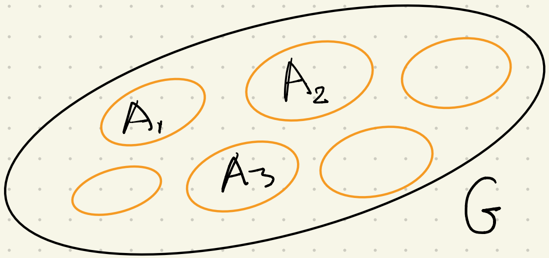
Modularity + Sampling

with Colin McDiarmid

Modularity 'meas. of how well a graph can be clustered'



graph G, m edges. $\mathcal{A} = \{A_1, \dots, A_k\}$ vertex partition

score of partition $\mathcal{A}$, $q_{\mathcal{A}}(G) =$

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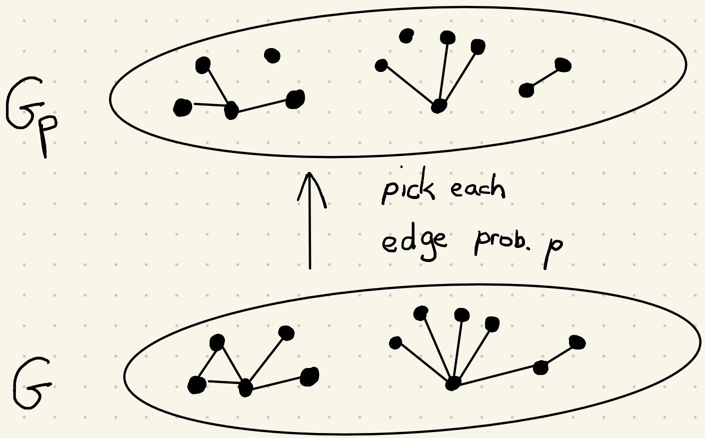
"high vals taken to indicate more community structure"

Sampling



$G_p = (V, E_p)$ $\quad E_p$ each edge kept indep. prob. p

$G = (V, E)$ fixed graph

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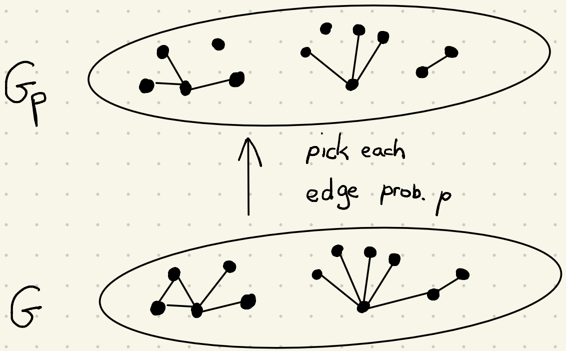

$G_p = (V, E_p)$ E_p each edge kept indep. prob. p

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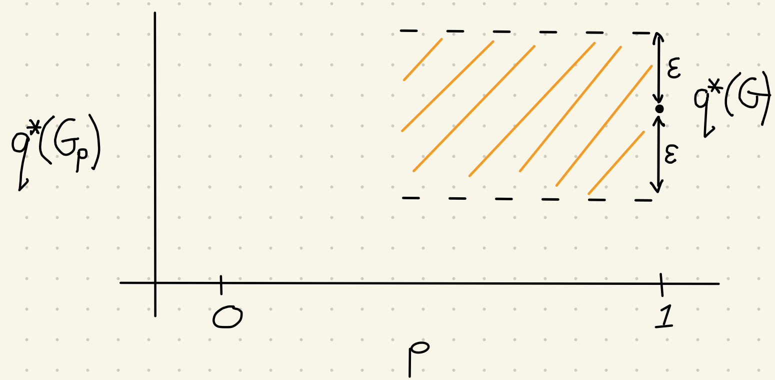
Q Given G, $\varepsilon > 0$ for which p is it true with prob $> 1 - \varepsilon$

