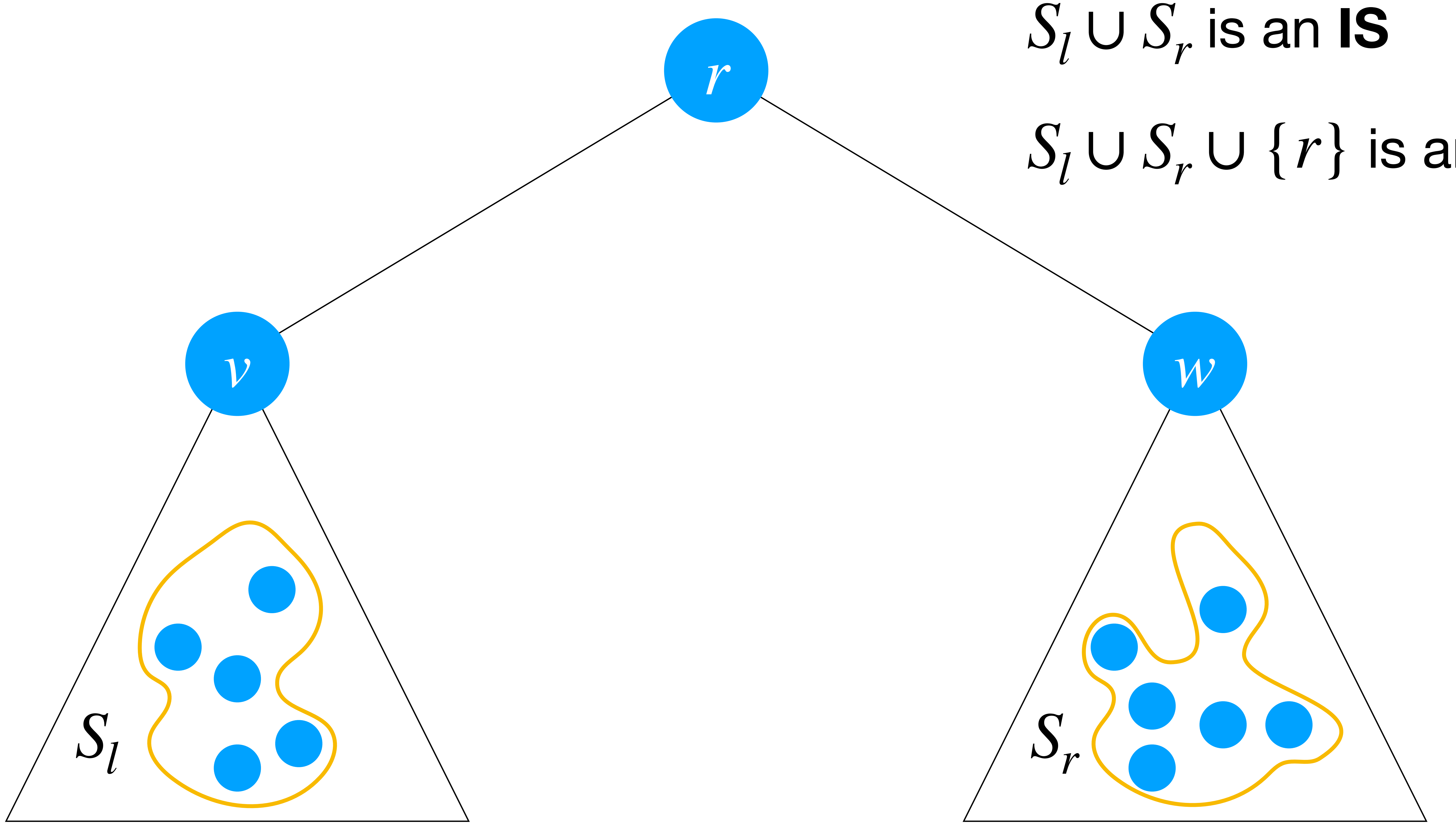

$S_l \cup S_r$ is an **IS**

$S_l \cup S_r \cup \{r\}$ is an **IS** if $v \notin S_l$ and $w \notin S_r$

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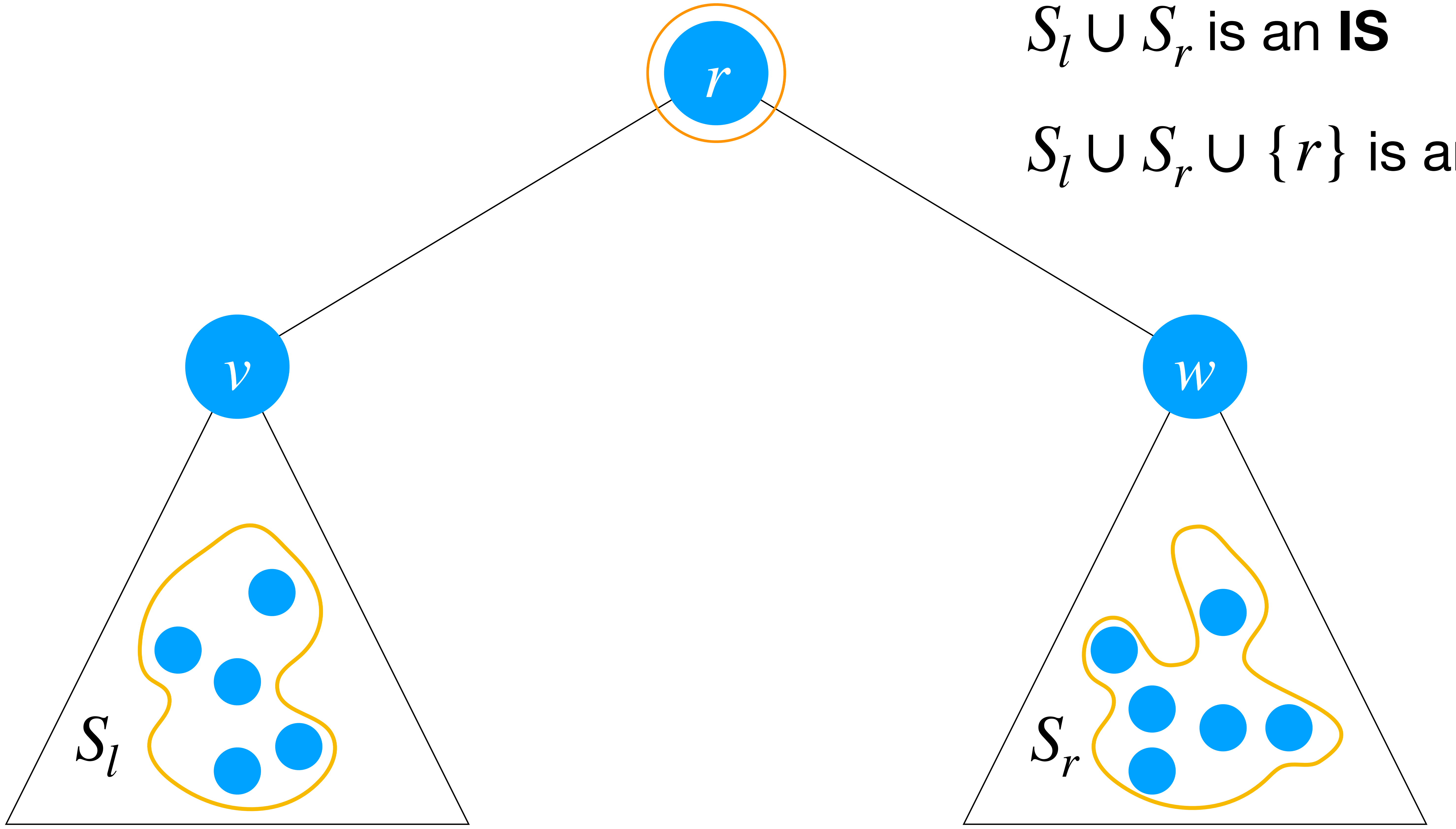

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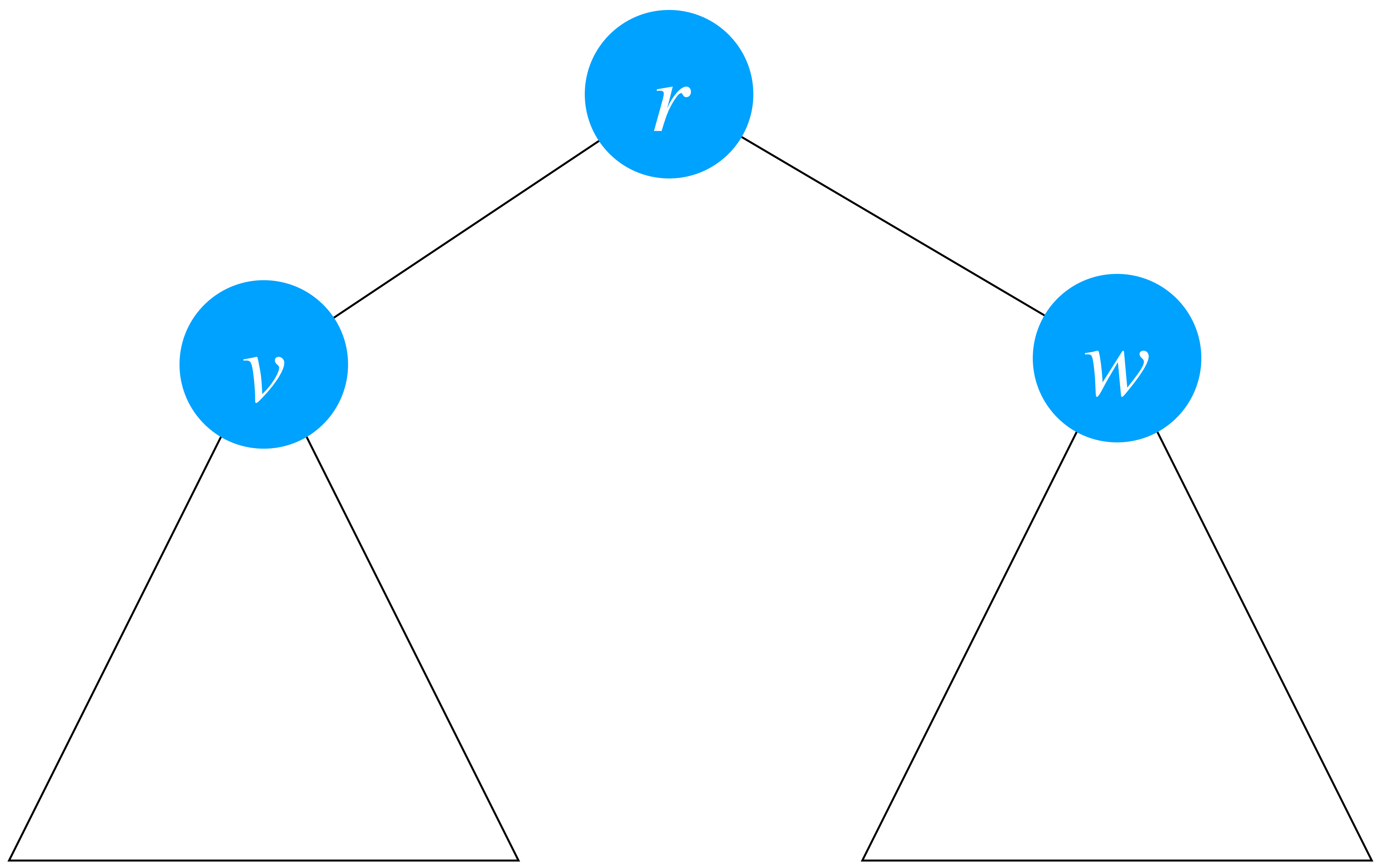
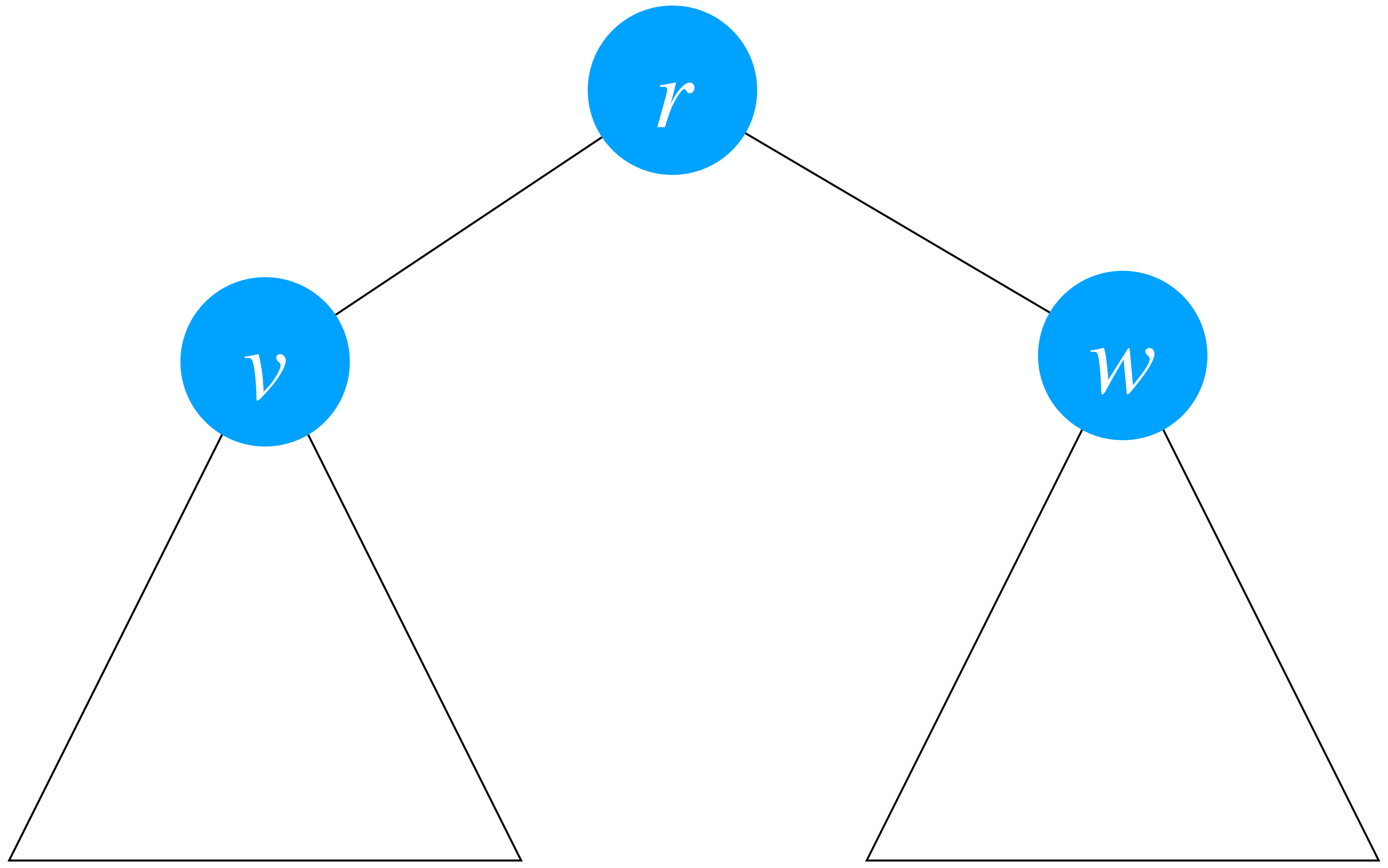


Independent Set on Trees

$n(u)$ max **IS** of the subtree **containing** u.



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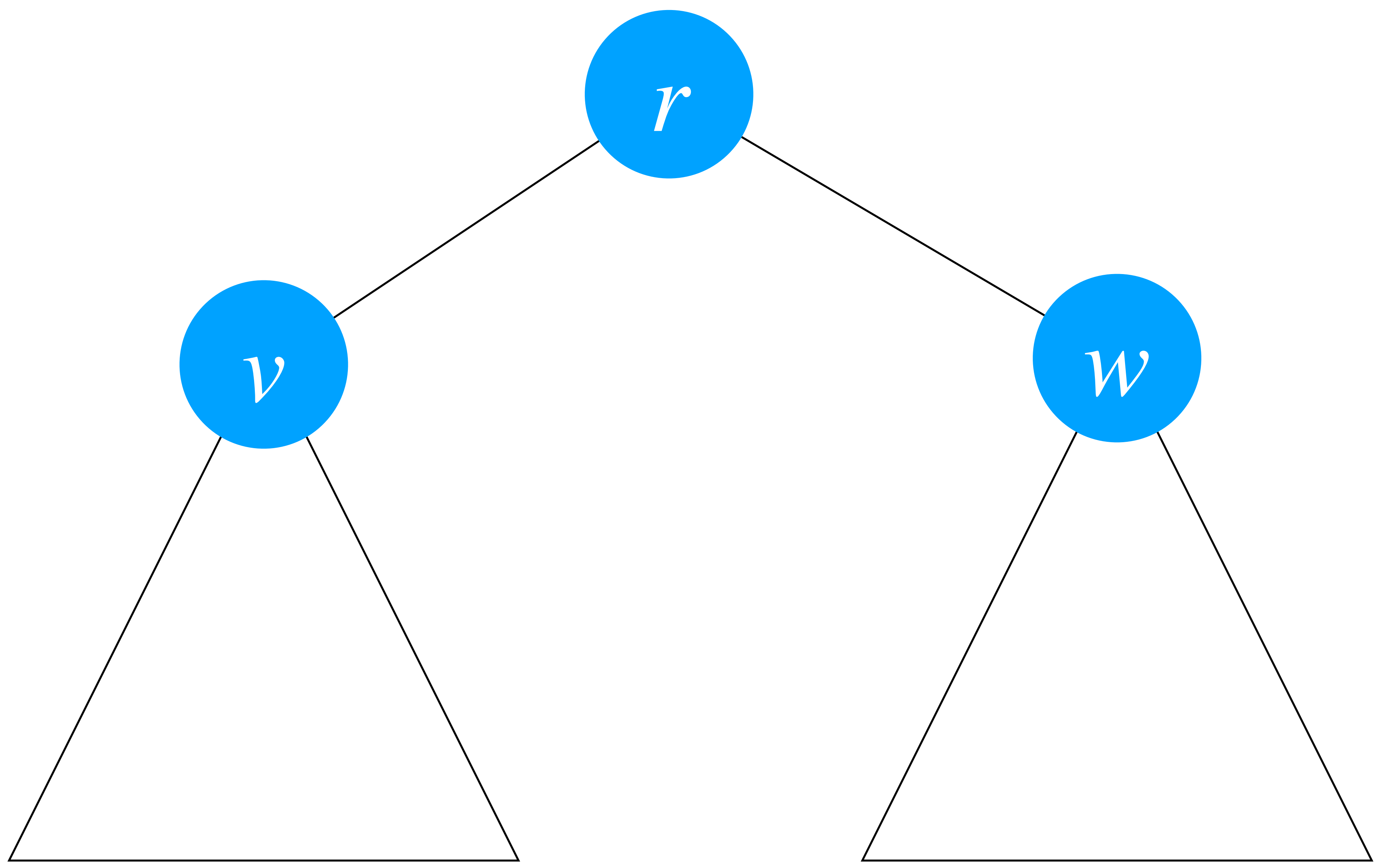

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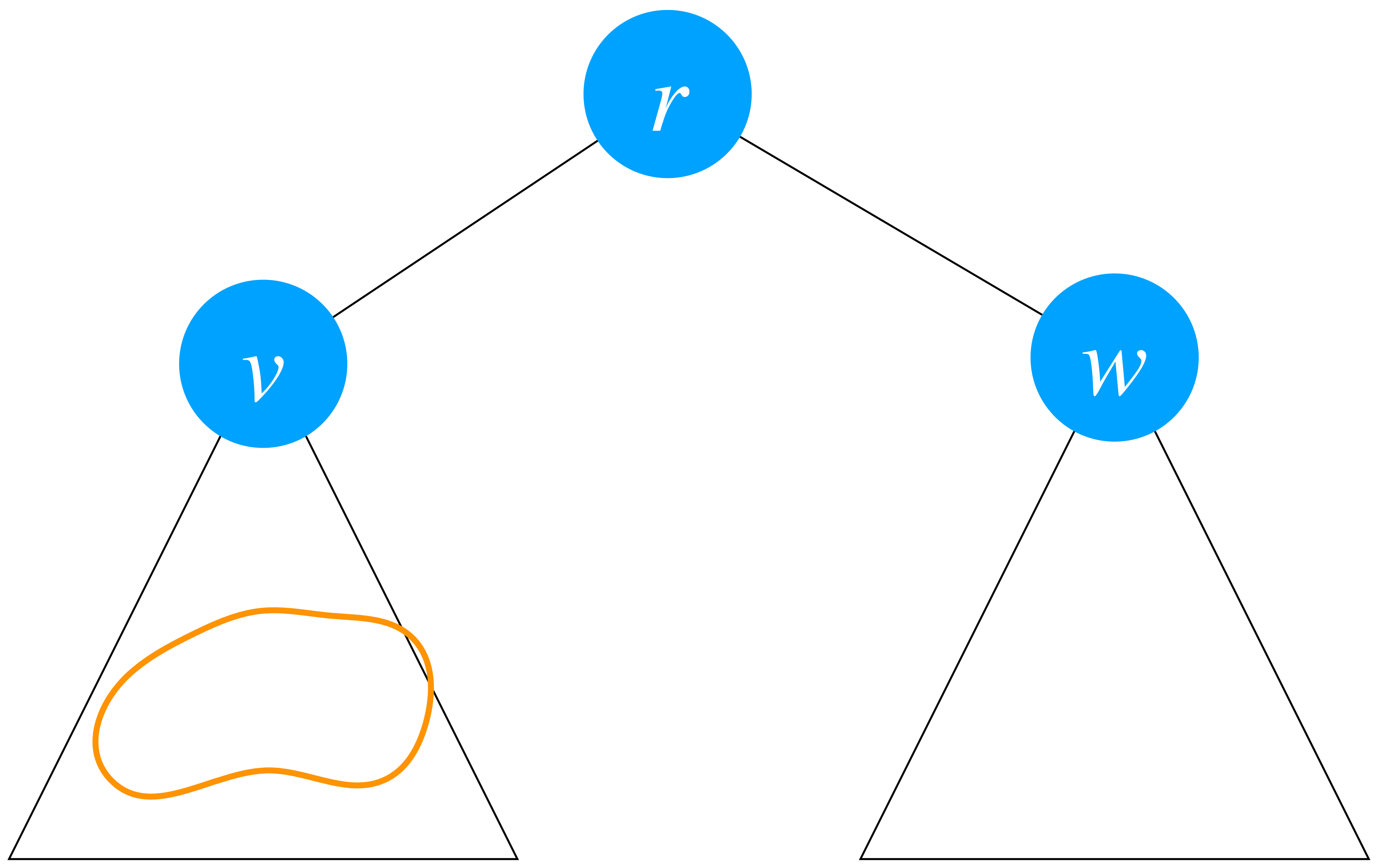

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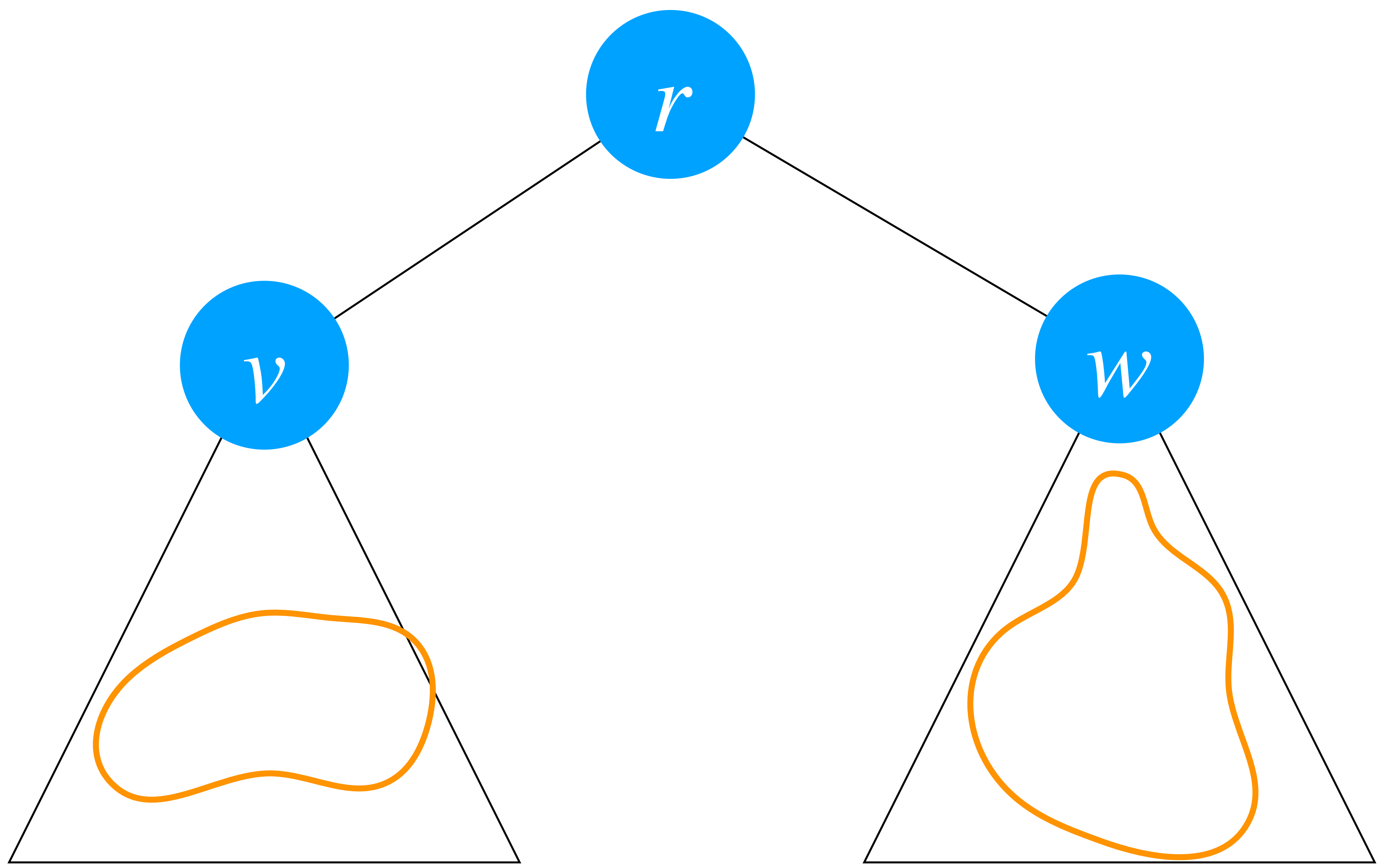

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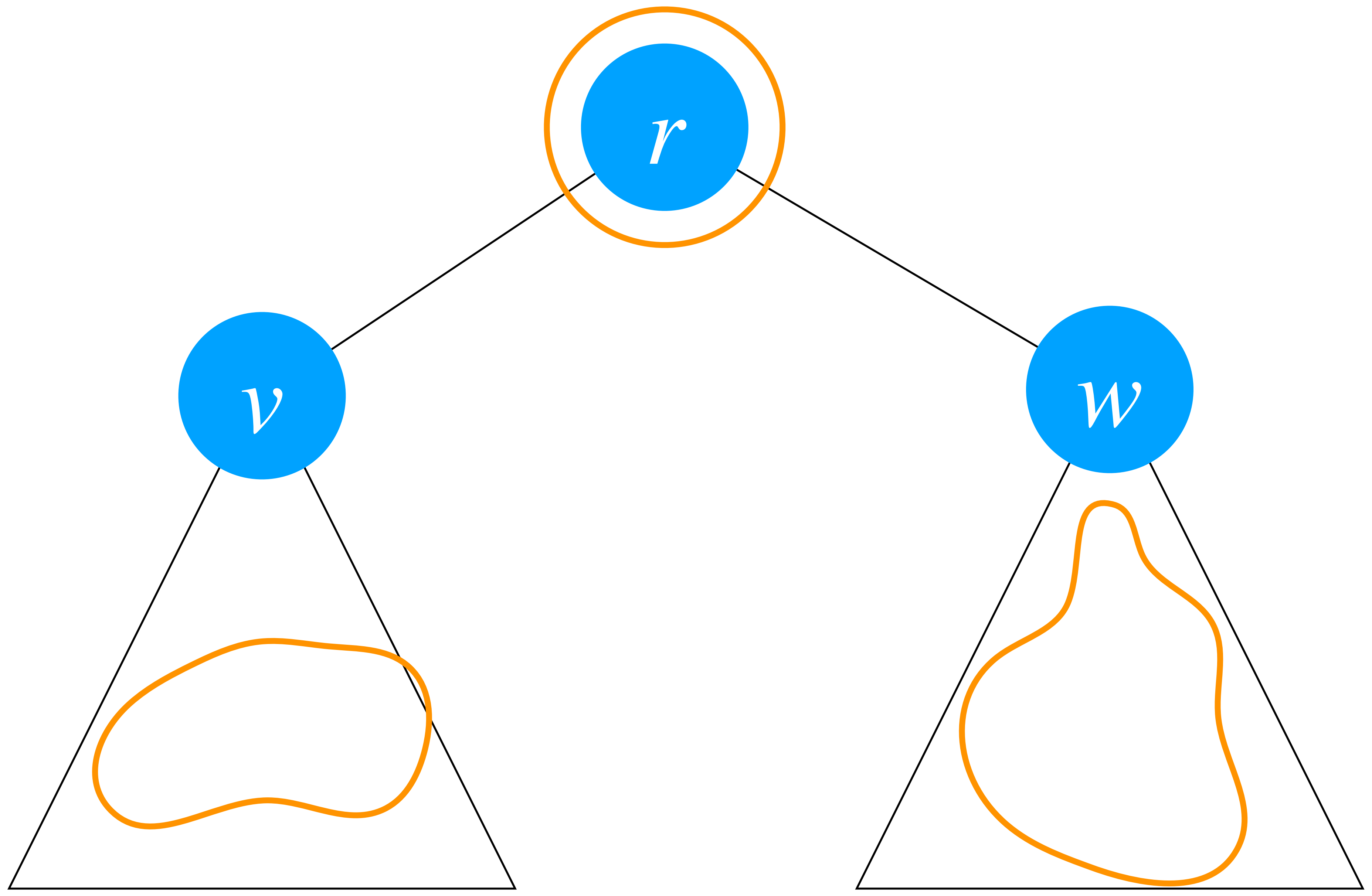

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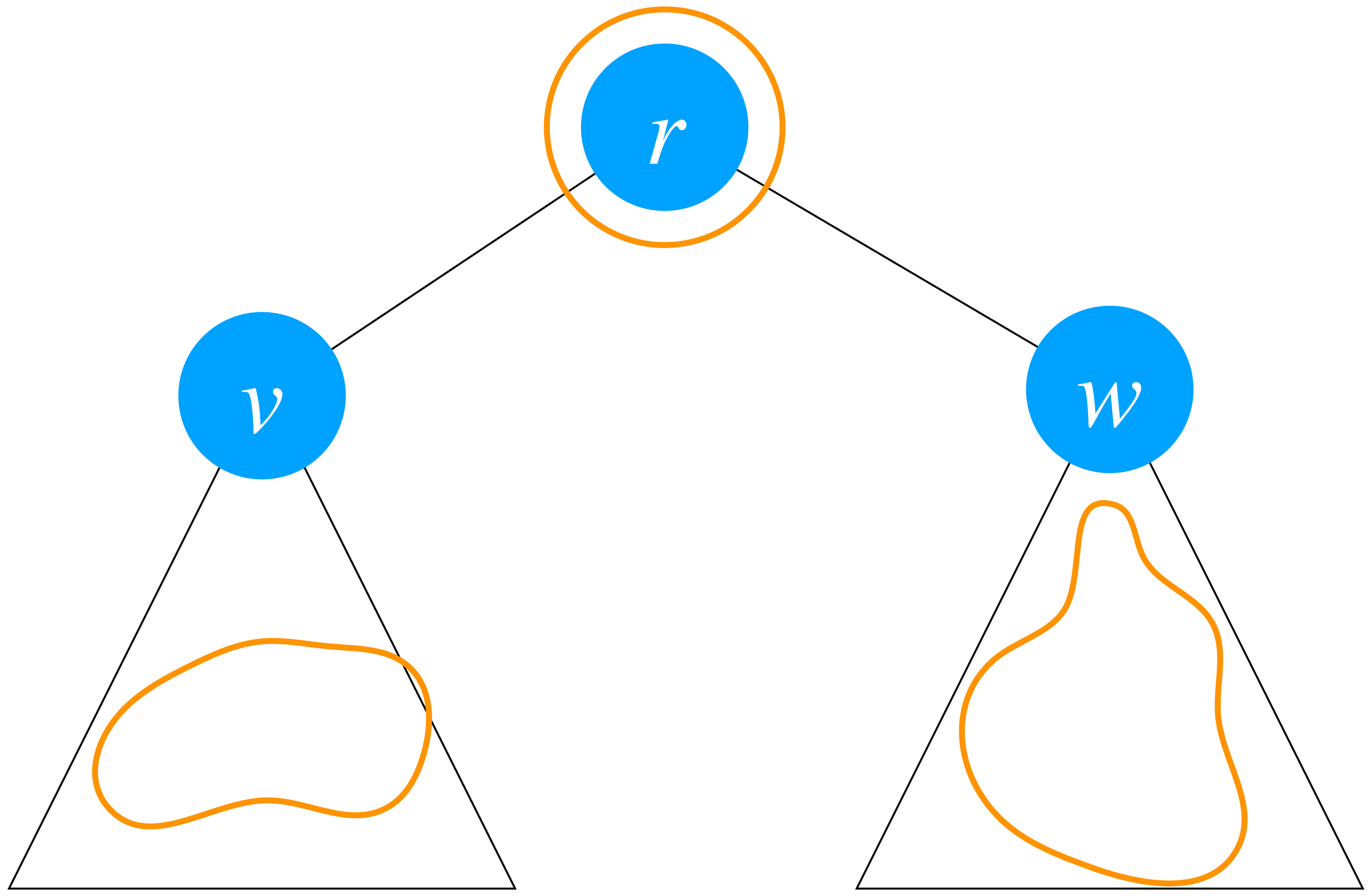

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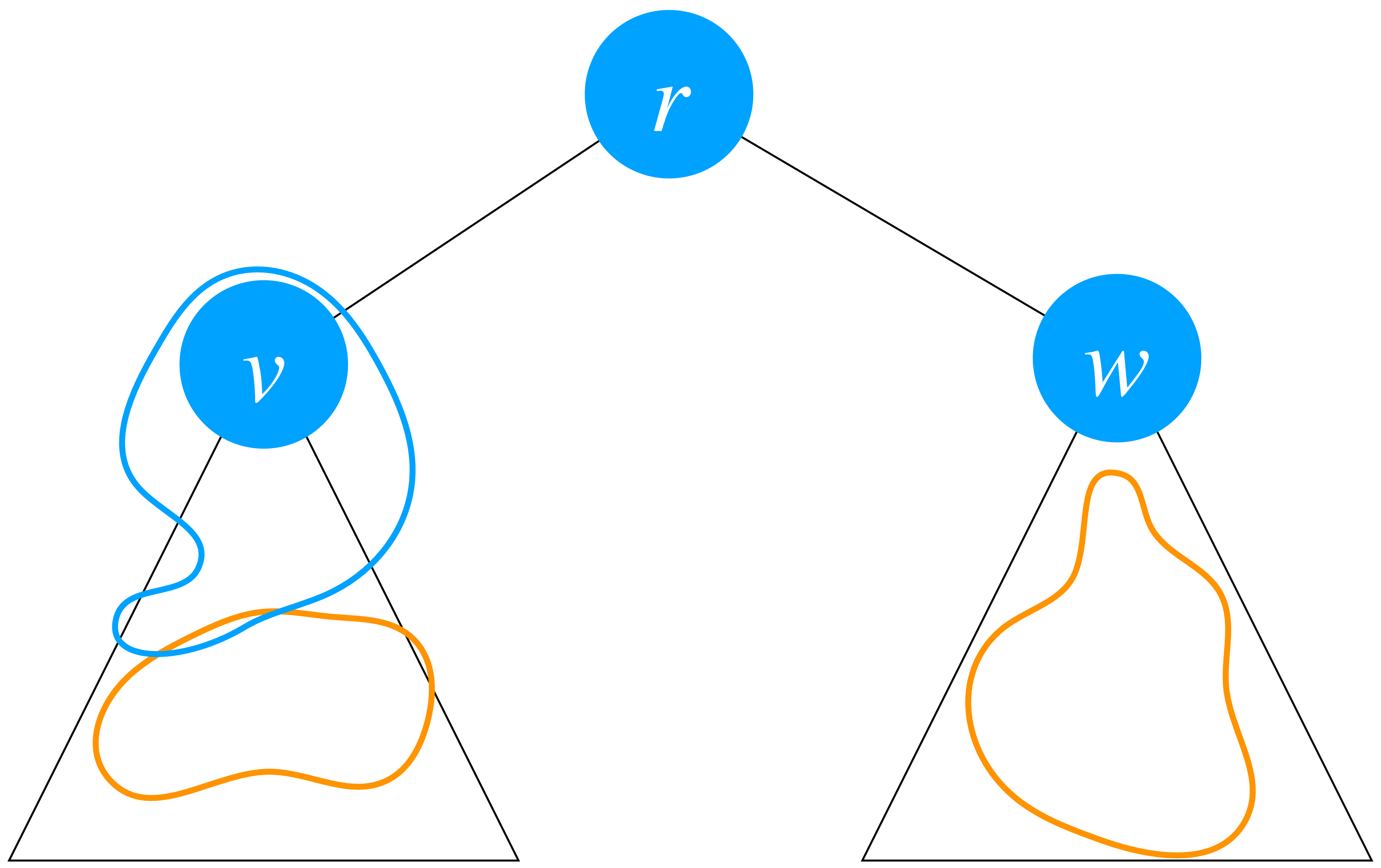

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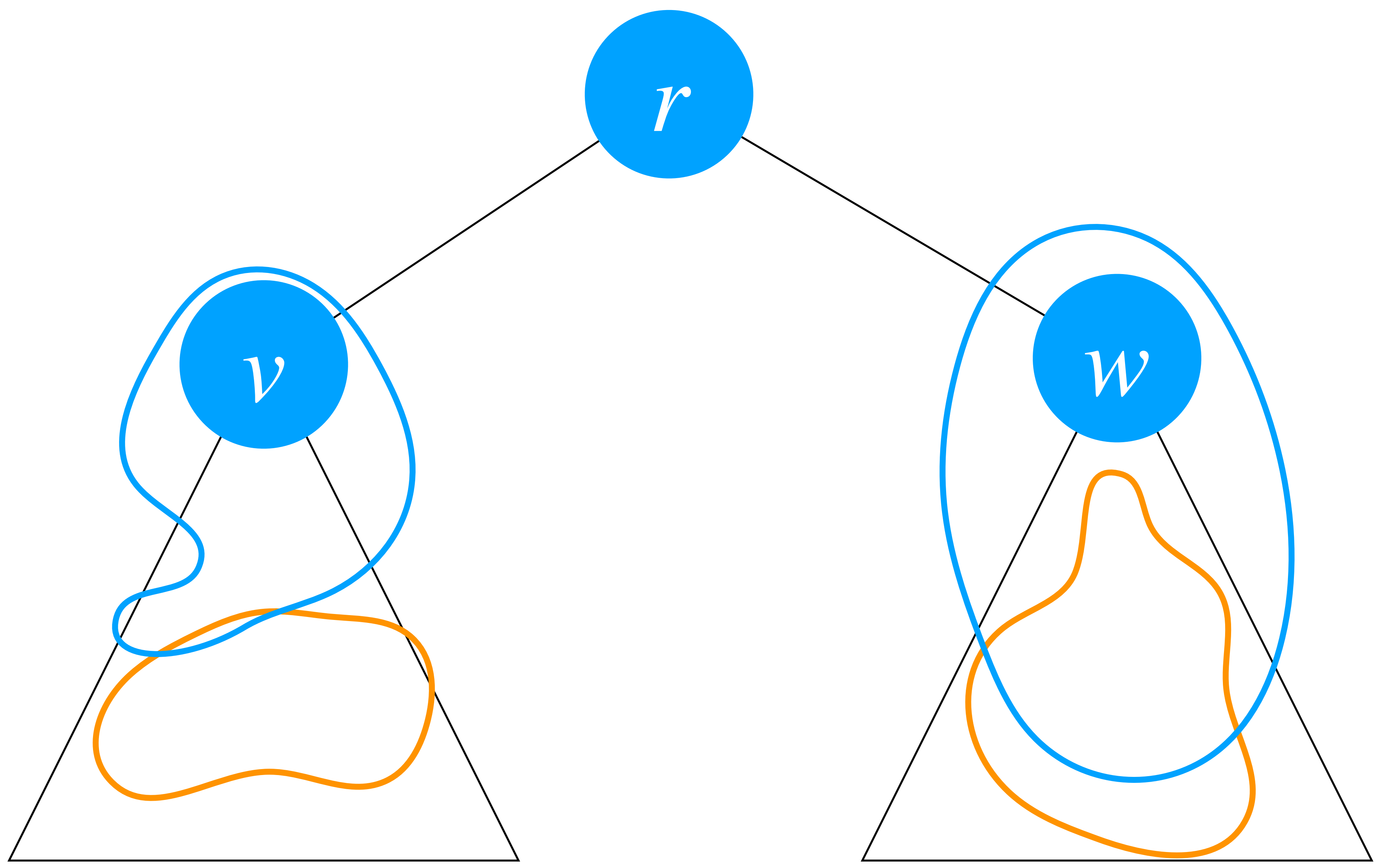

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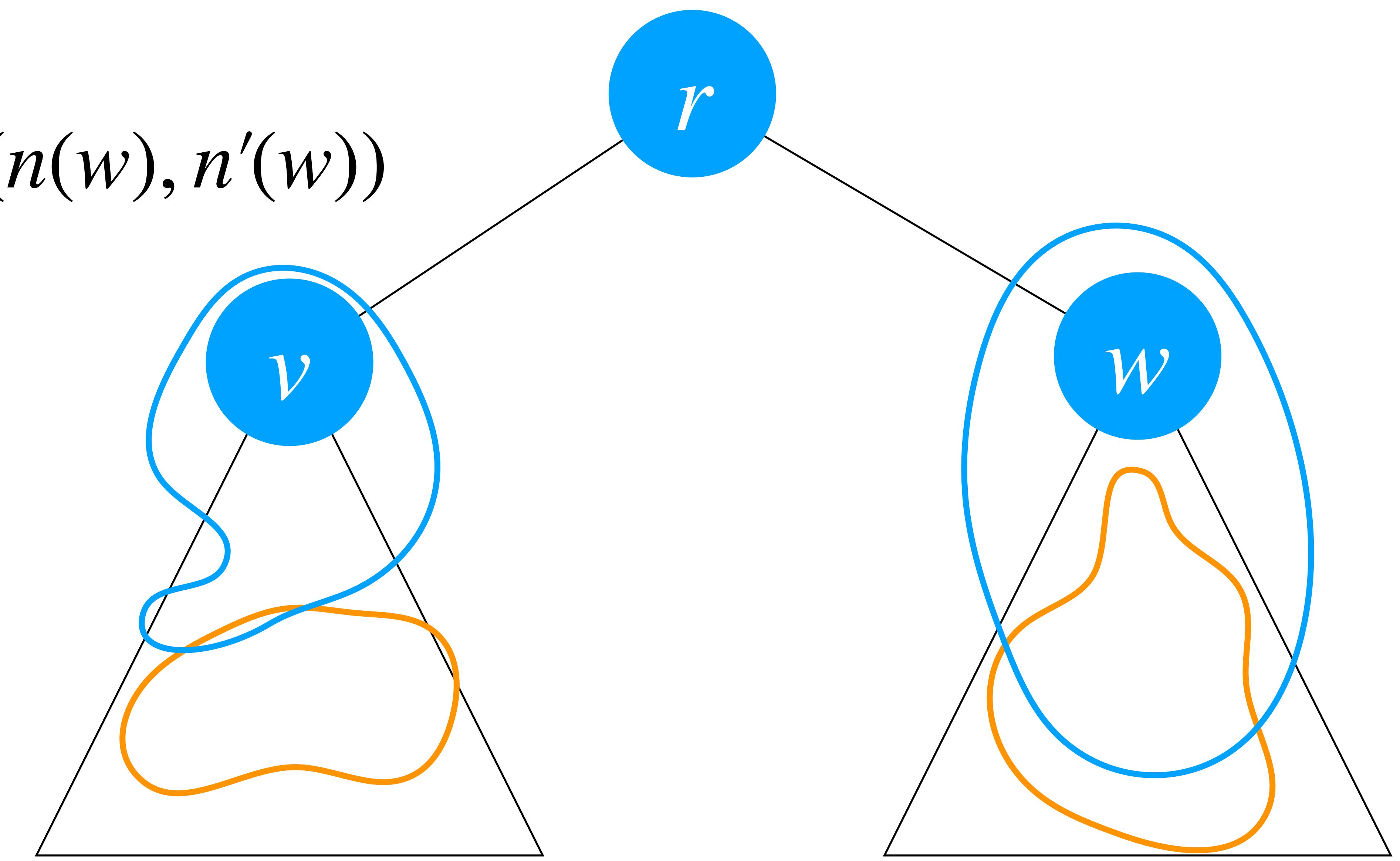

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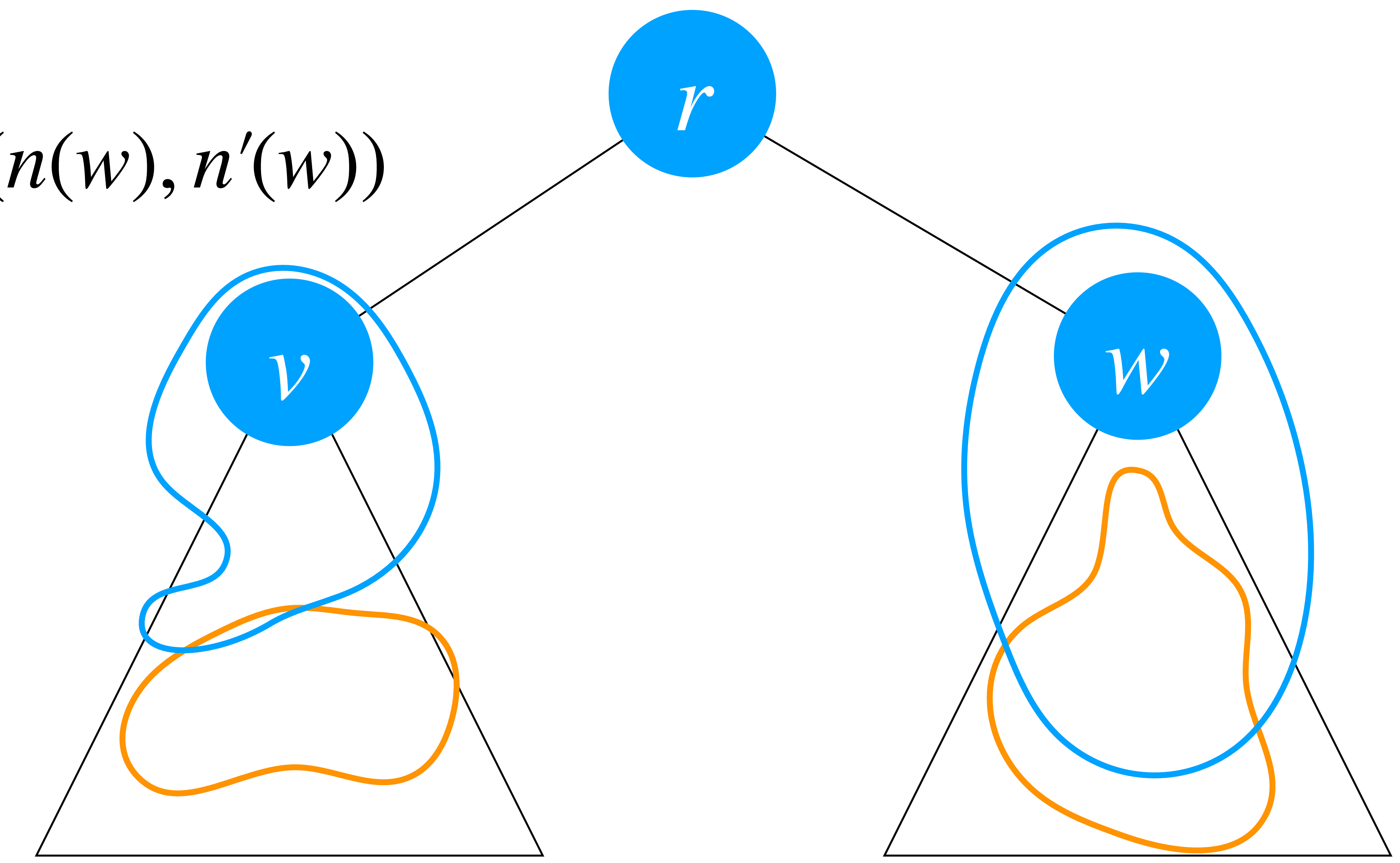
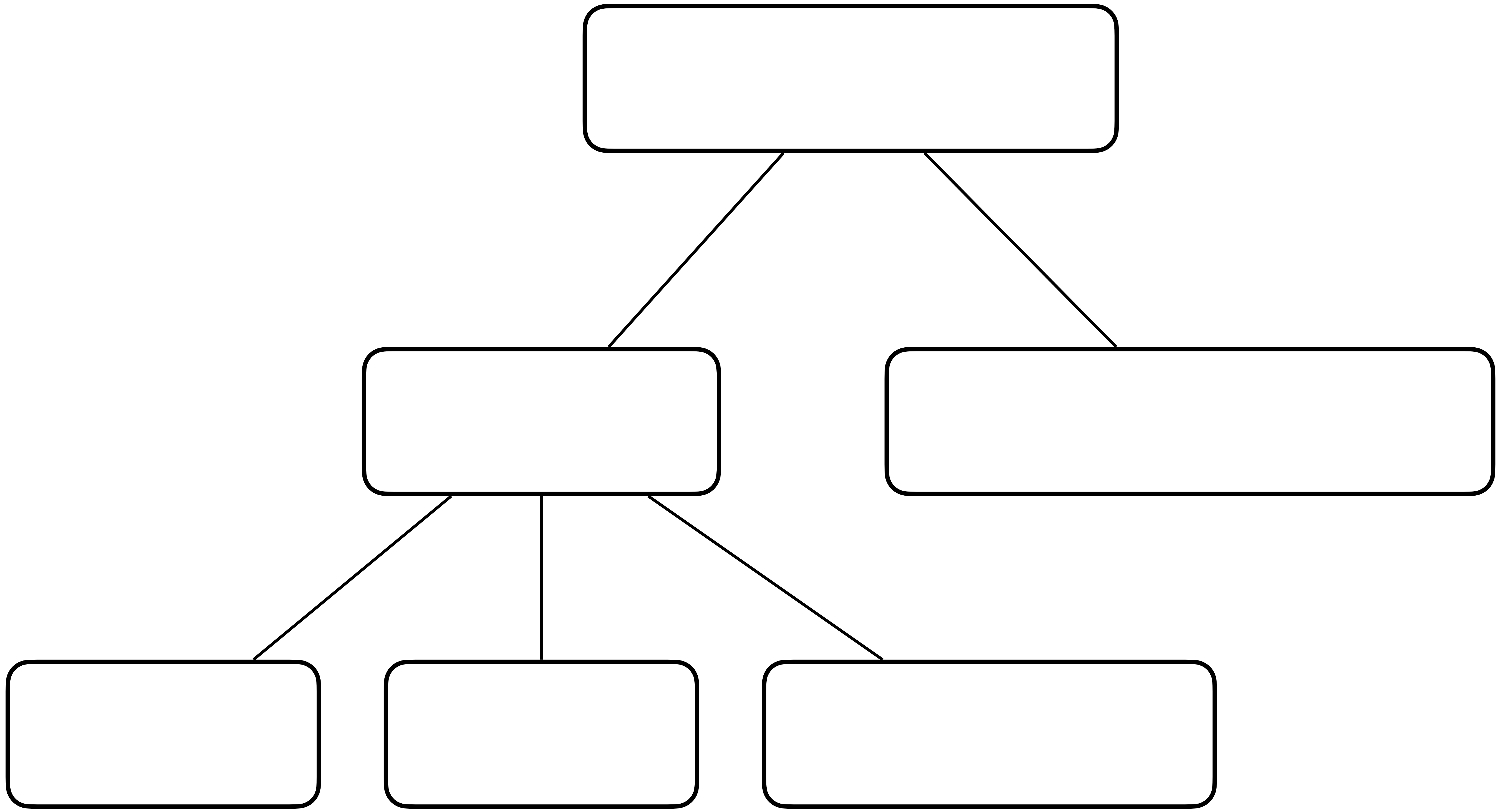


$\max(n(r), n'(r))$ is the size of the largest **IS**.

