Branching processes with cousin mergers and locality of hypercube's critical percolation

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joint work in progress with L. Eslava, S. Penington

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Making sense of formulas

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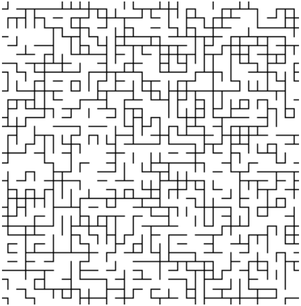
1. Percolation
    - Critical probability
    - Exploration of components
2. The structure of the hypercube
    - Critical probability expansions
    - The quest for a heuristic
3. Branching processes with mergers
    - Cousin mergers does not suffice
    - A refined collision model

Percolation

Given an *underlying graph*, keep each edge independently with prob. p

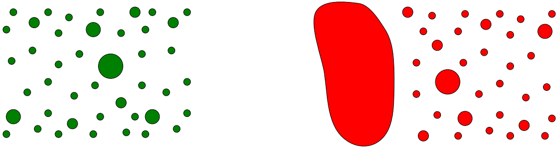


Critical Probability: *Edge density where an giant component appears*

- *Infinite graphs:* $p_c := \inf\{p : P(|C(0)| = \infty) > 0\}$

Phase transition for giant component Erdős-Rényi $G_{n,p}$

Critical Probability: *Edge density where a giant component appears*



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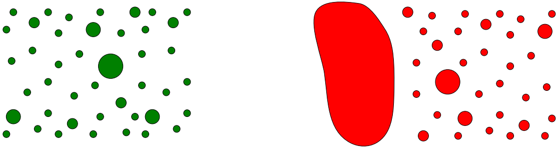

If $p = \frac{1}{n}(1+\epsilon)$, *whp largest component of size:*

- **Subcritical** $\epsilon^3 n \to -\infty$: $L_1(G_{n,p}) = O(\log n)$
- **Critical** $\epsilon^3 n \to a \in \mathbb{R}$: $L_1(G_{n,p}) = \Theta(n^{2/3})$
- **Supercritical** $\epsilon^3 n \to \infty$: $L_1(G_{n,p}) = \Theta(n)$

The critical window is of order $O(n^{-4/3})$.

Phase transition for giant component Erdős-Rényi $G_{n,c/n}$

Exploration on $G(n, c/n)$.
Count size of component of v

Exploration tracks:

- Explored vertices D_k
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$$D_0 = \emptyset, \qquad X_0 = \{v\}$$

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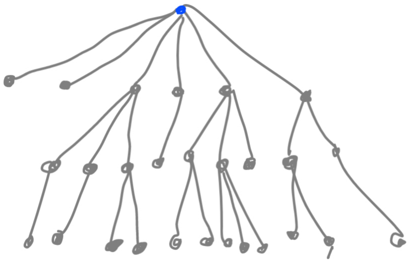

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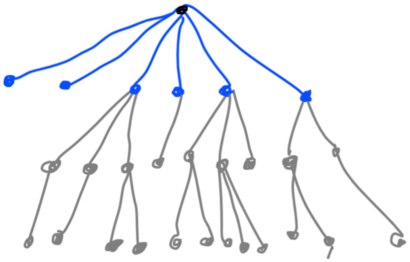

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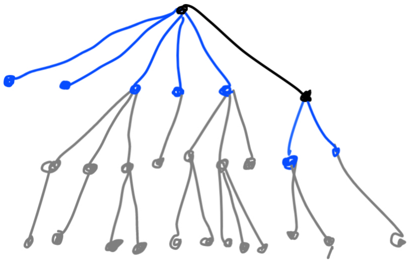

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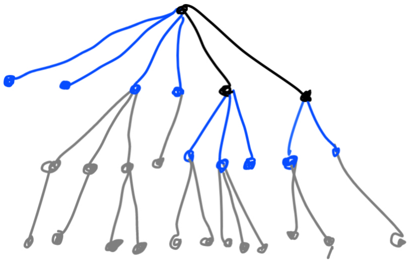

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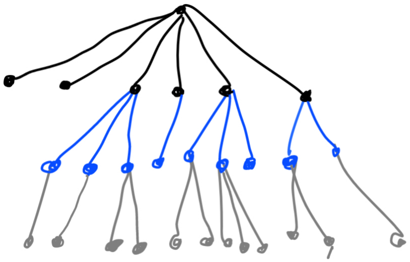

