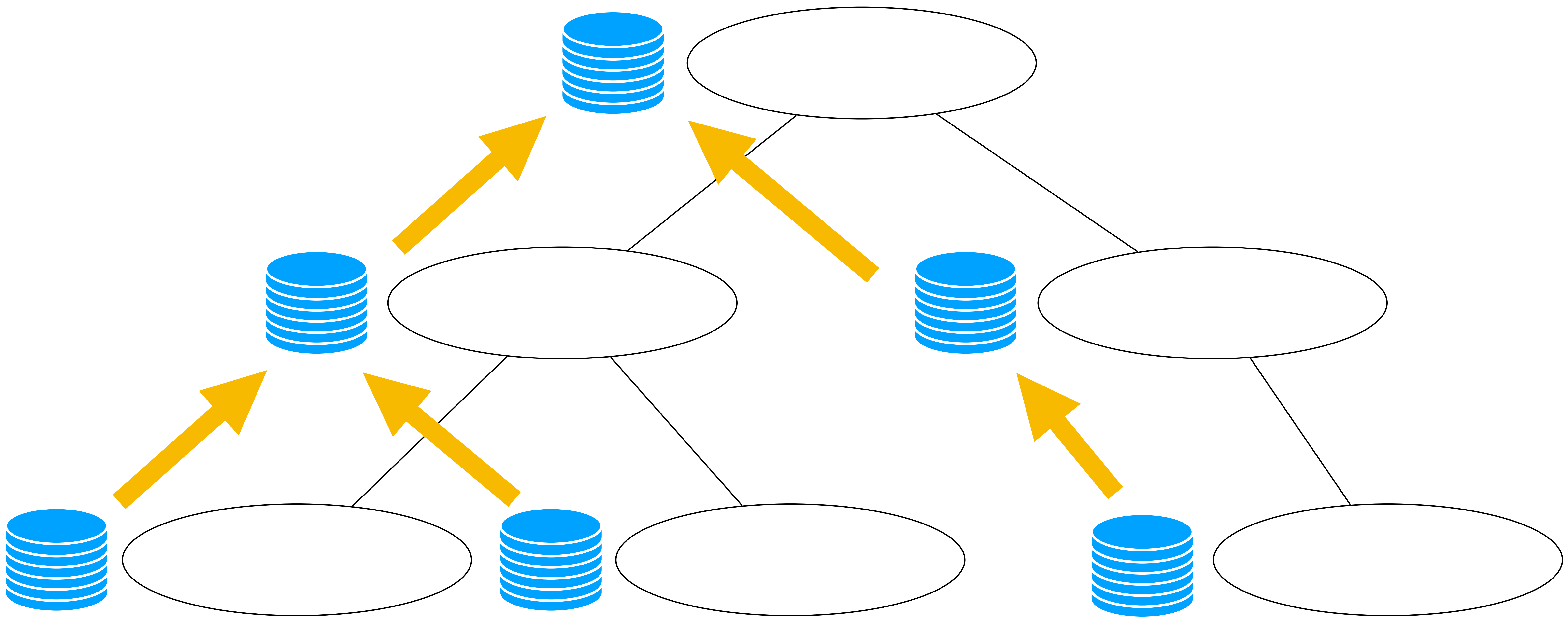
Courcelle’s Theorem

Disadvantages of Dynamic Programming



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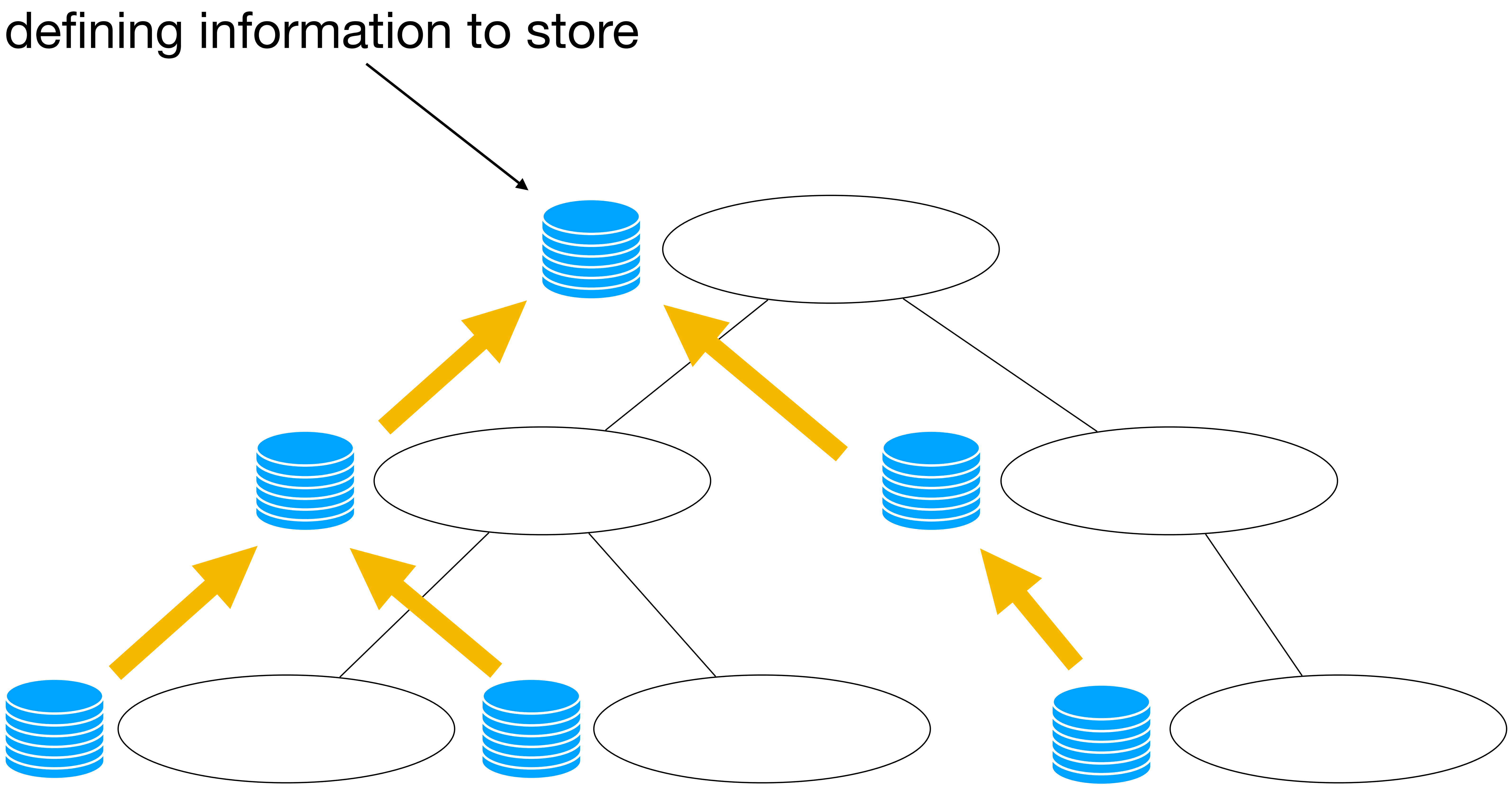

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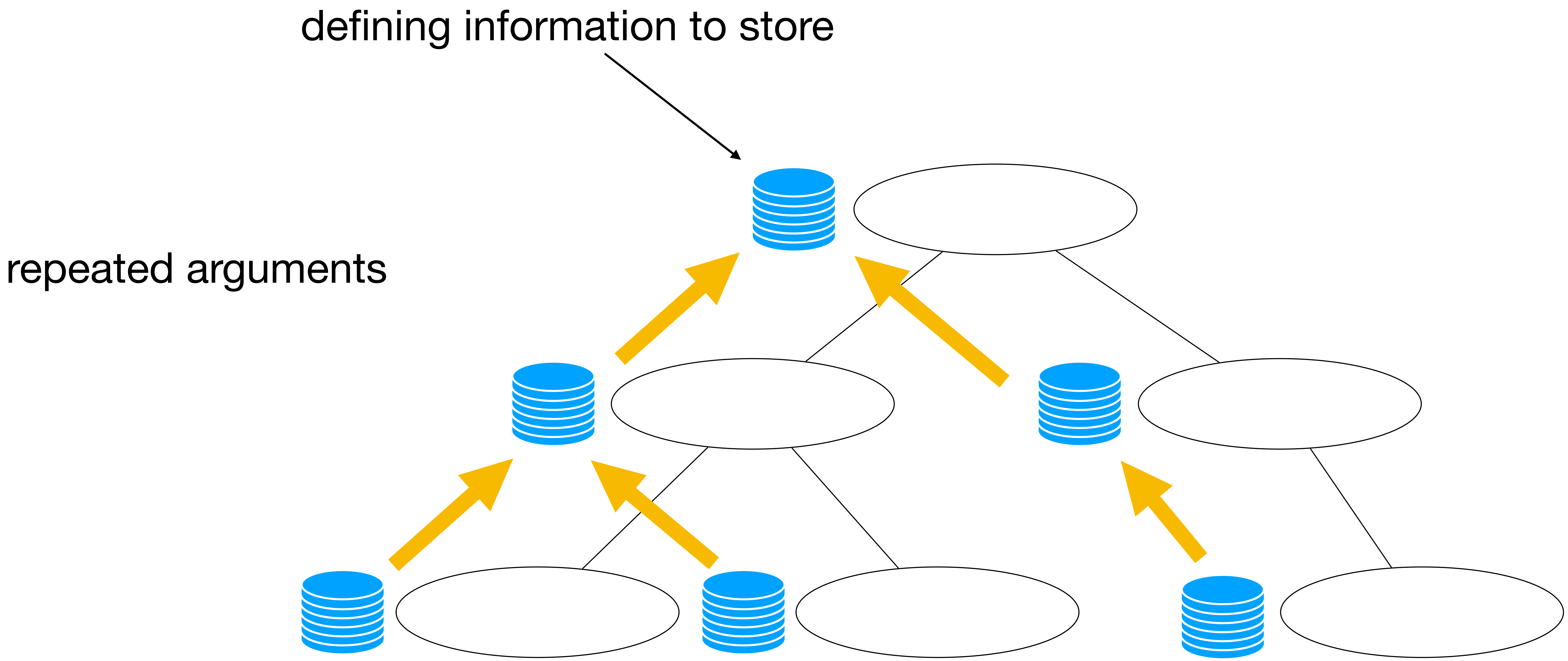
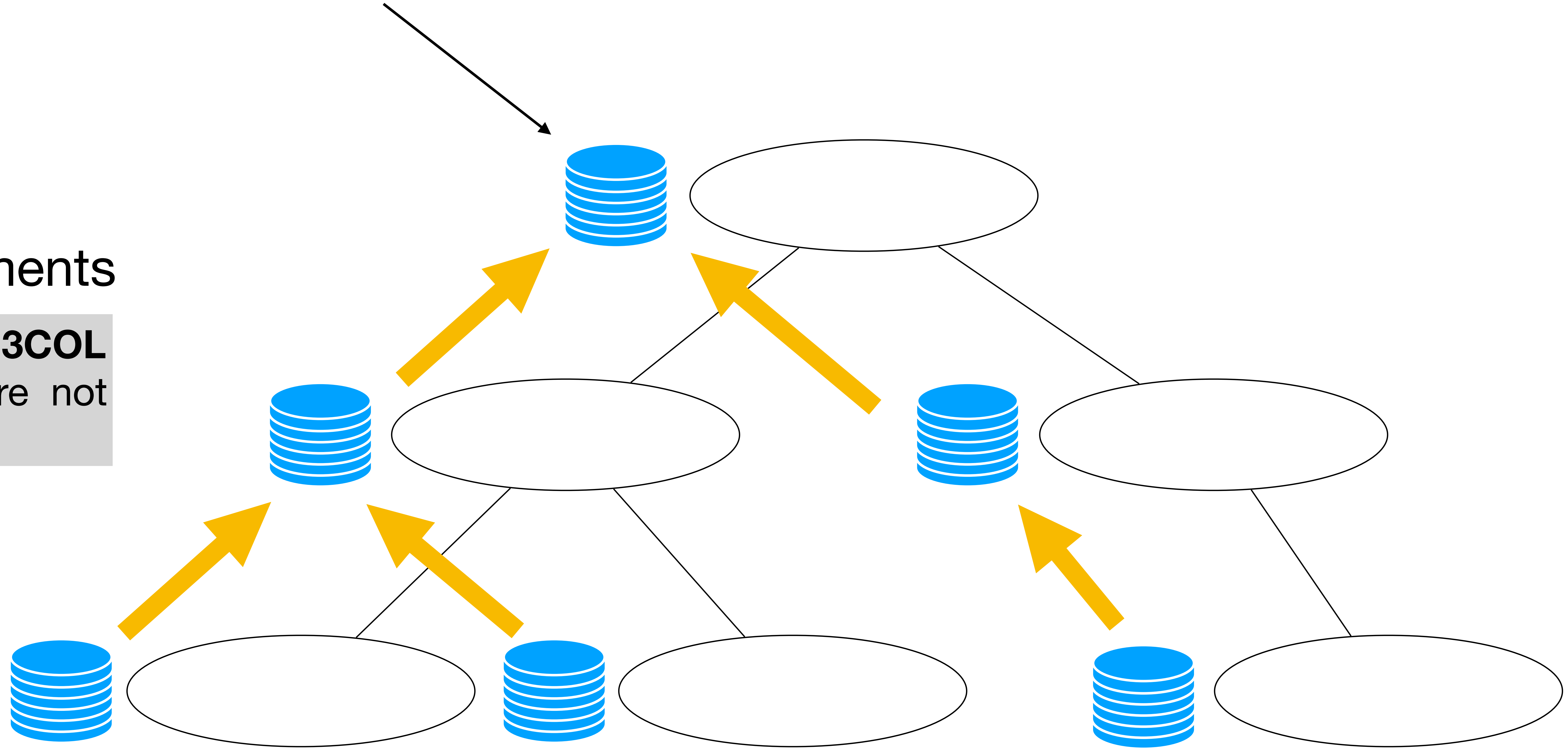


Disadvantages of Dynamic Programming

defining information to store



repeated arguments

DynProg for **IS** and **3COL** look similar, but are not exactly the same.