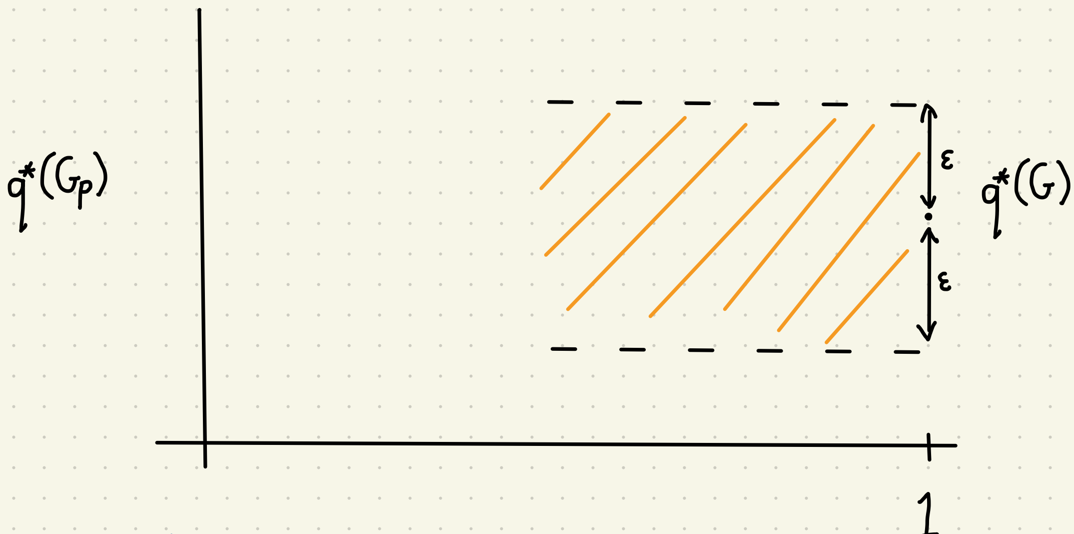
$$|q^*(G_p) - q^*(G)| < \varepsilon$$



Thm [McD+S] $\forall \varepsilon > 0 \;\; \exists c = c(\varepsilon)$

for graph G
constant p

- if then w. prob $> 1-\varepsilon$ $\quad q^*(G_p) > q^*(G) - \varepsilon$
- if " " $\quad q^*(G_p) < q^*(G) + \varepsilon$



$$q_{\mathcal{A}}(G) = \sum_{A \in \mathcal{A}} \frac{e(A)}{m} - \frac{\mathrm{vol}(A)^2}{(2m)^2}$$

"edge contrib." "degree tax"

Thm [McD+S] $\forall \varepsilon > 0 \ \exists c = c(\varepsilon)$

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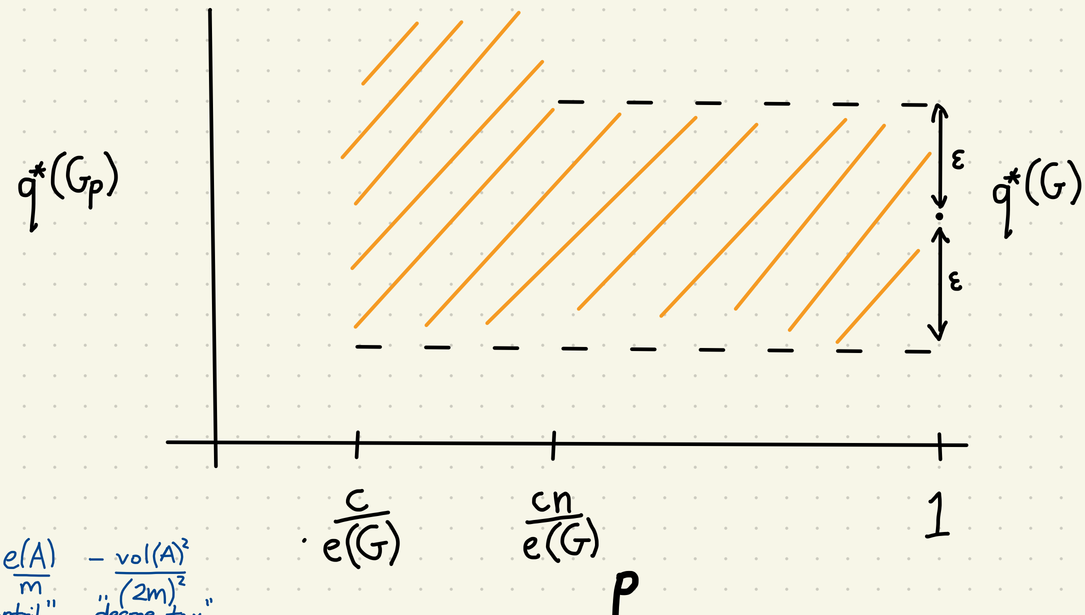
- if $e(G)p \geqslant c$ then w. prob $> 1-\varepsilon$ $q^*(G_p) > q^*(G) - \varepsilon$
- if $e(G)p \geqslant cn$ " " $q^*(G_p) < q^*(G) + \varepsilon$



