Contents

Chapter1 Basic Concepts on Matrices and Vectors

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1 Matrix

1.1 Definitions

A **matrix** is a rectangular array of scalars.
If the matrix has m rows and n columns, we say the **size** of matrix is **m by n**, written $m \times n$.
The matrix is called square if $m = n$.
The scalar in ith row and jth column is called (i, j)**-entry** of the matrix.

1.2 Notation

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \in M_{m\times n} \tag{1}$$

$M_{m\times n}$ denotes the set that contains all matrices whose size is $m \times n$.

1.3 Equality of matrices

1.3.1 Definitions

We say two matrices A and B are equal if

1. they have the same size.
2. they have equal corresponding entries.

Let $A, B \in M_{m\times n}$
Then $A = B \iff a_{ij} = b_{ij}, \forall i = 1, \ldots, m, j = 1, \ldots, n$

1.4 Submatrices

1.4.1 Definitions

A submatrix is obtained by deleting from a matrix entire rows and/or columns.

1.4.2 Sample

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} is\ a\ submatrix\ of \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix} \tag{2}$$

1.5 Matrix addition

1.5.1 Definitions

Let A and B be in matrix $m \times n$. We define the sum of A and B, denoted $A + B$, to be a $m \times n$ matrix obtained by adding the corressponding entries of A and B; that is the $m \times n$ matrix whose $(i, j) - entry$ is $a_{ij} + bij$.

1.5.2 Notation

Let $A, B \in M_{m\times n}$
Then $A + B = [a_{ij} + b_{ij}], \forall i = 1, \dots, m, j = 1, \dots, n$

1.5.3 Theorem

Commutative $A + B = B + A$

Associative $(A + B) + C = A + (B + C)$

1.6 Scalar Multiplication

1.6.1 Definitions

Let A be an $m \times n$ matrix and c be a scalar. The scalar multiple cA of matrix A is defined to be the $m \times n$ matrix whose $(i, j) - entries$ is $c \times a_{ij}$.

1.6.2 Notation

Let $A \in M_{m\times n}$ and $c \in R$
Then $cA = [c \times a_{ij}], \forall i = 1, \dots, m, j = 1, \dots, n$

1.6.3 Theorem

Associative $(st)A = s(tA)$, $(s, t \in R)$

Distributive $s(A + B) = sA + sB$ or $(s + t)A = sA + tA$,$(s \in R)$

1.7 Zero Matrices

1.7.1 Definitions

A **zero matrix** with all zero entries, denoted by O(any size) or $O_{m\times n}$.

1.7.2 Properties

1. $A = O + A, \forall A \in M_{m\times n}$
2. $0 \cdot A = O, \forall A \in M_{m\times n}$

1.8 Matrix Substraction

1.8.1 Definitions

We define the matrix $-A$ to be $-1(A)$. The matrix substraction of the two matrix A and B is define to be as

$$A - B = A + (-B)$$

1.8.2 Theorem

1. $A - A = O, \forall A \in M_{m\times n}$

1.9 Matrix Transpose

1.9.1 Definitions

The transpose of a $m \times n$ matrix A is the $n \times m$ matrix A^T whose $(i, j) - enrty$ is the $(j, i) - enrty$ of A.

1.9.2 Properties

$$A \in M_{m\times n} \Rightarrow A^T \in M_{n\times m}$$

1.9.3 Theorem

associative $(sA)^T = s(A^T)$, $\forall s \in R$

Distributive $(A + B)^T = A^T + B^T$

$(A^T)^T = A$

2 Vector

2.1 Definitions

Vector can refer to either a **row vector** or a **column vector**.
Row vector is a matrix with **one** row.

$$\underline{v} = [a_1, \cdots, a_n]$$

Column vector is a matrix with **one** column.

$$\underline{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \; or \; \underline{v}^T = [a_1, \cdots, a_m] \tag{3}$$

So row vector can transpose to column vector, vice versa.

2.2 Notation

$$\underline{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [a_i] \in R^n, \forall i = 1, \cdots, n \tag{4}$$

We denote the set of all column vectors with n components by R^n. That is $R^n = M_{n\times 1}$.

2.3 Vector Addition/Substraction, Scalar Multiplication and Zero Vector

Follow those for matrices.

Zero Vector is denoted $\mathbf{0}$

A matrix is often regarded as a stack of row vectors or a cross list of column vectors.
Let $C \in M_{m\times n}$

$$C = [\underline{c_1}, \; \cdots \; , \underline{c_j}, \; \cdots \; , \underline{c_n}]$$

$$\underline{c_j} = \begin{bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{bmatrix} = [c_i] \in R^m, \forall i = 1, \cdots, m \tag{5}$$

3 Linear Combinations

3.1 Definitions

A linear combination of vectors $\underline{u_1}, \cdots, \underline{u_n}$ is a vector of the form

$$\underline{u} = c_1\underline{u_1} + \cdots + c_n\underline{u_n}$$

Where $c_1, \cdots, c_n$ are scalars which called **coefficients** of the linear combination.