Tree Decomposition

Tree Decomposition



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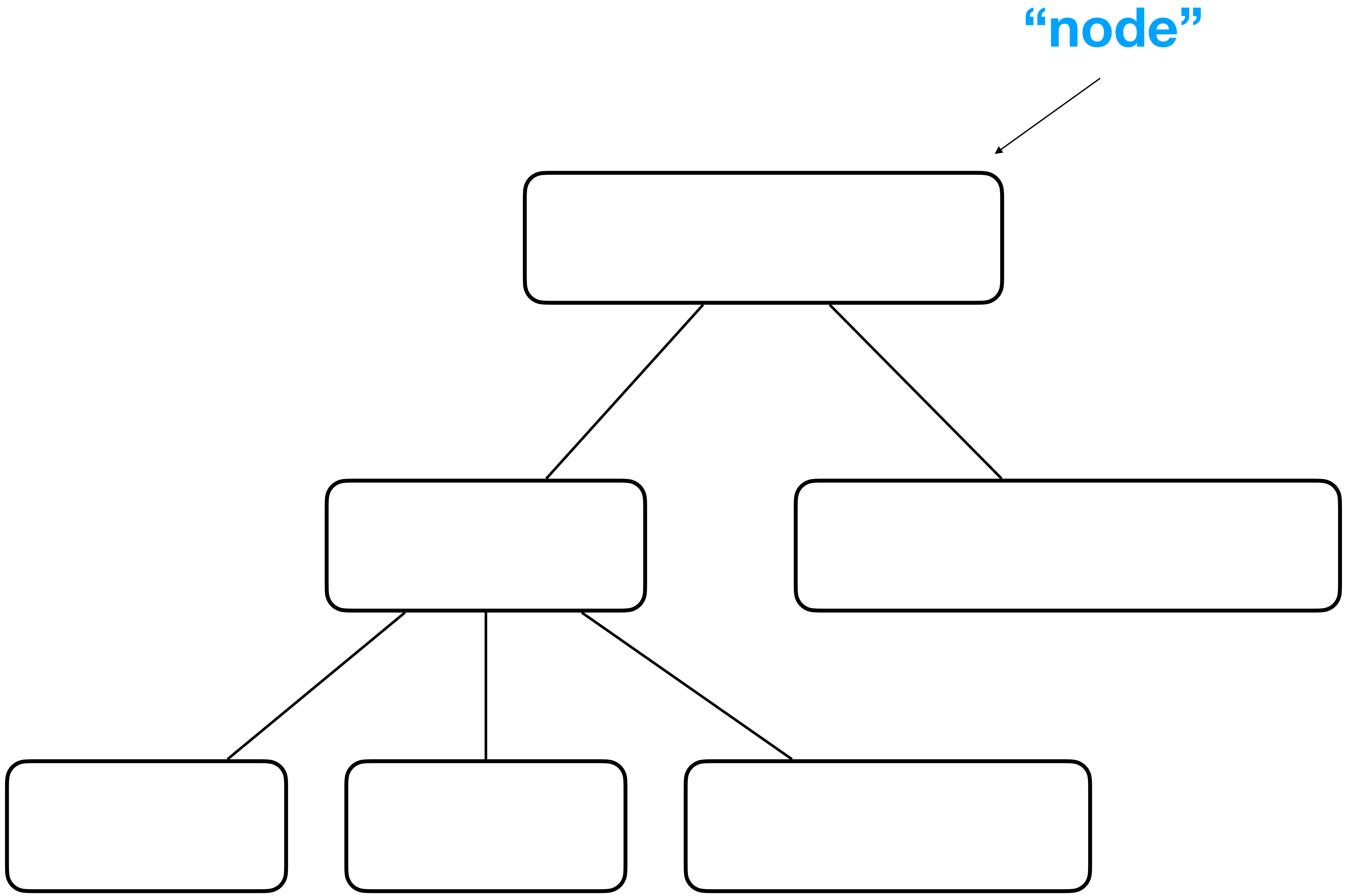

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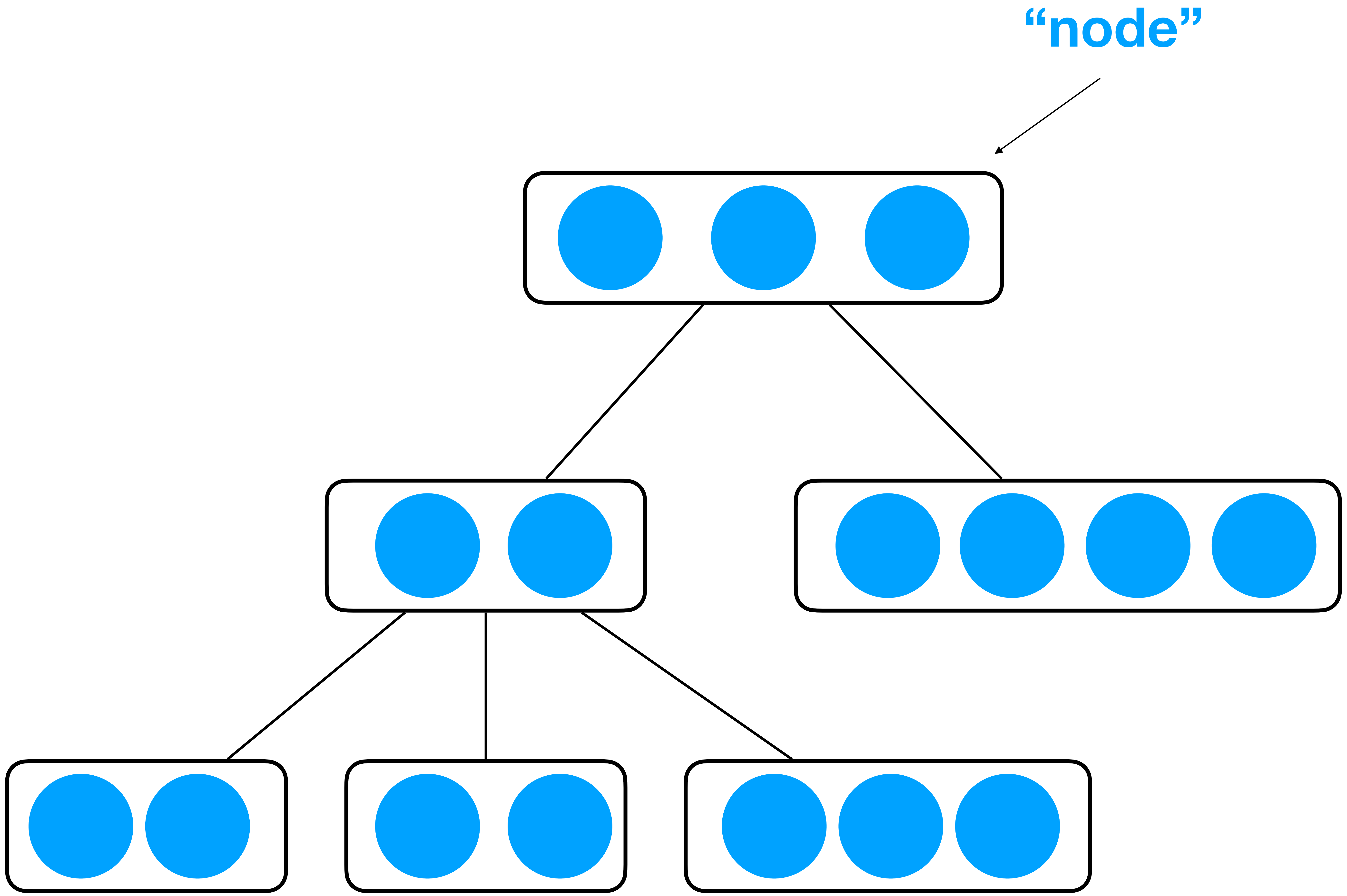

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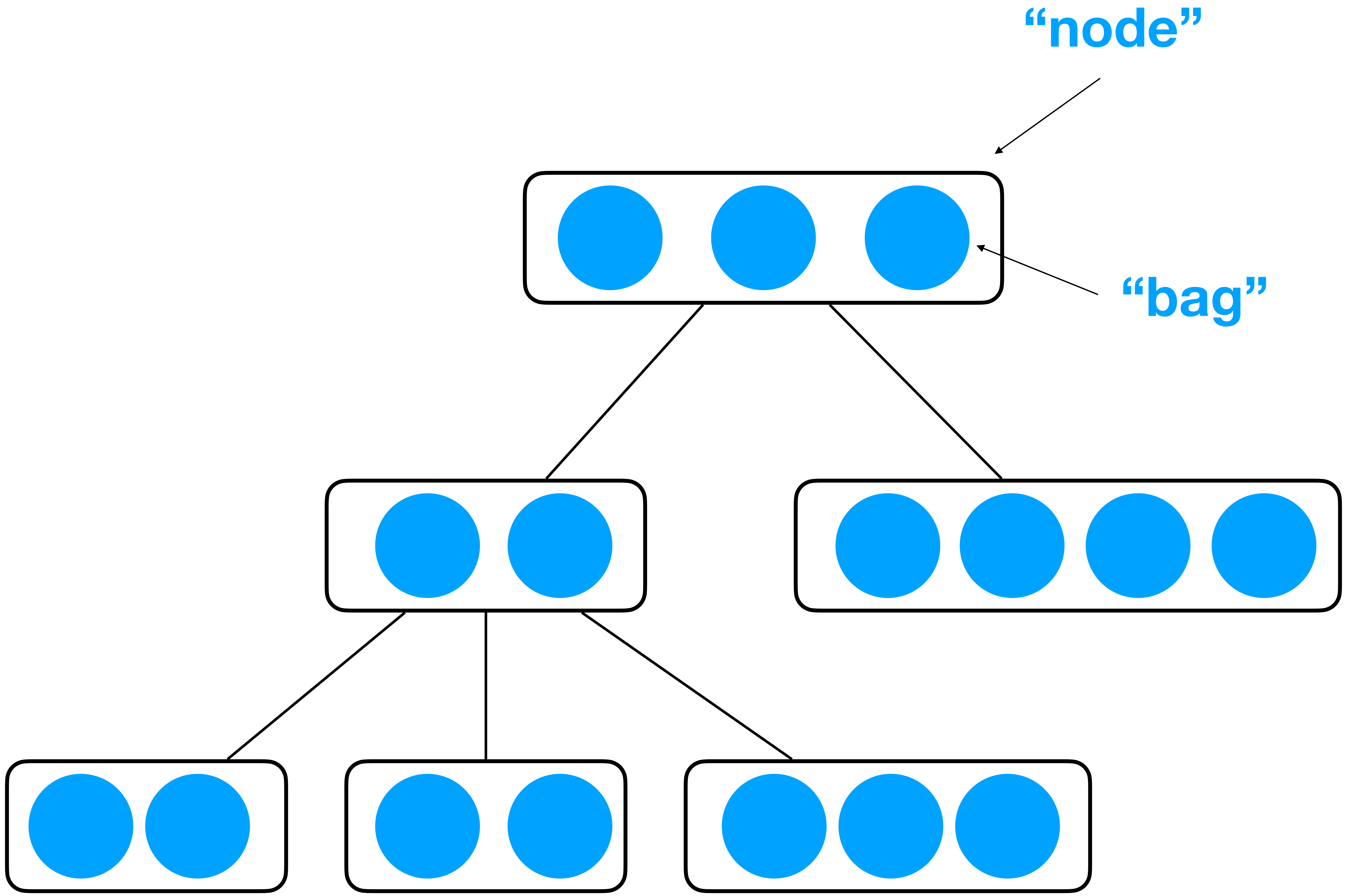
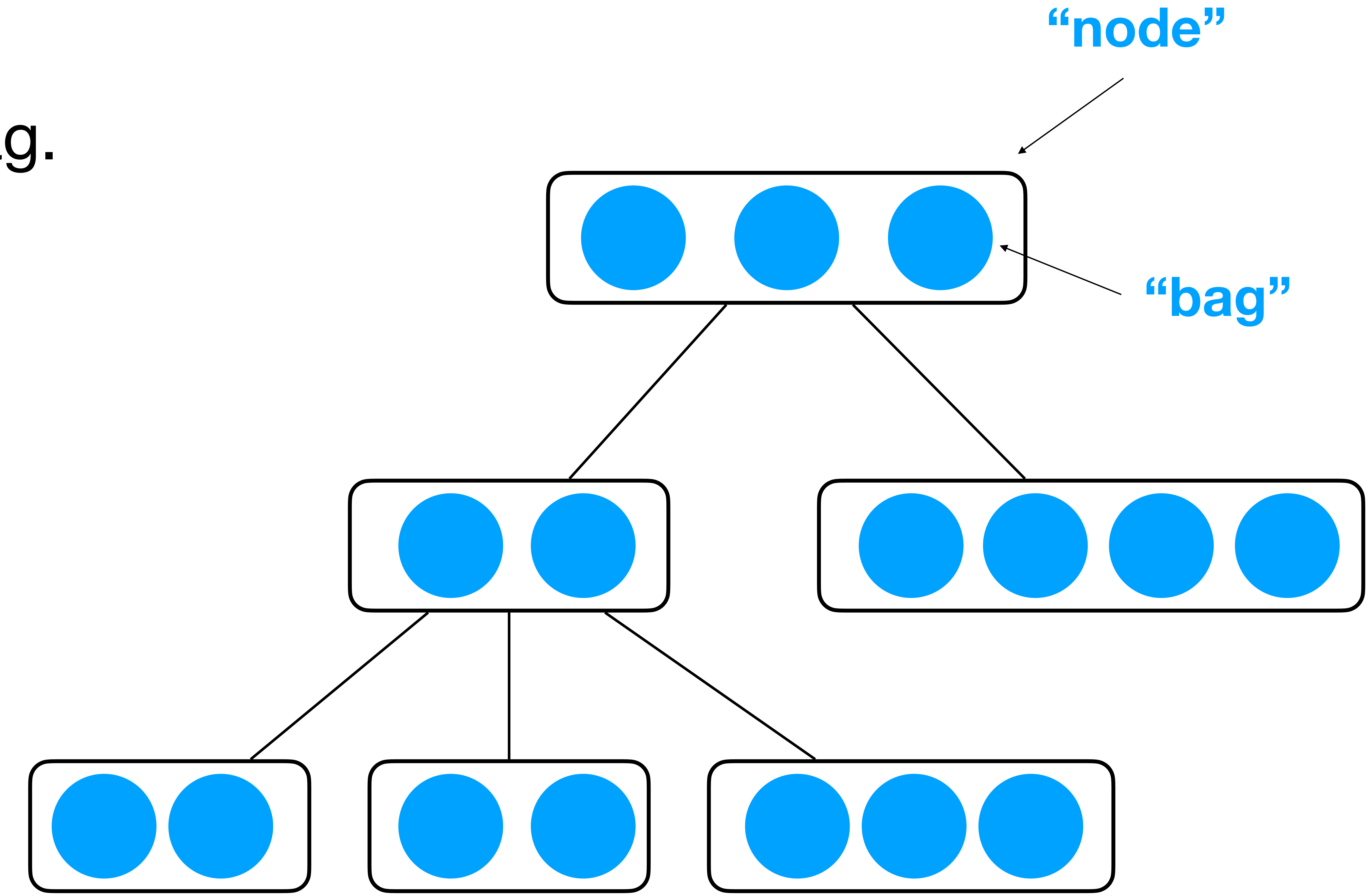


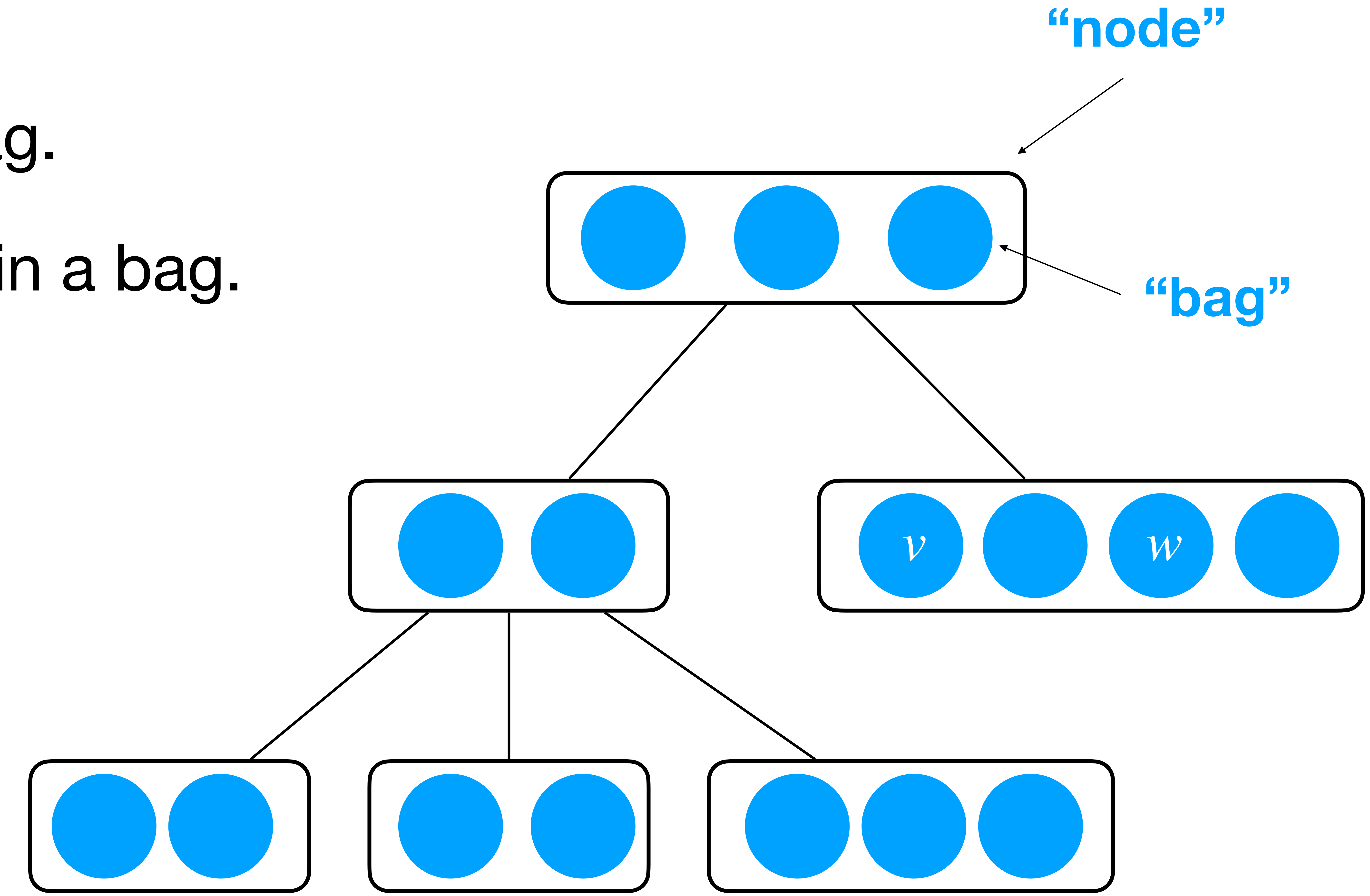
Tree Decomposition

1. Each **vertex** appears in a bag.



Tree Decomposition

1. Each **vertex** appears in a bag.
2. Each **edge** vw is contained in a bag.



Tree Decomposition

1. Each **vertex** appears in a bag.
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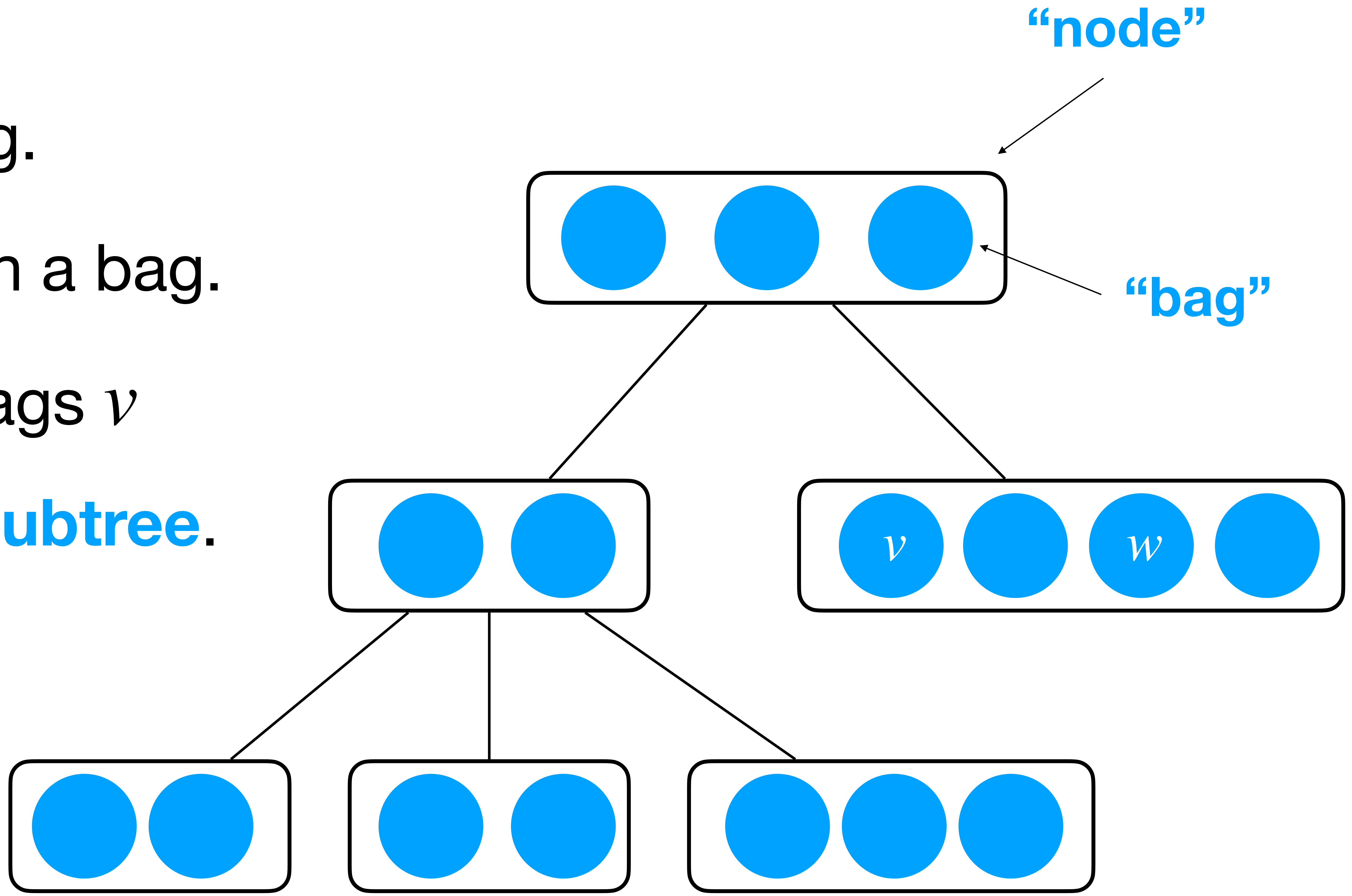
3. The set of nodes in whose bags v appears form a **connected subtree**.



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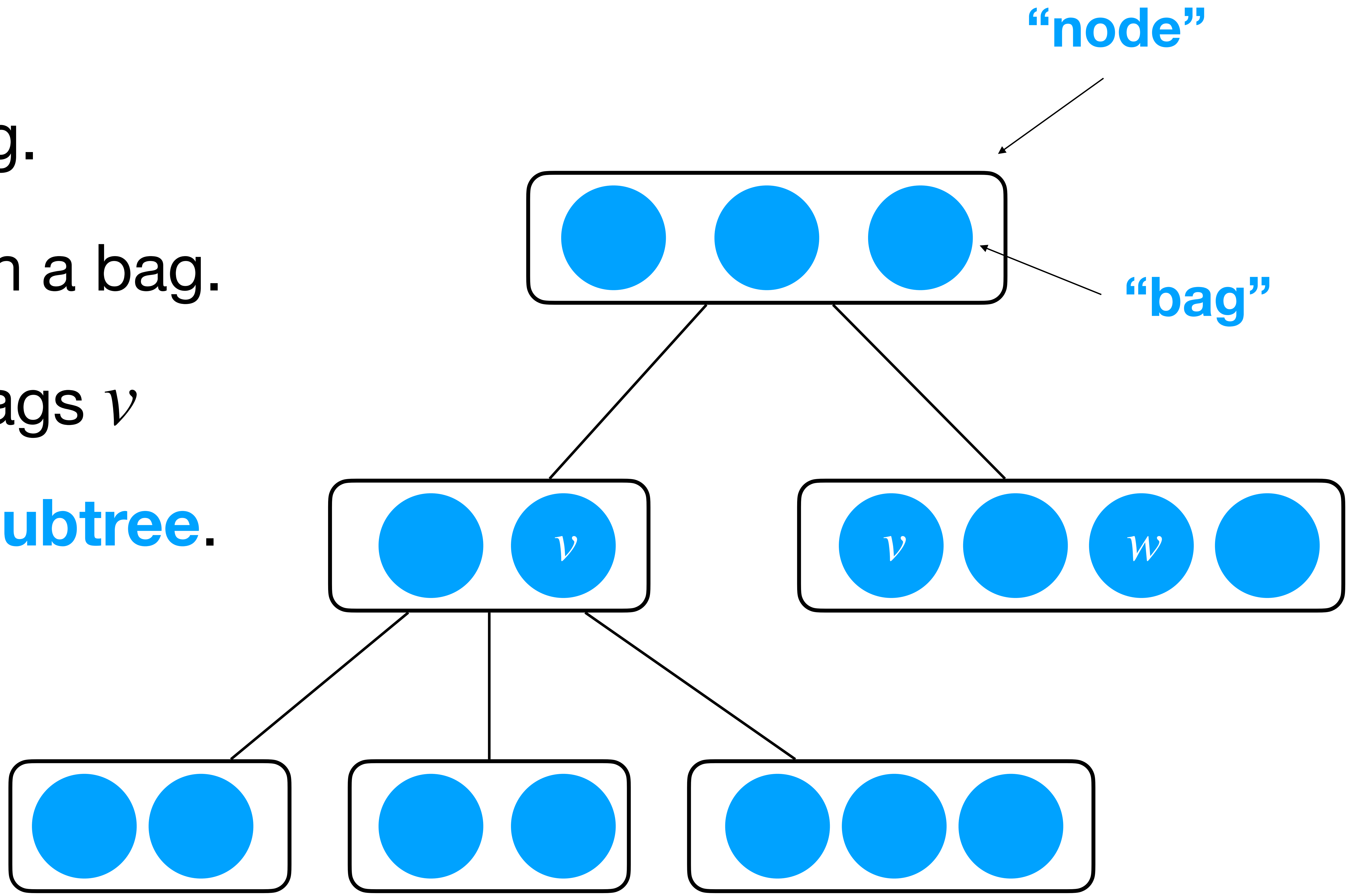

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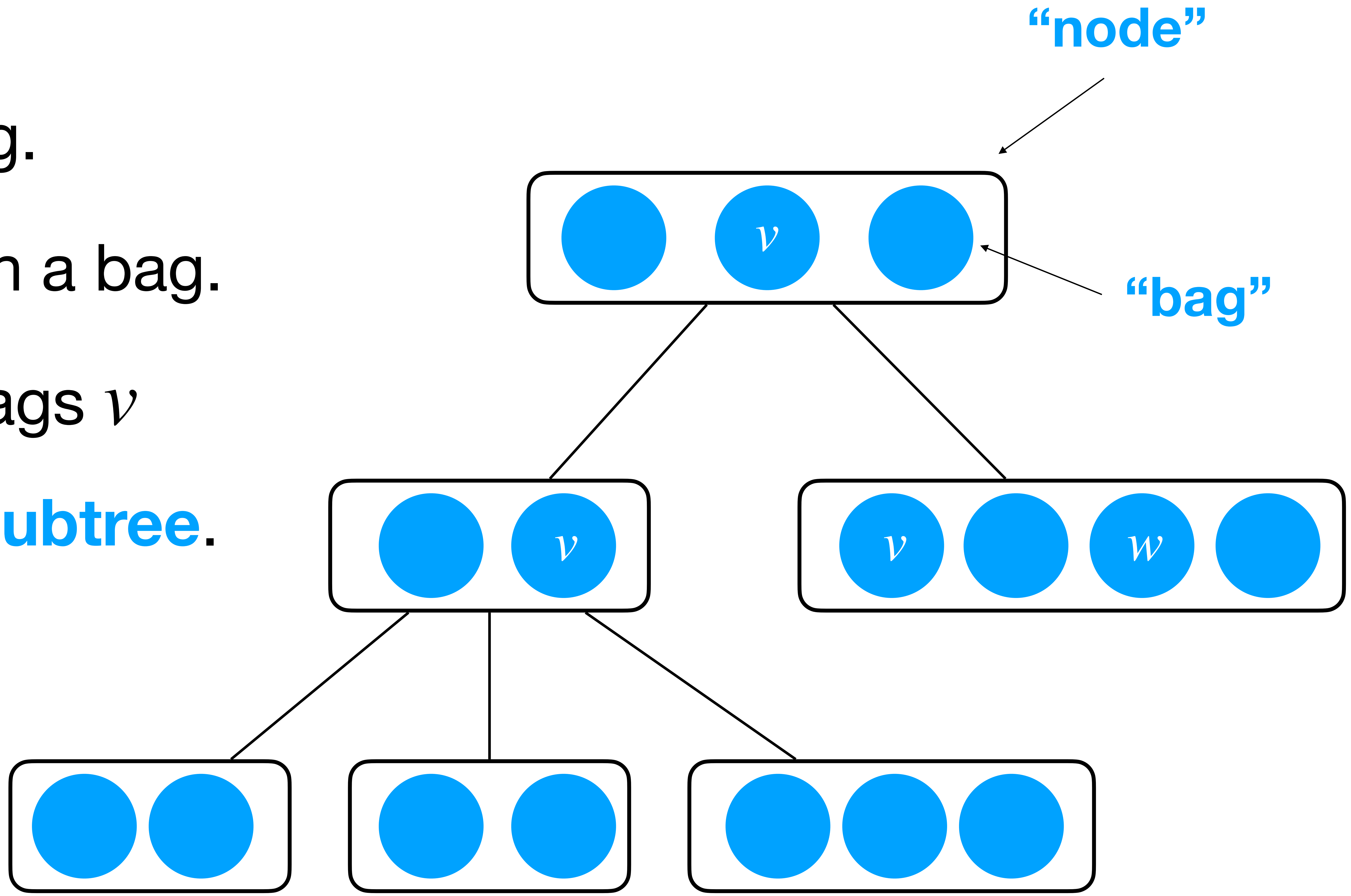

Tree Decomposition

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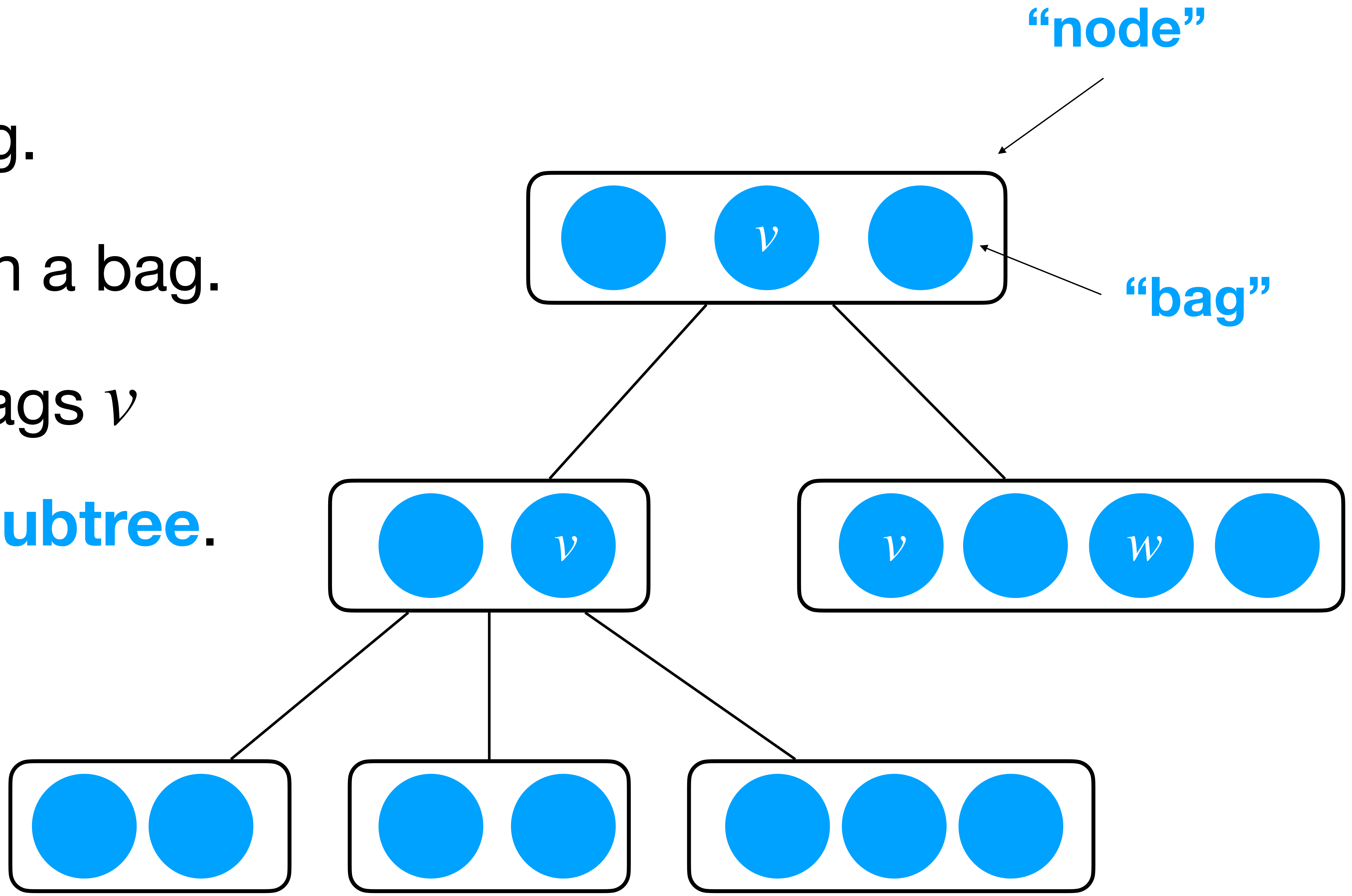
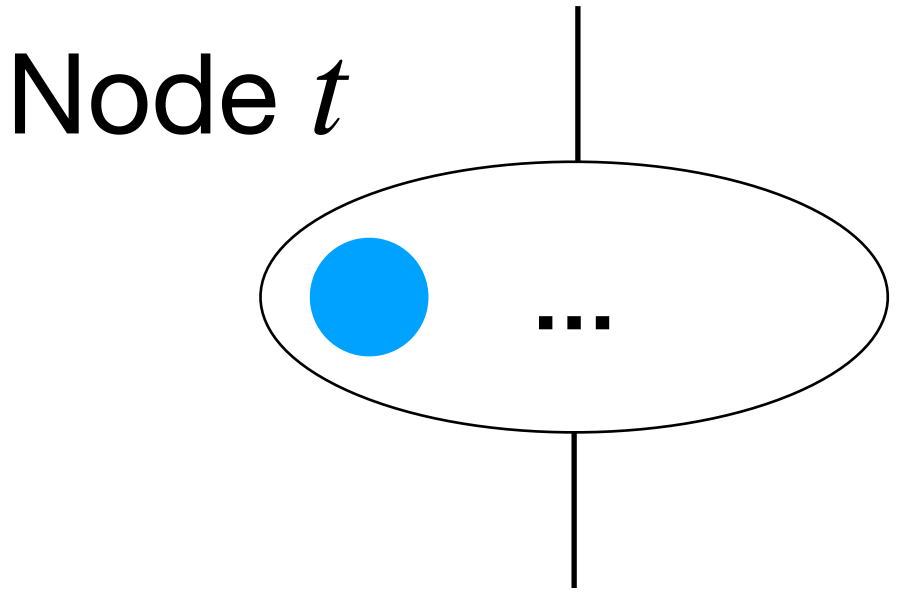
The **width** of a tree decomposition is the size of its largest bag - 1.



Independent Set with Tree Decompositions



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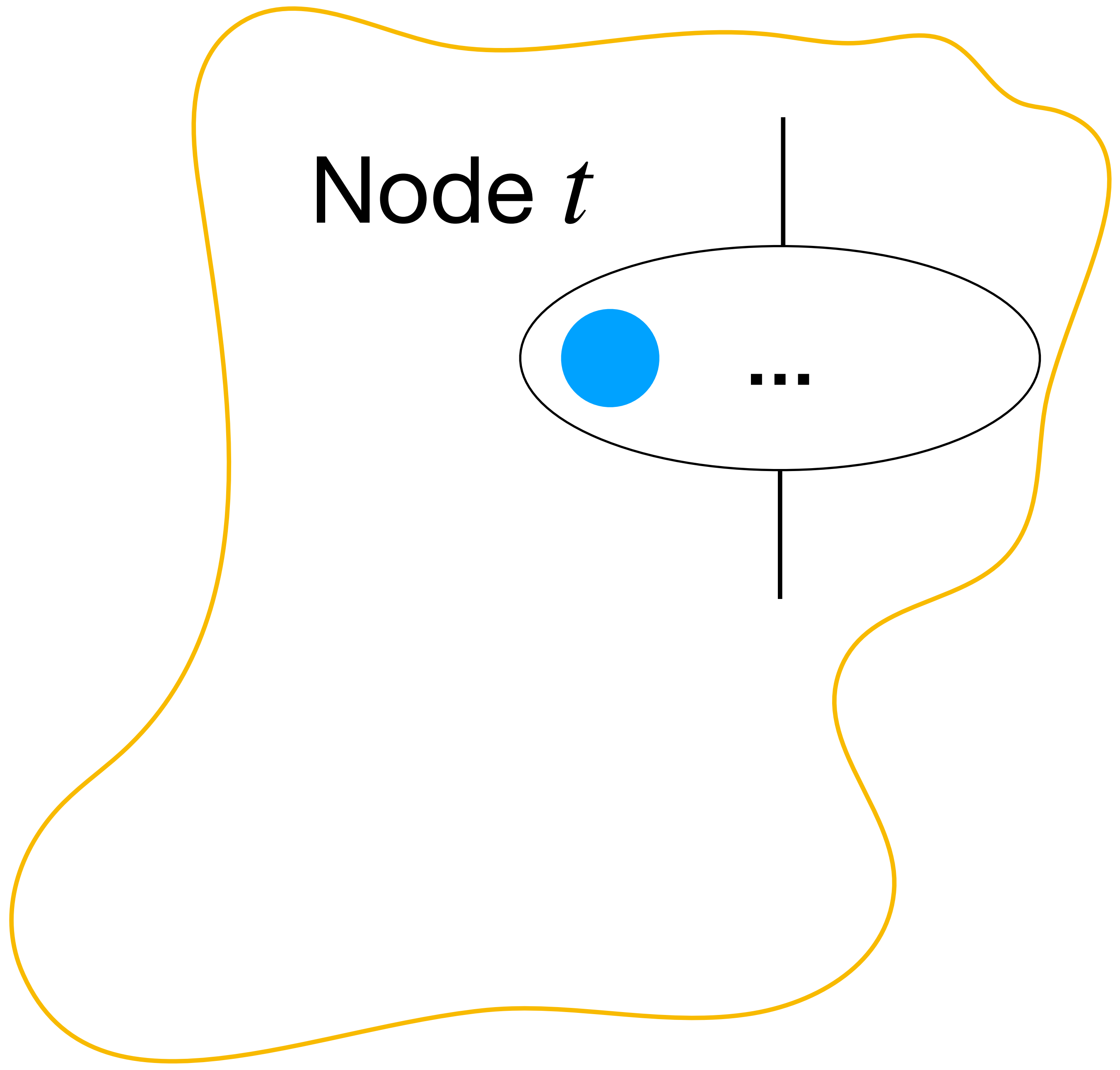

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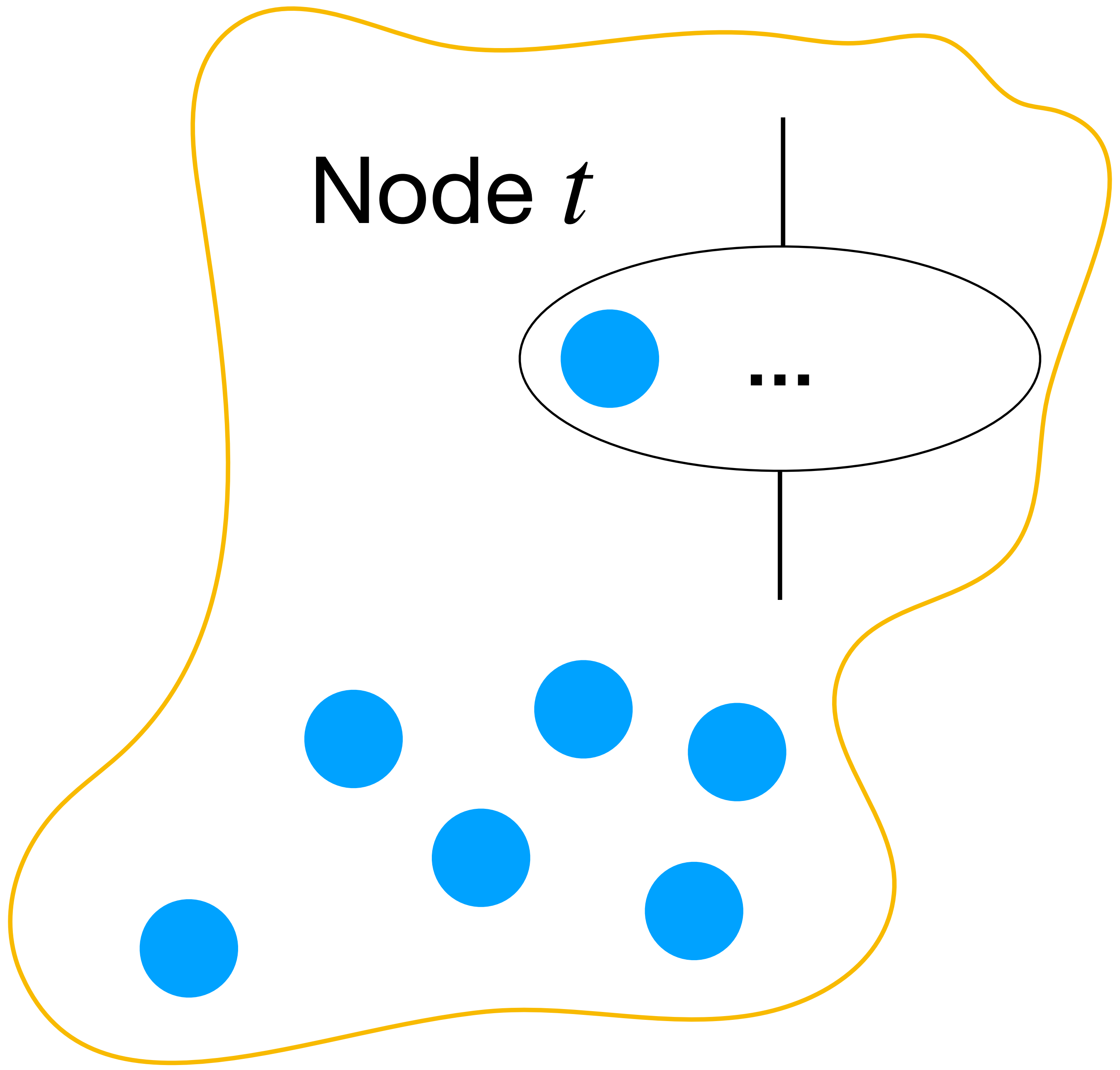

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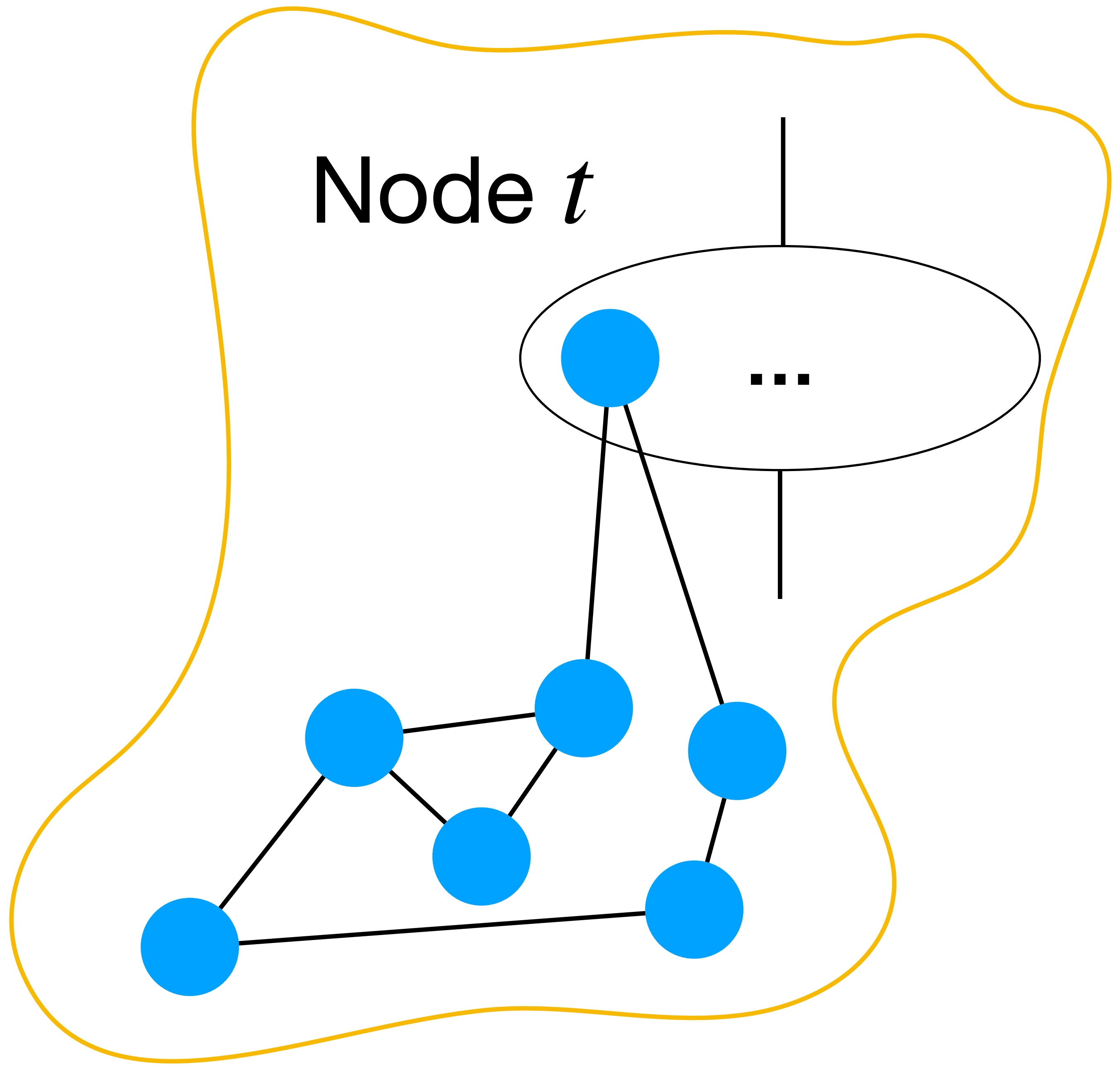

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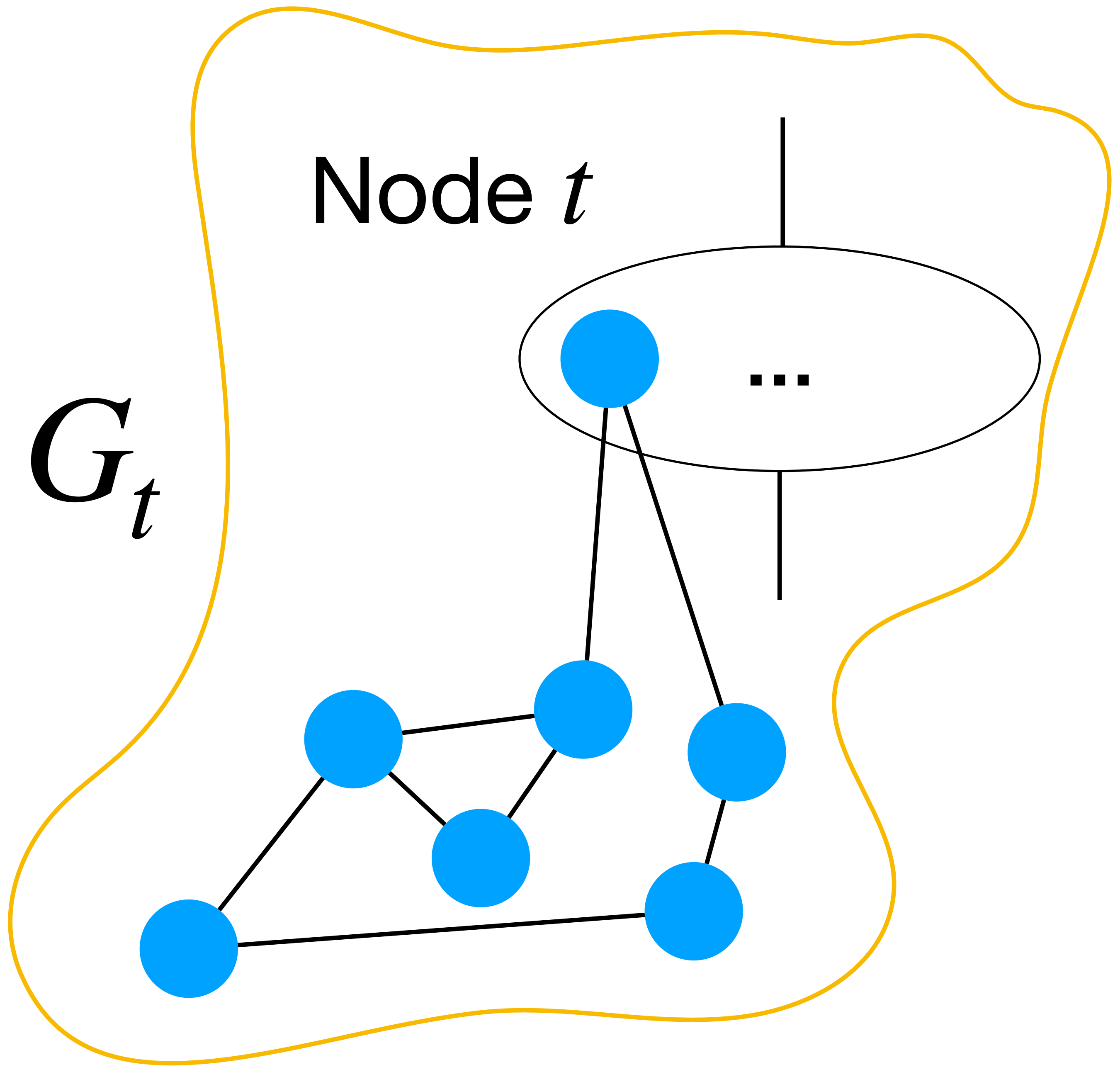

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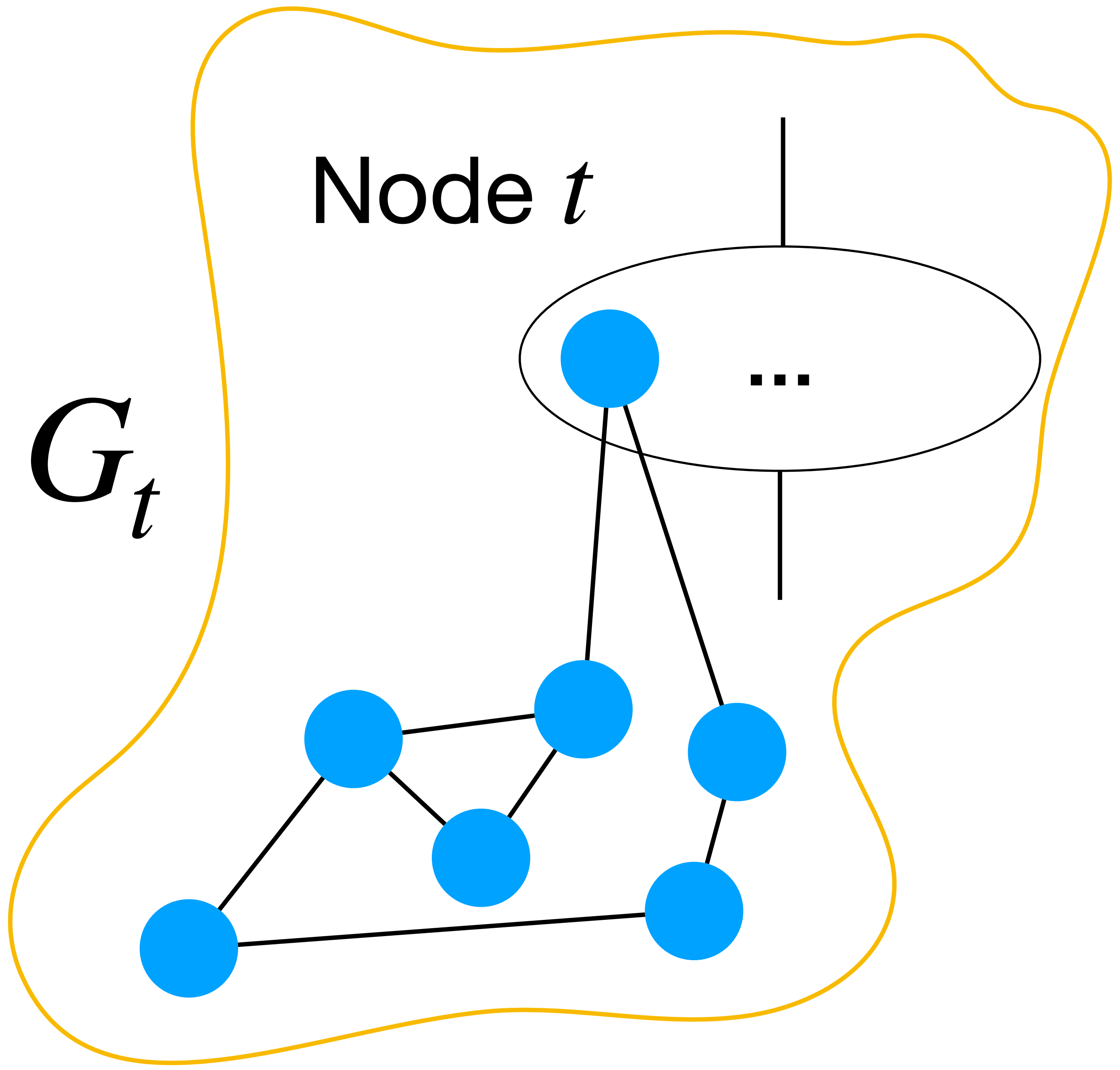
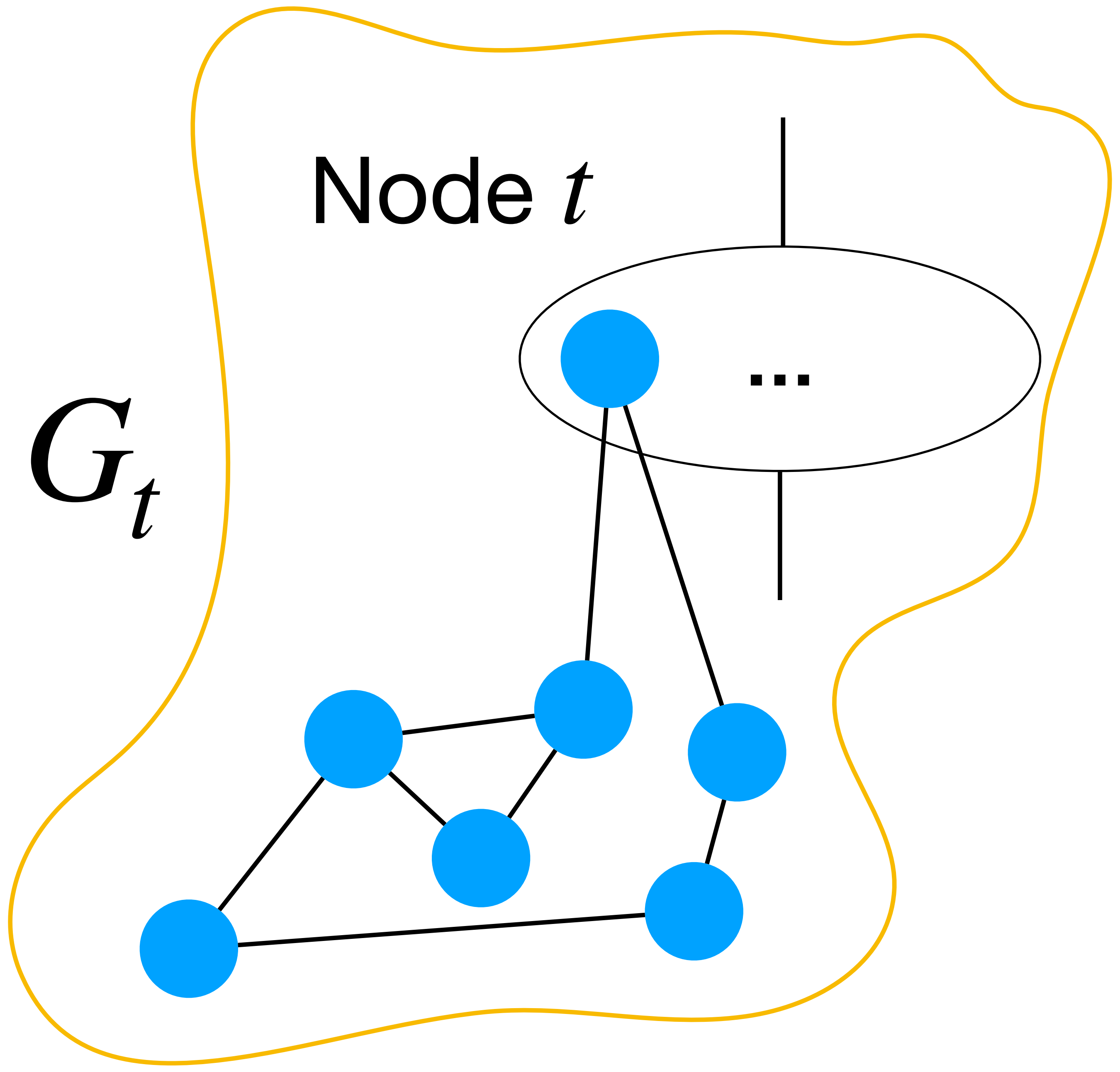


For each subset $S \subseteq \chi(t)$ of bag vertices,
$n_t(S)$ = | max. **IS** of G_t containing S |.

Independent Set with Tree Decompositions



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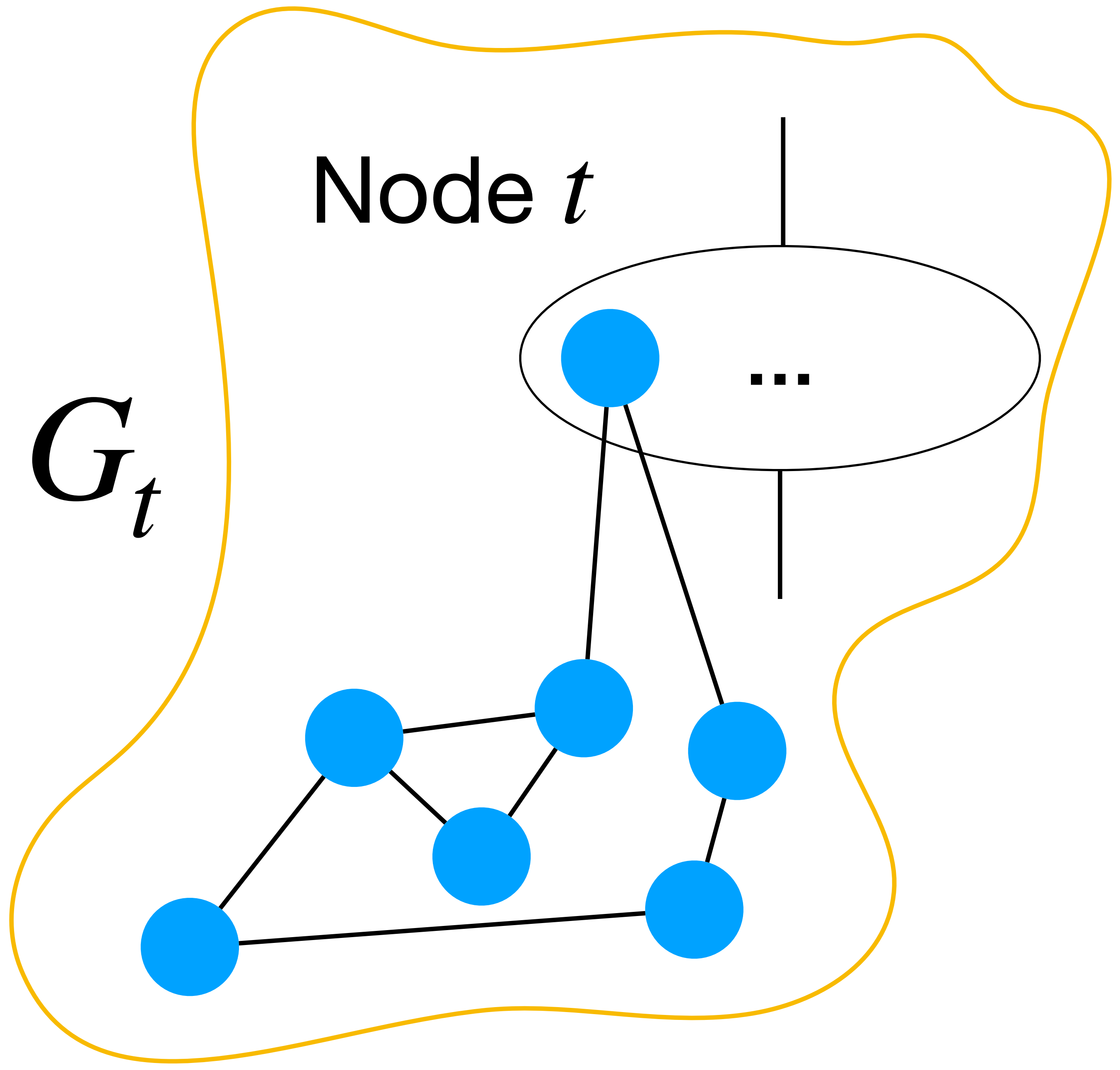

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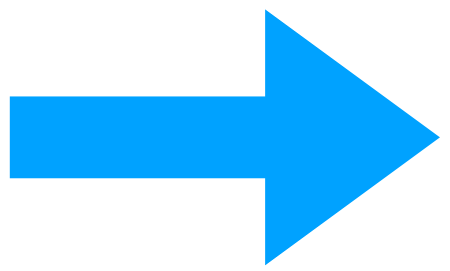
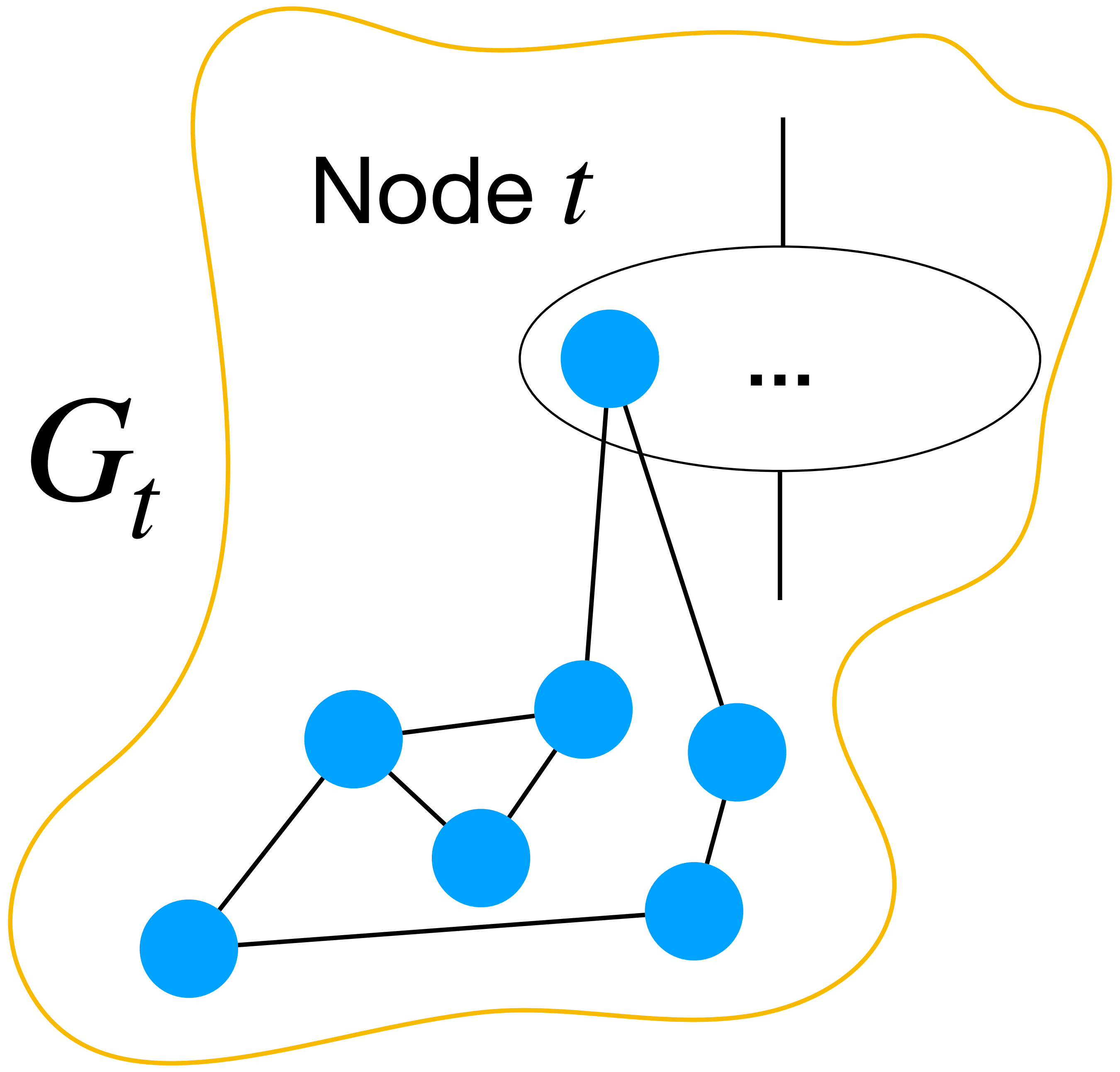
Treewidth k



Independent Set with Tree Decompositions

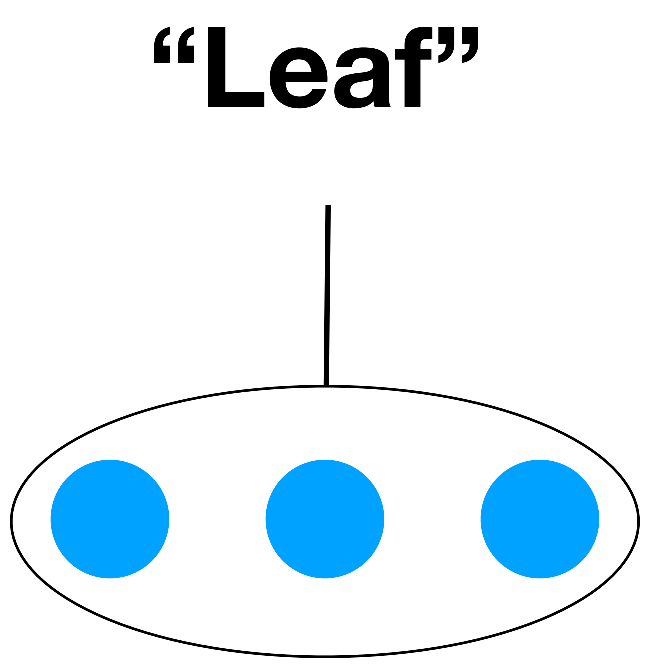


For each subset $S \subseteq \chi(t)$ of bag vertices,
$n_t(S)$ = | max. **IS** of G_t containing S |.

