CE 5155
Finite Element Analysis of Structural Systems

Lecture 1

- Overview of Finite Element Method and Historical Background
- Overview of Application of FEM

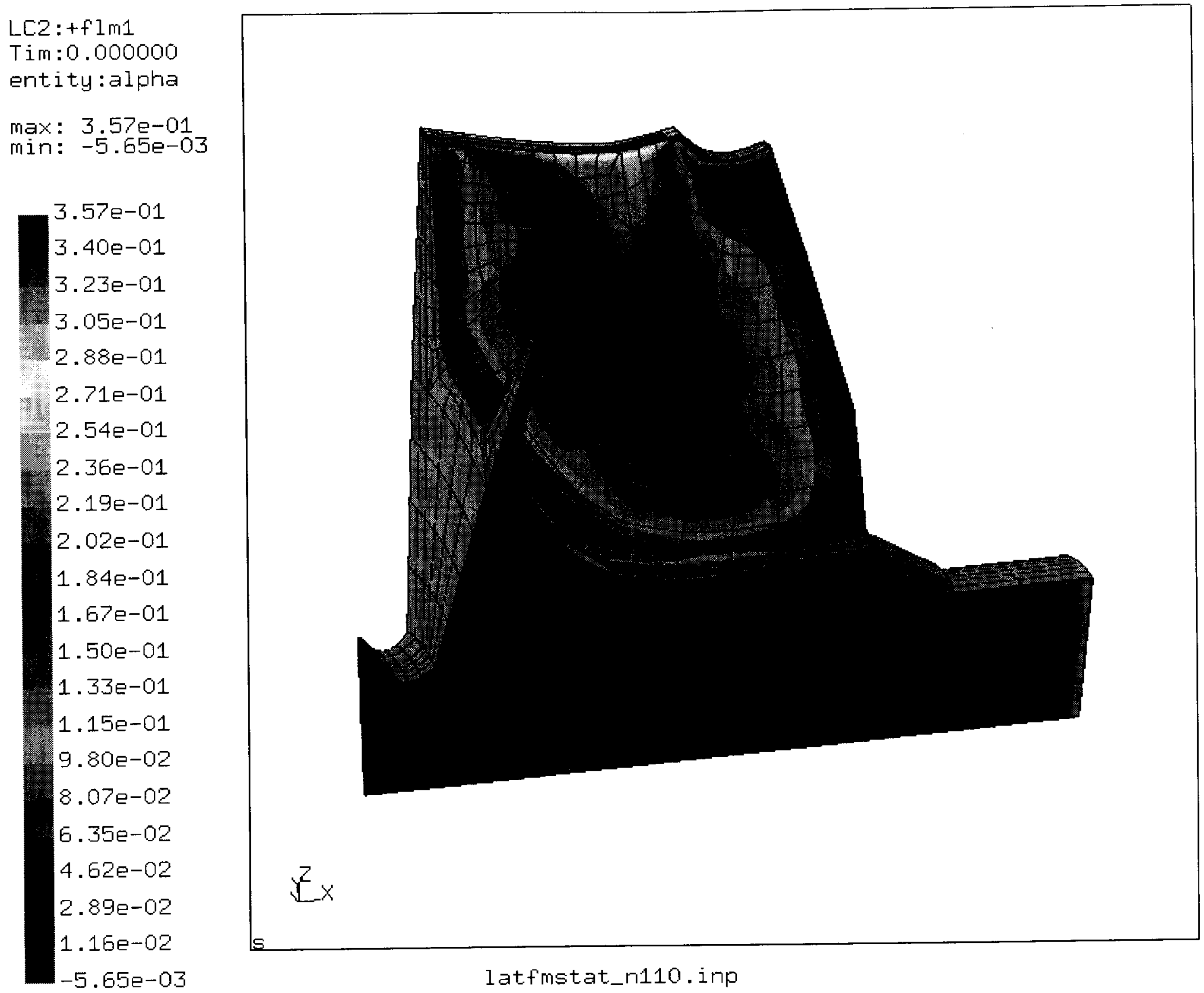
Review of Continuum Mechanics
- Definitions of Stresses & Strains
- Equations governing 1-D Elasticity Problems
- Equations governing 2-D Elasticity Problems.

Overview of Finite Element Method

Finite Element Method is a method for obtaining numerical solutions to mathematical equations governing problems of interest such as, Problems of stress analysis, heat transfer, fluid flow, electric fields etc.

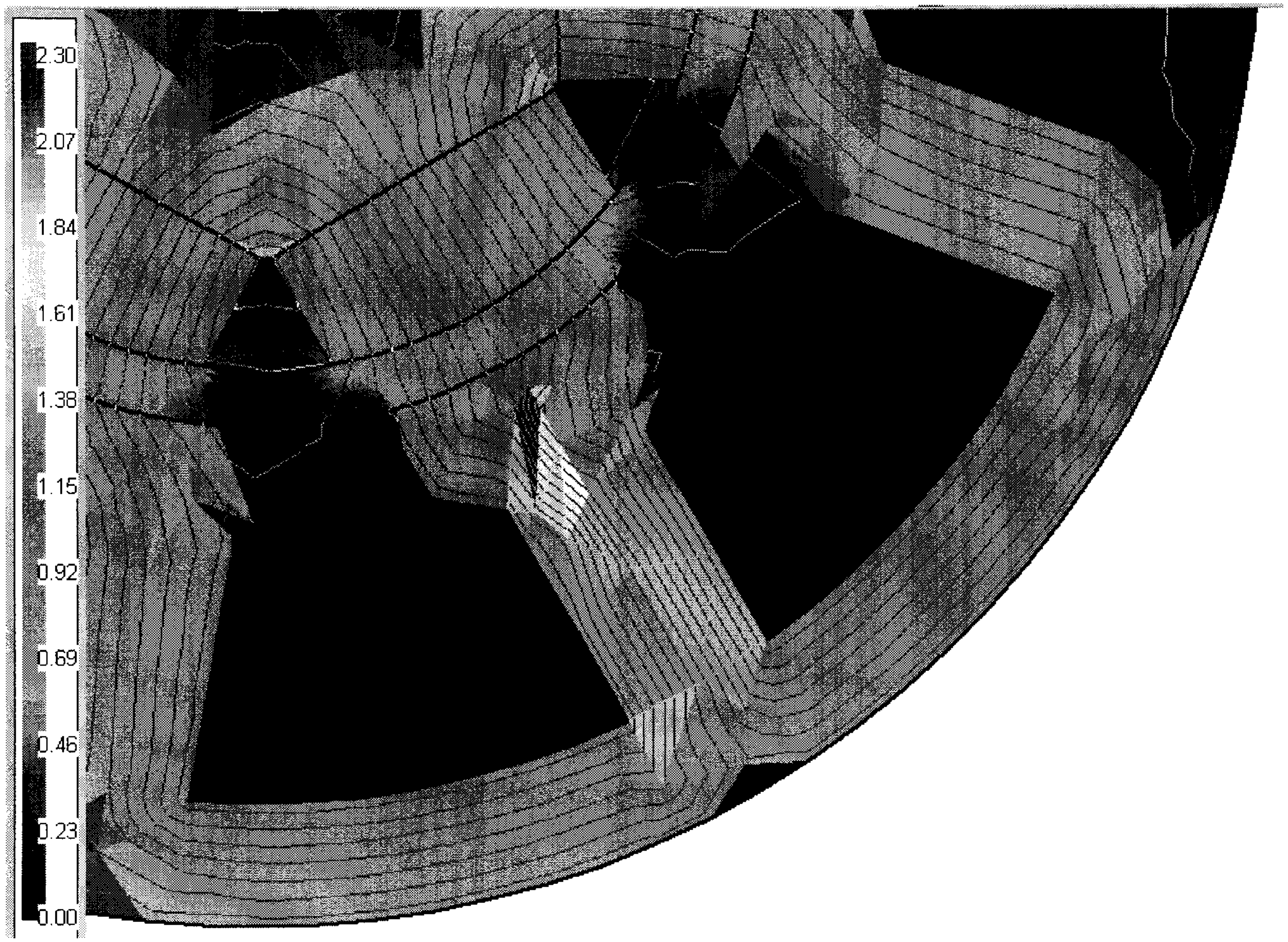
The common feature in finite element analysis of problems of all types is discretization of the spatial domain of the posed problem into "finite sized elements". Piecewise approximation of function of interest (ϕ) over the domain of the problem is obtained by means of polynomials, each defined over a finite element and expressed in terms of nodal values of the function at element nodes.

