$$
\begin{aligned}
H_I &= \sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ \\
H_{DST} &= H_I \otimes \mathbb{1} + \mathbb{1} \otimes H_I \\
\left|\psi_T\right\rangle &= cos(\theta/2)\left|0\right\rangle + e^{i\phi} sin(\theta/2)\left|1\right\rangle \\
\left|\dot{\psi}_T\right\rangle &= -sin(\theta/2)\dot{\theta}/2\left|0\right\rangle + e^{i\phi}(cos(\theta/2)\ \dot{\theta}/2 + i\dot{\phi}\ sin(\theta/2))\left|1\right\rangle \\
\frac{1}{2}\frac{dW}{dt} &= -\operatorname{Im}\left\langle\dot{\psi}_T\middle|\psi_T^{\lightning}\right\rangle = \operatorname{Im}\left\langle\dot{\psi}_T\middle| iH_T\left|\psi_T\right\rangle = \operatorname{Im}\left\langle\psi_D\right|\left\langle\psi_S\right|\left\langle\dot{\psi}_T\middle| iH_{DST}\left|\psi_D\right\rangle\left|\psi_S\right\rangle\left|\psi_T\right\rangle \\
&= \operatorname{Im}\left\langle\psi_D\right|\left\langle\psi_S\right|\left\langle\dot{\psi}_T\middle| iH_{DS}\otimes\mathbb{1}\left|\psi_D\right\rangle\left|\psi_S\right\rangle\left|\psi_T\right\rangle + \operatorname{Im}\left\langle\psi_D\right|\left\langle\psi_S\right|\left\langle\dot{\psi}_T\middle| i\mathbb{1}\otimes H_{ST}\left|\psi_D\right\rangle\left|\psi_S\right\rangle\left|\psi_T\right\rangle \\
&= \operatorname{Im} i\left\langle H_{DS}\right\rangle\left\langle\dot{\psi}_T\middle|\psi_T\right\rangle + \operatorname{Im}\left\langle\psi_S\right|\left\langle\dot{\psi}_T\middle| iH_{ST}\left|\psi_S\right\rangle\left|\psi_T\right\rangle \\
&= 0\ \ + \operatorname{Im}\left\langle\psi_S\right|\left\langle\dot{\psi}_T\middle| i\sigma_+\otimes\sigma_-\left|\psi_S\right\rangle\left|\psi_T\right\rangle + \operatorname{Im}\left\langle\psi_S\right|\left\langle\dot{\psi}_T\middle| i\sigma_-\otimes\sigma_+\left|\psi_S\right\rangle\left|\psi_T\right\rangle \\
&= -\operatorname{Im} i\left\langle\psi_S\right|\sigma_+\left|\psi_S\right\rangle e^{i\phi} sin^2(\theta/2)\dot{\theta}/2 + \operatorname{Im} i\left\langle\psi_S\right|\sigma_-\left|\psi_S\right\rangle e^{-i\phi}(cos^2(\theta/2)\dot{\theta}/2 - i\dot{\phi}\ sin(\theta/2)) \\
&= -\operatorname{Im} i\left\langle\psi_S\right|\sigma_+\left|\psi_S\right\rangle e^{i\phi} sin^2(\theta/2)\dot{\theta}/2 + \operatorname{Im}[-i\left\langle\psi_S\right|\sigma_+\left|\psi_S\right\rangle e^{i\phi}(cos^2(\theta/2)\dot{\theta}/2 + i\dot{\phi}\ sin(\theta/2))]^* \\
&= -\operatorname{Im} i\left\langle\psi_S\right|\sigma_+\left|\psi_S\right\rangle e^{i\phi} sin^2(\theta/2)\dot{\theta}/2 + \operatorname{Im} i\left\langle\psi_S\right|\sigma_+\left|\psi_S\right\rangle e^{i\phi}(cos^2(\theta/2)\dot{\theta}/2 + i\dot{\phi}\ sin(\theta/2)) \\
