The Hénon Map and its Strange Attractor

An Example of Two-Dimensional Discrete Dynamical System Exhibiting Chaotic Behavior

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Contents

Chapter 1

Dynamical systems that exhibit chaotic behavior

Discrete dynamical systems are mathematical models used to describe the evolution of systems over time. They are characterized by a set of state variables that change according to a set of rules, or equations. These systems can exhibit a wide range of behaviors, from simple and regular to complex and chaotic. In this paper, we focus on chaotic dynamics of the Hénon map, a widely studied mathematical model that exhibits chaotic behavior. We will explore the properties of the Hénon map and its behavior.

1.1 Order and chaos in discrete dynamical systems

Chaos has traditionally been thought of as a state in which natural systems do not obey laws. However, observations of natural systems show that there is a well-defined path from order to chaos. Several dynamical systems can produce chaos, and we focus on the quadratic transformation in particular because it illustrates many key phenomena that occur in dynamical systems. Additionally, the properties of chaos can be fully analyzed mathematically using the quadratic transformation.
To introduce our topic, we begin by studying chaos in two-dimensional dynamical systems. In particular, we are considering dissipative dynamical systems, in which there is a loss of energy, because in these systems we can observe strange attractors. Indeed, the long-term behavior of dissipative systems could run into complex patterns that show signs of chaos.

1.2 Attractors

The focus of this discussion is an important example of the behavior of dynamical systems in multiple dimensions: **attractors**. An attractor is a set toward which most other points tend under iteration. More precisely, an attractor is a set of states of a dynamic system that the system approaches asymptotically. From a geometrical perspective, we can see it as a region of state space toward which all trajectories of a set of initial states converge.

1.2.1 Strange attractors

Within the complex patterns that we can find in the long-term behavior of dynamic systems, there are the so-called **strange attractors**. We call them *strange* because, although they have fractal geometric patterns, they are chaotic as dynamical objects. For this reason, strange attractors are not well-defined mathematical notions. One of the many fascinating facts about strange attractors is that they provide us with a new understanding of non-linear effects. The first strange attractor was discovered in 1962 by Edward Lorenz, an American mathematician and meteorologist.
We focused our attention on the Hénon's attractor, a strange attractor suggested by Michel Hénon (a French mathematician and astronomer) in 1976 as a simplified model for the dynamics of the Lorenz system.
A first attempt at a formal definition of a *strange attractor* might be:

Definition (Strange attractor, [1])**.** Let $T(x, y)$ be a given transformation in the plane and let A be a bounded subset of the plane. A is a **strange attractor** for the transformation T if there exists a set R with the following properties.

1. A is an *attractor*. An attractor is an invariant set to which all nearby orbits converge. [2]
2. A is a *chaotic attractor*: orbits starting in R show a sensitive dependence on the initial conditions.
3. A is a *strange attractor*: the attractor has a fractal structure.
4. A cannot be split into two different attractors: there are starting points in R with orbits that arbitrarily approach any point of A.

Chapter 2

Hénon map and its strange attractor

We can define the Hénon transformation H as a two-parameter family of maps of the plane:

$$H(x, y) = (y + 1 - ax^2, bx) \qquad \text{with } a, b \in \mathbb{R}$$

Despite this map is one of the simplest nonlinear maps in higher dimensions (it has only one nonlinear term), there are many parameter values for which the map is still not well understood.
We can create an orbit of this system with a starting point (x_0, y_0) and its iterated images:

$$\begin{cases} x_{k+1} = y_k + 1 - ax_k^2 \\ y_{k+1} = bx_k \end{cases} \qquad k \in \mathbb{N}, \quad a, b \in \mathbb{R}$$

The dynamic of the system depends both on the choice of a, b and on the choice of the starting point.

Decomposition of the Hénon transformation

If we partition the transformation H into three steps, we can see how it works: it starts with a non-linear bending in the y-direction ($H_1(x, y) = (x, y + 1 - ax^2)$), then follows a contraction in the x-direction ($H_2(x, y) = (bx, y)$) and eventually a reflection at the diagonal ($H_3(x, y) = (y, x)$). Hence, we have that

$$H(x, y) = H_3(H_2(H_1(x, y)))$$

We can see step-by-step transformation in Figure 2.1.

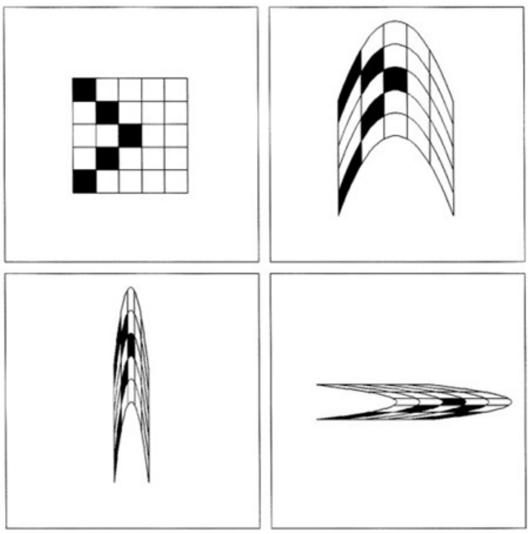

Figure 2.1: Decomposition of H transformation, image from [1]

2.1 Hénon strange attractor

We now consider the Hénon map with $a = 1.4$ and $b = -0.3$:

$$\begin{cases} x_{k+1} &= 1 + y_k - 1.4x_k^2 \\ y_{k+1} &= 0.3x_k \end{cases}$$

After some iterations it can be noticed its strange behavior (Figure 2.2): trajectories appear to randomly skip around an attractor. In fact we can verify that, given an infinite set of points representing possible initial conditions, these all approach the attractor asymptotically after repeated iterations of this map.