Logical Meta Theorems: A Teaser

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Problem in Monadic Second Order Logic

Logical Meta Theorems: A Teaser

Problem in Monadic Second Order Logic ⇨

Logical Meta Theorems: A Teaser

Problem in Monadic Second Order Logic ⇨ Meta Theorem

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Problem in Monadic Second Order Logic ⇨ Meta Theorem ⇨

Logical Meta Theorems: A Teaser

Problem in Monadic Second Order Logic ⇨ Meta Theorem ⇨ **FPT parameterized by treewidth**

set $P \subseteq A$ or $Q \subseteq B$, and then the duplicator answers by a subset $Q \subseteq B$ or $P \subseteq A$, respectively. After m moves, elements $a_1, \ldots, a_r$ and subsets $P_1, \ldots, P_s$ in A, and corresponding elements $b_1, \ldots, b_r$ and subsets $Q_1, \ldots, Q_s$ in B (with $m = r + s$) have been chosen. The duplicator wins if $\overline{a} \mapsto \overline{b} \in \mathrm{Part}((\mathcal{A}, P_1, \ldots, P_s), (\mathcal{B}, Q_1, \ldots, Q_s))$.

Theorem 3.1.1 $\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$ *iff the duplicator wins* MSO-G$_m(\mathcal{A}, \mathcal{B})$.

The following exercise leads to a proof of this theorem (along the lines of the proof of the corresponding result 2.2.8). □

Exercise 3.1.2 Given $\mathcal{A}$, $\overline{a}\,(= a_1 \ldots a_r)$ in A, and $\overline{P}\,(= P_1 \ldots P_s)$ a sequence of subsets of A, define the formulas $\psi^j_{\overline{a},\overline{P}}$ similar to the j-isomorphism type $\varphi^j_{\overline{a}}$, but now taking into account also the second-order set quantifiers:

$$\psi^0_{\overline{a},\overline{P}} :=$$

$$\bigwedge\{\varphi(v_1, \ldots, v_r, V_1, \ldots, V_s) \mid \varphi \text{ atomic or negated atomic}, \mathcal{A} \models \varphi[\overline{a}, \overline{P}]\}$$

Monadic Second Order Logic (MSO)

Syntax (MSO_1)

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Individual variables for vertices $v, w, \ldots$

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Atomic formulas $v = w$ or Evw

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Compound formulas φ, ψ

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$\exists v \, \varphi$

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Variables for sets of vertices $X, Y, Z, \ldots$

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MSO

Semantics

Semantics

Graph $G = (V, E)$

Semantics

Graph $G = (V, E)$

MSO_1-formula φ

Semantics

Graph $G = (V, E)$ MSO₁-formula φ **Interpretation** I

Semantics

Graph $G = (V, E)$ | MSO$_1$-formula φ | **Interpretation** I

$I(v) \in V$

Semantics

Graph $G = (V, E)$

MSO₁-formula φ

Interpretation I

$I(v) \in V \quad I(X) \subseteq V$

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$G, I \vDash Xx$	$\Leftrightarrow$	$I(x) \in I(X)$
$G, I \vDash \varphi \odot \psi$	$\Leftrightarrow$	$G, I \vDash \varphi \odot G, I \vDash \psi$
$G, I \vDash \exists X \varphi$	$\Leftrightarrow$	there is an $W \subseteq V$ such that $G, I' \vDash \varphi$

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$G \models \varphi$ ← for **closed** φ

Model Checking

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MSO MODEL CHECKING

Input: A graph G and an MSO sentence φ.

Question: Does $G \models \varphi$ hold?

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n number of vertices

k size of input sentence

fixed-parameter linear with respect to (F, φ) where φ is the input sentence.

(2) For every triple of finite, pairwise disjoint sets of labels (C, K, Λ), the problem of checking whether $\lceil val(t) \rceil_C \models \varphi$ for $t \in T(F^{\mathrm{HR}}_{C,[K,\Lambda]})$ and $\varphi \in \mathrm{CMS}_2(\mathcal{R}_{\mathrm{m},C,[K,\Lambda]}, \emptyset)$ is fixed-parameter linear with respect to (C, K, Λ, φ).

(3) For every triple of finite, pairwise disjoint sets of labels (C, K, Λ), the problem of checking whether $\lfloor val(t) \rfloor_C \models \varphi$ for $t \in T(F^{\mathrm{VR}}_{C,[K,\Lambda]})$ and $\varphi \in \mathrm{CMS}(\mathcal{R}_{\mathrm{s},C,[K,\Lambda]}, \emptyset)$ is fixed-parameter linear with respect to (C, K, Λ, φ).

Courcelle's Theorem

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Theorem (Courcelle)

Let G be an n-vertex graph and let φ be an MSO formula. There exists an algorithm that, given a tree decomposition of G of width k, determines whether $G \models \varphi$ in time $f(|\varphi|, k) \cdot n$ for some computable f.

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Theorem

MSO MODEL CHECKING is FPT parameterized by the treewidth of G and $|\varphi|$.

Applications

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3-COLORABILITY

Input: A graph G.

Question: Does G have a vertex-coloring with 3 colors?

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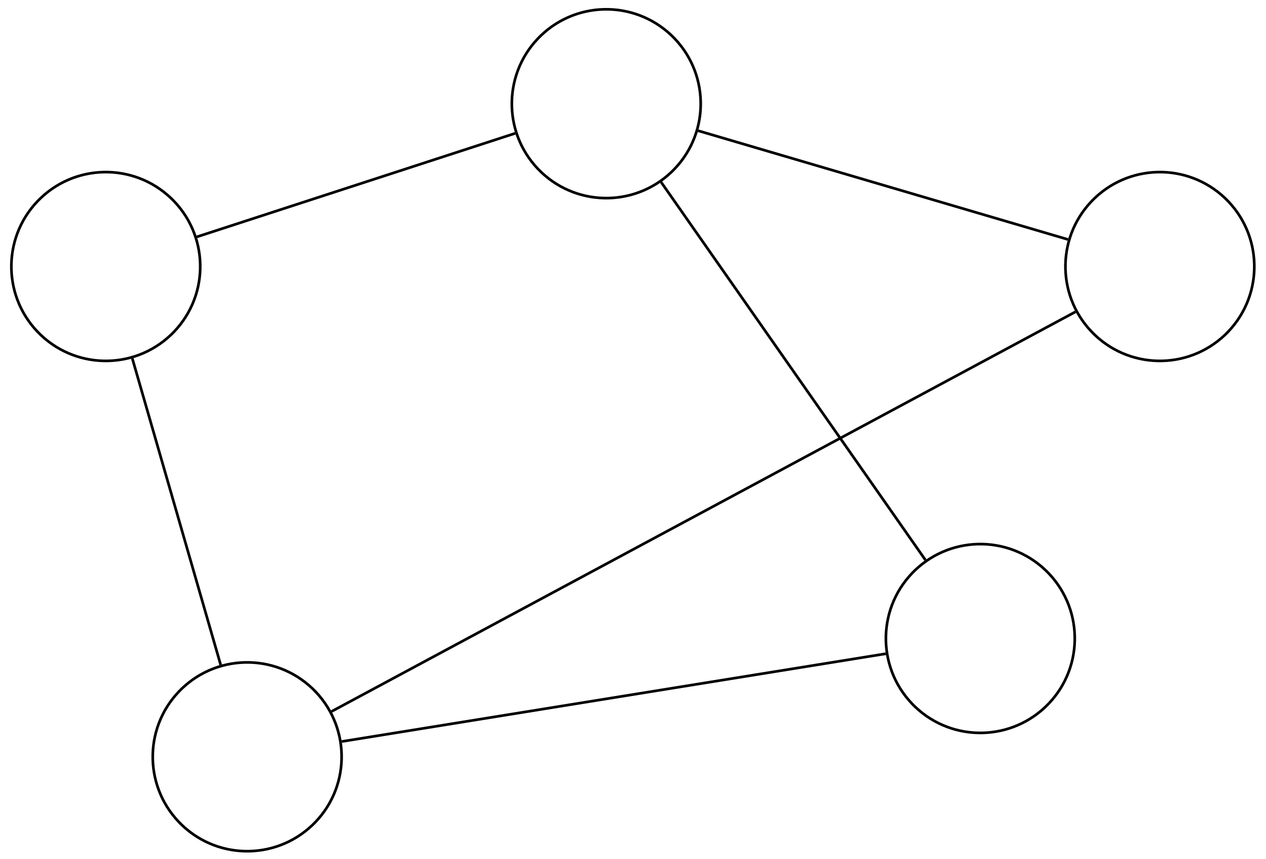

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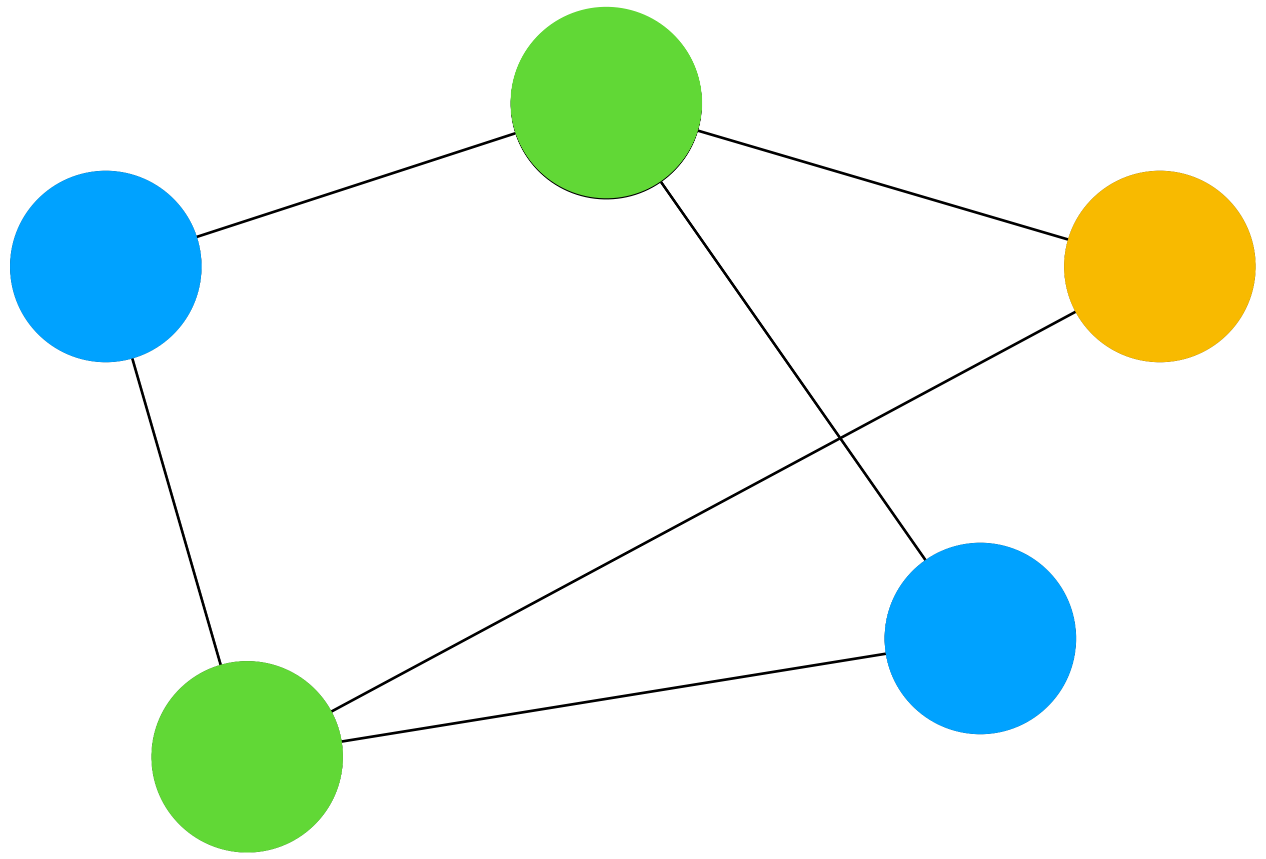