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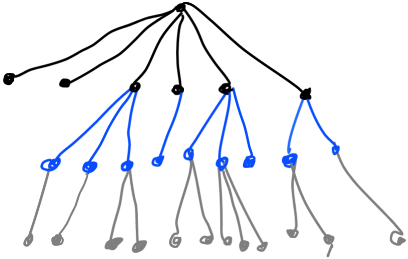

degree of $v \sim \mathrm{Bin}(n-1, p)$
$\mathbb{P}(d_v = i) = \binom{n-1}{i}\left(\frac{c}{n}\right)^i\left(1-\frac{c}{n}\right)^{n-1-i} \sim \frac{c^i}{i!e^c}$.

degree of $u \sim \mathrm{Bin}(n-7, p)$

Phase transition for giant component Erdős-Rényi $G_{n,c/n}$

Galton-Watson trees with child distribution X

- Generation zero $\bullet$.
- Generation $n+1$, each individual v in Generation n has random number $X_v \sim X$ children independently from rest.

Galton-Watson Survival

Average children $\mathbb{E}[X] = (1+\epsilon)$ determines

- $\epsilon \leq 0$: Extinction w.p. 1
- $\epsilon > 0$: Survival with positive prob.

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Markovian process: Each generation only depends on previous one.
$Z_n = \#$indiv. at generation n

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{v_i}$$

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GW tree with Po(c) child distribution

For $c < 1$, extinction whp. For $c > 1$ get $\alpha_c > 0$ probability of suvival.

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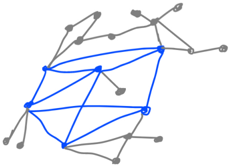
Giant component in $G(n, c/n)$

For $c < 1$ whp no giant component.
For $c > 1$ whp giant component of size $a_c n(1 + o(1))$.

Phase transition for 3-core in Erdős-Rényi $G_{n,p}$

Pittel, Spencer, Wormald ['96]; Riordan ['07]

The 3-core of G is the maximal connected subgraph of G with minimum degree at least 3.



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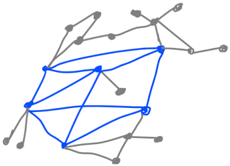

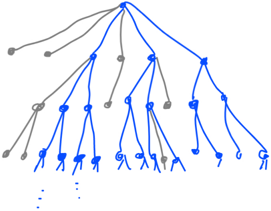

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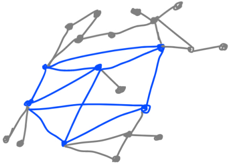

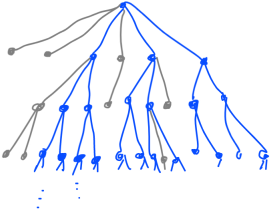

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Then $p_c = c^*/n$ where $c^* = \inf\{c \,:\, \alpha_c > 0\}$ ($c^* \sim 3.35$).
And for $c > c^*$ whp size of 3-core is $\alpha(c)n(1 + o(1))$.

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Pittel, Spencer, Wormald *Branching Poison Process Connection:*
Here is an admittedly loose attempt of such an analysis that suggests - some serious gaps and leaps of faith notwithstanding - an intuitive explanation of why the [3-core] appears when the [edge probability] passes through [c^*/n].

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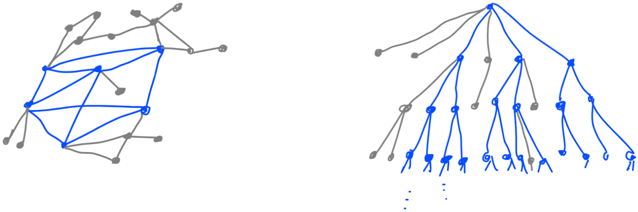

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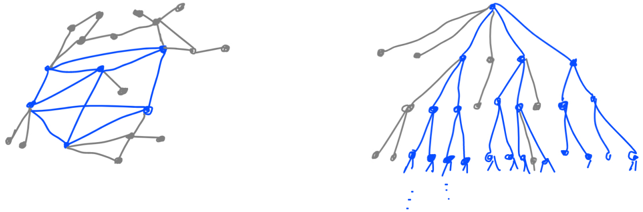

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The case of the Hypercube Q^n

Borgs et al. ['05, '06]; Hosftad, Slade ['05, '06], Hofstad, Nachmias ['12,'14]

