Exercise 5. Show whp $np \to \infty$ implies whp G_n contains $\triangle$ i.e. a 3-cycle[4].

Let Y_n count the number of $\triangle$ in G_n and for any 3-subset of vertices $S \subset V(G)$ let A_S be the event that G_n restricted to the vertices S is a $\triangle$.

(a) Show by linearity of expectation that:

$$\mathbb{V}(Y_n) = \sum_{S,T \in \binom{[n]}{3}} \Big(\mathbb{P}(A_S \ \&\ 1_{A_T}) - \mathbb{P}(A_S)\mathbb{P}(A_T) \Big).$$

(b) Notice that when the sets of vertices S and T don't intersect that the events A_S and A_T are independent. What about when they intersect on one vertex? Using (a) show that:

$$\mathbb{V}(Y_n) \leq \sum_{|S \cap T| = \{2,3\}} \mathbb{P}(A_S \ \&\ A_T).$$

(c) After some case analysis and from (b) show: $\mathbb{V}(Y_n) \leq n^4 p^5 + n^3 p^3$.

(d) From (c) conclude that whp $Y_n > 0$. *Hint: use Chebyshev's inequality.*

Exercise 6. Given $k \in \mathbb{N}$, let $\mathcal{P}_k$ be the set of graphs which have a path on k vertices as a subgraph. Find the threshold function for $\mathcal{P}_3$ (containing the path $\wedge$ as a subgraph) and for $\mathcal{P}_4$. Can you find the threshold for $\mathcal{P}_k$ in terms of k and n?

[4] This exercise demonstrates a different way to prove the second part of Theorem 1.6. In the proof we showed that whp $e(G_n) \geq n$ for $np \to \infty$ and from this and Q 2a we concluded that $np \to \infty$ implies whp G_n has a cycle.