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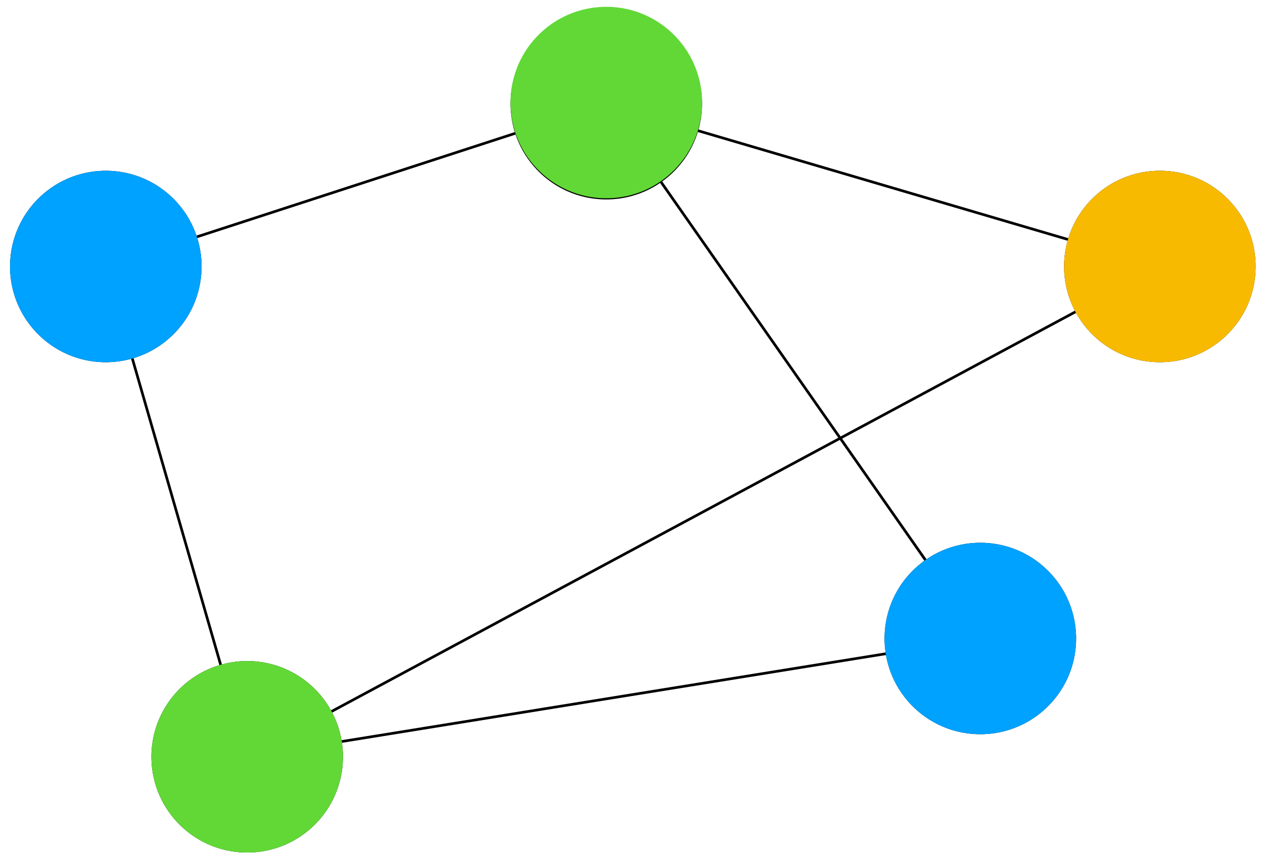

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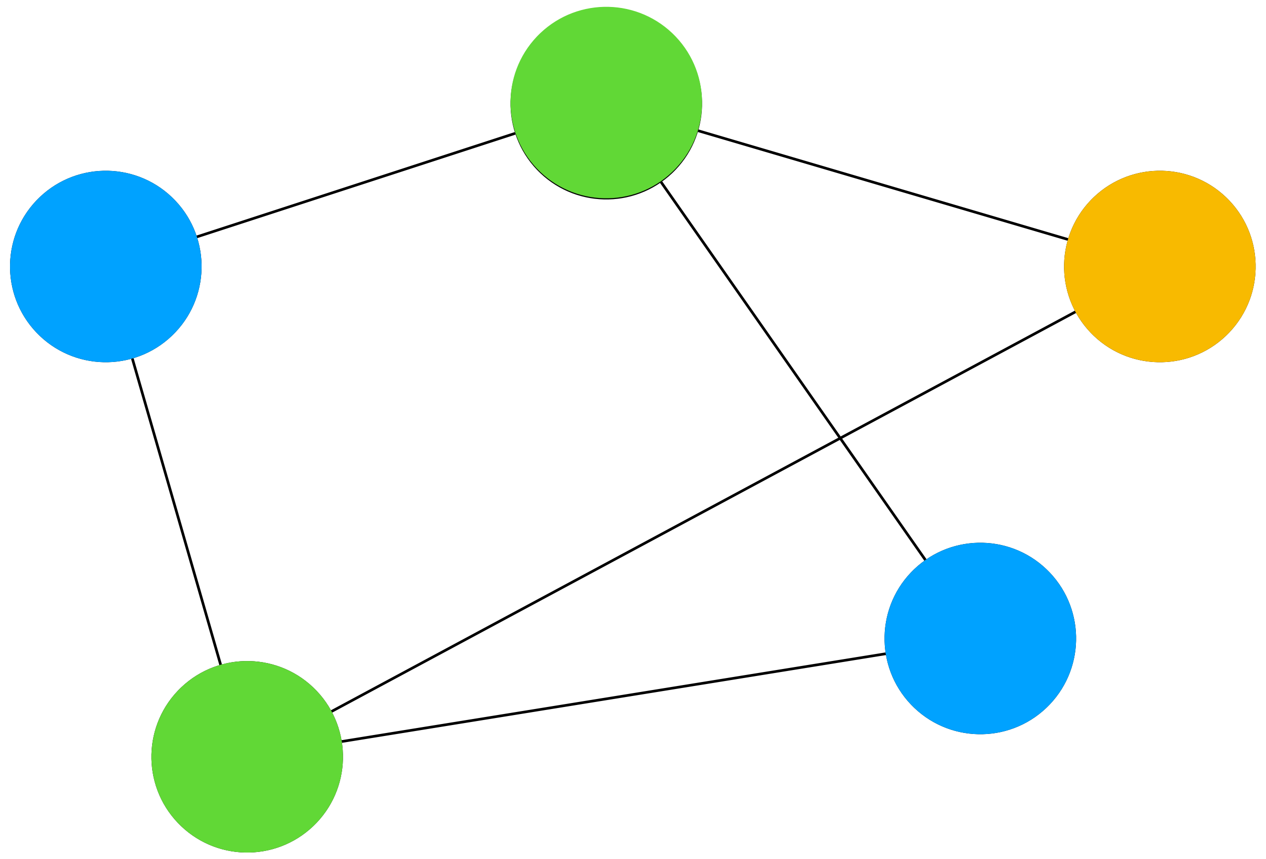

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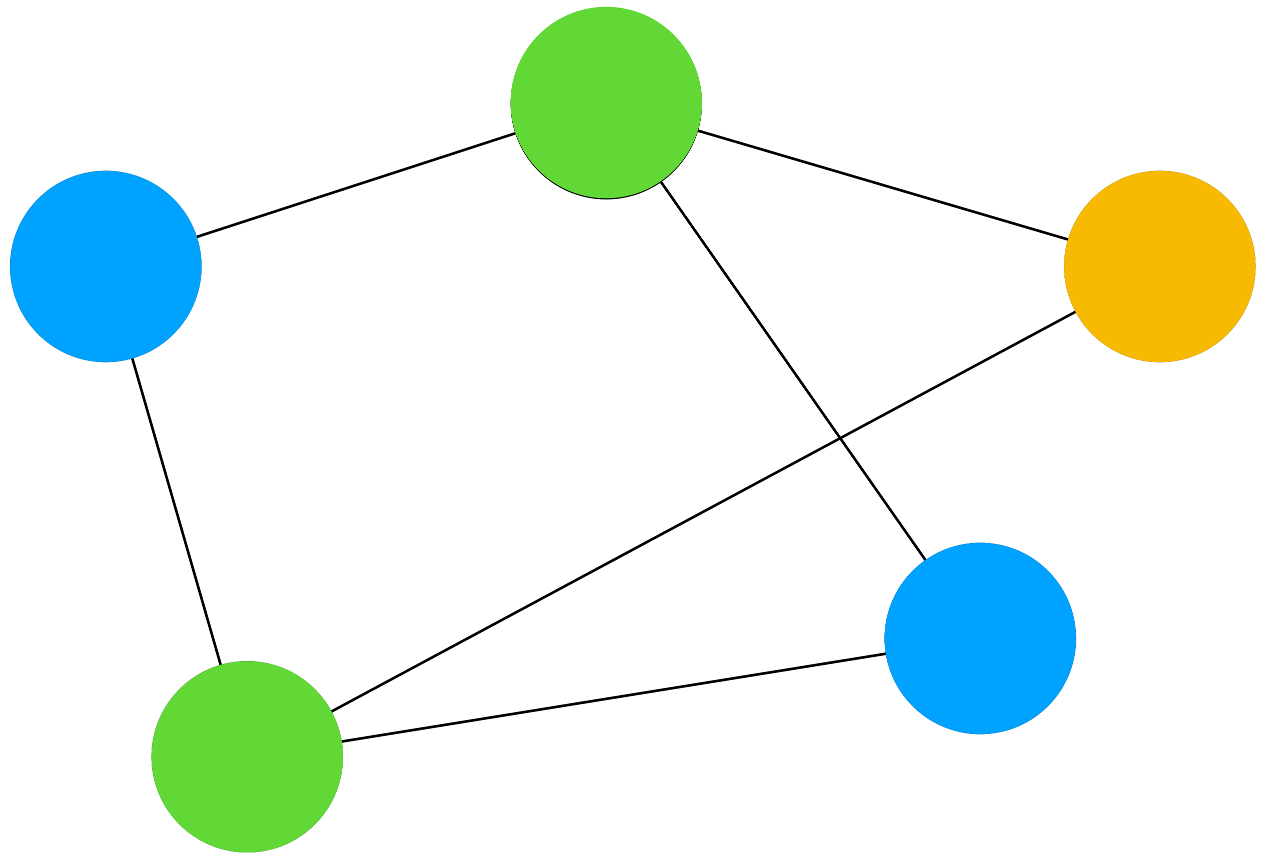

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$$\forall x \forall y\, Exy \rightarrow (\neg(Xx \wedge Xy) \wedge \neg(Yx \wedge Yy) \wedge \neg(Zx \wedge Zy))$$

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HAMILTONIAN CYCLE

Input: A graph G.

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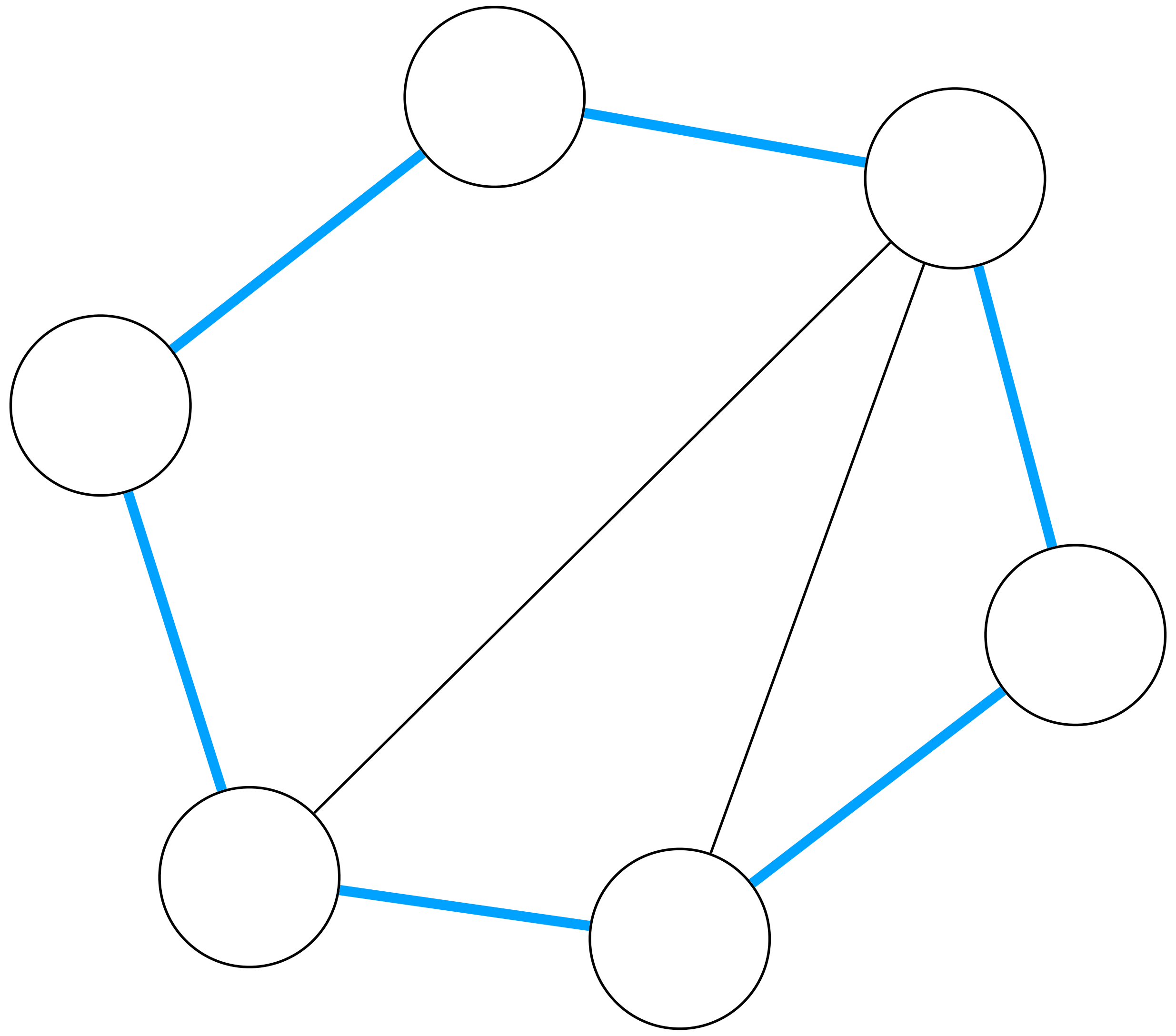