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- if $e(G)p \geq cn$ given $\mathcal{A}$, can construct $\mathcal{A}'$
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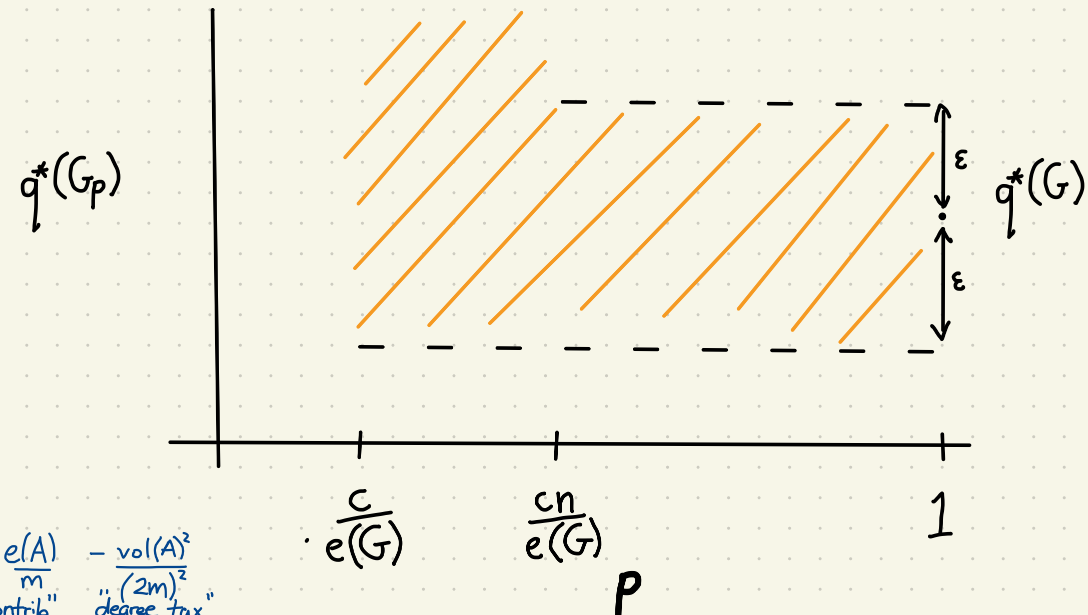

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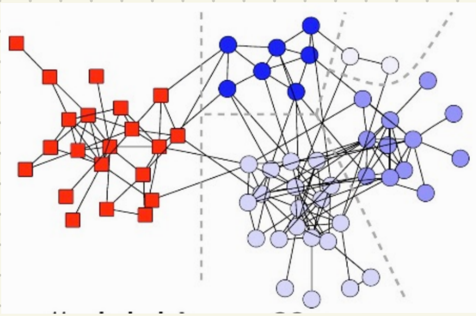
Simulations

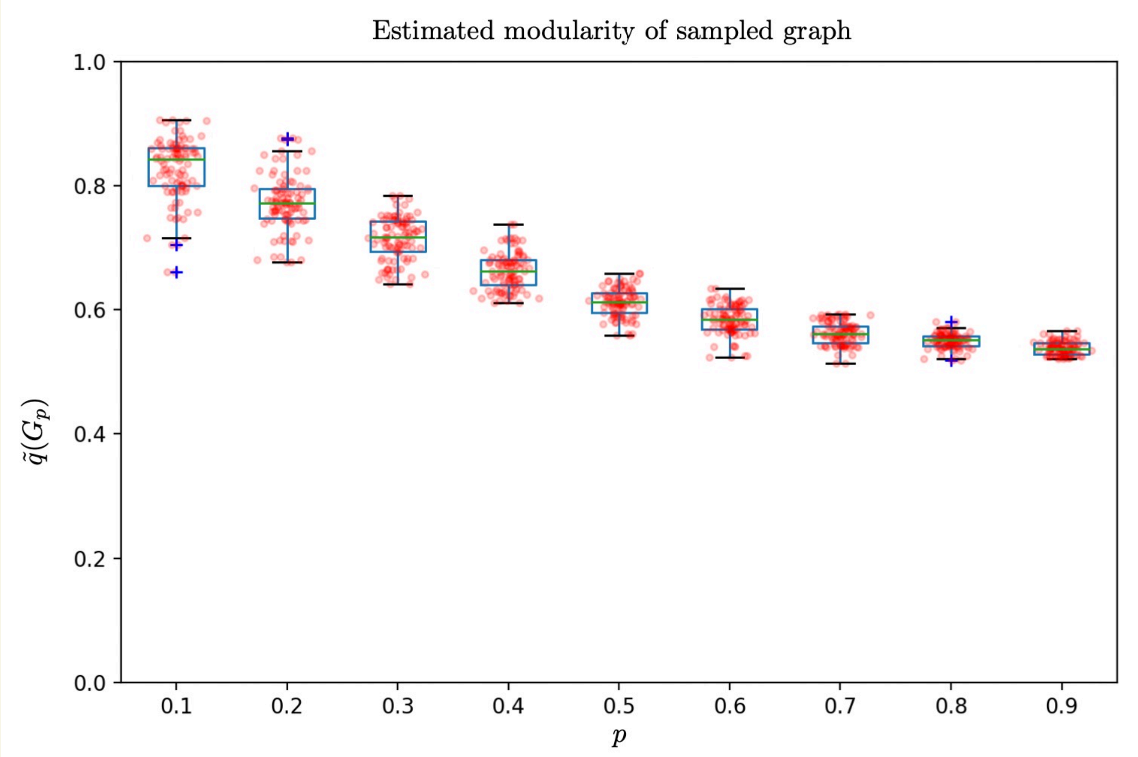
Dolphin Network [LUSSEAU]