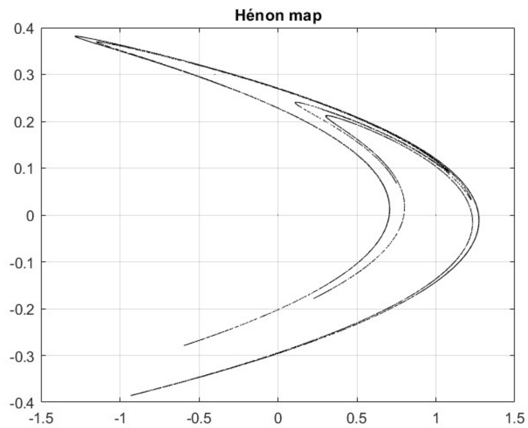

Figure 2.2: The first 10000 iterations of the Hénon map with $a = 1.4$ and $b = 0.3$

2.2 Fixed points

In two-dimensional discrete dynamical systems, a fixed point is a point that does not change under the action of the system. Formally, this can be defined as a point (x, y) in the plane that satisfies the equation

$$f(x, y) = (x, y)$$

where f is the function that defines the dynamics of the system. In other words, a fixed point is a point that remains unchanged after one iteration of the system.
Hence the fixed points for the Hénon map satisfy

$$\begin{cases} \overline{x} = 1 - a\overline{x}^2 + \overline{y} \\ \overline{y} = b\overline{x} \end{cases}$$

from which we can get

$$a\overline{x}^2 + (1 - b)\overline{x} - 1 = 0.$$

This quadratic equation has two real roots for $\Delta = (1 - b)^2 + 4a > 0$, so in this case we have the two fixed points and they are

$$\begin{cases} \overline{x}_1 = \frac{-(1-b)+\sqrt{(1-b)^2+4a}}{2a} \\ \overline{y}_1 = \frac{-(1-b)+\sqrt{(1-b)^2+4a}}{2a} \end{cases} \qquad \begin{cases} \overline{x}_2 = \frac{-(1-b)-\sqrt{(1-b)^2+4a}}{2a} \\ \overline{y}_2 = \frac{-(1-b)-\sqrt{(1-b)^2+4a}}{2a} \end{cases}$$

2.2.1 Analysis of fixed points

Existence of fixed points for $b = 0.3$

The study of the discriminant Δ allows us to see whether there are fixed points. In fact, we have no fixed points if $\Delta < 0$ while we have two fixed points if $\Delta \geq 0$ (the two fixed points are coincident if $\Delta = 0$).
We then have the following condition for the existence of fixed points:

$$(1 - b)^2 + 4a > 0 \quad \Leftrightarrow \quad a > a_0 = -\frac{(1 - b)^2}{4} = -0.1225$$

a_0 is the threshold value that divides the two cases we can have: when a grows above a_0 there are two fixed points while for values of a lower than a_0 there are no fixed points.

Stability of fixed points for selected values of a and b

For $a = 1.4$ and $b = 0.3$, we have the fixed points $A = (0.6314, 0.1894)$ and $B = (-1.1314, -0.3394)$, as we can see in Figure 2.3.
To classify the fixed points of a nonlinear discrete dynamical system we use the **linearization method**. Therefore we start from the following discrete system

$$\begin{cases} x(t+1) = y(t) + 1 - ax^2(t) \\ y(t+1) = bx(t) \end{cases} \quad \rightarrow \quad \begin{cases} x(t+1) = y(t) + 1 - 1.4 \cdot x^2(t) \\ y(t+1) = 0.3 \cdot x(t) \end{cases}$$

and we compute its Jacobian matrix:

$$J_H = \begin{bmatrix} 2ax(t) & 1 \\ b & 0 \end{bmatrix} \quad \rightarrow \quad J_H = \begin{bmatrix} 2.8 \cdot x(t) & 1 \\ 0.3 & 0 \end{bmatrix}$$

Evaluating J_H in the fixed point A we get:

$$J_H(A) = \begin{bmatrix} 1.7679 & 1 \\ 0.3 & 0 \end{bmatrix}$$

We now apply the theory of two-dimensional discrete dynamical systems (asymptotic stability test based on trace and determinant):

$$\begin{cases} |\text{tr } J_H| = 1.7679 < (1 + \det J_H) = 0.7 \\ \det J_H = -0.3 < 1 \end{cases} \quad \Rightarrow A \text{ is stable}$$

Now we consider the Jacobian matrix evaluated in B:

$$J_H(B) = \begin{bmatrix} -3.1679 & 1 \\ 0.3 & 0 \end{bmatrix}$$

Applying a criterion based on the study of the trace we obtain

$$|\text{tr } J_H| = 3.1679 > n = 2 \quad \Rightarrow B \text{ is unstable}$$

where $n = 2$ is the order of the system, i.e. the size of the state vector (in this system we have two variables in the state vector: it is a two-dimensional system).
A is located within the attractor itself and is *stable*. This means that if a point is plotted close to A, the next iterated points will stay close to the fixed point.
This can be seen in the graph on the right, where the previous iteration is plotted together with the current iteration. In this graph we can see precisely how the next iteration of the points close to A stays close to A.
The second fixed point, B, is *unstable* and it is located outside of the bounds of the attractor. An unstable fixed point is not stable under the dynamics of the system, meaning that small perturbations to the initial conditions of the system can cause the point to move away from its original position, and the system will not return to that point in the future.

