Toric Varieties - Geometria Algebrica F

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Introduction

Syllabus

The first part of the course deals with:

- Algebraic Tori, their actions and representations
- Affine toric varieties (with monoids) $\leftrightarrow$ cones in some $\mathbb{R}^n$
- Projective toric varieties $\leftrightarrow$ polytopes in some $\mathbb{R}^n$
- General toric varieties $\leftrightarrow$ fans in $\mathbb{R}^n$

We will then deal with (subject to change)

- Divisors/line bundles on toric varieties
- Cox ring of a toric variety
- Cohomology of divisors
- Toric morphisms and resolution of singularities
- and more...?

The main reference for this course —"Toric varieties" by Cox, Little, Schenck [CLS11] —is available in the same folder as this PDF.

What is the course about?

We will work over an algebraically closed field (and we will be lax about the characteristic of the field). In [CLS11] the authors work over $\mathbb{C}$ but many results hold more generally.

The main goal of the course is understanding toric varieties:

Definition 0.1 (Toric variety)**.** An n-dimensional toric variety X is a (normal) k-variety equipped with an open immersion of an n-dimensional torus $T \subseteq X$, where $T \cong (k^*)^n$, and an action $T \times T \to T$ which extends to the whole of X [a].

[a] that is, it extends to a $T \times X \to X$

Remark 0.2. Normality is a standard assumption that we'll make at some point but some things work without it.

We'll see that the geometry of such an object is encoded in a combinatorial gadget, converting problems in algebraic geometry to problems in combinatorics, which is sometimes convenient.

The opposite reduction is also possible and has been used historically. One of the main examples of a combinatorial problem being solved via the geometry of toric varieties is

Example 0.3 (McMullen's "g-conjecture")**.** The then conjecture, and now theorem, is a characterization of the f-vectors of simple polytopes[1].

Definition 0.4 (f-vectors)**.** If P is a polytope, its f**-vector** is

$$(f_0(P), \cdots, f_d(P)), \quad \text{where } d = \dim P$$

and $f_i(P)$ is the number of i-dimensional faces. We may set $f_{-1}(P) = 1$.

It's reasonable to ask ourselves which f-vectors can appear. We may define the h**-vector** by setting

$$\sum_{i=0}^{d} f_i(t-1)^i = \sum_{i=0}^{d} h_i t^i, \quad \text{i.e. } h_i = \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} f_j, \ h_{-1} = 0.$$

It was a known theorem that the h-vector of a simple polytope is palindromic (i.e. $h_i = h_{d-i}$). From the h-vector we obtain the g**-vector** by setting $g_i = h_i - h_{i-1}$.

The conjecture was that

Theorem 0.5 (g-conjecture)**.** $f = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1}$ is the f-vector of a simple polytope if

1. $h_i = h_{d-1}$ for all $0 \leq i \leq \lfloor d/2 \rfloor$
2. $g_i \geq 0$ for all $0 \leq i \leq \lfloor d/2 \rfloor$
3. $(g_1, \cdots, g_{\lfloor d/2 \rfloor})$ is a "Macauly vector" if, when we write (uniquely)

$$g_i = \binom{n_i}{i} + \cdots + \binom{n_{r_i}}{r_i}$$

with $n_i > n_{i-1} > \cdots > n_{r_i}$ then

$$g_{i+1} \leq \binom{n_i + 1}{i+1} + \cdots + \binom{n_{r_i} + 1}{r_i + 1}$$

Stanley proved necessity using toric varieties. He proved that the g-vector of a simple polytope is the vector of dimensions for some cohomology ring of the associated toric variety.

Later McMullen found a completely combinatorial proof but for some time the only proof of this combinatorial fact passed through the geometry of toric varieties.

[1] for now a simple polytope is the convex hull of a finite subset of $\mathbb{R}^n$

Part I

Geometry of toric varieties

Chapter 1

Algebraic tori and their actions

1.1 Basic definitions

Definition 1.1 (Algebraic group)**.** An **algebraic group** G is a k-variety equipped with the the structure of a "group object" in the category of k-varieties, i.e. we have two morphisms and a *closed* point

$$m : G \times G \to G, \quad i : G \to G, \quad e \in G$$

that satisfy the usual group axioms "diagrammatically".

