The Language of Set Theory

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1 Basic

1.1 Empty Set

$S_1 = \{\}$ or $S_1 = \emptyset$

1.2 "is in" and "is not in"

Let $S_1 = \{a, b, c\}$. a is an element of set S_1, which denoted as $a \in S_1$. However d is not in the set S_1 which denoted as $d \notin S_1$.

1.3 Subset

Let $S_1 = \{a, b, c\}$, a, $S_2 = \{b, c\}$ and $S_3 = \{a, b, c\}$. S_2 is the subset of S_1, which denoted as $S_2 \subset S_1$, and S_1 is the subset of S_3 which denoted as $S_3 \subseteq S_1$.

1.4 Equal Sets

Let $S_3 = \{a, b, c\}$,
Then $S_1 = S_3 \Leftrightarrow S_1 \subset S_3$ and $S_1 \supset S_3$.

1.5 Union Set

$S_1 \cup S_2$

1.6 Intersection Set

$S_1 \cap S_2$

1.7 Difference Set

$S_1 \setminus S_2$

2 Span of a Set of Vectors

Definition For a nonempty set $S = \{\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}\}$ of vectors in R^n, we define the span of S to be the set of all linear combinations of $\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}$ in R^n. This set is denoted by *Span* S or *Span* $\{\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}\}$.

$$Span\ S = \{c_1\underline{u_1} + c_2\underline{u_2} + \cdots + c_k\underline{u_k} | \forall c_1, c_2, \cdots, c_k \in R\}$$

or

$$Span\ S = \{A\underline{v} | v \in R^k, A = [\underline{u_1}\ \cdots\ \underline{u_k}]\}$$

When we want to say vector $\underline{v}$ is the linear combination of vectors $\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}$, we could denote it by

$$\underline{v} \in Span\ S$$

2.1 Properties

- $Span\ \{0\} = \{0\}$
- $Span\ \{\underline{u}\}$ is the set of all scalar multiples of vector $\underline{u}$.
- If S contains a nonzero vector, then *Span* S has infinitely many vectors.

2.2 $\underline{v} \in Span\ S$ or not?

Let $\underline{v} = [a_1\ \cdots\ a_k]^T$ and $S = [\underline{u_1}\ \cdots\ \underline{u_k}]$
then $\underline{v} \in Span\ S \Leftrightarrow$ the solution set of $[\underline{u_1}\ \cdots\ \underline{u_k}\ \underline{v}]$ is consistent.

2.3 Definition of Generating Set

If $S, V \subset R^n$ and $Span\ S = V$, then we say "S is a generating set for V" or "S generates V".

If we don't know the actual vector $\underline{v}$, how to determine?
The answer is resolve it by $rank\ A$.
For any $\underline{v} \in R^n$, let $[R\ \underline{c}]$ whose $R \in M_{m\times n}$ and $\underline{c} \in R^n$ be the reduced row echelon form of $[A\ \underline{v}], A \in M_{m\times n}, \underline{v} \in R^n$.
If the $rank\ R = n$ where n is the number of columns of R, then A is the generating set of R^n(i.e. $Span\ \{a_1, \cdots, a_n\} = R^n | [a_1\ \cdots\ a_n] = A$).

Theorem1 The following statements about an $m \times n$ matrix A are equivalent.

- The span of the columns of A is R^m.
- The equation $A\underline{x} = \underline{b}$ has at least one solution.(i.e. $A\underline{x} = \underline{b}$ is consistent, for each $\underline{b} \in R^m$)

- The rank of A is m, the number of rows of A.
- The reduced row echelon form of A has no zero rows.
- There is a pivot position in each row of A.

Theorem2 Let $S = \{\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}\}$ be a set of vectors from R^n and let $\underline{v}$ be a vector in R^n. Then $Span\ S = Span\ (S \cup \{\underline{v}\})$ if and only if $\underline{v}$ belongs to the span of S. The S is the smallest generating set for $Span\ S$.

Proof Since $Span\ S \subseteq Span\ (S \cup \{\underline{v}\})$, only need to show

$$Span\ (S \cup \{\underline{v}\}) \subseteq Span\ S \Leftrightarrow \underline{v} \in Span\ S$$

$$\begin{aligned} Let\ \underline{v} \in Span\ S \Rightarrow \underline{v} = c_1\underline{u_1} + \cdots + c_n\underline{u_n} \mid c_1, \cdots, c_n \in R\ and\ \underline{u_1}, \cdots, \underline{u_n} \in S \\ Then\ \forall x \in Span\ (S \cup \underline{v}) \Rightarrow d_1\underline{u_1} + \cdots + d_n\underline{u_n} + d_k\underline{v} \mid d_1, \cdots, d_k \in R\ and\ \underline{u_1}, \cdots, \underline{u_n} \in S \\ = (d_1 + c_1d_k)\underline{u_1} + \cdots + (d_n + c_nd_k)\underline{u_n} \\ = k_1\underline{u_1} + \cdots + k_n\underline{u_n} \mid k_1, \cdots, k_n \in R \\ = Span\ S \end{aligned}$$

3 Linear Dependence and Linear Independence

3.1 Linear Dependence(aka. L.D.)

A set of k vectors $\{\underline{u_1}, \cdots, \underline{u_k}\}$ in R^n is called **linearly dependent** if there exist scalars $c_1, \cdots, c_k$, not all zero, such that

$$c_1\underline{u_1} + \cdots + c_k\underline{u_k} = 0$$

In this case, we also say that the vectors $\underline{u_1}, \cdots, \underline{u_k}$ are linearly dependent.