Treewidth k ➡ $\leq 2^{k+1}$ subsets

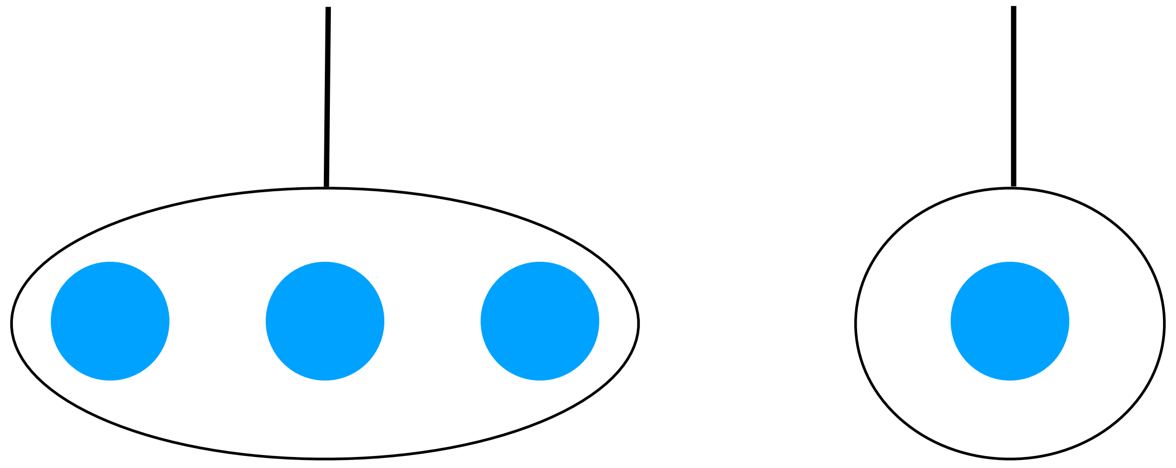
Nice Tree Decompositions

Nice Tree Decompositions



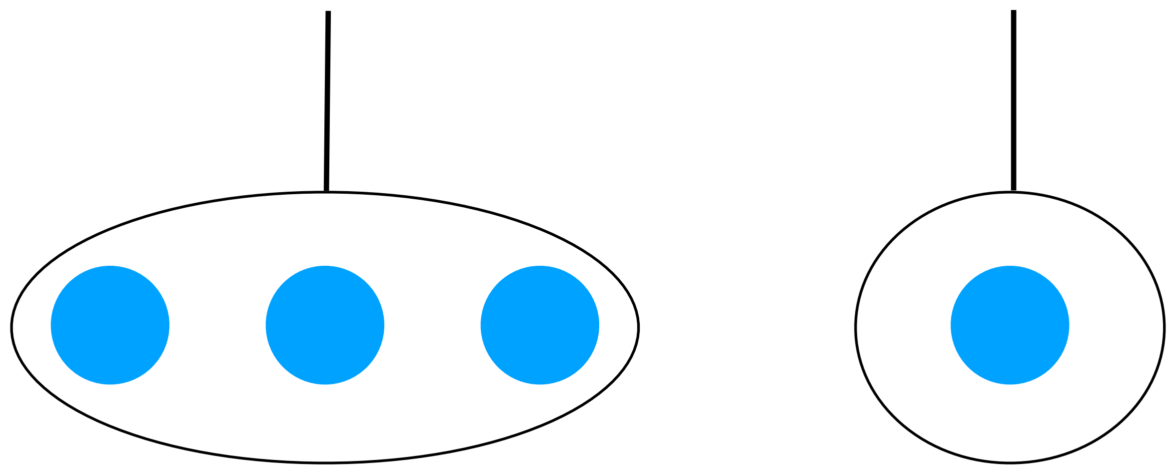
Nice Tree Decompositions

“Leaf”

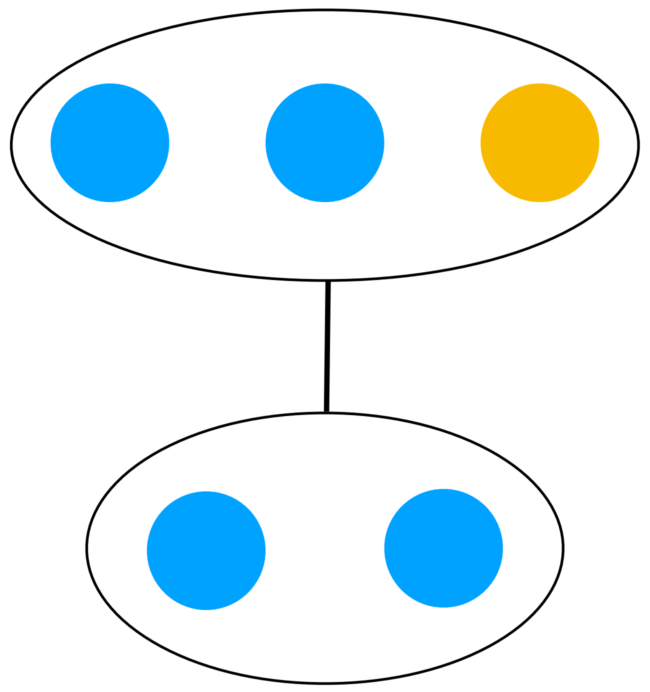


Nice Tree Decompositions

“Leaf”

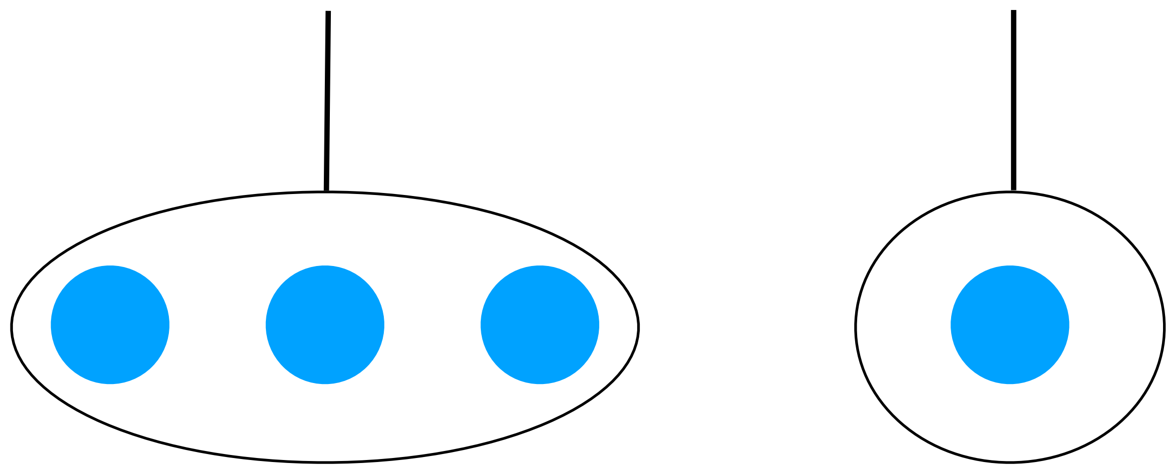


“Introduce”

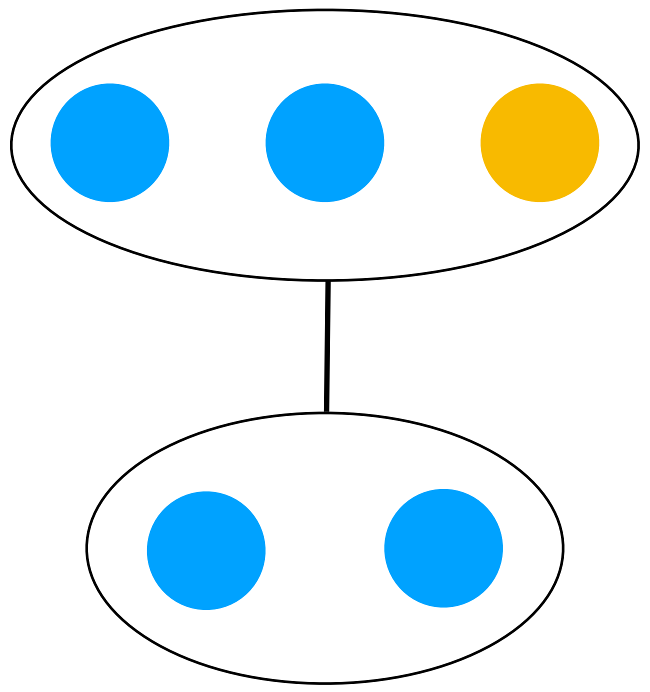


Nice Tree Decompositions

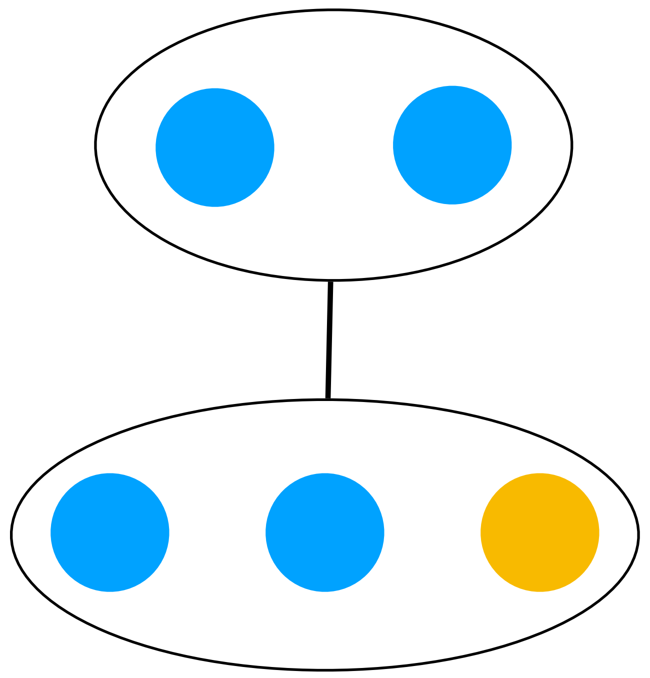
“Leaf”



“Introduce”



“Forget”



Nice Tree Decompositions

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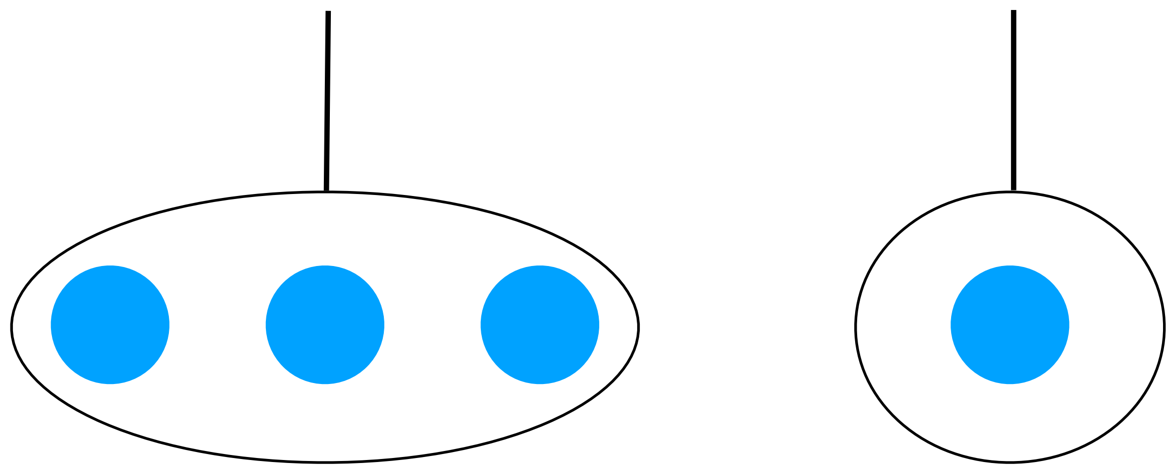

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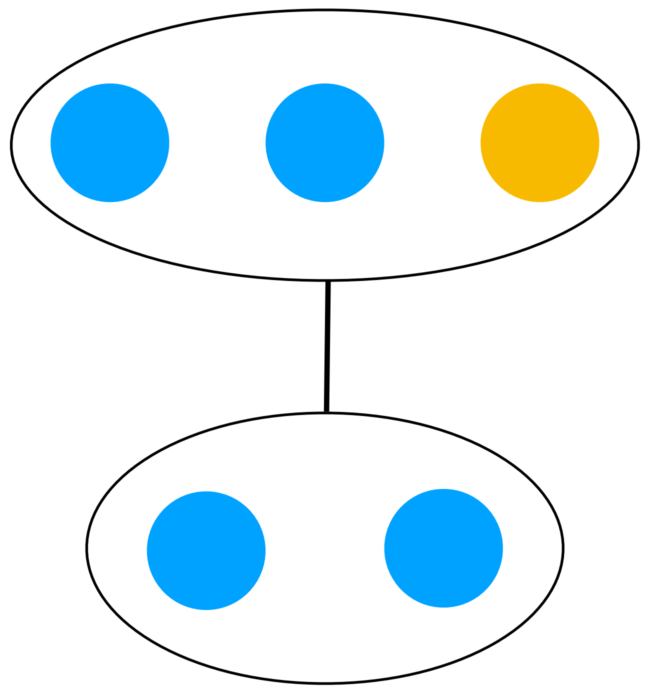

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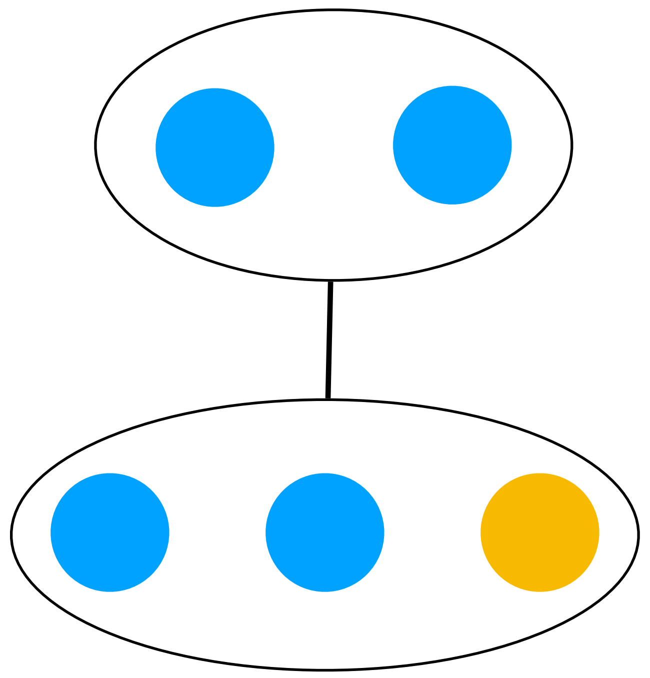
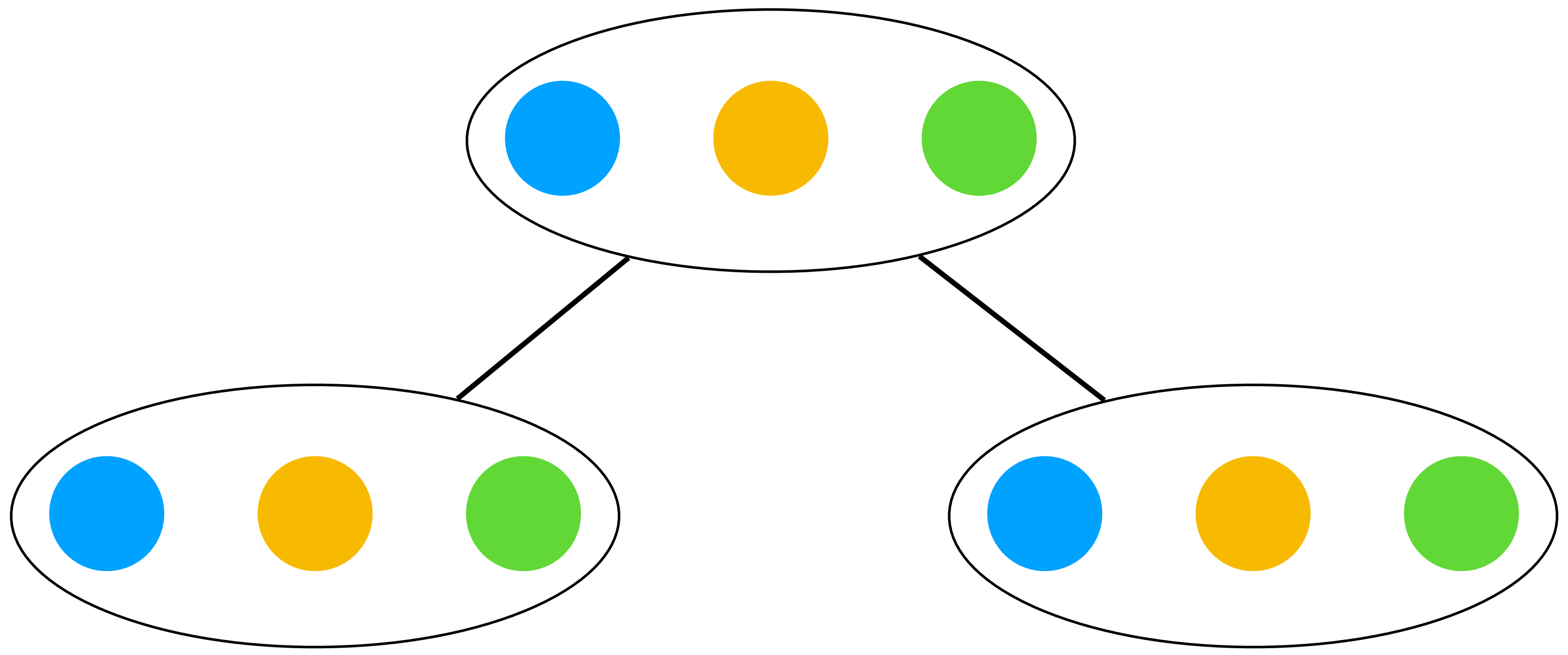
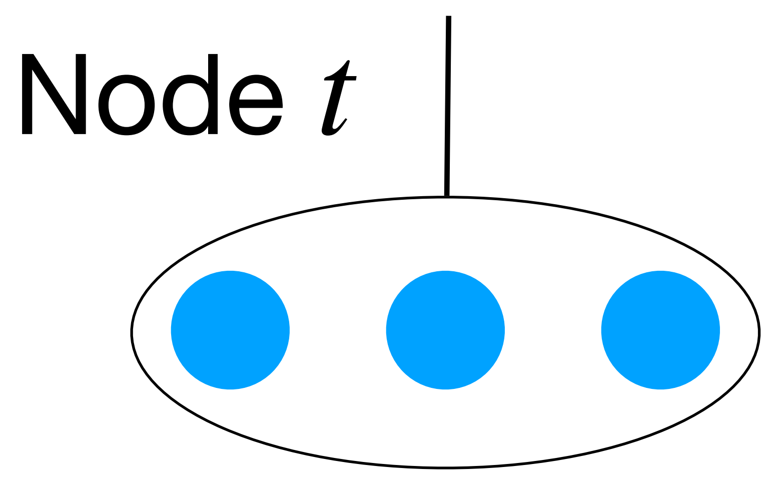


“Join”



Leaf Nodes



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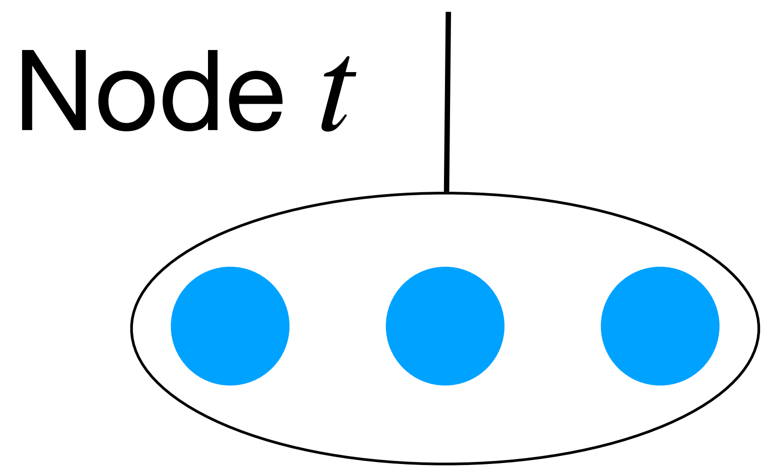
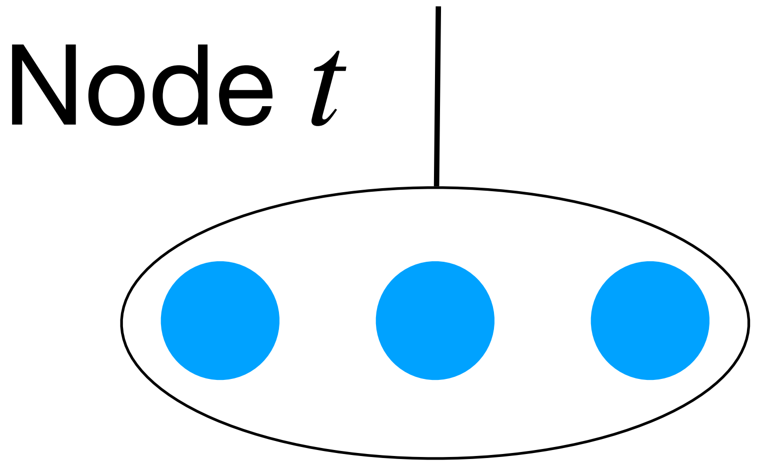


for $S \subseteq \chi(t)$**:**

Leaf Nodes

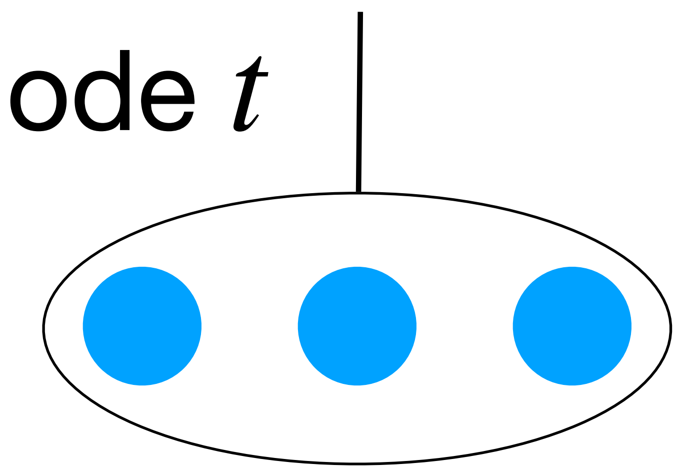


for $S \subseteq \chi(t)$:

  if S is an independent set

Leaf Nodes

Node t

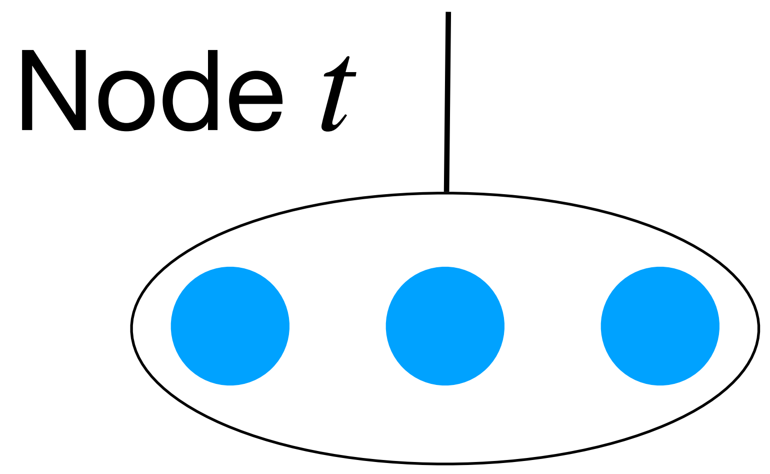


for $S \subseteq \chi(t)$:

if S is an independent set

then $n_t(S) := |S|$

Leaf Nodes



for $S \subseteq \chi(t)$:

  if S is an independent set

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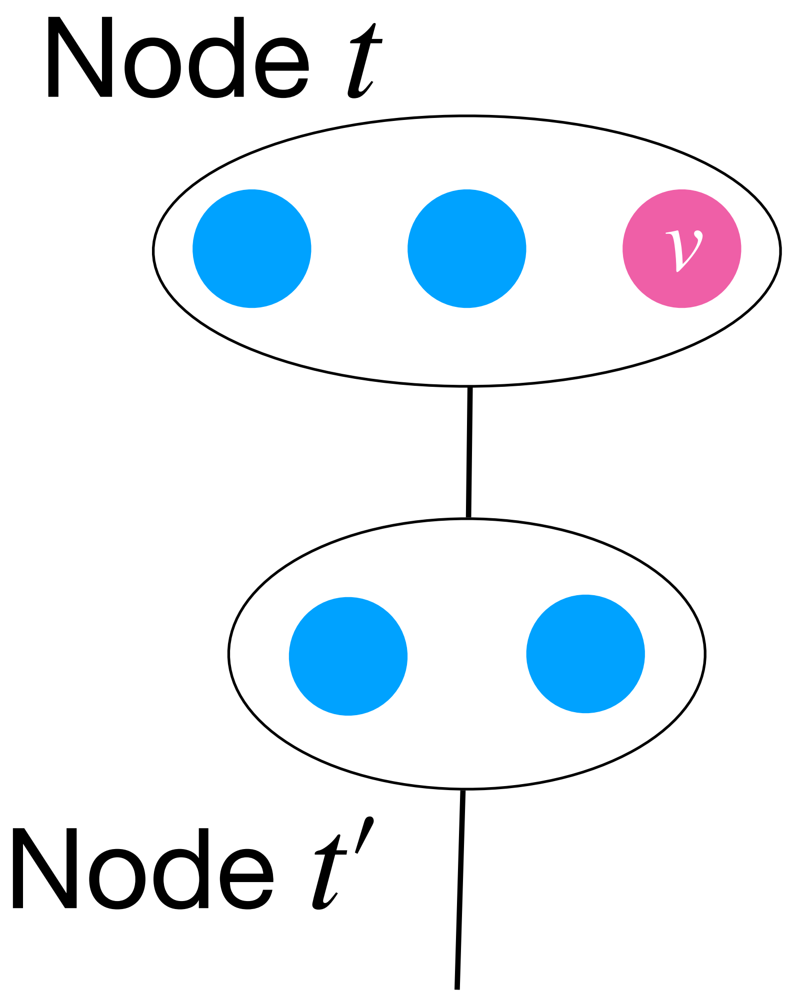
    else $n_t(S) := 0$

Introduce Nodes



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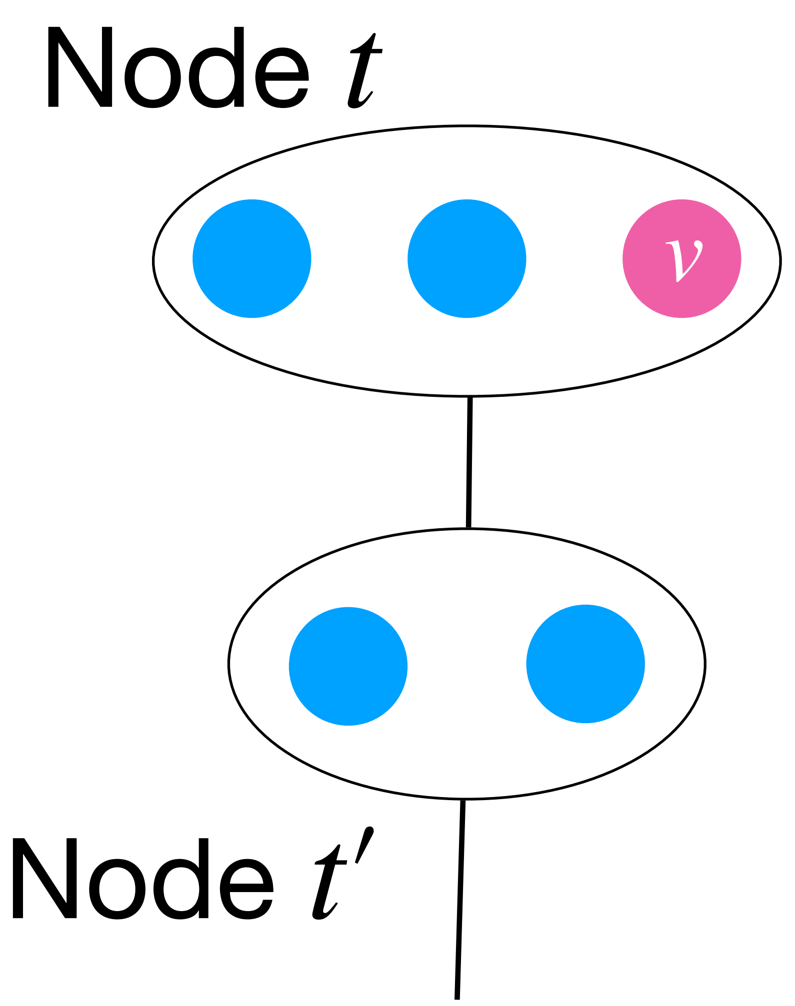

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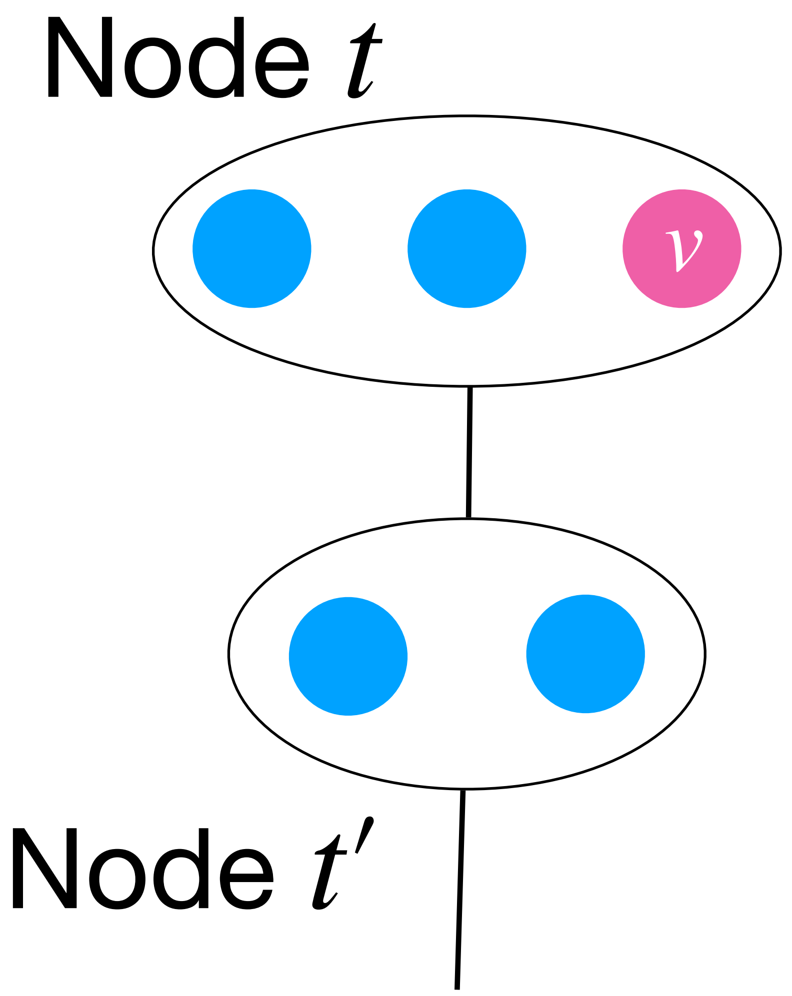

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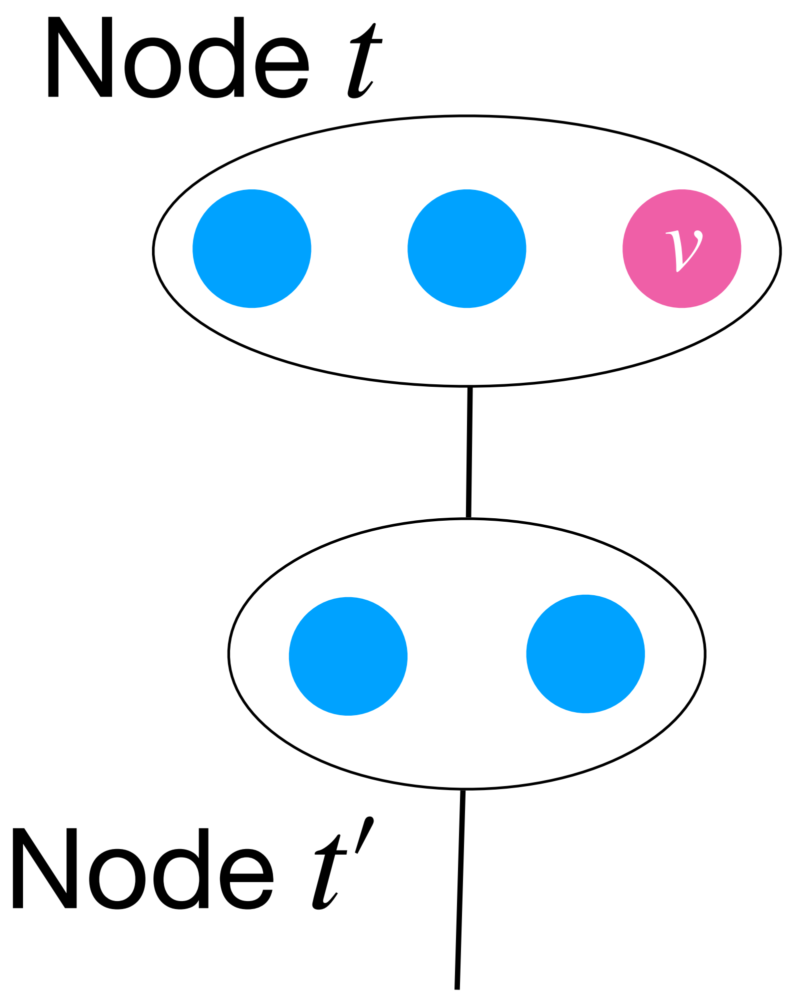
Introduce Nodes



for $S \subseteq \chi(t)$:

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then $n_t(S) := n_{t'}(S)$

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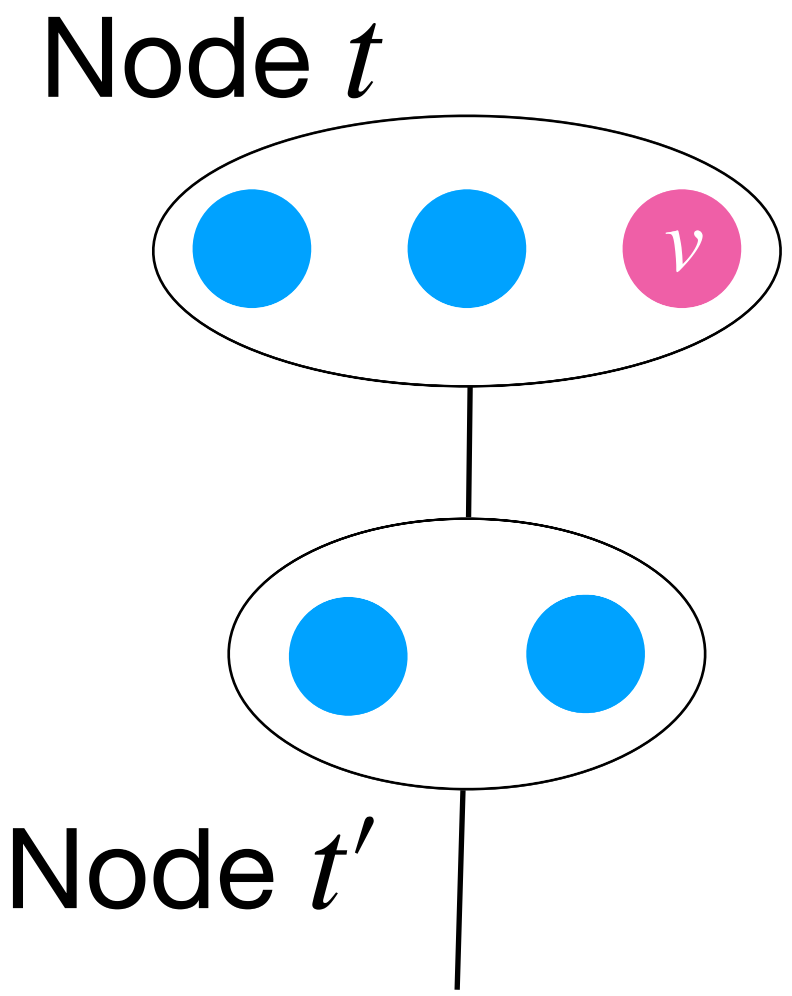

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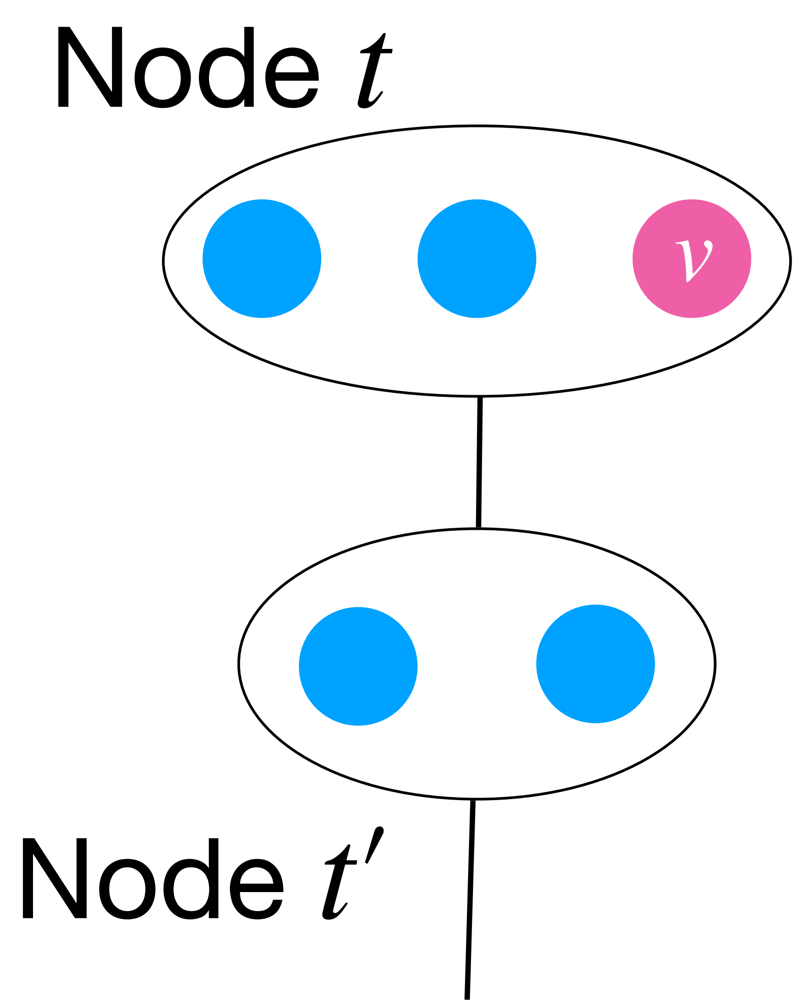

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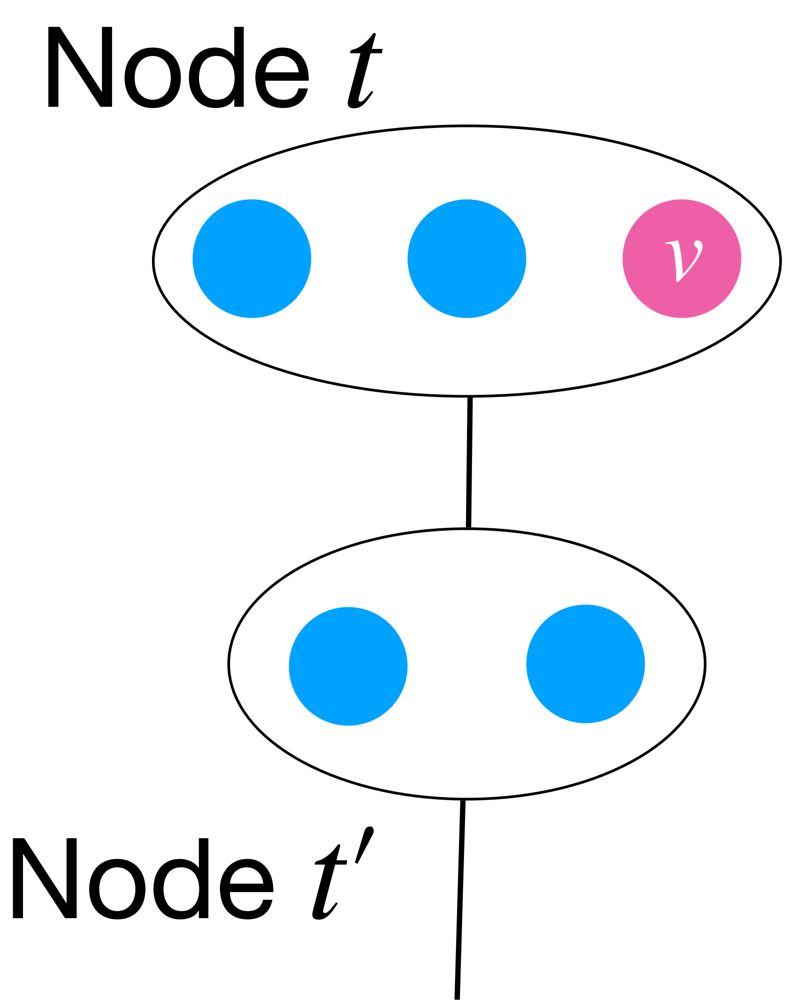

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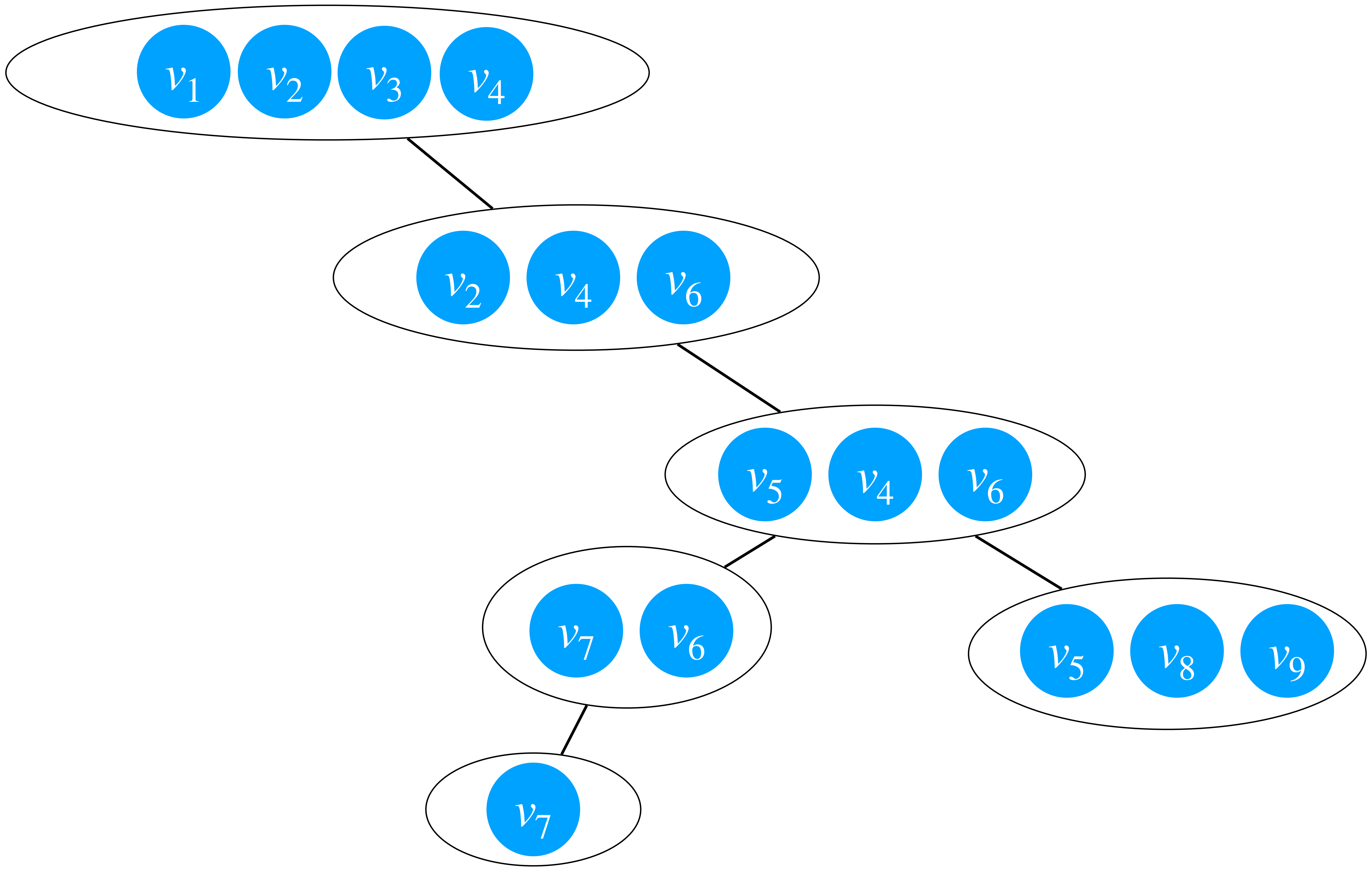
Bags are Separators



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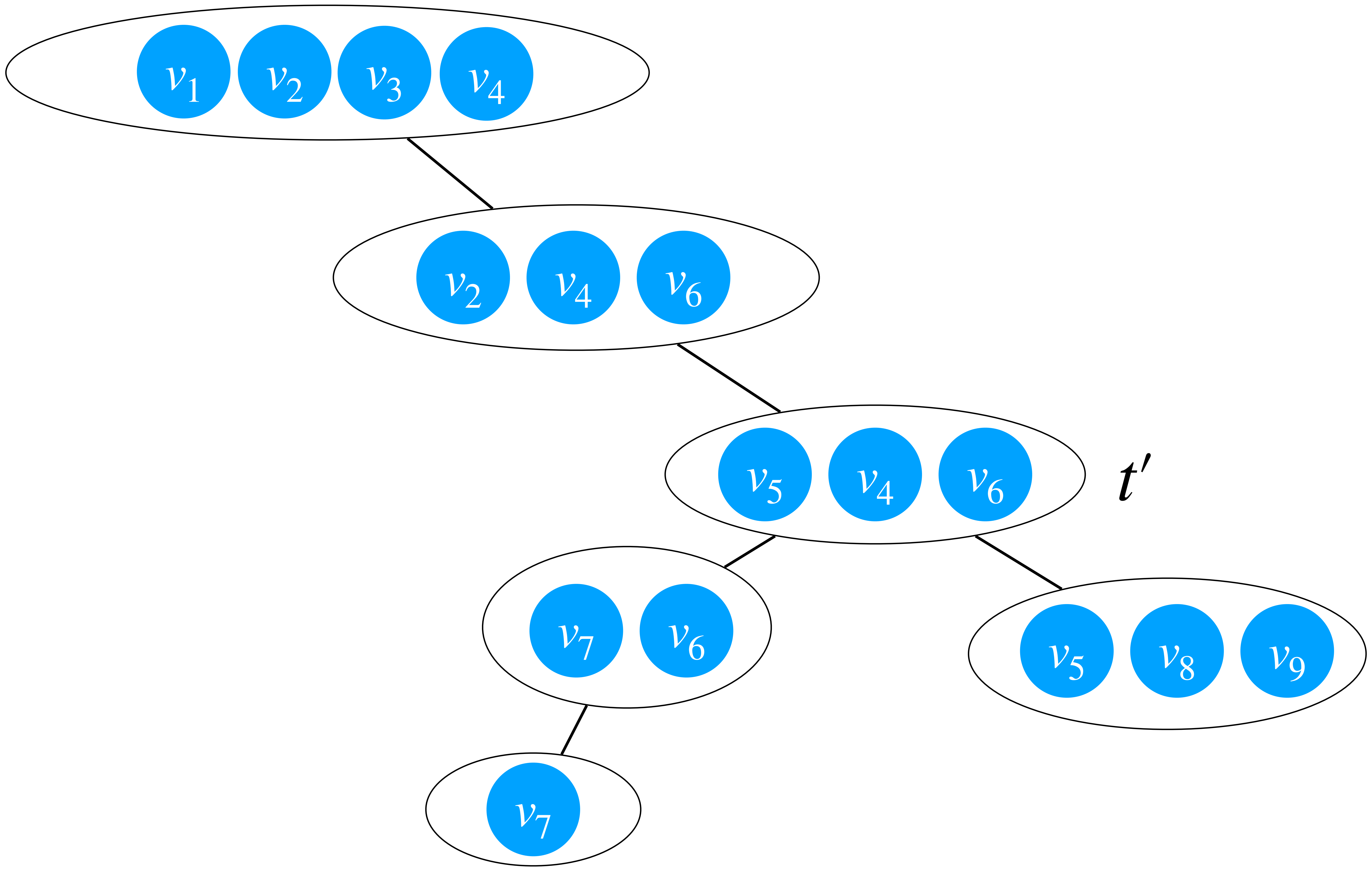

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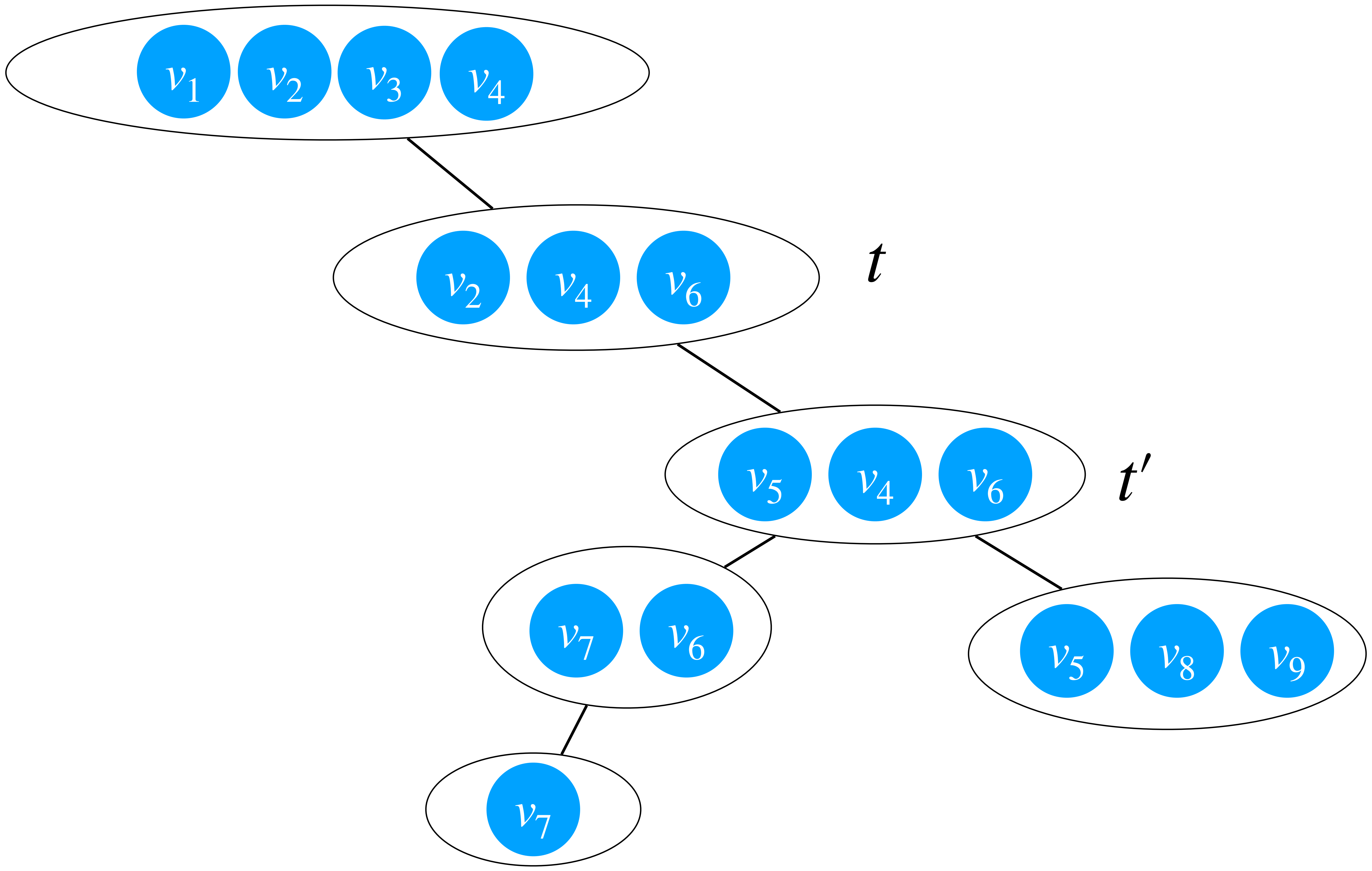

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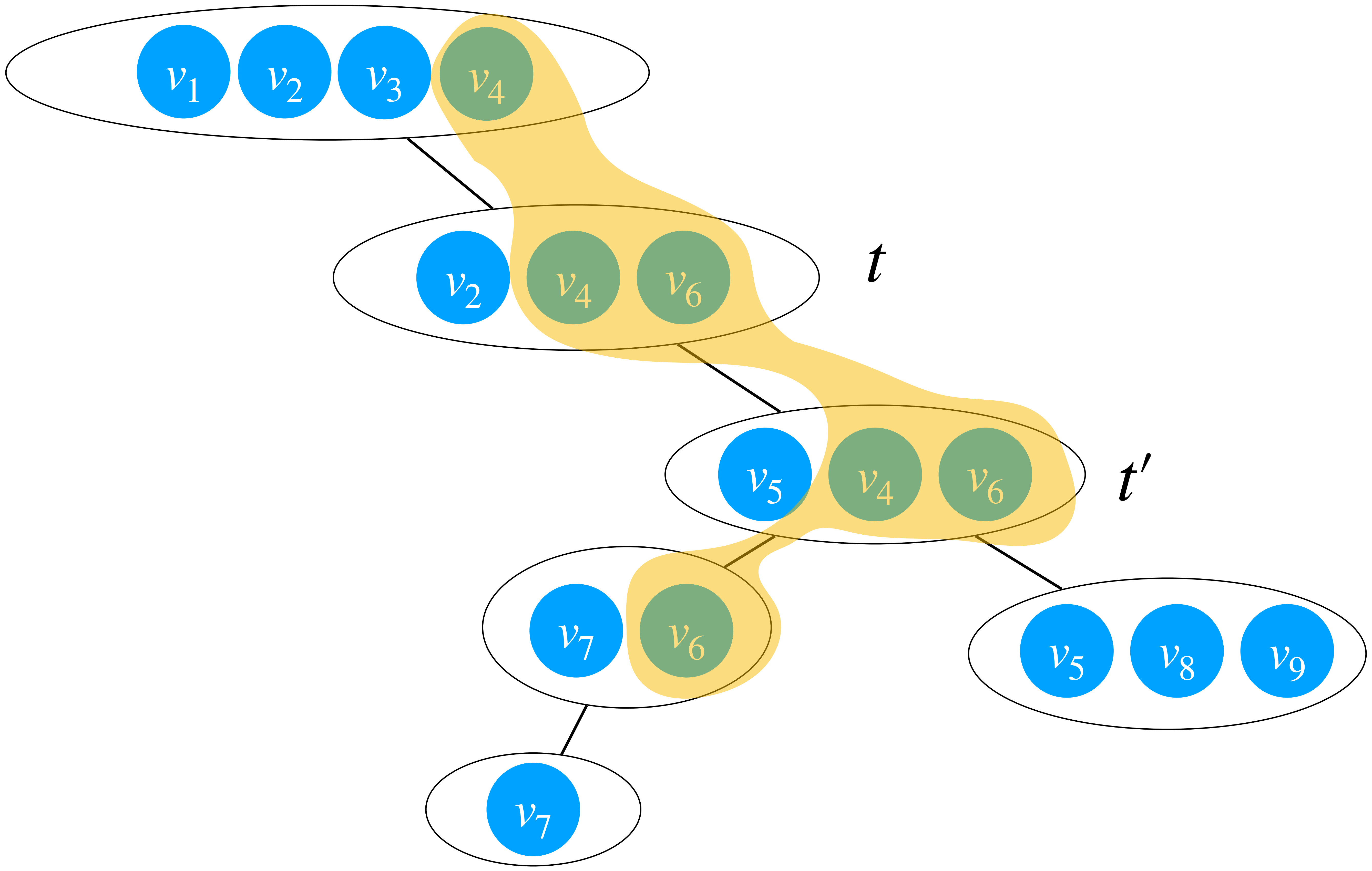