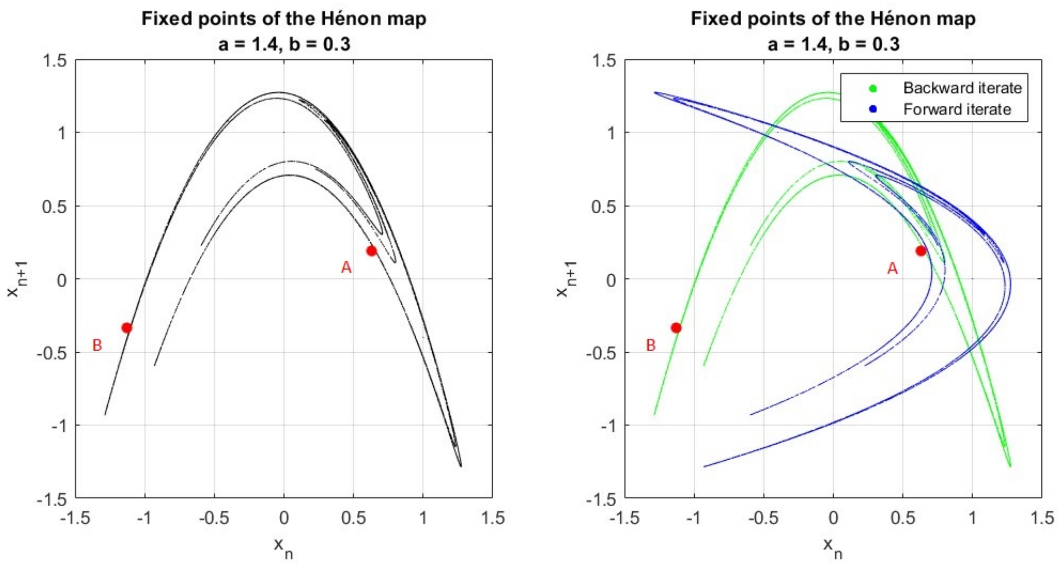

Figure 2.3: Fixed points for Hénon map with $a = 1.4$ and $b = 0.3$

2.3 Trapping Region and Basin of Attraction

Iteration of H will not lead to the attractor for any randomly chosen starting point: each attractor has a trapping region within which all starting points have orbits leading to the attractor.
If we start iterating for example from $(x_0, y_0) = (3, 0)$, we can see that the orbit diverges rapidly.

$$\begin{array}{ll} x_0 = 3 & y_0 = 0 \\ x_1 = -11.6 & y_1 = 0.9 \\ x_2 = -186.484 & y_2 = -3.48 \\ x_3 = -48689.275 & y_3 = -55.945 \\ x_4 = -3318903776.576 & y_4 = -14606.783 \end{array}$$

We can define a *trapping region* R from which no orbit can escape. The orbits starting from the trapping region must converge to some limit set.

Definition (Trapping region, [2])**.** A closed region $R \subset \mathbb{R}^n$ is a trapping region for H if $H(R)$ is contained in the interior of R.

We can define the trapping region R for the Hénon's transformation H as the quadrilateral with vertices $P_1 = (1.32, 0.133)$, $P_2 = (1.32, 0.133)$, $P_3 = (1.245, -0.14)$ and $P_4 = (-1.06, -0.5)$ (calculated explicitly by Hénon and reported in his paper [3]).
The image of the region R obtained from one application of H does lie entirely within the trapping region:

$$H(R) \subset R$$

It follows that by repeatedly applying H we always obtain subsets of the region, in this sense no orbit can escape from the trapping region and also the Hénon attractor belongs to it.
We can also define

$$A = \bigcap_{k=0}^{\infty} H^k(R)$$

where H^k means the $k-$fold composition of H.
Therefore, on one side we have escaping points and on the other we have a trapping region which leads to the attractor. The set of all points which have orbits that are eventually caught by the trapping region is called **basin of attraction** of A.

