Local optimisation of extractable work, $\rho_0 = |+\rangle\langle+|$. We start in an Eigenstate of the Drive Hamiltonian, without loss of generality let $H_{DS} \propto \sigma_x$. By local optimisation we find $H'_{ST} = -\frac{\sigma_x}{2}, H_{ST} = \frac{\sigma_x}{2}$. This extracts the maximal $dW = 1$.

We now examine the following step, where state and transducer are now anti-parallel on the Bloch sphere. We can generalise this to an arbitrary state with zero y-component $\rho = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$. The current transducer Hamiltonian is given by $H_{ST} = -\frac{\sigma_x}{2}$. The total system Hamiltonian is given by

$$\begin{aligned} H &= H_{DS} + H_{ST} = \delta\sigma_+ + \delta^*\sigma_- - \frac{\sigma_x}{2} \\ &= \alpha\sigma_+ + h.c. = \mathrm{Re}\{\alpha\}\sigma_x + \mathrm{Im}\{\alpha\}\sigma_y \\ \delta &= \sin\theta_D e^{i\phi_D}/2. \end{aligned}$$

The evolved state ρ' can be stated as

$$\begin{aligned} \rho' &= e^{-iH\Delta\mathrm{T}}\rho\, e^{iH\Delta\mathrm{T}} \\ &= \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \end{aligned}$$

with

$$\begin{aligned} \vec{r} &= (\ \mathrm{Re}\{a\}, \mathrm{Im}\{a\}, b/2 + \cos(2\alpha\Delta\mathrm{T})\cos\theta\ ), \\ a &= -\frac{\alpha}{|\alpha|}i\ \sin(2|\alpha|\Delta\mathrm{T})\cos\theta + \frac{\alpha}{\alpha^*}\sin\theta\ \sin^2(|\alpha|\Delta\mathrm{T}) + \sin\theta\ \cos^2(|\alpha|\Delta\mathrm{T}) \\ b &= -\frac{1}{\alpha}\sin(2|\alpha|\Delta\mathrm{T})\sin(\theta)\,\mathrm{Im}\{\alpha\}. \end{aligned}$$

The step work output dW is then given by

$$\begin{aligned} dW &= \mathrm{Tr}\{\rho'(H_{ST} - H'_{ST})\} = \mathrm{Tr}\Big\{\rho'(-\frac{\sigma_x}{2} - \mathrm{Re}\{\tau\}\sigma_x - \mathrm{Im}\{\tau\}\sigma_y)\Big\}, \\ &= -\frac{1}{2}\mathrm{Re}\{a\} - \mathrm{Re}\{\tau\}\,\mathrm{Re}\{a\} - \mathrm{Im}\{\tau\}\,\mathrm{Im}\{a\}, \\ \tau &= \frac{1}{2}\sin\theta_T e^{i\phi_T}. \end{aligned}$$

dW has a maximum for $\theta_T = \frac{\pi}{2}, \phi_T = \arctan\frac{\mathrm{Im}\{a\}}{\mathrm{Re}\{a\}} + \pi$, giving

$$dW = \frac{1}{2}\left(\sqrt{\mathrm{Re}\{a\}^2 + \mathrm{Im}\{a\}^2} - \mathrm{Re}\{a\}\right) \geq 0\ \forall a.$$

H'_{ST} is again anti-parallel to ρ'_S and thus the prerequisite (after a passive transformation) for the above is given.