Example 1.2. Associativity can be expressed "diagrammatically" as

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(id_G, m)} & G \times G \\ {\scriptstyle (m, id_G)}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array}$$

Remark 1.3. If $G = \operatorname{Spec} A$ is an affine variety, a structure of algebraic group is equivalent to a structure of **Hopf algebra** on A:

$$\begin{aligned} m : G \times G \to G \quad &\longleftrightarrow \quad \Delta : A \to A \otimes_k A \\ i : G \to G \quad &\longleftrightarrow \quad S : A \to A \\ e : \operatorname{Spec} k \to G \quad &\longleftrightarrow \quad \varepsilon : A \to k \end{aligned}$$

and the homomorphisms Δ, S, ε satisfy the diagrammatic group axioms with the arrows reversed.

Remark 1.4. If G and H are algebraic groups, $G \times H$ is also naturally an algebraic group. For example

$$m_{G \times H} : \begin{array}{ccc} (G \times H) \times (G \times H) & \longrightarrow & G \times H \\ ((g_1, h_1), (g_2, h_2)) & \longmapsto & (m_G(g_1, g_2), m_H(h_1, h_2)) \end{array}.$$

Definition 1.5 (Homomorphism between Algebraic groups)**.** If G, H are algebraic groups over k then a homomorphism $f : G \to H$ is a morphism of k-varieties such that

$$\begin{array}{ccc} G \times G & \xrightarrow{(f,f)} & H \times H \\ {\scriptstyle m}\big\downarrow & & {\scriptstyle m}\big\downarrow \\ G & \xrightarrow[f]{} & H \end{array}$$

Remark 1.6. If G and H are affine, the axioms of homomorphism dualize to what a homomorphism of Hopf algebras should be.

Remark 1.7. All algebraic subgroups of an algebraic group are closed subvarieties.

The first example of algebraic group we present is the multiplicative group

Definition 1.8 (Multiplicative group)**.** The **multiplicative group**, denoted $\mathbb{G}_m$, is the k-variety $\mathbb{A}^1 \setminus \{0\}$ equipped with the morphisms

$$m : \begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ (a, b) & \longmapsto & ab \end{array}$$

$$i : \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ a & \longmapsto & 1/a \end{array}$$

$$e = 1 \in \mathbb{A}^1 \setminus \{0\}$$

(we are identifying $\mathbb{G}_m = k^*$).

Remark 1.9. $\mathbb{G}_m$ is affine, indeed $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and $\mathbb{A}^1 \setminus \{0\} = \mathbb{A}^1 \setminus V(x) = D(x)$, thus

$$D(x) = \operatorname{Spec}(k[x])_x = \operatorname{Spec}(k[x, x^{-1}]) = \operatorname{Spec} k[x^{\pm 1}].$$

If you are uncomfortable with "x^{-1}" appearing you may simply think of this coordinate ring as

$$\frac{k[x, y]}{(xy - 1)}.$$

Remark 1.10. The multiplication $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ can be described as the morphism corresponding to the k-algebra homomorphism

$$\begin{array}{ccc} k[x^{\pm 1}] & \longrightarrow & k[y^{\pm 1}] \otimes_k k[z^{\pm 1}] \\ x & \longmapsto & y \otimes z \end{array}.$$

Similarly, the inverse corresponds to

$$\begin{array}{ccc} k[x^{\pm 1}] & \longrightarrow & k[y^{\pm 1}] \\ x & \longmapsto & y^{-1} \end{array}$$

and the neutral element corresponds to[1]

$$\begin{array}{rcl} k[x^{\pm 1}] & \longrightarrow & k \\ x & \longmapsto & 1 \end{array}$$

Definition 1.11 (Algebraic tori)**.** The **standard n-dimensional algebraic torus over k** is $\mathbb{G}_m^n$. An **algebraic torus**[a] is an algebraic group T which is isomorphic to $\mathbb{G}_m^n$ for some n.

[a]we may simply say "torus" if no confusion can occur.

Remark 1.12. If $k = \mathbb{C}$ then $\mathbb{G}_m^n = (\mathbb{C}^*)^n$, which is homotopy equivalent to $(S^1)^n$. This $(S^1)^n$ is the "maximal compact subgroup" and is the reason why these groups are called tori in the first place.