$|V| = 62 \qquad |E| = 152$

$q^*(G) = 0.\underline{529}\ldots$ (3 dec places)

[BRANDE + '08]



Modularity

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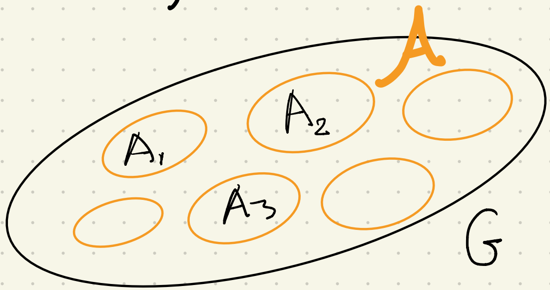

graph G, m edges. $\mathcal{A} = \{A_1, \ldots, A_k\}$ vertex partition

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d_u = # edges incident to u.

$e(A)$ = # edges in set A

$\mathrm{vol}(A) = \sum_{u \in A} d_u$

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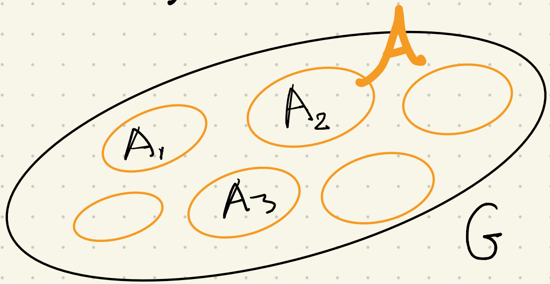

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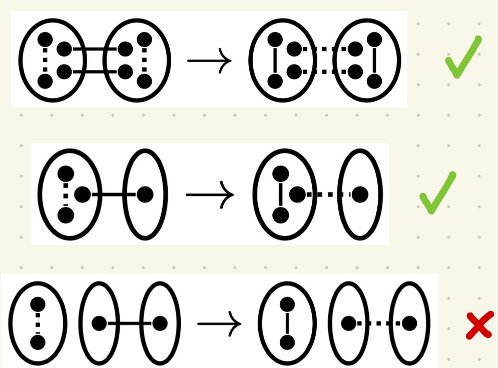

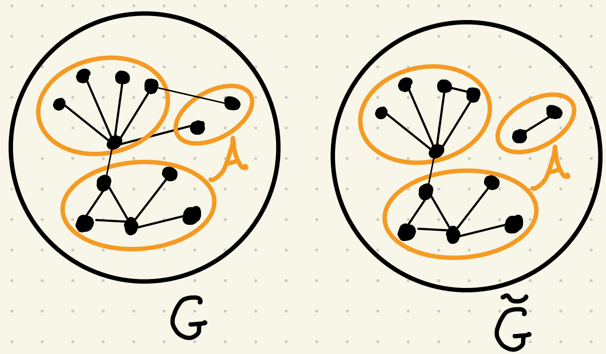

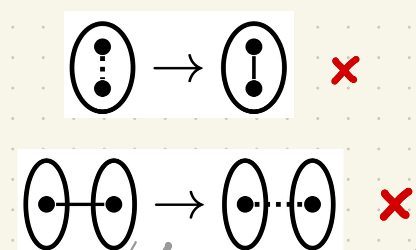

(✓ if $|A| = 2$, $q_A(G) \geq 0$)

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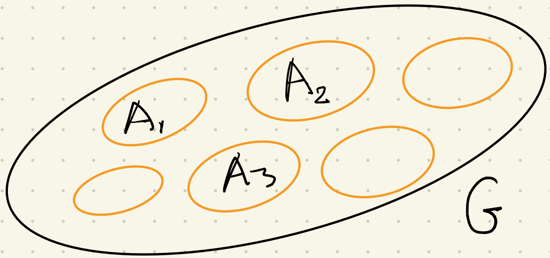

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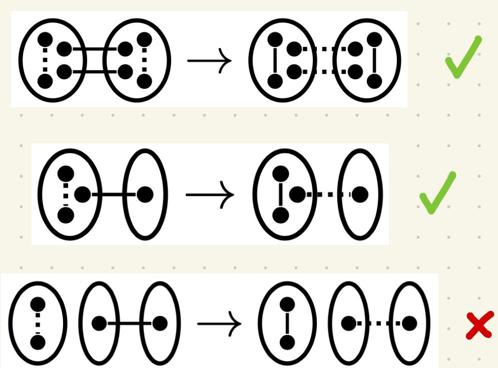
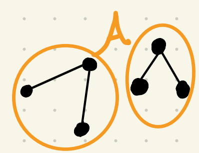
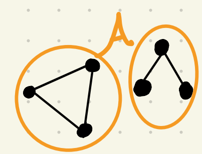
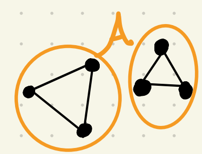