Application in Heat Transfer



Thermal Analysis of a Turbine Blade

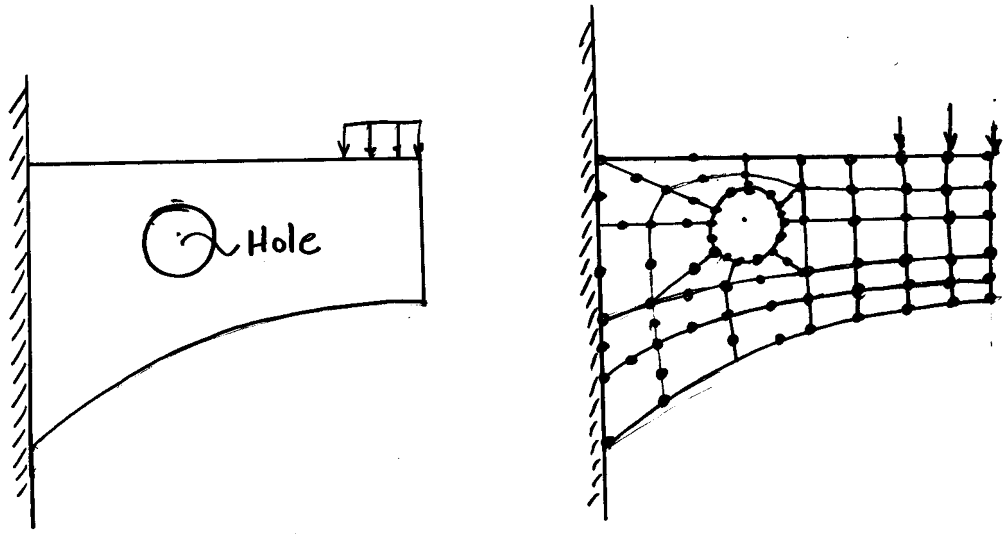
Application in Electromagnetics



FE Analysis of Flux in an Electric Motor

Versitility of FEM

The mathematical eqs. governing a problem can be complex and finding a closed form exact solution that satisfies the governing eqs. over the entire complex domain of the problem is impossible in many cases.



A Problem with complex domain and its discretization.

- Problems with complex domains, boundary condition and load patterns can be solved.

- shortcoming: The numerical results obtained are not closed form and are specific to the problem solved. A change in parameters of the problem such as geometry, loading boundary conditions, require a complete re-analysis of the problem.

History of FEM

- Proposition of "Lattice Analogy" to solve problems of Continuum Mechanics in 1906.

- Presention of idea of piece-wise polynomial interpolation over triangular subregions to obtain approximate numerical solutions by Courant in 1943
"R. Courant, "Variational Methods for solution of Problems of Equilibrium and Vibrations", Bulletin of the American Mathematical Society, Vol 49, 1943, pp 1-23"

- Courant's work was forgotten till engineers developed the method independently.

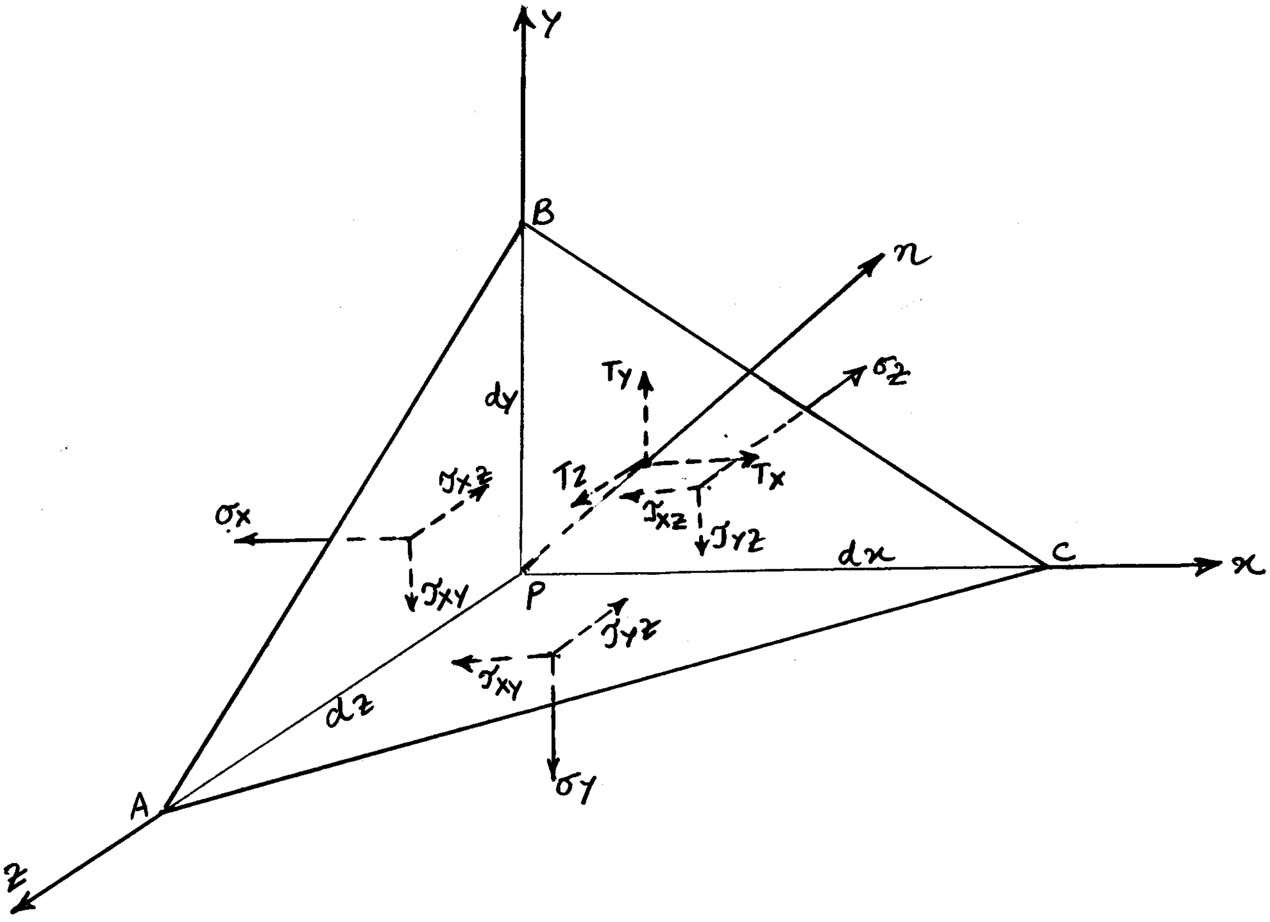
- Arival of digital computers in 1953 and usage of the stiffness method to solve problems

- Classic Paper by Turner, Clough, Martin & Topp in 1956 in which the term "Finite Element" was used for the first time.
" M.J Turner, R.W. Clough, H.C Martin and L.J. Topp, "Stiffness and Deflection Analysis of Complex Structures, Journal of Aeronautical Sciences, Vol. 23 No. 9, 1956 pp 805-823.

- Important Contributions were later made by J.H. Argris, O.C Zienkiewicz, and Y.K. Cheung. in 1960s

Review of Continuum Mechanics

Three-Dimensional Stress at a Point



Consider a small Tetrahedron of a continuum as shown. TX, TY, TZ are Cartesian components of Traction force T acting on Area ABC. It is required to relate the stresses acting on perpendicular planes intersecting at origin P to the applied tractions acting on Area ABC

The orientation of Plane ABC may be defined in terms of the angle between the unit normal "n" to the plane and the x, y, z directions.

$$\cos(n, x) = l$$
$$\cos(n, y) = m$$
$$\cos(n, z) = n$$

The areas of Perpendicular planes PAB, PAC & PBC may now be expressed in terms of area ABC and the direction cosines:

$$A_{PAB} = A_x = \vec{A} \cdot \vec{i} = A(l i + m j + n k) \cdot i = A l$$

$$A_{PAC} = A_y = \vec{A} \cdot \vec{j} = A(l i + m j + n k) \cdot j = A m$$

$$A_{PBC} = A_z = \vec{A} \cdot \vec{k} = \quad = A n$$

From Equilibrium of Forces in x, y, z directions we have

$$\Sigma F_x = 0 \Rightarrow T_x \cdot A = \sigma_x A l + \tau_{xy} A m + \tau_{xz} A n$$

$$\Rightarrow \underline{T_x = \sigma_x l + \tau_{xy} m + \tau_{xz} n}$$

$$\Sigma F_y = 0 \Rightarrow T_y \cdot A = \tau_{xy} \cdot A l + \sigma_y \cdot A m + \tau_{yz} A n$$

$$\underline{T_y = \tau_{xy} l + \sigma_y m + \tau_{yz} n}$$

$$\Sigma F_y = 0 \Rightarrow \underline{T_z = \tau_{xz} l + \tau_{yz} m + \sigma_z n}$$

Summarizing we have:

$$\boxed{\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}}$$

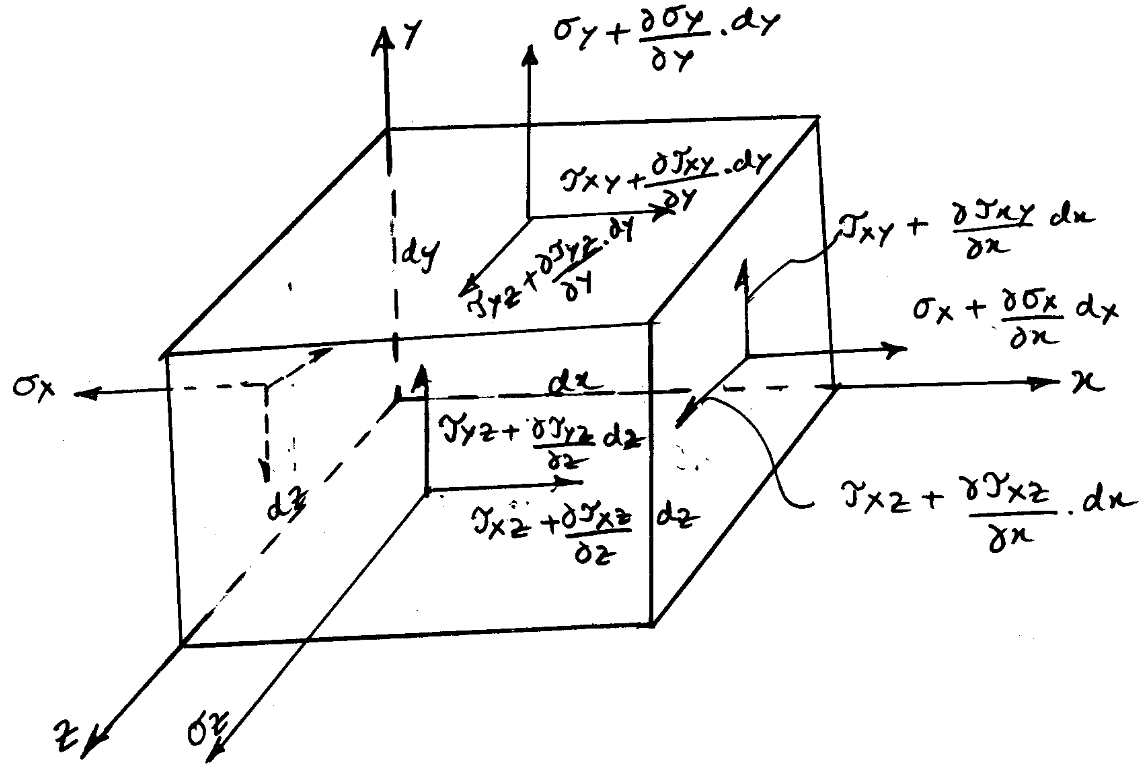
or in indicial Notation Form

$$\boxed{T_i = \sigma_{ij} n_j} \qquad i = 1,2,3 \quad , j = 1,2,3.$$

or in Matrix Form

$$\boxed{T = \sigma \cdot n}$$

Equations of Equilibrium at a point



Consider a rectangular element as shown in figure above acted upon by Body Forces F_x, F_y and F_z and stresses as shown on element faces.

From equilibrium in x direction we have

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx - \sigma_x\right) dy\, dz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} . dy - \tau_{xy}\right) dx\, dz$$

$$+ \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz - \tau_{xz}\right) dx\, dy \quad + F_x\, dx dy dz = 0$$

$$\Rightarrow \quad \frac{\partial \sigma_x}{\partial x} dx\, dy dz + \frac{\partial \tau_{xy}}{\partial y} dx\, dy dz + \frac{\partial \tau_{yz}}{\partial z} dx dy dz + F_x dy dy dz = 0$$

$$\Rightarrow \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_x = 0$$

Similarly considering equilibrium in y and z directions two additional Equations of equilibrium can be derived.

$$\sum F_x = 0 \Rightarrow \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$

$$\sum F_y = 0 \Rightarrow \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0$$

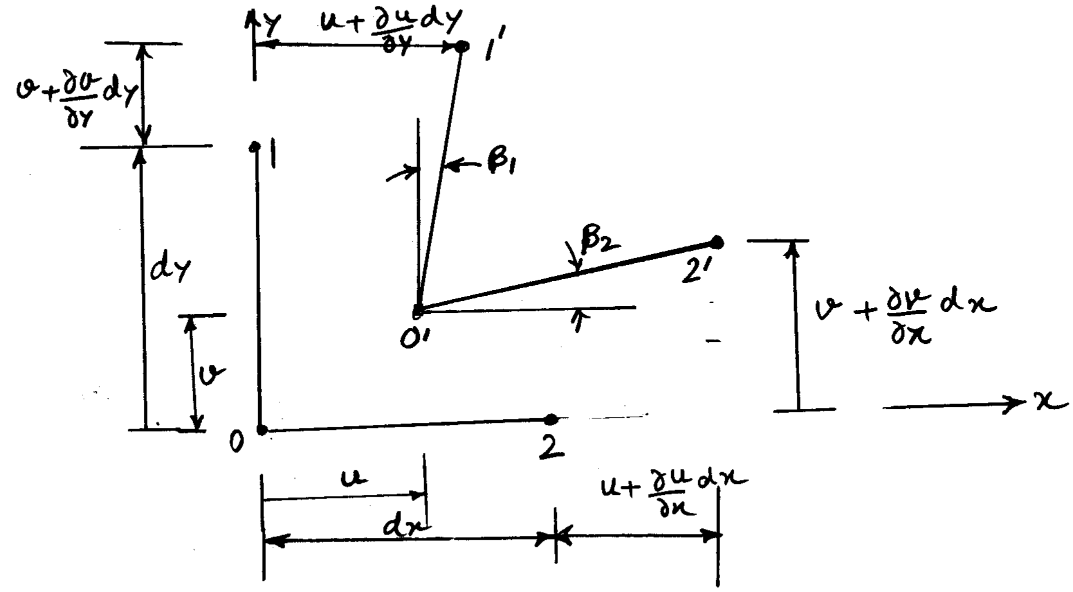
$$\sum F_z = 0 \Rightarrow \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\tau_{yz}}{\partial y} + F_z = 0.$$

In indicial Notation Form we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_j = 0 \qquad i, j = 1, 2, 3.$$

Definition of Strains at a point

Consider a configuration 012 which after deformation changes to 0'1'2' as shown in figure



$$\varepsilon_x = \frac{L_{0'2'} - L_{02}}{L_{02}} = \frac{\left[dx + u + \frac{\partial u}{\partial x}dx - u\right] - dx}{dx}$$

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{L_{0'1'} - L_{01}}{L_{01}} = \frac{\left[dy + v + \frac{\partial v}{\partial y}dy - v\right] - dy}{dy}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

Engineering shear strain γ_{xy} is equal to

$$\gamma_{xy} = \beta_1 + \beta_2 = \frac{u + \frac{\partial u}{\partial y}dy - u}{dy} + \frac{v + \frac{\partial v}{\partial x}dx - v}{dx}$$

$$\gamma_{xy} = \quad = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Definition of Strain @ a Point

For 2-D case we have

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}}$$

For 2-D and 3-D case we have strain Definitions in Matrix Form as Follows

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

In Indicial Notation we have

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i,j = \overset{x}{1}, \overset{y}{2}, \overset{z}{3}$$

Equations of Compatibility

The definitions of Strain in 3-D connect 6 Components of Strain to only 3 displacements u, v, w Components. Therefore one cannot arbitrarily specify all the strains as functions of x, y & z. This indicates that the strains are related to each other. as we will see below.

Differentiating ε_x twice w.r.t y
& " ε_y " " x
& " γ_{xy} w.r.t. x & y results in

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2};$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right)$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial^2 y} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}}$$

This is the Compatibility Condition for a 2-D problem expressed in terms of strains.

The six compatibility equations for 3-D Problem are as follows:

$$\boxed{\begin{array}{ll}
\dfrac{\partial^2 \varepsilon_x}{\partial y^2} + \dfrac{\partial^2 \varepsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y}, & 2\dfrac{\partial^2 \varepsilon_x}{\partial y \partial z} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{xz}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z}\right) \\[2ex]
\dfrac{\partial^2 \varepsilon_y}{\partial z^2} + \dfrac{\partial^2 \varepsilon_z}{\partial y^2} = \dfrac{\partial^2 \gamma_{yz}}{\partial y \partial z}, & 2\dfrac{\partial^2 \varepsilon_y}{\partial z \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{xz}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z}\right) \\[2ex]
\dfrac{\partial^2 \varepsilon_z}{\partial x^2} + \dfrac{\partial^2 \varepsilon_x}{\partial z^2} = \dfrac{\partial^2 \gamma_{xz}}{\partial z \partial x}, & 2\dfrac{\partial^2 \varepsilon_z}{\partial x \partial y} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{xz}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z}\right)
\end{array}}$$

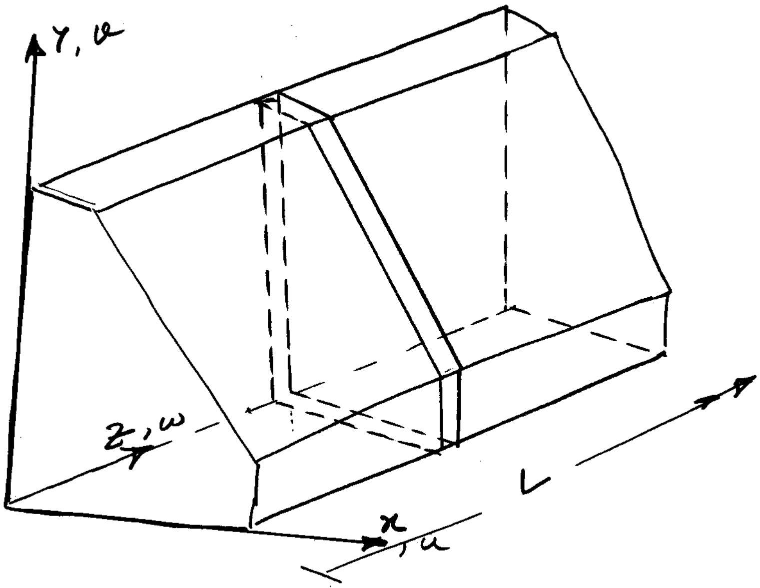
2-D Problems in Elasticity

2-D Problems in Elasticity are of 2 Types

- Plane Strain Problems
- Plane Stress Problems

Plane Strain Problems

In case of a long Dam the state of strain on a slice taken through the cross-section of the dam would be under plane strain condition.