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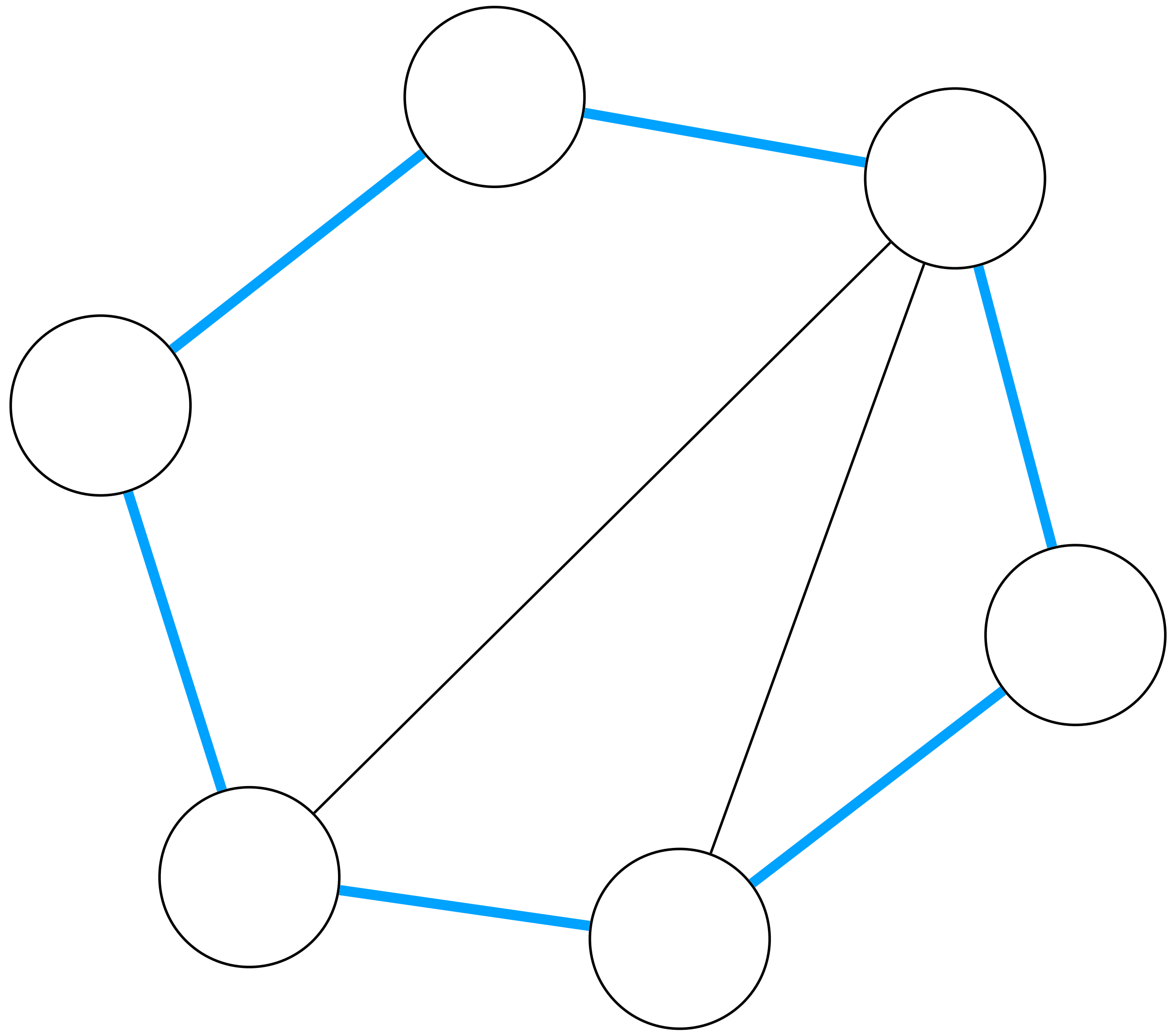

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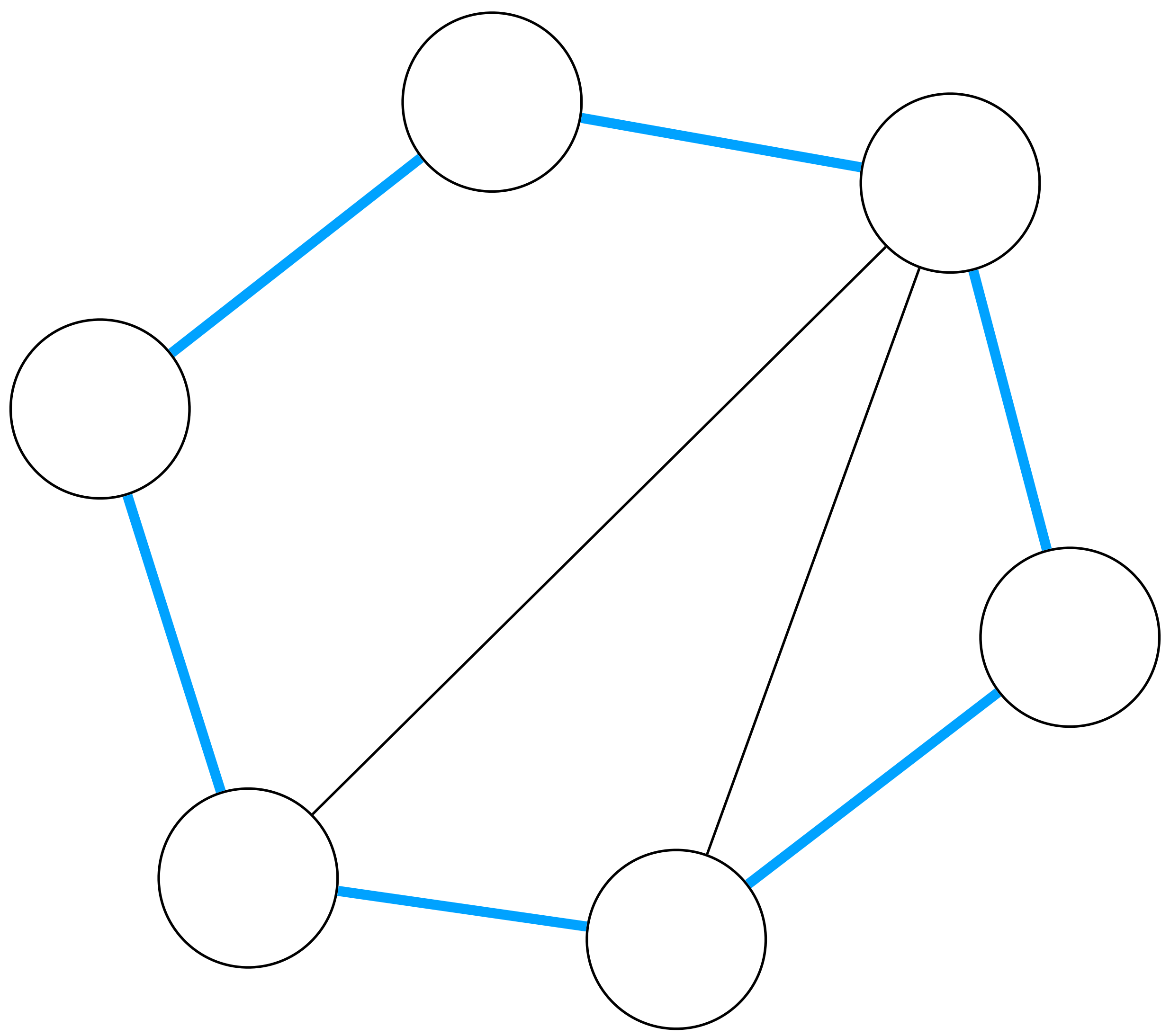

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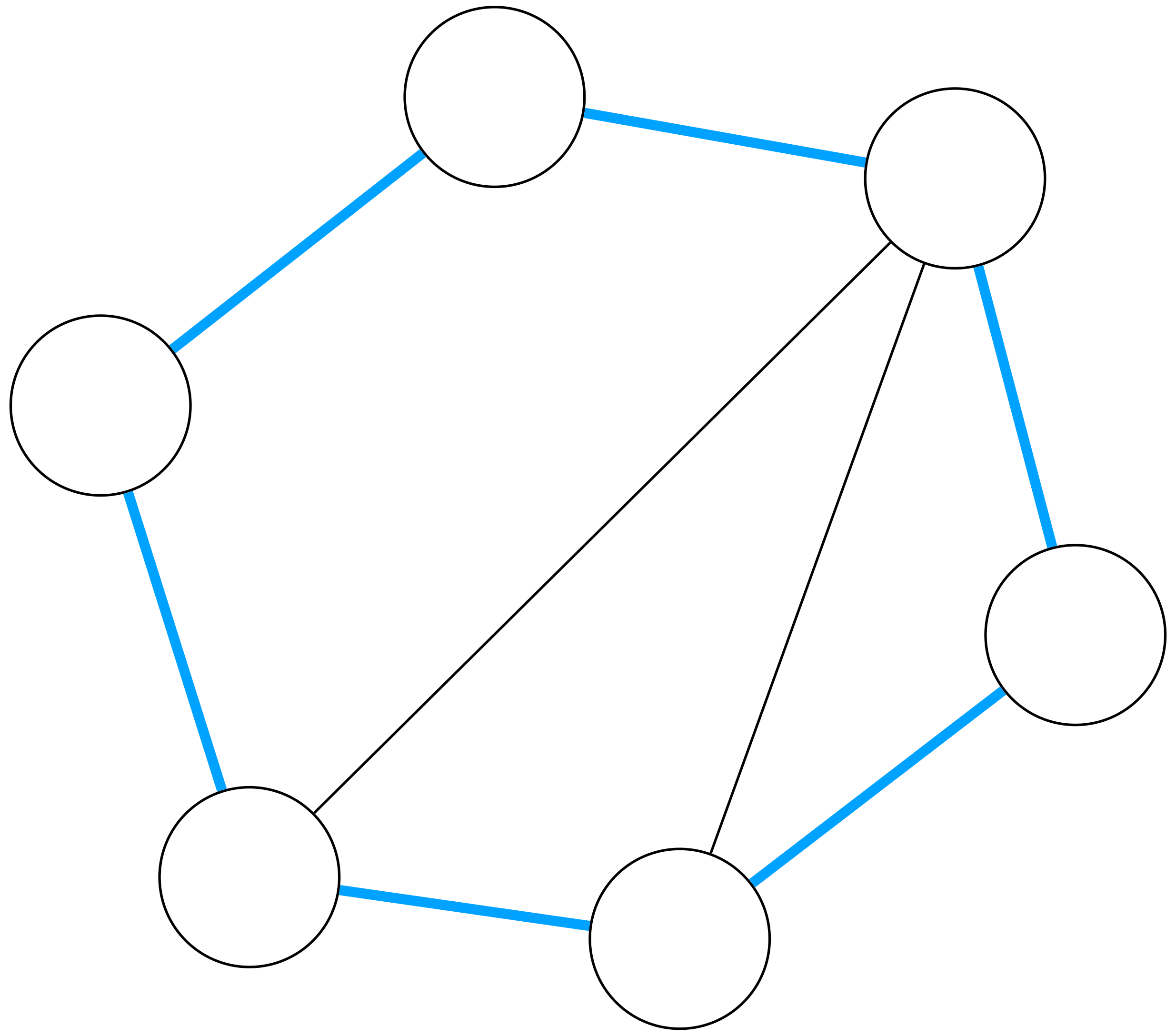

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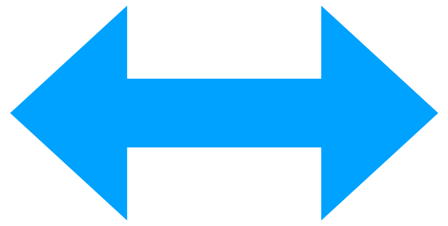
$G \models \varphi$ 

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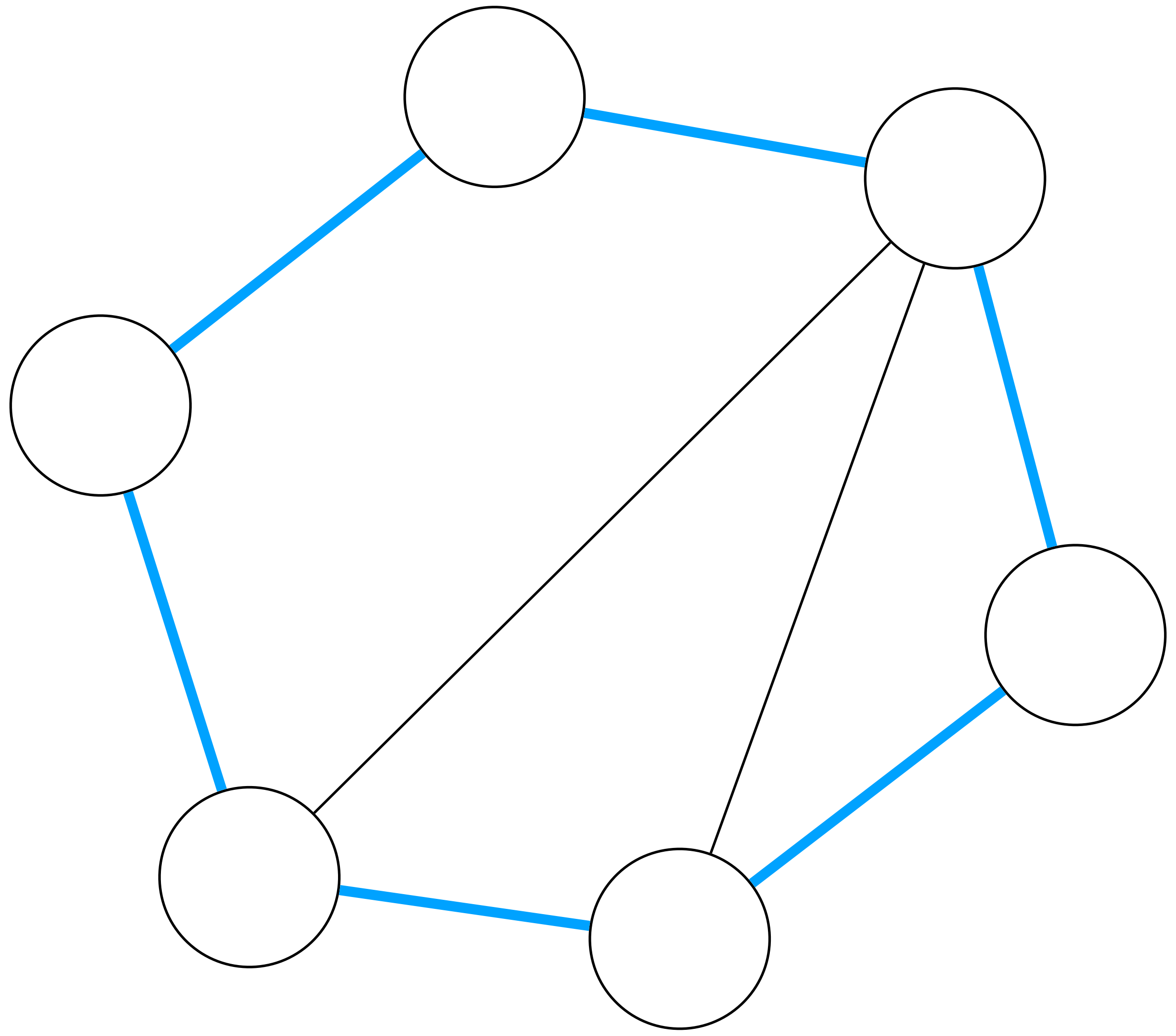