$q_A = \frac{1}{2}$



$q_A = \frac{12}{25}$



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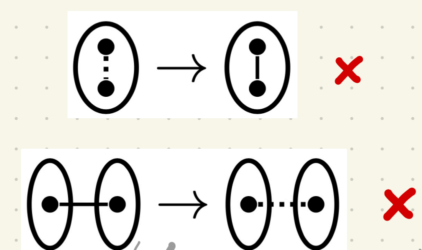

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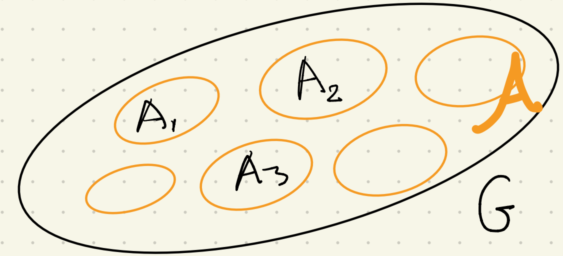
Modularity Properties



- Robust to small perturbations in edge set

$$|q^*(G) - q^*(G \setminus E')| < \frac{2|E'|}{e(G)}$$

$q^*(G) > 1 - \varepsilon$ if any of following hold

- Connected components in G all $< \varepsilon e(G)$ edges
- $\exists \mathcal{A}$ with #edges between parts $< \frac{\varepsilon}{2} e(G)$

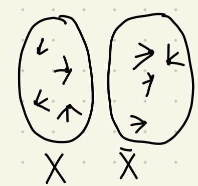
  and $\forall A \in \mathcal{A}$ $\mathrm{vol}(A) < \frac{\varepsilon}{2} \mathrm{vol}(G)$
- no subgraph $H \subseteq G$, $\bar{H}$ δ-expander, $e(H) > \delta e(G)$

(+ LOUF)

$q^*(G) < \varepsilon$ if any of following hold

- $\bar{\lambda}(G) < \varepsilon$ where $\bar{\lambda}(G)$ is spectral gap of Laplacian of G

  NB: r-regular G, $\bar{\lambda}(G) = \frac{1}{r} \max_{i \neq 0} |\lambda_i(A_G)|$
- $\forall X \subseteq V(G)$, $\frac{e(X, \bar{X})}{e(G)} > 2(1-\varepsilon) \frac{\mathrm{vol}(X)}{\mathrm{vol}(G)} \frac{\mathrm{vol}(\bar{X})}{\mathrm{vol}(G)}$



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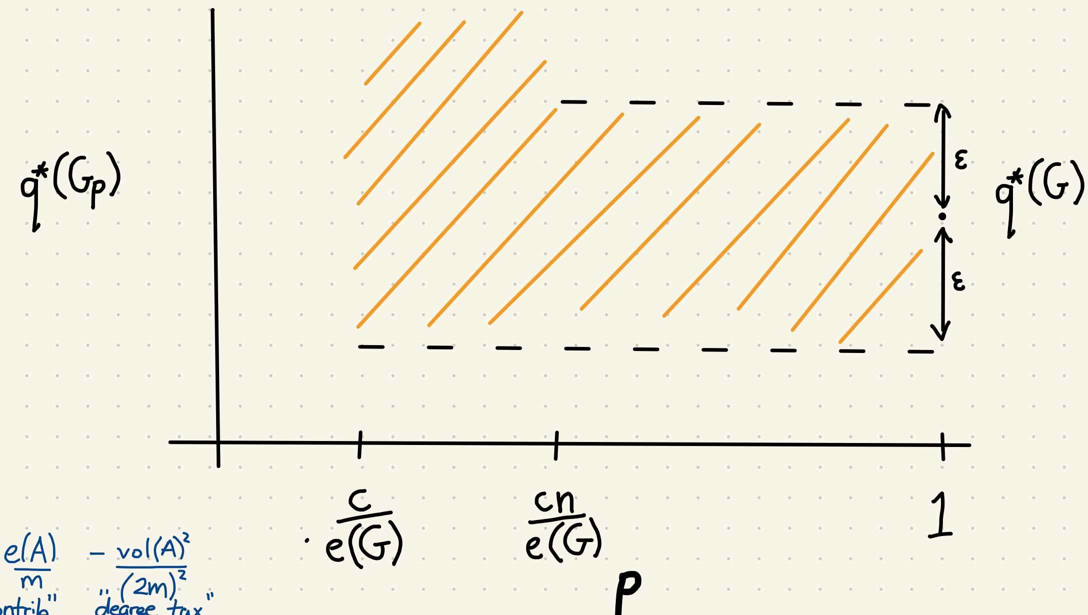