Theorem Vector $\underline{u_1}, \cdots, \underline{u_k}$ in R^n are linearly dependent if and only if $\underline{u_1} = \underline{0}$ or there exists an $i \geq 2$ such that $\underline{u_i}$ is a linear combination of the preceding vectors $\underline{u_1}, \cdots, \underline{u_k}$.

Proof for "only if($\Rightarrow$)":
Since $\{\underline{u_1}, \cdots, \underline{u_k}\}$ is L.D., $\exists\ c_1, \cdots, c_k$ not all zero, such that $c_1\underline{u_1} = \underline{0}$
Let $i = max\{j : c_j \neq 0\}$(i.e. $c_{i+1} = c_{i+2} = \cdots = 0$)
Case(1): $i = 1 \Rightarrow c_1\underline{u_1} = \underline{0} \Rightarrow \underline{u_1} = \underline{0}$
Case(2): $i > 1 \Rightarrow c_1\underline{u_1} + \cdots + c_i\underline{u_i} = \underline{0}$
$\Rightarrow \underline{u_i} = \dfrac{-c_1}{c_i}\underline{u_1} + \cdots + \dfrac{-c_{i-1}}{c_i}\underline{u_{i-1}}$

3.2 Linear Independence(aka. L.I.)

A set of k vectors $\{\underline{u_1}, \cdots, \underline{u_k}\}$ in R^n is called **linearly independent** if the only scalars $c_1, \cdots, c_k$ such that

$$c_1\underline{u_1} + \cdots + c_k\underline{u_k} = 0$$

are $c_1 = \cdots = c_k = 0$. In this case, we also say that the vectors $\underline{u_1}, \cdots, \underline{u_k}$ are linearly independent.

3.3 Properties

- Any finite set $S = \{\underline{0}, \underline{u_1}, \cdots, \underline{u_k}\}$ that contains the zero vector is L.D. since $1\underline{0} + 0\underline{u_1} + \cdots + 0\underline{u_k} = \underline{0}$
- If the number of elements of a finite set S is greater than the number of entry of the vector element, then the set is L.D. .
- If finite set S is L.I. such that $c_1\underline{u_1} + \cdots + c_k\underline{u_k} = 0$, then there exists one and only one solution. And $c_1\underline{u_1} + \cdots + c_k\underline{u_k} = \underline{b}$ where $\underline{b}$ is nonzero vector has no more than one solution.
- If $rank\ S = n$ where n is the number of columns of S or $nullity\ S = 0$, then S is L.I. .
- For a 1-vector set, $\{\underline{u}\}$ is L.I. as long as $\underline{u} \neq \underline{0}$. The set $\underline{0}$ is L.D.
- For a 2-vector set $\{\underline{u_1}, \underline{u_2}\}$ is L.D.

  $\Leftrightarrow$ $\underline{u_1} = \underline{0}$, or $\underline{u_2}$ is a multiple of $\underline{u_1}$.

  $\Leftrightarrow$ one vector is a multiple of the other.
- Let $S = \{\underline{u_1}, \cdots, \underline{u_k}\}$ be a linear independent subset of R^n, and $\underline{v}$ be in R^n. Then

$$\underline{v} \notin Span\ S \Leftrightarrow S \cup \{\underline{v}\} \text{is L.I.}$$

- No vector can be removed from a set $S \subset R^n$ without changing its span $\Rightarrow$ S is L.I.

Theorem The following statements about an $m \times n$ matrix A are equivalent:

- The columns of A are linearly independent.
- The equation $A\underline{x} = \underline{b}$ has at most one solution for each $\underline{b}$ in R^n.
- The nullity of A is zero.
- The rank of A is n, the number of columns of A.
- The columns of the reduced row echelon form of A are distinct standard vectors in R^n.
- The only solution of $A\underline{x} = \underline{0}$ is $\underline{0}$.
- There is a pivot position in each columns of A.

3.4 Homogeneous System of Linear Equations

Definition A system of linear equations $A\underline{x} = \underline{b}$ is called homogeneous if $\underline{b} = \underline{0}$.

Properties

- always consistent, since $\underline{x} = \underline{0}$ is a solution;
- if it has nonzero solution, then columns of A is L.D.;
- if the number of variables is greater than the number of equations, then it has nonzero solutions, since free variables exist.(the number of variables is the same as the number of columns of matrix, the number of equations is the same as the number of rows of matrix)

4 Summary

Let $A_{m\times n}$ and $m \neq n$

$rank\ A = m$ $\Rightarrow$ every row of R which is the reduced row echelon form of A contains a pivot position.
$\Rightarrow$ exists free variables.
$\Rightarrow$ $A\underline{v} = \underline{0}$ has infinitely many solutions.
$\Rightarrow$ $A\underline{v} = \underline{b} \mid \underline{b} \in R^m$ has infinitely many solutions. $\Rightarrow$ the columns of A are the generating set for R^m but not the smallest one.
$\Rightarrow$ the columns of A are L.D. .

$rank\ A = n$ $\Rightarrow$ every column of R which is the reduced row echelon form of A contains a pivot position.
$\Rightarrow$ no variables.
$\Rightarrow$ $A\underline{v} = \underline{0}$ has exactly one solution.
$\Rightarrow$ $A\underline{v} = \underline{b} \mid \underline{b} \in R^m$ has at most one solution.$\Rightarrow$ the columns of A are the smallest generating set for $R'^m \subset R^m$.
$\Rightarrow$ the columns of A are L.I. .