There exists rational coefficients a_k such that

$$p_c = \sum_{k=1}^{K} a_k n^{-k} + O(n^{-K-1})$$

In particular,

$$\begin{aligned} p_c &= \frac{1}{n} + \frac{1}{n^2} + \frac{7}{2n^3} + O(n^{-4}) \\ &= \frac{1}{n-1} + \frac{5}{2}(n-1)^{-3} + O(n^{-4}) \end{aligned}$$

- Based on lace expansion and triangle condition verification
- Window too small $O(n^{-1}2^{-n/3})$ to neglect any expansion term

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If $\epsilon = O(n^{-1/3})$ then there is $\epsilon' = O(n^{-1/3})$ with

$$p = \frac{1}{n}(1+\epsilon) = \frac{1}{n-1}(1+\epsilon')$$

Exploration on lattice-like graphs

Goal: Count size of a vertex v component

Exploration tracks:

- Explored vertices D_k
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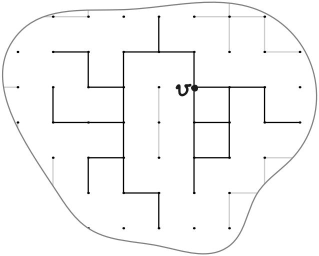

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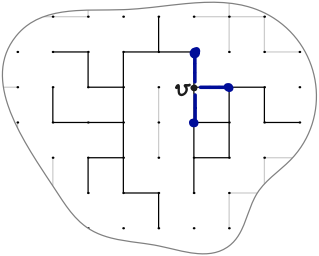

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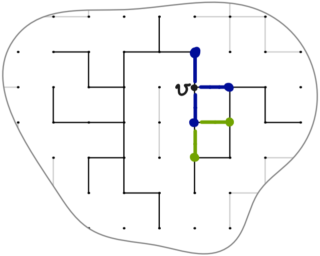

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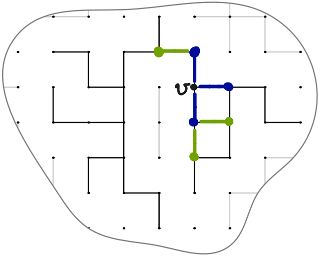

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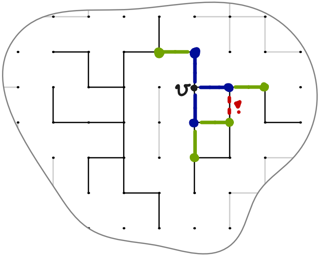


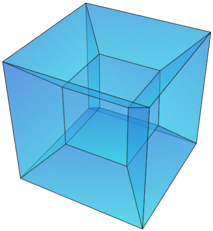
Branching process - collisions
$\approx$
Component size

The hypercube's local structure

$\{0,1\}^n$ **Representation**

- Sequences with n entries
- Crossing edge changes **one** entry:

  $(0, 0, 1, 0, \ldots, 0)$
- Shortest cycle has length 4



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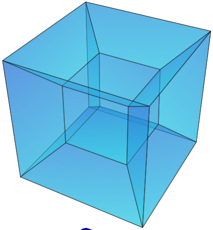

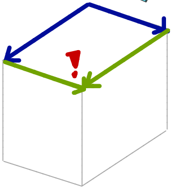

$\{0,1\}^n$ Representation

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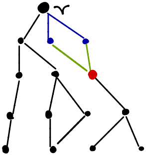
Simple collisions

- Parent: $(1,0,0,0,\ldots,0)$
- Possible children:

$$(1,0,1,0,\ldots,0)$$
$$(1,1,0,0,\ldots,0)$$

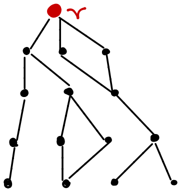
- Possible grandkid: $(1,1,1,0,\ldots,0)$



Branching process with cousin mergers

Project: Heuristic to recover $p_c = 1/n - 1 + \frac{5}{2}(n-1)^{-3}$

- Indiv. have $X_v \sim \mathrm{Poi}(1+\epsilon)$ children
- Independently with probability q, each pair of cousins becomes a single indiv.
- Multiple mergers allowed
- Probability $q \sim (n-1)^{-2}$.