The strain quantities having derivatives w.r.t z coordinate and displacement w is = 0 in this case. Thus we have for strains and Equilibrium

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_z = 0$$

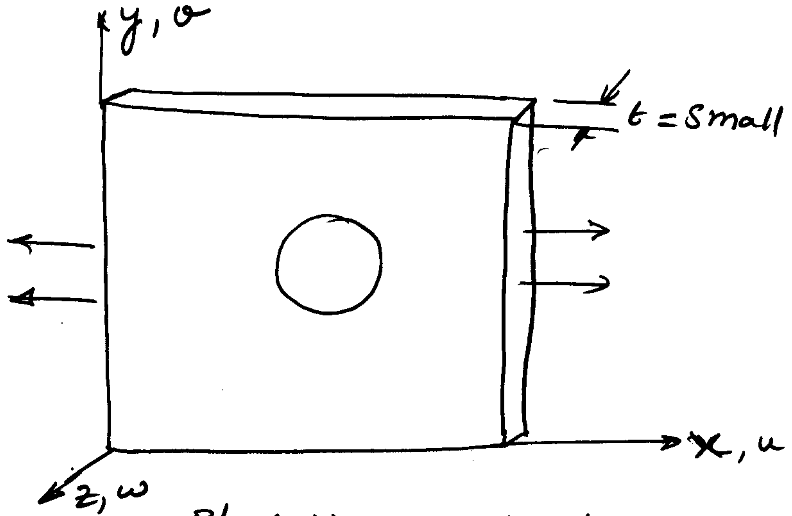
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \sigma_z \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \varepsilon_z = 0 \end{Bmatrix}$$

2-D Problems in Elasticity

Plane Stress Problems



Stretching of thin plate

$\sigma_z = 0$

$\gamma_{xz} = \gamma_{yz} = 0$

$\varepsilon_z \neq 0$

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

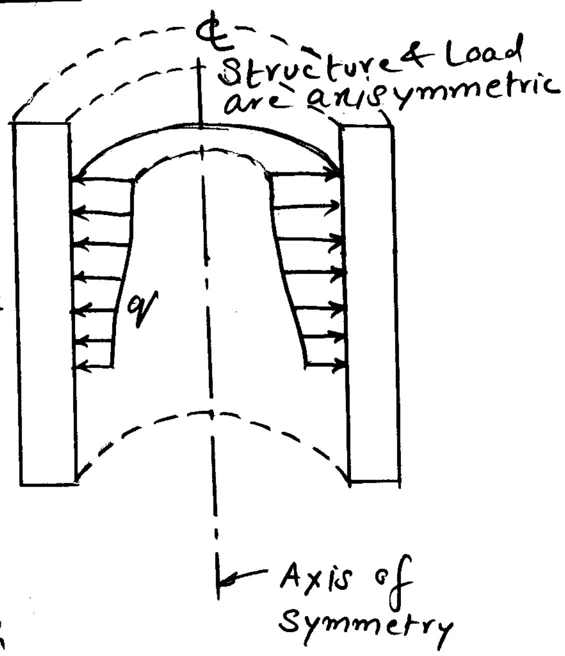
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

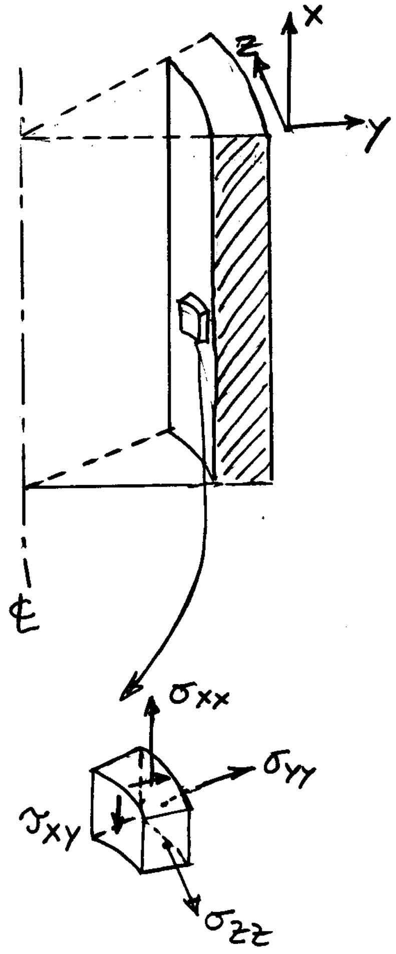
$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

2-D Problems in Elasticity

Axisymmetric Problem

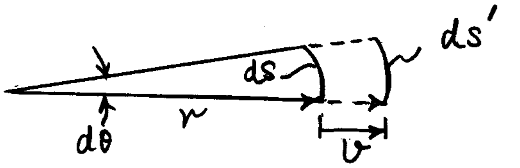
Both the structure and the applied Load are axisymmetric with respect to an axis of symmetry





The analysis of such a structure can be carried out by analyzing a radial segment of the structure

Governing equations in this case are as follows



$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_z = \frac{v}{r}$$

$$\varepsilon_z = \frac{ds' - ds}{ds}$$

$$\varepsilon_z = \frac{(r+v)d\theta - r\,d\theta}{r\,d\theta} = \frac{v}{r}$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \sigma_z \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \varepsilon_z \end{Bmatrix}$$

Introduction to Calculus of Variations

The Potential Energy associated with a system is defined as the sum of strain energy stored in it and the potential of the external forces acting on the system.

$$\Pi = \mathcal{U} - \mathcal{W}$$

where Π = Total Potential Energy

$\mathcal{U}$ = Strain Energy stored in the system or Body

$\mathcal{W}$ = Potential of loads acting on the sytem or Body

Principle of Minimum Potential Energy

The Principle of Minimum Potential Energy states that:

"Among all admissible configurations of a conservative system, those that satisfy the equations of equilibrium make the Potential Energy stationary with respect to small variations of displacement. If the stationary condition is a minimum, the equilibrium state is stable"

A Structure in conjunction with the forces acting on it is said to constitute a "system"

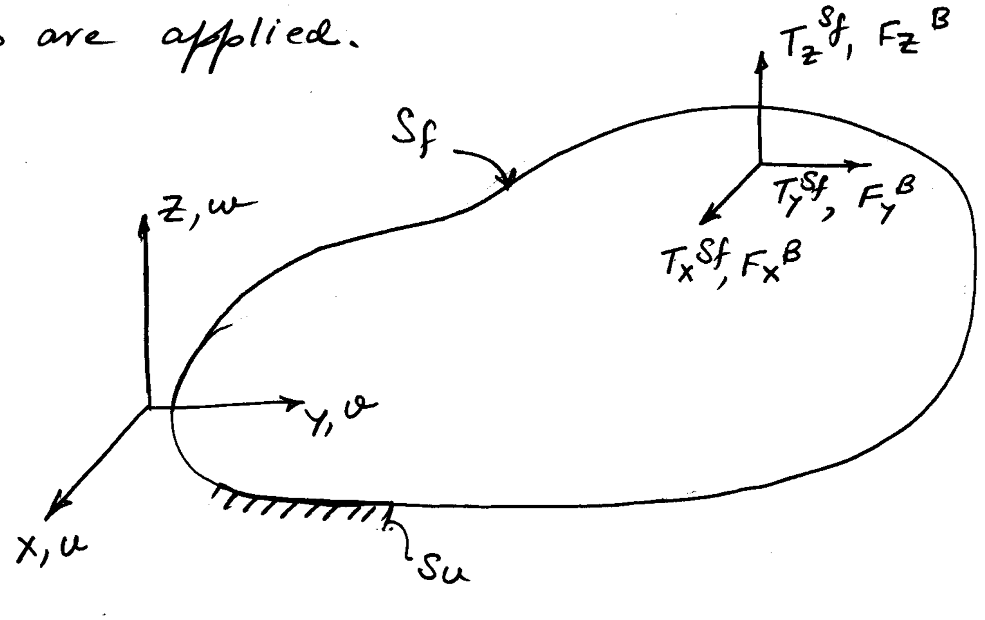
A system is said to be "conservative" if, when the system is displaced from any configuration and then led back to it again, the forces do zero net work regardless of the path taken. For a conservative system the current potential energy depends upon the current configuration of the system, and not on how the system got to the current state.

"Admissible Configuration" of a system is defined as a configuration of a system that it violates neither the internal compatibility conditions nor the essential Boundary conditions.

Boundary Conditions are of two types: Essential, and Non-essential (or Natural)

Essential Boundary conditions are displacement conditions that are prescribed on the body that must be met by any valid admissible displacement field. These are also referred to as geometric boundary conditions.

Natural or Non-essential boundary conditions correspond to displacements at the locations where forces or stresses are applied.



Consider the equilibrium of a three-dimensional body above. The body is supported on area S_u with prescribed displacements U^{Su}. It is subjected to surface tractions T^{Sf} (forces per unit area) on the surface area S_f. In addition it is acted upon externally applied body forces F^B (forces per unit Volume)

The principle of minimum potential energy states that for stable equilibrium

$$\delta \Pi = \delta(\mathcal{U} - \mathcal{W}) = 0 \qquad \text{——— (a)}$$

where $\delta \Pi$ is the change in potential energy as a result of introducing a small virtual displacement from the state of equilibrium configuration such that the virtual displacement field is admissible i.e satisfies the essential boundary conditions.