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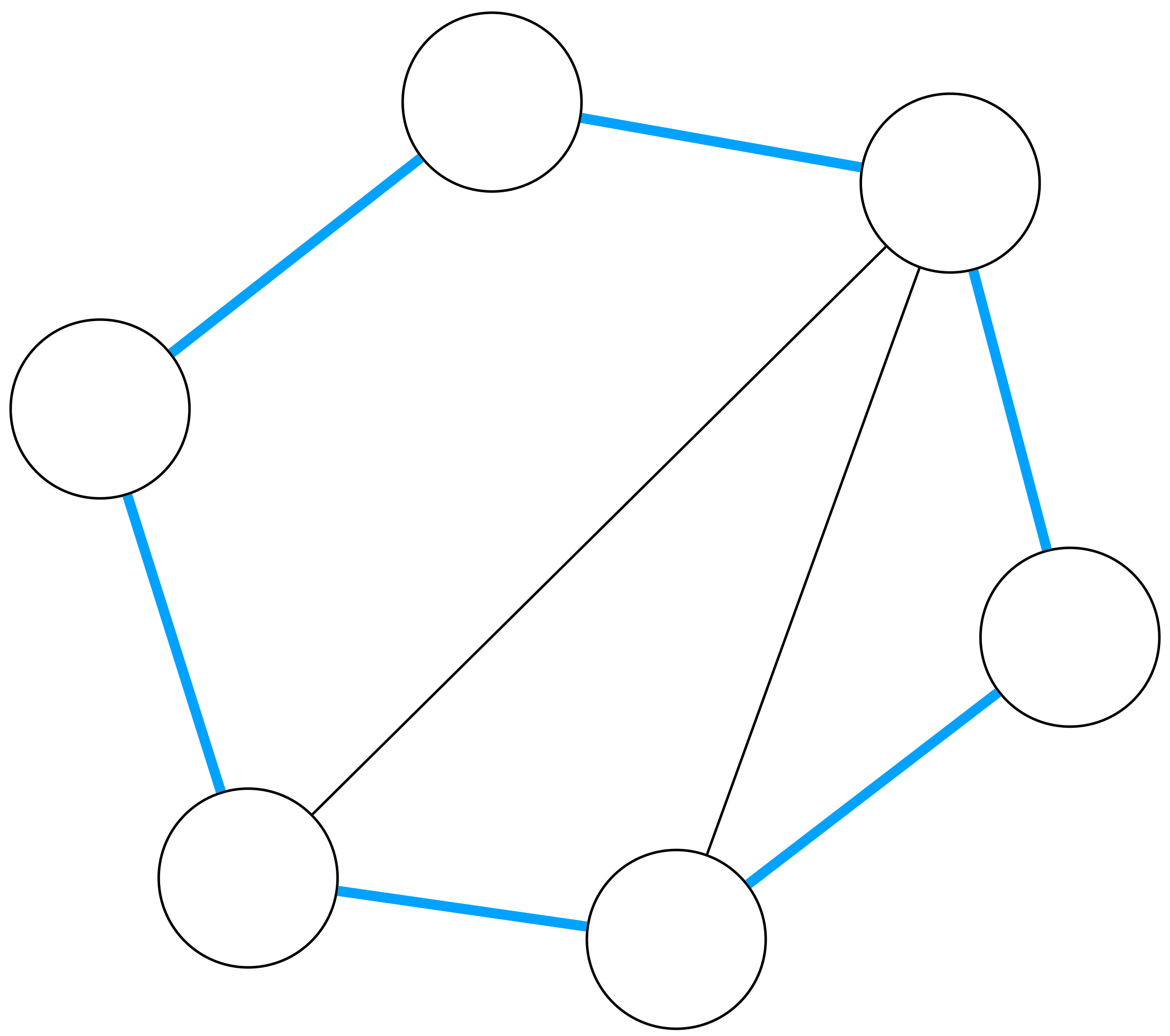

Can we say “there is a Hamiltonian Cycle” in MSO_1?

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Fact

There is no such MSO_1-sentence.

Syntax (MSO_2)

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Atomic formulas $v = w$ or Vw or Ive

$\odot \in \{ \wedge , \vee , \rightarrow , \leftrightarrow \}$

Compound formulas φ, ψ $\quad \neg\varphi \quad \varphi \odot \psi$

$\exists v\, \varphi \quad \forall v\, \varphi$

Variables for sets of vertices **and edges** $X, Y, Z, \ldots$

Atomic formulas Xx $\quad \exists X\, \varphi \quad \forall X\, \varphi$

Semantics

Semantics

Graph $G = (V, E)$

Semantics

Graph $G = (V, E)$

MSO_2-formula φ

Semantics

Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

Semantics

Graph $G = (V, E)$

MSO$_2$-formula φ

Interpretation I

$I(x) \in V \cup E$

Semantics

Graph $G = (V, E)$

MSO₂-formula φ

Interpretation I

$I(x) \in V \cup E \qquad I(X) \subseteq V \cup E$

Semantics

Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

$$G, I \models \varphi$$

$I(x) \in V \cup E$ $I(X) \subseteq V \cup E$

Semantics

Graph $G = (V, E)$ MSO₂-formula φ **Interpretation** I

$$G, I \models \varphi$$

$I(x) \in V \cup E$ $I(X) \subseteq V \cup E$

$G, I \models Vw$

Semantics

Graph $G = (V, E)$

MSO₂-formula φ

Interpretation I

$$G, I \vDash \varphi$$

$I(x) \in V \cup E \qquad I(X) \subseteq V \cup E$

$G, I \vDash Vw \quad \Leftrightarrow$

Semantics

Graph $G = (V, E)$

MSO$_2$-formula φ

Interpretation I

$$G, I \vDash \varphi$$

$I(x) \in V \cup E$ $\quad$ $I(X) \subseteq V \cup E$

$G, I \vDash Vw \quad \Longleftrightarrow \quad I(w) \in V$

Semantics

Graph $G = (V, E)$

MSO$_2$-formula φ

Interpretation I

$$G, I \models \varphi$$

$I(x) \in V \cup E$ $\quad$ $I(X) \subseteq V \cup E$

$G, I \models Vw$ $\iff$ $I(w) \in V$

$G, I \models Ive$

Semantics

Graph $G = (V, E)$ MSO2-formula φ **Interpretation** I

$G, I \vDash \varphi$

$I(x) \in V \cup E \quad I(X) \subseteq V \cup E$

$G, I \vDash Vw \iff I(w) \in V$

$G, I \vDash Ive \iff$

Semantics

Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

$$G, I \models \varphi$$

$I(x) \in V \cup E$ $I(X) \subseteq V \cup E$

$G, I \models Vw \iff I(w) \in V$

$G, I \models Ive \iff I(v)$ is an endpoint of $I(e)$

Semantics

Graph $G = (V, E)$ | MSO_2-formula φ | **Interpretation** I

$$G, I \vDash \varphi$$

$I(x) \in V \cup E$ $\quad$ $I(X) \subseteq V \cup E$

$G, I \vDash Vw \iff I(w) \in V$

$G, I \vDash Ive \iff I(v)$ is an endpoint of $I(e)$

$G, I \vDash Xx$

Semantics

Graph $G = (V, E)$ MSO₂-formula φ **Interpretation** I

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Semantics

Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

$$G, I \models \varphi$$

$I(x) \in V \cup E$ $I(X) \subseteq V \cup E$

$G, I \models Vw \Leftrightarrow I(w) \in V$

$G, I \models Ive \Leftrightarrow I(v)$ is an endpoint of $I(e)$

$G, I \models Xx \Leftrightarrow I(x) \in I(X)$

$G, I \models x = y$

Semantics

Graph $G = (V, E)$

MSO$_2$-formula φ

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$$G, I \vDash \varphi$$

$I(x) \in V \cup E \qquad I(X) \subseteq V \cup E$

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Semantics

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$G, I \vDash x = y \iff I(x) = I(y)$

$G, I \vDash \varphi \odot \psi$

Semantics

Graph $G = (V, E)$

MSO$_2$-formula φ

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$I(x) \in V \cup E$ $\quad I(X) \subseteq V \cup E$

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Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

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$G, I \vDash x = y \Leftrightarrow I(x) = I(y)$

$G, I \vDash \varphi \odot \psi \Leftrightarrow G, I \vDash \varphi \odot G, I \vDash \psi$

Semantics

Graph $G = (V, E)$ MSO₂-formula φ **Interpretation** I

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$G, I \vDash x = y \iff I(x) = I(y)$

$G, I \vDash \varphi \odot \psi \iff G, I \vDash \varphi \odot G, I \vDash \psi$

$G, I \vDash \exists X \varphi$

Semantics

Graph $G = (V, E)$ MSO₂-formula φ **Interpretation** I

$$G, I \vDash \varphi \qquad I(x) \in V \cup E \qquad I(X) \subseteq V \cup E$$

$G, I \vDash Vw \iff I(w) \in V$

$G, I \vDash Ive \iff I(v)$ is an endpoint of $I(e)$

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$G, I \vDash \varphi \odot \psi \iff G, I \vDash \varphi \odot G, I \vDash \psi$

$G, I \vDash \exists X\varphi \iff$

Semantics

Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

$G, I \vDash \varphi$ $I(x) \in V \cup E$ $I(X) \subseteq V \cup E$

$G, I \vDash Vw$	⟺	$I(w) \in V$
$G, I \vDash Ive$	⟺	$I(v)$ is an endpoint of $I(e)$
$G, I \vDash Xx$	⟺	$I(x) \in I(X)$
$G, I \vDash x = y$	⟺	$I(x) = I(y)$
$G, I \vDash \varphi \odot \psi$	⟺	$G, I \vDash \varphi \odot G, I \vDash \psi$
$G, I \vDash \exists X\varphi$	⟺	there is an $W \subseteq V \cup E$ such that $G, I' \vDash \varphi$

Semantics

Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

$G, I \vDash \varphi$ $I(x) \in V \cup E$ $I(X) \subseteq V \cup E$

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$G, I \vDash Ive \iff I(v)$ is an endpoint of $I(e)$

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$G, I \vDash x = y \iff I(x) = I(y)$

$G, I \vDash \varphi \odot \psi \iff G, I \vDash \varphi \odot G, I \vDash \psi$

$G, I \vDash \exists X\varphi \iff$ there is an $W \subseteq V \cup E$ such that $G, I' \vDash \varphi$ where $I'(X) = W$.

Semantics

Graph $G = (V, E)$ MSO$_2$-formula φ **Interpretation** I

$G, I \models \varphi$

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$G, I \models Ive \iff I(v)$ is an endpoint of $I(e)$

$G, I \models Xx \iff I(x) \in I(X)$

$G \models \varphi$

$G, I \models x = y \iff I(x) = I(y)$

$G, I \models \varphi \odot \psi \iff G, I \models \varphi \odot G, I \models \psi$

$G, I \models \exists X \varphi \iff$ there is an $W \subseteq V \cup E$ such that $G, I' \models \varphi$ where $I'(X) = W$.

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$G, I \vDash \varphi \odot \psi \iff G, I \vDash \varphi \odot G, I \vDash \psi$

$G, I \vDash \exists X\varphi \iff$ there is an $W \subseteq V \cup E$ such that $G, I' \vDash \varphi$ where $I'(X) = W$.

$G \vDash \varphi$ ← for **closed** φ

Applications

Applications

HAMILTONIAN CYCLE

Input: A graph G.

Question: Does G have a Hamiltonian Cycle?