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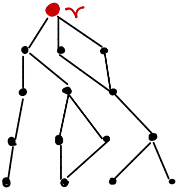
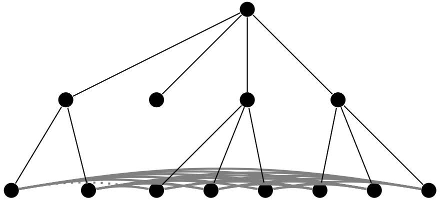


Difficulties:

Non-Markovian Process: $Z_0, Z_1, \ldots, Z_n$ not enough to obtain Z_{n+1}

Non-monotonicity: No straightforward coupling gives monotonicity of survival

Branching process with cousin mergers - definition



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Generation $n+1$

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- Each pair of cousins merges independently with probability q.

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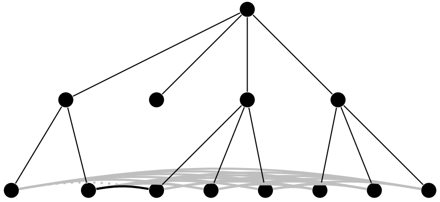

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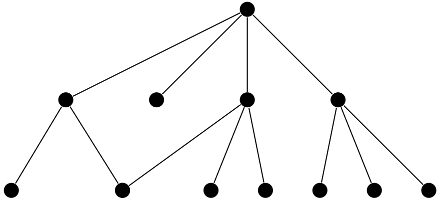

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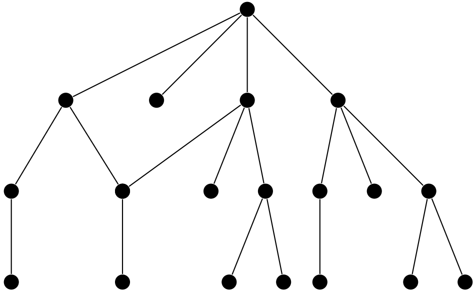

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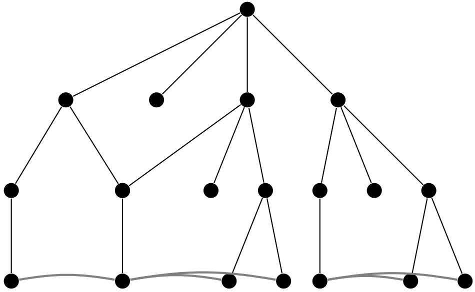

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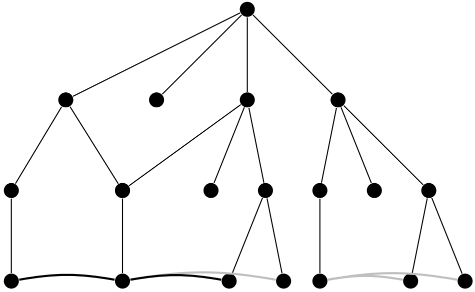

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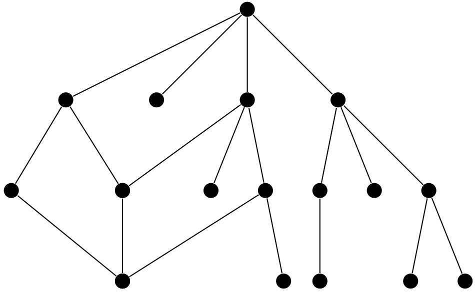

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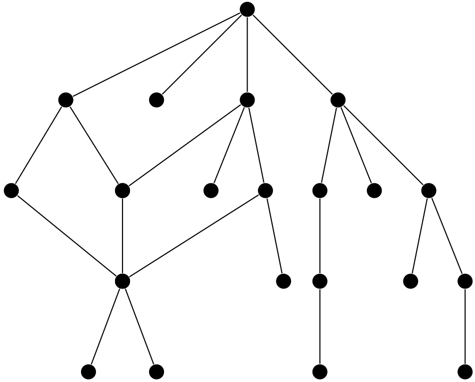

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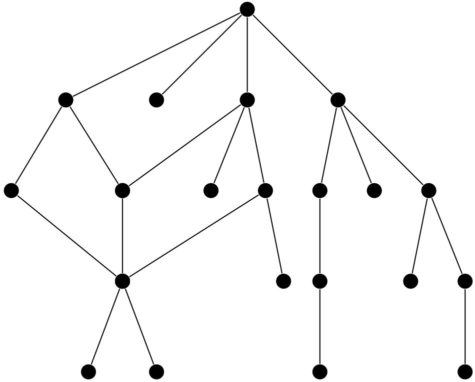