The strain energy of a continuum is defined as:

$$\boxed{U = \int_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz})\, dV}$$

The strain Energy corresponding to a virtual straining is given by:

$$\delta U = \int_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz})\, dV$$

The virtual work done by external forces is defined as follows. Assuming that the body experiences virtual displacements $\delta u, \delta v$, and δw. The work done by the Body Forces F^B and the Surface Tractions T^{Sf} is:

$$\delta W = \int_V (F_x^B \delta u + F_y^B \delta v + F_z^B \delta w)\, dV + \int_{S_f} (T_x^{Sf} \delta u + T_y^{Sf} \delta v + T_z^{Sf} \delta w)\, dA$$

As virtual displacements result in no geometric alteration of the body we may write eqn (a) as:

$$\boxed{\delta \Pi = \delta \left[U - \int_{S_f} (T_x^{Sf} u + T_y^{Sf} v + T_z^{Sf} w)\, dA + \int_V (F_x^B u + F_y^B v + F_z^B w)\, dV = 0\right.}$$

Consider now a case in which the loading system consists only of forces applied at points on the surface of the body, denoting each point force by P_i and the displacement in the direction of the force by u_i (corresponding to the equilibrium state)

From stationarity of Potential Energy we have

$$\delta(U - W) = 0$$

$$\delta(U - P_i u_i) = 0 \qquad \text{if virtual displacement is introduced only at location of } P_i$$

$$\delta U = P_i \delta u_i$$

$$\Rightarrow \boxed{\frac{\partial U}{\partial u_i} = P_i} \qquad \text{——— (e)}$$

meaning that the partial derivative of the strain energy w.r.t. to a displacement u_i equals to the force acting in the direction of u_i. Equation (e) is known as "Castigliano's First Theorem" as is applicable to linear and non-linear systems.

Principle of Virtual Work

For a three-dimensional body in equilibrium considered previously, the principle of Virtual Work states that The virtual work done by body forces (F_X, F_Y, F_Z) and surface tractions (T_X, T_Y, T_Z) is equal to the increment in the strain energy acquired by the body i.e

$$\boxed{\delta W = \delta U}$$

or $$\boxed{\delta(W - U) = 0}$$

where δW is the external work done by body forces and surface tractions as the body undergoes virtual displacements, and δU is the increase in strain energy of the body due to virtual displacements.

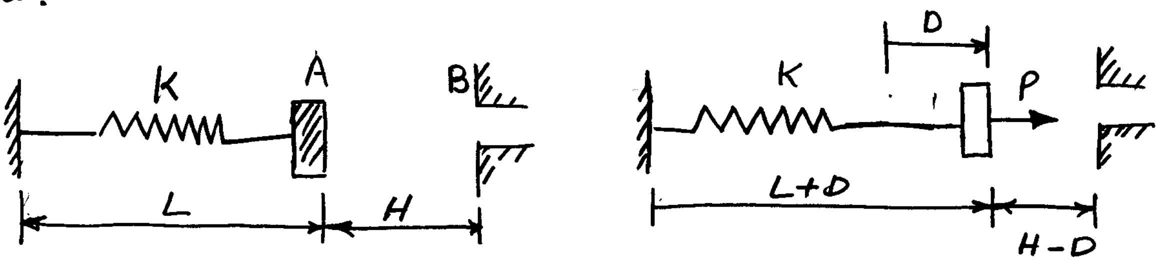
$$\delta W = \int_V \left(F_X^B \delta u + F_Y^B \delta v + F_Z^B \delta w\right) dV + \int_{S_f} \left(T_X^{S_f} \delta u + T_Y^{S_f} \delta v + T_Z^{S_f} \delta w\right) dA$$

and

$$\delta U = \int_V \left(\sigma_X \delta \varepsilon_X + \sigma_Y \delta \varepsilon_Y + \sigma_Z \delta \varepsilon_Z + \tau_{XY} \delta \gamma_{XY} + \tau_{XZ} \delta \gamma_{XZ} + \tau_{YZ} \delta \gamma_{YZ}\right) dV$$

Example of Principle of minimum Potential Energy applied to Single Degree of Freedom System

Consider a conservative system as shown in figure below:



Two valid expressions for potential energy may be written as follows which can differ by a constant term

$$\Pi_P = \frac{1}{2}KD^2 + P(H-D), \quad \Pi_P = \frac{1}{2}KD^2 - PD$$

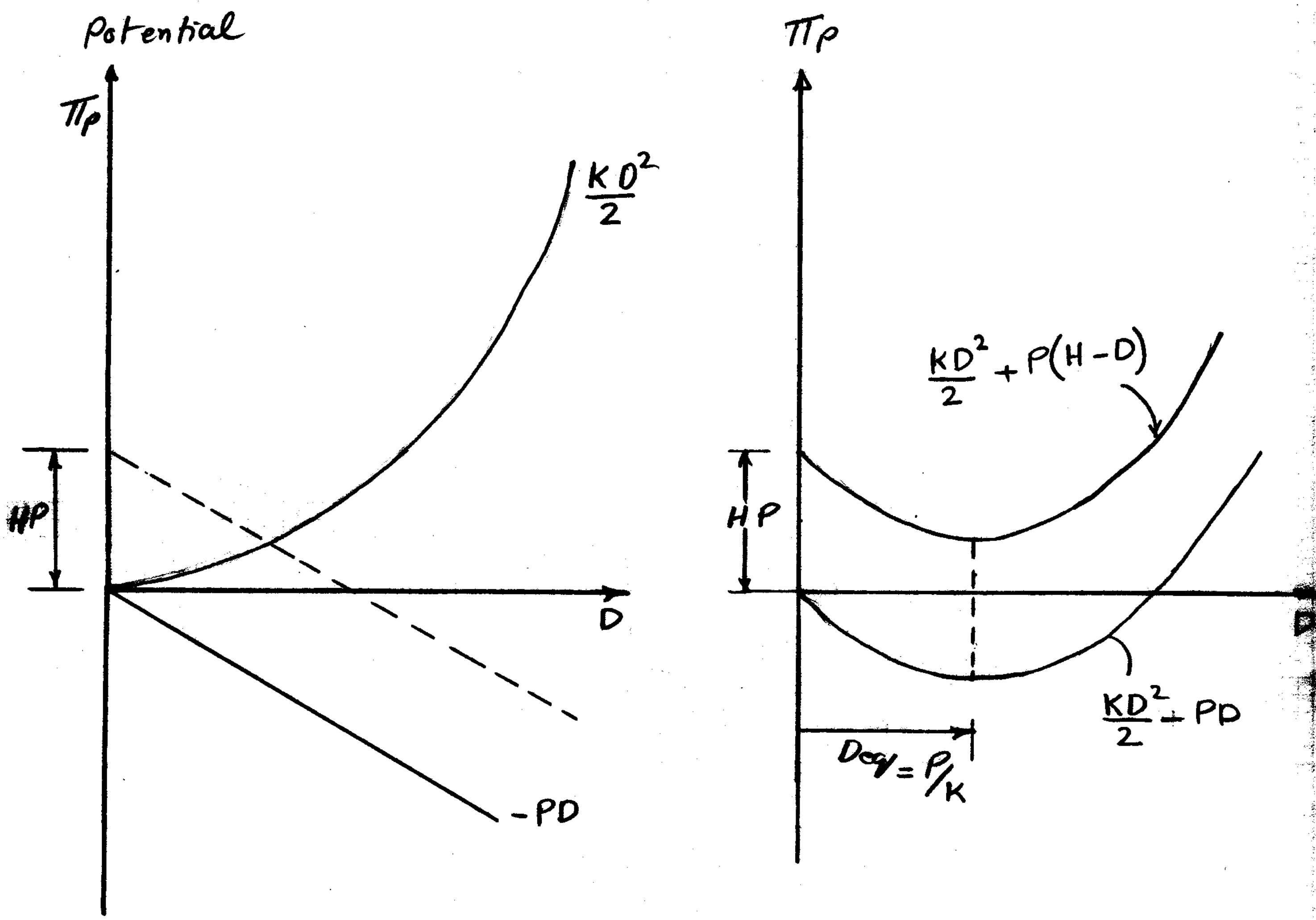
In the first expression $P(H-D)$ represent the potential of the force P to do additional work before it hits the stop, and in the second expression if P and D are positive in the same sense, then $-PD$ represent the reduction in capacity of P to do work

Invoking stationarity of Π_P (potential energy) for equilibrium we have

$$\frac{d\Pi_P}{dD} = 0 \quad \Rightarrow \quad KD - P = 0$$

$$\therefore D = \frac{P}{K} \quad \underline{\text{Solution.}}$$

$$\frac{d^2\Pi_P}{dD^2} = K > 0 \Rightarrow$$ Solution obtained is a minima point.


Potential
Π_p
$\frac{KD^2}{2}$
HP
D
-PD
Π_p
$\frac{KD^2}{2} + P(H-D)$
HP
$\frac{KD^2}{2} - PD$
$D_{eq} = P/K$

Principle of minimum potential energy applied to Multiple Degree of Freedom Systems

If a system has n degrees of freedom. The Potential Energy Function or the Potential Π_p is a function of these degrees of freedom D_i, $i = 1 \cdots n$

$$\Pi_p = \Pi_p(D_1, D_2, \cdots D_n)$$

The total differential of Π_p can be written as:

$$d\Pi_p = \frac{\partial \Pi_p}{\partial D_1} dD_1 + \frac{\partial \Pi_p}{\partial D_2} dD_2 + \cdots \frac{\partial \Pi_p}{\partial D_n} dn$$

$$= \left\{\frac{\partial \Pi_p}{\partial D}\right\}^T \left\{dD\right\} \quad \text{in matrix form.}$$

The principle of stationarity of Π_p states that for for equilibrium D_i must define a configuration for which

$$d\Pi_p = 0$$

For any choice of dD_i $\quad i = 1, \cdots n \quad d\Pi_p = 0$

This is possible only when

$$\frac{\partial \Pi_p}{\partial D_1} = \frac{\partial \Pi_p}{\partial D_2} = \cdots = \frac{\partial \Pi_p}{\partial D_n} = 0$$

or $\boxed{\dfrac{\partial \Pi_p}{\partial D_i} = 0}$

The above expression is reffered to as principle of stationarity of Potential energy for a multiple degree of freedom system. This yields n system of equations for n unknown degrees of freedom.

