The Language of Set Theory

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1 Basic

1.1 Empty Set

$S_1 = \{\}$ or $S_1 = \emptyset$

1.2 "is in" and "is not in"

Let $S_1 = \{a, b, c\}$. a is an element of set S_1, which denoted as $a \in S_1$. However d is not in the set S_1 which denoted as $d \notin S_1$.

1.3 Subset

Let $S_1 = \{a, b, c\}$, a, $S_2 = \{b, c\}$ and $S_3 = \{a, b, c\}$. S_2 is the subset of S_1, which denoted as $S_2 \subset S_1$, and S_1 is the subset of S_3 which denoted as $S_3 \subseteq S_1$.

1.4 Equal Sets

Let $S_3 = \{a, b, c\}$,
Then $S_1 = S_3 \Leftrightarrow S_1 \subset S_3$ and $S_1 \supset S_3$.

1.5 Union Set

$S_1 \cup S_2$

1.6 Intersection Set

$S_1 \cap S_2$

1.7 Difference Set

$S_1 \setminus S_2$

2 Span of a Set of Vectors

Definition For a nonempty set $S = \{\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}\}$ of vectors in R^n, we define the span of S to be the set of all linear combinations of $\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}$ in R^n. This set is denoted by $Span\ S$ or $Span\ \{\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}\}$.

$$Span\ S = \{c_1\underline{u_1} + c_2\underline{u_2} + \cdots + c_k\underline{u_k} | \forall c_1, c_2, \cdots, c_k \in R\}$$

or

$$Span\ S = \{A\underline{v} | v \in R^k, A = [\underline{u_1}\ \cdots\ \underline{u_k}]\}$$

When we want to say vector $\underline{v}$ is the linear combination of vectors $\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}$, we could denote it by

$$\underline{v} \in Span\ S$$

2.1 Properties

- $Span\ \{0\} = \{0\}$
- $Span\ \{\underline{u}\}$ is the set of all scalar multiples of vector $\underline{u}$.
- If S contains a nonzero vector, then $Span\ S$ has infinitely many vectors.

2.2 $\underline{v} \in Span\ S$ or not?

Let $\underline{v} = [a_1\ \cdots\ a_k]^T$ and $S = [\underline{u_1}\ \cdots\ \underline{u_k}]$
then $\underline{v} \in Span\ S \Leftrightarrow$ the solution set of $[\underline{u_1}\ \cdots\ \underline{u_k}\ \underline{v}]$ is consistent.

2.3 Definition of Generating Set

If $S, V \subset R^n$ and $Span\ S = V$, then we say "S is a generating set for V" or "S generates V".

If we don't know the actual vector $\underline{v}$, how to determine?
The answer is resolve it by $rank\ A$.
For any $\underline{v} \in R^n$, let $[R\ \underline{c}]$ whose $R \in M_{m\times n}$ and $\underline{c} \in R^n$ be the reduced row echelon form of $[A\ \underline{v}], A \in M_{m\times n}, \underline{v} \in R^n$.
If the $rank\ R = n$, then A is the generating set of R^n(i.e. $Span\ \{a_1, \cdots, a_n\} = R^n | [a_1\ \cdots\ a_n] = A$).

Theorem1 The following statements about an $m \times n$ matrix A are equivalent.

- The span of the columns of A is R^m.
- The equation $A\underline{x} = \underline{b}$ has at least one solution.(i.e. $A\underline{x} = \underline{b}$ is consistent, for each $\underline{b} \in R^m$)

- The rank of A is m, the number of rows of A.
- The reduced row echelon form of A has no zero rows.
- There is a pivot position in each row of A.

Theorem2 Let $S = \{\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}\}$ be a set of vectors from R^n and let $\underline{v}$ be a vector in R^n. Then $Span\ S = Span\ (S \cup \{\underline{v}\})$ if and only if $\underline{v}$ belongs to the span of S. The S is the smallest generating set for $Span\ S$.

Proof Since $Span\ S \subseteq Span\ (S \cup \{\underline{v}\})$, only need to show

$$Span\ (S \cup \{\underline{v}\}) \subseteq Span\ S \Leftrightarrow \underline{v} \in Span\ S$$

$$\begin{aligned}
Let\ \underline{v} \in Span\ S \Rightarrow \underline{v} &= c_1\underline{u_1} + \cdots + c_n\underline{u_n} \mid c_1, \cdots, c_n \in R\ and\ \underline{u_1}, \cdots, \underline{u_n} \in S \\
Then\ \forall x \in Span\ (S \cup \underline{v}) \Rightarrow & d_1\underline{u_1} + \cdots + d_n\underline{u_n} + d_k\underline{v} \mid d_1, \cdots, d_k \in R\ and\ \underline{u_1}, \cdots, \underline{u_n} \in S \\
&= (d_1 + c_1 d_k)\underline{u_1} + \cdots + (d_n + c_n d_k)\underline{u_n} \\
&= k_1\underline{u_1} + \cdots + k_n\underline{u_n} \mid k_1, \cdots, k_n \in R \\
&= Span\ S
\end{aligned}$$

3 Linear Dependence and Linear Independence