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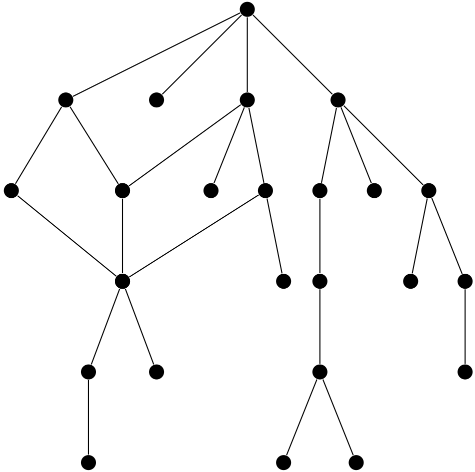

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BP with cousin mergers (Eslava, Penington, S., '20^{+})

If $\xi_v \sim \mathrm{Poi}(1+\epsilon)$, $\epsilon > 0$ suff. small, then merger prob. q determines

- $\epsilon \leq \frac{1}{2}\epsilon - K\epsilon^2$: Extinction w.p. 1
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Partial Idea: Estimate average growth per generation

$$\mathbb{E}[Z_{n+1}] \approx \left(1+\epsilon-\frac{q}{2}\right)\mathbb{E}[Z_n]$$
$$\approx (1+\epsilon)\mathbb{E}[Z_n] - \frac{q}{2}\mathbb{E}[Z_{n-1}]$$

Zoom-in on idea

Partial Idea: Estimate average growth per generation

$$\mathbb{E}[Z_{n+1}] \approx (1+\epsilon)\mathbb{E}[Z_n] - \frac{q}{2}(1+\epsilon)^4\mathbb{E}[Z_{n-1}] + O(\epsilon^2)$$

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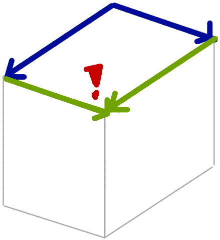

$\mathbb{E}[\#$ pairs of cousins per grandparent$]$

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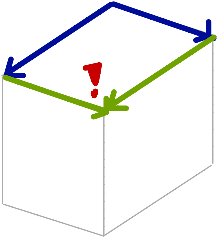

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Partial Idea: Estimate average growth per generation

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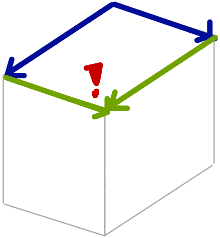

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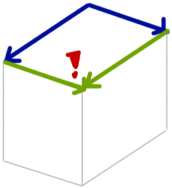
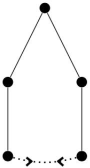
What went wrong?

Offspring distribution: Not all vertices can explore $n-1$ new edges.





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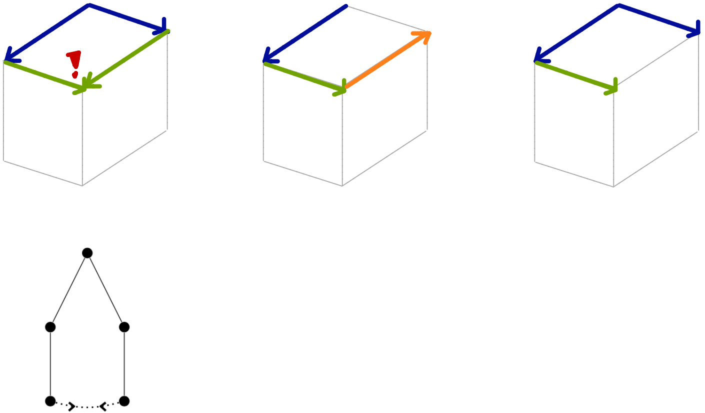

