Average-case complexity and statistical inference

Exercise Sheet 2

Please choose some questions below amounting to at least (4) points. Deadline 30th May, email to me `fiona.skerman@math.uu.se` or put a physical copy in my pigeon-hole.

Several questions will relate to the stochastic block model, *stochastic block model* so we define it here.

For Q2 (Definition - Stochastic Block Model - vanilla model.)
Let $SBM(n,p,q)$ be the model constructed as follows. For each vertex $v \in [n]$ independently let $v \in S^*$ with probability 1/2. Let $\sigma_v = 1$ if $v \in S^*$ and $\sigma_v = -1$ if $v \notin S^*$. Construct G by choosing each edge to be present independently with probability

$$\mathbb{P}(uv \in E \mid \sigma_u, \sigma_v) = \begin{cases} p & \text{if } \sigma_u\sigma_v = 1 \\ q & \text{otherwise.} \end{cases}$$

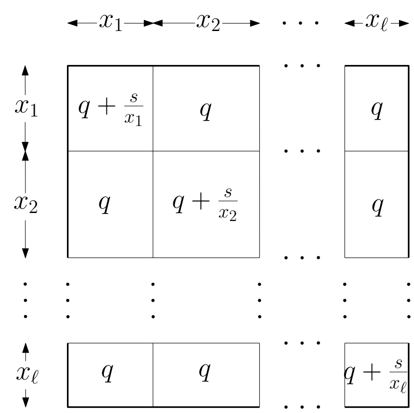


Figure 1: Stochastic Block Model (SBM). General model for many communities of unequal sizes.

We also consider fixed size version $SBM'(n,p,q)$ which is as above except we take $S^* \in \binom{[n]}{n/2}$, i.e. let S^* be a set of $n/2$ vertices chosen uniformly from all sets of that size in $[n]$. For this model we assume n is even.

For Q1 (Definition - Stochastic Block Model many unequal size parts) - see Figure 1.
Let $SBM(n,q,s,(x_1,x_2,\dots,x_\ell))$ be the model constructed as follows. For each vertex $v \in [n]$, $\sigma(v) \in \{1,\dots,k\}$, we independently choose $\sigma(v) = i$ with probability x_i. Construct G by choosing each edge to be present independently with probability

$$\mathbb{P}(uv \in E \mid \sigma_u, \sigma_v) = \begin{cases} q + \frac{s}{x_i} & \text{if } \sigma_u = \sigma_v = i \\ q & \text{otherwise.} \end{cases}$$

1. (1) We want to show that counts of a small subgraph will distinguish the stochastic block model with equal size parts from the stochastic blockmodel with non-equal sized parts.

Let $x \neq 1/2$. Distinguishing $H_1 : SBM(n, p, q, (x, 1-x))$ and $H_0 : SBM(n, p, q, (1/2, 1/2))$, see Figure 2

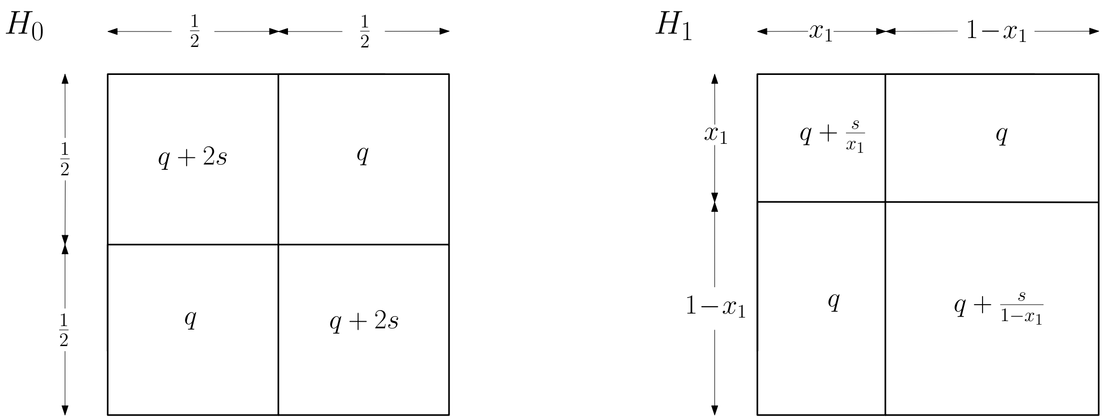
Denote the adjacency matrix of the observed graph by A, it may be easier to count triangles, $\#\triangle = \sum_{i,j,k} A_{ij}A_{ik}A_{jk}$ or signed triangles $\#\triangle_s = \sum_{i,j,k}(A_{ij}-q)(A_{ik}-q)(A_{jk}-q)$.

(a) (1) Show that triangles (or signed triangles) will not work. i.e. show that

$$\mathbb{E}_0[\#\triangle] = \mathbb{E}_1[\#\triangle].$$

(b) (1) Find a small subgraph H (or the signed version) such that $\mathbb{E}_0[\#H] \neq \mathbb{E}_1[\#H]$.

(c) (1) (Bonus) For a subgraph H satisfying (b) characterise which distributions it can not distinguish.



(a) $H_0 = SBM(n, p, s, (\frac{1}{2}, \frac{1}{2}))$ (b) $H_1 = SBM(n, p, s, (x_1, 1-x_1))$

Figure 2: The distinguishing problem in the question 1.

2. (1) **Prove, disprove or salvage if possible.** In the SBM two distinct nodes the probability that they have common neighbours is independent of whether they share an edge or not.

   Feel free to consider either SBM or SBM' and to change the wording slightly, e.g. to consider expected number of common neighbours etc.

3. (1) Prove Lemma 5.3 in the notes, i.e. prove the following.

   If P, P_1 and P_2 be three probability spaces, and $\mathcal{A}_1$ and $\mathcal{A}_2$ algorithms such that

$$P \xrightarrow{\mathcal{A}_1}_{\varepsilon_1} P_1 \quad \text{and} \quad P_1 \xrightarrow{\mathcal{A}_2}_{\varepsilon_2} P_2.$$

   Then

$$P \xrightarrow{\mathcal{A}_2 \circ \mathcal{A}_1}_{\varepsilon_1+\varepsilon_2} P_2.$$

4. (1) Find another example of a worst-case to average-case reduction and write down a clear explanation of the reduction and why it works.

5. (1) Find a gap in your knowledge and write about it. It could be a detail skipped in the lecture or reading a section of the lecture notes of Lugosi or Wu and Xu and explaining it in your own words (and equations), or something else.