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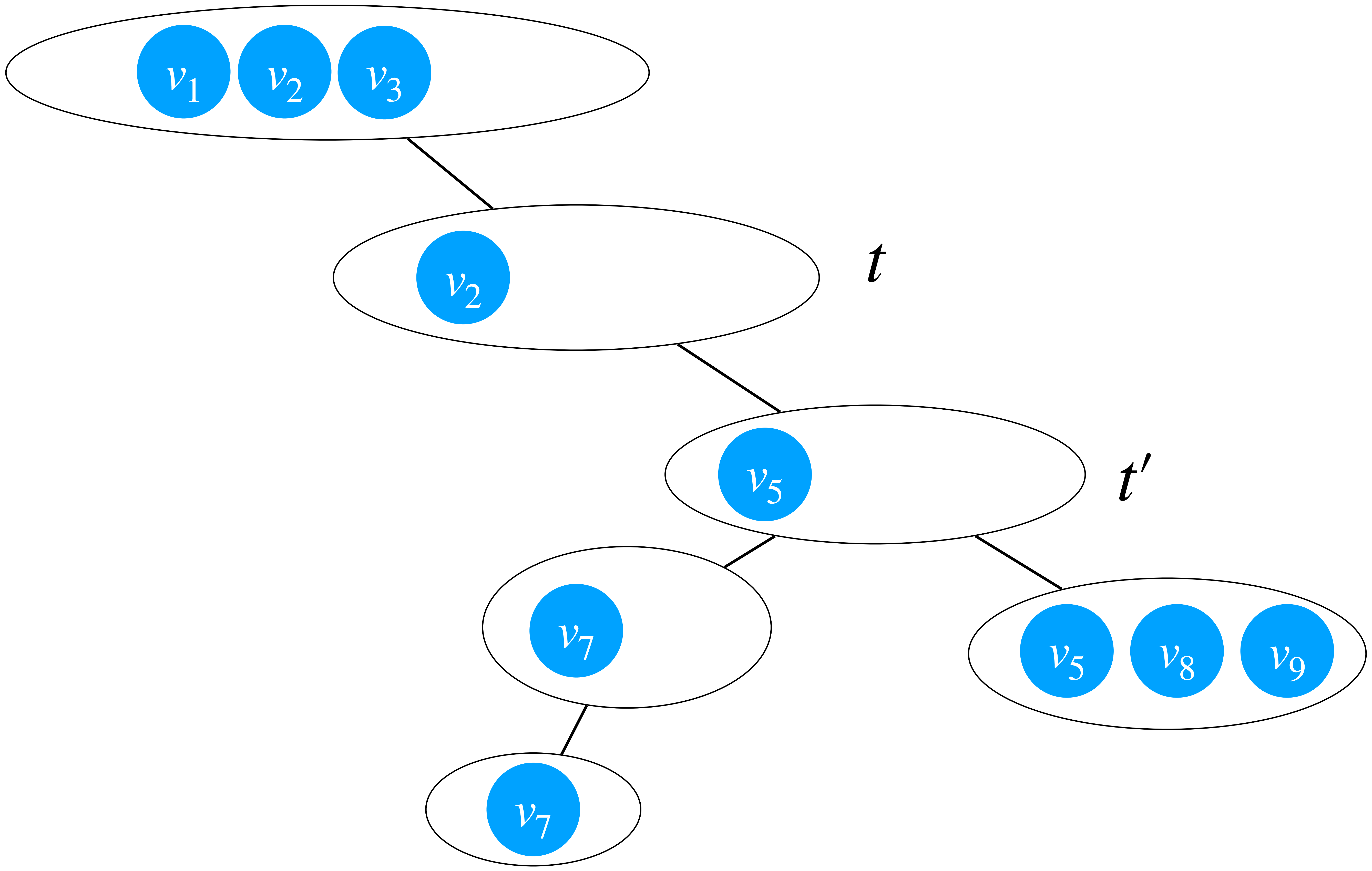

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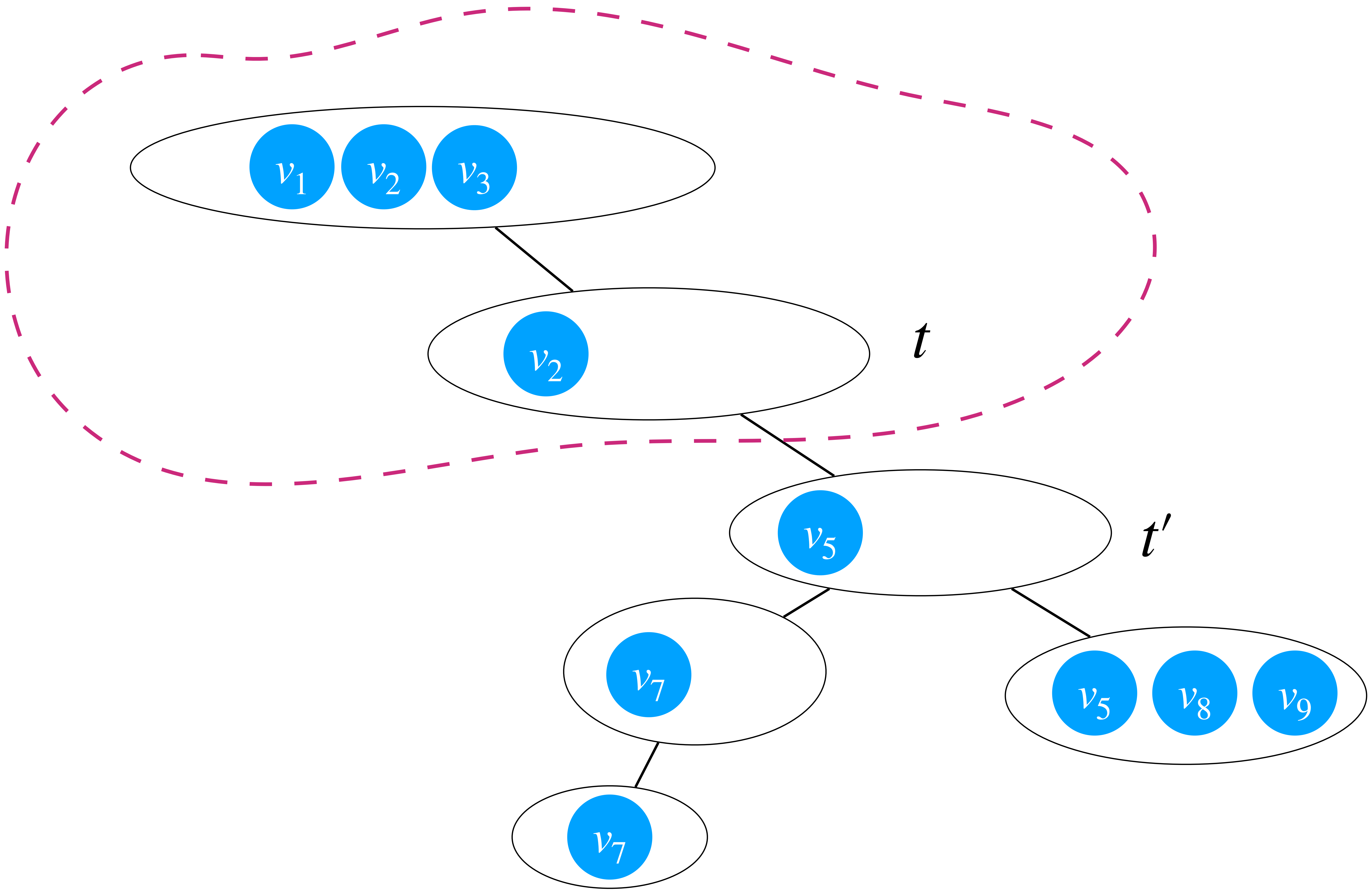

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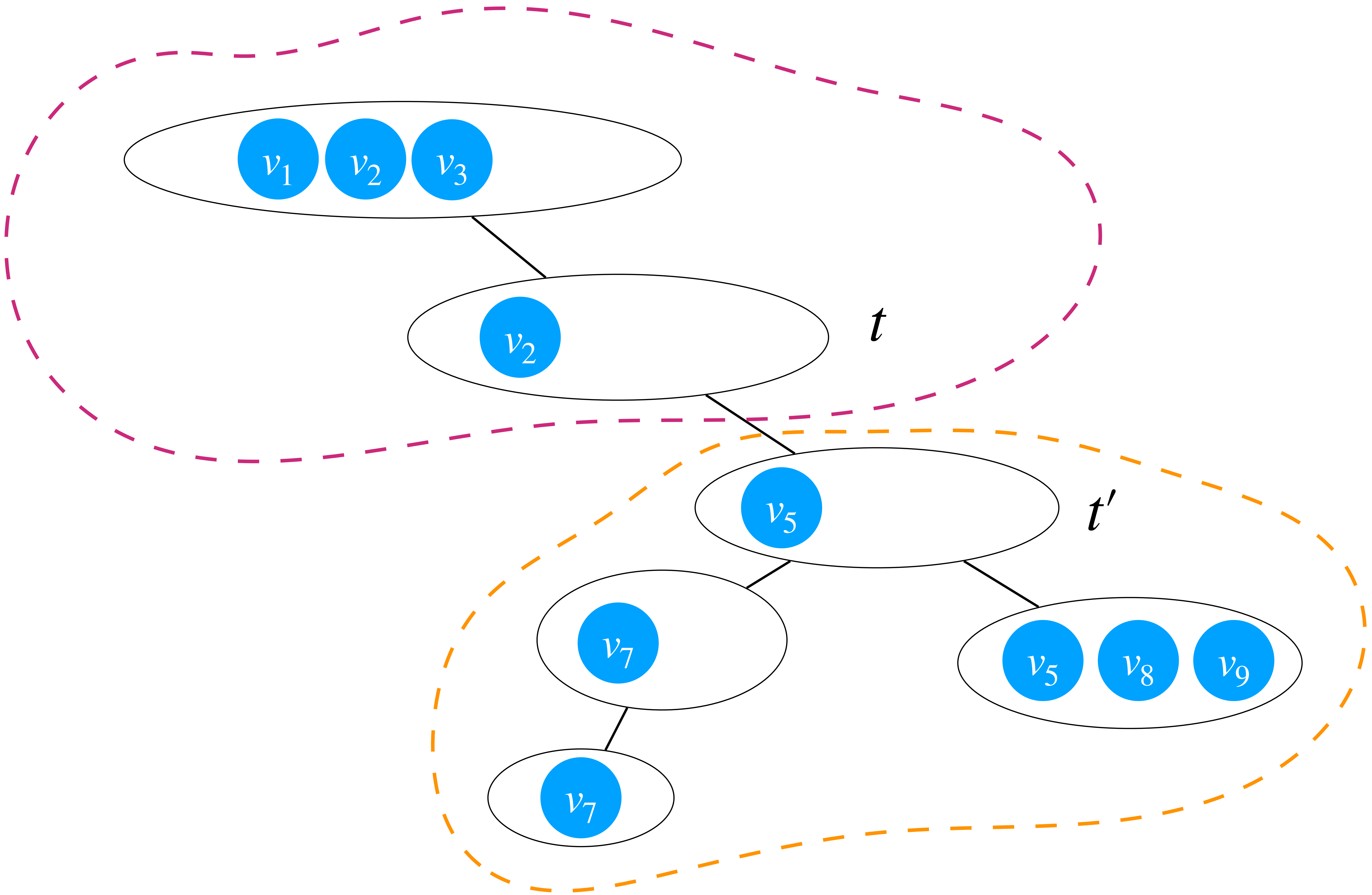

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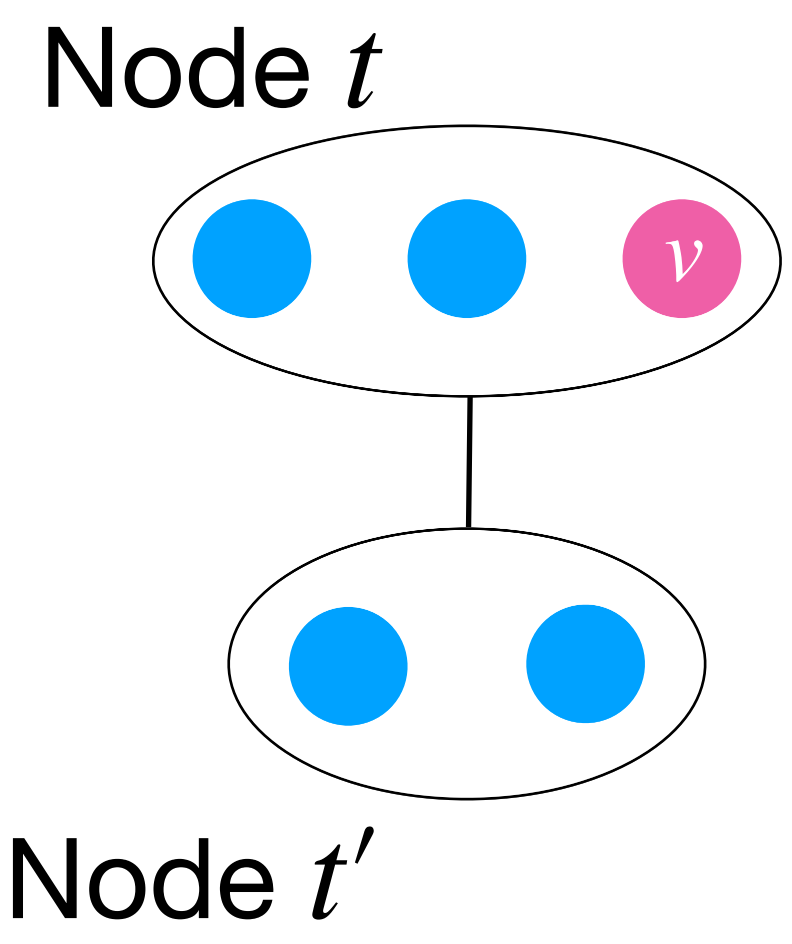

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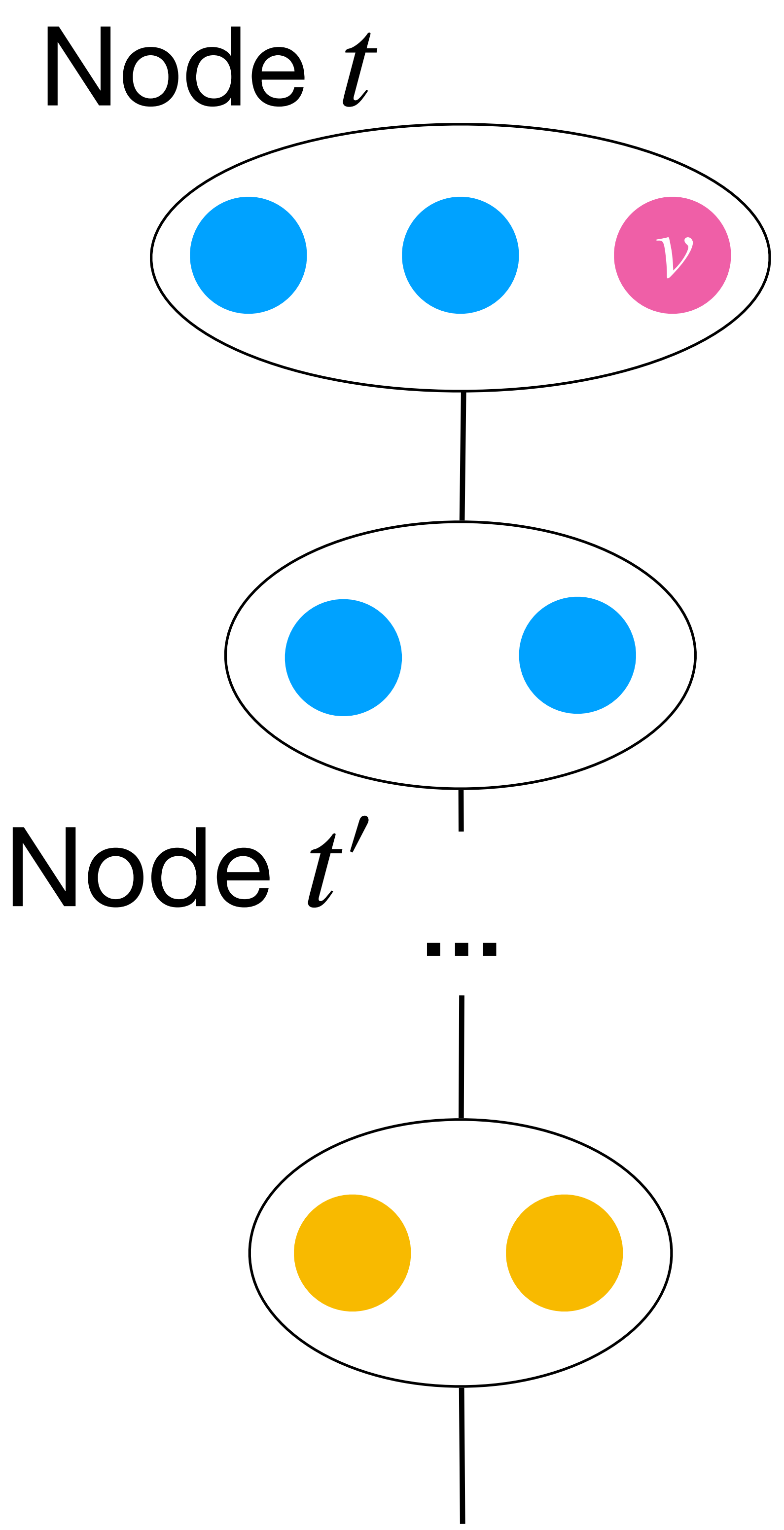

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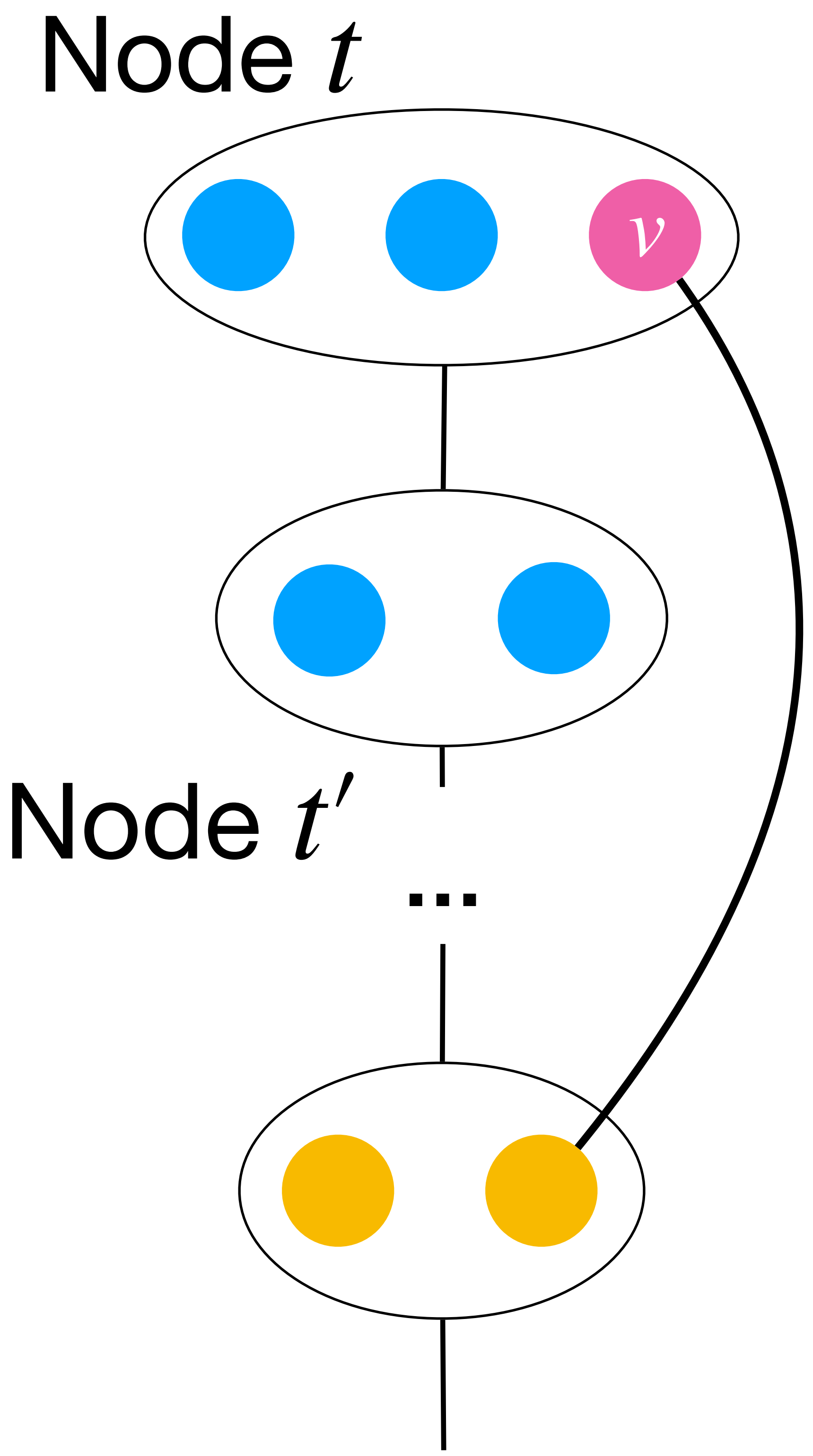

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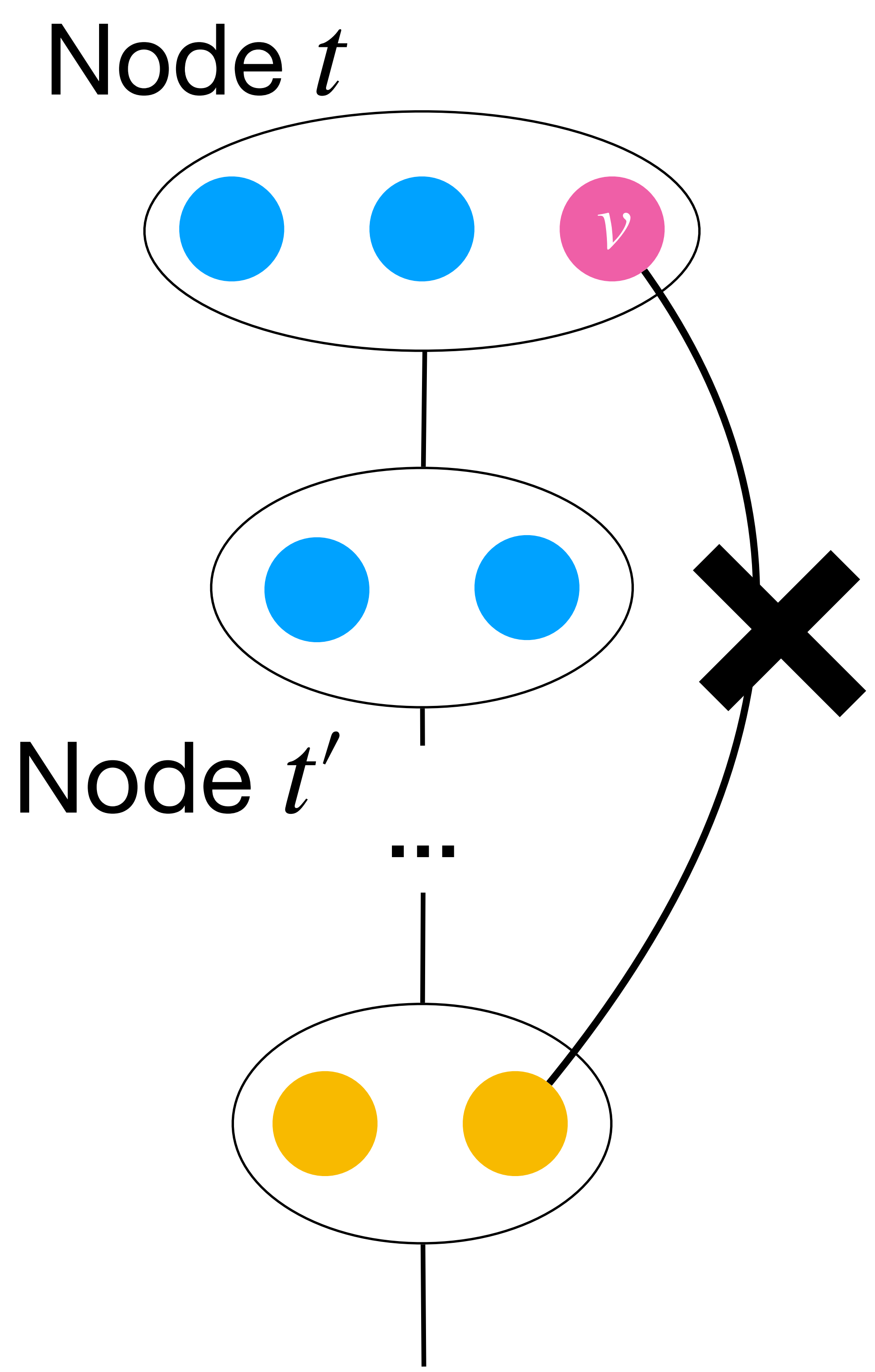

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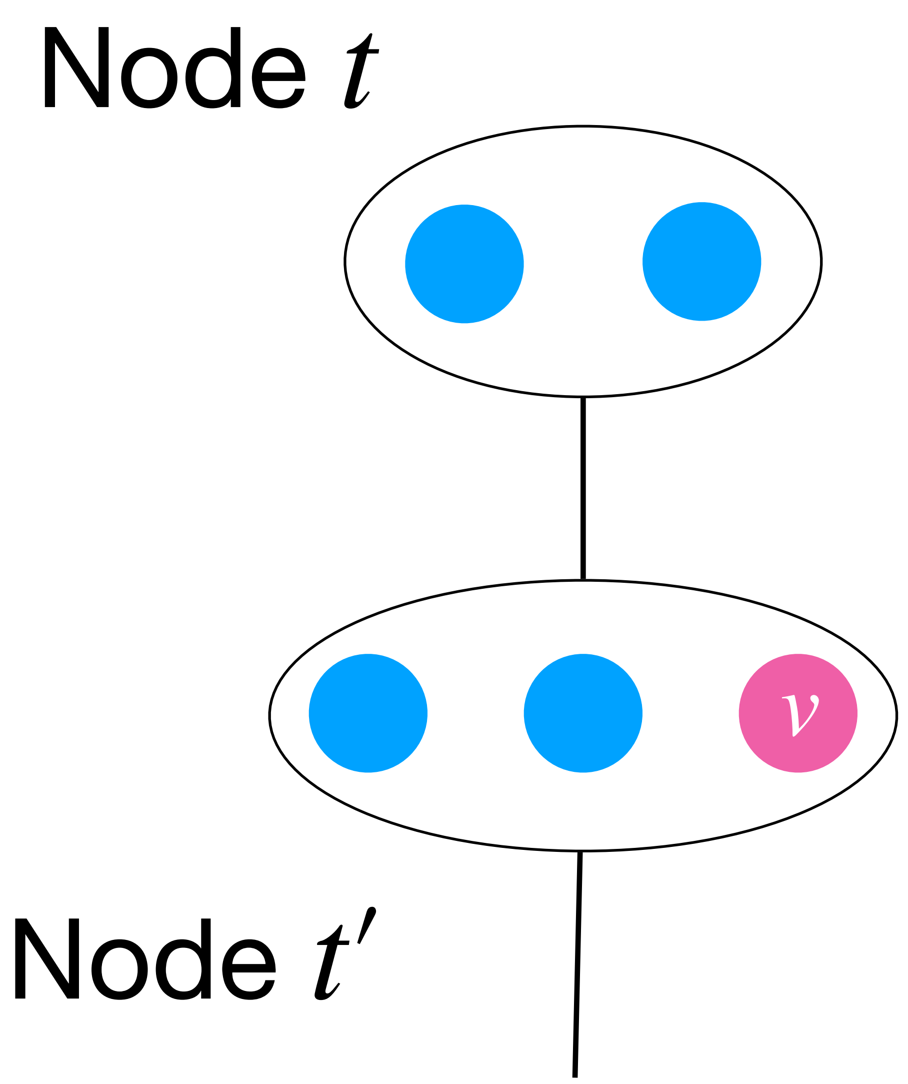
Forget Nodes



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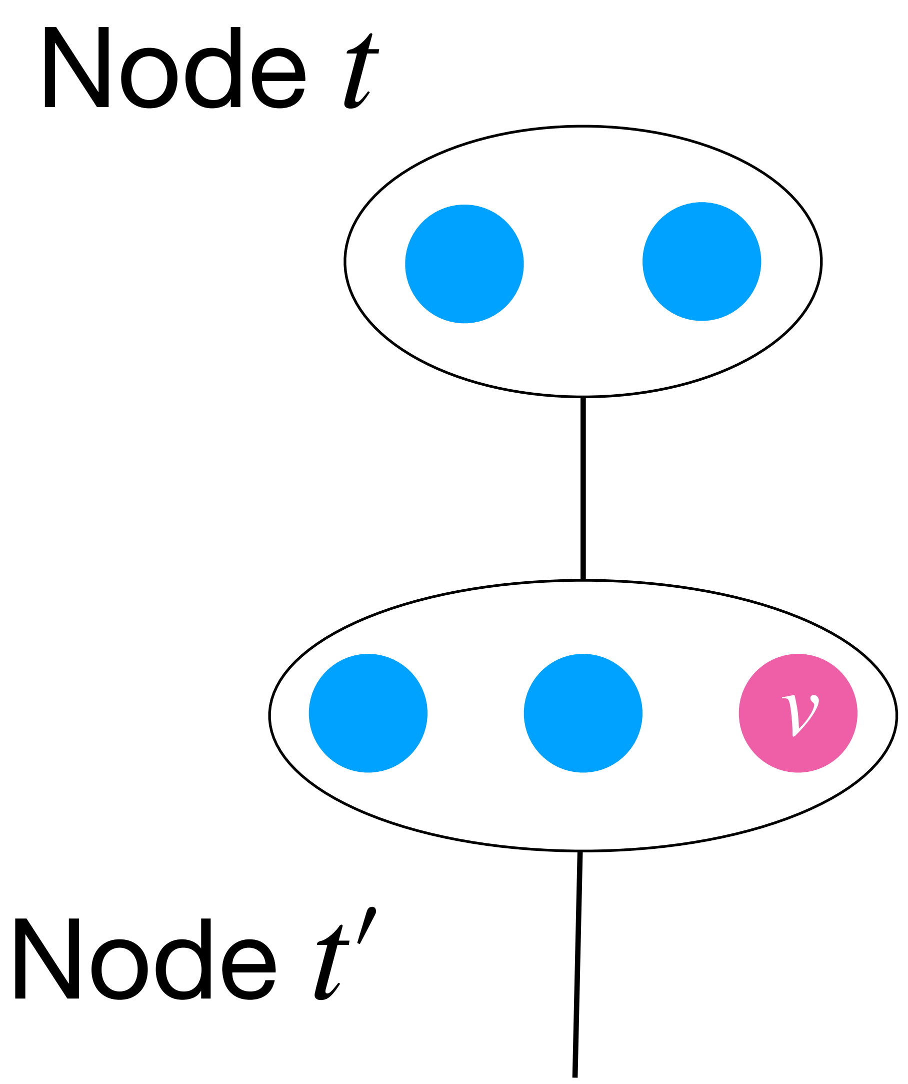
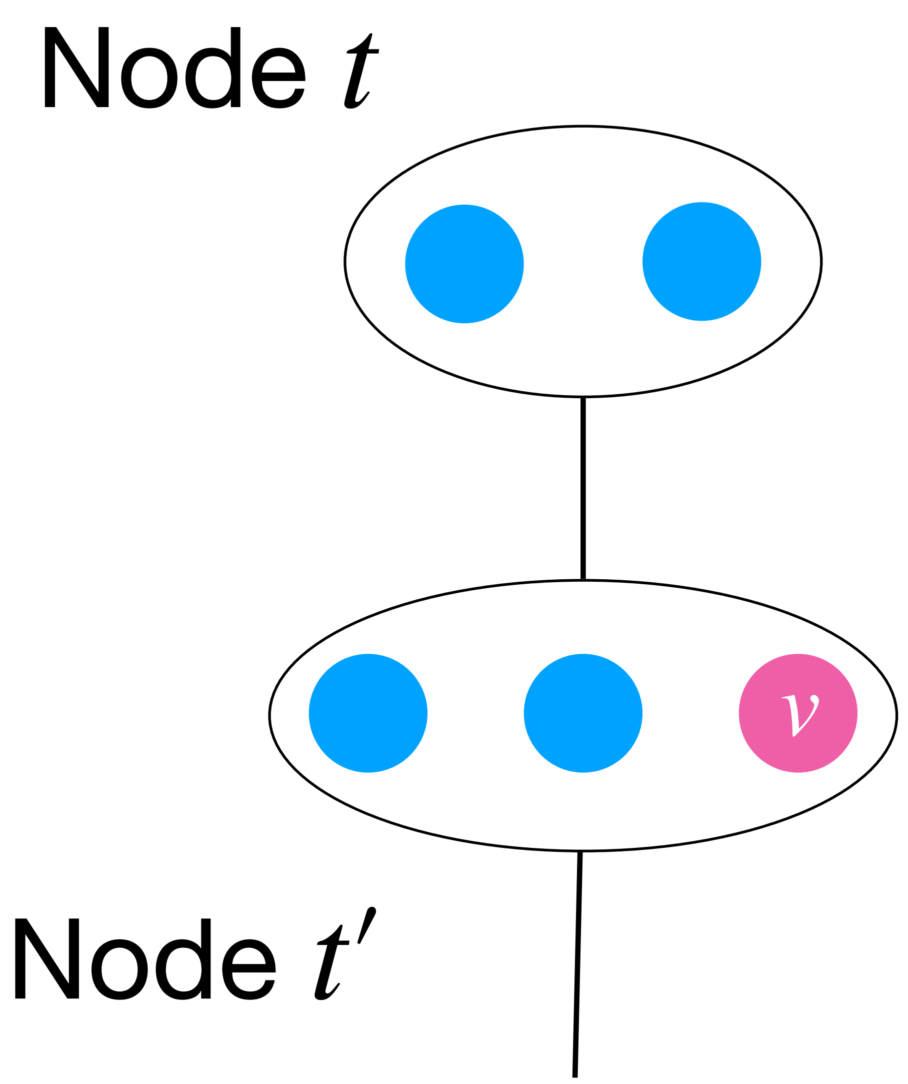


for $S \subseteq \chi(t)$

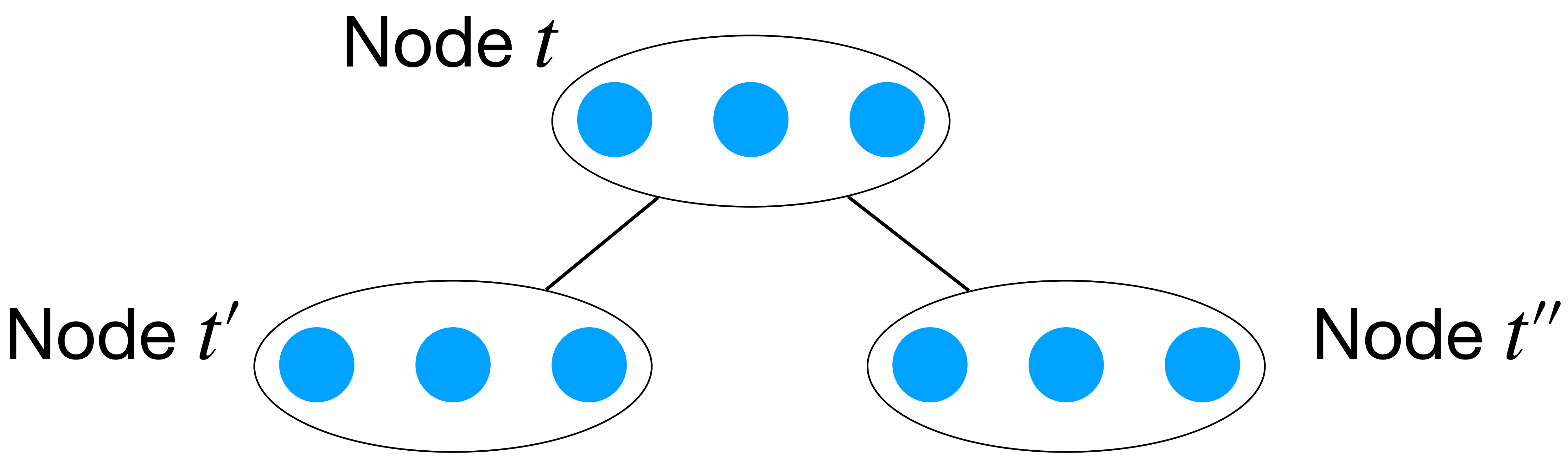
Forget Nodes



for $S \subseteq \chi(t)$

$$n_t(S) := \max(n_{t'}(S), n_{t'}(S \cup \{v\}))$$

Join Nodes



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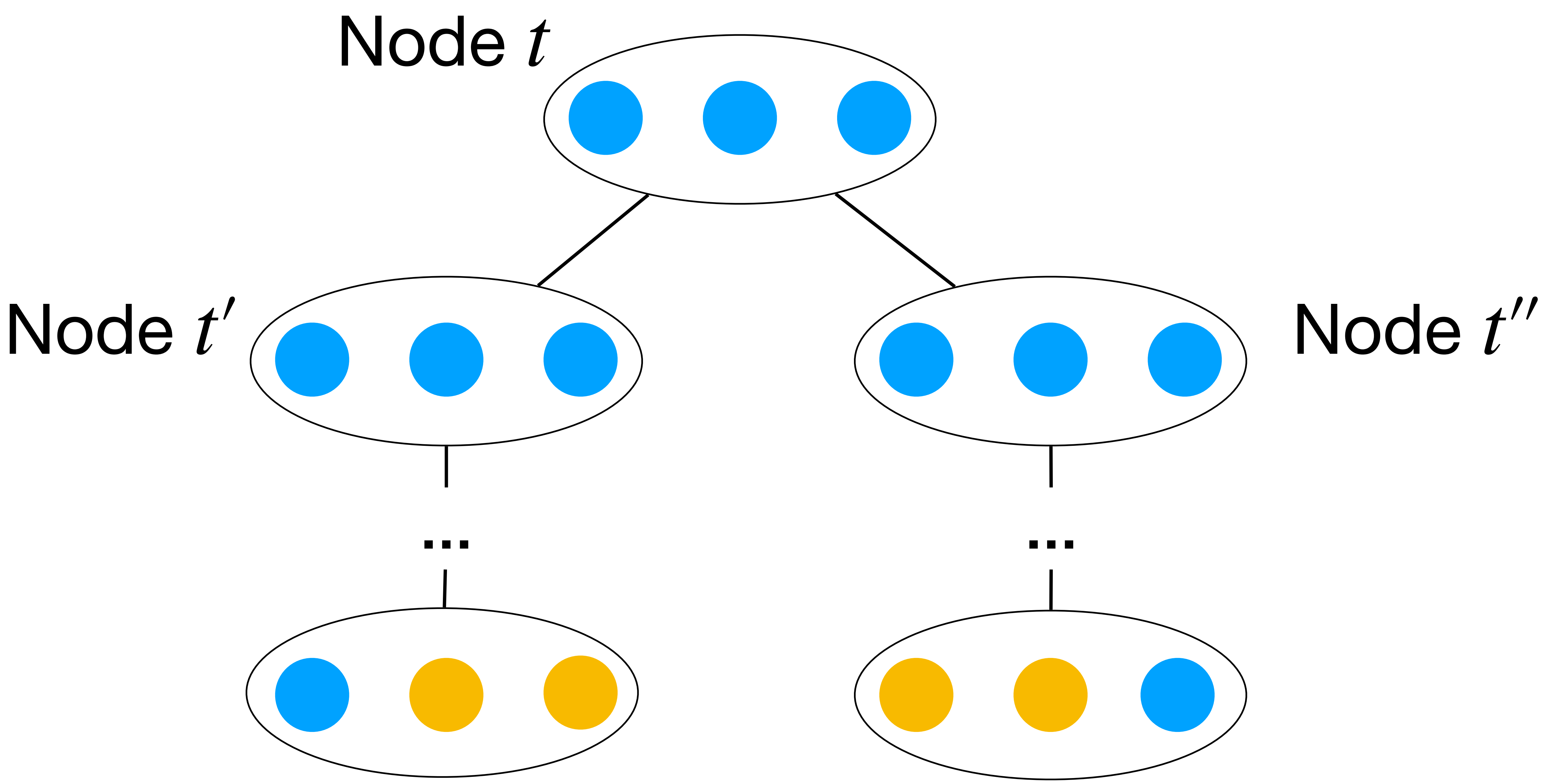

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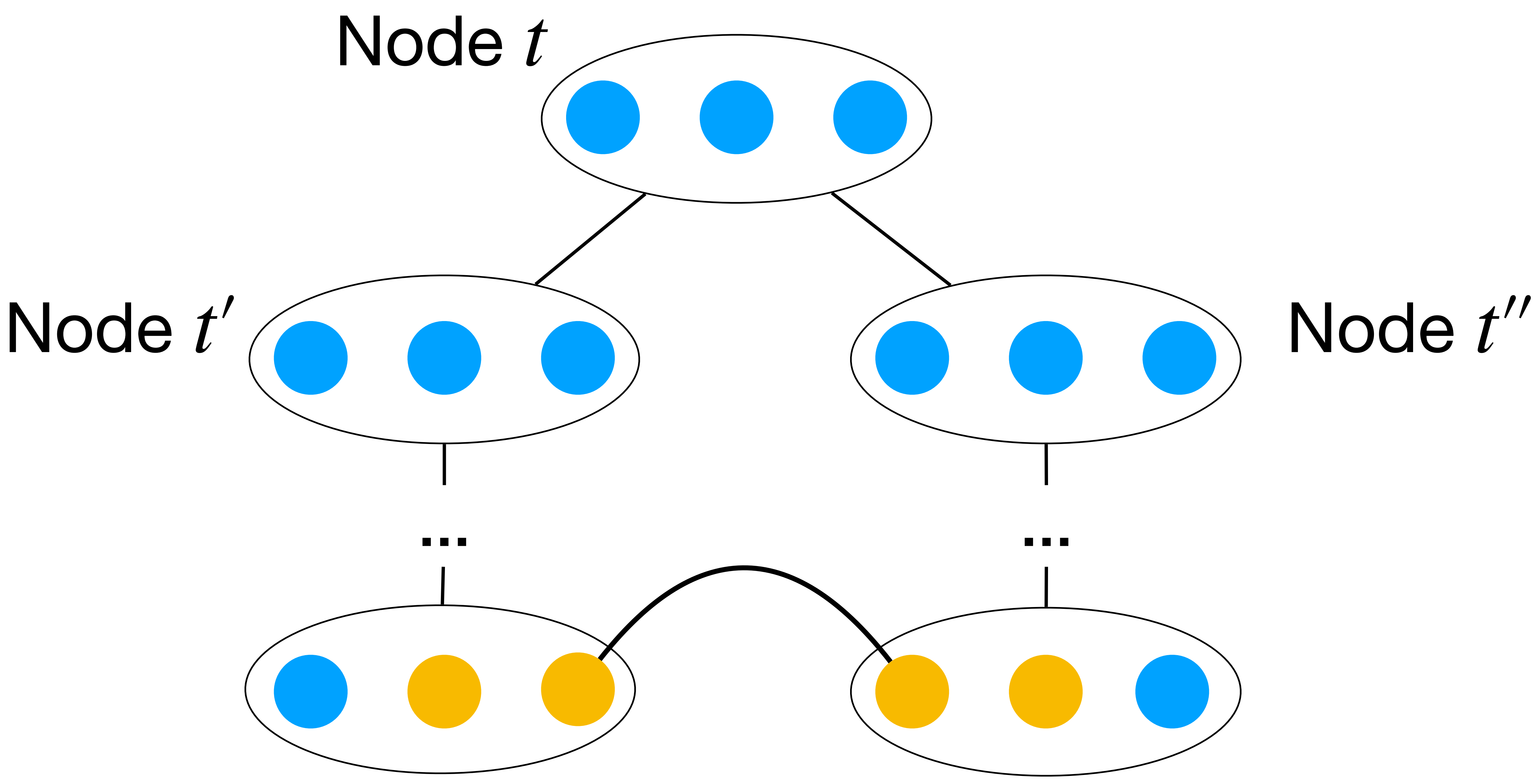

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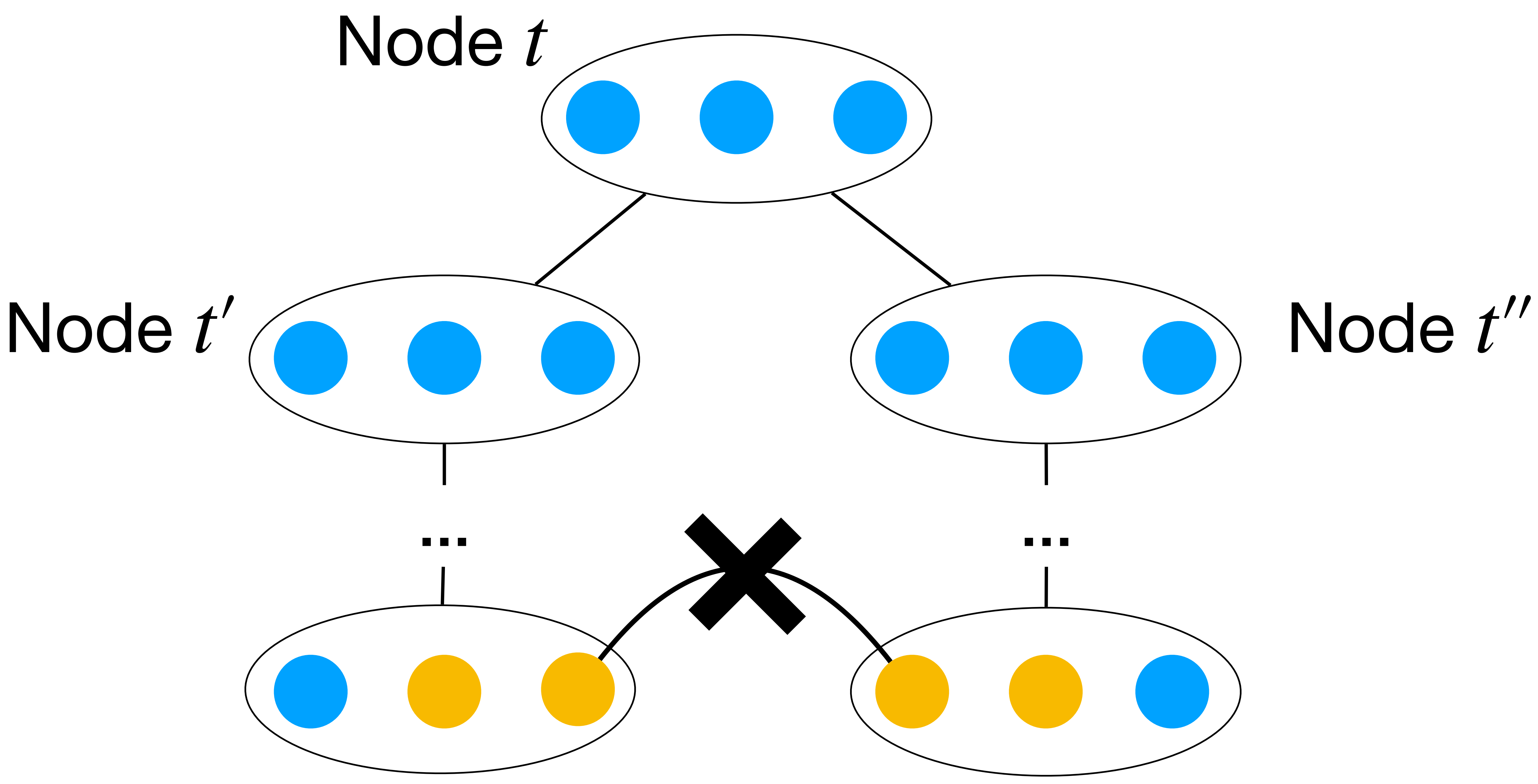

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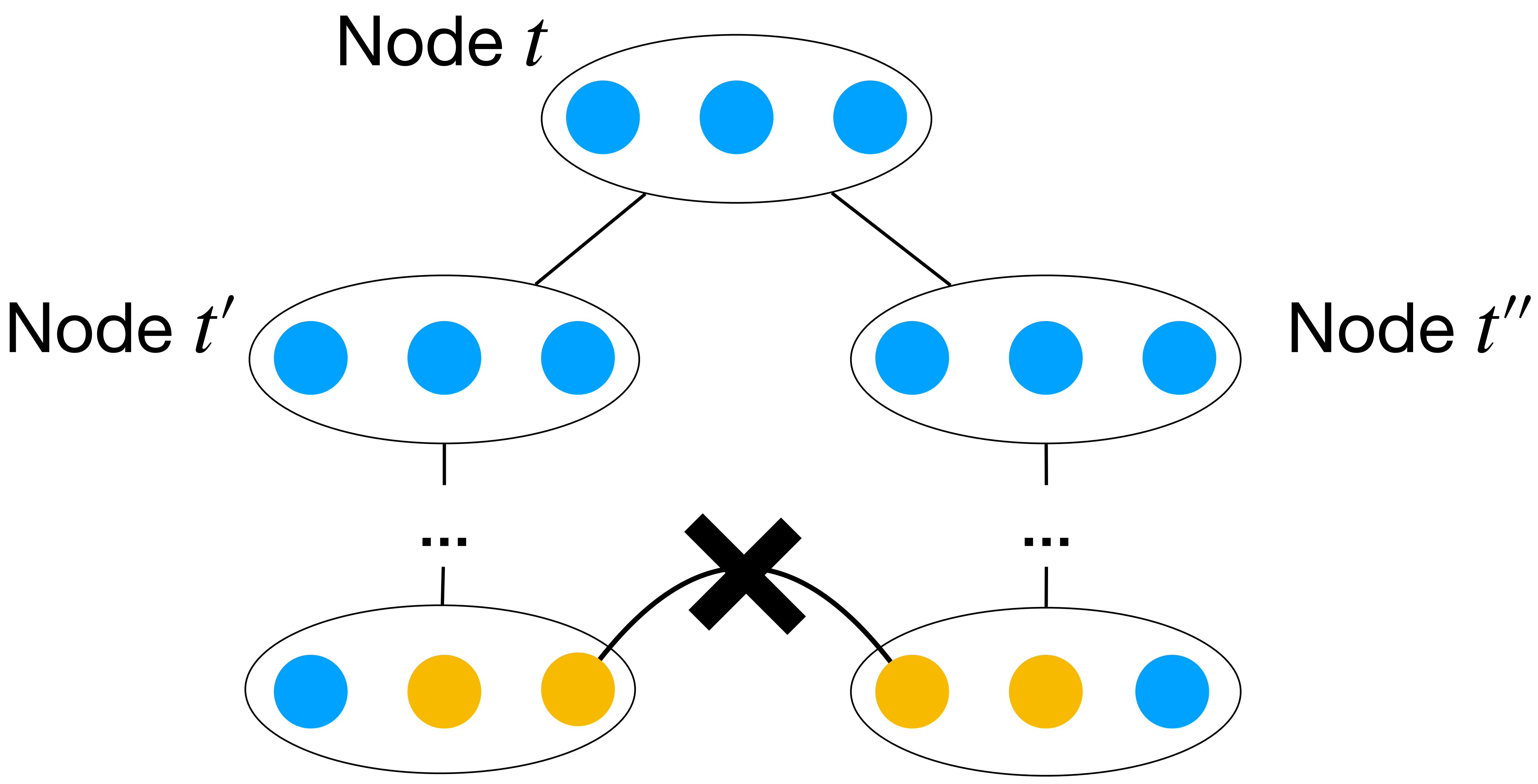
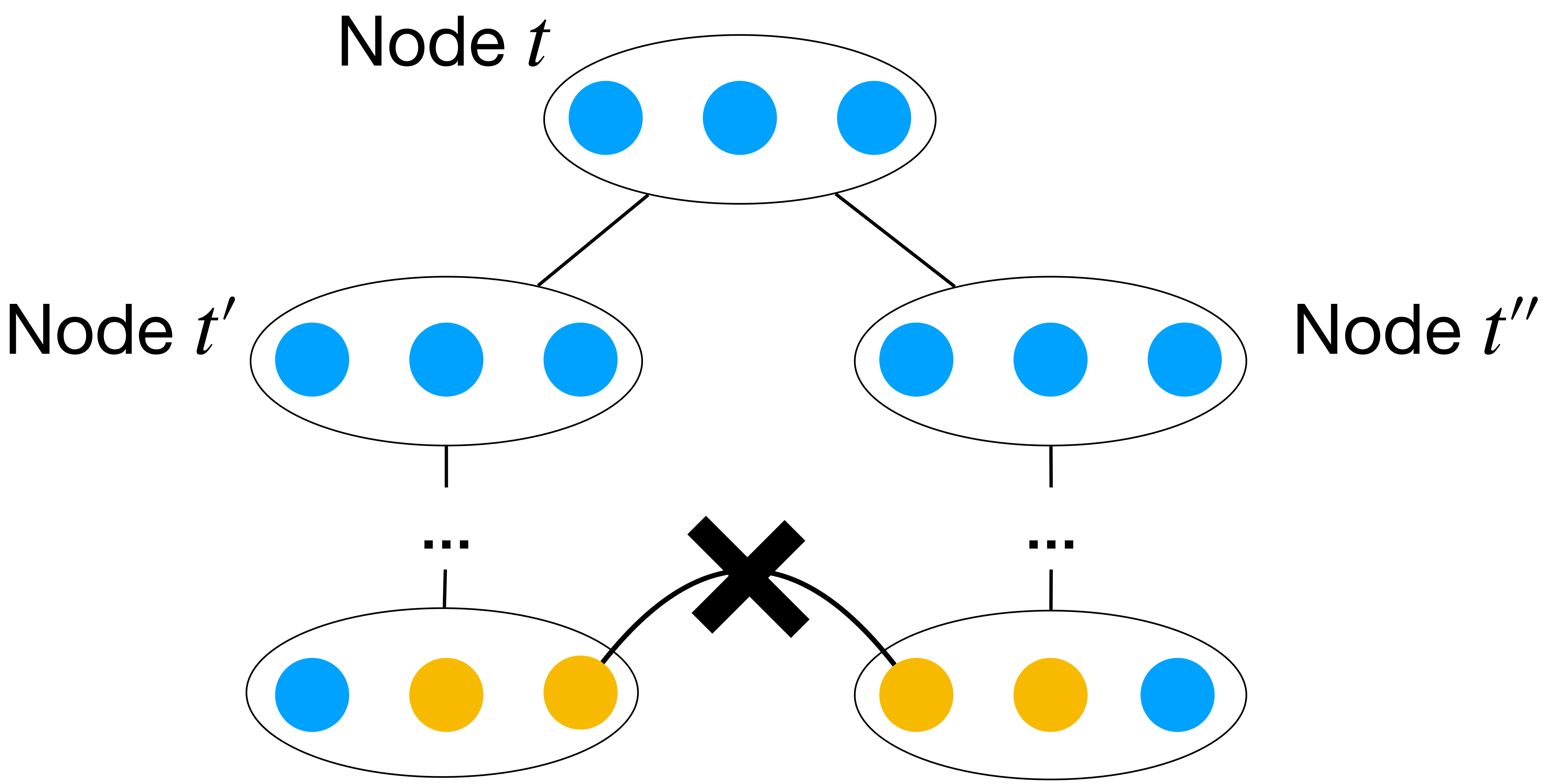


for $S \subseteq \chi(t)$

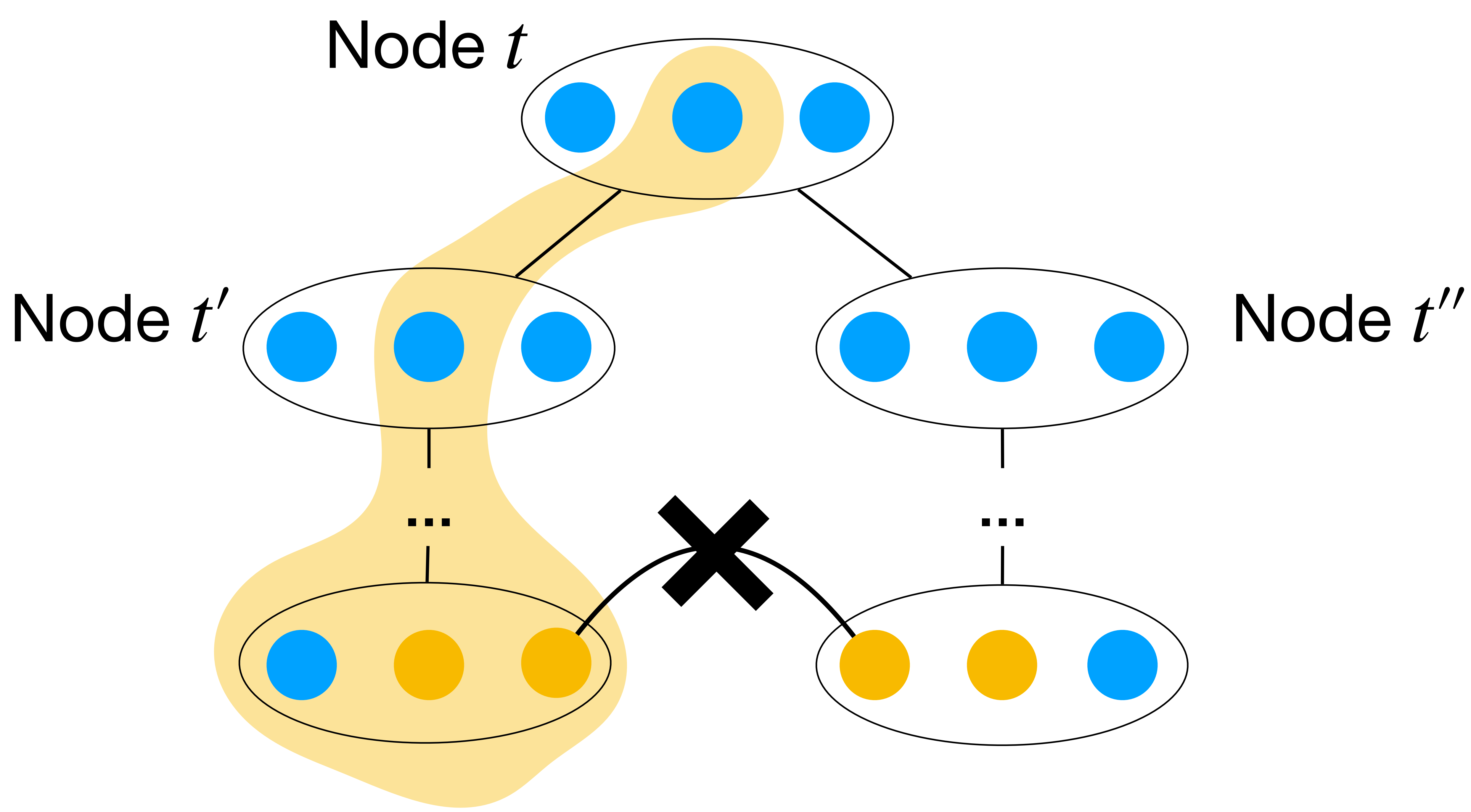
Join Nodes



for $S \subseteq \chi(t)$

$$n_t(S) := n_{t'}(S) + n_{t''}(S)$$

Join Nodes



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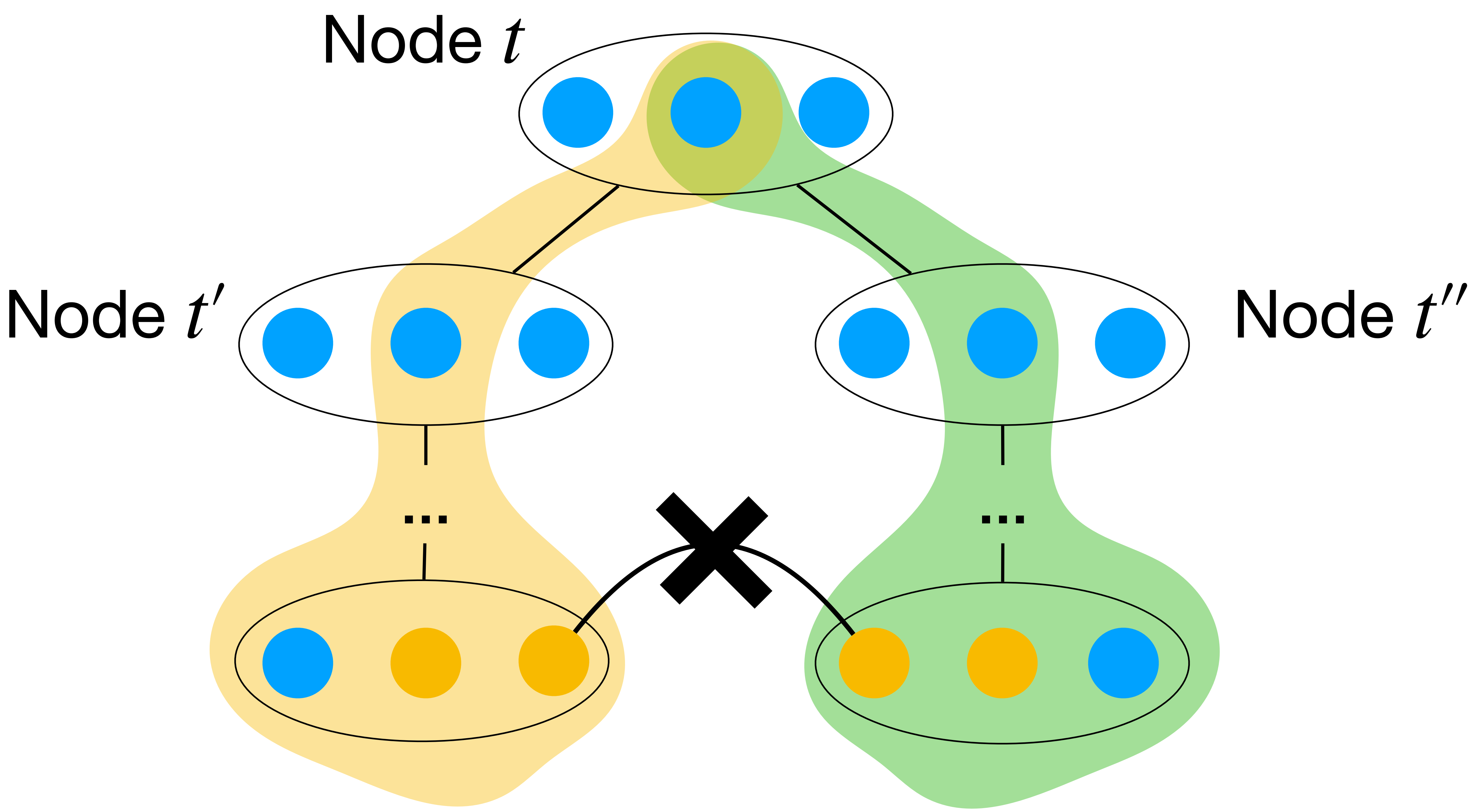