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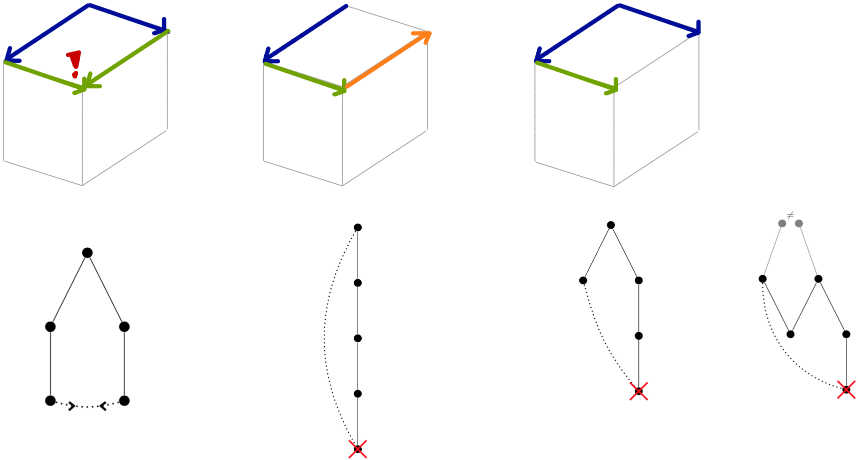
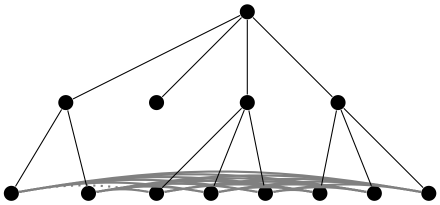


All these pairs give collisions with probability $(n-1)^{-2} \sim q$.

Refined branching process with cousin mergers - definition



Generation zero $\bullet$.

Generation $n+1$

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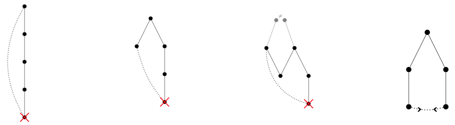

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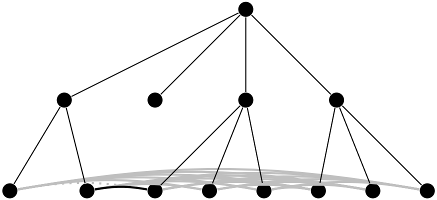

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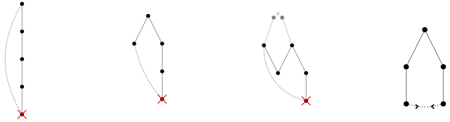

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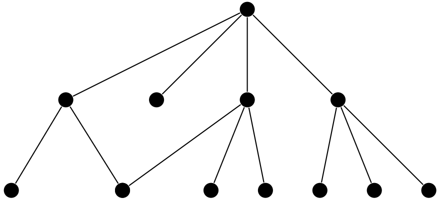

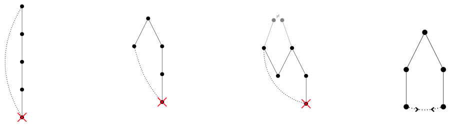

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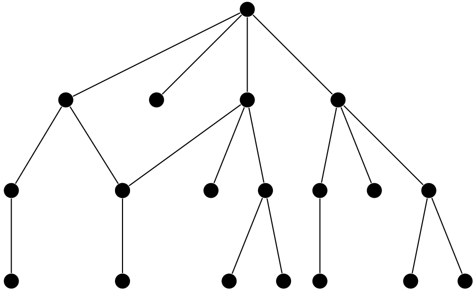

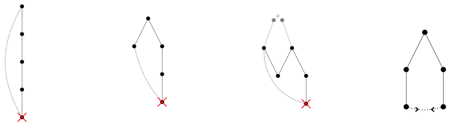

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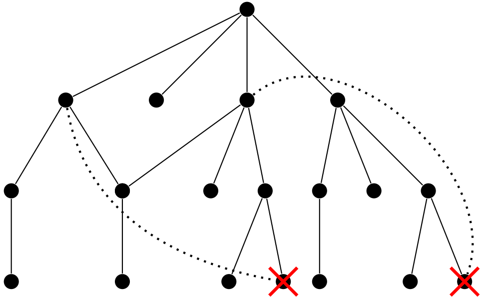

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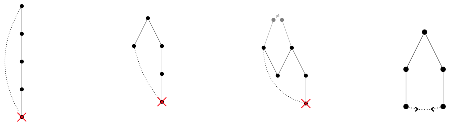

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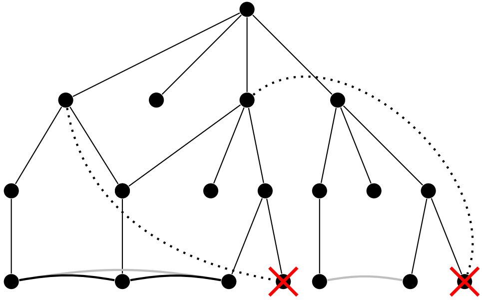