1.2 Cartier duality

In this section we will define an equivalence of categories between finitely generated abelian groups[2] and a specific type of algebraic groups. Under this correspondence, tori will be "dual" to finitely generated free abelian groups.

1.2.1 Group algebra and Cartier dual

The first step is transforming general (finitely generated) abelian groups into (finite type reduced) algebras over k, the way we do this is via the

Definition 1.13 (Associated group algebra)**.** If M is a finitely generated abelian group, the **k-group algebra of** M, denoted by $k[M]$, is the k-vector space spanned formally by the basis $\{t^m \mid m \in M\}$ together with the multiplication induced by $t^m t^{m'} = t^{m+m'}$.

Example 1.14. If $M = \mathbb{Z}^n$ then

$$k[\mathbb{Z}^n] = k[t^{(1,0,\cdots,0)}, t^{(-1,0,\cdots,0)}, \cdots, t^{(0,\cdots,0,-1)}] = k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}],$$

which is the coordinate ring of $(\mathbb{G}_m)^n$.

[1]recall that a k-point e of the variety G can be seen as a morphism $\operatorname{Spec} k \to G$ with set-theoretic image e.

[2]with no p-torsion if $p = \operatorname{char} k \neq 0$

Fact 1.15. These formulas give $k[M]$ a Hopf algebra structure for all finitely generated abelian groups M

$$\Delta : \begin{array}{ccc} k[M] & \longrightarrow & k[M] \otimes_k [M] \\ t^m & \longmapsto & t^m \otimes t^m \end{array}$$

$$S : \begin{array}{ccc} k[M] & \longrightarrow & k[M] \\ t^m & \longmapsto & t^{-m} \end{array}$$

$$\varepsilon : \begin{array}{ccc} k[M] & \longrightarrow & k \\ t^m & \longmapsto & 1 \end{array}$$

Remark 1.16. If we see $\mathbb{G}_m^n$ as $\operatorname{Spec} k[\mathbb{Z}^n]$ then the usual algebraic group structure is the one induced by the maps we just mentioned.

Remark 1.17. If M is finitely generated then $k[M]$ is of finite type over k. It turns out that it is also reduced when M has no elements of order divided by the characteristic of k.

Definition 1.18 (Cartier dual)**.** If M is a finitely generated abelian group, $D(M) := \operatorname{Spec} k[M]$ is the **cartier dual** of M.

Let us compute the cartier dual of another type of finitely generated abelian group:

Example 1.19. If $M = \mathbb{Z}/n\mathbb{Z}$ then the group algebra is

$$k[\mathbb{Z}/n\mathbb{Z}] = \frac{k[t]}{(t^n - 1)}.$$

$\operatorname{Spec} k[\mathbb{Z}/n\mathbb{Z}]$ then is the closed subvariety (and subgroup) of $\mathbb{G}_m$ described by the equation $t^n = 1$, i.e. the group of the n-th roots of unity μ_n

Definition 1.20 (Group of n-th roots of unity)**.** $\mu_n = D(\mathbb{Z}/n\mathbb{Z})$.

Remark 1.21. If $n = p = \operatorname{char} k$ then $(t^p - 1) = (t - 1)^p$, so μ_p would be a point. To get any interesting geometric information in this case you need to allow nilpotens, stumbling into the teorritory of group schemes.

Since we know the structure theorem for finitely generated abelian groups, let us consider the following

Exercise 1.22. $D(M \oplus N) = D(M) \times D(N)$.

Solution (Sketch).
It is enough to note that $k[M \oplus N] = k[M] \otimes k[N]$ and this follows from the fact that

$$t^{(m,n)} = t^{(m,0)} t^{(0,n)}.$$

□

It follows that

Proposition 1.23. For a general finitely generated abelian group

$$M = \mathbb{Z}^n \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$$

the Cartier dual is

$$D(M) \cong \mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_k}.$$

Since we hope to find an equivalence of categories, let us try to understand another way in which we can view these types of algebraic groups.

Remark 1.24. GL_n is an algebraic group: It is a variety when seen as[3] $\mathbb{A}^{n^2} \setminus V(\det)$ and it can be checked that matrix multiplication and inversion are morphisms of k-varieties.

