Random tree recursions: which fixed points correspond to tangible sets of trees?

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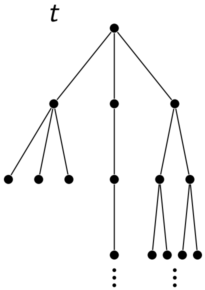
arxiv:1808.03019

Czech Academy of Sciences

T_λ Galton-Watson tree with offspring distribution $\mathrm{Po}(\lambda)$

EXAMPLE 1: Survival



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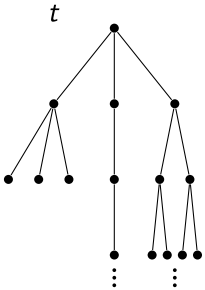

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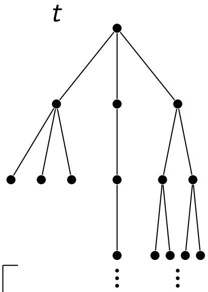

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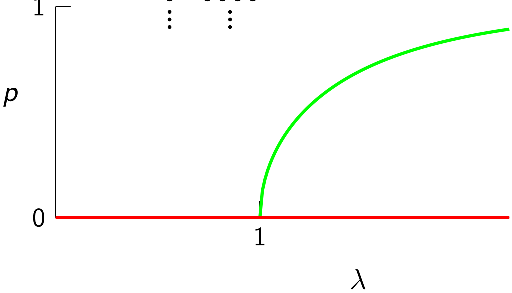

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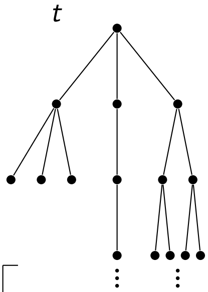

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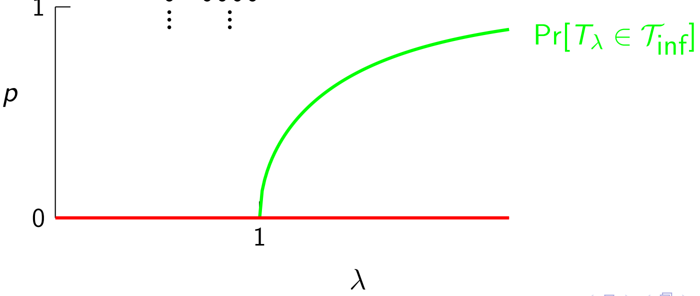

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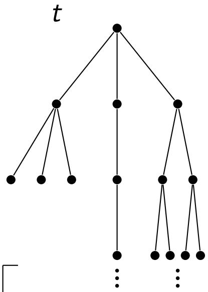

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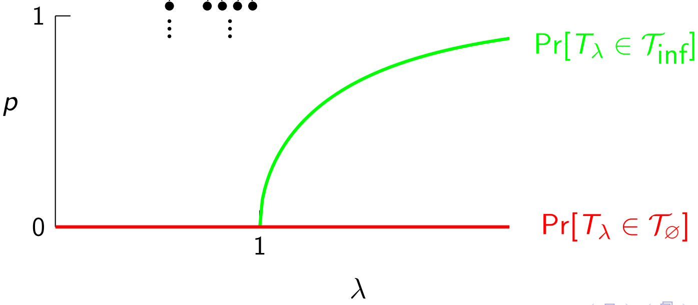

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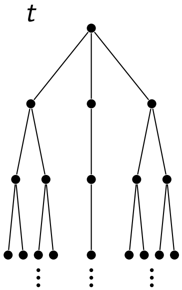

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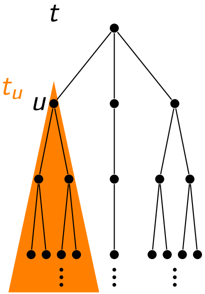

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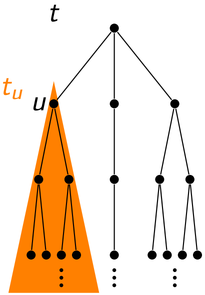