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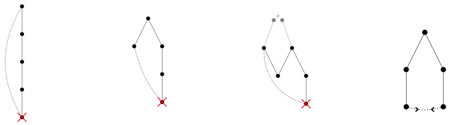

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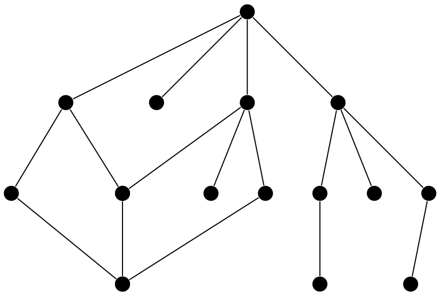

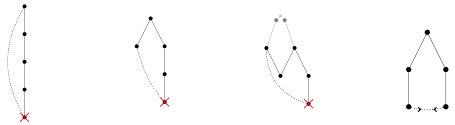

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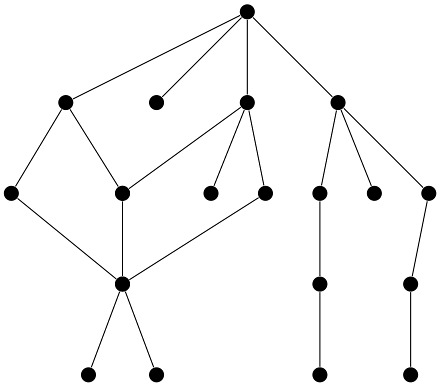

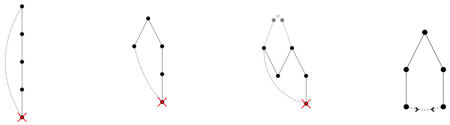

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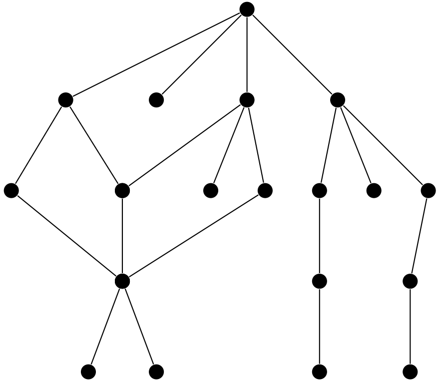

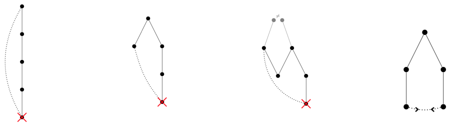

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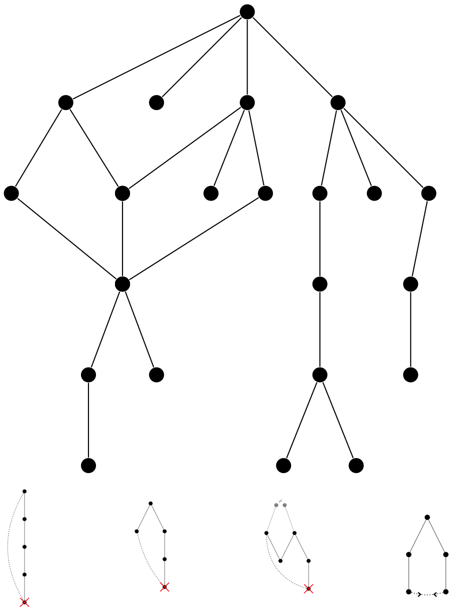

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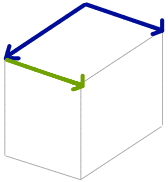
What went wrong?

Offspring distribution: Not all vertices can explore $n-1$ new edges

$$\mathbb{E}[Z_{n+1}] \approx (1+\epsilon)\mathbb{E}[Z_n] - \frac{q}{2}\mathbb{E}[Z_{n-1}] - q\mathbb{E}[Z_{n-2}] - q\mathbb{E}[Z_{n-3}]$$



$\mathbb{E}[\#$ pairs of aunt-niece per grandparent$]$

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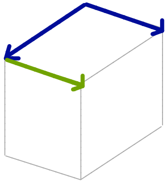

$$\begin{aligned} &\mathbb{E}[\# \text{ pairs of aunt-niece per grandparent}] \\ =&\mathbb{E}[\# \text{ pairs of children } (v_1, v_2)] \\ &\cdot \mathbb{E}[\# \text{ grandchildren of } v_1] \end{aligned}$$