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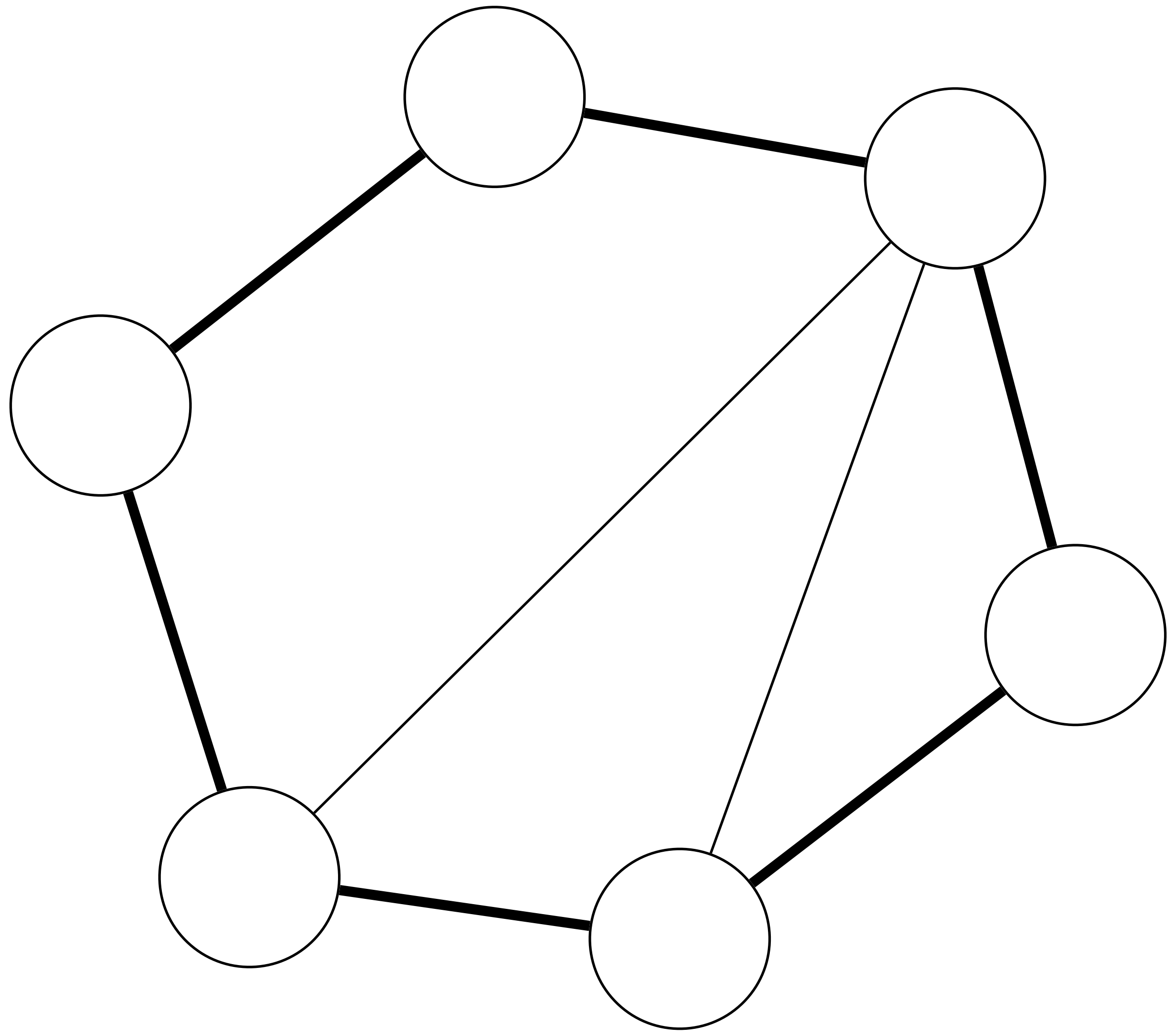

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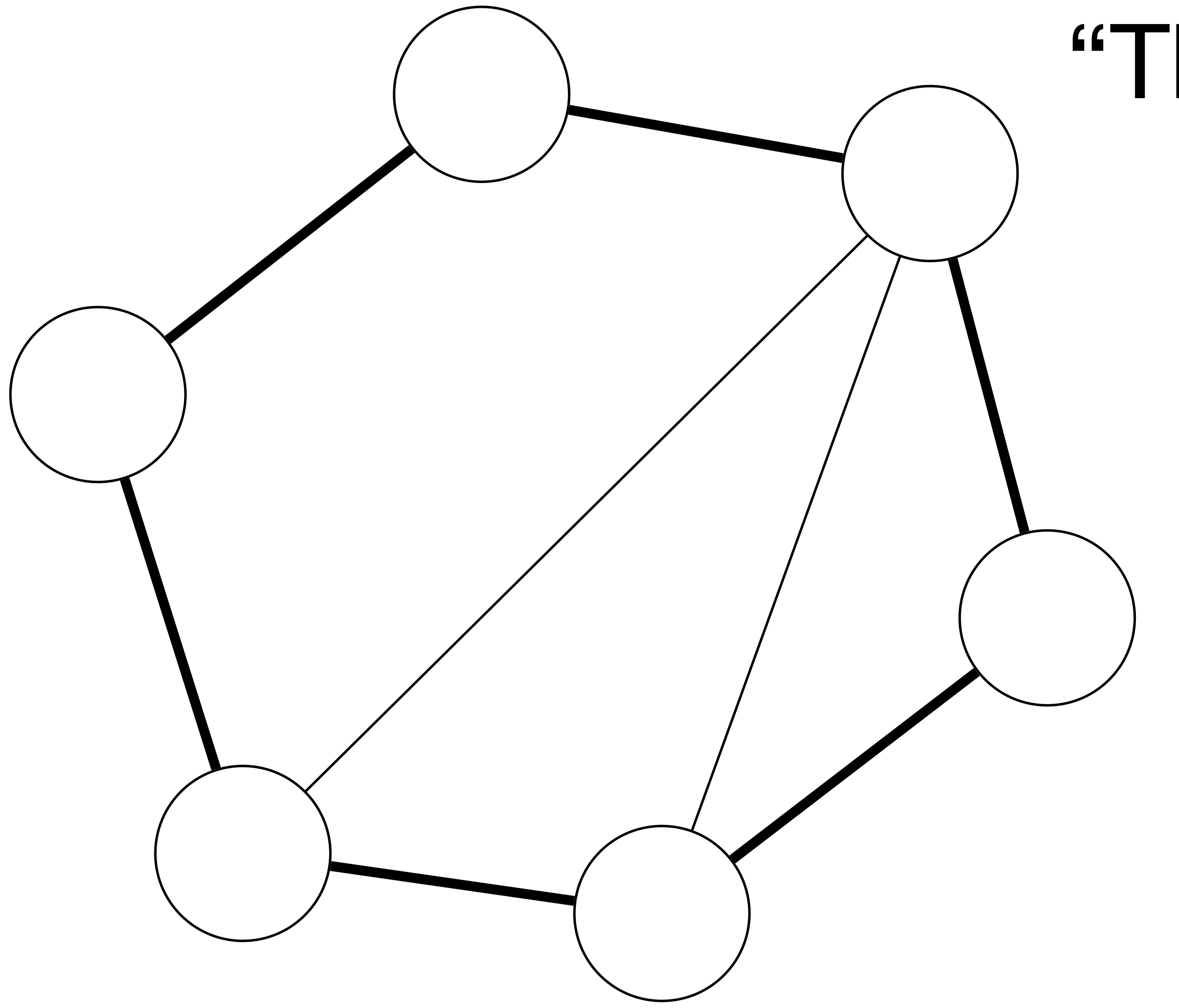

"There is a set of edges forming a Hamiltonian cycle" in **MSO$_2$**:

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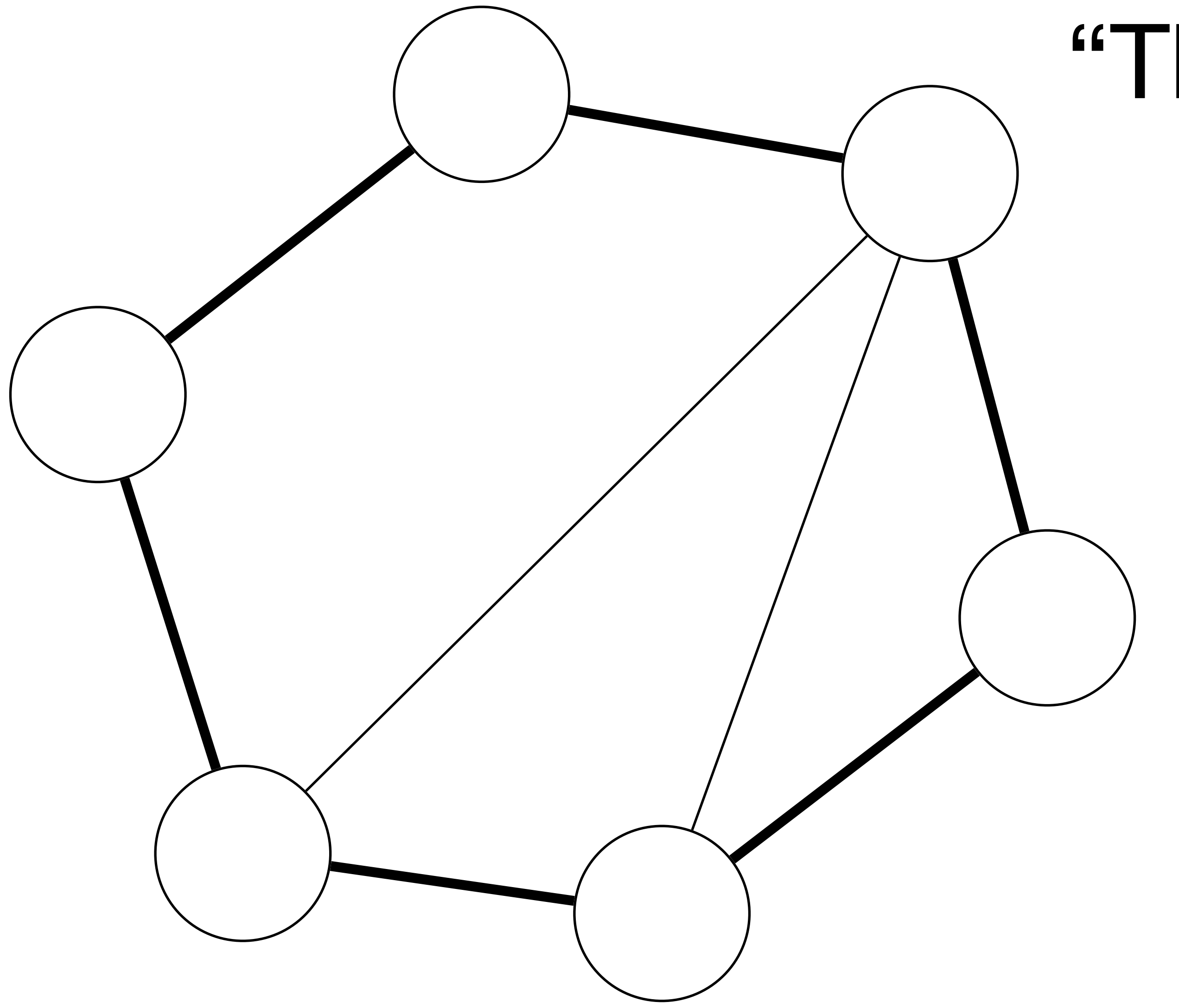

"There is a set of edges forming a Hamiltonian cycle" in **MSO₂**:

$$\exists_{Y \subseteq E}$$

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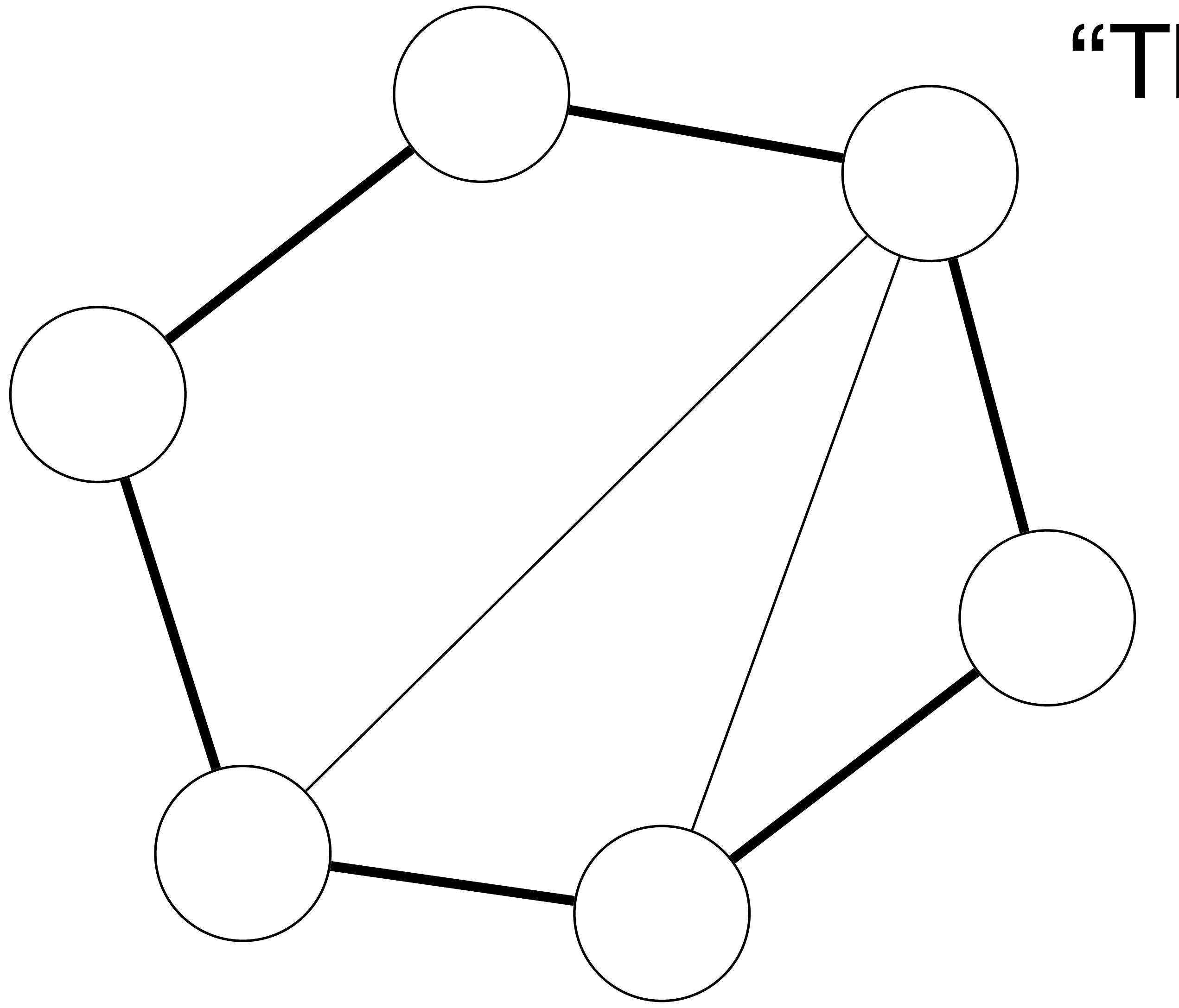