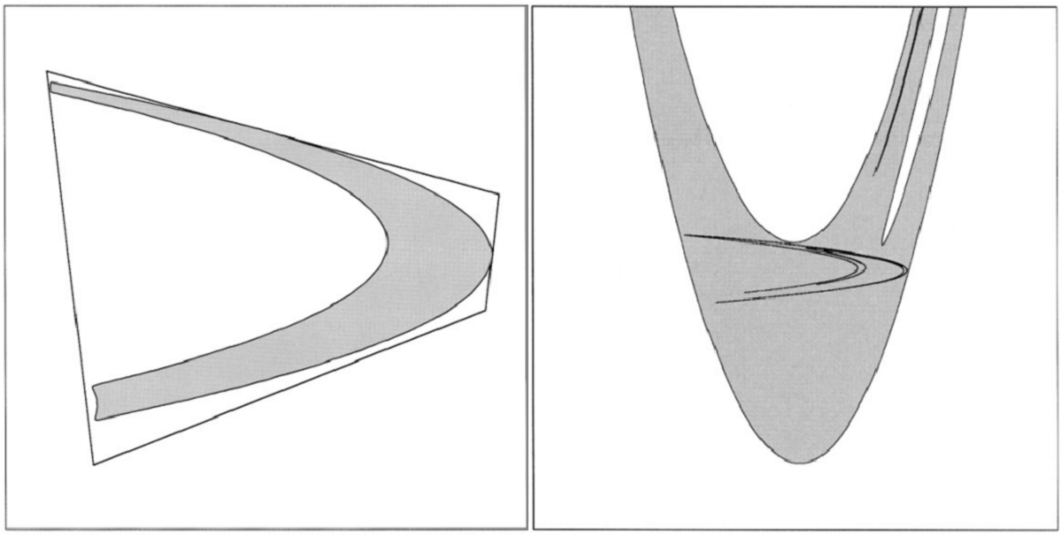

Figure 2.4: Trapping Region (on the left) and Basin of Attraction for Hénon's attractor (on the right), images from [1]

In Figure 2.4 (left) we can see how the area of the quadrilateral reduces by the factor $b = 0.3$ when the transformation is applied. Analytically, we can obtain that an initial area of 1 reduces to 0.3^k after k iterations. Thus the attractor A can only cover a subset of the plane with an area of 0 (this is because it must reside in all iterations of the region). On a general level, we call $|\det J_H|$ the previously calculated modulus of the determinant of the Jacobian matrix of H:

$$|\det J_H| = |b|$$

We have that $|\det J_H|$ is the reduction factor of the small portion of area close to a generic point $P = (x, y)$. In fact, the Hénon transformation is not linear but if we linearize the transformation we can obtain results that hold locally. Since in this case $|\det J_H|$ is a constant which does not depend on the location of P, the area changes uniformly by that factor.

2.4 Lozi's Piecewise Linear Model

The above definition of *strange attractor* does not negate speculations such as that the experimental observations are due to an attractive periodic orbit with a very long period.
The confirmation that the Hénon attractor is a strange attractor was provided by Michał Misiurewicz who proved an earlier conjecture by René Lozi.
Lozi suggested that the trasformation $\tilde{H}$, defined as

$$\tilde{H}(x, y) = (1 + y - a|x|, bx) \qquad a, b \in \mathbb{R},$$

admits a strange attractor for parameters $a = 1.7$ and $b = 0.5$ (Figure 2.5).

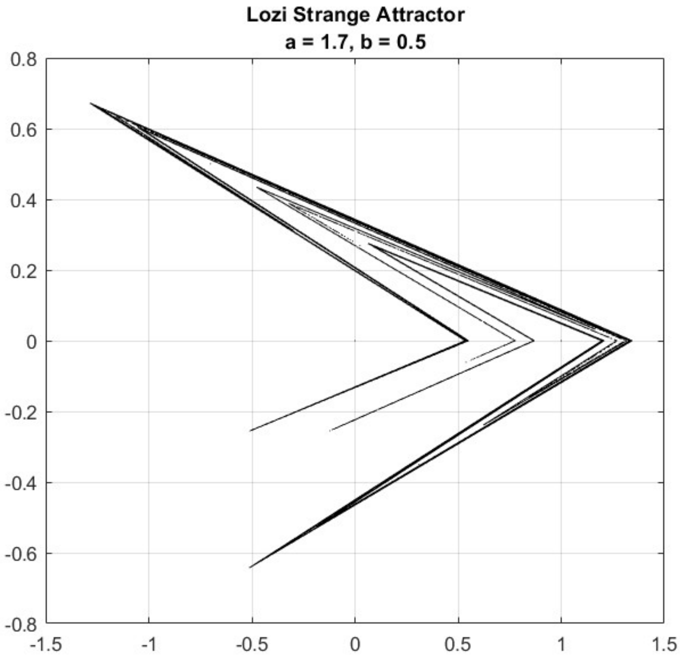

Figure 2.5: The Lozi Strange Attractor

This trasformation is very similar to the Hénon trasformation: the only difference between them is that the x^2 term is replaced by $|x|$. This change makes the transformation linear for $x > 0$ and $x < 0$ so it allows Misiurewicz to complete his proof.

2.5 Behavior of the Hénon attractor

2.5.1 Self-similarity

In Figure 2.6 we have a close-up sequence in the region $0.6 \leq x \leq 0.75$ of the Hénon map for $a = 1.4$ and $b = 0.3$. It can be seen that as we continue to enlarge the region, the lines that form the structure of the Hénon attractor always assume the same arrangement: we can say that this attractor has a self-similar structure. Furthermore, we can see that these lines become layers of increasingly blurred lines that seem to resemble the Cantor set.

Definition (Self-similarity, [1])**.** A structure is said to be **self-similar** if it can be broken down into arbitrarily small pieces, each of which is a small replica of the entire structure.

