Matrix and Linear Transformation

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1 Matrix Multiplication

Proposition Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then there is a **unique** $m \times p$ matrix C such that $C\underline{v} = A(B\underline{v})$ for every $p \times 1$ vector v. Furthermore,

$$C = \begin{bmatrix} A\underline{b_1} & \cdots & A\underline{b_n} \end{bmatrix}$$

Definition Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define a **(matrix) product** AB to be the $m \times p$ matrix whose jth column is $A\underline{b_j}$. That is

$$AB = \begin{bmatrix} A\underline{b_1} & \cdots & A\underline{b_n} \end{bmatrix}$$

1.1 Theorem

Let A and B be $k \times m$ matrices, C be an $m \times n$ matrix, and P and Q be $n \times p$ matrices. Then the following statements are true:

- $s(AC) = sAC = A(sC), \forall s \in R$.

Associative $ACP = A(CP)$

Not Commutative $AC \neq CA$

Right Distributive $(A + B)C = AC + BC$

Left Distributive $C(P + Q) = CP + CQ$

- $I_k A = A = AI_m$, I_k and I_m are indentity matrices.
- $A0 = 0A = 0$
- $(AC)^T = C^T A^T$.
- Let $k = m$, then $A^0 = I_m$ and $A^1 = A$

1.2 Augmented Matrix

If A and B are matrices with the same number of rows, then the augmented matrix $[A\ B]$ is the matrix whose columns are the columns of A followed by the columns of B.

Properties For any $P \in R^{m\times n}$ and matrices A and B with n rows

$$P[A\ B] = [PA\ PB]$$

1.3 Diagonal Matrix

$A = [a_{ij}] \in M_{n\times n}$ and $a_{ij} = 0$ for $i \neq j$, denoted by $A = diag[a_{11}\ \cdots\ a_{nn}]$
Sample:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{1}$$

Properties

$$A = diag[a_{11}\ \cdots\ a_{nn}], B = diag[b_{11}\ \cdots\ b_{nn}] \Leftrightarrow AB = diag[a_{11}b_{11}\ \cdots\ a_{nn}b_{nn}]$$

1.4 Symmetric Matrix

$A = [a_{ij}] \in M_{n\times n}$ and $a_{ij} = a_{ji}$ or $A = A^T$.
Sample:

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & 10 \end{bmatrix} \tag{2}$$

Properties For any $A \in M_{n\times n}$, AA^T and A^TA are square and symmetric:

$$(AA^T)^T = A^{TT}A^T = AA^T \text{ and } (A^TA)^T = A^TA^{TT} = A^TA$$