"There is a set of edges forming a Hamiltonian cycle" in **MSO₂**:

$$\exists_{Y \subseteq E}\, connectsgraph(Y)$$

Applications

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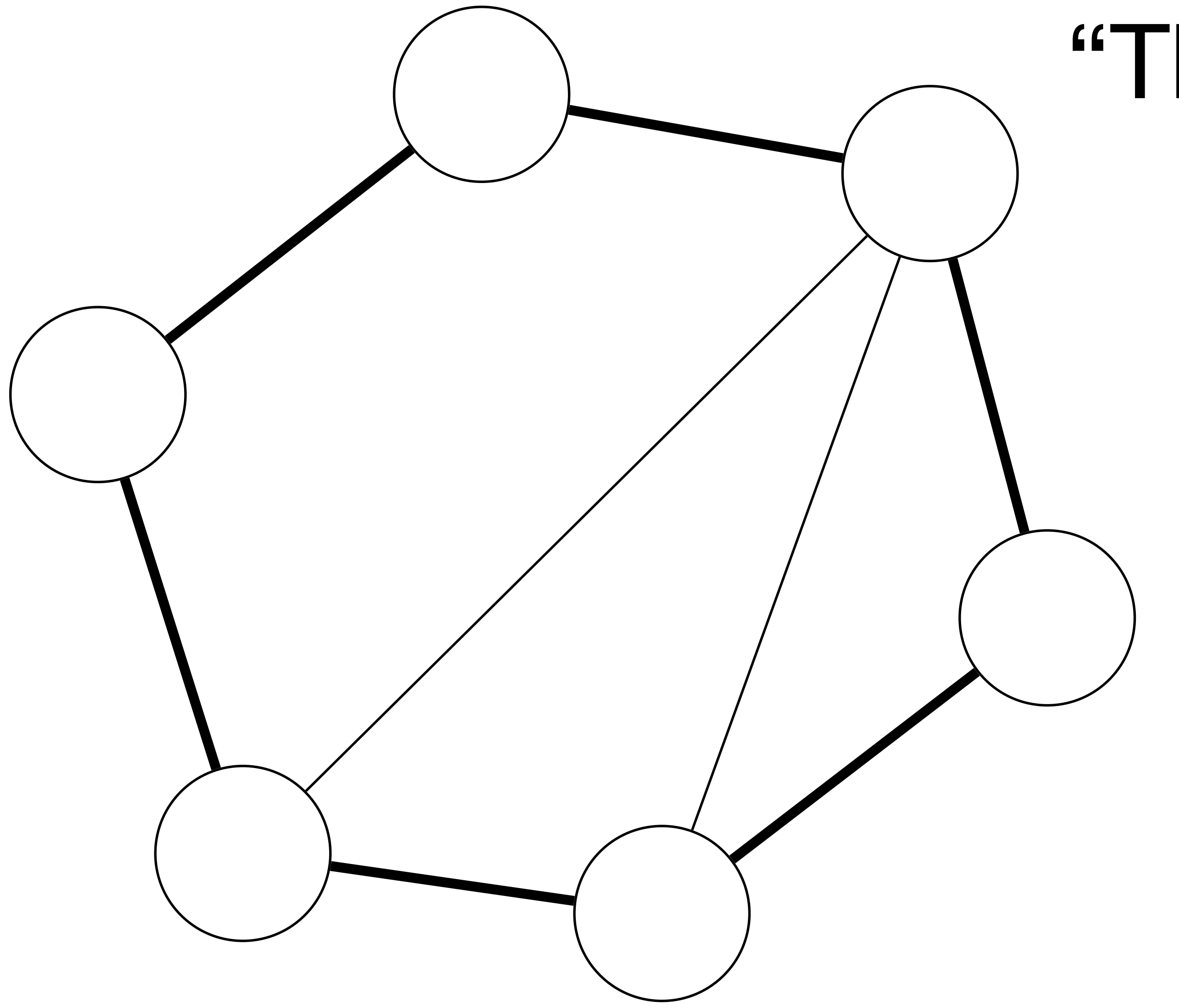

"There is a set of edges forming a Hamiltonian cycle" in **MSO₂**:

$$\exists_{Y \subseteq E} \, connectsgraph(Y) \land \forall_{v \in V} \, degree_2(v, Y)$$

Applications

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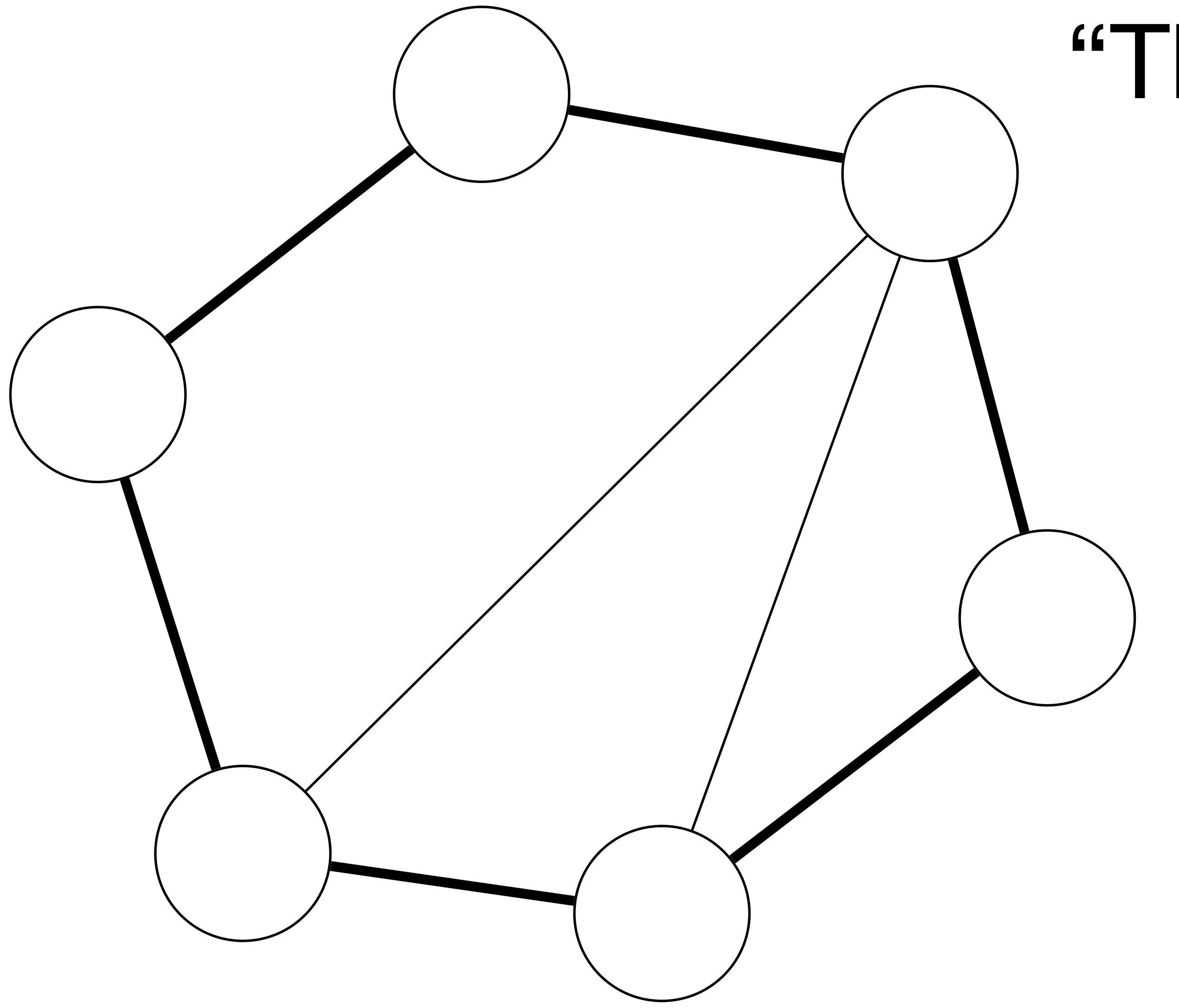

"There is a set of edges forming a Hamiltonian cycle" in **MSO₂**:

$$\exists_{Y \subseteq E}\, connectsgraph(Y) \wedge \forall_{v \in V}\, degree_2(v, Y)$$

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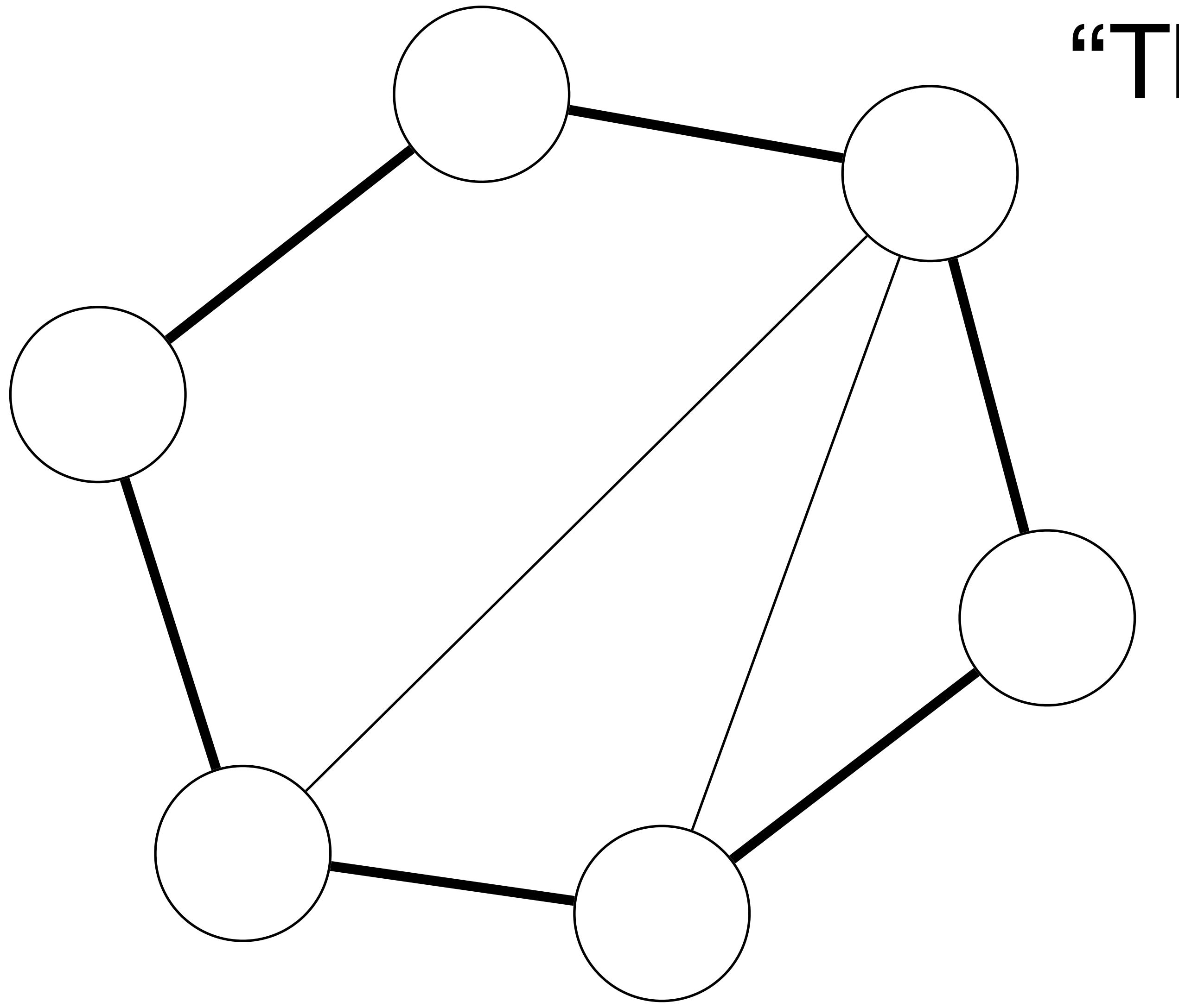

"There is a set of edges forming a Hamiltonian cycle" in **MSO₂**:

$$\exists_{Y \subseteq E}\, connectsgraph(Y) \wedge \forall_{v \in V}\, degree_2(v, Y)$$

$$\forall_{X \subseteq V} (\exists x Xx \wedge \exists x \neg Xx) \rightarrow \exists x \exists y Xx \wedge \neg Xy \wedge \exists_{e \in Y} Ixe \wedge Iye$$