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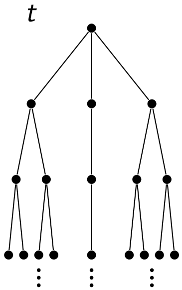

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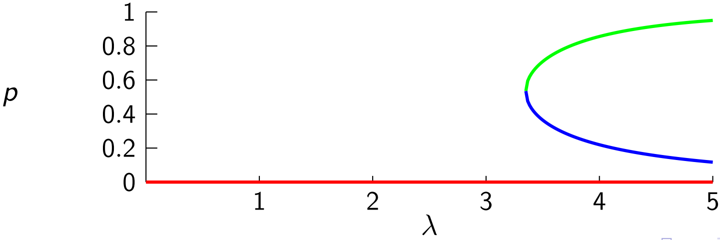

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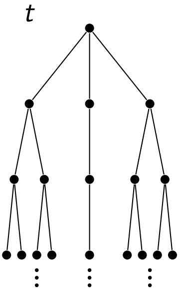

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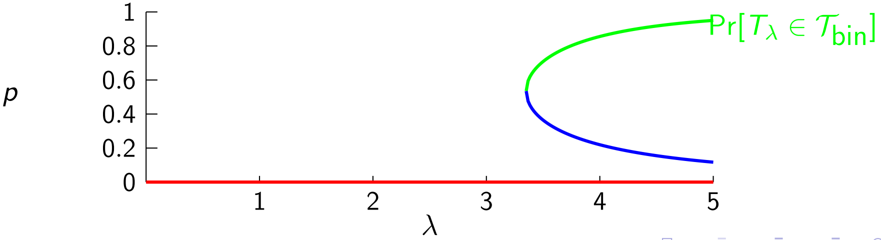

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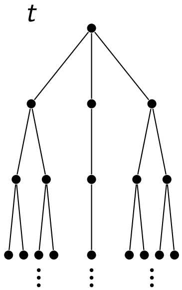

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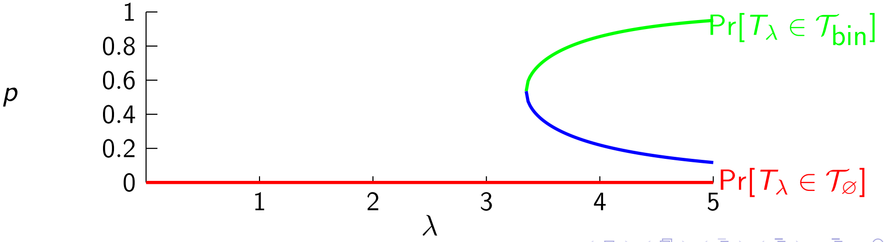

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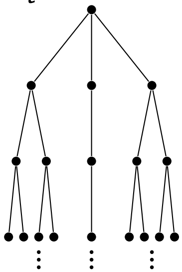

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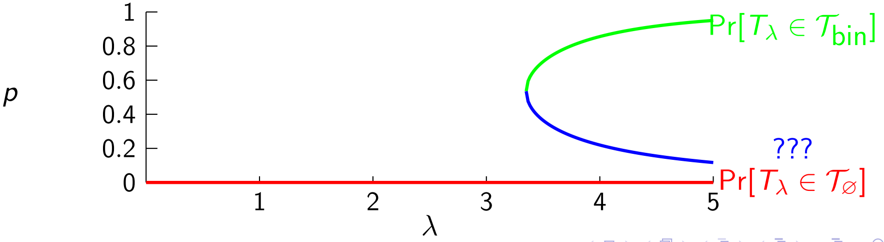

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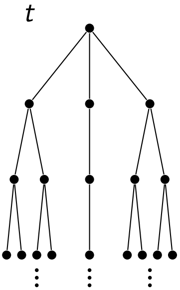