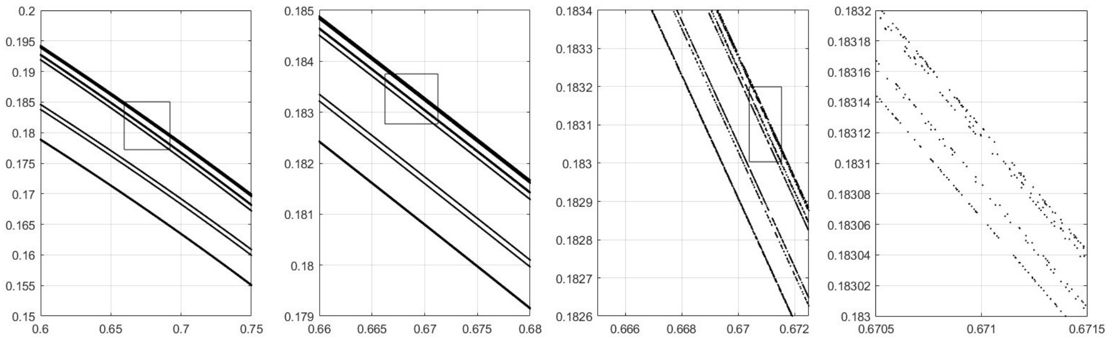

Figure 2.6: Self-similarity of the Hénon map

It is important to note that the small pieces of a fractal can be obtained from the entire structure by a similarity transformation: when we zoom in on the image, we get an identical, smaller structure, and if we continue to zoom in, we will always see the same structure reappear.
However, the concept of self-similarity is not exclusive to fractals: it would be a mistake to believe that if a structure is self-similar, then it is necessarily a fractal. In fact, there are examples of structures that are not fractals but can be broken into small copies that are obtained by similarity transformations (for example, a line segment, a square, or a cube). On the other hand, some fractals demonstrate self-similarity in its purest form: many fractals can be completely characterized and defined by their self-similarity properties. In the next section, we will see why the attractor discussed in this paper can be considered a fractal.

2.5.2 Particular characteristics that make the Hénon attractor a *strange attractor*

This attractor is called *strange* due to the sensitive dependence on initial conditions coupled with its fractal structure.

Sensitivity on initial conditions

To obtain an image of the attractor, it is sufficient to calculate a single orbit of an initial point. In fact, the visual result does not change if we choose a different random starting point and ignore the first few iterations until the orbit is close enough to the attractor. For almost all arbitrarily chosen random points close to the initial value, there is no correlation between two different orbits, even if they generate the same limit set and even if the initial points are very close to each other. The

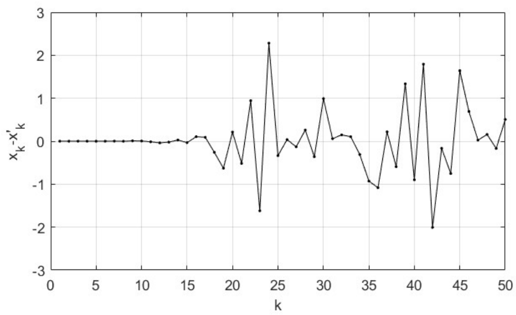

Figure 2.7: Sensitivity of the Hénon map

sensitive dependence on initial conditions can be seen choosing two initial points very close to each other and observe the orbits they generate: even if the orbits have the same limit set, there is no correlation between them.
In Figure 2.7 we took two sets of iterations of the Hénon transformation with slightly different starting points ($x_0 = (0, 0)$ and $x'_0 = (0.0001, 0.0001)$) and plotted the difference $x_k - x'_k$ for each k. It can be seen that, after a few iterations, the difference between the two series increases rapidly and without any particular pattern or periodicity.

Fractal structure

Through the analysis of the conformation of the Hénon attractor, it appears to be formed by an infinite number of parabola-like layers. If we assume that at least one parabola-like curve exists in the attractor, we can see that every iteration generates new layers that belong to the attractor.
Due to the folding in the transformation, a cross-section of the attractor looks similar to a Cantor set: sections of the Hénon attractor are cross products of an interval with a Cantor set. In Figure 2.8 we can see two successive enlargements of the Hénon attractor that show the Cantor-like structure of parallel lines. The attractor is definitely a fractal.

Definition (Cantor-like set, [4])**.** A nonempty set $E \subseteq \mathbb{R}$ is called a **Cantor-like set** if

(i) E is closed and bounded;

(ii) E is totally disconnected, that is it contains no nontrivial interval;

(iii) E has no isolated point, that is if $p \in E$, then $\forall\, r > 0, \quad (E \setminus \{p\}) \cap (p - r, p - r) \neq \emptyset$.

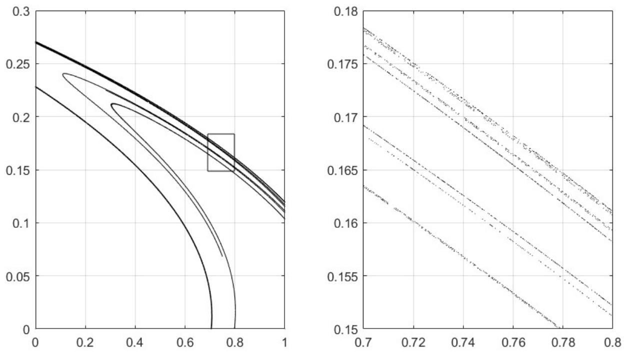

Figure 2.8: Enlargements of the Hénon attractor revealing its Cantor-like structure

2.5.3 Final-state diagram

To see the final state of the iteration for fixed a and b we have chosen an initial value (x_0, y_0) and carried out 10000 iterations (computing $x_1, \ldots, x_{10000}$). Then we plotted the final 9000 iterations in order to visualise the strange attractor to which they converge. With a slight variation of the starting point, we can see that the points produced by the last 9000 iterations are different even if the shape of the attractor remains unchanged. This derives from the high sensitivity of the attractor to the initial conditions and is a clear sign of chaos.