&= \operatorname{Im} i\left\langle\psi_S\right|\sigma_+\left|\psi_S\right\rangle e^{i\phi}(cos(\theta)\dot{\theta}/2 + i\dot{\phi}\ sin(\theta/2))
\end{aligned}
$$

$$
\begin{aligned}
H_S(\theta_D(t),\phi_D(t),\theta_T(t),\phi_T(t)) &= \frac{1}{2}\left[sin(\theta_D(t))e^{i\phi_D(t)} + sin(\theta_T(t))e^{i\phi_T(t)}\right]\sigma_+ + h.c. \\
&= \alpha\sigma_+ + h.c. \\
\rho_S(t+1) &= U\rho_S(t)U^\dagger, U = e^{-iH_S(t)\Delta t} \\
dW(t) &= \mathrm{Tr}\ \rho(t)dH \\
dH = H_S(\theta_D(t),\phi_D(t),\theta_T(t+1),\phi_T(t+1)) &- H_S(\theta_D(t),\phi_D(t),\theta_T(t),\phi_T(t)) \\
= \frac{1}{2}(sin(\theta_T(t+1))e^{i\phi_T(t+1)} &- sin(\theta_T(t))e^{i\phi_T(t)})\sigma_+ + h.c. =: (\tau' - \tau)\sigma_+ + h.c. \\
&\text{For N=2 only one jump, analytical solution is possible:} \\
U = e^{-iH_S\Delta t} = exp\begin{pmatrix} 0 & -i\alpha^*\Delta t \\ -i\alpha\Delta t & 0 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} \frac{\alpha^*}{|\alpha|} & -\frac{\alpha^*}{|\alpha|} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i|\alpha|\Delta t} & 0 \\ 0 & e^{i|\alpha|\Delta t} \end{pmatrix}\begin{pmatrix} \frac{|\alpha|}{\alpha^*} & 1 \\ -\frac{|\alpha|}{\alpha^*} & 1 \end{pmatrix} \\
&= \begin{pmatrix} cos(|\alpha|\Delta t) & -i\frac{\alpha^*}{|\alpha|}sin(|\alpha|\Delta t) \\ -i\frac{|\alpha|}{\alpha^*}sin(|\alpha|\Delta t) & cos(|\alpha|\Delta t) \end{pmatrix} \\
&\text{With } \rho_0 = |0\rangle\langle 0|\text{, the time evolved state } \rho = U\rho_0 U^\dagger \text{is} \\
\rho &= \begin{pmatrix} cos^2(|\alpha|\Delta t) & i\frac{|\alpha|}{2\alpha}sin(2|\alpha|\Delta t) \\ -i\frac{|\alpha|}{2\alpha^*}sin(2|\alpha|\Delta t) & sin^2(|\alpha|\Delta t) \end{pmatrix} \\
W = \mathrm{Tr}\begin{pmatrix} 0 & \tau'^* - \tau^* \\ \tau' - \tau & 0 \end{pmatrix} &\begin{pmatrix} cos^2(|\alpha|\Delta t) & i\frac{|\alpha|}{2\alpha}sin(2|\alpha|\Delta t) \\ -i\frac{|\alpha|}{2\alpha^*}sin(2|\alpha|\Delta t) & sin^2(|\alpha|\Delta t) \end{pmatrix} \\
&= -sin(2|\alpha|\Delta t)\,\mathrm{Im}\left\{(\tau' - \tau)\frac{|\alpha|}{\alpha}\right\}
\end{aligned}
$$

$$
\begin{aligned}
H_S^i &= \langle\psi_D^i|\langle\psi_T^i| H_{DST} |\psi_D^i\rangle|\psi_T^i\rangle \\
\rho^{i+1} &= U^i\rho^i U^{i\dagger} \\
U^i &= e^{-iH_S^i\Delta t} \\
W &= -\Sigma_i \mathrm{Tr}\rho^i dH^i \\
dH^i = \langle\psi_D^i|\langle\psi_T^{i+1}| H_{DST} |\psi_D^i\rangle|\psi_T^{i+1}\rangle &- \langle\psi_D^i|\langle\psi_T^i| H_{DST} |\psi_D^i\rangle|\psi_T^i\rangle
\end{aligned}
$$