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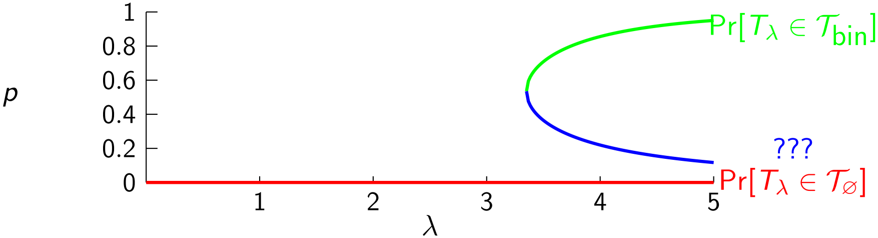

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EXAMPLE 2 (AGAIN): start with metaproperty

metaproperty:

tree $t \in \mathcal{T}^*$ *iff children* u, v *&* $t_u, t_v \in \mathcal{T}^*$

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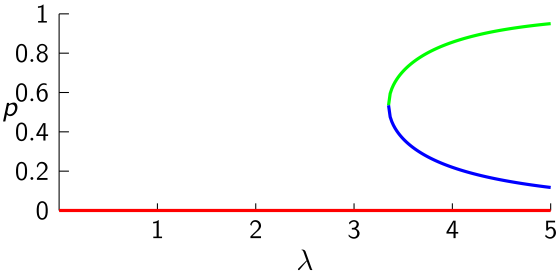
fixed points



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metaproperty: subsets of trees

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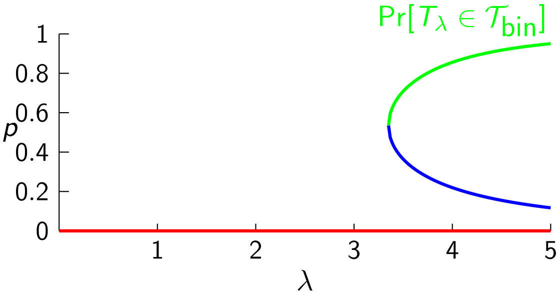

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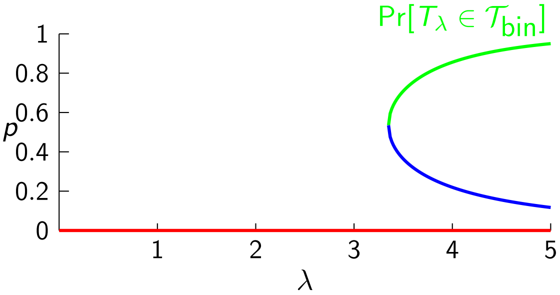

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subsets of trees

$\mathcal{T}_{\text{bin}}$

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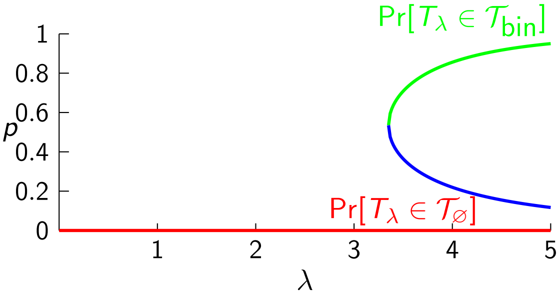

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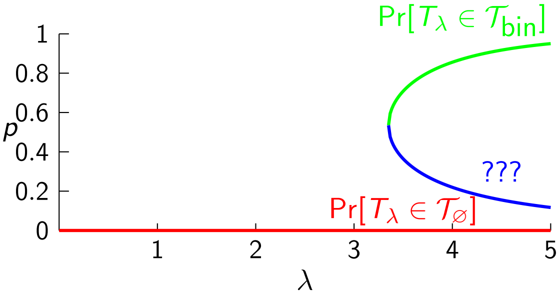

QUESTION:

Is there set of trees $\mathcal{T}'$ satisfying metaproperty with $\Pr(T_\lambda \in \mathcal{T}') =$ blue line