Applications

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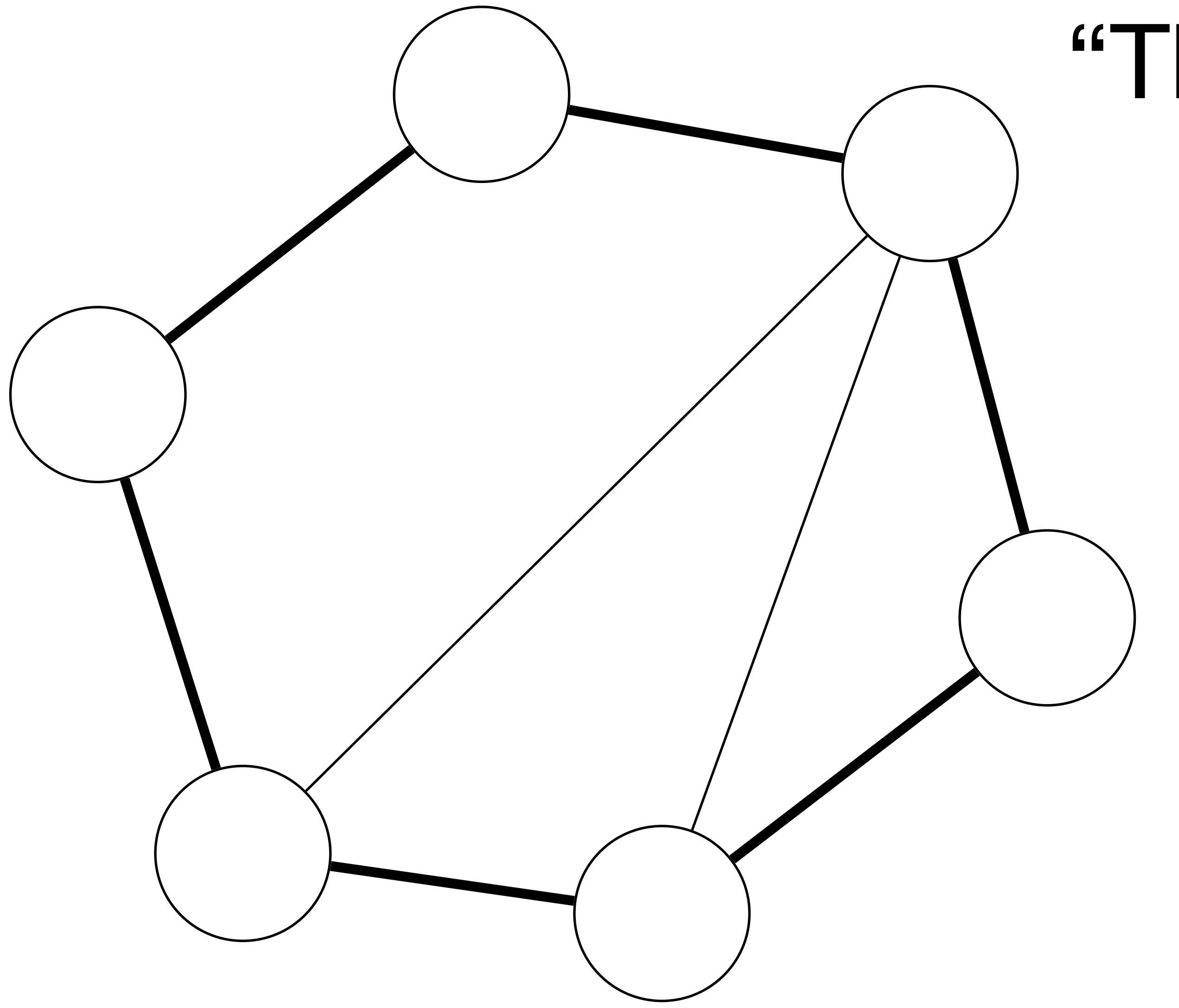

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$$\exists_{e_1 \in Y} \exists_{e_2 \in Y} Ive_1 \wedge Ive_2$$

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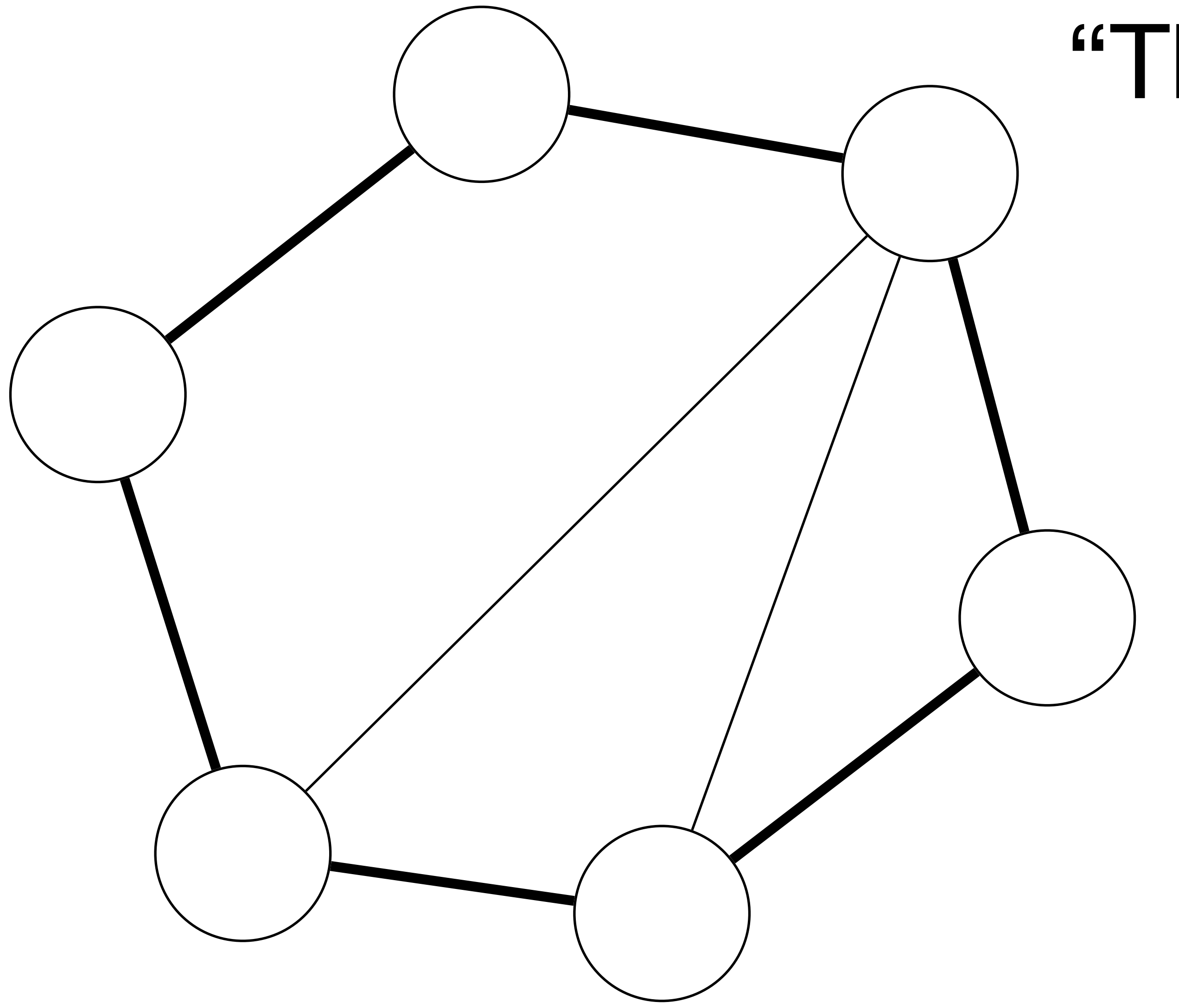

"There is a set of edges forming a Hamiltonian cycle" in **MSO₂**:

$$\exists_{Y \subseteq E}\, connectsgraph(Y) \wedge \forall_{v \in V}\, degree_2(v, Y)$$

$$\forall_{X \subseteq V}(\exists x Xx \wedge \exists x \neg Xx) \rightarrow \exists x \exists y Xx \wedge \neg Xy \wedge \exists_{e \in Y} Ixe \wedge Iye$$

$$\exists_{e_1 \in Y} \exists_{e_2 \in Y} Ive_1 \wedge Ive_2 \wedge \forall_{e \in Y} Ive \rightarrow (e = e_1 \vee e = e_2)$$

A Generalization of Courcelle's Theorem to Optimization Problems

MSO and Cardinalities

INDEPENDENT SET

Input: A graph G and an integer k.

Question: Does G have an **IS** of size k?

MSO and Cardinalities

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Input: A graph G and an integer k.

Question: Does G have an **IS** of size k?

$$\exists v_1 \ldots \exists v_k \bigwedge_{1 \le i < j \le k} \neg E v_i v_j$$

MSO and Cardinalities

INDEPENDENT SET

Input: A graph G and an integer k.

Question: Does G have an **IS** of size k?

$$\exists v_1 \ldots \exists v_k \bigwedge_{1 \leq i < j \leq k} \neg E v_i v_j$$

Theorem

MSO MODEL CHECKING is FPT parameterized by the treewidth of G and the length $|\varphi|$ of φ.

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Theorem

MSO MODEL CHECKING is FPT parameterized by the treewidth of G and the length $|\varphi|$ of φ.

Corollary

INDEPENDENT SET is FPT parameterized by the treewidth of G and k.

Linear EMS Problems

Syntax

open MSO_2 formula $\varphi(X_1, \ldots, X_l)$

Linear EMS Problems

Syntax

open MSO_2 formula $\varphi(X_1, \ldots, X_l)$

linear function $g(x_{11}, \ldots, x_{ml}, y_1, \ldots, y_t)$

Linear EMS Problems

Syntax

Semantics

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Semantics

graph G

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Semantics

graph G

$f_i^G : V \cup E \to \mathbb{Q}$

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$1 \leq i \leq m$

Linear EMS Problems

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linear function $g(x_{11}, \ldots, x_{ml}, y_1, \ldots, y_t)$

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Linear EMS Problems

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open MSO$_2$ formula $\varphi(X_1, \ldots, X_l)$

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graph G

$f_i^G : V \cup E \to \mathbb{Q}$

$1 \leq i \leq m$

$r_1, \ldots, r_t \in \mathbb{Q}$

$$\max_{\substack{A_1, \ldots, A_l \subseteq V \cup E \\ G \models \varphi[A_1, \ldots, A_l]}} g\Big(\sum_{a \in A_1} f_1^G(a), \ldots, \sum_{a \in A_l} f_m^G(a), r_1, \ldots, r_t\Big)$$

Meta-Theorem

Meta-Theorem

LinEMSO Maximization

Input: A graph $G = (V, E, f_1^G, \ldots, f_m^G, r_1, \ldots, r_t)$, an MSO formula $\varphi(X_1, \ldots, X_l)$, and a linear EMS evaluation term $g(x_{11}, \ldots, x_{ml}, y_1, \ldots, y_t)$.

Output: $\max\limits_{\substack{A_1, \ldots, A_l \subseteq V \cup E \\ G \models \varphi[A_1, \ldots, A_l]}} g(\sum_{a \in A_1} f_1^G(a), \ldots, \sum_{a \in A_l} f_m^G(a), r_1, \ldots, r_t)$

Meta-Theorem

LinEMSO Maximization

Input: A graph $G = (V, E, f_1^G, \ldots, f_m^G, r_1, \ldots, r_t)$, an MSO formula $\varphi(X_1, \ldots, X_l)$, and a linear EMS evaluation term $g(x_{11}, \ldots, x_{ml}, y_1, \ldots, y_t)$.

Output:

$$\max_{\substack{A_1, \ldots, A_l \subseteq V \cup E \\ G \models \varphi[A_1, \ldots, A_l]}} g\Big(\sum_{a \in A_1} f_1^G(a), \ldots, \sum_{a \in A_l} f_m^G(a), r_1, \ldots, r_t\Big)$$

Theorem (Arnborg, Lagergren, Seese)

There is a computable function $f(\cdot, \cdot)$ such that **LinEMSO Maximization** can be solved in time $f(|\varphi| + |g|, k) \cdot ||G||$ given a width k tree decomposition of G.

Applications

Applications

INDEPENDENT SET

Applications

INDEPENDENT SET

$$\varphi(X) = \forall_{v \in V} \forall_{w \in V} (Xv \land Xw \land v \neq w) \rightarrow \neg Evw$$

Applications

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$$\varphi(X) = \forall_{v \in V} \forall_{w \in V} (Xv \wedge Xw \wedge v \neq w) \rightarrow \neg Evw$$

$$g = x_{11}$$

Applications

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$$\varphi(X) = \forall_{v \in V} \forall_{w \in V} (Xv \land Xw \land v \neq w) \to \neg Evw$$

$$g = x_{11}$$

$$f_1^G : v \mapsto 1$$

Applications

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$$\varphi(X) = \forall_{v \in V} \forall_{w \in V} (Xv \land Xw \land v \neq w) \rightarrow \neg Evw$$

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VERTEX COVER

Applications

INDEPENDENT SET

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$$f_1^G : v \mapsto 1$$

VERTEX COVER

$$\varphi(X) = \forall_{e \in E} \exists_{v \in V} (Xv \wedge Ive)$$

Applications

INDEPENDENT SET

$$\varphi(X) = \forall_{v\in V}\forall_{w\in V}(Xv \wedge Xw \wedge v \neq w) \rightarrow \neg Evw$$

$g = x_{11}$

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VERTEX COVER

$$\varphi(X) = \forall_{e\in E}\exists_{v\in V}(Xv \wedge Ive)$$

$g = 0 - x_{11}$

Applications

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WEIGHTED DOMINATING SET

Applications

INDEPENDENT SET

$$\varphi(X) = \forall_{v \in V} \forall_{w \in V} (Xv \wedge Xw \wedge v \neq w) \rightarrow \neg Evw$$

$$g = x_{11}$$

$$f_1^G : v \mapsto 1$$

VERTEX COVER

$$\varphi(X) = \forall_{e \in E} \exists_{v \in V} (Xv \wedge Ive)$$

$$g = 0 - x_{11}$$

$$f_1^G : v \mapsto 1$$

WEIGHTED DOMINATING SET

$$\varphi(X) = \forall_{v \in V} (\neg Xv \rightarrow (\exists_{w \in V} Xw \wedge Evw))$$

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