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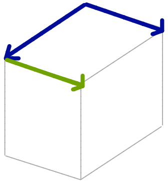

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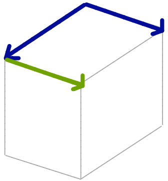

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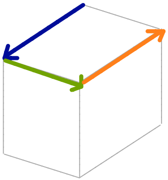

$$\mathbb{E}[\#\text{ greatgrandchildren per indiv.}] = (1+\epsilon)^4$$

What went wrong?

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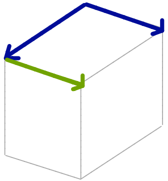
$$\mathbb{E}[Z_{n+1}] \approx (1+\epsilon)\mathbb{E}[Z_n] - \frac{q}{2}\mathbb{E}[Z_{n-1}] - q\mathbb{E}[Z_{n-2}] - q\mathbb{E}[Z_{n-3}]$$



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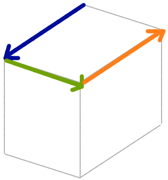
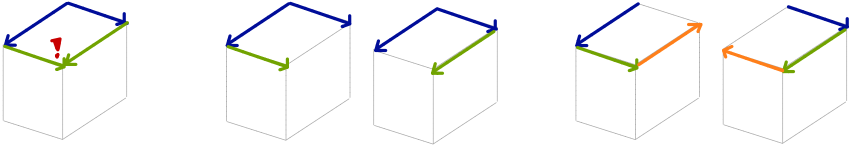


$$\mathbb{E}[\#\text{ greatgrandchildren per indiv.}] = (1+\epsilon)^4$$

All these pairs give collisions with probability

$$(n-1)^{-2} \sim q$$

Refining the cousin mergers model



Process construction

From generation n to $n+1$:

1. Reproduction: Indiv. at generation n have children.
2. Deletions: Keep *authentic* children of v w.p. $(1-q)^{k_v}$
   - where k_v is the number in earlier generations at distance 3 from v.
3. Collisions: Each pair of cousins flip biased coin,
4. Identification: of pairs of cousins.

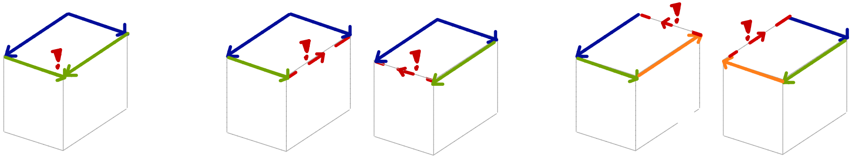
Survival gets the mysterious coefficients!

Refined BP with collisions (Eslava, Penington, S., '20$^+$)

If $\xi_v \sim \mathrm{Poi}(1+\epsilon)$, $\epsilon > 0$ suff. small, then collision prob. q determines

- $\epsilon \leq \frac{5}{2}q - Kq^2$: Extinction w.p. 1
- $\epsilon \geq \frac{5}{2}q + Kq^2$: Survival w. positive prob.

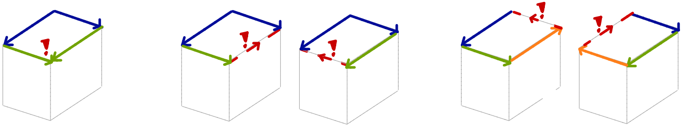
Partial Idea: There are collisions occurring 4 times as often



$$\mathbb{E}[Z_{n+1}] \approx \left(1 + \epsilon \quad -\frac{1}{2}q \quad -\frac{2}{2}q \quad -\frac{2}{2}q\right)(1 + O(\epsilon))\mathbb{E}[Z_n].$$

Summary

- We obtain a survival threshold for a variant of a branching process that mimics hypercube's exploration near criticality



- This sheds light on structures determining its critical probability

Refined BP with collisions (Eslava, Penington, S., '20$^+$)

If $\xi_v \sim \mathrm{Poi}(1+\epsilon)$, $\epsilon > 0$ suff. small, then collision prob. q determines

- $\epsilon \leq \frac{5}{2}q - Kq^2$: Extinction w.p. 1
- $\epsilon \geq \frac{5}{2}q + Kq^2$: Survival w. positive prob.