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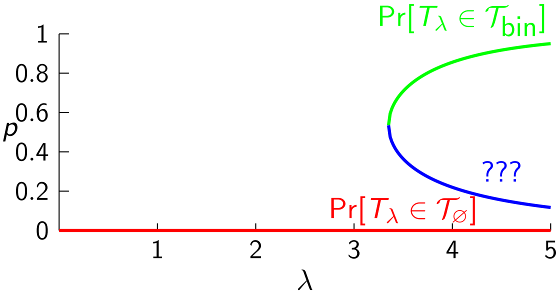

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Tree Automata

- nodes states ‘0’ and ‘1’
- automata A $\sim$ rules to determine the state of a parent from the number of children with states ‘0’, ‘1’.
- examples, n_i number of children state i

at-least-one $A(n_0, n_1) = \mathbf{1}[n_1 \geq 1]$

at-least-two $A(n_0, n_1) = \mathbf{1}[n_1 \geq 2]$

three musketeers
$A(n_0, n_1) = \mathbf{1}[((n_0 = 0) \wedge (n_1 = 3)) \vee ((n_0 = 3) \wedge (n_1 = 0))]$

Distribution map for automaton A

- $\Phi_A(p)$ probability a root has state '1' after applying automaton A to children which have indep. state '1' with prob. p, '0' otherwise.
- $\Phi_A(p) = \Pr[A(\mathrm{Po}((1-p)\lambda), \mathrm{Po}(p\lambda)) = 1]$

Interpretation for automaton A

- intuitively (indicator of) set of trees satisfying the automaton.
- $\iota : \mathcal{T} \to \{0,1\}$ measurable map. ι interpretation if $\forall t \notin \mathcal{T}_{\text{bad}}$, assigned states $\iota(v) = \iota(t_v)$ compatible with A for some exceptional set $\Pr(T_\lambda \in \mathcal{T}_{\text{bad}}) = 0$

Question

Given a two state automaton A which fixed pts of Φ_A have an interpretation?

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EXAMPLE 2 (AGAIN AGAIN): start with automaton

let $A(n_0, n_1) = \mathbf{1}[n_1 \geq 2]$

states '0':$\circ$ and '1':$\bullet$

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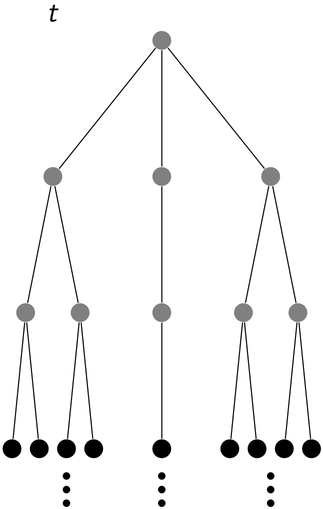

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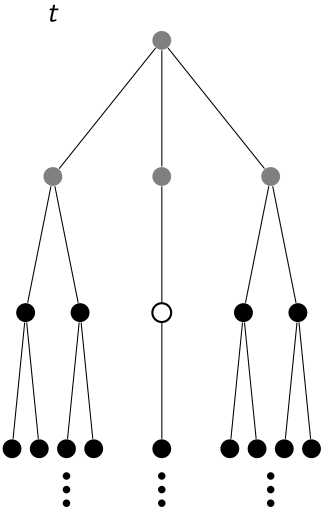

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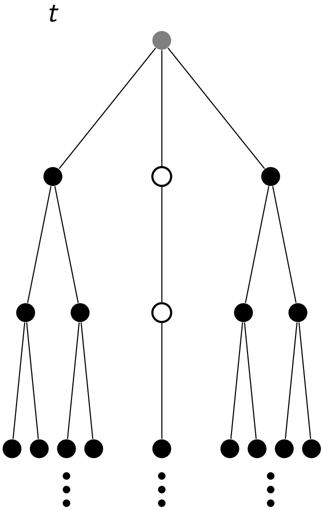

