

$$\begin{aligned}
H_I &= \sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ \\
H_{DST} &= H_I \otimes \mathbb{1} + \mathbb{1} \otimes H_I \\
|\psi_T\rangle &= \cos(\theta/2) |0\rangle + e^{i\phi} \sin(\theta/2) |1\rangle \\
|\dot{\psi}_T\rangle &= -\sin(\theta/2) \dot{\theta}/2 |0\rangle + e^{i\phi} (\cos(\theta/2) \dot{\theta}/2 + i\dot{\phi} \sin(\theta/2)) |1\rangle \\
\frac{1}{2} \frac{dW}{dt} &= -\text{Im} \left\langle \dot{\psi}_T \left| \dot{\psi}_T^z \right. \right\rangle = \text{Im} \left\langle \dot{\psi}_T \left| iH_T \right| \psi_T \right\rangle = \text{Im} \langle \psi_D | \langle \psi_S | \left\langle \dot{\psi}_T \left| iH_{DST} \right| \psi_D \right\rangle | \psi_S \rangle | \psi_T \rangle \\
&= \text{Im} \langle \psi_D | \langle \psi_S | \left\langle \dot{\psi}_T \left| iH_{DS} \otimes \mathbb{1} \right| \psi_D \right\rangle | \psi_S \rangle | \psi_T \rangle + \text{Im} \langle \psi_D | \langle \psi_S | \left\langle \dot{\psi}_T \left| i\mathbb{1} \otimes H_{ST} \right| \psi_D \right\rangle | \psi_S \rangle | \psi_T \rangle \\
&= \text{Im} i \langle H_{DS} \rangle \left\langle \dot{\psi}_T \left| \psi_T \right. \right\rangle + \text{Im} \langle \psi_S | \left\langle \dot{\psi}_T \left| iH_{ST} \right| \psi_S \right\rangle | \psi_T \rangle \\
&= 0 + \text{Im} \langle \psi_S | \left\langle \dot{\psi}_T \left| i\sigma_+ \otimes \sigma_- \right| \psi_S \right\rangle | \psi_T \rangle + \text{Im} \langle \psi_S | \left\langle \dot{\psi}_T \left| i\sigma_- \otimes \sigma_+ \right| \psi_S \right\rangle | \psi_T \rangle \\
&= -\text{Im} i \langle \psi_S | \sigma_+ | \psi_S \rangle e^{i\phi} \sin^2(\theta/2) \dot{\theta}/2 + \text{Im} i \langle \psi_S | \sigma_- | \psi_S \rangle e^{-i\phi} (\cos^2(\theta/2) \dot{\theta}/2 - i\dot{\phi} \sin(\theta/2)) \\
&= -\text{Im} i \langle \psi_S | \sigma_+ | \psi_S \rangle e^{i\phi} \sin^2(\theta/2) \dot{\theta}/2 + \text{Im} [-i \langle \psi_S | \sigma_+ | \psi_S \rangle e^{i\phi} (\cos^2(\theta/2) \dot{\theta}/2 + i\dot{\phi} \sin(\theta/2))]^* \\
&= -\text{Im} i \langle \psi_S | \sigma_+ | \psi_S \rangle e^{i\phi} \sin^2(\theta/2) \dot{\theta}/2 + \text{Im} i \langle \psi_S | \sigma_+ | \psi_S \rangle e^{i\phi} (\cos^2(\theta/2) \dot{\theta}/2 + i\dot{\phi} \sin(\theta/2)) \\
&= \text{Im} i \langle \psi_S | \sigma_+ | \psi_S \rangle e^{i\phi} (\cos(\theta) \dot{\theta}/2 + i\dot{\phi} \sin(\theta/2))
\end{aligned}$$

$$H_S(\theta_D(t), \phi_D(t), \theta_T(t), \phi_T(t)) = \frac{1}{2} \left[\sin(\theta_D(t)) e^{i\phi_D(t)} + \sin(\theta_T(t)) e^{i\phi_T(t)} \right] \sigma_+ + h.c.$$

$$= \alpha \sigma_+ + h.c.$$

$$\rho_S(t+1) = U \rho_S(t) U^\dagger, U = e^{-iH_S(t)\Delta t}$$

$$dW(t) = -\text{Tr } \rho(t) dH$$

$$dH = H_S(\theta_D(t), \phi_D(t), \theta_T(t+1), \phi_T(t+1)) - H_S(\theta_D(t), \phi_D(t), \theta_T(t), \phi_T(t))$$

$$= \frac{1}{2} (\sin(\theta_T(t+1)) e^{i\phi_T(t+1)} - \sin(\theta_T(t)) e^{i\phi_T(t)}) \sigma_+ + h.c. =: (\tau' - \tau) \sigma_+ + h.c.$$

For N=2 only one jump, analytical solution is possible:

$$U = e^{-iH_S \Delta t} = \exp \begin{pmatrix} 0 & -i\alpha^* \Delta t \\ -i\alpha \Delta t & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{\alpha^*}{|\alpha|} & -\frac{\alpha^*}{|\alpha|} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-i|\alpha| \Delta t} & 0 \\ 0 & e^{i|\alpha| \Delta t} \end{pmatrix} \begin{pmatrix} \frac{|\alpha|}{\alpha^*} & 1 \\ -\frac{|\alpha|}{\alpha^*} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(|\alpha| \Delta t) & -i \frac{\alpha^*}{|\alpha|} \sin(|\alpha| \Delta t) \\ -i \frac{|\alpha|}{\alpha^*} \sin(|\alpha| \Delta t) & \cos(|\alpha| \Delta t) \end{pmatrix}$$

With $\rho_0 = |0\rangle \langle 0|$, the time evolved state $\rho = U \rho_0 U^\dagger$ is

$$\rho = \begin{pmatrix} \cos^2(|\alpha| \Delta t) & i \frac{|\alpha|}{2\alpha^*} \sin(2|\alpha| \Delta t) \\ -i \frac{|\alpha|}{2\alpha^*} \sin(2|\alpha| \Delta t) & \sin^2(|\alpha| \Delta t) \end{pmatrix}$$

$$W = -\text{Tr} \begin{pmatrix} 0 & \tau'^* - \tau^* \\ \tau' - \tau & 0 \end{pmatrix} \begin{pmatrix} \cos^2(|\alpha| \Delta t) & i \frac{|\alpha|}{2\alpha^*} \sin(2|\alpha| \Delta t) \\ -i \frac{|\alpha|}{2\alpha^*} \sin(2|\alpha| \Delta t) & \sin^2(|\alpha| \Delta t) \end{pmatrix}$$

$$= \frac{1}{|\alpha|} \sin(2|\alpha| \Delta t) \text{Im}\{(\tau' - \tau) \alpha^*\}$$

$$H_S^i = \langle \psi_D^i | \langle \psi_T^i | H_{DST} | \psi_D^i \rangle | \psi_T^i \rangle$$

$$\rho^{i+1} = U^i \rho^i U^{i\dagger}$$

$$U^i = e^{-iH_S^i \Delta t}$$

$$W = -\Sigma_i \text{Tr} \rho^i dH^i$$

$$dH^i = \langle \psi_D^i | \langle \psi_T^{i+1} | H_{DST} | \psi_D^i \rangle | \psi_T^{i+1} \rangle - \langle \psi_D^i | \langle \psi_T^i | H_{DST} | \psi_D^i \rangle | \psi_T^i \rangle$$