3.2 Samples

3.2.1 Given coefficients, computes linear combination

Let $\underline{u_1}^T = [-1, -3, 4]$, $\underline{u_2}^T = [-4, 1, 2]$, $c_1 = 3$ and $c_2 = 2$

$$Then\ c_1\underline{u_1}^T + c_2\underline{u_2}^T = \begin{bmatrix} -3 & -9 & 12 \end{bmatrix} + \begin{bmatrix} -12 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -15 & -6 & 18 \end{bmatrix} \tag{6}$$

$$\because\ c_1\underline{u_1}^T + c_2\underline{u_2}^T = (c_1\underline{u_1} + c_2\underline{u_2})^T \tag{7}$$

$$\therefore\ c_1\underline{u_1} + c_2\underline{u_2} = \begin{bmatrix} -15 \\ -6 \\ 18 \end{bmatrix} \tag{8}$$

3.2.2 Given linear combination, computes coefficients

Which could be transform to solve a system of linear equations. But there are three solution.

Unique solution when $s\underline{c_1}$ and $\underline{c_2}$ are not **collinear vectors**.

Infinitely many solutions when $s\underline{c_1}$, $\underline{c_2}$ are **collinear vectors** and $\underline{c} = s\underline{c_1}$, $\underline{c}$ is linear combination

No solutions when $s\underline{c_1}$, $\underline{c_2}$ are **collinear vectors** and $\underline{c} \neq s\underline{c_1}$, $\underline{c}$ is linear combination

3.3 Parallel/Collinear Vectors

Let $\underline{a}$ and $\underline{b}$ not be in zero vector. We define $\underline{a}$ is parallel with $\underline{b}$ when $s\underline{a} = \underline{b}$, $s \in R$.

3.3.1 Notations

Let $\forall \underline{a}, \underline{b} \in R^n$

Then $\underline{a} \parallel \underline{b} \Rightarrow s\underline{a} = \underline{b}\ , s \in R$

$\underline{a} \parallel \underline{b}$ denotes the vectors $\underline{a}$, $\underline{b}$ are the parallel or collinear vector

4 Standard Vectors

4.1 Definitions

The standard vectors of R^n are defined as

$$\underline{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ \underline{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \ \cdots \ \underline{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{9}$$

4.2 Properties

Every vector in R^n may be uniquely linearly combined by standard vectors.

5 Matrix-Vector Product

5.1 Definitions

Let A be an $m \times n$ matrix and $\underline{v}$ be an $n \times 1$ vector. We define the **matrix-vector product** of A and $\underline{v}$, denoted by $A\underline{v}$, to be the linear combination of the columns of A whose coefficients are the corresponding components of $\underline{v}$. That is

$$A\underline{v} = v_1\underline{a_1} + v_2\underline{a_2} + \cdots + v_n\underline{a_n}$$

5.2 Cautions

1. the size of $\underline{v}$ must be equal to the count of columns of A.

2. the solution of matrix-vector product is a new vector whose size is same with the count of rows of A.

5.3 Theorems

Distributive $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v}$ and $(A+B)\underline{u} = A\underline{u} + B\underline{u}$

Associative $A(c\underline{u}) = c(A\underline{u}) = (cA)\underline{u},\ \forall\ c \in R$

$A\underline{0} = \underline{0}\ and\ O\underline{v} = \underline{0}\ , \forall A\ \in M_{m\times n}\ and\ \underline{v} \in R^n$

$A\underline{e_j} = \underline{a_j}\ ,\ A = [\underline{a_1}\ \cdots\ \underline{a_m}]$

$B\underline{w} = A\underline{w} \Rightarrow A = B\ ,\ \forall\ B, A \in M_{m\times n}\ and\ \forall\ \underline{w} \in R^n.$

6 Identity Matrix

6.1 Definitions

For each positive integer n, the $n\times n$ **identity matrix** I_n is the $n\times n$ matrix whose respective columns are the standard vectors $\underline{e_1},\ \underline{e_2},\ \cdots\ ,\underline{e_n}$ in R^n
Sometime I_n is simply written as I.

6.2 Sample

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

6.3 Properties

$$I_n\underline{v} = \underline{v}\ , \forall\ \underline{v} \in R^n$$

7 Stochastic Matrix

7.1 Definitions

An matrix $A \in M_{m\times n}$ is called a stochastic matrix if all entries of A is nonnegative and the sum of all entries in each columns is unity.