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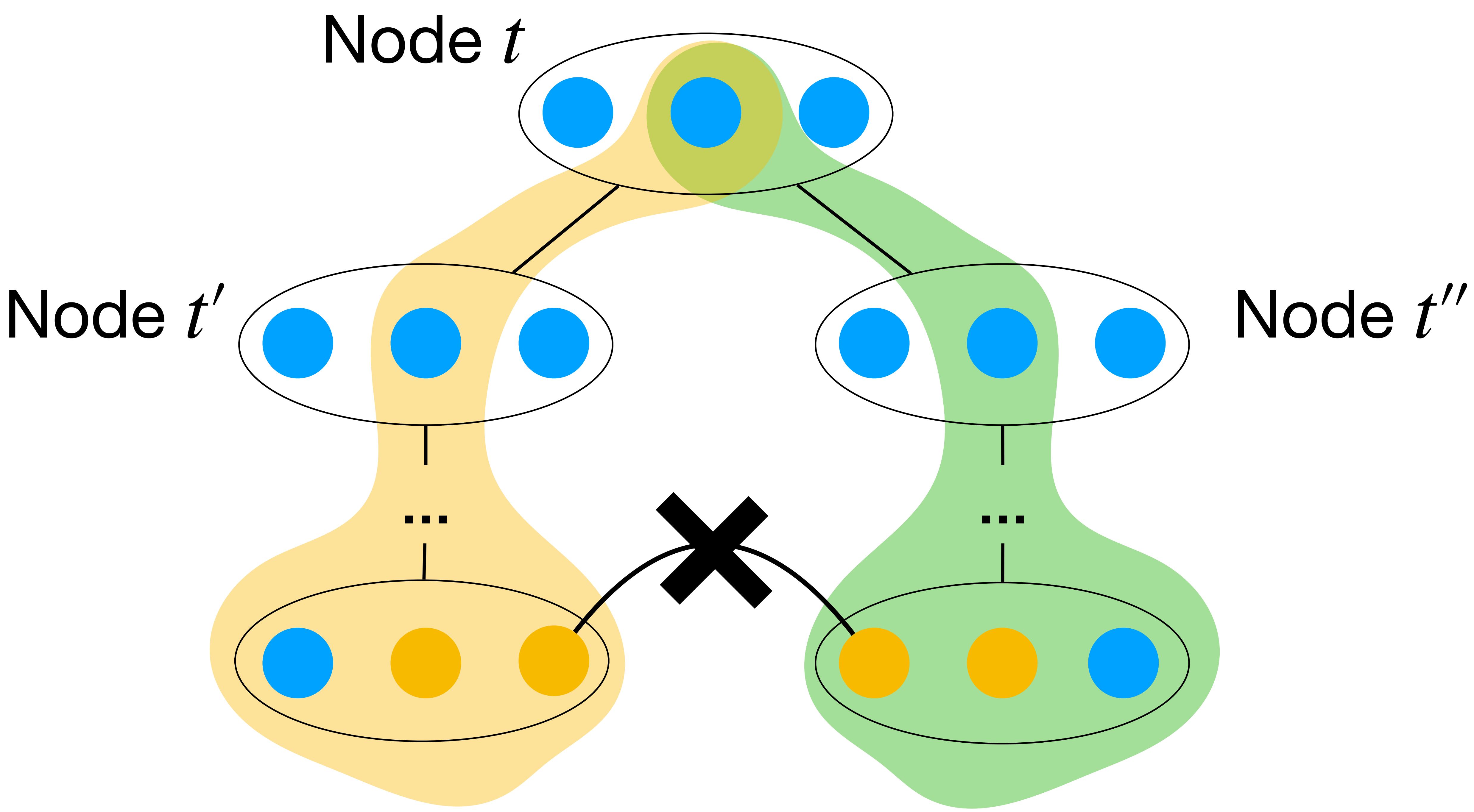

for $S \subseteq \chi(t)$

$$n_t(S) := n_{t'}(S) + n_{t''}(S) - |S|$$

The Algorithm

The Algorithm

1. Compute a nice tree decomposition (T, χ) of G

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Leaf Nodes

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for each $U \subseteq \chi(t)$

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Leaf Nodes

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$$n_t(U) := |U|$$

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$n_t(U) := |U|$ if U is an IS

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Introduce Nodes

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Introduce Nodes

for each $U \subseteq \chi(t)$

if $v \notin U$

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Introduce Nodes

for each $U \subseteq \chi(t)$

if $v \notin U$ $n_t(U) := n_{t'}(U)$

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if $v \in U$ $n_t(U) := n_{t'}(U \backslash \{v\}) + 1$ if U is an IS

The Algorithm

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Leaf Nodes

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$n_t(U) := |U|$ if U is an IS

$n_t(U) := 0$ otherwise

Introduce Nodes

for each $U \subseteq \chi(t)$

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for each $U \subseteq \chi(t)$

$$n_t(U) := \max(n_{t'}(U), n_{t'}(U \cup \{v\}))$$

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for each $U \subseteq \chi(t)$

$$n_t(U) = n_{t'}(U) + n_{t''}(U) - |U|$$

The Algorithm

1. Compute a nice tree decomposition (T, χ) of G
2. For each $t \in T$ and $U \subseteq \chi(t)$, initialize $n_t(U)$
3. Update $n_t(U)$ by dynamic programming …
4. Output $\max\{n_r(U) \mid U \subseteq \chi(r)\}$

Leaf Nodes

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Independent Set Parameterized by Treewidth

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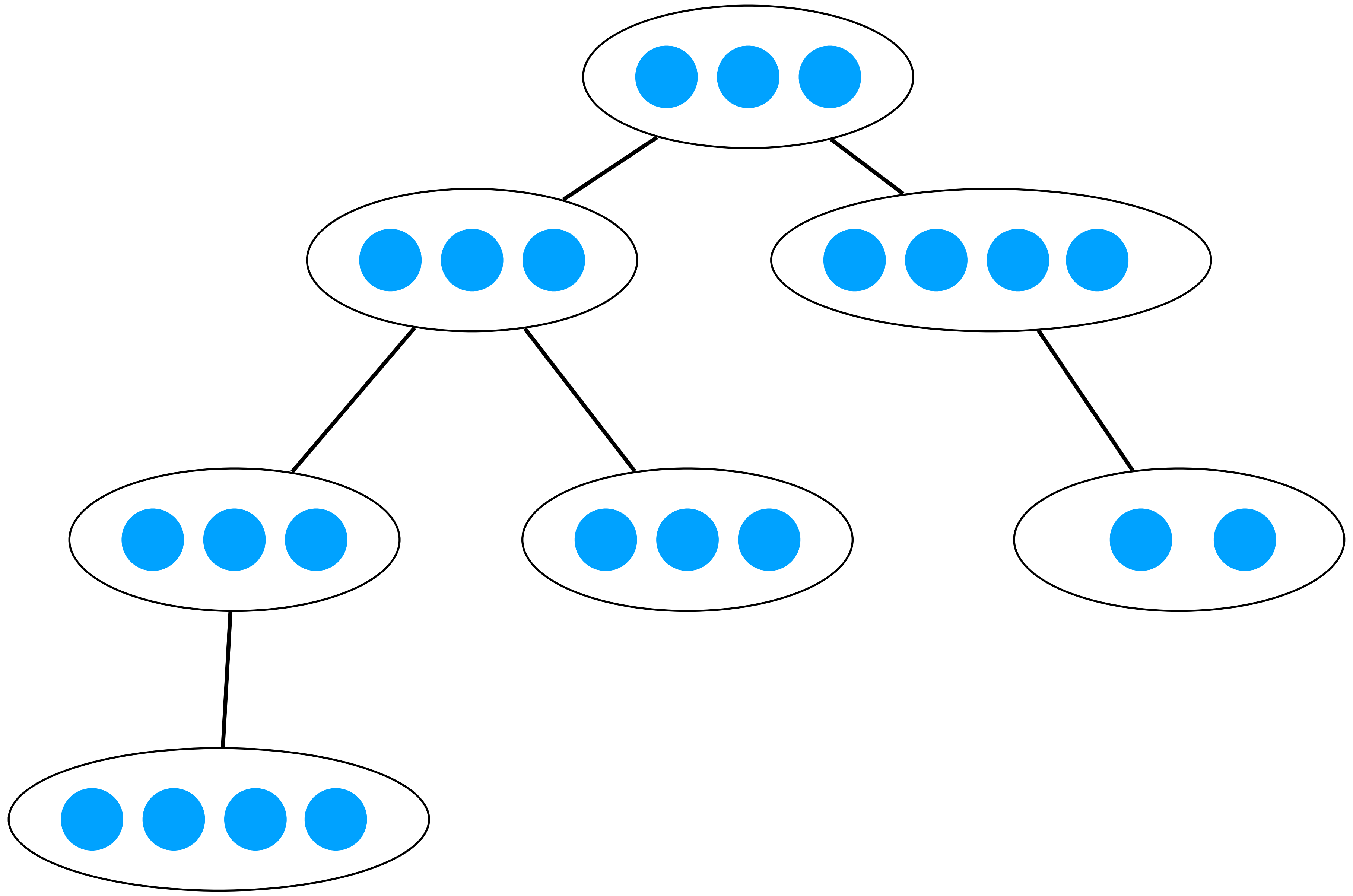
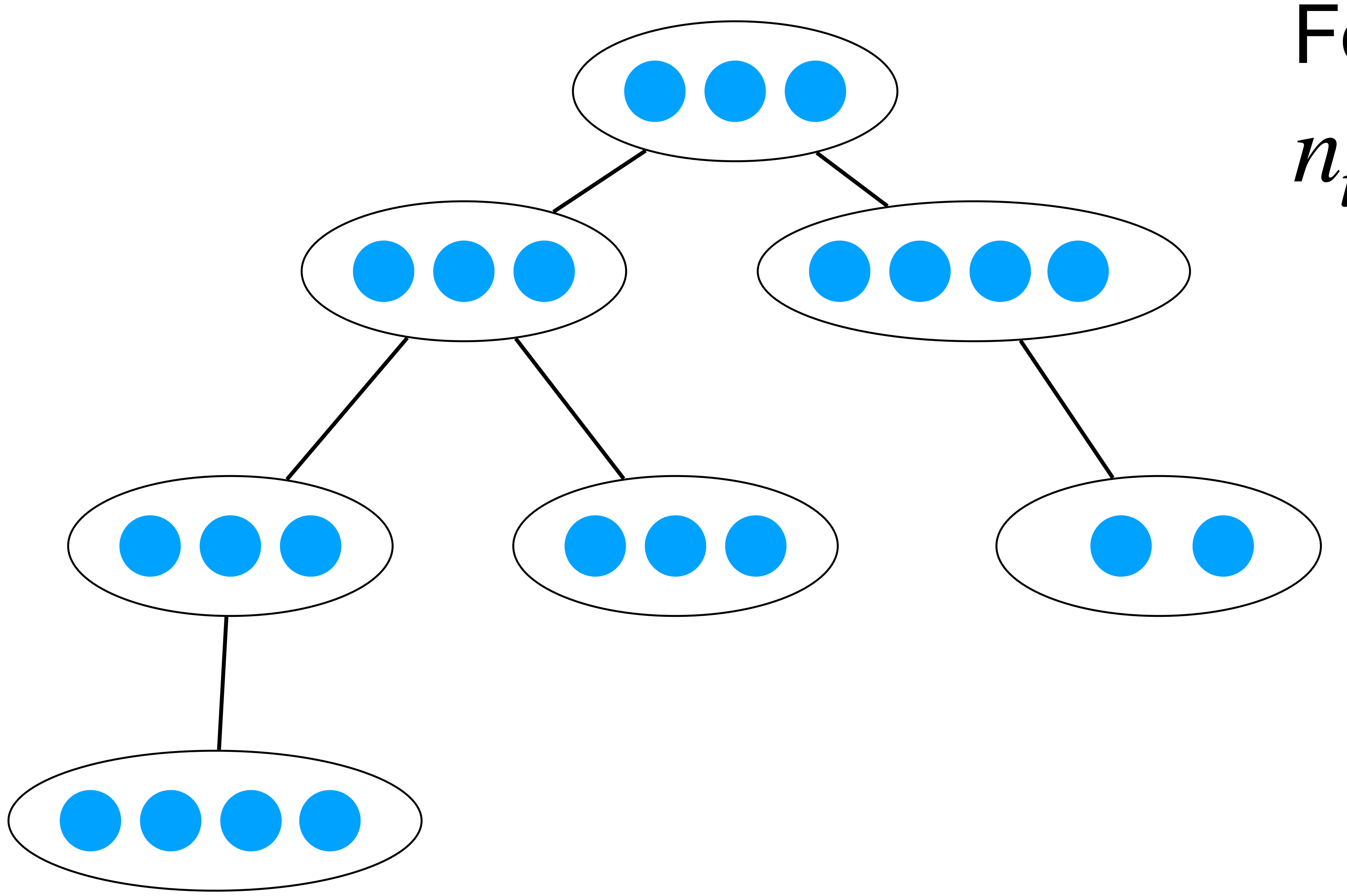
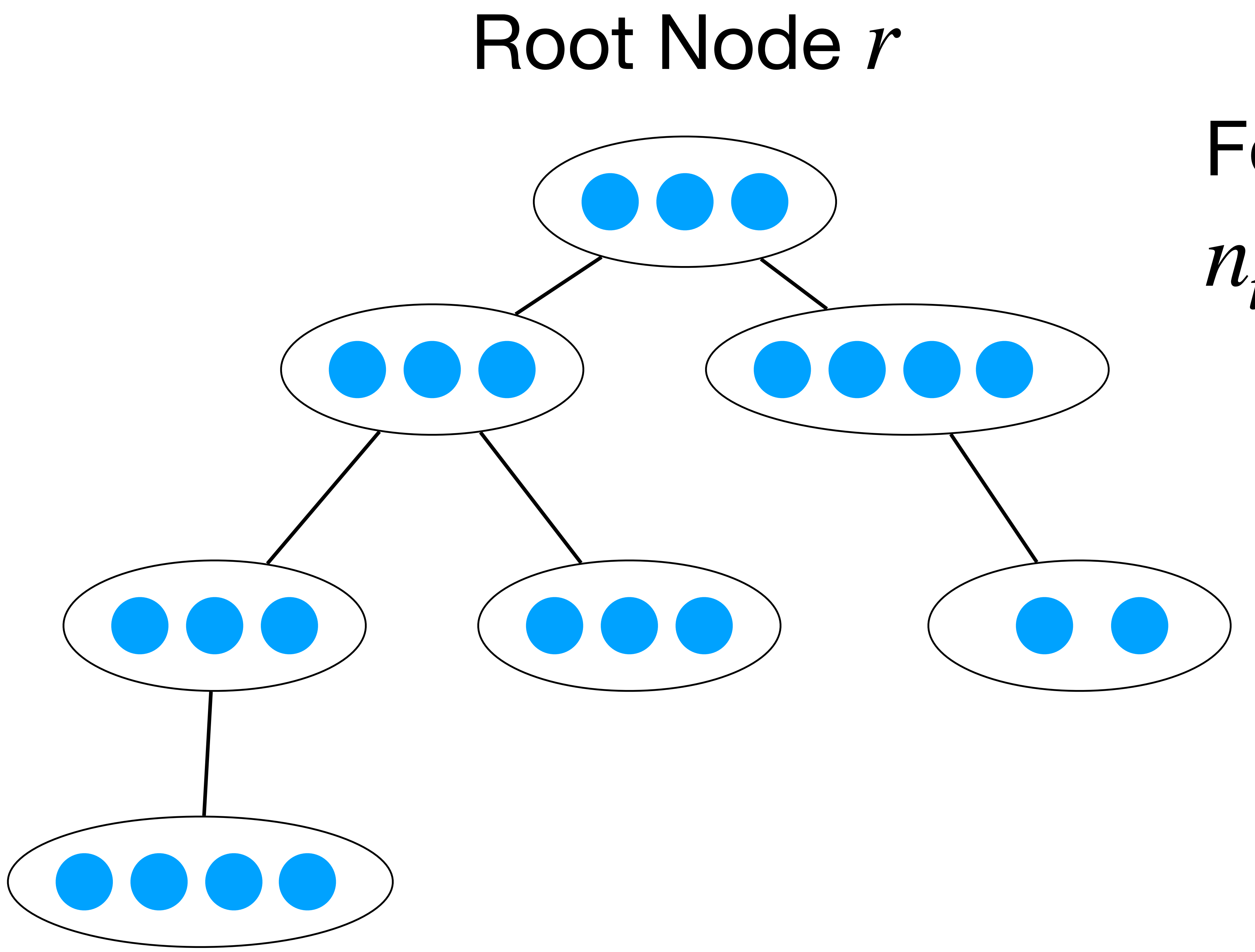


Independent Set Parameterized by Treewidth



For each subset $S \subseteq \chi(t)$ of bag vertices,
$n_t(S)$ = | max. **IS** of G_t containing S |.

Independent Set Parameterized by Treewidth



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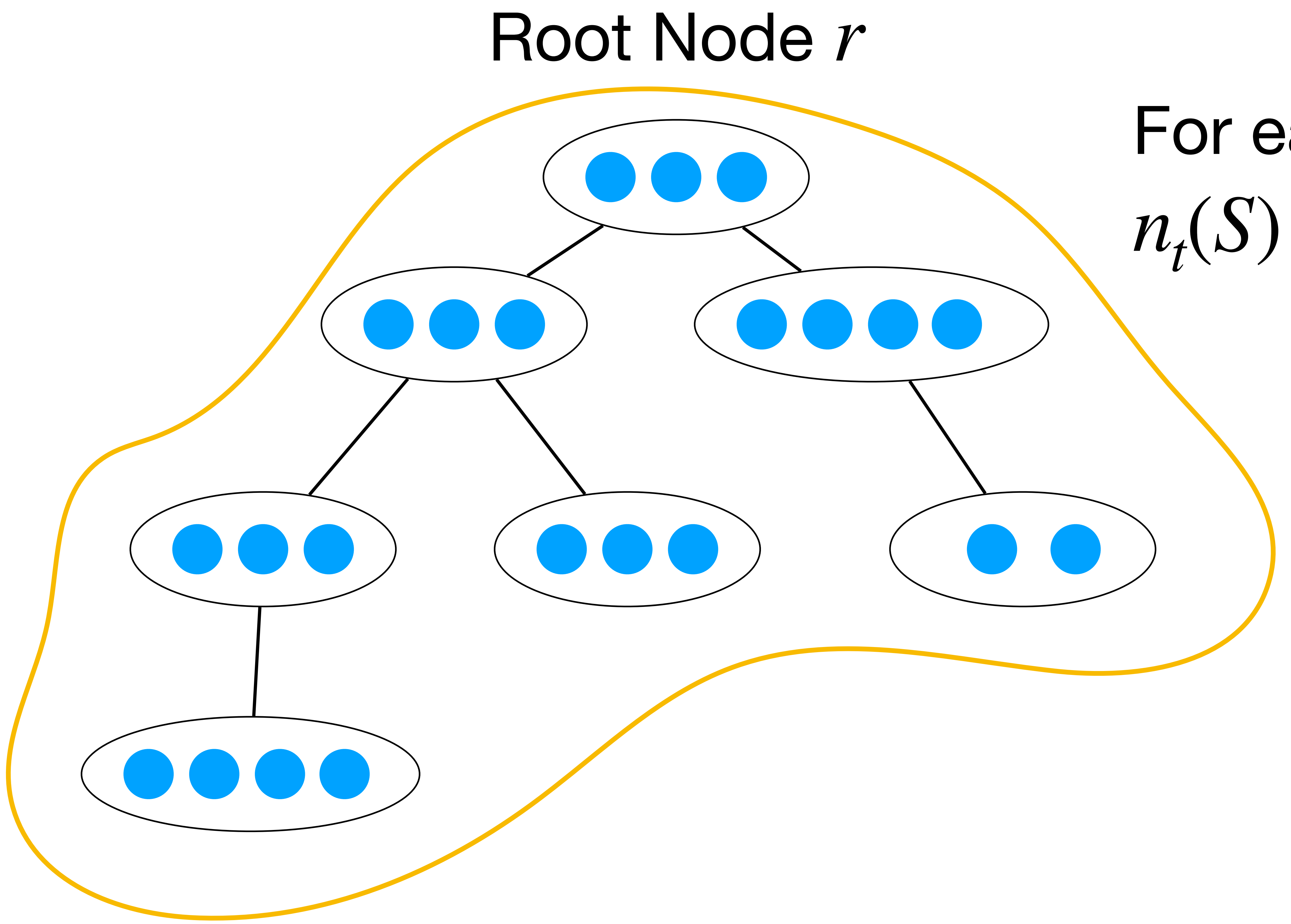

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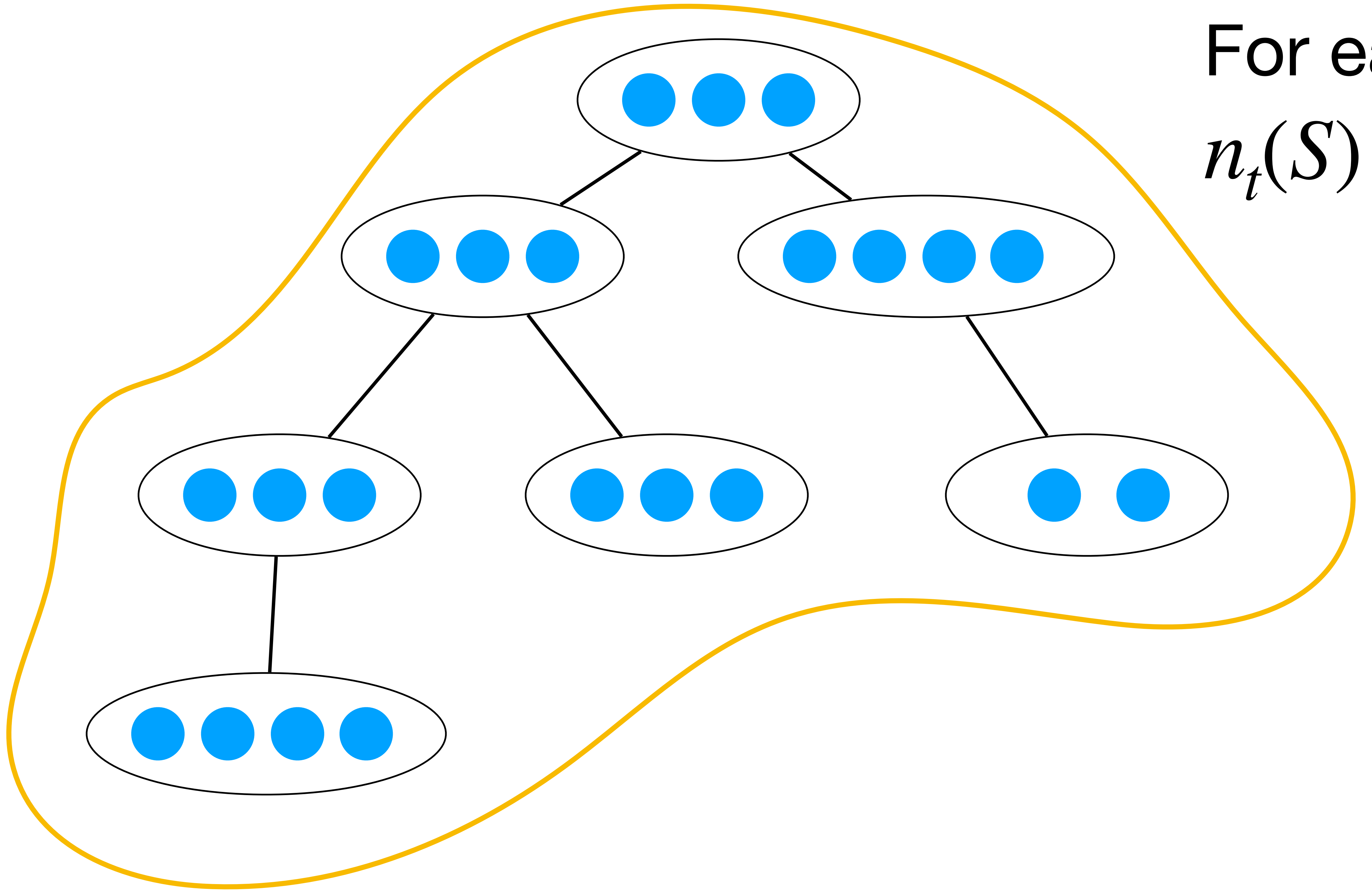
Independent Set Parameterized by Treewidth

Root Node r



For each subset $S \subseteq \chi(t)$ of bag vertices,
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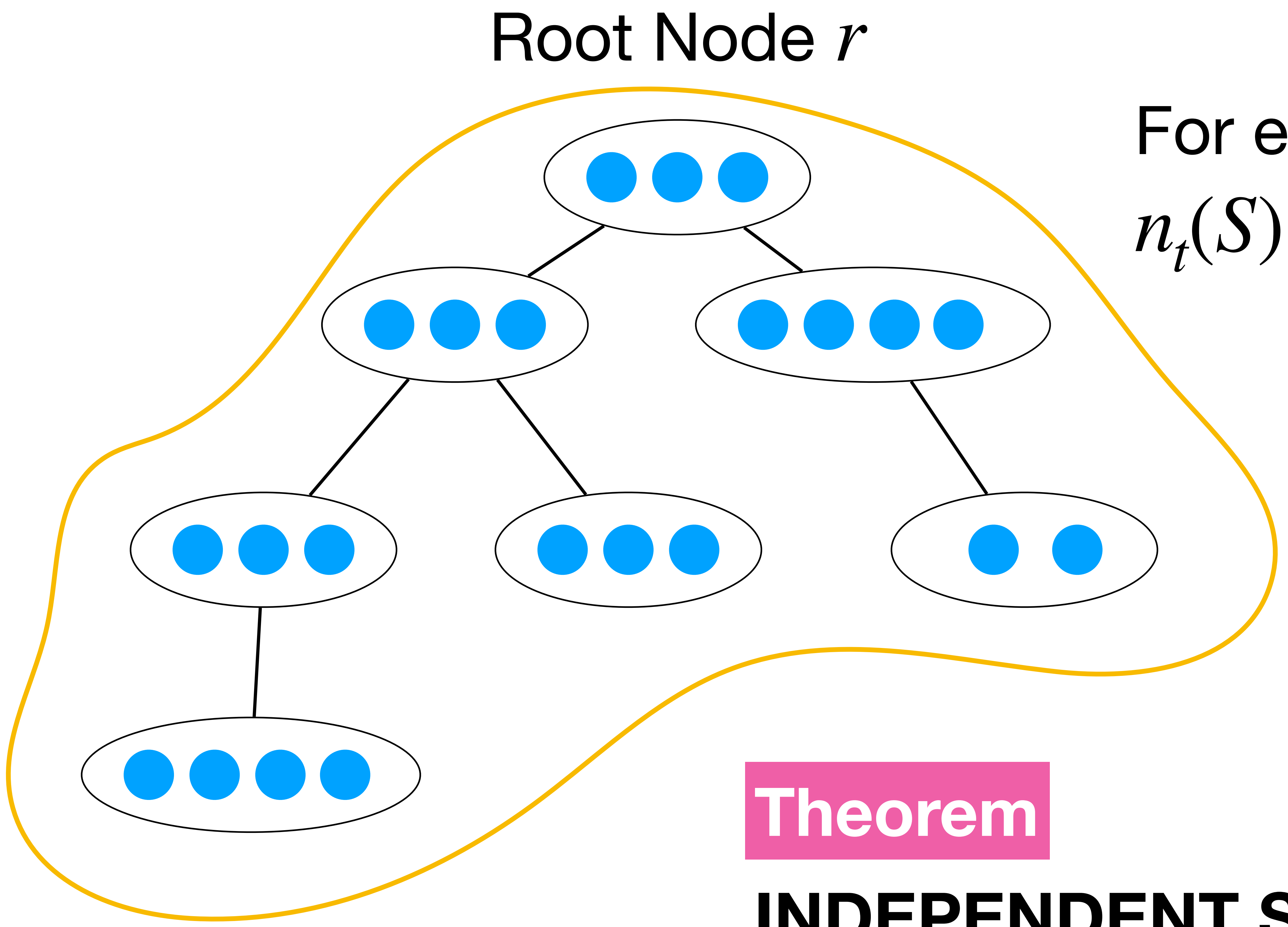
$$G_r = G$$

Independent Set Parameterized by Treewidth



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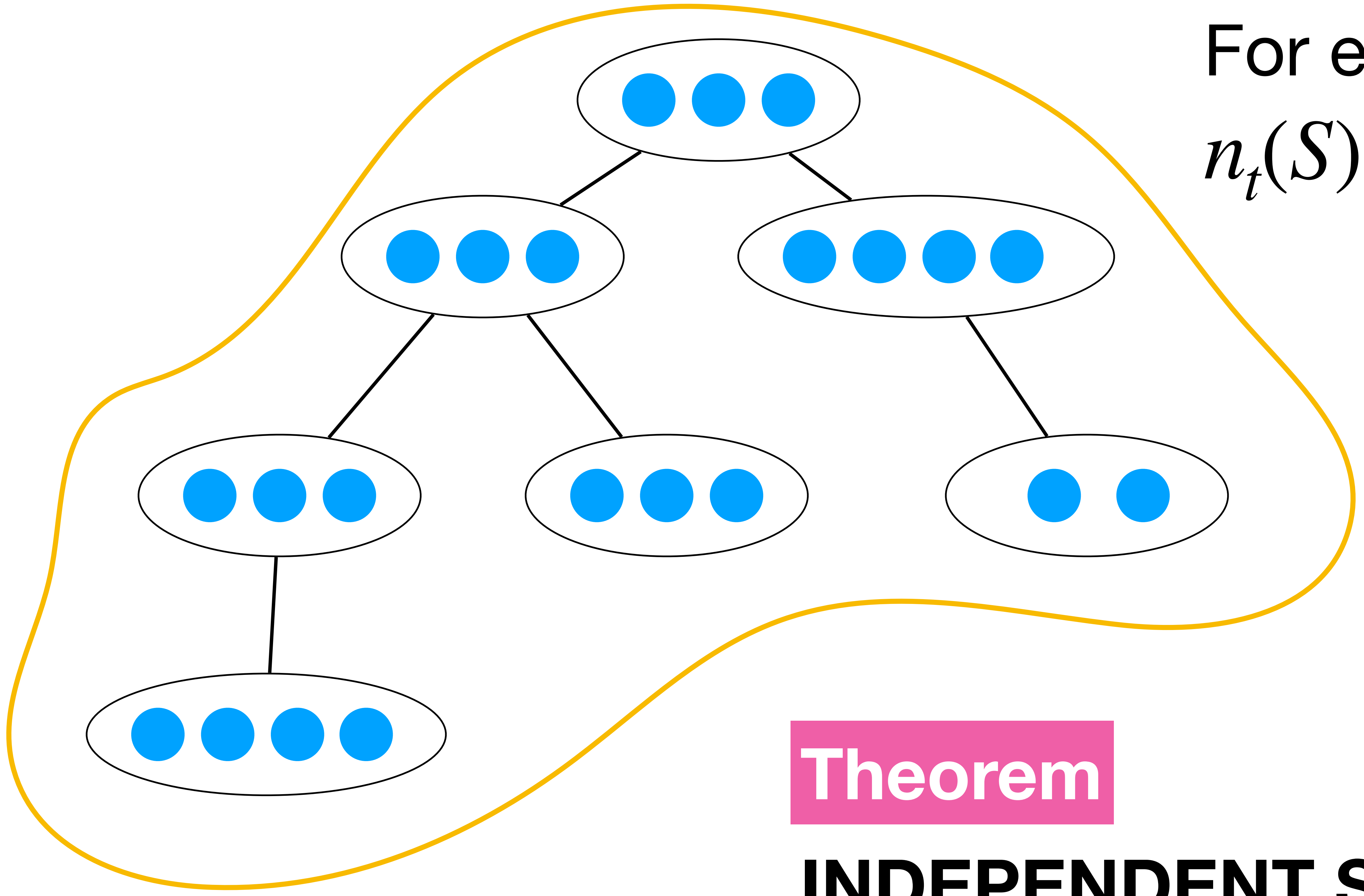
Theorem

INDEPENDENT SET is **FPT** parameterized by the treewidth of the input graph.*

Independent Set Parameterized by Treewidth

Root Node r



For each subset $S \subseteq \chi(t)$ of bag vertices,
$n_t(S)$ = | max. **IS** of G_t containing S |.

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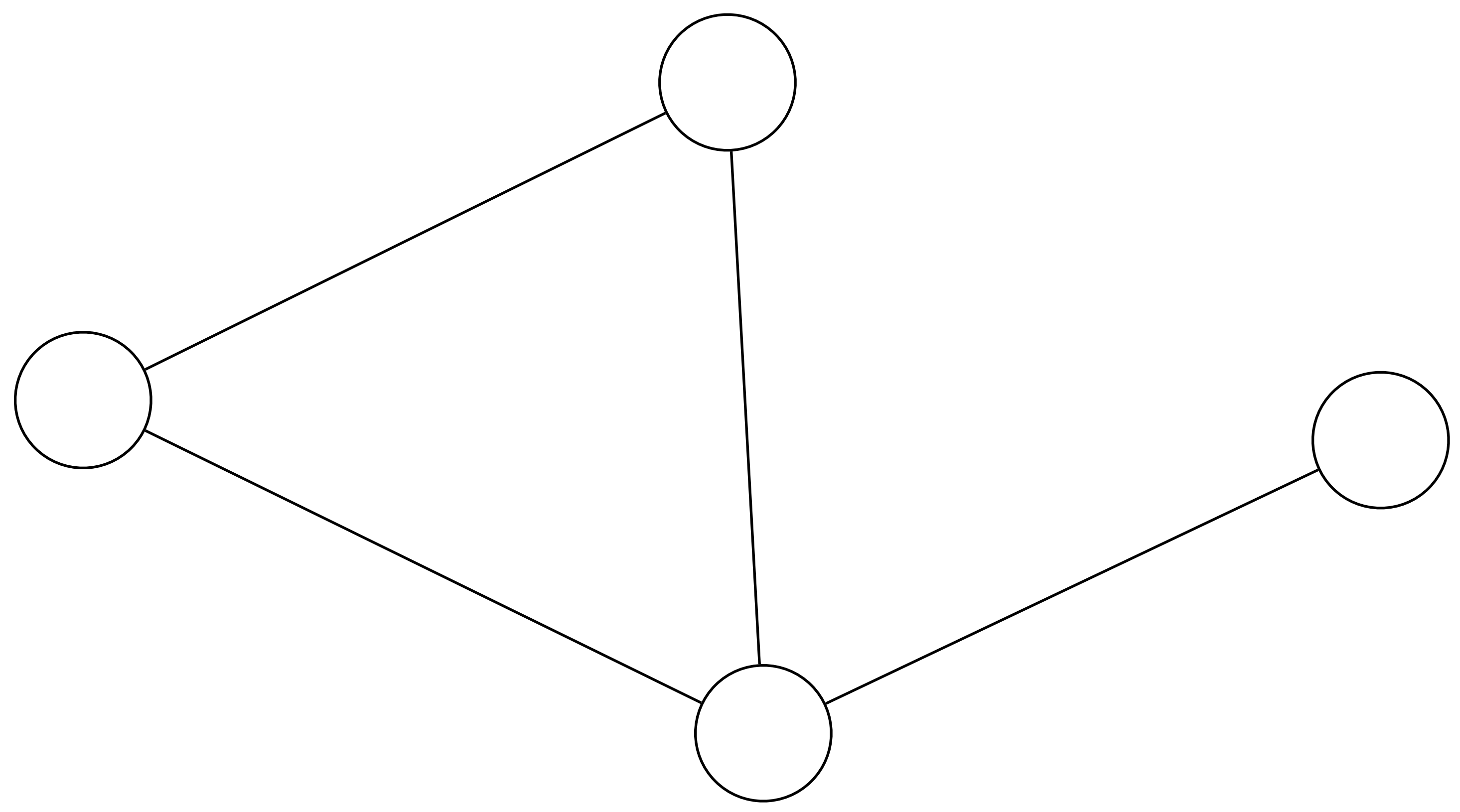
Theorem

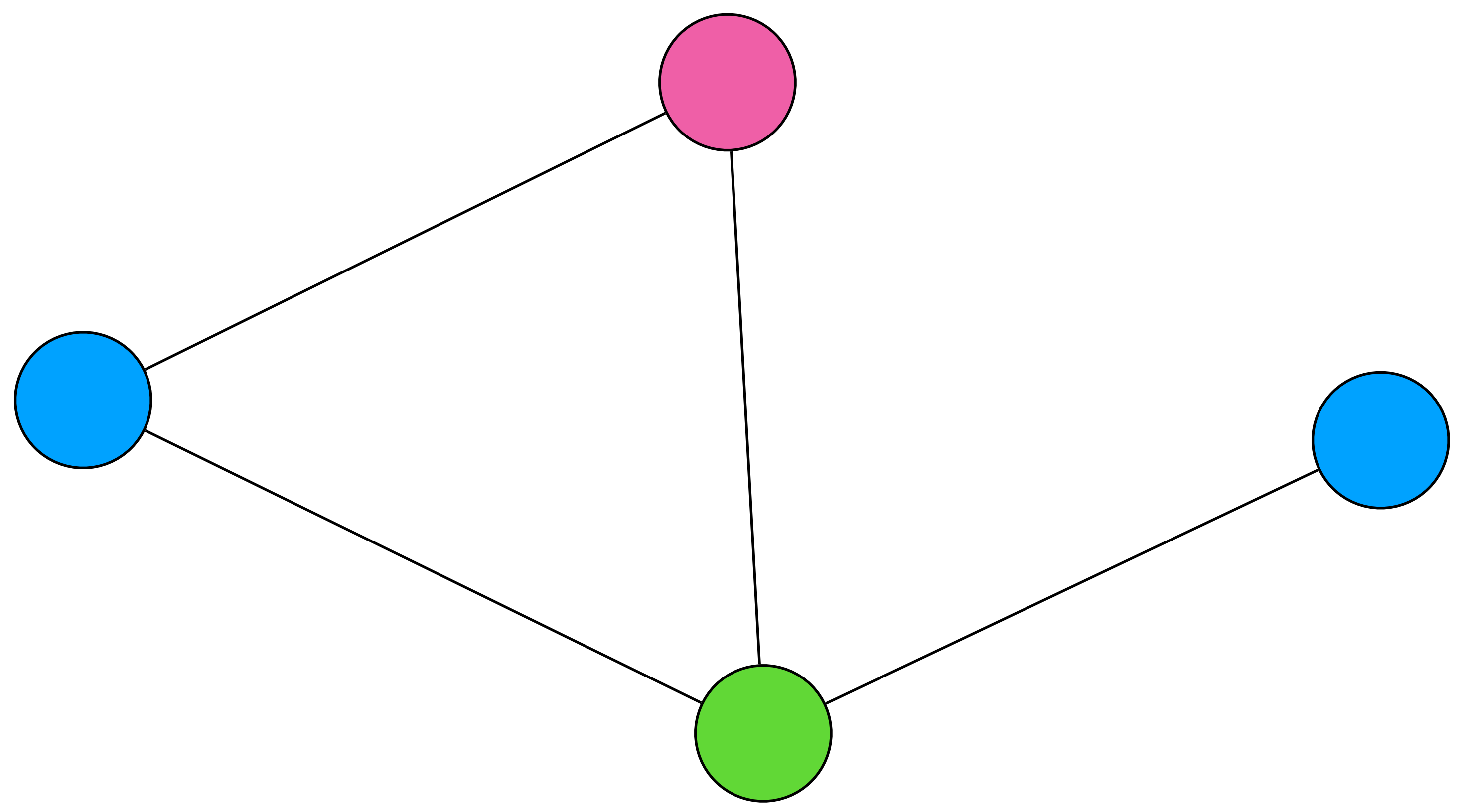
INDEPENDENT SET is **FPT** parameterized by the treewidth of the input graph.*

*If we can compute a nice tree decomposition of width $g(k)$ in FPT time.

3-Colorability



3-Colorability



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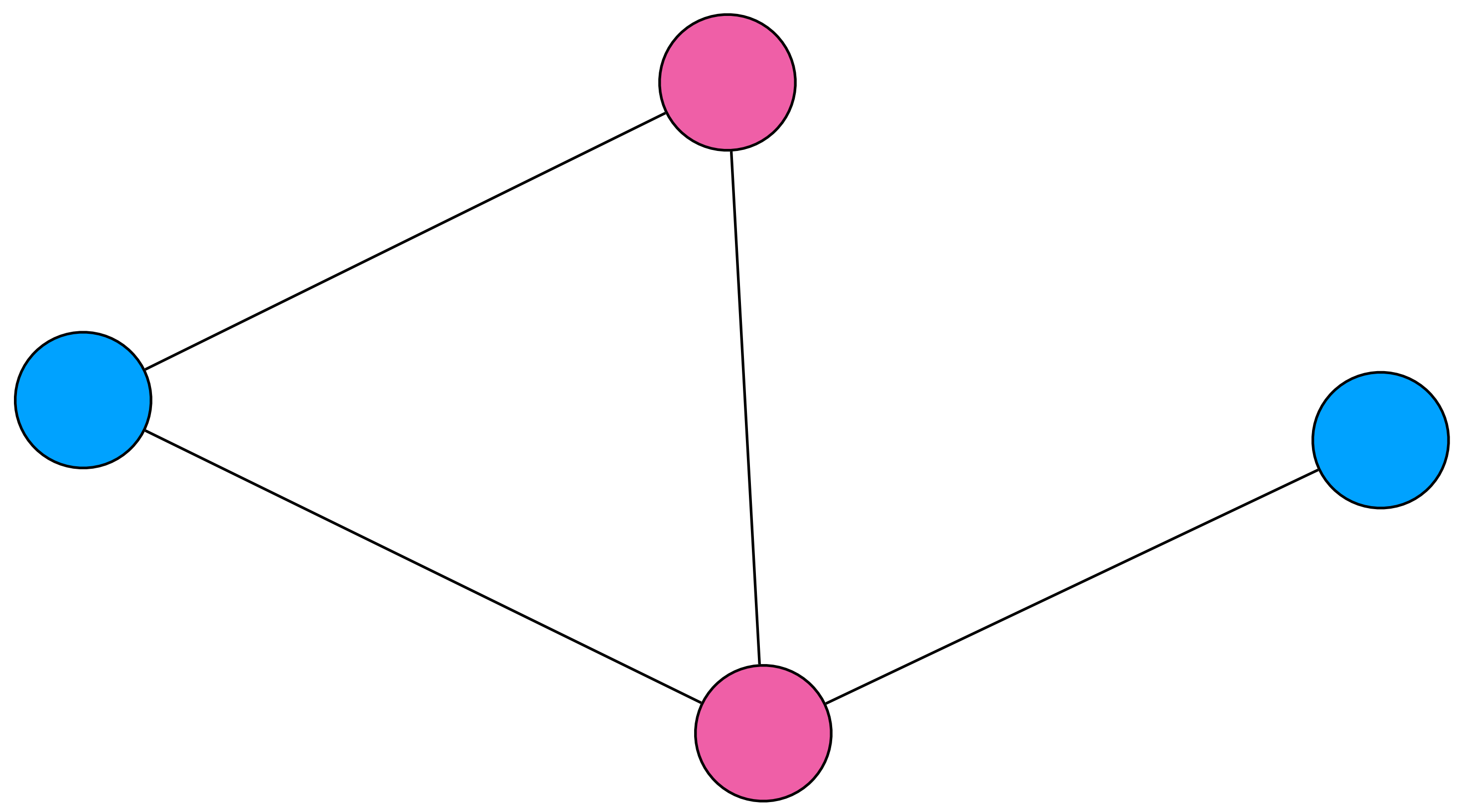

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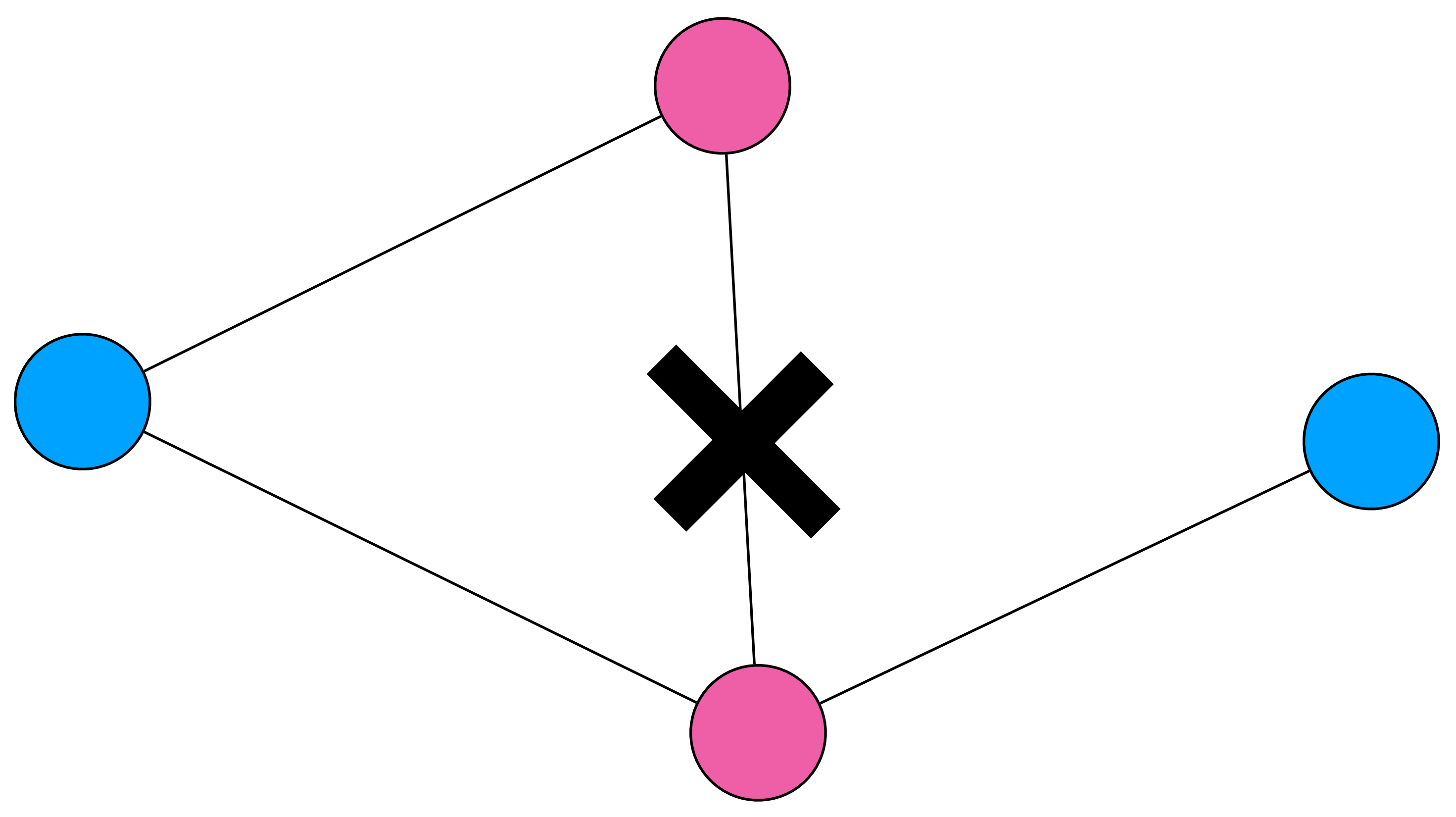

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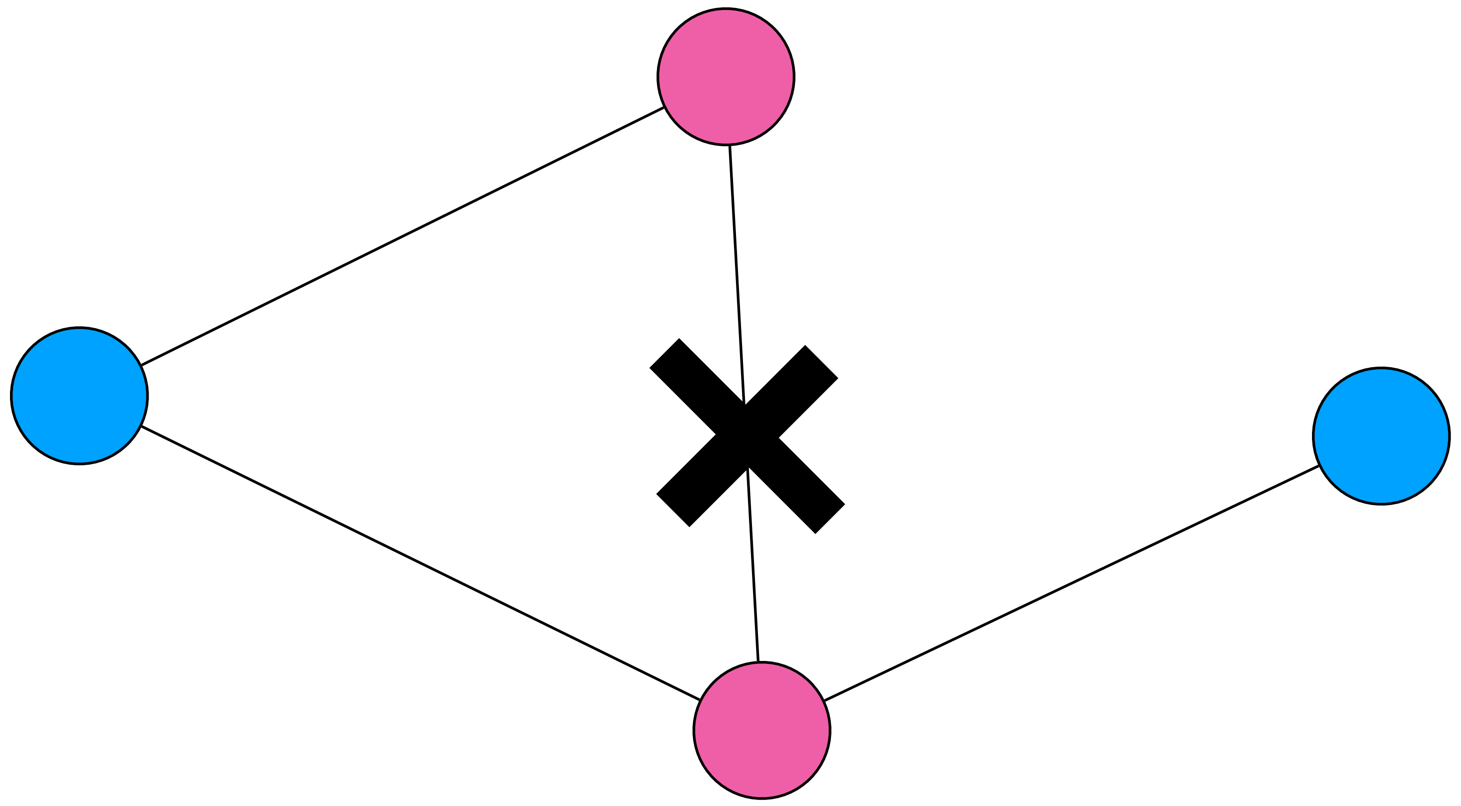
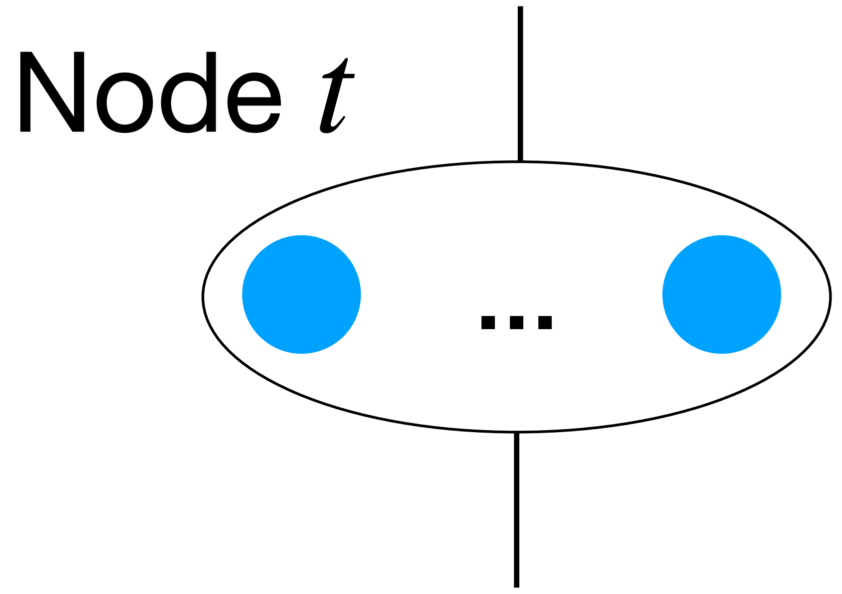


3-COLORABILITY

Input: A graph G.

Question: Does G have a vertex-coloring with 3 colors?