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OPEN - LIMITS: let $\bar{q}(n,c) = \mathbb{E}(q^*(G_{n,\frac{c}{n}}))$

Conj [McD+S]: $\forall c > 1$, $\bar{q}(n,c)$ tends to a limit $\bar{q}(c)$ as $n \to \infty$

If conj holds then

(i) for $0 < c \leq 1$,

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(ii) $0 < \bar{q}(c) < 1$ for $c > 1$

(iii) $\bar{q}(c) = \Theta(c^{-\frac{1}{2}})$ as $c \to \infty$

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⇐ Erdös-Renyi Results

⇐ Sampling Thm

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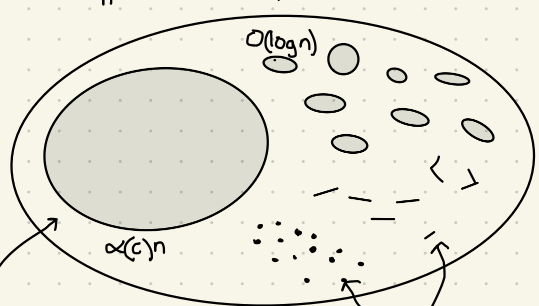
Conj [Reichardt+Bornholdt '06]: $\max_{|A|=5} q_A(G_{n,\frac{c}{n}}) = q^*(G_{n,\frac{c}{n}})$ whp

$G_{n,\frac{c}{n}}$ $c>1$ whp:



Giant comp on $\alpha(c)n$ vert with $\beta(c)m$ edges

linearly many isolated vertices + " edges

$\alpha(c) \to 1$ as $c \to \infty$

$\beta(c) \to 1$

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But! $\forall c\ \exists \delta(c)$ $\max_{|A| \leq k} q_A(G_{n,\frac{c}{n}}) \leq q^*(G_{n,\frac{c}{n}})(1 - \delta(c))$ whp

By Dinh + Thai $\max_{|A| \leq k} q_A(G_{n,\frac{c}{n}}) \geq q^*(G_{n,\frac{c}{n}})(1 - \frac{1}{k})$

Conj [McD+S]: $\exists k$ ($k=5$?) s.t. $\forall \varepsilon > 0$ $\exists c_0$ and $\forall c \geq c_0$

$$\max_{|A| \leq k} q_A(G_{n,\frac{c}{n}}) \geq q^*(G_{n,\frac{c}{n}})(1 - \varepsilon)$$

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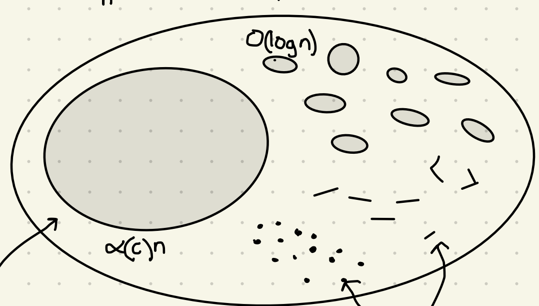

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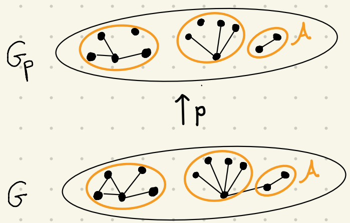
IDEA OF PROOF: approx opt partition with 'well-behaved' partition



Thm [Dinh+Thai]: for k positive integer

$$\max_{|\mathcal{A}| \leq k} q_{\mathcal{A}}(G) \geq q^*(G)\left(1 - \frac{1}{k}\right)$$

$\mathcal{A}$ is η-fat for G if $\text{vol}(A) \geq \eta \, \text{vol}(G)$

Fattening Lemma: Given $\mathcal{A}$ partition of G can construct $\mathcal{A}'$ η-fat s.t. $\mathcal{A}$ refinement of $\mathcal{A}'$

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ASIDE: LOAD BALANCING

$\vec{x} = (x_1, \dots, x_n) \quad x_i \geq 0 \quad \sum_i x_i = 1$

let $\gamma(\vec{x}) = \max_{I \subseteq [n]} \min \left\{ \sum_{i \in I} x_i, \; 1 - \sum_{i \in I} x_i \right\}$

observe

$$\gamma(\vec{x}) \geq \frac{1}{2} - \max_i x_i$$

Q what is the largest α s.t.

$$\gamma(\vec{x}) \geq \alpha\left(1 - \sum_i x_i^2\right) \quad ?$$

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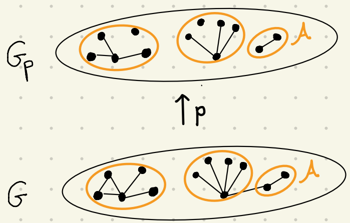

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$\alpha \leq \frac{1}{2}$

$$\gamma\left(\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)\right) = \tfrac{1}{3} = \tfrac{1}{2}\left(1 - \tfrac{1}{3}\right)$$