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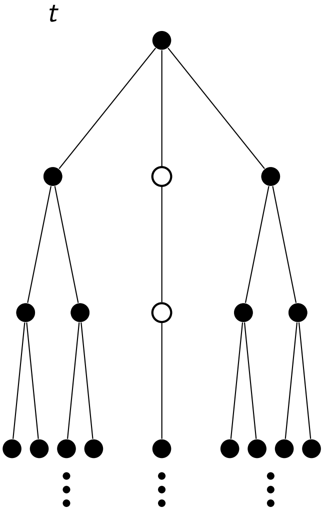

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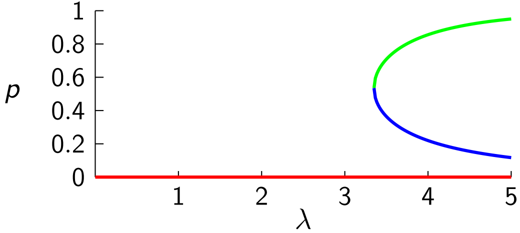

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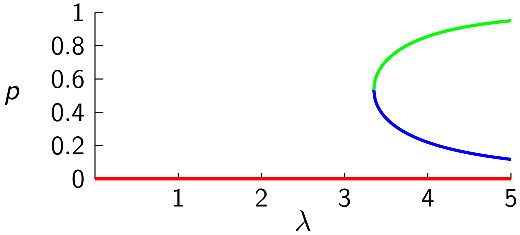


interpretations for fixed pts

$\iota_1(v) = \mathbf{1}[t_v \in \mathcal{T}_{\mathrm{bin}}]$

$\iota_\varnothing(v) = 0$

RESULT: there is NO interpretation for blue fixed pts

PIVOTAL VERTICES

- a vertex is pivotal if switching its colour and applying automaton to its ancestors switches the colour at the root.
- parent of a pivotal node is pivotal

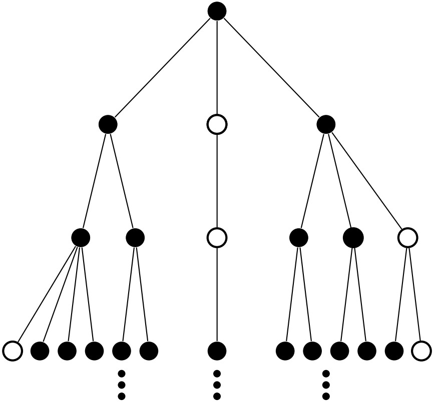
PIVOT TREE for zero-one labelled tree (t, ω)

- t_{piv} is the subgraph induced by pivotal vertices
- Observe t_{piv} is a tree from the root

at-least-two

$A(n_0, n_1) = \mathbf{1}[n_1 \geq 2]$

states '0':$\circ$ and '1':$\bullet$



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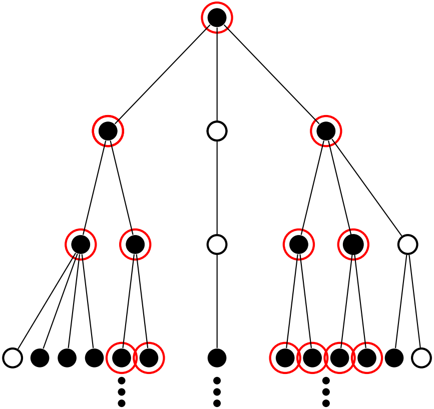

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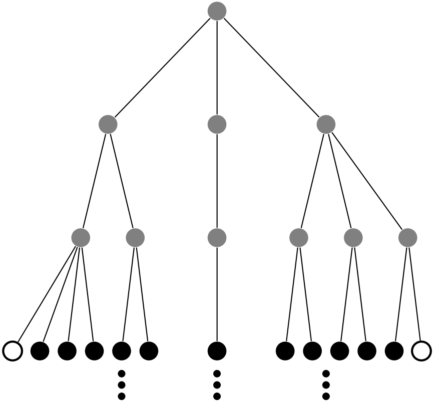