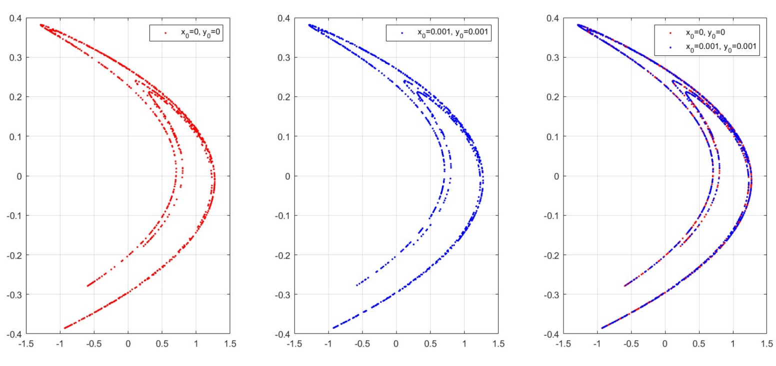

Figure 2.9: Comparison of the last 9000 iterations of the Hénon transformation for two different initial points

2.6 Bifurcation diagrams

Let us now study how the system evolves when we change the parameters a and b of Hénon's map.
A bifurcation diagram is a graph that shows the evolution of a dynamic system as a parameter is varied and can be used to visualize the different states that the system can exhibit. In the case of the Hénon map, the bifurcation diagram can be used to visualize how the behavior of the system changes as the parameter is varied. For example, the bifurcation diagram could show how the number or stability of fixed points changes as the parameter is varied. The bifurcation diagram can also show the appearance of bifurcations, where the behavior of the system changes dramatically, such as the emergence of chaotic behavior. By examining the bifurcation diagram of the Hénon map, we can gain insight into the behavior of the system and how it changes as the parameter is varied.

2.6.1 Bifurcation diagram for a when $b = 0.3$

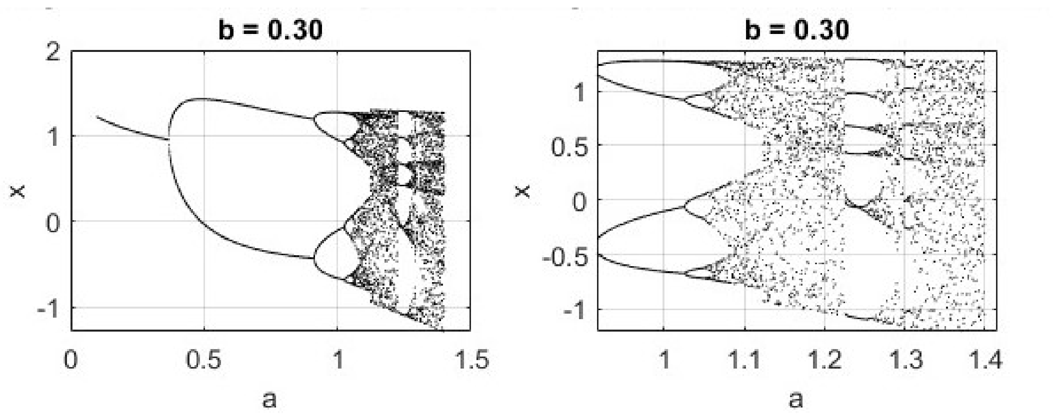

Figure 2.10: Bifurcation diagram for Hénon map with $b = 0.3$

The diagram (Figure 2.10) shows the bifurcation phenomenon affecting this dynamical system as a function of the parameter a: it shows all the possible configurations that our system can assume (the values of x that the system can take). We have plotted a series of convergent values of x for different values of the parameter a by fixing the value of $b = 0.3$ and the initial condition $x_0 = y_0 = 0$.
For $0 \leq a \leq 1$ we can observe the so called *period-doubling bifurcations*. We have in fact that a slight variation in the parameters of a system causes a new periodic trajectory to emerge with a doubled period from the original. We can also see how increasing the value of a leads into a period-doubling cascade (i.e. an infinite sequence of period-doubling bifurcations).

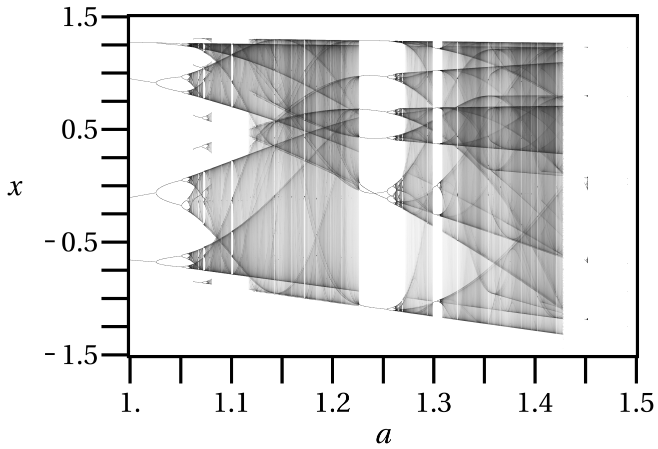

Figure 2.11: Hénon bifurcation diagram with $b = 0.3$ created by Jordan Pierce with Mathematica

In Figure 2.11 we can see a more clearer bifurcation diagram with focus on the region $1 \leq a \leq 1.4$, made with a symbolic mathematical computation program (Wolfram Mathematica).
The darker region indicates a higher density of points, therefore a greater probability that the variable x acquires that value for the given value of a.
A stable periodic orbit in a bifurcation diagram can be recognized by the presence of a closed loop in the diagram. This

loop indicates that the system undergoes repeated cycles of behavior, with each point on the loop corresponding to a different state of the system. The stability of the periodic orbit can be determined by the shape of the loop: a loop that is closed and not intersecting itself indicates a stable periodic orbit, while a loop that is open or intersecting itself indicates an unstable periodic orbit. In particular, this bifurcation diagram shows that for $a = 1.25$, the Hénon map has a stable periodic orbit as an attractor.