THE RAYLEIGH-RITZ METHOD

Consider an elastic solid for which we are to find displacements when subjected to applied loads.
The displacement at any point within the body may be defined in terms of displacements at finite number of locations within the body using interpolation functions. as given below

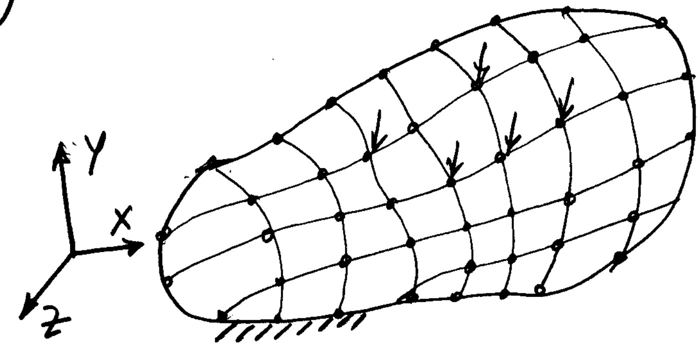
$$u = \sum a_i f_i \quad \text{where } f_i = f_i(x,y,z), \quad i = 1 \ldots l$$
$$v = \sum b_i \cdot g_i \quad \text{where } g_i = g_i(x,y,z), \quad i = 1 \ldots m$$
$$w = \sum c_i h_i \quad \text{where } h_i = h_i(x,y,z) \quad i = 1 \ldots n$$

a_i, b_i, c_i are displacement at finite locations within the body and f_i, g_i, h_i are interpolation functions representing an admissible displacement field.
Then the displacement at finite locations can be found by the principle of stationarity of potential energy as follows.

$$\boxed{\frac{\partial \Pi_P}{\partial a_i} = 0, \quad \frac{\partial \Pi_P}{\partial b_i} = 0, \quad \frac{\partial \Pi_P}{\partial c_i} = 0} \quad \text{(a)}$$

The above expression yields $l+m+n$ equations yielding solution of a_i, b_i and c_i degrees of freedom.

3-Dimensional Body with $l+m+n$ DOFs at i number of pts with nodal loads



RAYLEIGH-RITZ METHOD (CONTD.)

With the determination of a_i, b_i, c_i nodal Dofs the complete displacement field is known as these nodal values can be used in conjunction with the interpolation functions f_i, g_i, h_i to find displacements u, v, w anywhere in the body. from which strains and then stresses can be determined by differentiation of displacements and use of constitutive equations.

Since for solution of nodal displacements stationarity of potential energy has been invoked, the displacement field obtained as solution would satisfy equilibrium conditions. However, the equilibrium will be satisfied only in an approximate sense since only a finite set of degrees of freedom have been used, whereas a continuum body has infinite degrees of freedom.

Never the less, the error in satisfaction of equilibrium conditions would reduce significantly if more degrees of freedom are used. to define the displacement field.

The Rayleigh-Ritz method in essence the finite element method.

Equations ⓐ yield the equilibrium equations for the discretized system in the following matrix form

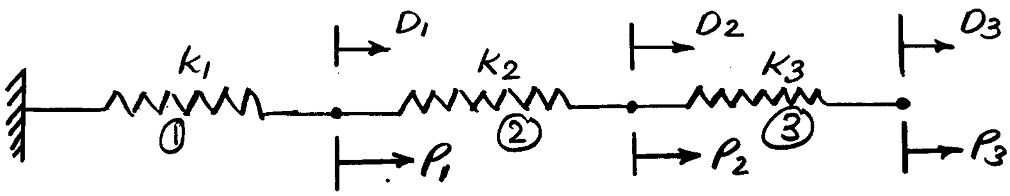
$$[K]\{D\} = \{P\}$$

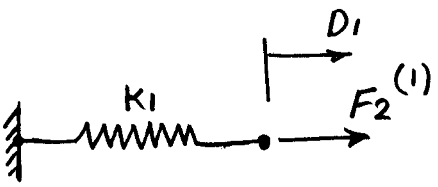
where $\{P\}$ is the vector of nodal forces.

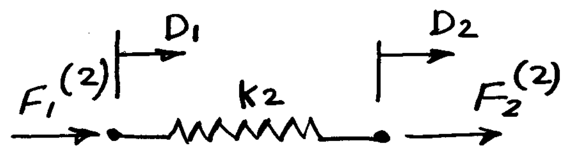
and $\{D\}$ is the vector of nodal displacements.

Example Problem

Formulate the equilibrium equations for the system shown below by direct stiffness method and the Rayleigh-Ritz Method.







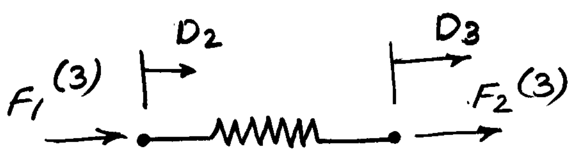
$$K_1 \; D_1 = F_2^{(1)}$$

$$K_2 D_1 - K_2 D_2 = F_1^{(2)}$$

$$-K_2 D_1 + K_2 D_2 = F_2^{(2)}$$

$$K_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix}$$

Matrix Form of Element Equilibrium Eqns for element 2



$$K_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{Bmatrix}$$

From Nodal equilibrium considerations we have

$$\left.\begin{aligned} F_2^{(1)} + F_1^{(2)} &= P_1 \\ F_2^{(2)} + F_1^{(3)} &= P_2 \\ F_2^{(3)} &= P_3 \end{aligned}\right\} \text{—— } (B)$$

Substituting in nodal equilibrium equations (B) the element equilibrium equations we have:

$$K_1 D_1 + K_2 D_1 - K_2 D_2 = P_1$$

$$-K_2 D_1 + K_2 D_2 + K_3 D_2 - K_3 D_3 = P_2$$

$$-K_3 D_2 + K_3 D_3 = P_3$$

$$(K_1 + K_2) D_1 - K_2 D_2 = P_1$$

$$-K_2 D_1 + (K_2 + K_3) D_2 - K_3 D_3 = P_2$$

$$-K_3 D_2 + K_3 D_3 = P_3$$

In Matrix Form we have the following Equilibrium Equations:

$$\underbrace{\begin{bmatrix} (K_1 + K_2) & -K_2 & 0 \\ -K_2 & (K_2 + K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix}}_{\text{Structure Stiffness Matrix}} \underbrace{\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}}_{\substack{\text{Nodal} \\ \text{Displacements}}} = \underbrace{\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}}_{\substack{\text{Nodal} \\ \text{Forces.}}} \quad \text{— (C)}$$

Now we derive the same equilibrium equations from Rayleigh-Ritz Method.

$$\Pi_P = \frac{1}{2} K_1 D_1^2 + \frac{1}{2} K_2 (D_2 - D_1)^2 + \frac{1}{2} K_3 (D_3 - D_2)^2$$

$$- P_1 D_1 - P_2 D_2 - P_3 D_3$$

From $\dfrac{\partial \Pi_P}{\partial D_1} = 0$ we have

$$K_1 D_1 - K_2 (D_2 - D_1) - P_1 = 0$$

From $\dfrac{\partial \Pi_P}{\partial D_2} = 0$ we have

$$K_2 (D_2 - D_1) - K_3 (D_3 - D_2) - P_2 = 0$$

From $\dfrac{\partial \Pi_P}{\partial D_3} = 0$ we have

$$K_3 (D_3 - D_2) - P_3 = 0$$

D

Writing Equations D in matrix form we have

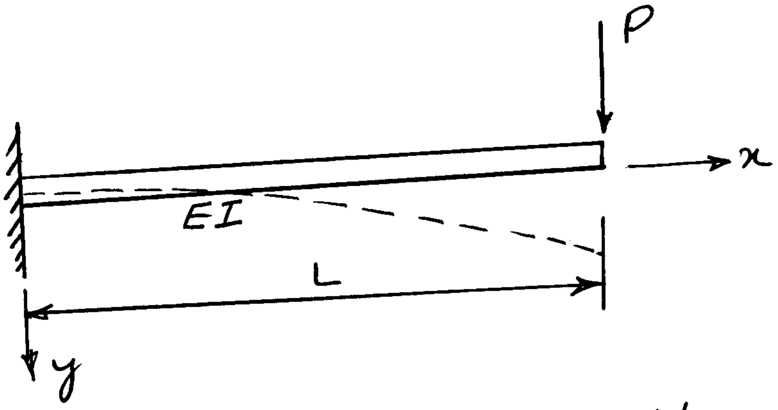
$$\underbrace{\begin{bmatrix} (K_1 + K_2) & -K_2 & 0 \\ -K_2 & (K_2 + K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix}}_{\text{Structure Stiffness Matrix}} \underbrace{\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}}_{\text{Nodal Displacements}} = \underbrace{\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}}_{\text{Nodal Forces}} \quad \text{— E}$$

It is seen that the above equations E obtained from Rayleigh-Ritz method are the same as Equations C obtained earlier from equilibrium considerations.