$\alpha \geq \frac{1}{2}$

greedy alg

- $x_1 \geq \dots \geq x_n$
- init. $A, B = \emptyset$
- step i add x_i to smaller heap

achieves $\alpha \geq \frac{1}{2}$

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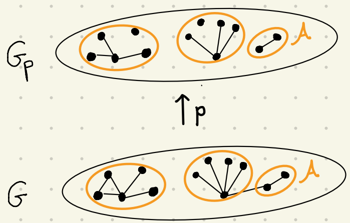

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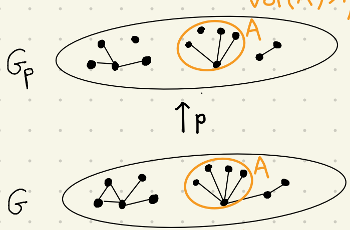
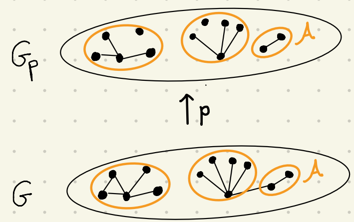
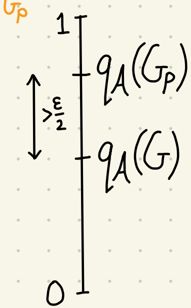
BAD EVENTS

B_0: $\exists A \subseteq V$ s.t.



B_1: $\exists \mathcal{A}$ s.t. η-fat for G_p





$$q_{\mathcal{A}}(G) = \sum_{A\in\mathcal{A}} \underbrace{\frac{e(A)}{m}}_{\text{"edge contrib"}} - \underbrace{\frac{\mathrm{vol}(A)^2}{(2m)^2}}_{\text{"degree tax"}}$$

ASIDE: LOAD BALANCING

$\vec{x} = (x_1, \dots, x_n)$ $\quad x_i \geq 0 \quad \sum_i x_i = 1$

let $\gamma(\vec{x}) = \max_{I\subseteq[n]} \min\left\{ \sum_{i\in I} x_i,\ 1-\sum_{i\in I} x_i \right\}$

observe

$$\gamma(\vec{x}) \geq \frac{1}{2} - \max_i x_i$$

Q what is the largest α s.t.

$$\gamma(\vec{x}) \geq \alpha\left(1-\sum_i x_i^2\right) \quad ?$$

$\alpha \leq \frac{1}{2}$

$$\gamma\left(\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)\right) = \tfrac{1}{3} = \tfrac{1}{2}\left(1-\tfrac{1}{3}\right)$$

$\alpha \geq \frac{1}{2}$

greedy alg

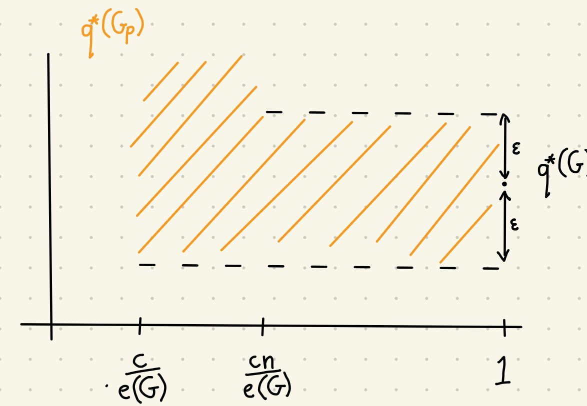
- $x_1 \geq \dots \geq x_n$
- init $A, B = \emptyset$
- step i add x_i to smaller heap

achieves $\alpha \geq \frac{1}{2}$

Thm [McD+S] $\forall \varepsilon > 0 \ \exists c = c(\varepsilon)$

for graph G
constant p

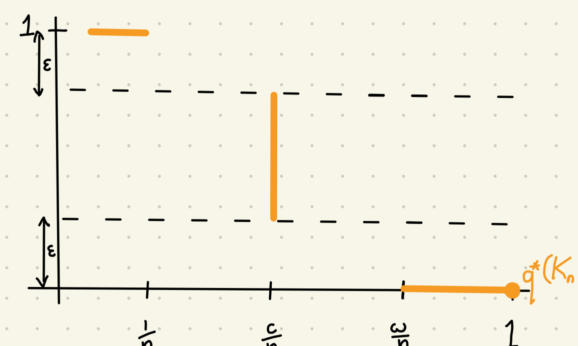
- if $e(G)p \geq c$ then w. prob $> 1-\varepsilon$ $\quad q^*(G_p) > q^*(G) - \varepsilon$
- if $e(G)p \geq cn$ " " $\quad q^*(G_p) < q^*(G) + \varepsilon$
- if $e(G)p \geq cn$ given A, can construct A'
  w. prob $> 1-\varepsilon$ $\quad q_{A'}(G) > q_A(G_p) - \varepsilon$