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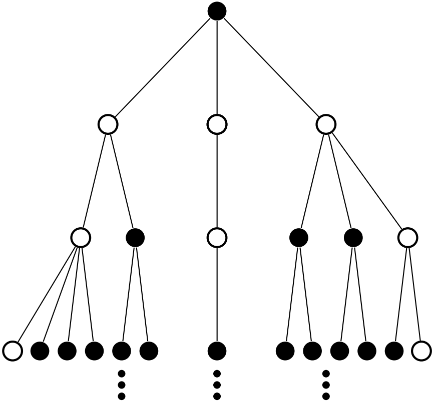

multiple-that-agree

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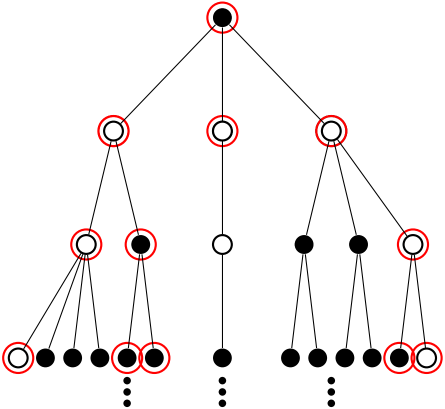


RANDOM STATE TREE (T, ω) for automaton A and p a fixed pt of Φ_A

- (i) $T \sim T_\lambda$
- (ii) for every n the conditional distribution of $(\omega(v) : d(v) = n)$ given $T|_n$ is i.i.d. Ber(p).
- (iii) ω almost surely compatible with A

THM (JOHNSON, PODDER, S. 2018+)

Given automata A and p a fixed point of Φ_A. Let $T_{\text{piv}}(\lambda)$ denote the pivot tree of the random state tree of A and p. The fixed point p admits an interpretation iff T_{piv} is subcritical or critical.

Boolean functions and influence

function $f : \{0,1\}^m \to \{0,1\}$

say $\sigma = \sigma_1 \ldots \sigma_m$ pivotal at i if flipping i-th bit flips value of f

$I_i(f) =$ influence of i is probability the i-th co-ordinate pivotal

$I(f) = \sum_i I_i(f)$

DICTATOR $f(\sigma) = \sigma_1$
$I_1(f) = 1$ and $i > 1$ $I_i = 0$ so $I(f) = 1$

MAJORITY $f(\sigma) = \mathbf{1}[\sum_i \sigma_i > m/2]$
$I_i(f)$ is probability $\sigma \backslash \sigma_i$ has same #'1's and #'0's, order $m^{-\frac{1}{2}}$. $I(f) \sim m^{1/2}$.

PARITY $f(\sigma) = (-1)^{\sum_i \sigma_i}$
$I_i(f) = 1$ for each i. $I(f) = m$.

Thm (BKKKL)

$\exists c$ such that:
Given $g : \{0,1\}^m \to \{0,1\}$, $x = \mathbb{P}\big[g(S_1, \dots, S_m) = 1\big]$,
$S_i \sim \mathrm{Ber}(p)$ independent we have

$$I(g) \geq c \min\{1-x, x\} \log \frac{1}{\max_i I_i(g)}.$$

'if total influence and max influence small then g nearly constant'

SKETCH T_{piv} subcritical implies p interpretable

SET-UP Condition on $T|n$ and colour n-level $S_v \sim \mathrm{Ber}(p)$, $g = \omega(\mathrm{root}\big| T|_n)$

$I_v(g)$ is probability that $v \in T_{piv}$.

$\max_v I_v(g) \leq$ the probability T_{piv} survives to height n.

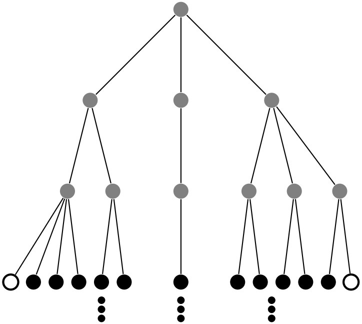
$p = \mathbb{P}[\omega(\mathrm{root}) = 1 \mid T|_n] \to \{0,1\}$ almost surely.