2.6.2 Bifurcation diagram for b when $a = 1.4$

If we fix the value of b and increase the parameter a, we can obtain complex dynamics of H (Figure 2.12): for a small there are no periodic points, while on the contrary for a large there are infinitely many.

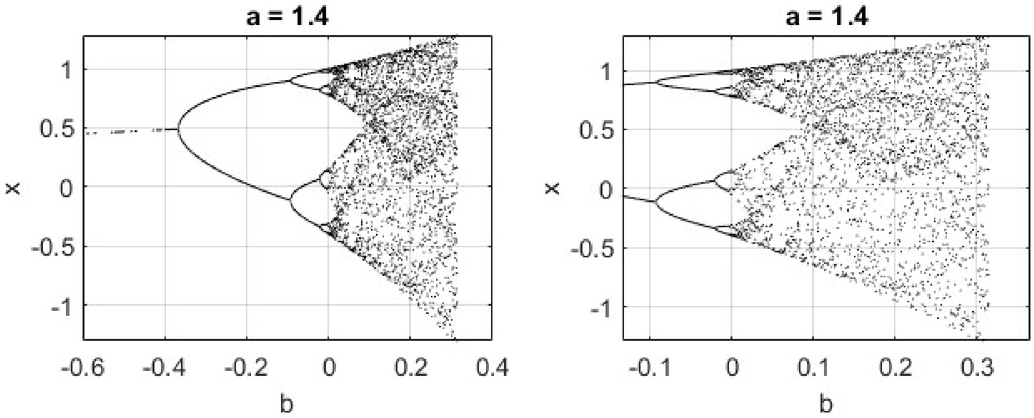

Figure 2.12: Bifurcation diagram for Hénon map with $a = 1.4$

2.7 Some of the possible evolutions of the Hénon map with *a* fixed and *b* sliding in an interval

In Figure 2.13, we have selected some of the many possible configurations that the Hénon map can assume.[1]
To produce these images, we plotted the first 10000 iterations of the Hénon map for each value of parameter *b* in the given range (we chose 100 equispaced points for each range).
The intervals in which *b* is selected are carefully chosen to prevent the map from diverging to infinity.