3-Colorability on Tree Decompositions



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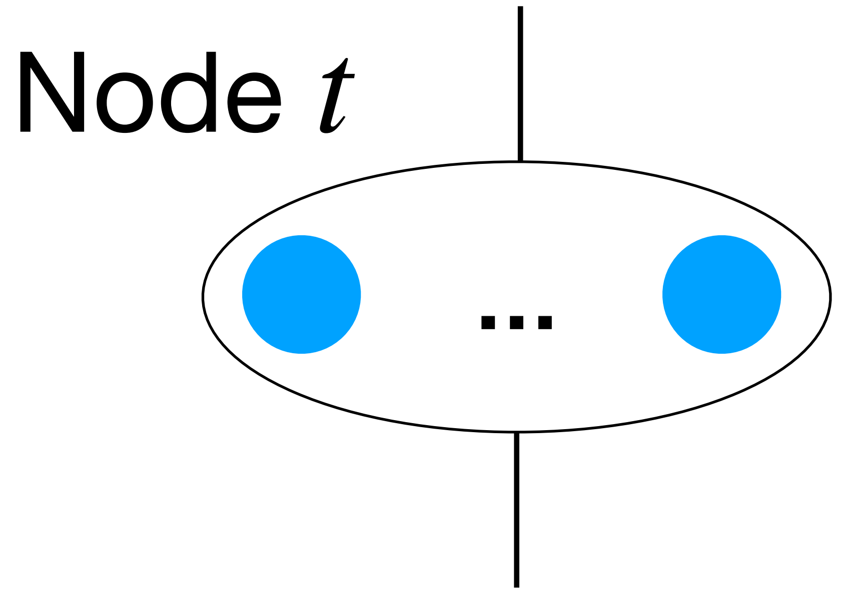

for $\sigma : \chi(t) \rightarrow \{red, green, blue\}$

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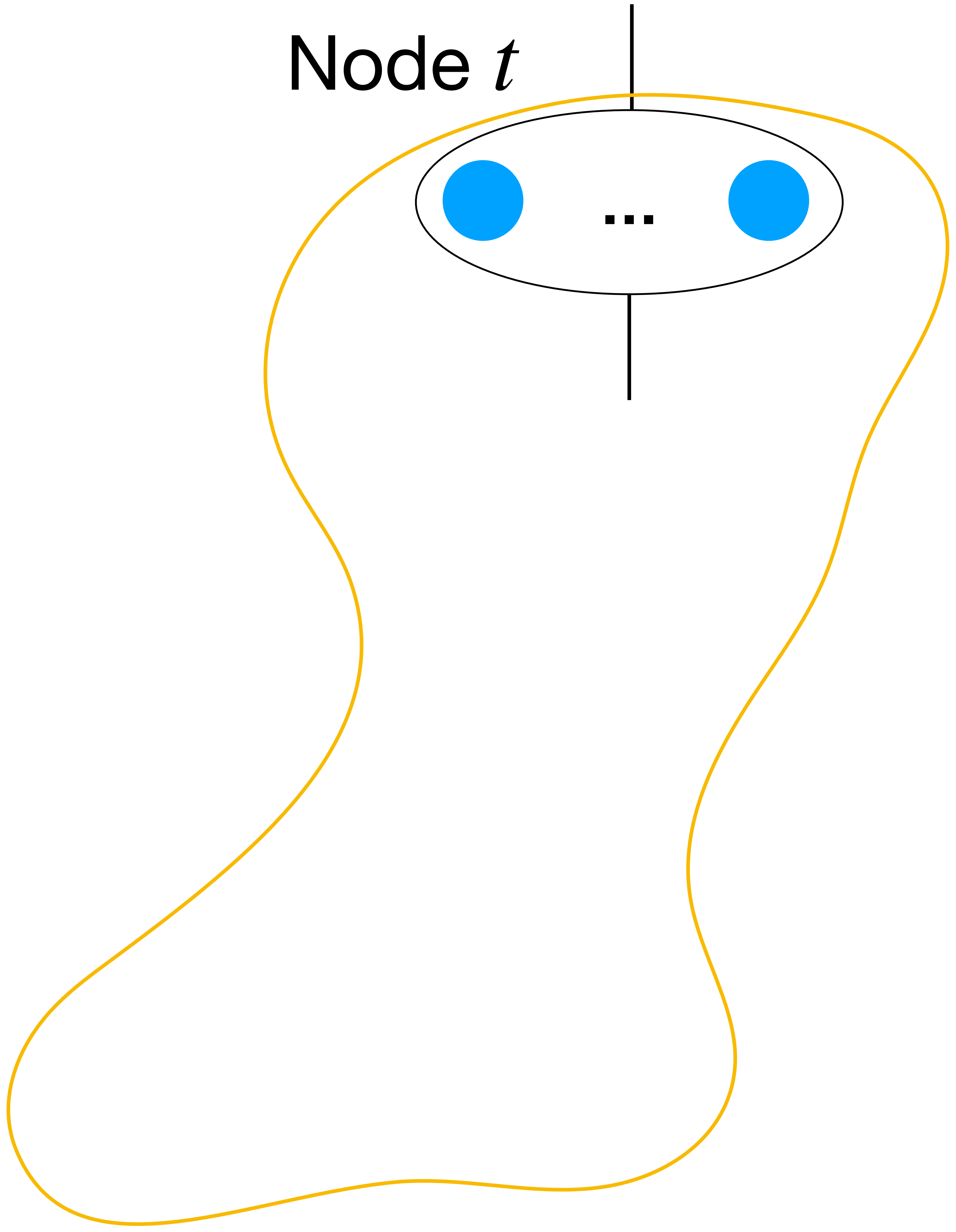

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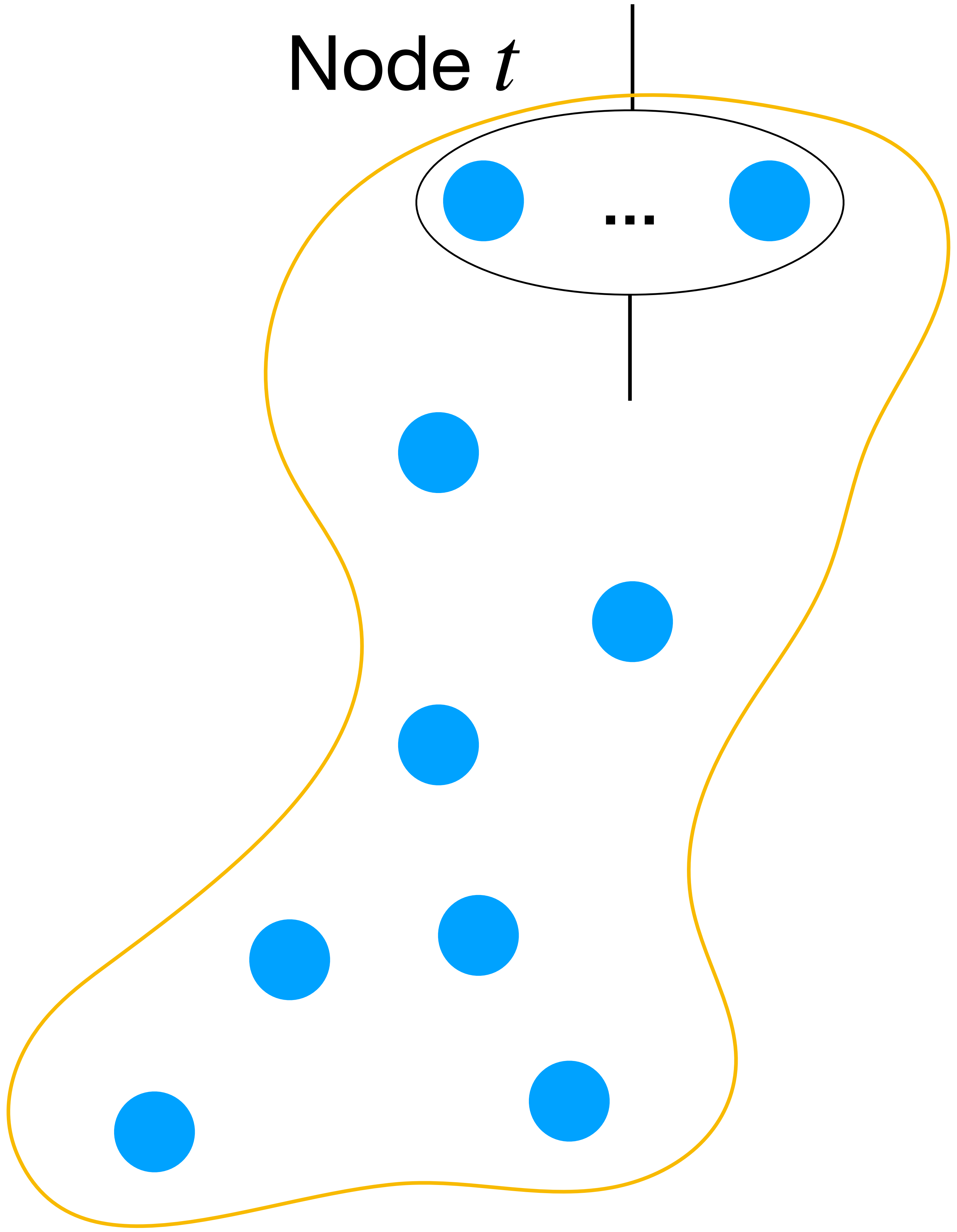

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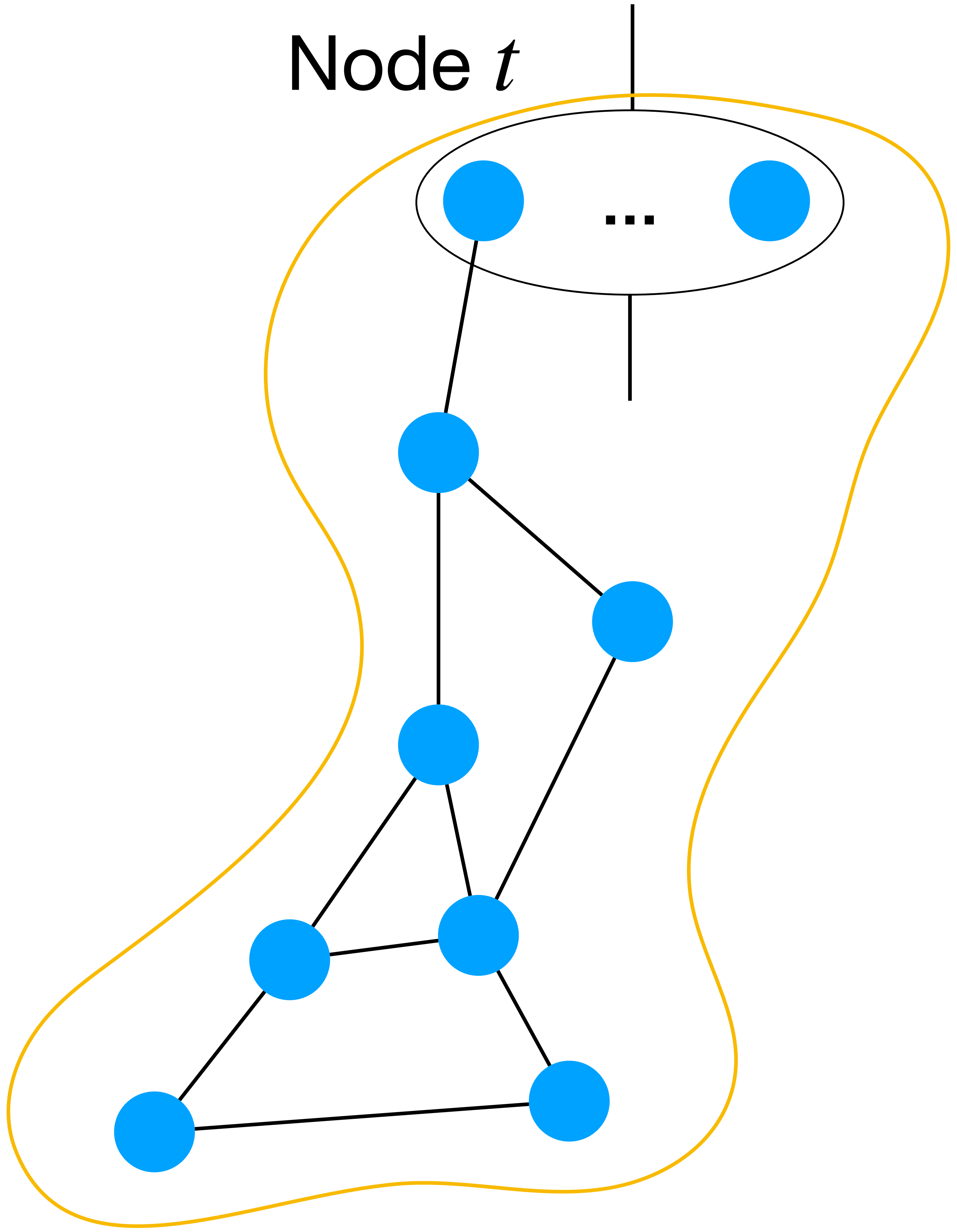

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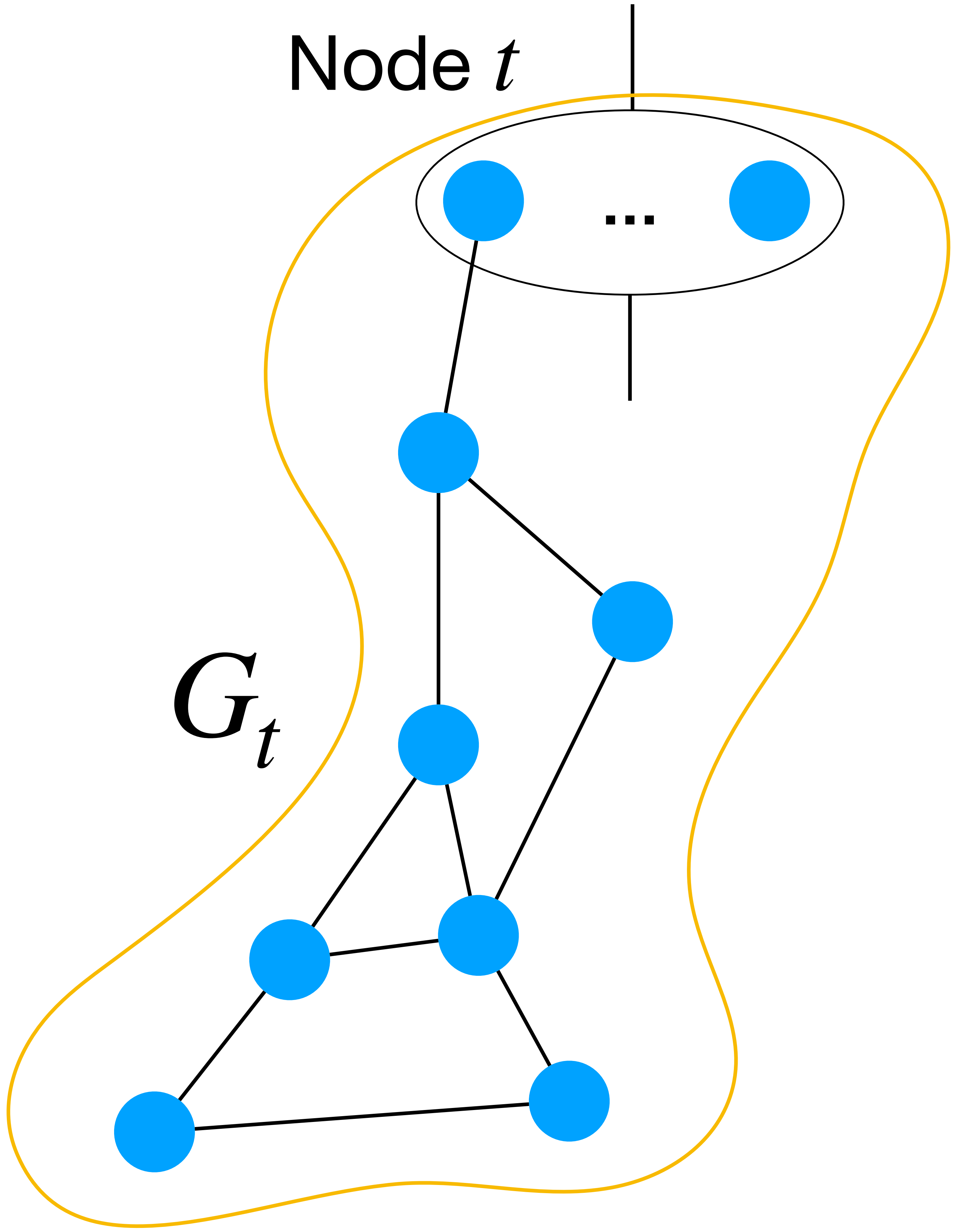

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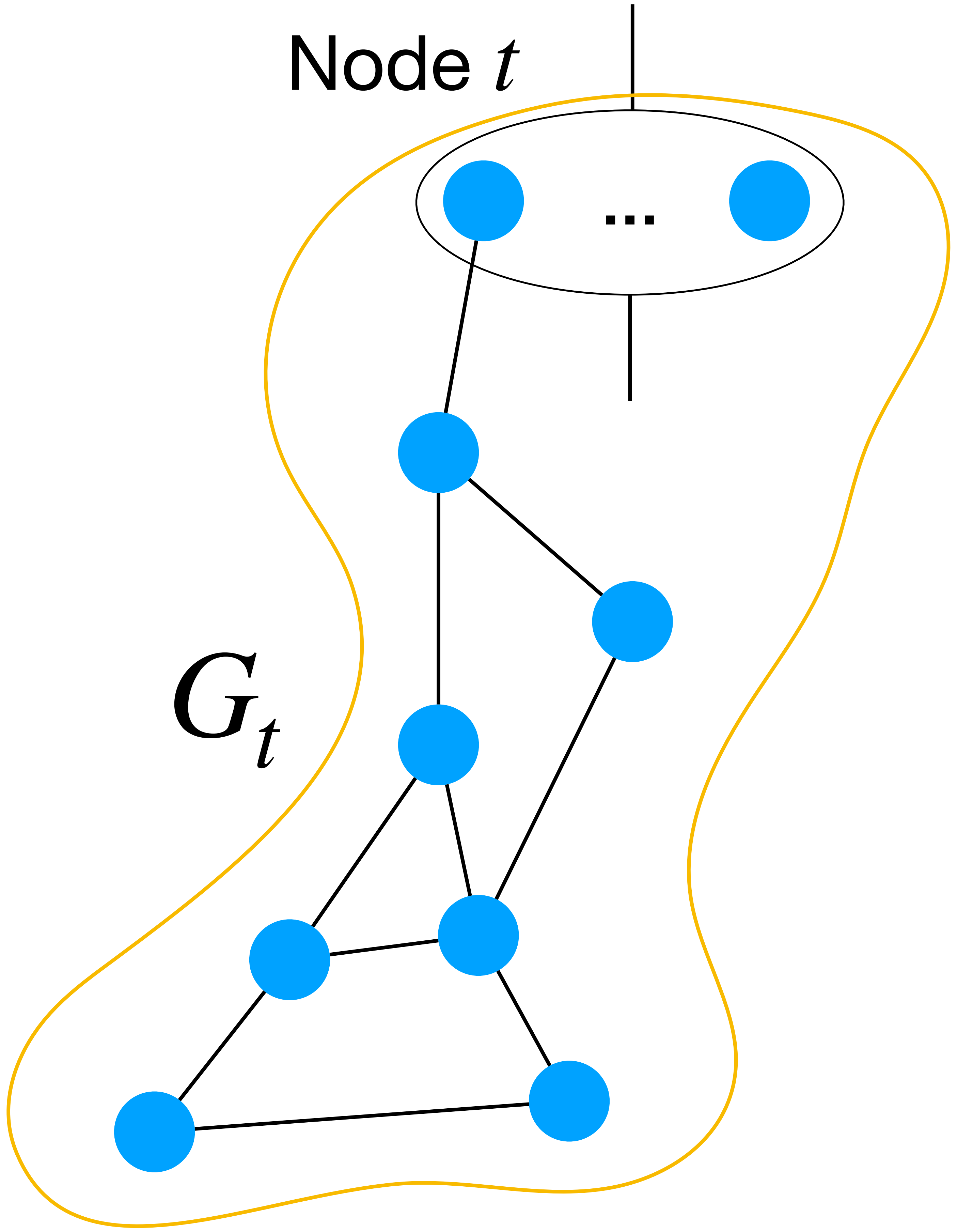

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$$n_t(\sigma) =$$

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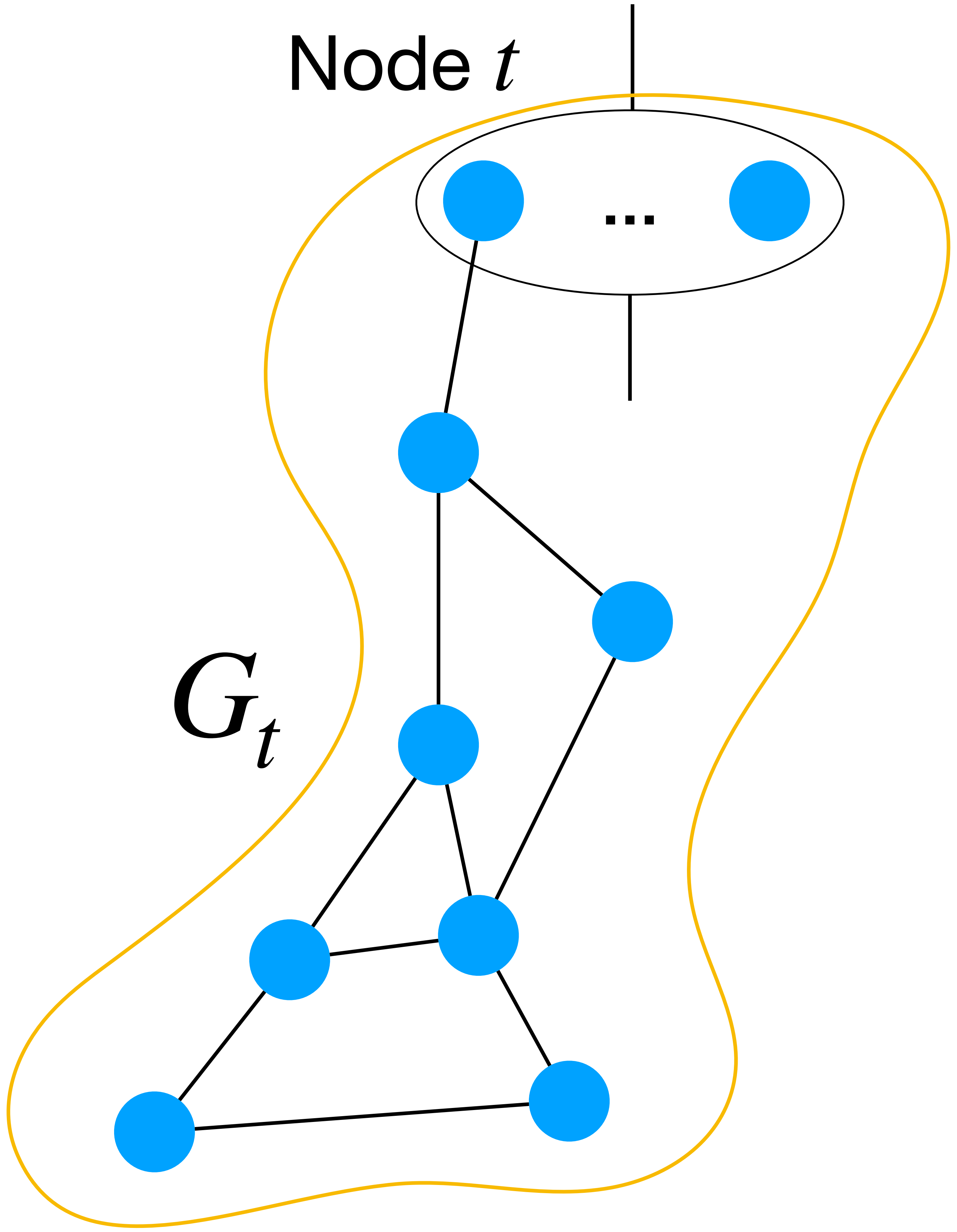

for $\sigma : \chi(t) \to \{red, green, blue\}$

$n_t(\sigma) =$ **true**, if σ can be extended to G_t

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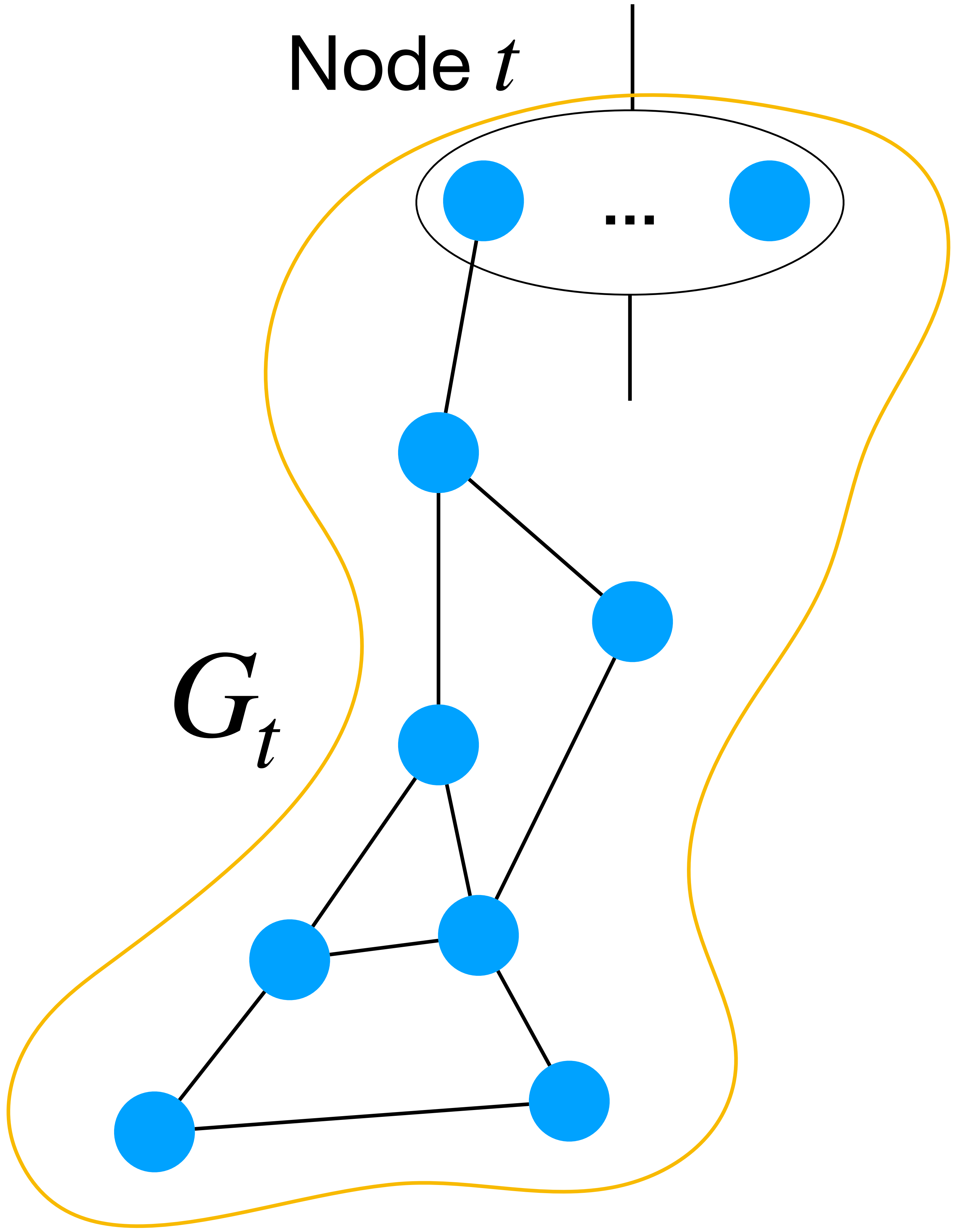

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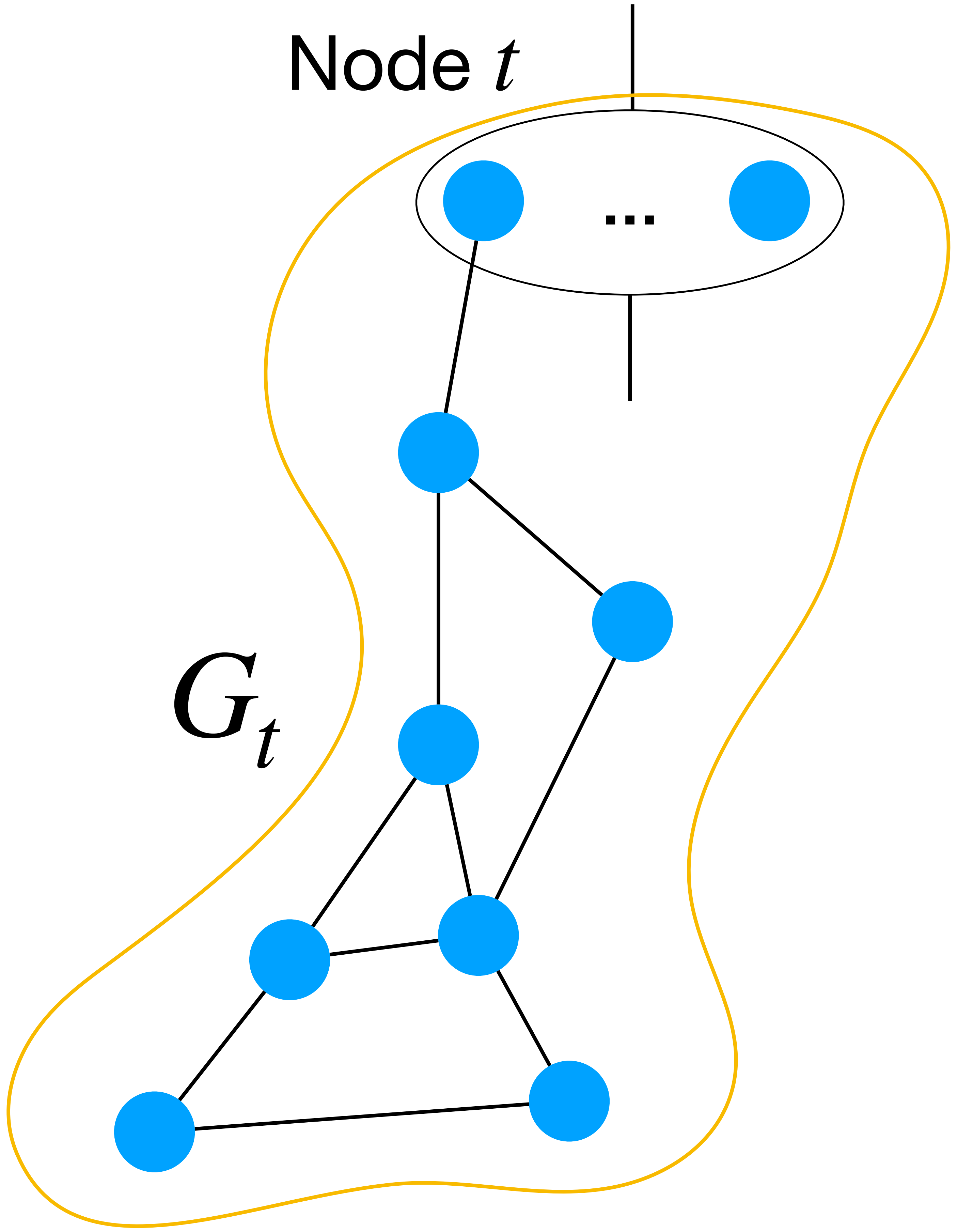

for $\sigma : \chi(t) \rightarrow \{red, green, blue\}$

$$n_t(\sigma) = \begin{cases} \textbf{true}\text{, if } \sigma \text{ can be extended to } G_t \\ \textbf{false}\text{, otherwise} \end{cases}$$

width k

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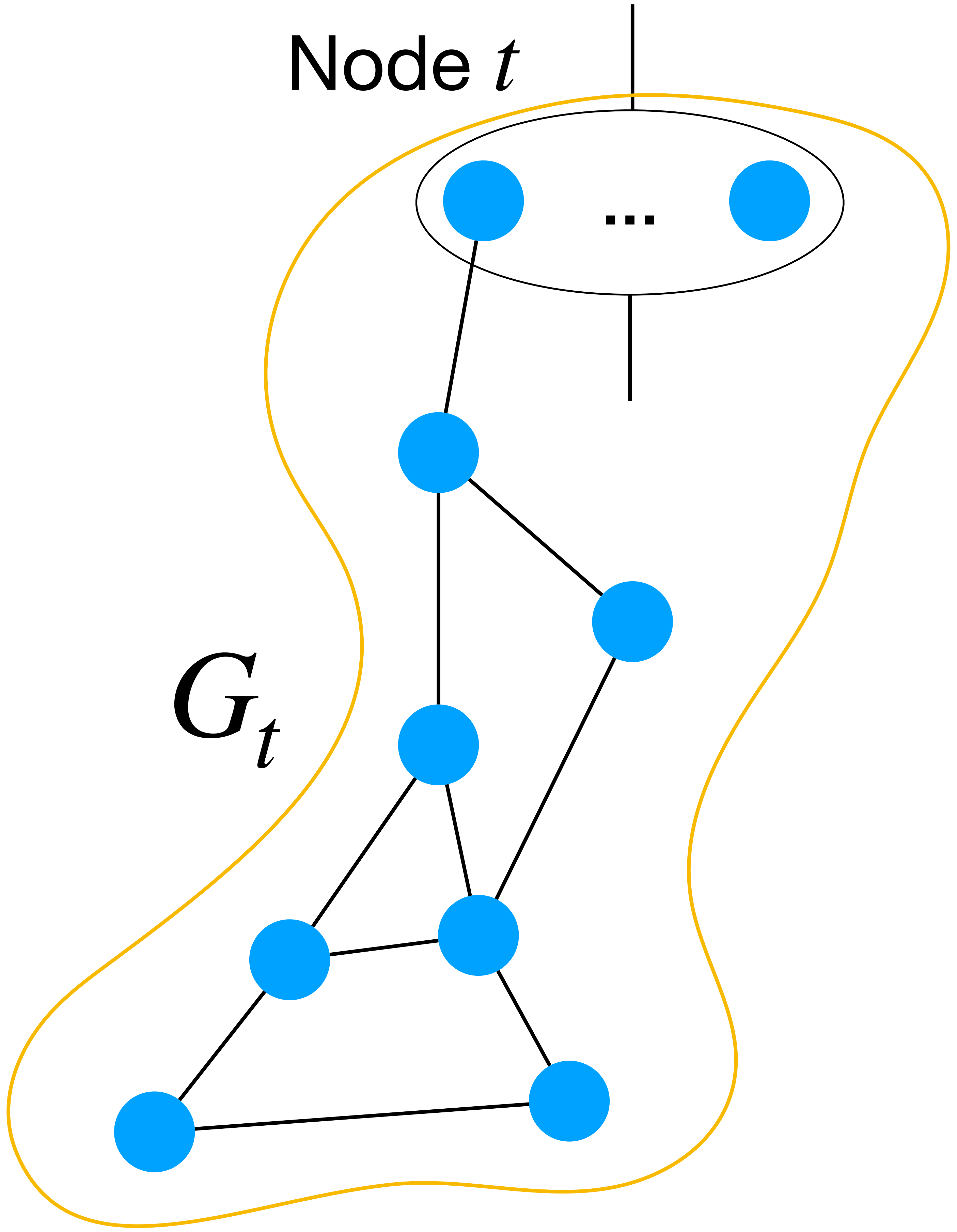

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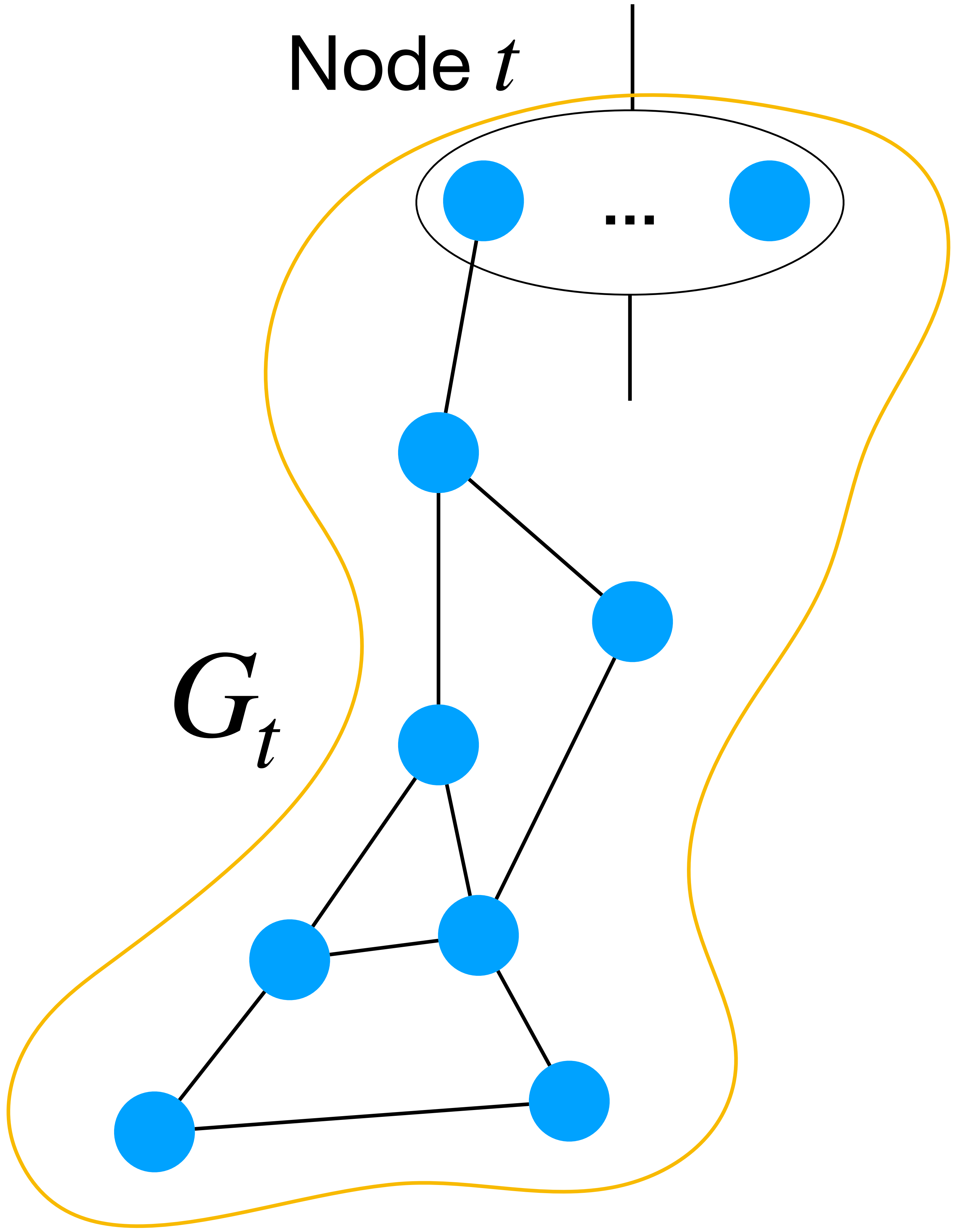

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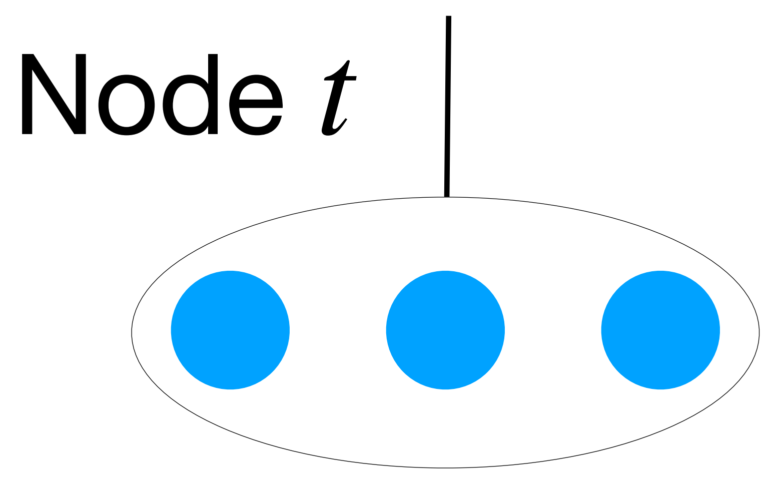
false, otherwise

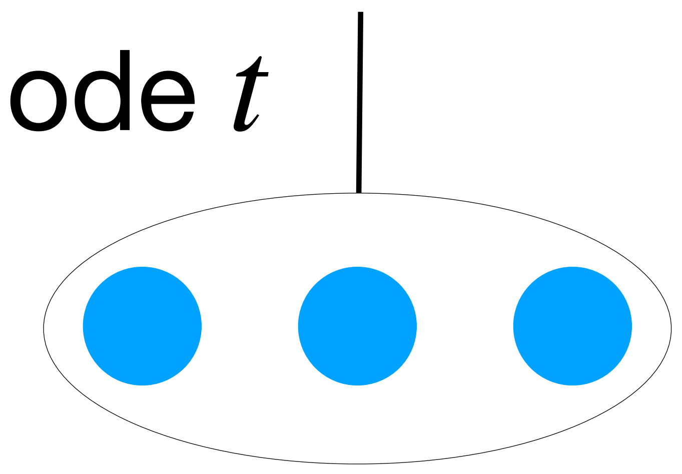
width k ➡ $\leq 3^{k+1}$ colorings

Leaf Nodes



Leaf Nodes

Node t



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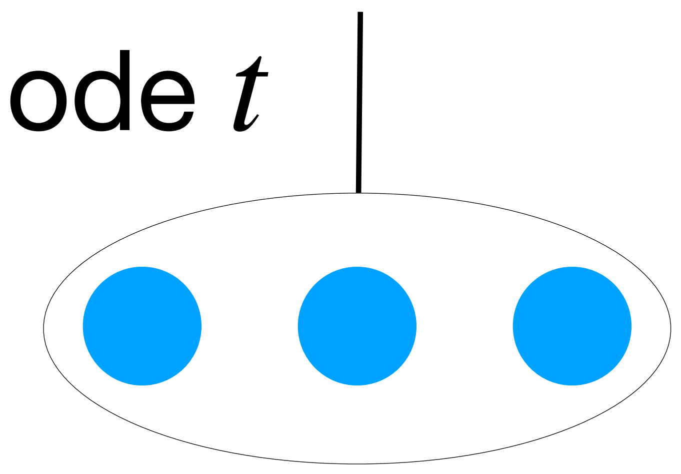

for $\sigma : \chi(t) \rightarrow \{red, green, blue\}$

if σ is a proper coloring of G_t

Leaf Nodes

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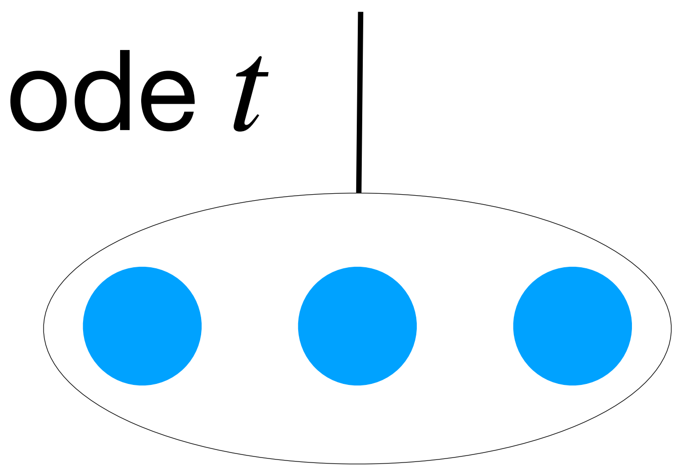

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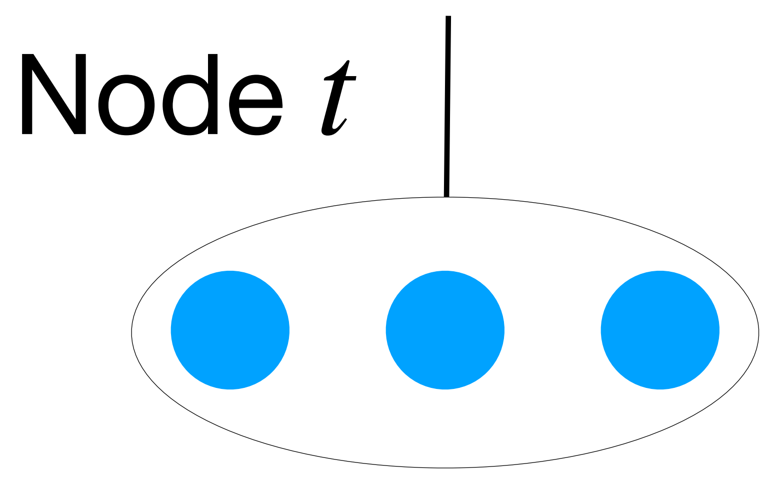
if σ is a proper coloring of G_t

then $n_t(\sigma)$:= **true**

Leaf Nodes



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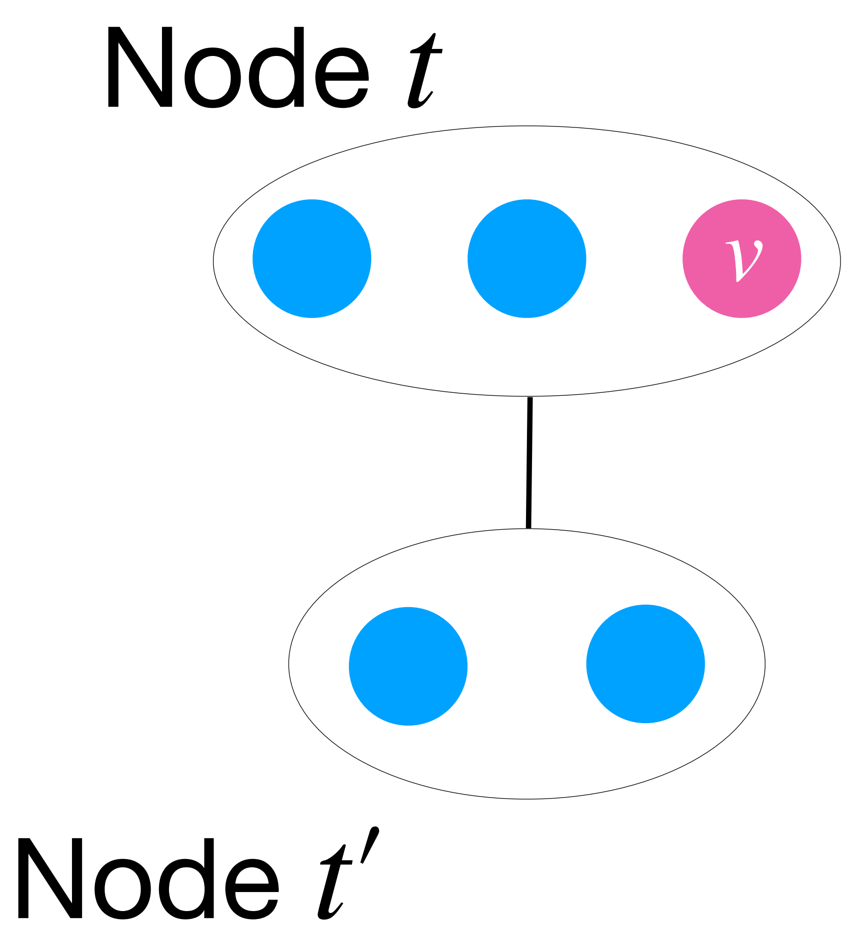

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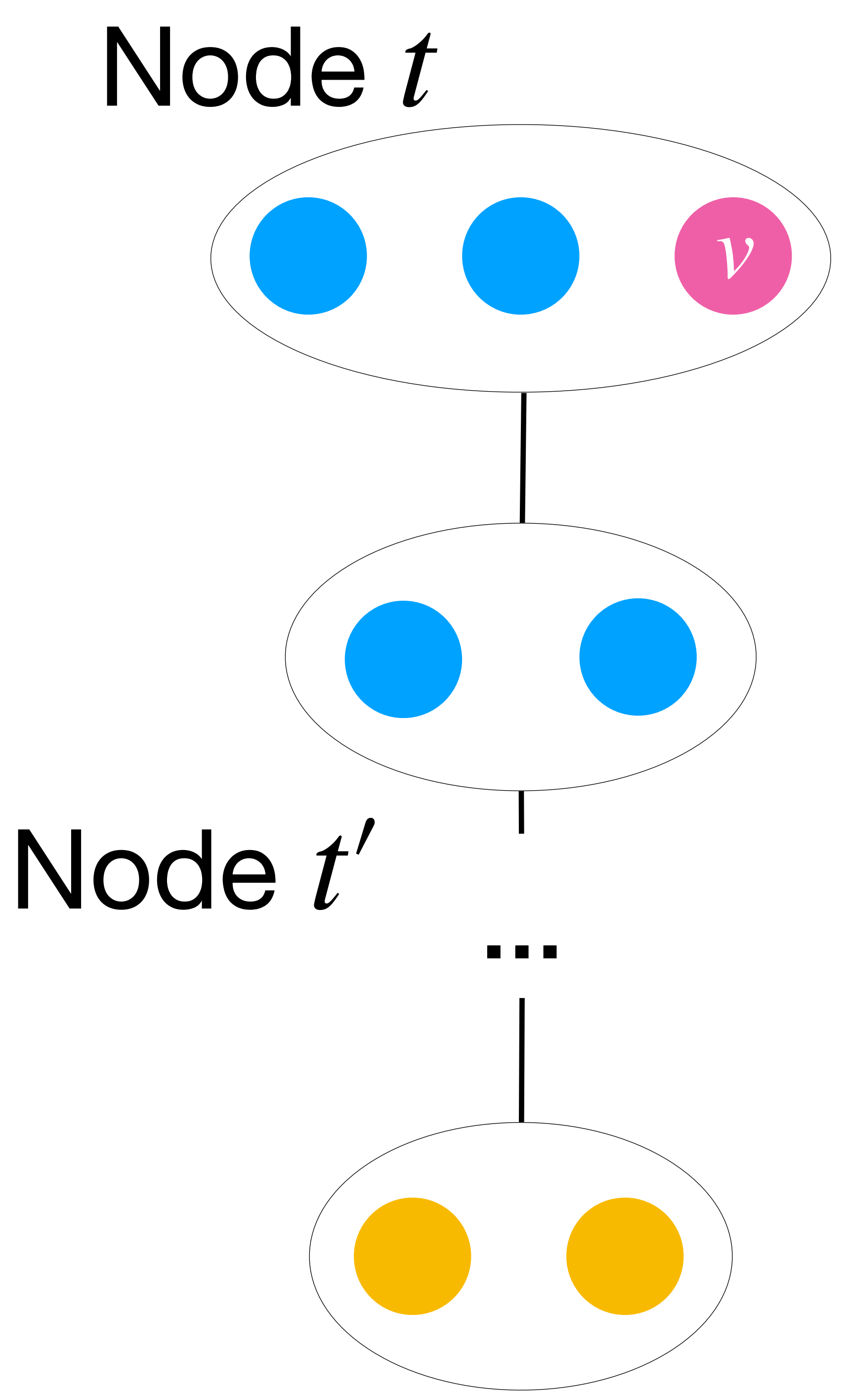

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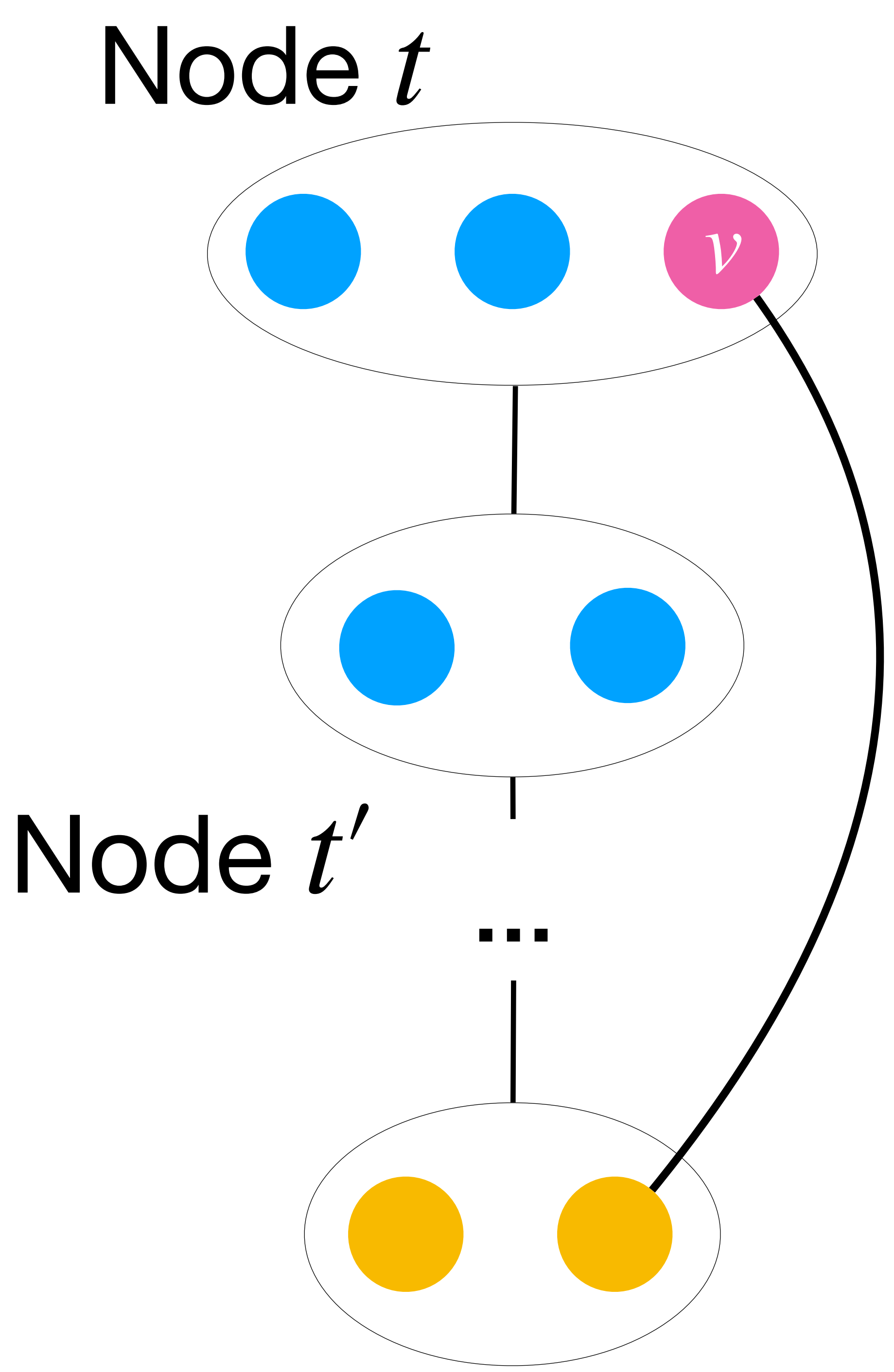

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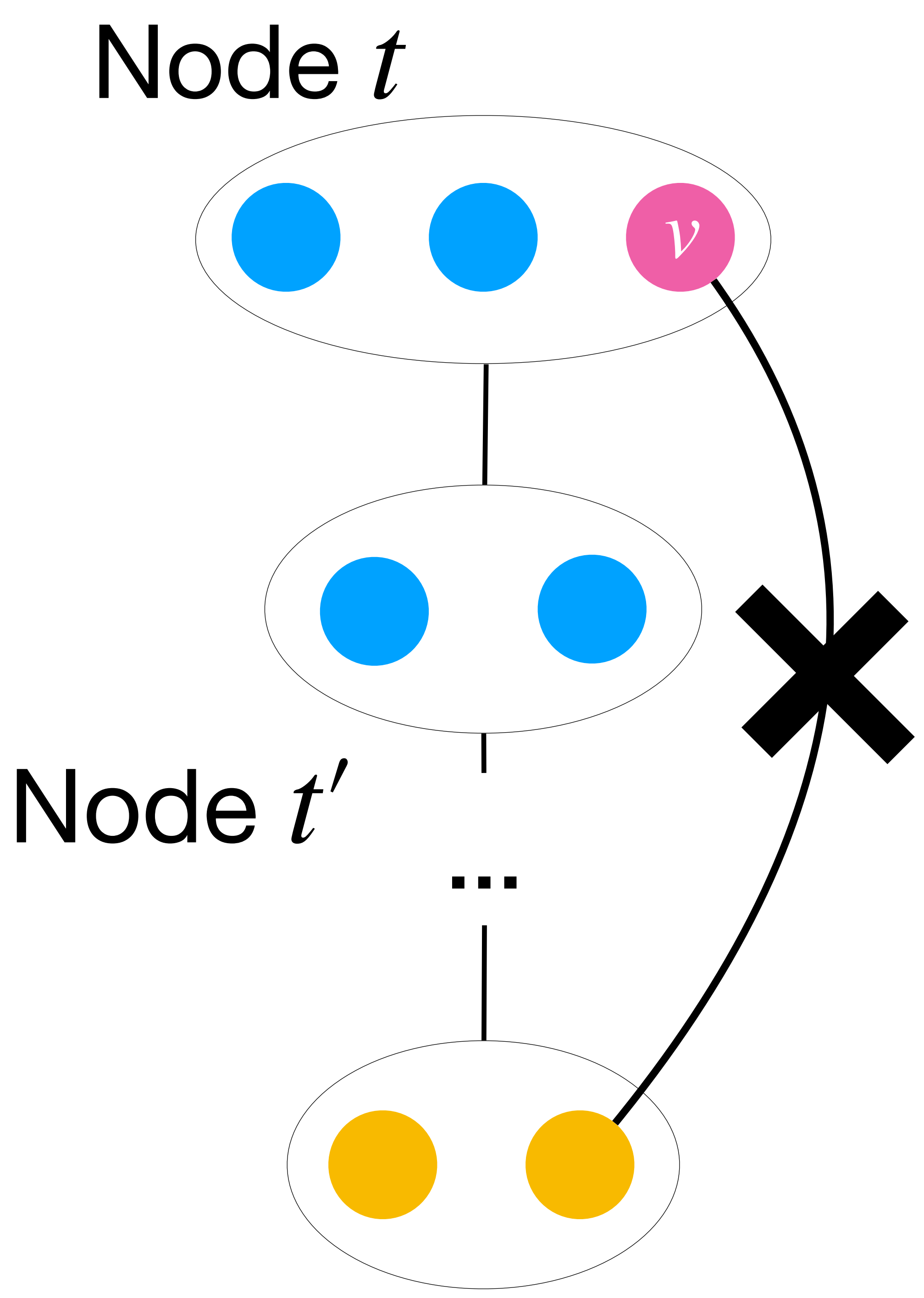

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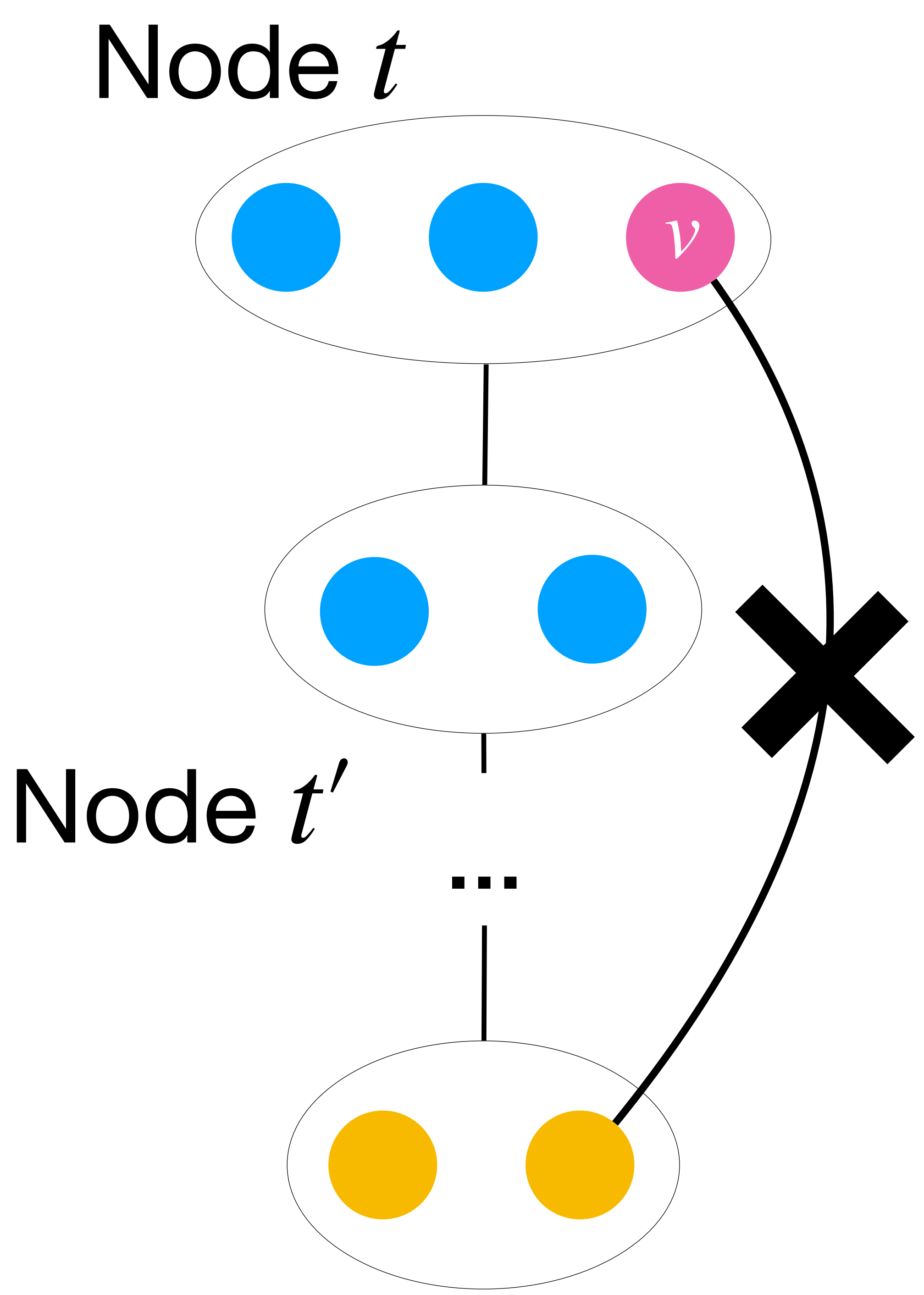

for $\sigma : \chi(t) \to \{red, green, blue\}$

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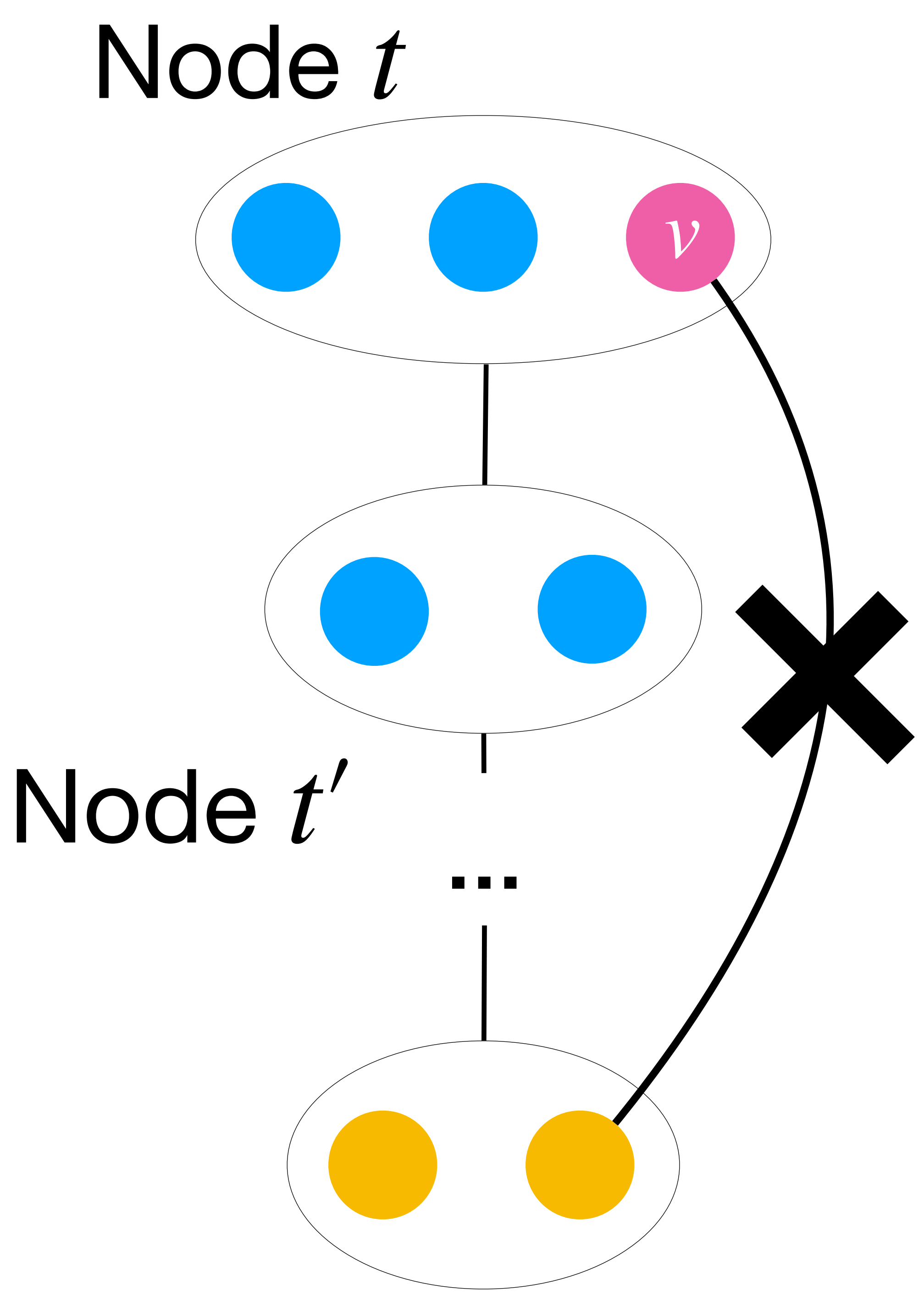