SKETCH SWITCHING: T_{piv} supercritical implies non-interpretable.

multiple-that-agree

$A(n_0, n_1) = \mathbf{1}[((n_0 = 0) \wedge (n_1 \geq 2)) \vee ((n_0 \geq 2) \wedge (n_1 = 0))]$

Generate a 1-state tree, by taking 0-state tree and switching along path in T_{piv}.



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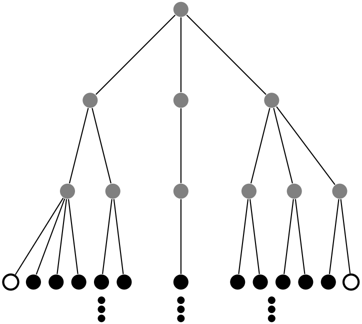

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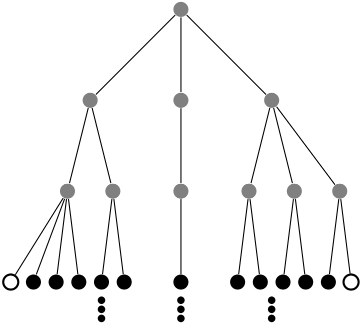

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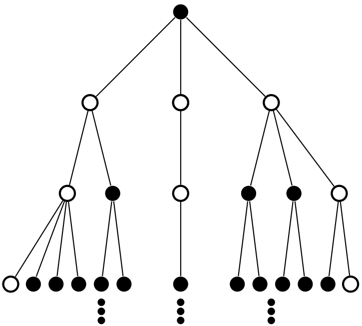

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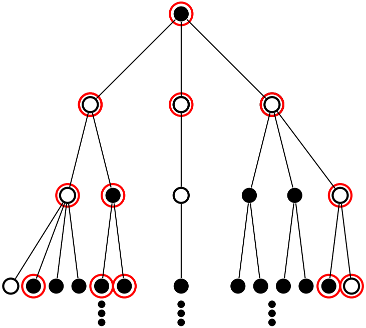

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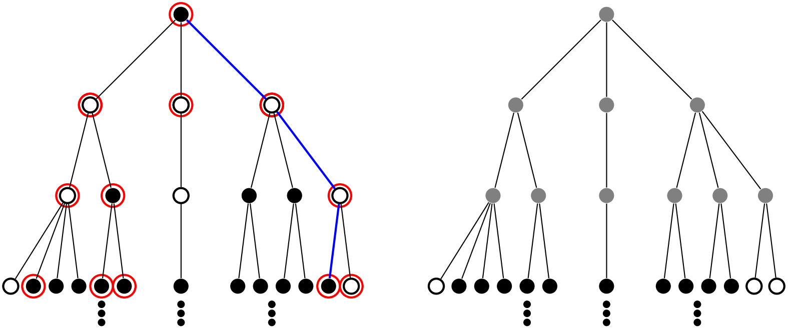

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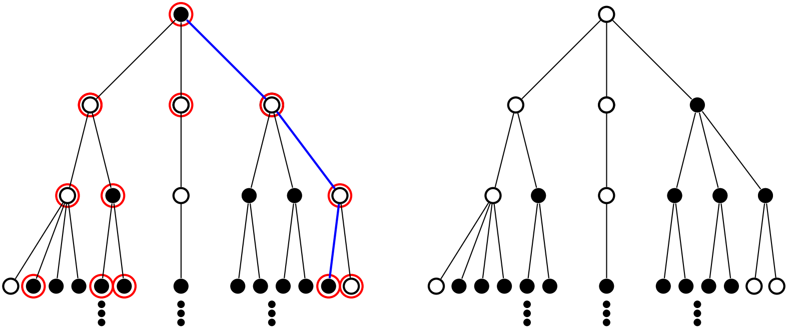