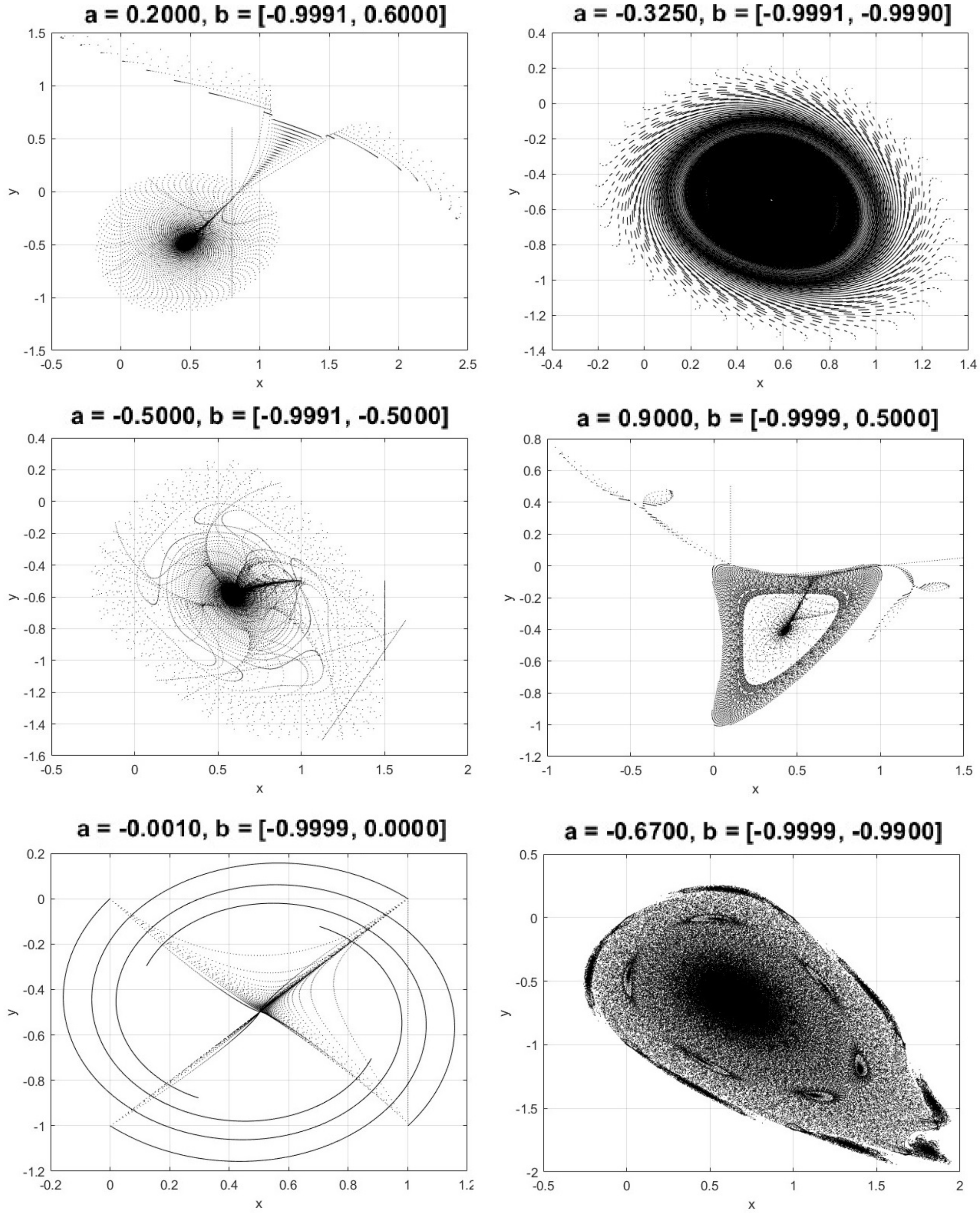

Figure 2.13: Some of the possible evolutions of the Hénon map with *a* fixed and *b* sliding in an interval

[1]Here is a video showing how the map evolves as *b* changes: `https://youtu.be/u2EpN9EX4Qo`

2.8 Hénon map with different values of *a* and *b*

In Figure 2.14 we have plotted the Hénon map for selected values of parameters *a* and *b* (*b* is now fixed and does not slide in an interval). We can see how this map can take several fascinating forms depending on the selected parameter.

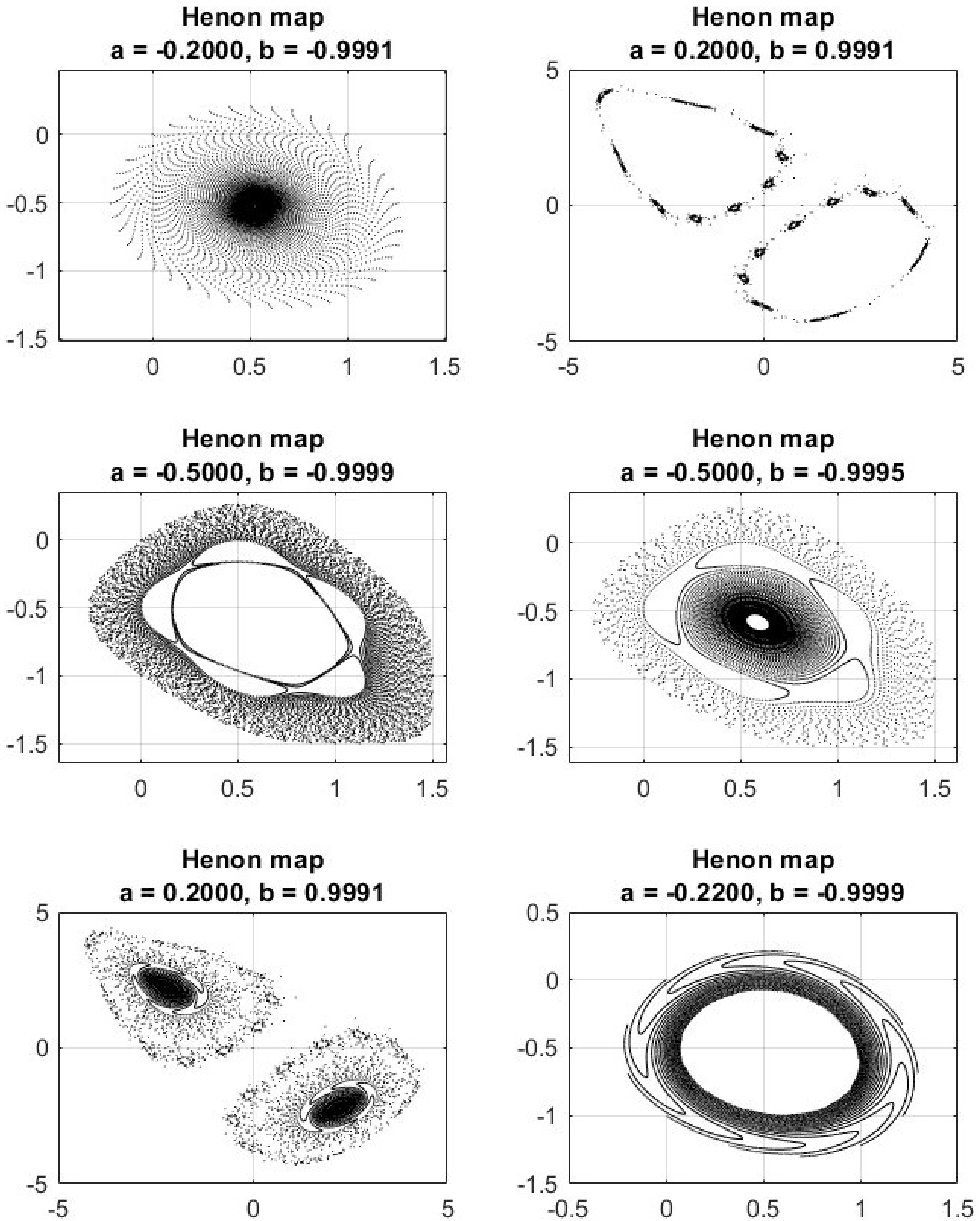

Figure 2.14: Hénon map with different values of *a* and *b*

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