Thm [McD+S] whp

- SPARSE $\quad e(K_n)p \to \infty$, $n^2p \to \infty$ $\quad e(K_n)p \leq n + o(n)$, $np \leq 1 + o(1)$ $\quad q^*(G_{n,p}) = 1 + o(1)$
- CRITICAL $\quad \forall c > 1 \ \exists \varepsilon = \varepsilon(c)$ $\quad e(K_n)p \to cn$, $np \to c$ $\quad \varepsilon < q^*(G_{n,p}) < 1 - \varepsilon$
- DENSE $\quad \frac{1}{n}e(K_n)p \to \infty$, $np \to \infty$ $\quad q^*(G_{n,p}) = o(1)$

$q^*(G_{n,p}) = q^*((K_n)_p)$



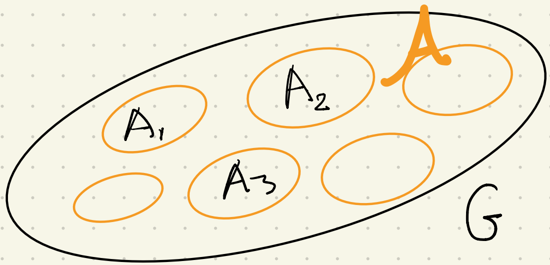
$\exists a,b: \ \frac{1}{n} \leq p \leq 0.99 \cdot$ whp $\quad \frac{a}{\sqrt{np}} < q^*(G_{n,p}) < \frac{b}{\sqrt{np}}$

fix $\varepsilon > 0$:
$C_0 = \frac{a^2}{\varepsilon^2} \qquad e(K_n) \cdot p = C_0 n$

$q^*(K_n) = 0$
$q^*((K_n)_p) > \varepsilon$ whp

$$q_A(G) = \sum_{A \in \mathcal{A}} \underbrace{\frac{e(A)}{m}}_{\text{"edge contrib"}} - \underbrace{\frac{\mathrm{vol}(A)^2}{(2m)^2}}_{\text{"degree tax"}}$$

Modularity 'meas. of how well a graph can be clustered'



graph G, m edges. $\mathcal{A} = \{A_1, \dots, A_k\}$ vertex partition

score of partition $\mathcal{A}$, $q_{\mathcal{A}}(G) =$

modularity of G $\quad q^*(G) = \max_{\mathcal{A}} q_{\mathcal{A}}(G)$

"high vals taken to indicate more community structure"

$$q_{\mathcal{A}}(G) = \underbrace{\sum_{A \in \mathcal{A}} \frac{e(A)}{m}}_{\text{"edge contrib."}} - \underbrace{\frac{\mathrm{vol}(A)^2}{(2m)^2}}_{\text{"degree tax"}} = \frac{1}{2m} \sum_{A \in \mathcal{A}} \sum_{u,v \in A} \mathbb{1}[u \sim v] - \frac{d_u d_v}{2m}$$

$d_u = \#$ edges incident to u

$e(A) = \#$ edges in set A

$\mathrm{vol}(A) = \sum_{u \in A} d_u$

Obs

- G regular, $\mathcal{A} = \{A_1, \dots, A_k\}$ with $|A_i| = \frac{n}{k}$

$$q_{\mathcal{A}}(G) = \sum_{A \in \mathcal{A}} \frac{e(A)}{m} - \frac{1}{k} \quad \Rightarrow \quad q^*(G) \geq \frac{1}{2} - \frac{1}{m} bw(G)$$

$$bw(G) = \min_{|U| = \lfloor \frac{n}{2} \rfloor} e(U, V \setminus U)$$

- $\mathcal{A}$ optimal partition of G i.e. $q_{\mathcal{A}}(G) = q^*(G)$
  - $\forall A \in \mathcal{A}$ $\quad G[A]$ conn. (+ isolated vert)
  - " $\quad |A| > 1$ $\quad$ Ex!
  - S min vertex cover $\Rightarrow \mathcal{A}$ has $\leq |S|$ parts

S vertex cover if $uv \in E \Rightarrow (v \in S) \vee (u \in S)$

OPEN - HIGH GENUS MAPS

$M_n(\theta)$: map on n edges with genus θn

Conj [LouF+S] $\forall \varepsilon\ \exists \delta, \delta'$ whp

- SPARSE $\quad \theta < \delta \quad q^*(M_n) > 1 - \varepsilon$
- CRITICAL $\quad \delta' < \theta < \frac{1}{2} - \delta' \quad \varepsilon < q^*(M_n) < 1 - \varepsilon$
- DENSE $\quad \theta > \frac{1}{2} - \delta \quad q^*(M_n) < \varepsilon$

Furthermore $\exists f$ cts non-increasing st whp

$$q^*(M_n(\theta)) = f(\theta)(1 + o(1))$$