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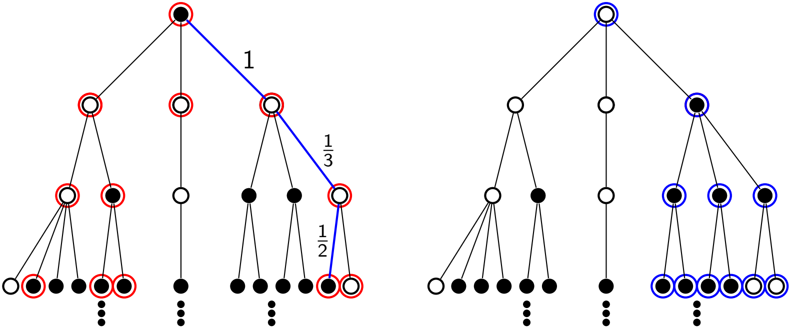
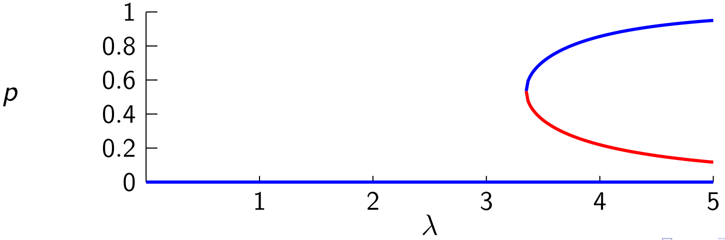


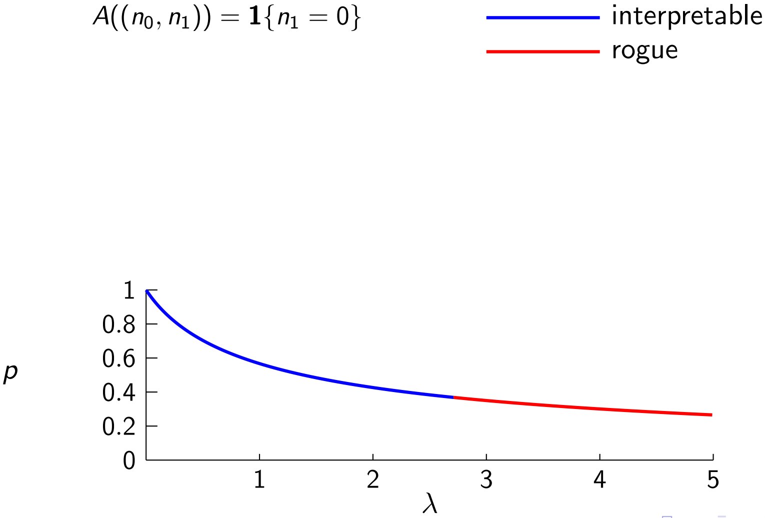
$A((n_0, n_1)) = \mathbf{1}\{n_1 \geq 2\}$

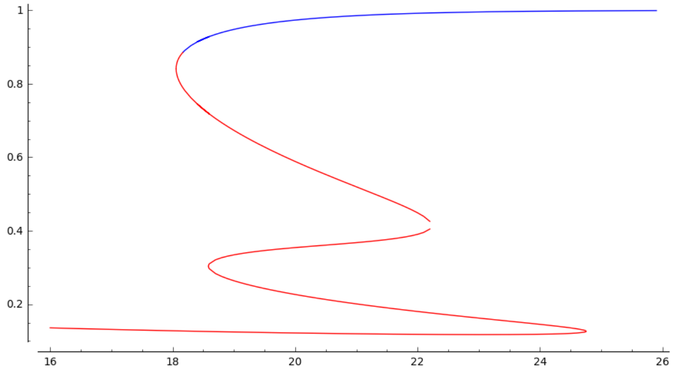
interpretable

rogue



$$A((n_0, n_1)) = \mathbf{1}\{n_1 = 0\}$$





$$A((n_0, n_1)) = \mathbf{1}\{(n_1 \in \{0, 6, 7\}) \vee (n_1 \geq 12)\}$$

interpretable

rogue