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Assignment 2
August 2020

Project 2: Modeling the spread of a disease

The idea of this assignment is to make use of SEIR model (with modifications) to predict the spread of a virus that is propagated by direct contact (let us call it Virus-Z). You will be asked to think a bit about this (simple) model, implement it, include new variables and extract conclusions from your simulation. You must write a report answering the questions and including any additional material that supports your study (pictures, codes, etc).

We will start by considering the following SEIR model:

$$\begin{aligned}
\dot{S} &= -\frac{\beta}{N} IS, \\
\dot{E} &= \frac{\beta}{N} IS - \kappa E, \\
\dot{I} &= -\gamma I + \kappa E, \\
\dot{R} &= \gamma I,
\end{aligned}$$

where each variable (S, E, I, R, N) is the amount of individuals in each class: S for susceptible, E for exposed, I for infectious, R for recovered, and N to the total. Notice that $N = S + E + I + R$ is a constant variable.

The model has some parameters: β is the average contact rate ($1/\beta$ is the average time between contacts), γ is the recovery rate ($1/\gamma$ is the average time to recover), κ is the incubation rate ($1/\kappa$ is the average time to incubate). These parameters are obtained from available data.

Notice that:

- You will need to repeat computations for different values of parameters and study the dependence of the output on those parameters. It is recommended that you implement your codes passing parameters in order to make this easy for you!

- Do not attach only numerical answers to the exercise. For example, if the exercise asks for the value of S after 10 weeks, the expected answer is not a number. The expected answer is a short code and explanations that allow me to reproduce the result (and give also the number of course!). As a rule of thumb in any report that contains matematical modelling, it is very important that the reader can reproduce your computations.

- Do not submit any separate file. All the information (explanations, figures, tables, codes, etc) must be included in a single PDF file. Always keep a copy of your codes for later use!
- Do not fill pages with irrelevant informations and large amount of figures without explanations. Everything that you include should have a purpose.

Remark 1. *You can submit this assignment in teams of 1 to 4 people. Each team will be asked to submit a single pdf with the names of all the team members in the first page. You are only allowed to work on this assignment with members of your team.*

1 Prepare initial conditions

Notice that it is very interesting to make our computations independent on the total size of the population. On the one hand, we do not want to repeat the experiments for populations of different sizes. On the other hand, we want to avoid roundoff errors when the population is large. This will be the purpose of the first exercise:

Question 1. *Given any real number $Z > 0$, write the system of ODEs associated to the new (scaled) variables*

$$S_z = ZS, \qquad E_z = ZE, \qquad I_z = ZI, \qquad R_z = ZS.$$

Use this computation to justify the use of initial conditions:

$$S(0) = 100 - \epsilon, \quad E(0) = 0, \quad I(0) = \epsilon, \quad R(0) = 0. \tag{1}$$

Assume that we want to model the spread of a disease for a total population of 14000 *inhabitants, starting with* 7 *infectious cases. Which value of ϵ should we take in* (2)*? Notice that we can do this because the nonlinear term is scaled by N.*

2 First simulation

The first exercise is to choose a numerical integrator and perform some tests. I recommend that you use an explicit integrator for this problem. Solve the equations asking for a Relative tolerance of 10^{-8} and an Absolute tolerance of 10^{-10}.

Question 2. *Assume the unit of t is one week. Use a SEIR model with parameters (obtaining suitable parameters for a new virus is a complex problem itself):*

- *Initially we have 1% of infectious individuals* ($\epsilon = 1$).
- $\beta = 0.96$.
- $\kappa = 1.12$ *(the average incubation period is* 6.25 *days).*
- $\gamma = 0.4$ *(the mean recovery time is* 17.5 *days).*

Compute the evolution of $S(t), E(t), I(t), R(t)$. *What is the value of these quantities after 10 weeks? Show the evolution of the spread for time* $t \in (0, 30)$. *When do we expect to have a maximum for* I? *What happens with the value of* S *at the end? Find an explanation for the last observation.*

Question 3. *Answer again Question 1 assuming that we start with exposed individuals:*

$$S(0) = 100 - \epsilon, \quad E(0) = \epsilon, \quad I(0) = 0, \quad R(0) = 0. \tag{2}$$

What qualitative changes do you observe?

In the following exercise you will be asked to approximate the model using Euler method naively and compare the result. Notice the important of controlling the integration error.