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if σ is a proper coloring of $G[\chi(t)]$

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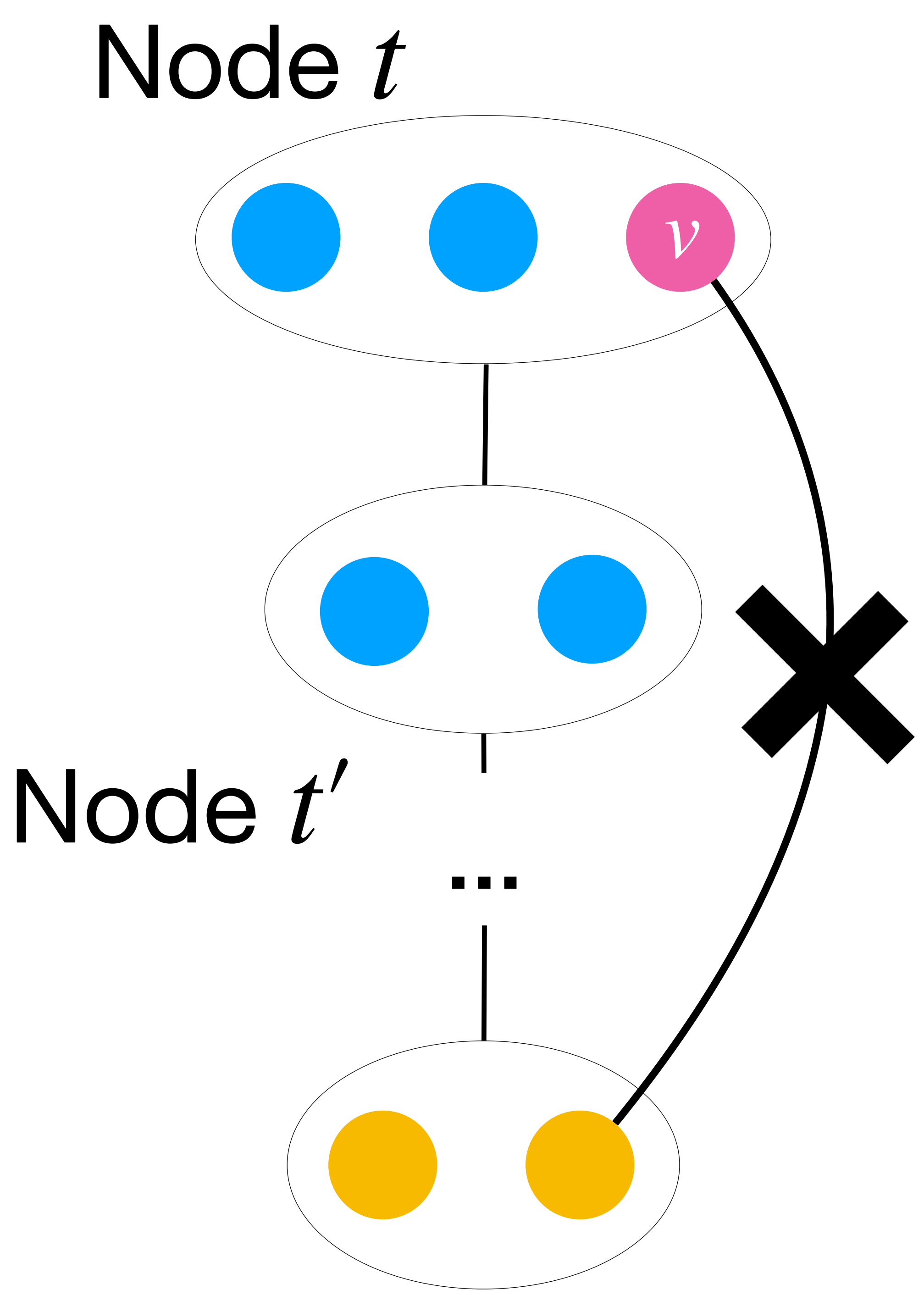

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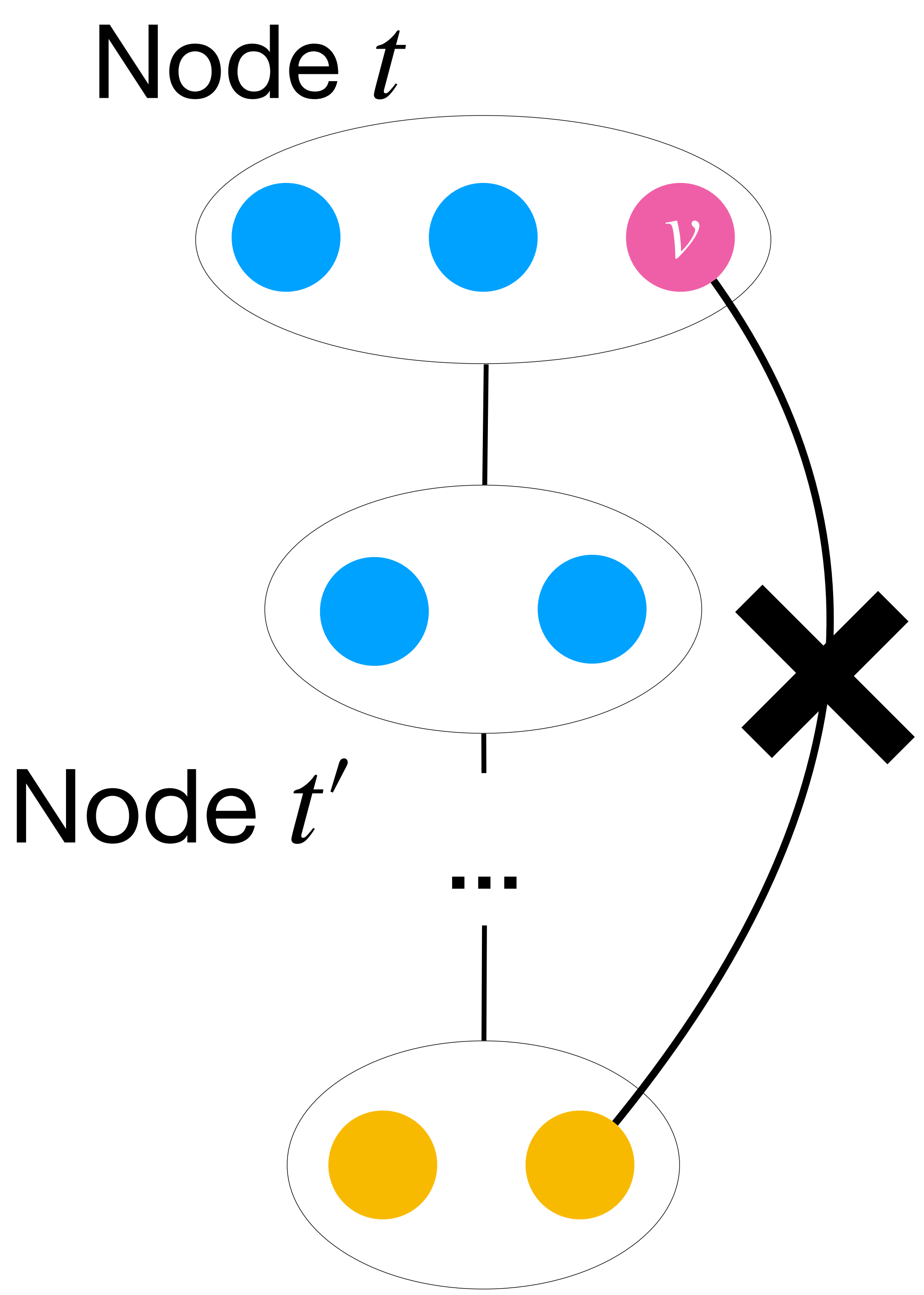
then $n_t(\sigma) := n_{t'}(\sigma')$

Introduce Nodes



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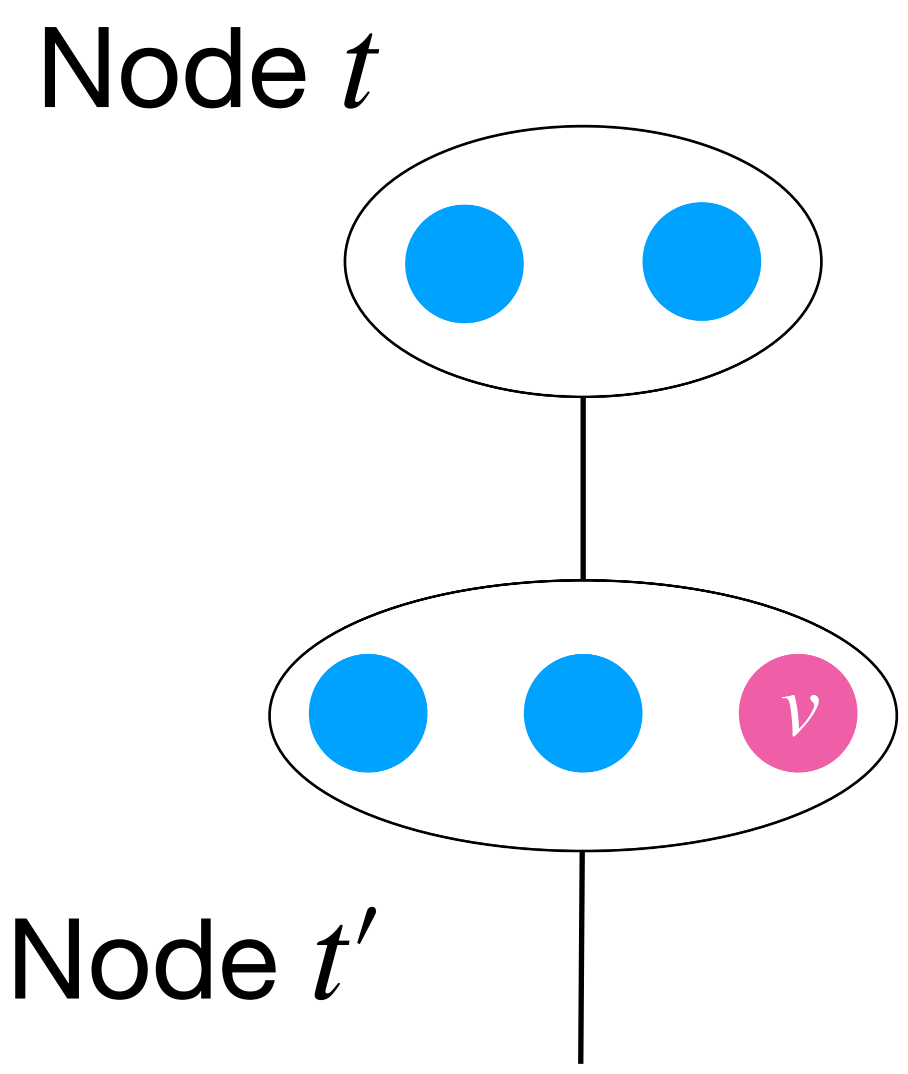
Forget Nodes



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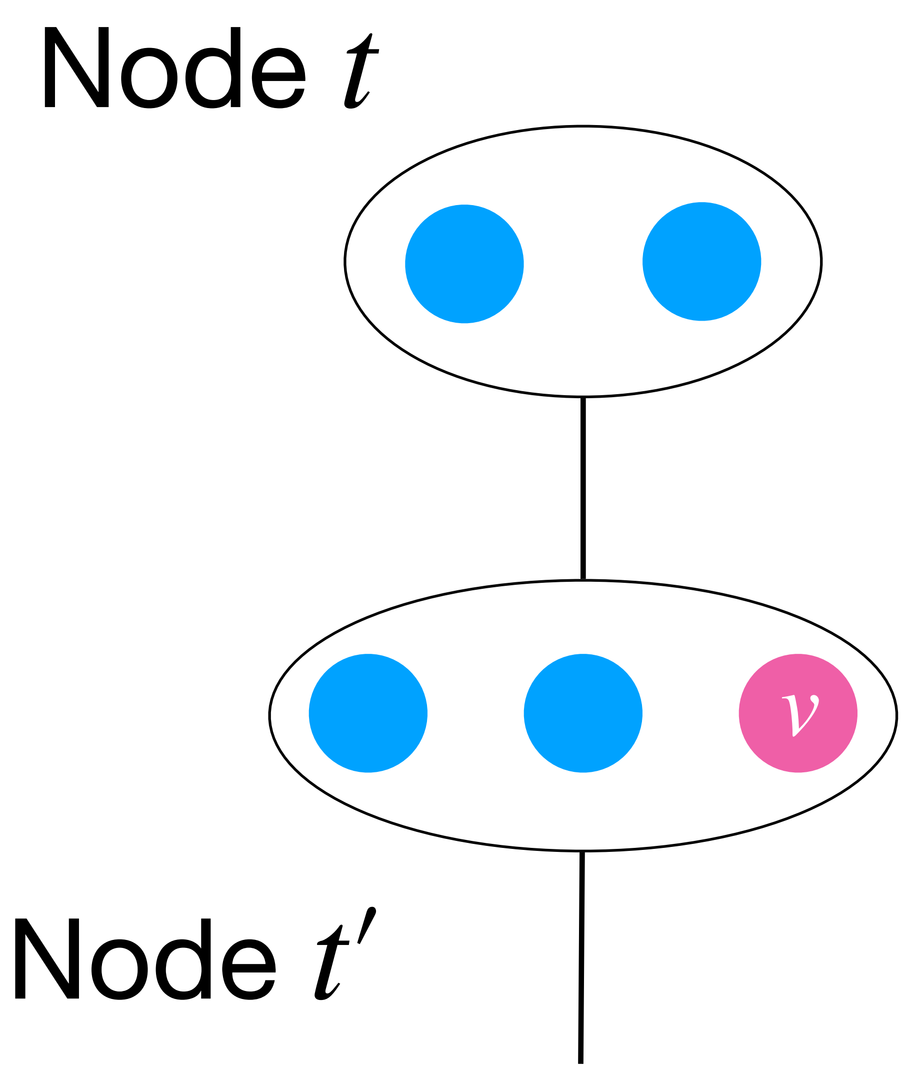

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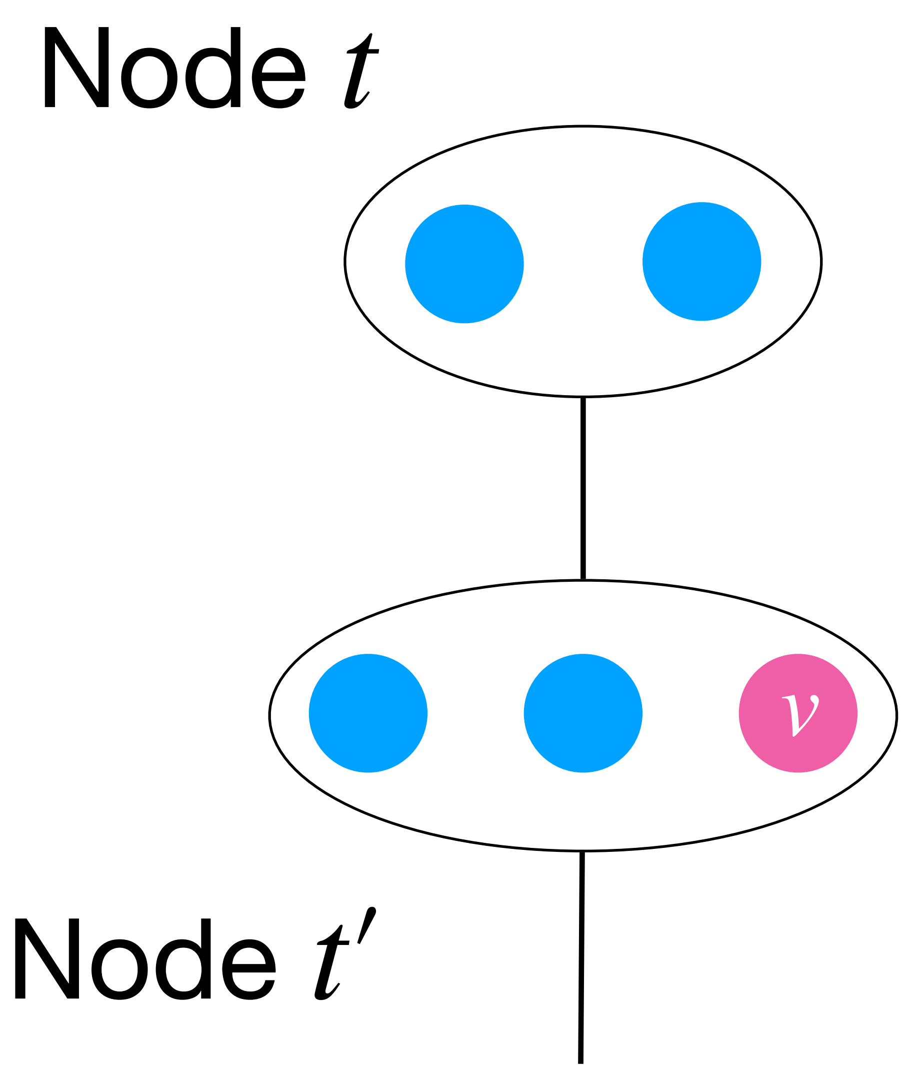
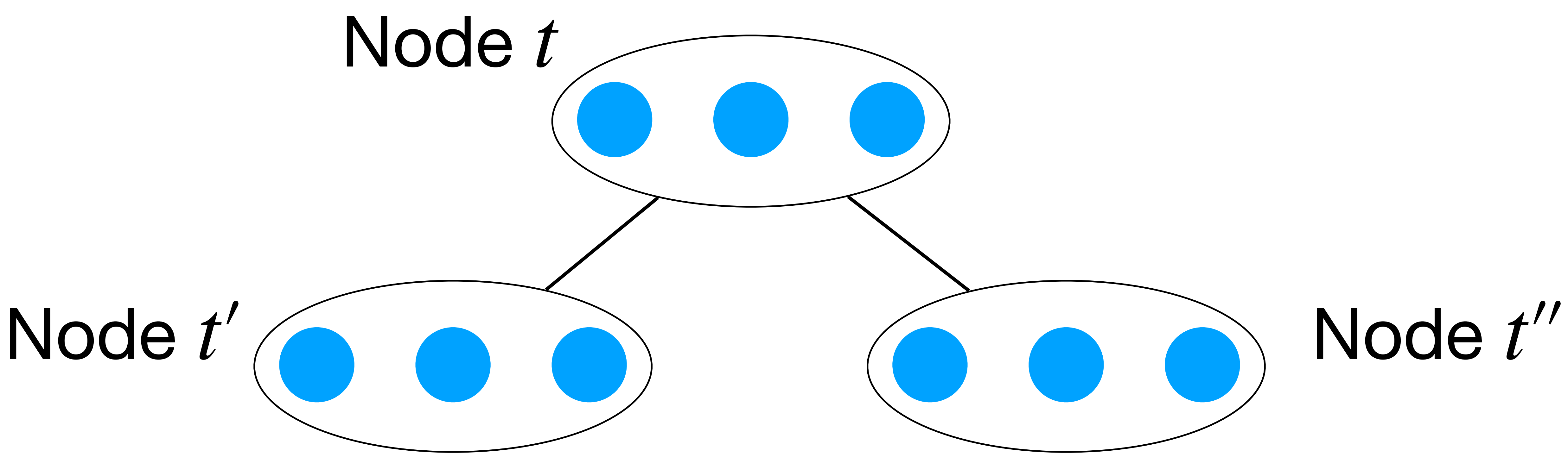
Forget Nodes



for $\sigma : \chi(t) \rightarrow \{red, green, blue\}$

then $n_t(\sigma) := \bigvee_{c \in red, green, blue} n_{t'}(\sigma' \cup \{v \mapsto c\})$

Join Nodes



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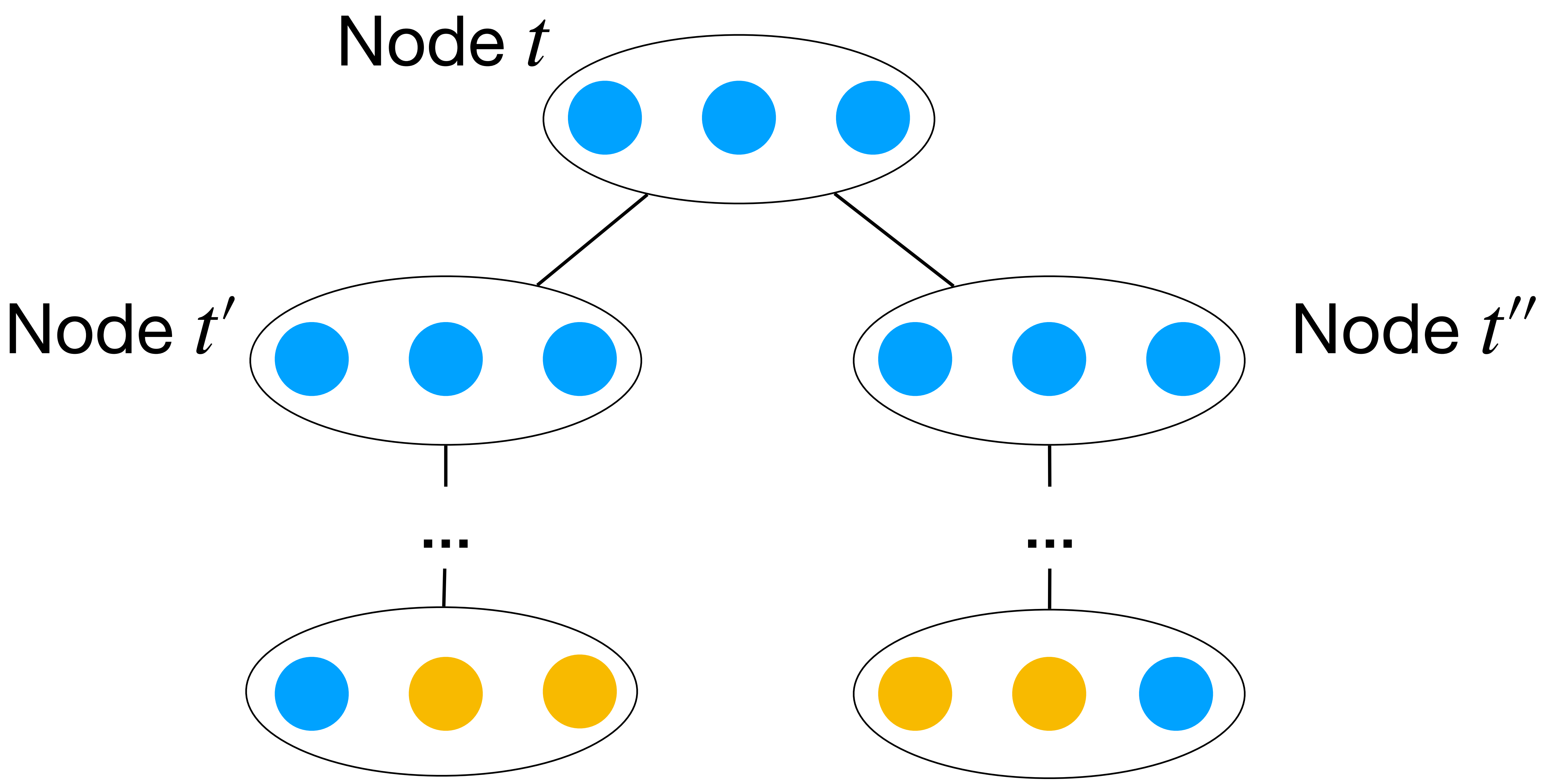

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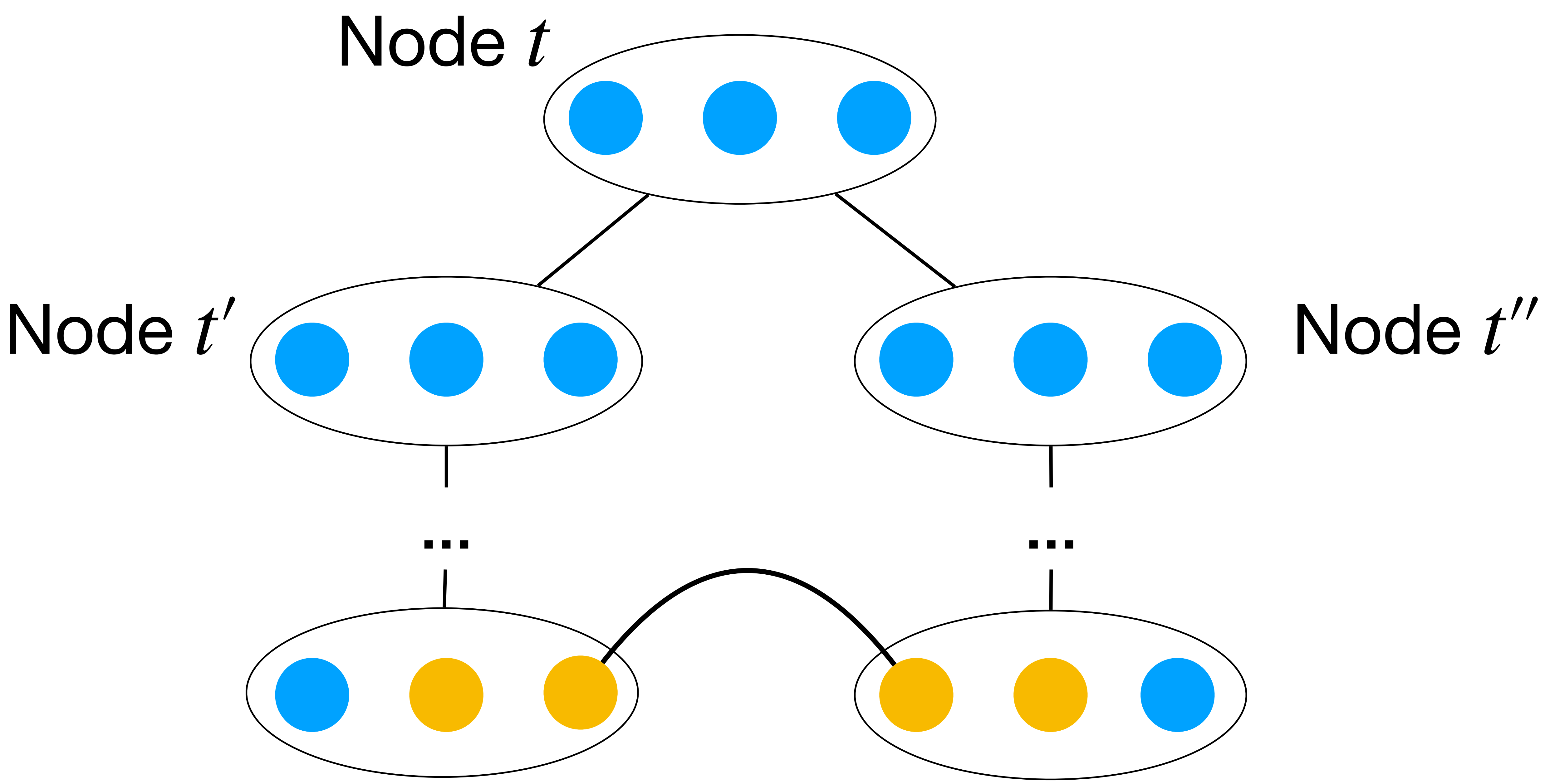

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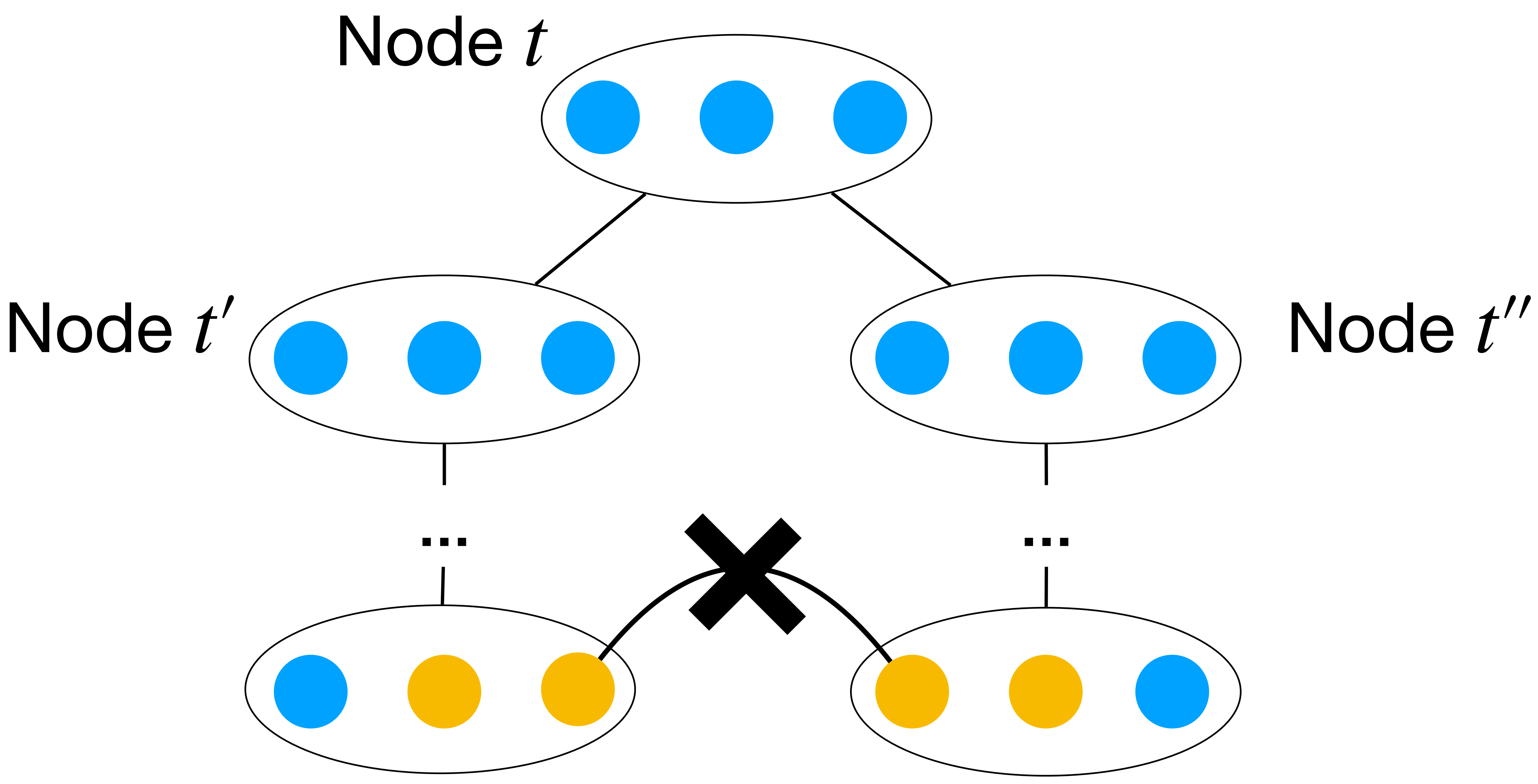

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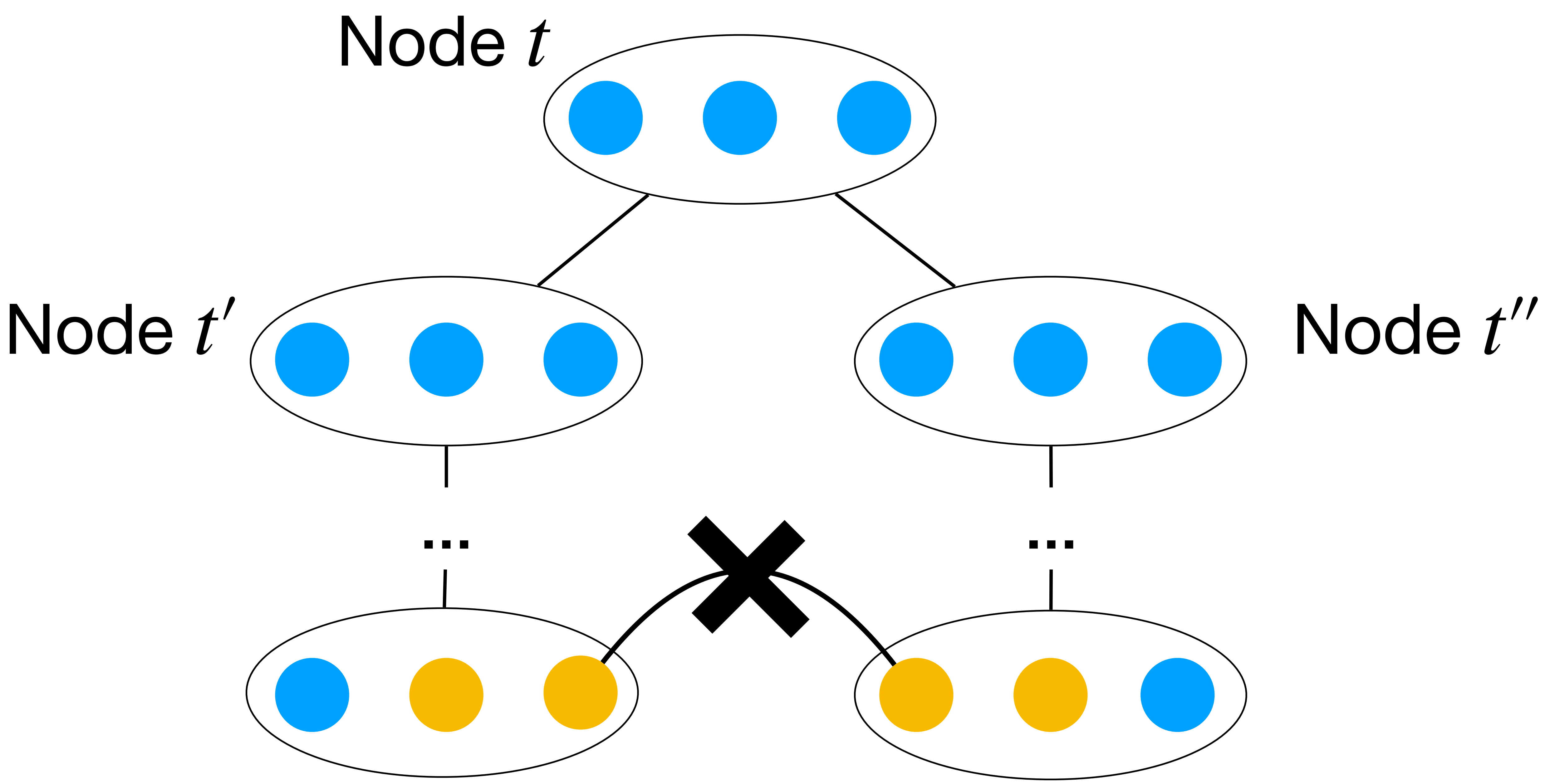

for $\sigma : \chi(t) \to \{red, green, blue\}$

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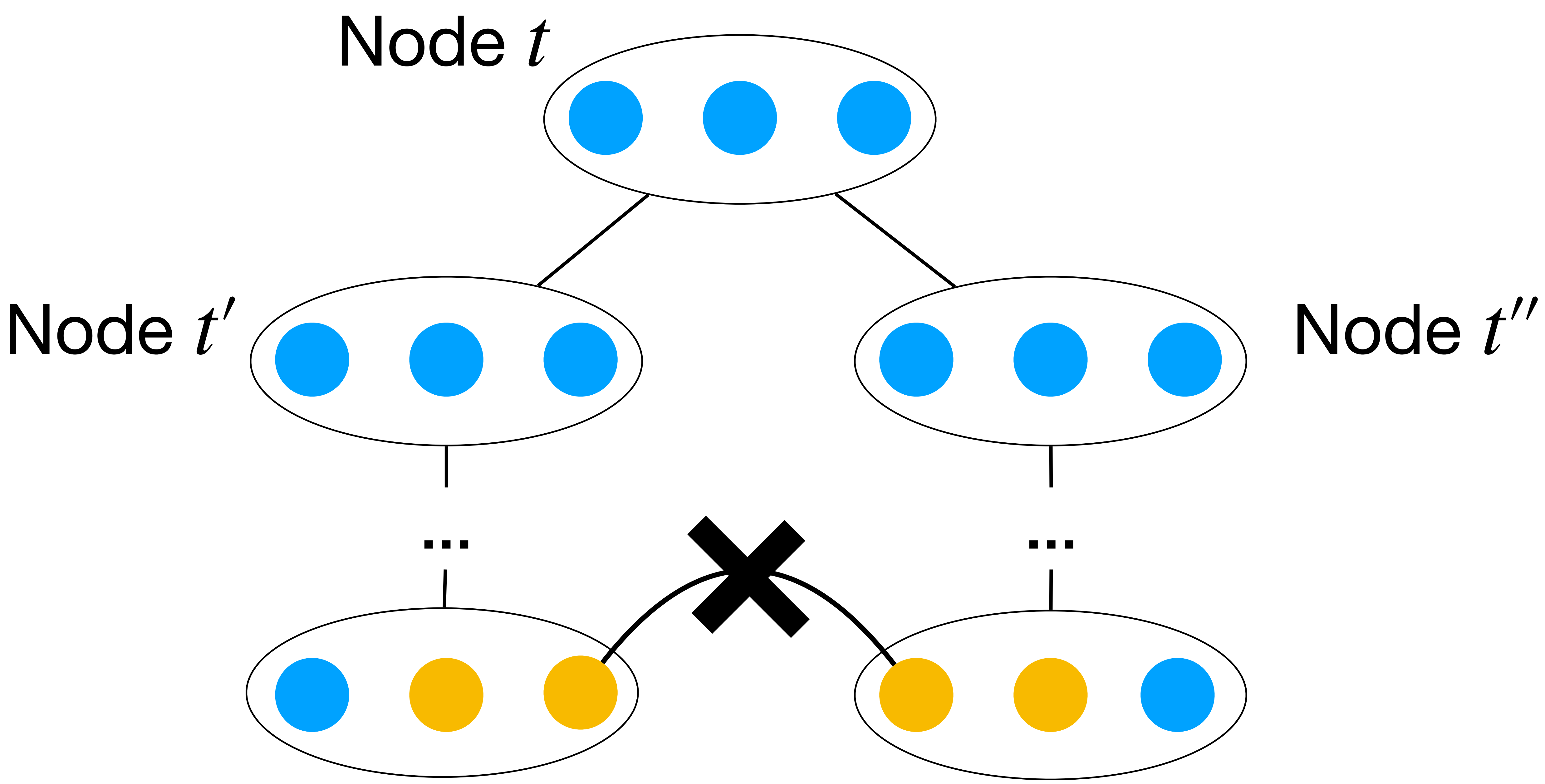

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$$n_t(\sigma) := n_{t'}(\sigma) \wedge n_{t''}(\sigma)$$

3-Colorability

Theorem

3-COLORABILITY is **FPT** parameterized by the treewidth of the input graph.*

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*If we can compute a nice tree decomposition of width $g(k)$ in FPT time.

Pros and Cons

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For **Dominating Set**, see

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Cygan et al., **Parameterized Algorithms**

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Pro: hand-crafted dynamic programming algorithm yields best running time

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Con: involves (repeating) tedious arguments

Con: coming up with the right notion of “partial solution” ($n_t(U)$) is difficult

Computing Treewidth

Treewidth is FPT

TREEWIDTH

Input: A graph G and an integer k.

Question: Does G have treewidth $\leq k$?

Parameter: k

Treewidth is FPT

TREEWIDTH

Input: A graph G and an integer k.

Question: Does G have treewidth $\leq k$?

Parameter: k

Theorem (Bodlaender)

There is a function f and an algorithm A that computes a tree decomposition of a graph G in time $f(k)\,|\,V(G)\,|$ or decides that its treewidth is greater than k.

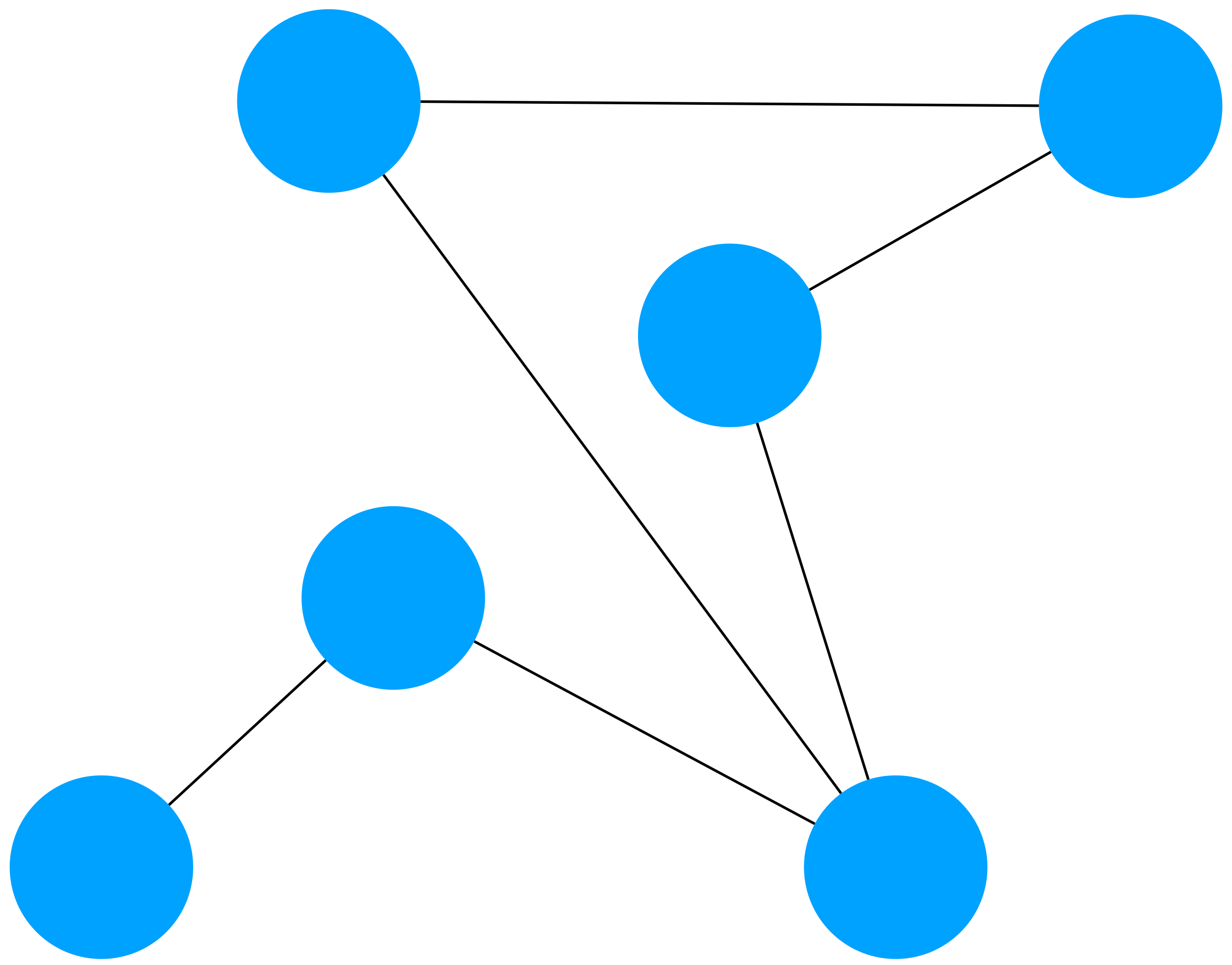
FPT 2-Approximation

Theorem (Korhonen 2021)

There is an algorithm that, given an n-vertex graph G and an integer k, in time $2^{O(k)}n$ either outputs a tree decomposition of width at most $2k+1$ or determines that the treewidth of G is larger than k.

Heuristics

Elimination Orderings



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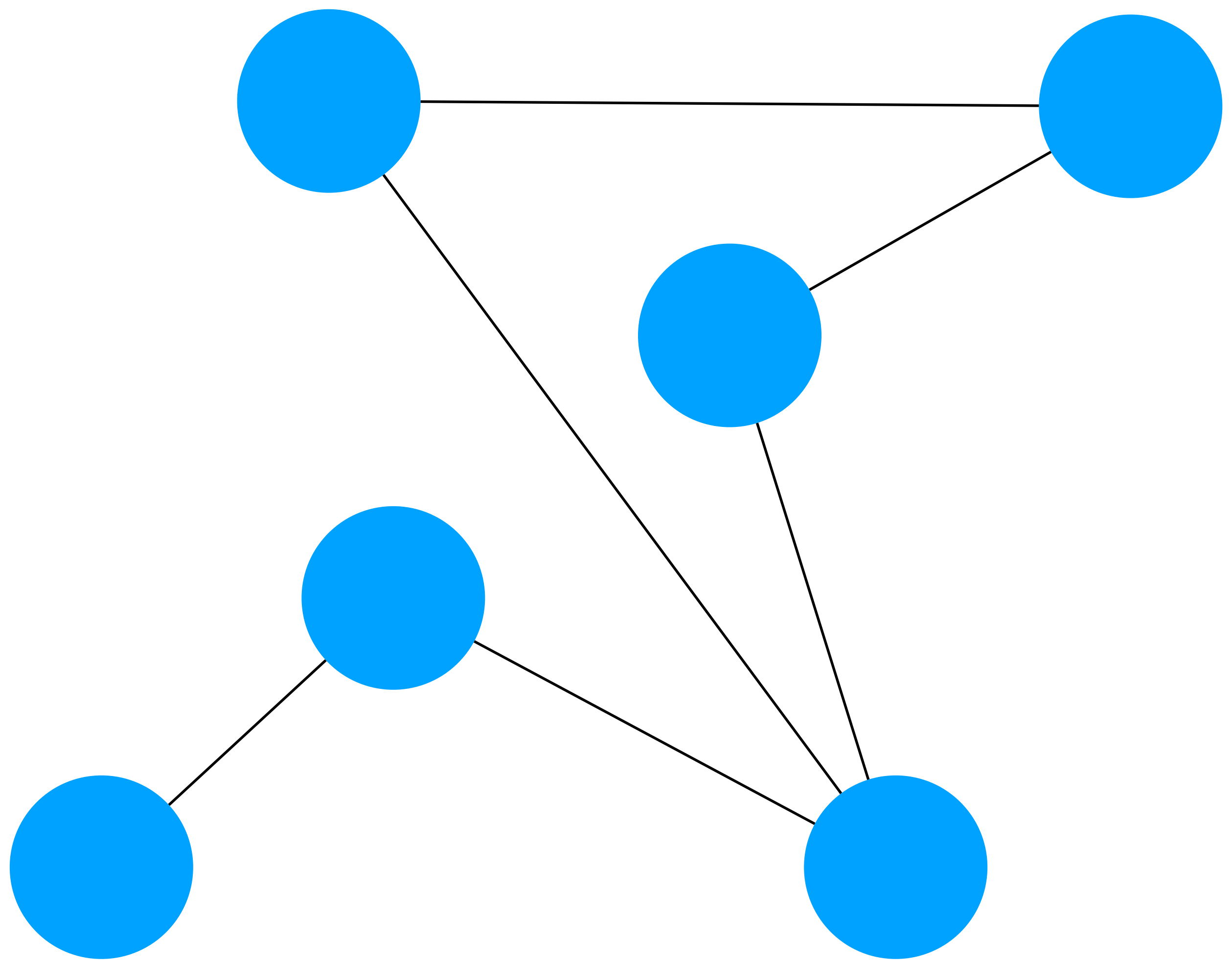

Definition

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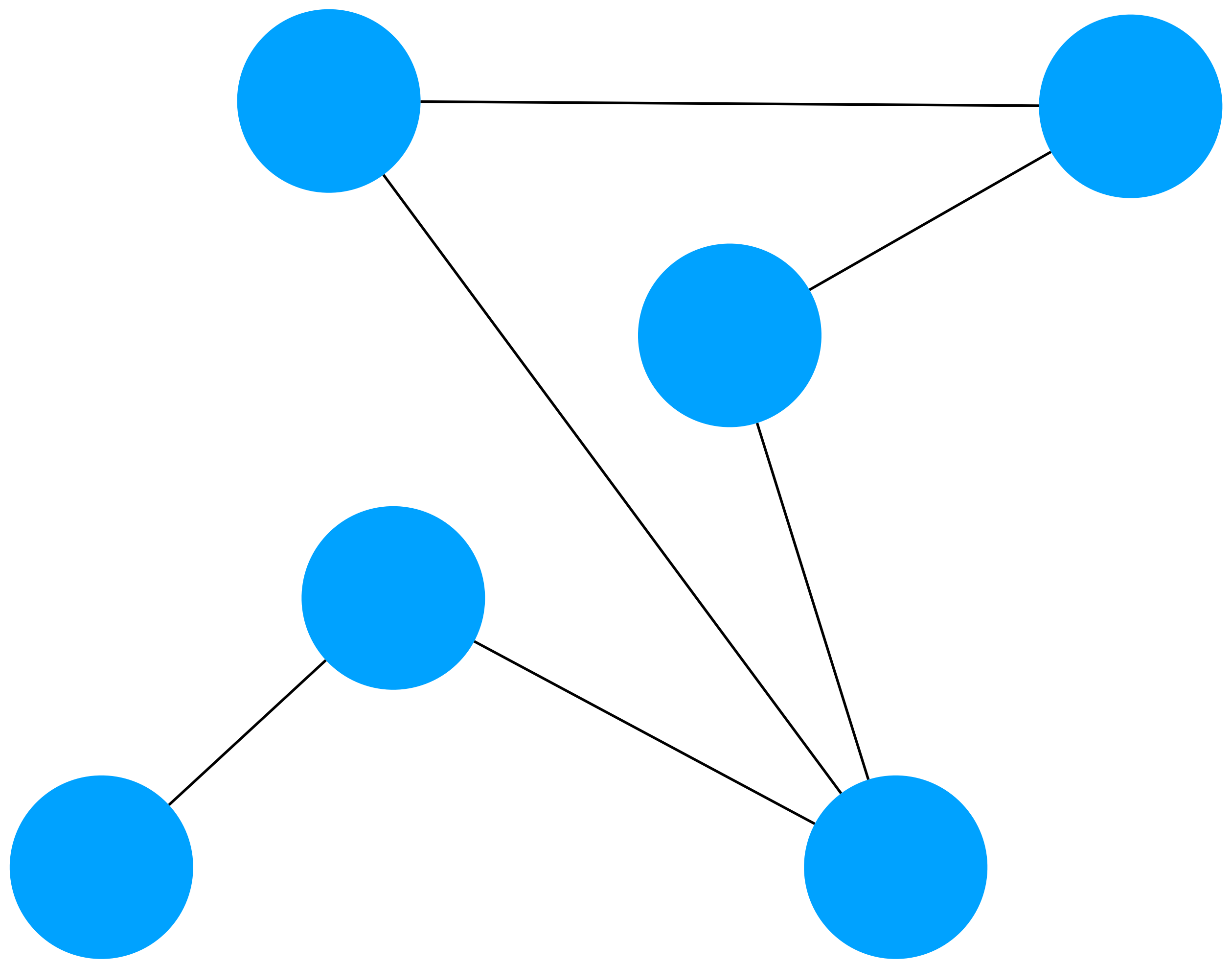

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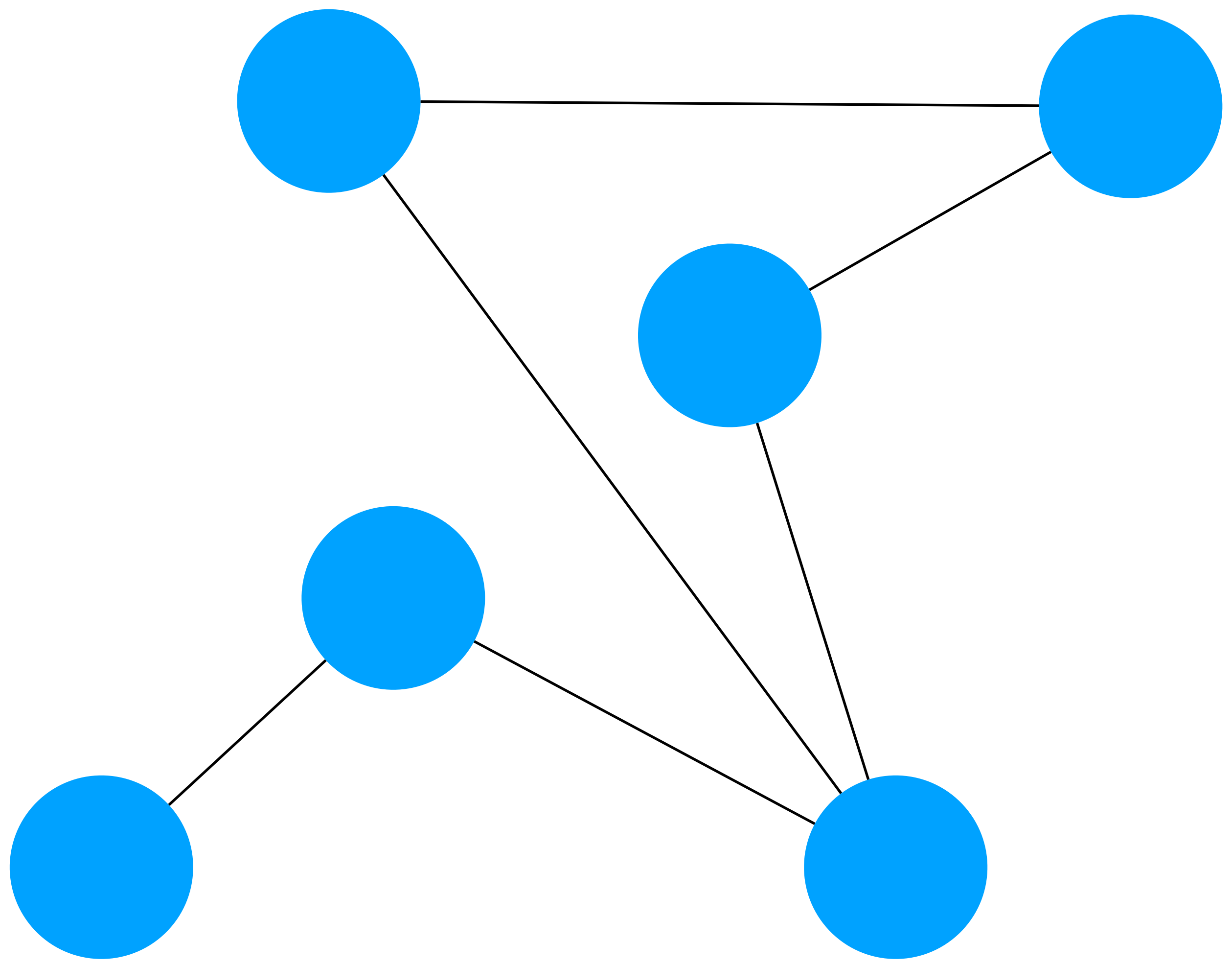

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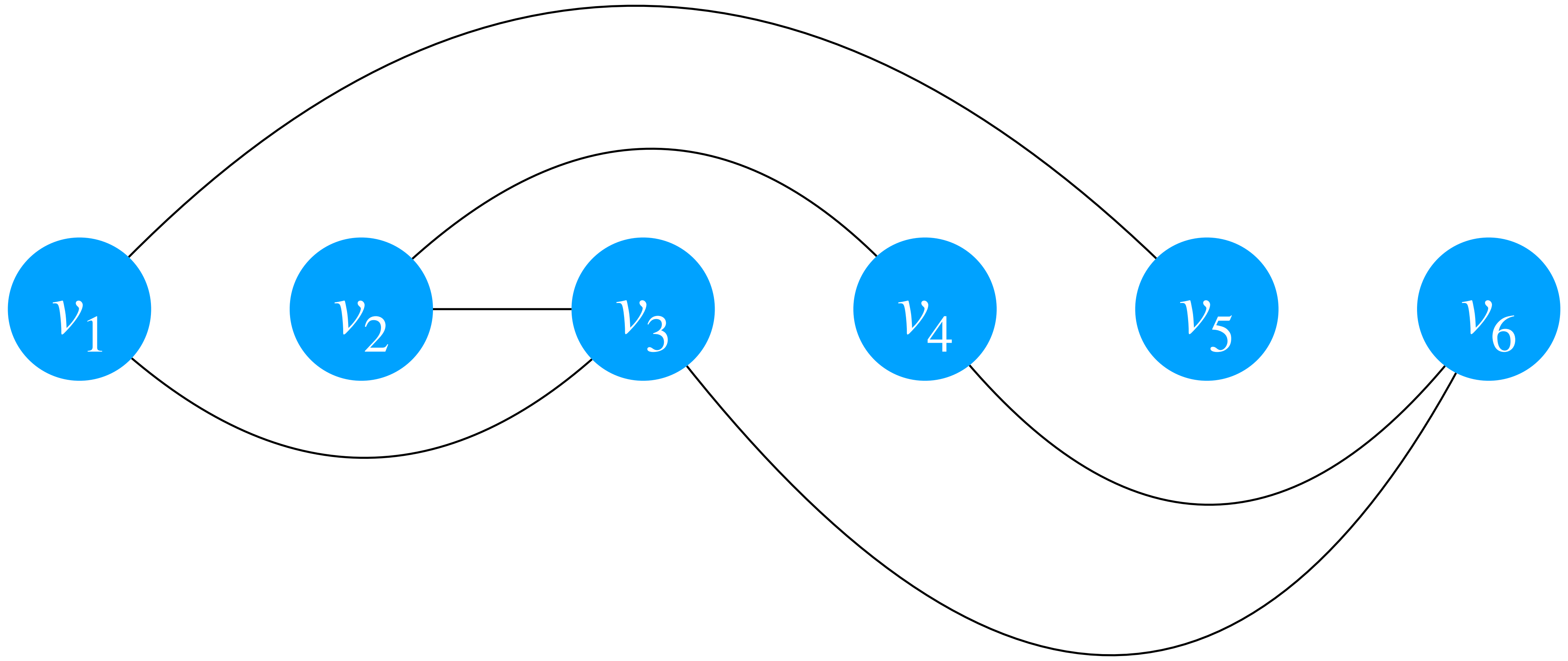