Example Problem

We present an example demonstrating that principle of stationarity of Potential Energy when applied to a continuous system yield the governing differential equations for the system.

Consider the cantilever beam shown in figure below acted upon by concentrated load at the free end.



The expression for potential energy of the cantilever beam system can be written as:

$$\Pi = \underbrace{\frac{1}{2} EI \int_0^L \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx}_{\text{Strain energy of Bending}} - \underbrace{P\,y(L)}_{\text{Potential of Applied Load}} = 0 \quad \text{—— (a)}$$

Strain energy density for the beam $= U_0 = \frac{1}{2}\sigma \cdot \varepsilon = \frac{1}{2}\sigma \cdot \frac{\sigma}{E} = \frac{\sigma^2}{2E}$

For bending $\sigma = \frac{M}{I} y$

$$\Rightarrow \qquad U_0 = \frac{M^2}{2EI^2} y^2$$

Integrating U_0 over the volume of the beam we have

$$U = \int_0^L \frac{M^2}{2EI^2} \left[\int_A y^2 \, dy\, dz \right] dx$$

$$U = \int_0^L \frac{M^2}{2EI^2} \cdot I \, dx$$

$$U = \int_0^L \frac{M^2}{2EI} \cdot dx$$

Now $M = -EI \frac{d^2y}{dx^2}$

$$\Rightarrow \boxed{U = \int_0^L \frac{EI}{2} \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 y}{\partial x^2}\right) dx}$$

which is the expression of strain energy used in expression for potential energy ⓐ

For convenience we rewrite the expression for potential energy Π_p as

$$\Pi_p = \frac{1}{2} EI \int_0^L (y'')^2 dx - P\, y(L)$$

Invoking stationarity of Π_p we have.

$$\delta \Pi_p = EI \int_0^L y'' \delta y'' \, dx - P\, \delta y(L) = 0$$

Integrating by parts once we have

$$\boxed{\int u v' dx = uv \Big| - \int u' v \, dx}$$

$$= EI\, y'' \delta y' \Big|_0^L - EI \int_0^L y''' \delta y' \, dx - P\, \delta y(L) = 0$$

Integrating by parts again we have

$$\delta \Pi_p = EI\, y'' \delta y' \Big|_0^L - EI\, y''' \delta y \Big|_0^L + EI \int_0^L y^{iv} \delta y \, dx$$

$$- P \delta y \Big|_L = 0$$

$$\delta \Pi_p = \underbrace{EI\, y'' \delta y' \Big|_L}_{①} - \underbrace{EI\, y'' \delta y' \Big|_0}_{②} - \underbrace{EI\, y''' \delta y \Big|_L}_{③}$$

$$+ \underbrace{EI\, y''' \delta y \Big|_0}_{④} + \underbrace{EI \int_0^L y^{iv} \delta y \, dx}_{⑤} - \underbrace{P \delta y \Big|_L}_{6} = 0$$

Now the imposed variations on y, y' must vanish at $x=0$ ie $\delta y|_0 ; \delta y'|_0 = 0$ in order to satisfy the essential boundary conditions on displacements

terms ②, ④ are equal to zero $= 0$

Since the variation on y and y' are arbitrary following equations must hold true.

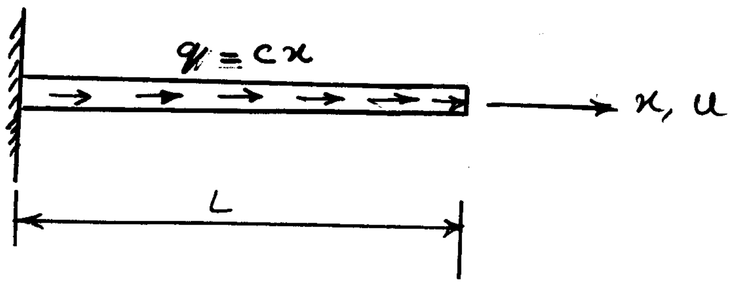
$$\boxed{\begin{array}{lll} y^{iv}(x) = 0 & \equiv \dfrac{\partial^4 y}{\partial x^4} = 0 & \text{— Differential Equation for system} \\ EI\, y''(L) = 0 & \equiv EI \dfrac{\partial^2 y}{\partial x^2}\Big|_L = 0 & \text{— BC } M\Big|_L = 0 \\ EI\, y'''(L) + P = 0 & \equiv EI \dfrac{\partial^3 y}{\partial x^3}\Big|_L + P = 0 & \text{— BC } \text{Shear}\Big|_L = P \end{array}}$$

*** The above are seen to be the governing differential equation and Boundary conditions for the system.

Example Problem

We solve a problem demonstrating the Rayleigh-Ritz method.

Consider a bar as shown in figure below acted upon by a distributed force $q = cx$. We will obtain solution to this problem using the Rayleigh-Ritz method.



The expression for Potential Energy of the system can be written as follows

$$\text{strain Energy density} = U_0 = \frac{1}{2}\sigma \cdot \varepsilon = \frac{E\varepsilon^2}{2}$$

$$U = \int_{Vol} \frac{E\varepsilon^2}{2}\, dv = \int_0^L \frac{E\,\varepsilon^2(x)}{2} A\, dx$$

$$= \int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx$$

The Potential of loads acting on the Beam is:

$$W = \int_0^L q(x)\, u(x)\, dx \quad \text{or} = \int_0^L q\, u\, dx$$

Potential Energy is then:

$$\boxed{\Pi_p = U - W = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L q\, u\, dx}$$

or

$$\Pi_p = \int_0^L \frac{EA}{2}(u_{,x})^2\, dx - \int_0^L q\, u\, dx$$

The simplest form of displacement function which satisfies the essential boundary conditions is :

$$u = a_1 x \qquad u\Big|_{x=L} = 0$$

satisfied.

substituting $u = a_1 x$ and $q = cx$ in expression for Potential Energy we have

$$\Pi_P = \int_0^L \frac{EA}{2} a_1^2 \, dx - \int_0^L (cx)(a_1 x) \, dx$$

$$= \frac{EAl\,a_1^2}{2} - \int_0^L c\,a_1 x^2 \, dx$$

$$= \frac{EAl\,a_1^2}{2} - \frac{c\,a_1 l^3}{3}$$

$$\Pi_P = \frac{EAl\,a_1^2}{2} - \frac{c\,l^3}{3} a_1$$

Using Rayleigh - Ritz relation.

$$\frac{\partial \Pi_P}{\partial a_1} = 0 \qquad \text{we have}$$

$$\frac{\Pi_P}{\partial a_1} = EAl\,a_1 - \frac{c\,l^3}{3} = 0$$

$$\Rightarrow \quad a_1 = \frac{c\,l^3}{3} \times \frac{1}{EAl} = \frac{c\,l^2}{3EA}$$

and $\boxed{u = a_1 x = \frac{c\,l^2}{3AE} x}$

$$\sigma_x = E \frac{\partial u}{\partial x} = E \frac{c\,l^2}{3AE}$$

$$\boxed{\sigma_x = \frac{c\,l^2}{3A}}$$

A better function describing the displacement field would be

$$u = a_1 x + a_2 x^2$$

$$\frac{\partial u}{\partial x} = a_1 + 2a_2 x$$

substituting in the expression for Potential Energy we have

$$\Pi_p = \int_0^l \frac{EA}{2}\left(a_1 + 2a_2 x\right)^2 dx - \int_0^l cx\left(a_1 x + a_2 x^2\right) dx$$

$$\Pi_p = \frac{EA}{2}\int_0^l a_1^2 + 4a_2^2 x^2 + 4a_1 a_2 x - \int_0^l \left(c a_1 x^2 + c a_2 x^3\right) dx$$

$$\Pi_p = \frac{EA}{2}\left[a_1^2 l + \frac{4}{3}a_2^2 l^3 + 2a_1 a_2 l^2\right] - \left[\frac{c\, a_1 l^3}{3} + \frac{c a_2 l^4}{4}\right]$$

$$\frac{\partial P}{\partial a_1} = 0$$

$$\Rightarrow EA\left[a_1 l + 0 + a_2 l^2\right] - \left[\frac{c\, l^3}{3}\right] = 0$$

$$EAl\begin{bmatrix} 1 & l \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \left\{\frac{c\, l^3}{3}\right\}$$

$$\frac{\partial P}{\partial a_2} = 0$$

$$\Rightarrow EA\left[a_1 l^2 \quad \frac{4}{3}a_2 l^3\right] - \left[\frac{c\, l^4}{4}\right] = 0$$

$$EAl\begin{bmatrix} l & \frac{4}{3}l^2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \left\{\frac{c\, l^4}{4}\right\}$$