Question 4. *Answer again Question 1 using an Euler method with a constant integration time of* $h = 1$. *Notice that this defines a discrete model*

$$\begin{aligned} S_{n+1} &= S_n - \frac{\beta}{N_n} I_n S_n, \\ E_{n+1} &= E_n + \frac{\beta}{N_n} I_n S_n - \kappa E_n, \\ I_{n+1} &= I_n - \gamma I_n + \kappa E_n, \\ R_{n+1} &= R_n + \gamma I_n, \end{aligned}$$

where $N_n = S_n + E_n + I_n + R_n$. *Does any of the conclusions change significantly?*

3 Impact of the spread

Now we will try to measure (naively) the impact of a disease in a population. To this end, consider the following parameters

- The population has 10 million individuals, and initially only 3 individuals are infectious. Scale this numbers according to (2).
- There is hospital capacity to take care of I_h individuals (in % of total population).

- Around 3%-5% of infections require hospitalization.
- Around 1.5%-2% of hospitalized individuals die instead of recovering.
- Around 45%-55% of infected that require hospitalization and are not able to be treated die.
- We have some vague idea to the parameters of the model $\beta \in (1.2, 1.5)$, $\gamma \in (0.3, 0.5)$ and $\kappa \in (1.4, 2.5)$.

Remark 2. *Notice that the above information gives you a lot freedom (to choose parameters) and uncertainty (in the output prediction). Make your own choices and decisions.*

To model the number of casualties, you can modify the equation for R, and include an additional variable D as follows:

$$\dot{R} = \gamma(1-\mu_h)\min\{\alpha I, I_h\} + \gamma(1-\mu_n)\max\{0, \alpha I - I_h\} + \gamma(1-\alpha)I,$$
$$\dot{D} = \gamma\mu_h \min\{\alpha I, I_h\} + \gamma\mu_n \max\{0, \alpha I - I_h\},$$

where $0 < \mu_h < 1$, $0 < \mu_n < 1$, $0 < \alpha < 1$. Remember to update $N = S + E + I + R + D$ in your code.

Question 5. *Select the parameters μ_h, μ_n, and α according to the previous data.*

Question 6. *Simulate the spread of the disease assuming that there is unlimited hospital capacity, i.e., $I_h = 100$ in your simulations. What is the total number of individuals that will die from this disease?*

Question 7. *Answer again Question 6 assuming that the hospital capacity is limited to attend* 1000 *people in ICU.*

Question 8. *Assume that we have a very bad estimation of ϵ, even orders of magnitude wrong. What can you say about the previous questions in this section?*

4 Measures to mitigate the effect of the disease

Assume that we have more information about the disease:

- 50% of infections are asymptomatic.
- Sanitary equipment allows to reduce by 95% the rate of contagion from a hospitalized individual.

Question 9. *Assume that you can implement a quarantine for those infected individuals that are symptomatic. Assume that you are able to isolate successfully 60% of the non-hospitalized individuals that present symptoms. Justify that this corresponds to replace*

$$\frac{\beta}{N} IS \longmapsto \frac{\beta}{N}(c_1 I + c_2 \min\{\alpha I, I_h\})S$$

in the previous model and find how to compute c_1 and c_2 (c_2 can be negative). Repeat the computations of Questions 6 and 7 according to this.

Question 10. *Rather than implementing a quarantine for those symptomatic individuals, assume that general social distance measures are introduced. Assume that you can reduce the mobility of all susceptible, exposed and non-hospitalized individuals by a certain %. What is the value of the rate required to get the same results as in Question 9? Notice that you do not need to perform any simulation here.*

Finally, assume that we can protect individuals against the disease (i.e. if there is a vaccine available):

- $\sigma = 0.25$ (the average period required to get immunity is 1 month).

To model the number of vaccinated individuals V, we can modify the equations for S and V as follows

$$\begin{aligned} \dot{S} &= -\frac{\beta}{N} IS - \delta_n(t)\sigma S, \\ \dot{V} &= \delta_n(t)\sigma S. \end{aligned}$$

We use a step function $\delta_n(t) = \{0 \text{ if } t \neq n, 1 \text{ if } t > n\}$ to take into account that the vaccine is available only after the n-th week.

Question 11. *Assume that you do not take any measure (regarding quarantine or social distance). Study the dependence of the result on n, the week where the vaccine is available. For which value of n it would be too late?*

5 Conclusions

Now it is the moment to summarize, extract your conclusions and reflect about them. Notice that this is not an exercise about epidemiology or political decisions, but about mathematical modeling. This means that you are asked to reflect about how this model is useful and about its limitations.

Question 12. *Play combining different scenarios and measures. Extract your own conclusions from this simulations and the use of this kind of models. What can we understand using this model? Describe the limitations of these kind of models (SIR, SEIR, etc).*

Question 13. *We have an important last minute update. It turns out that those individuals that die because of Virus-Z are not really dead. They become Zombies who try to eat S individuals! In order to check that we understand how SEIR-models work we are asked to modify the model taking into accound that we have another source of infection: D bites S so that S becomes E.*