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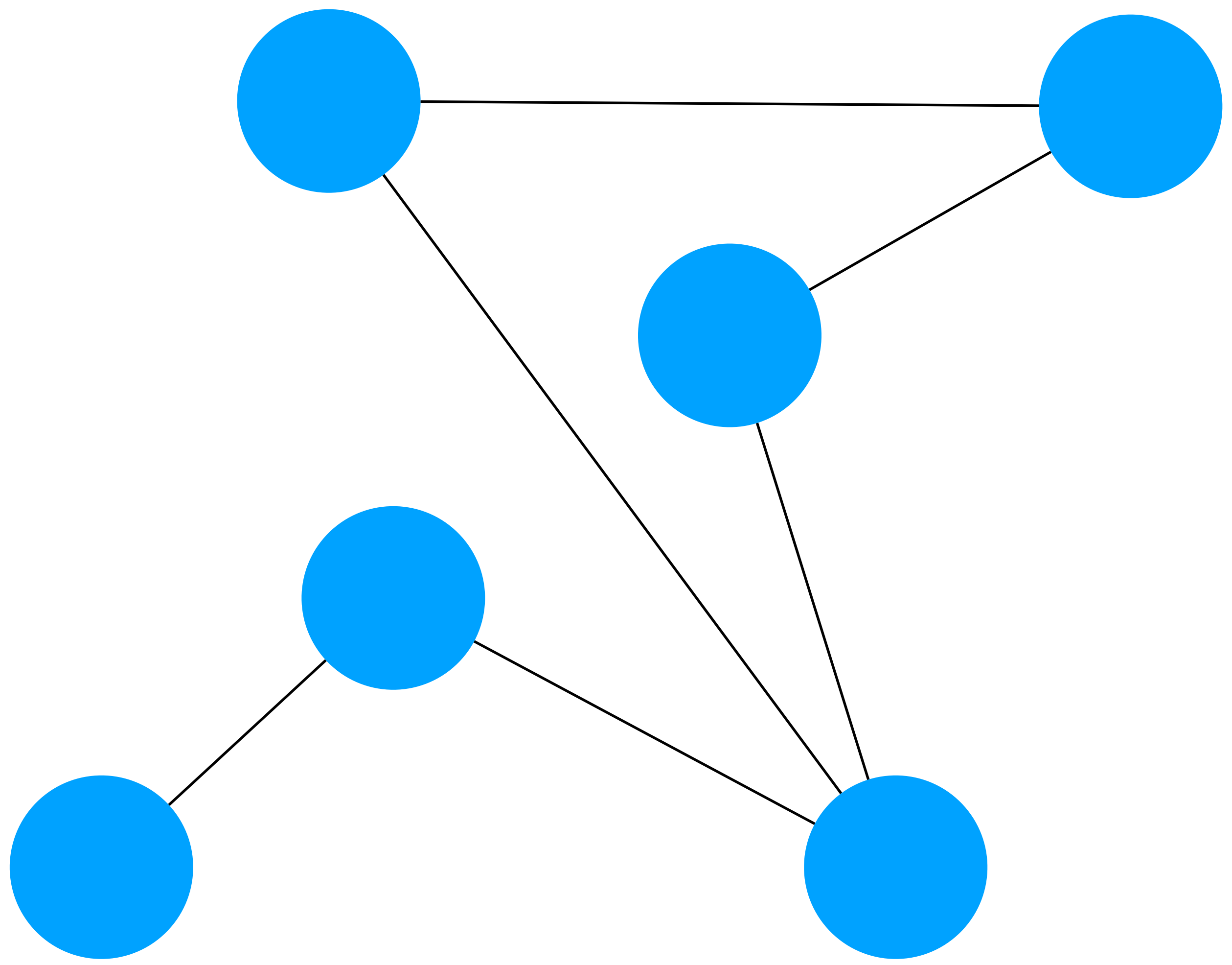

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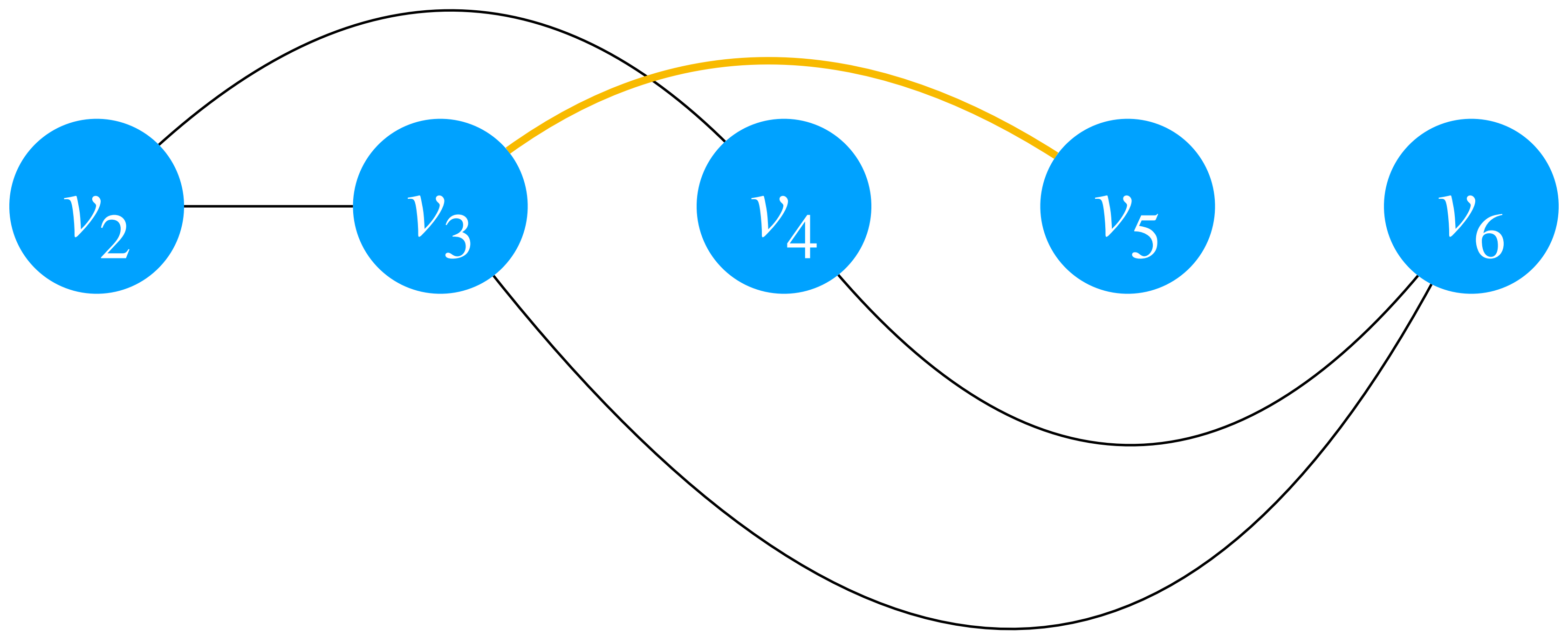

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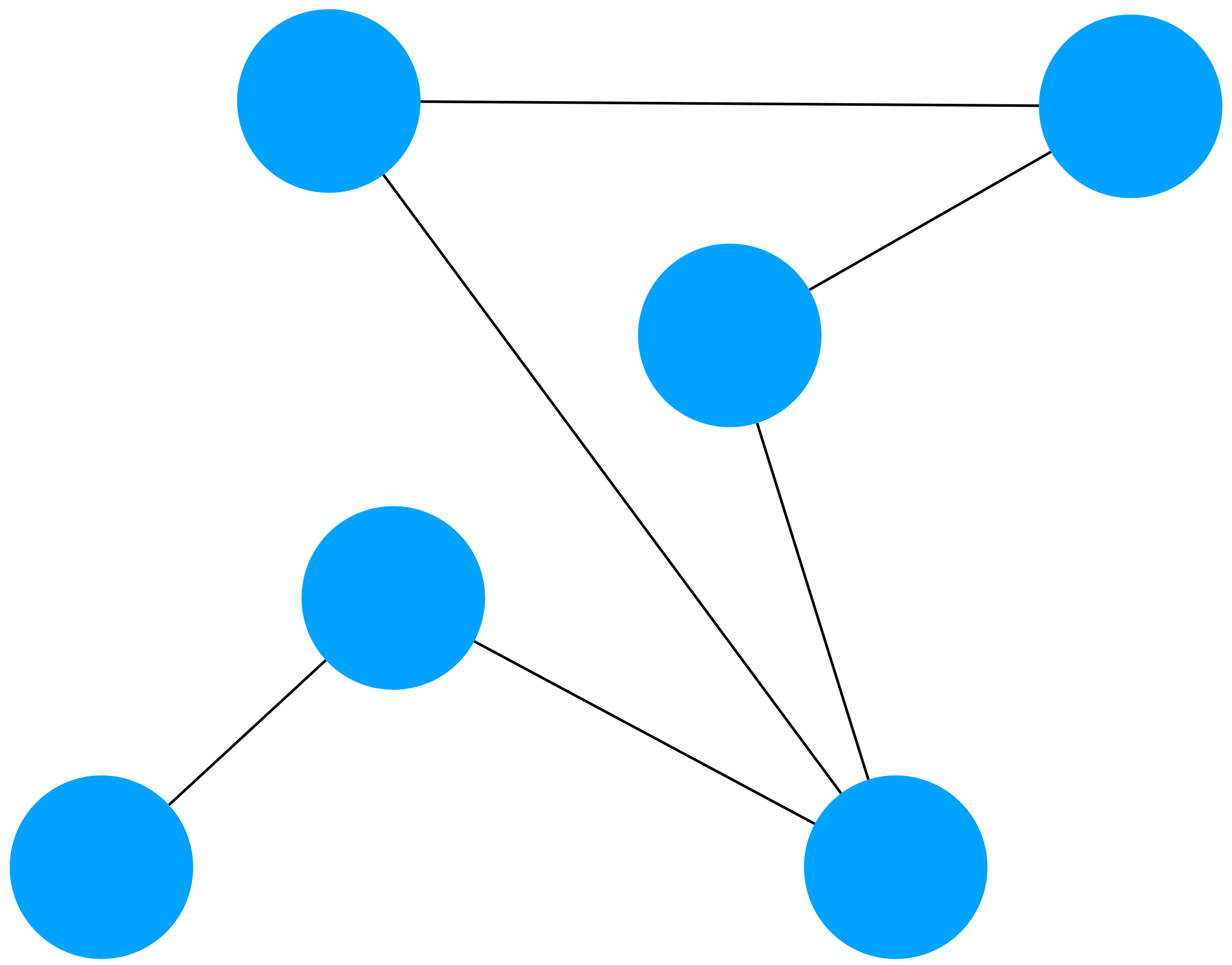

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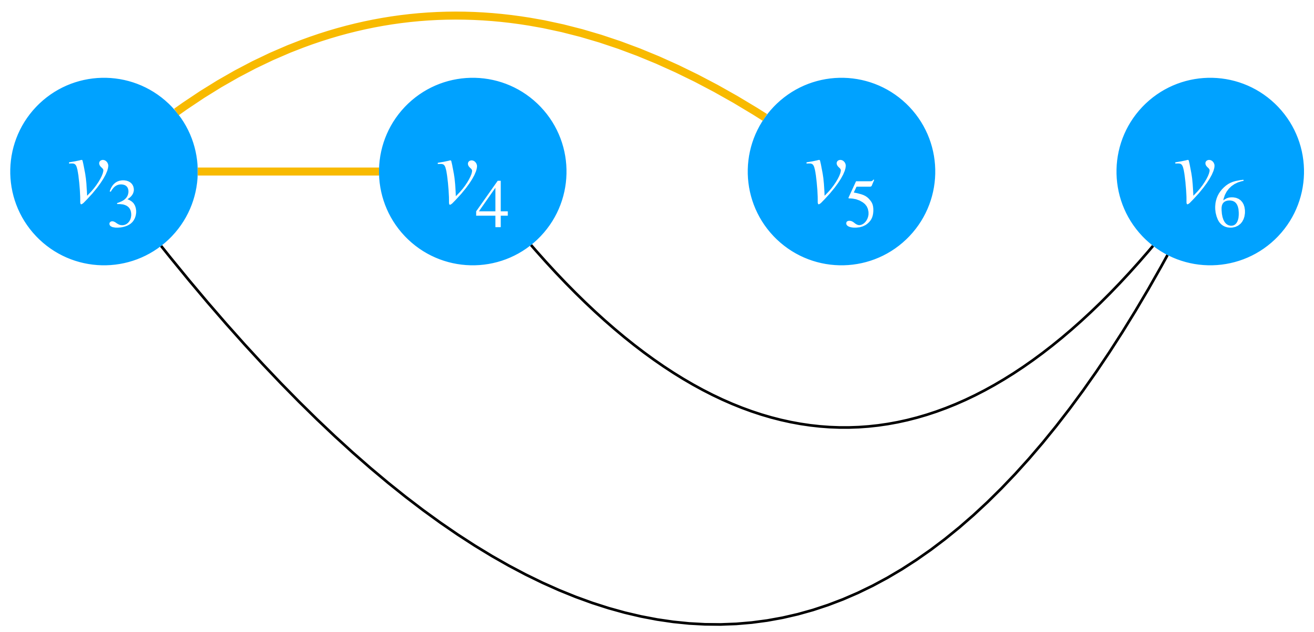

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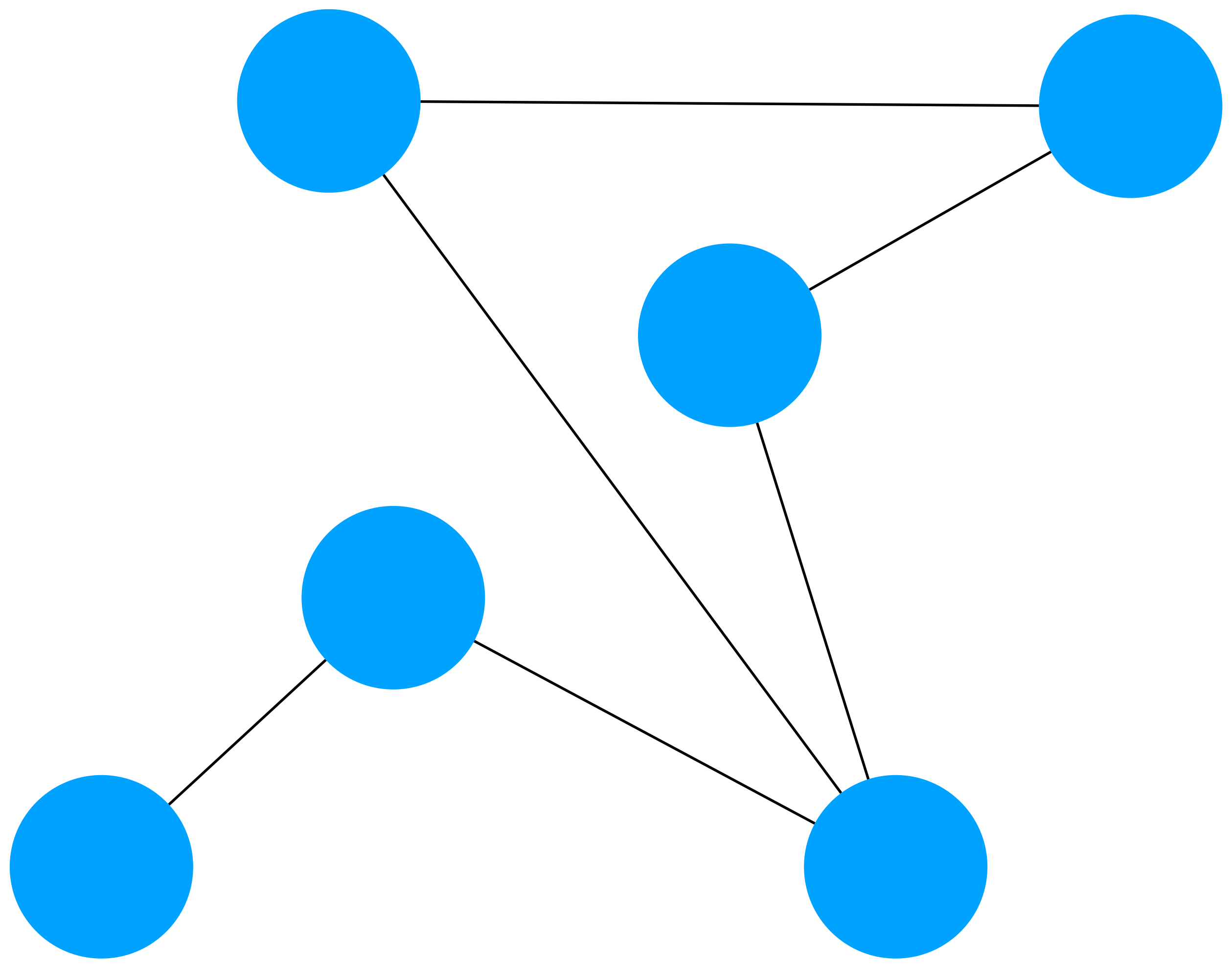

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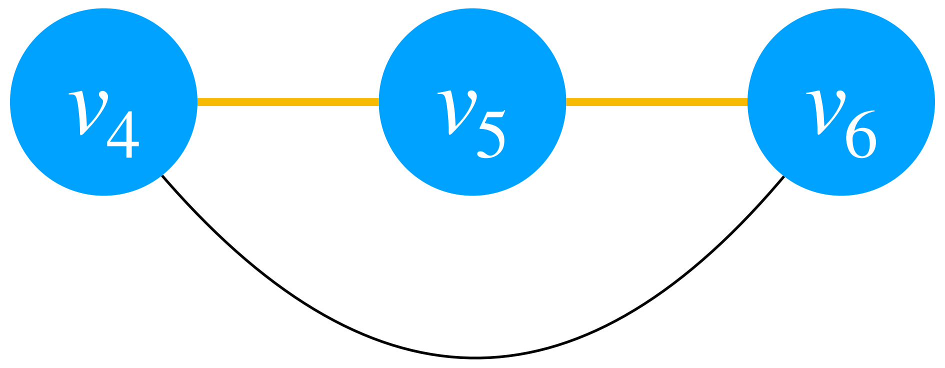

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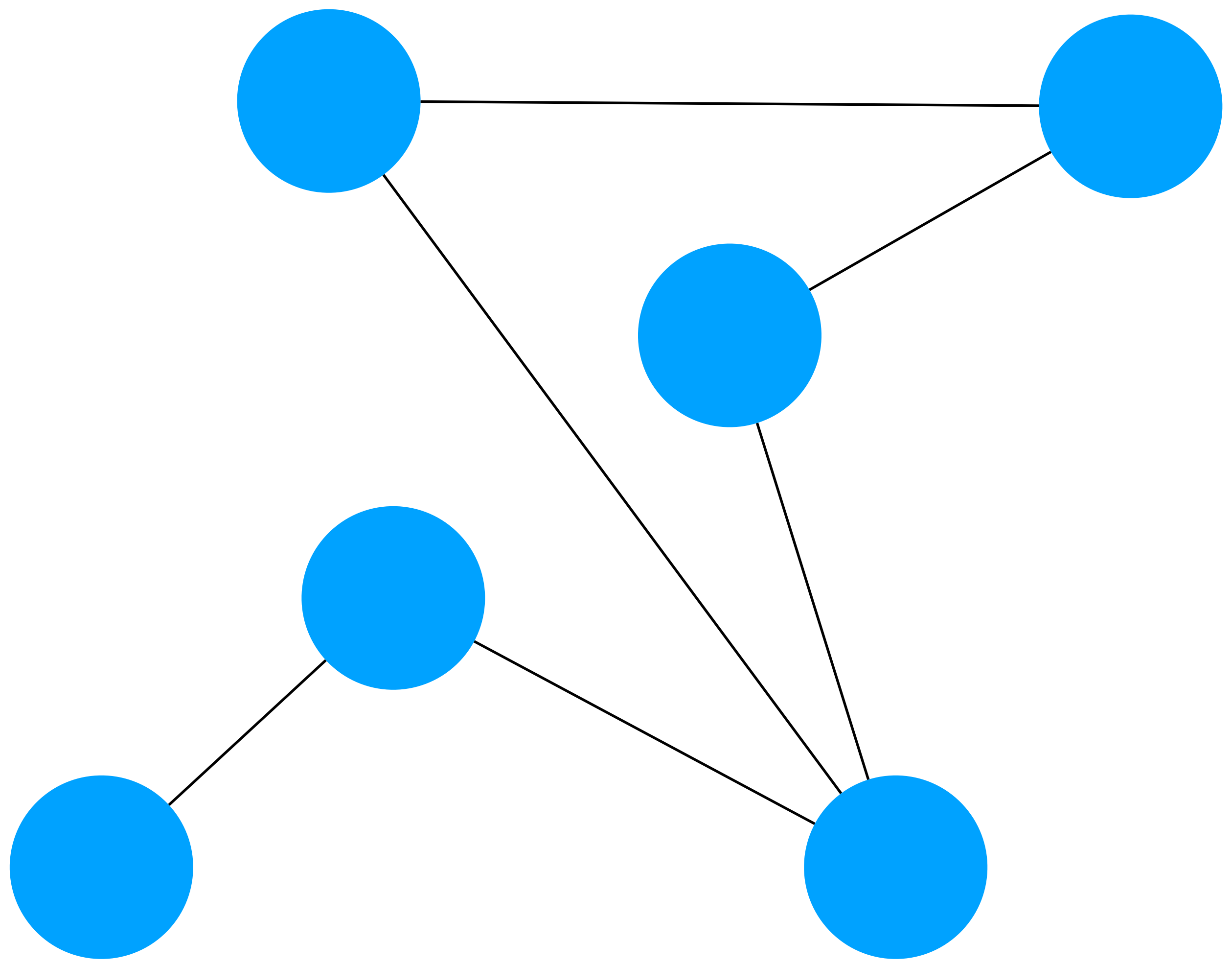

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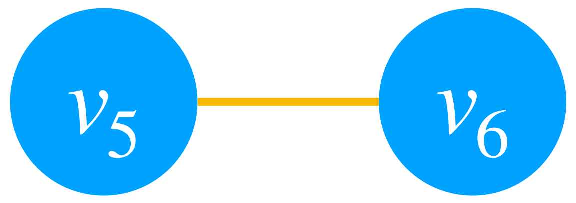

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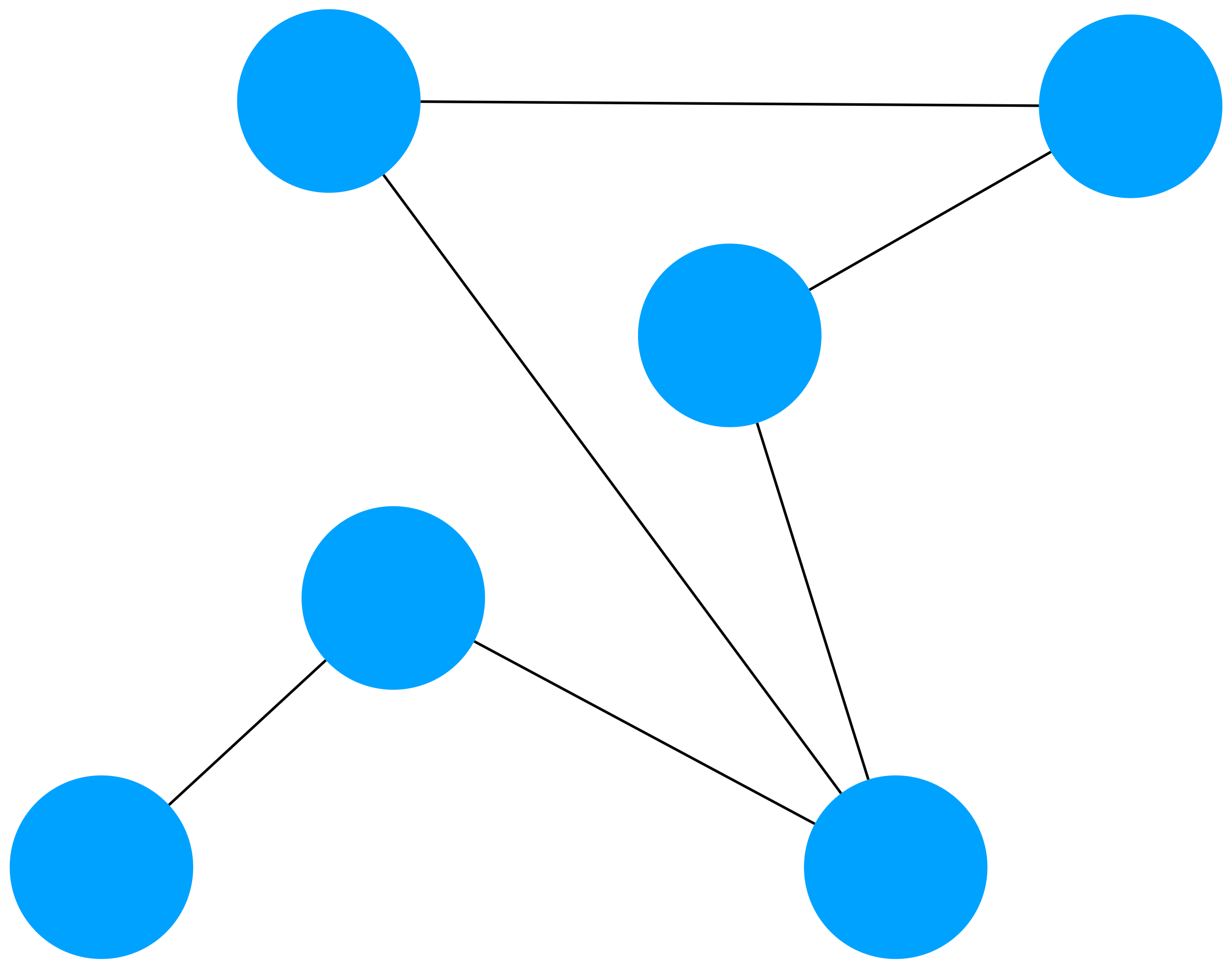

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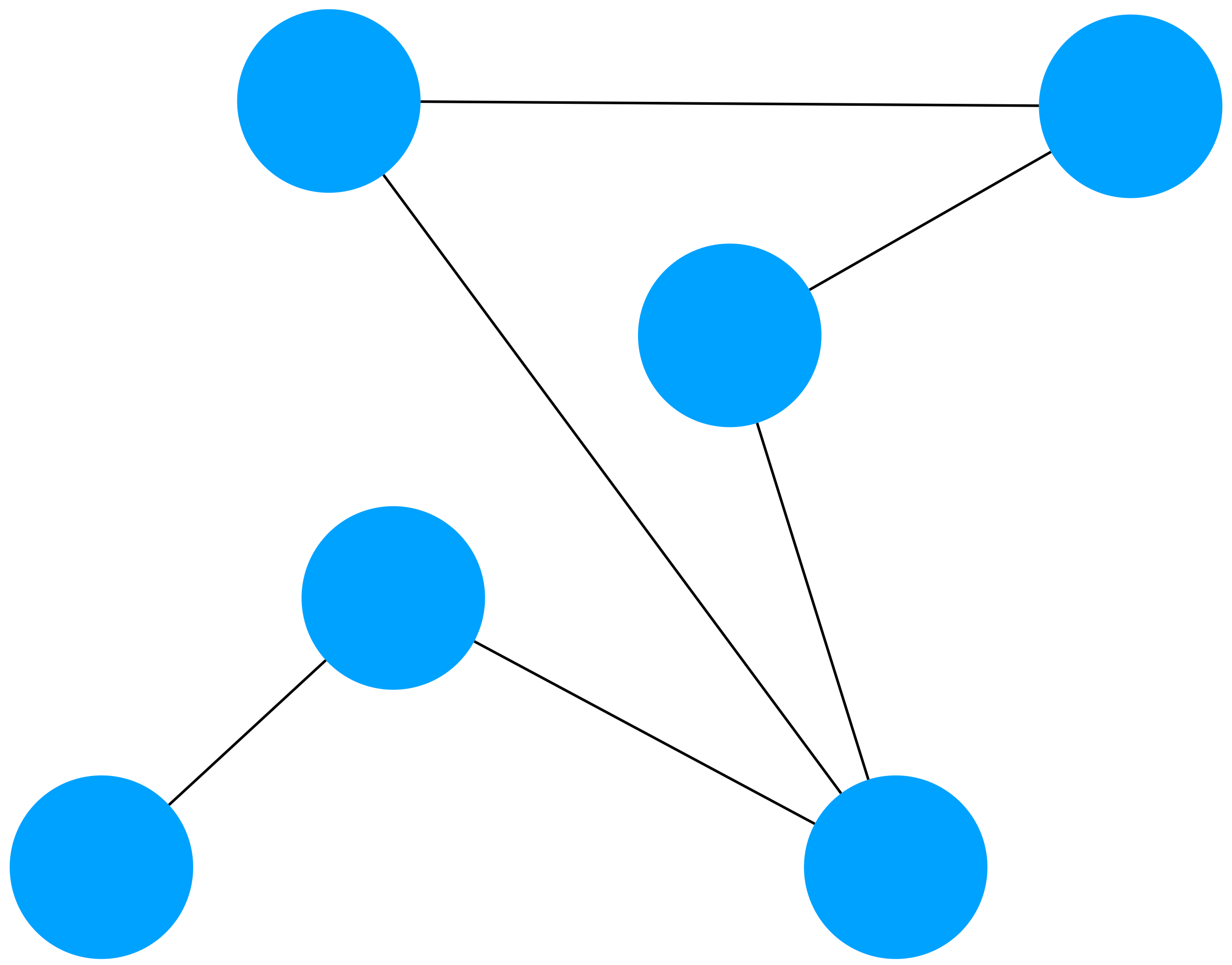

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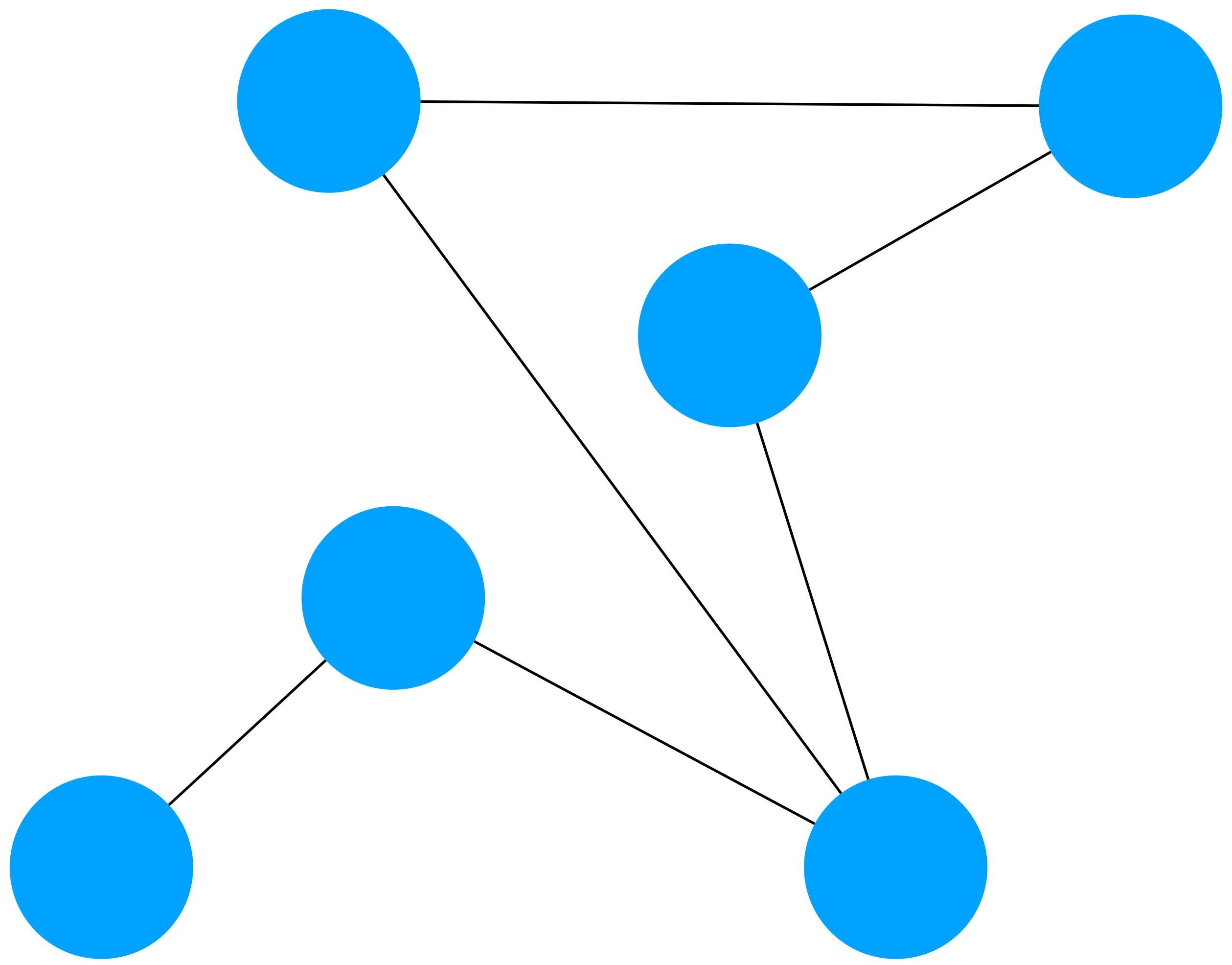

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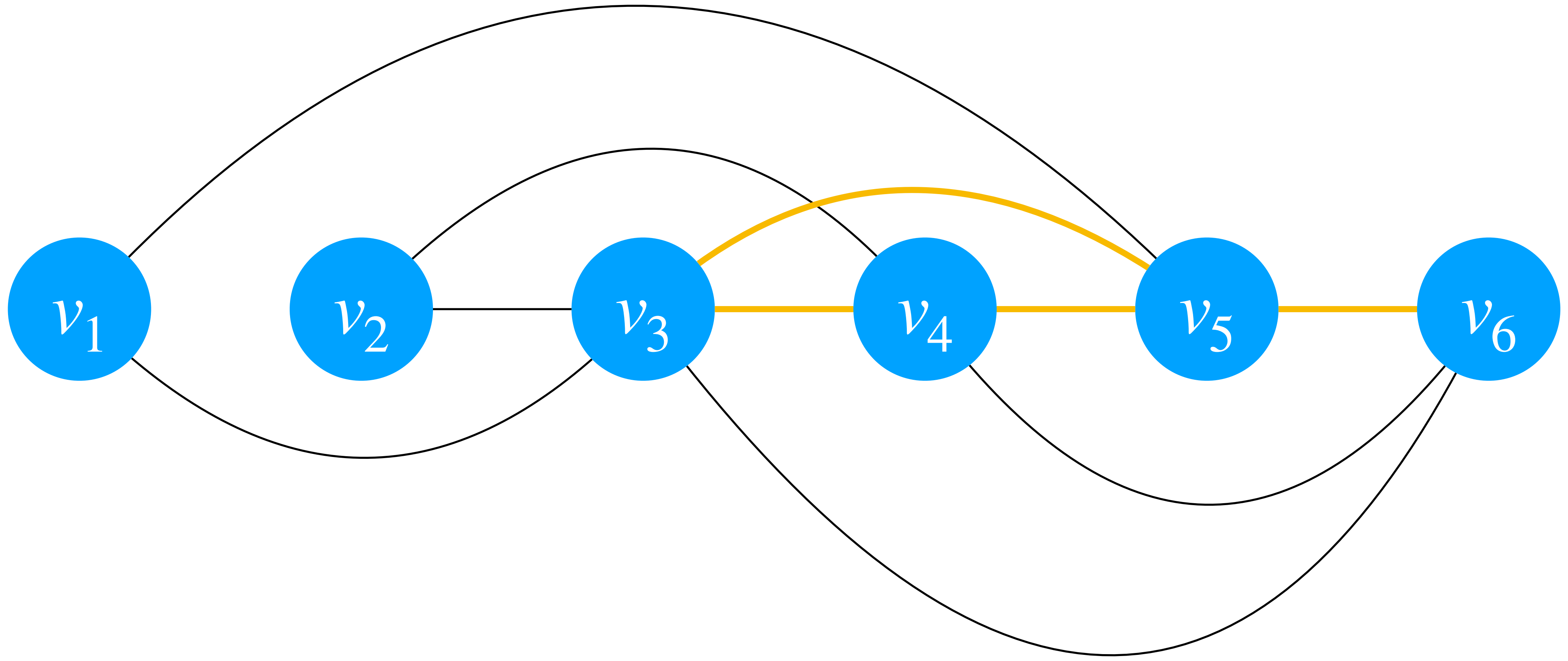

The **width** of σ is the maximum degree of a vertex upon elimination.

Tree Decompositions from Orderings

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return T', χ'

Greedy Heuristics

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1. **Min Degree/Size** minimum degree

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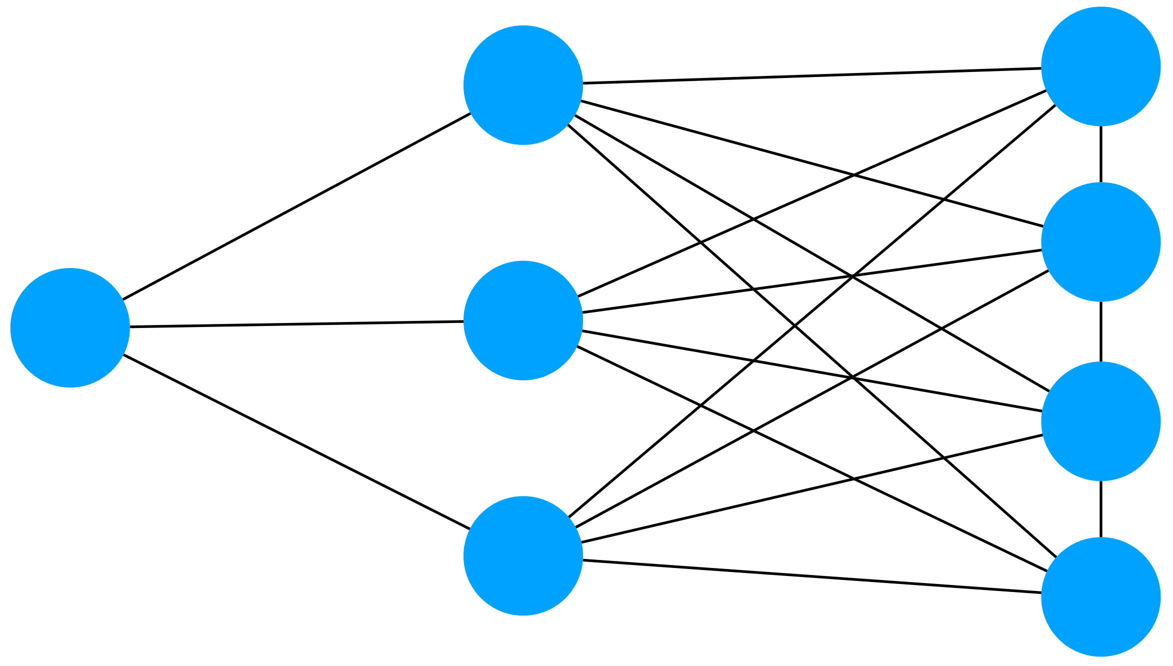
1. **Min Degree/Size** minimum degree
2. **Min Fill** fewest fill-in edges

Min Degree and Min Fill



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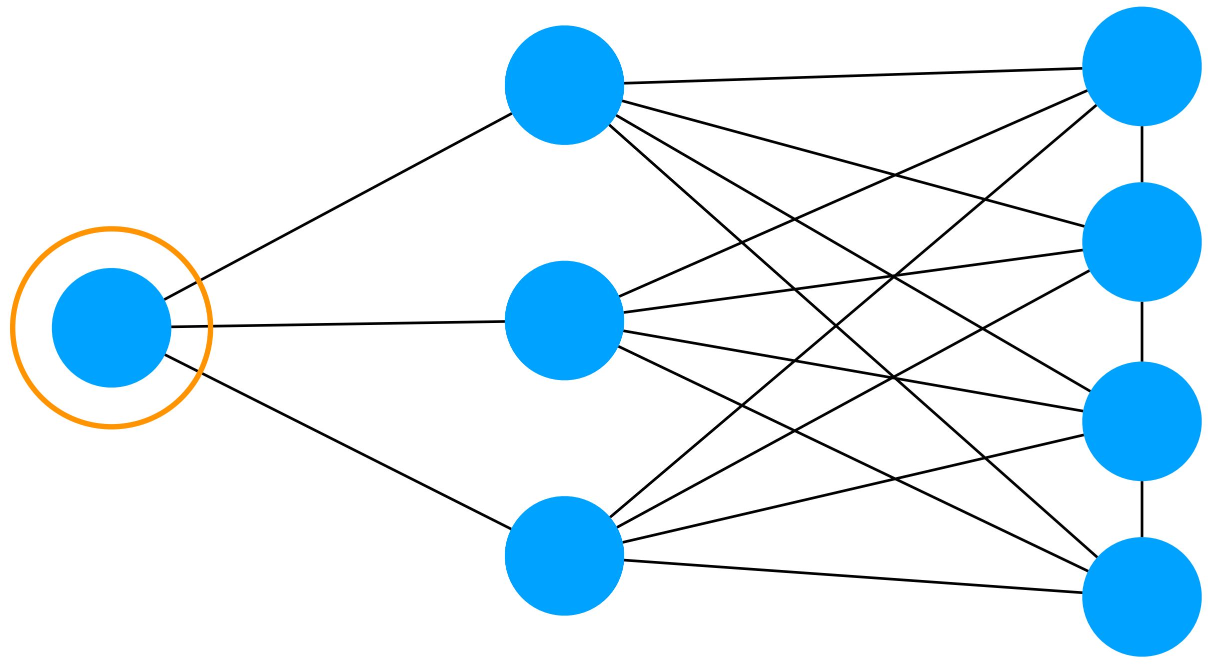

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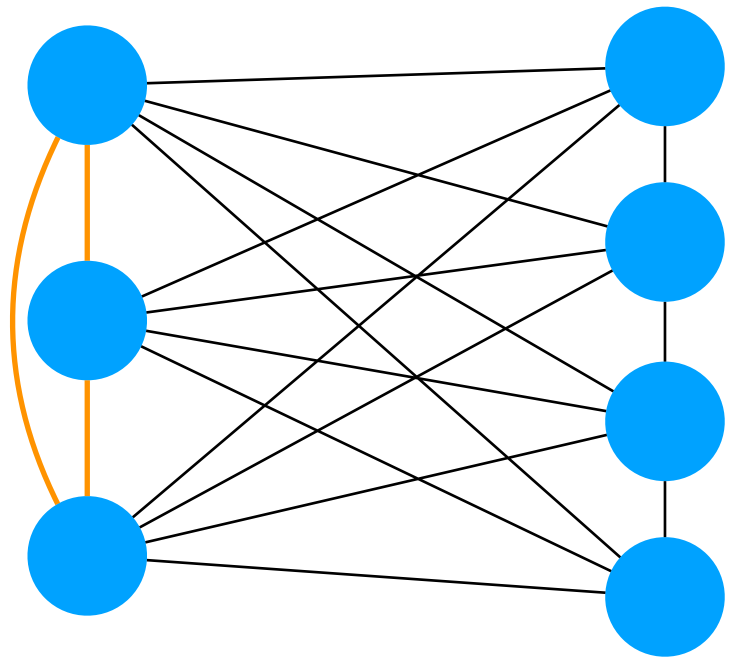

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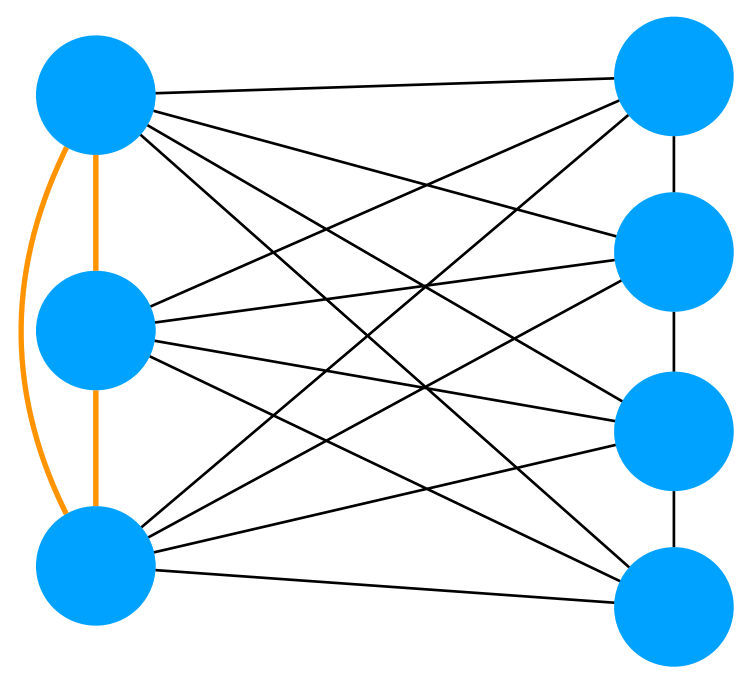

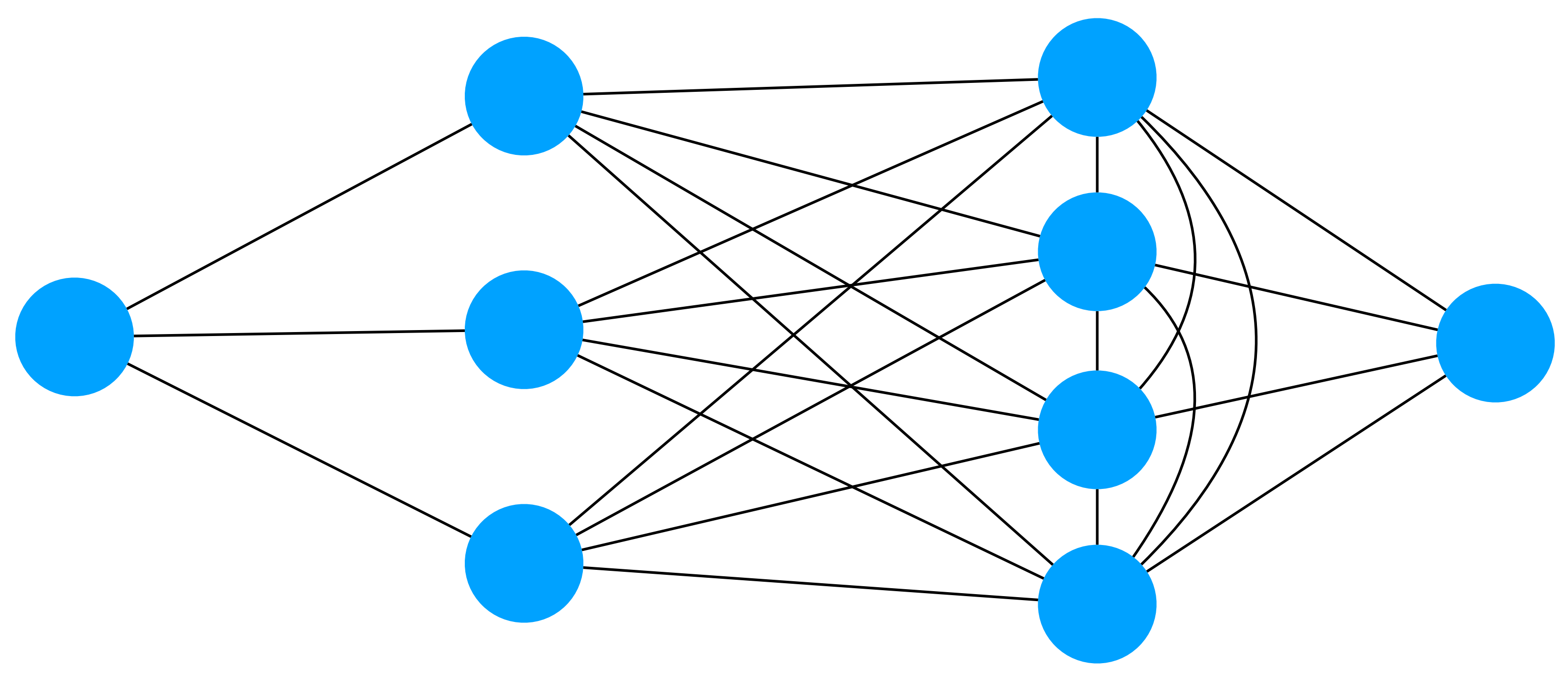

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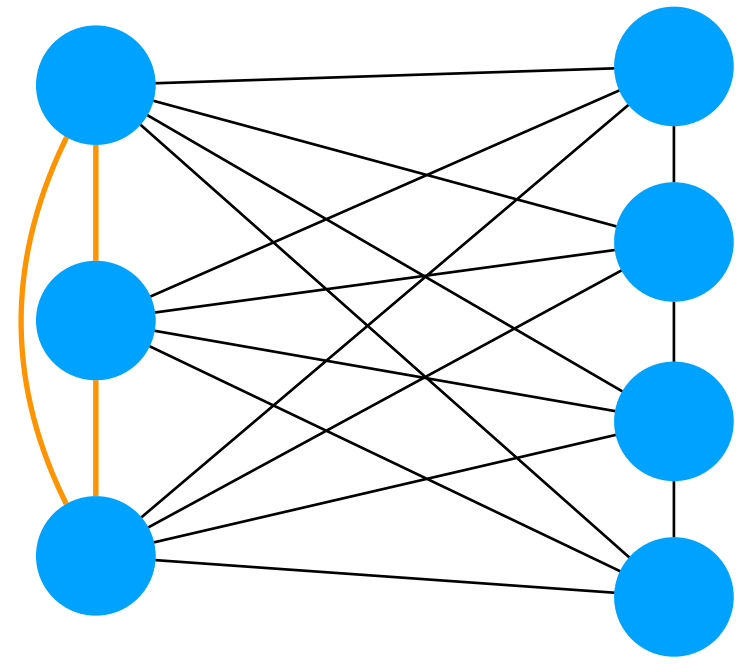

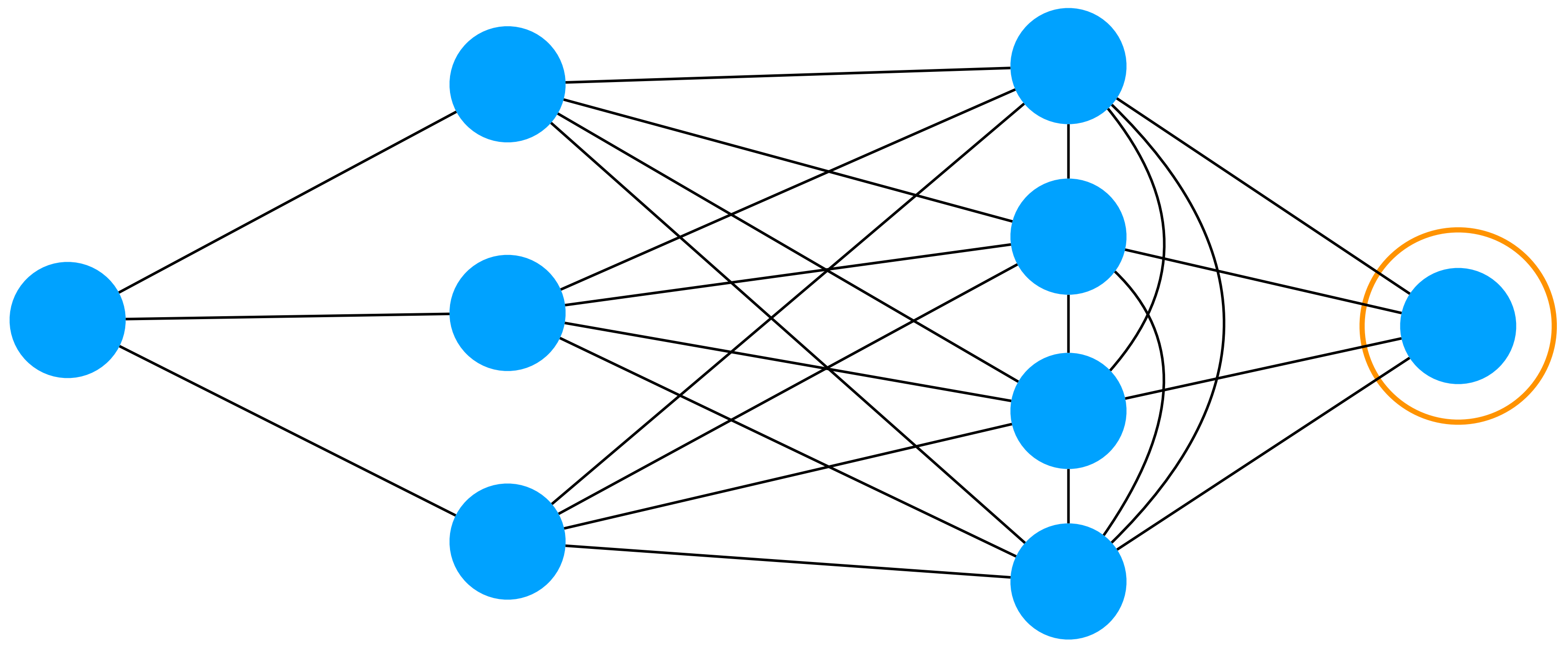

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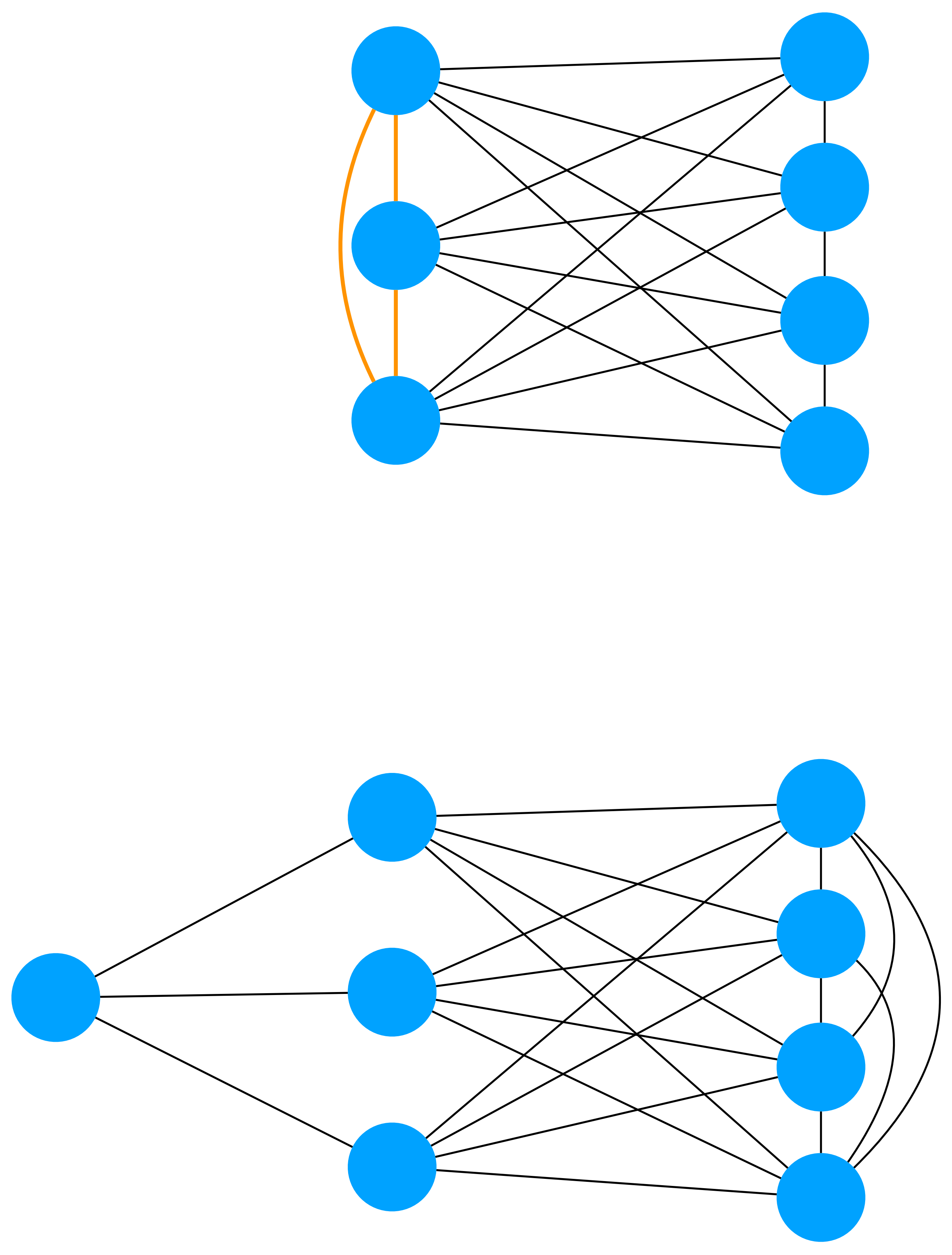

Maximum Cardinality Search

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Heuristics in Practice

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- **Min-Fill** generates better (in terms of width) orderings.
- Can generate good elimination orderings for large graphs.

Heuristics in Practice

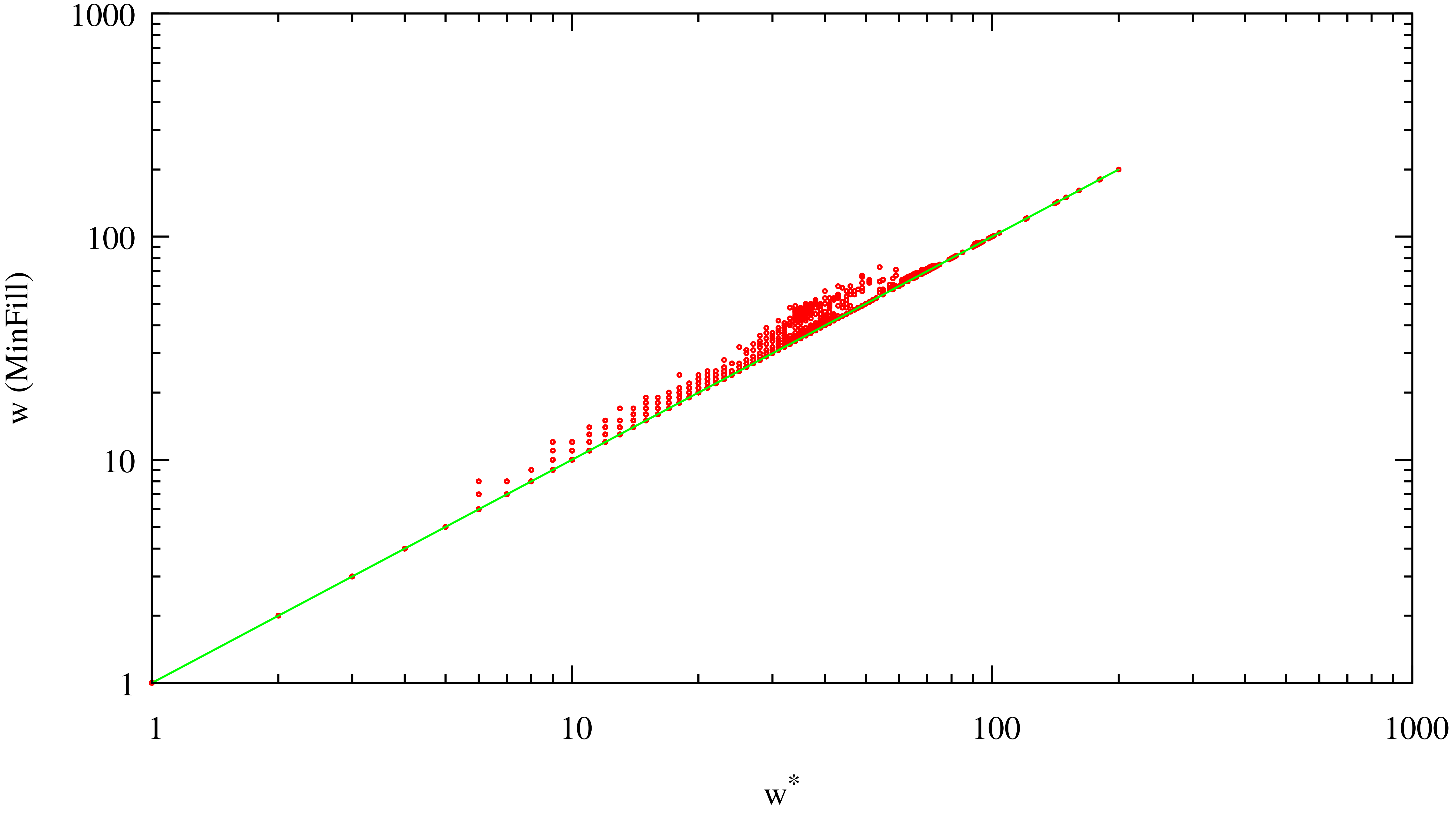

Fig. 2. The width obtained with MinFill vs the treewidth.

Jegou et al. ICTAI 2018