In Matrix form we can then write 2 simultaneous equations for a_1 and a_2

$$AE\ell \begin{bmatrix} 1 & \ell \\ \ell & \frac{4}{3}\ell^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{c\ell^3}{12} \begin{Bmatrix} 4 \\ 3\ell \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{c\ell^2}{12\,AE} \underbrace{\begin{bmatrix} 1 & \ell \\ \ell & \frac{4}{3}\ell^2 \end{bmatrix}}_{[A]}^{-1} \begin{Bmatrix} 4 \\ 3\ell \end{Bmatrix}$$

$$\text{Det}\,[A] = \frac{4\ell^2}{3} - \ell^2 = \frac{\ell^2}{3}$$

$$[A]^{-1} = \frac{1}{\text{Det}\,A}\,[\text{Adj}\,A]$$

$$[A]^{-1} = \frac{3}{\ell^2} \begin{bmatrix} \frac{4\ell^2}{3} & -\ell \\ -\ell & 1 \end{bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{c\ell^2}{12AE} \times \frac{3}{\ell^2} \begin{bmatrix} \frac{4\ell^2}{3} & -\ell \\ -\ell & 1 \end{bmatrix} \begin{Bmatrix} 4 \\ 3\ell \end{Bmatrix}$$

$$= \frac{c}{4\,AE} \begin{Bmatrix} \frac{16\ell^2}{3} - 3\ell^2 \\ -4\ell + 3\ell \end{Bmatrix} = \frac{c}{4AE} \begin{Bmatrix} \frac{7\ell^2}{3} \\ -\ell \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{c\ell}{4AE} \begin{Bmatrix} \frac{7}{3}\ell \\ -1 \end{Bmatrix} = \begin{Bmatrix} \frac{7c\ell^2}{12\,AE} \\ \frac{-c\ell}{4AE} \end{Bmatrix}$$

$$u = a_1 x + a_2 x^2$$

$$\Rightarrow \boxed{u = \frac{7c\ell^2}{12AE}x - \frac{c\ell}{4AE}x^2}$$

$$\sigma_x = E\frac{\partial u}{\partial x} = E\left(\frac{7cl^2}{12AE} - \frac{cl}{2AE}x\right)$$

$$\sigma_x = \left(\frac{7cl^2}{12A} - \frac{cl}{2A}x\right)$$

Exact Solution for the Problem is

$$\boxed{u = \frac{c}{6AE}\left(3l^2x - x^3\right), \quad \sigma_x = \frac{c}{2A}\left(l^2 - x^2\right)}$$

If more terms in the assumed displacement shape had been assumed according to the polynomial

$$u = a_1 + a_2x^2 + a_3x^3 + a_4x^4 \cdots a_nx^n$$

Ritz method would have yielded

$$a_1 = \frac{cl^2}{2AE}, \quad a_3 = -\frac{c}{6AE}, \quad a_2 = a_4 = a_5, \ldots a_n = 0$$

If we normalize the length of the bar by using generalized coordinate

$$\xi = \frac{x}{L}$$

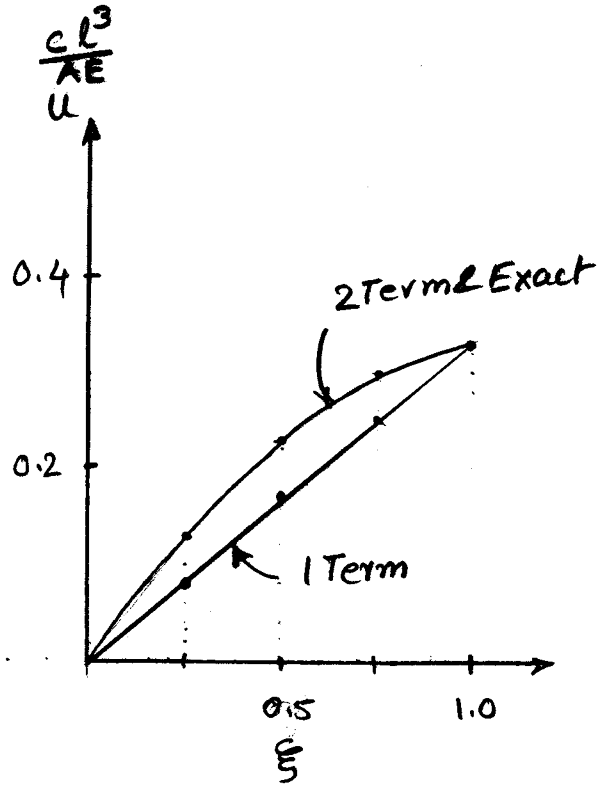
we can write the approximate and exact solutions as

Approximate	Exact
$u(\xi) = \frac{7cl^3}{12AE}\xi - \frac{cl^3}{4AE}\xi^2$	$u(\xi) = \frac{cl^3}{2AE}\xi - \frac{cl^3}{6AE}\xi^3$
$\sigma(\xi) = \frac{7cl^2}{12A} - \frac{cl^2}{2A}\xi$	$\sigma(\xi) = \frac{cl^2}{2A} - \frac{cl^2\xi^2}{2A}$

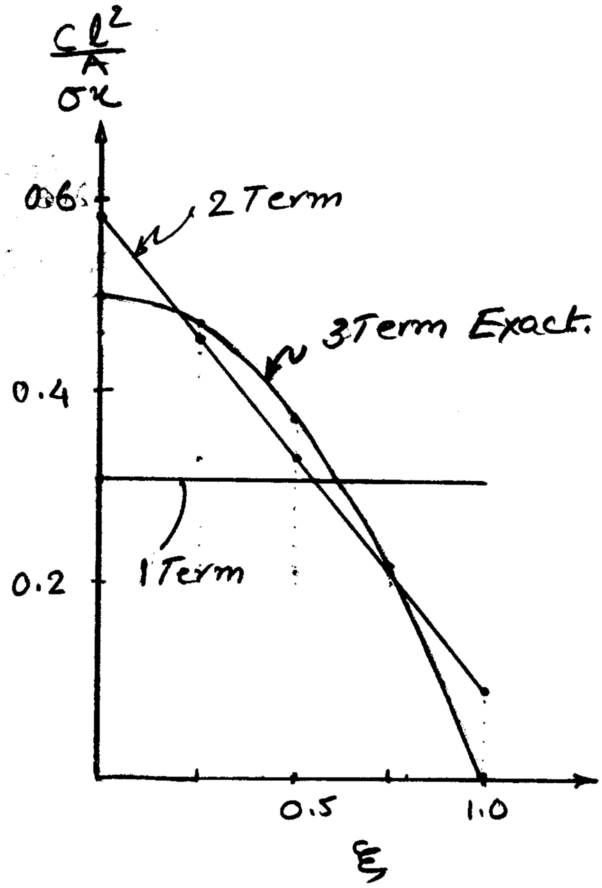
or

$u(\xi) = \frac{7cl^3}{12AE}\left(\xi - \frac{3}{7}\xi^2\right)$	$u(\xi) = \frac{cl^3}{2AE}\left(\xi - \frac{1}{3}\xi^3\right)$
$\sigma(\xi) = \frac{7cl^2}{12}\left(1 - \frac{6}{7}\xi\right)$	$\sigma(\xi) = \frac{cl^2}{2A}\left(1 - \xi^2\right)$

ξ	1 Term Soln		2 Term Soln		Exact	
	$u = \frac{cl^3}{3AE}\xi$	$\sigma_x = \frac{cl^2}{3A}$	u: $\frac{4cl^3}{12AE}\left(\xi - \frac{3}{7}\xi^2\right)$	σ_x: $\frac{7cl^2}{12A}\left(1 - \frac{6}{7}\xi\right)$	u: $\frac{cl^3}{2AE}\left(\xi - \frac{\xi^3}{3}\right)$	σ_x: $\frac{cl^2}{2A}\left(1 - \xi^2\right)$
0	0	$0.33\frac{cl^2}{A}$	0	$\frac{7cl^2}{12}$	0	$\frac{cl^2}{2A}$
0.25	$0.083\frac{cl^3}{AE}$	$\updownarrow$	0.13	0.458	0.122	0.469
0.50	0.167		0.229	0.333	0.229	0.375
0.75	0.25		0.297	0.208	0.305	0.219
1.0	0.33	0.33	0.333	0.083	0.333	0

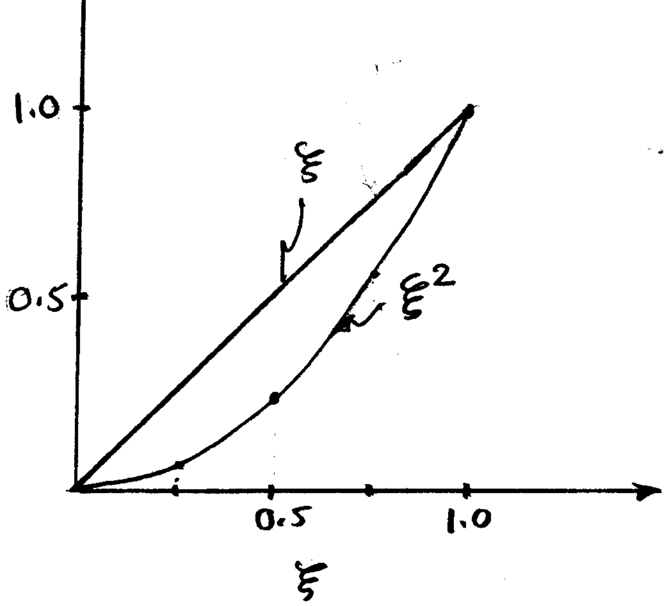


DISPLACEMENTS



STRESS

SHAPE FUNCTIONS
FOR 1 TERM & 2 TERM SOLUTION



$$u(\xi) = a_1 \xi + a_2 \xi^2$$

$$u(\xi) = \frac{7cl^3}{12AE}\xi - \frac{cl^3}{4AE}\xi^2$$

<u>2 TERM SOLN</u>

$$u(\xi) = \frac{cl^3}{3AE}\xi$$

$$u(\xi) = a_1 \xi$$

<u>1 TERM SOLN</u>