Definition 1.25 (Diagonizable group)**.** An algebraic group is called **diagonalizable** if it is isomorphic to a (closed) subgroup of $\mathrm{Diag}_n \subseteq \mathrm{GL}_n$ for some n

Remark 1.26. $\mathrm{Diag}_n \cong \mathbb{G}_m^n$ and the isomorphism is given by ignoring the entries which aren't on the diagonal.

Remark 1.27. $D(M)$ is diagonalizable, because

$$D(M) \cong \mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_k} \subseteq \mathbb{G}_m^{n+k} \cong \mathrm{Diag}_{n+k}.$$

Proposition 1.28. If $\varphi : M \to N$ is a group homomorphism

$$k[\varphi] : \begin{array}{ccc} k[M] & \longrightarrow & k[N] \\ t^m & \longmapsto & t^{\varphi(m)} \end{array}$$

is a k-algebra homomorphism and so $D(\varphi) = \mathrm{Spec}(k[\varphi]) : D(N) \to D(M)$ is a morphism of k-varieties.
This is actually a homomorphism of algebraic groups and the association is functorial.

Cartier duality is that statement that

$$D : (\text{fin.gen.AbGps}_{\text{no } p\text{-tors}})^{op} \to (\text{Diag.AlgGps}),$$

where $p = \operatorname{char} k$, is an equivalence of categories. To prove this fact we will build an inverse functor

[3]the determinant is a homogeneous polynomial of degree n

1.2.2 Character group

To find the "inverse" functor, we want to build a finitely generated abelian group from an algebraic group. The construction that will end up being what we want is the *group of characters*

Definition 1.29 (Character)**.** A **character** of an algebraic group G is a homomorphism $\chi : G \to \mathbb{G}_m$. We denote the set of all characters $X(G)$.

Remark 1.30. The characters of an algebraic group G form an abelian group via:

$$\chi_1 : G \to \mathbb{G}_m, \quad \chi_2 : G \to \mathbb{G}_m \quad \rightsquigarrow \quad \chi_1 \cdot \chi_2 : G \xrightarrow{(\chi_1,\chi_2)} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m.$$

From now on $X(G)$ will always also have the group structure.

Example 1.31. If $G = \mathbb{G}_m$ then for $k \in \mathbb{Z}$

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ a & \longmapsto & a^k \end{array}$$

is a character, which corresponds to

$$\begin{array}{ccc} k[x^{\pm 1}] & \longrightarrow & k[x^{\pm 1}] \\ x & \longmapsto & x^k \end{array}$$

Example 1.32. If $G = \mathbb{G}_m^n$ and $(k_1, \cdots, k_n) \in \mathbb{Z}^n$ then

$$\begin{array}{ccc} \mathbb{G}_m^n & \longrightarrow & \mathbb{G}_m \\ (a_1, \cdots, a_n) & \longmapsto & a_1^{k_1} \cdots a_n^{k_n} \end{array}.$$

We will see that these are all the characters on the torus.

Example 1.33. If $G = \mathrm{GL}_n$ the determinant is a character

$$\begin{array}{ccc} \mathrm{GL}_n & \longrightarrow & \mathbb{G}_m \\ M & \longmapsto & \det M \end{array}$$

Definition 1.34 (Group-like elements)**.** A **group-like element** in a Hopf algebra A is an $a \in A$ such that a is invertible and $\Delta(a) = a \otimes a$.

Lemma 1.35. If $G = \operatorname{Spec} A$ is an affine algebraic group, characters of G correspond to group-like elements of A.

Proof.
A character $\chi : \operatorname{Spec} A \to \mathbb{G}_m$ corresponds to a homomorphism of Hopf algebras $k[x^{\pm 1}] \to A$ which sends x to some $a \in A$. The homomorphism is uniquely determined by a so we just need to check which elements of A can be the image of x. Since x has an inverse, $a \in A^*$ and $\Delta(a) = a \otimes a$ because $\Delta(x) = x \otimes x$. On the other hand, an element which satisfies those properties does yield a Hopf-algebra homomorphism, so we are done. $\square$

1.2.3 Proof of Cartier duality

Remark 1.36. Constructing the character group extends to a functor

$$X : (\mathrm{AlgGps}) \to (\mathrm{AbGps})$$

via pullback, i.e. the map $f : G \to H$ becomes

$$X(f) \begin{array}{ccc} X(H) & \longrightarrow & X(G) \\ \chi & \longmapsto & \chi \circ f \end{array}$$

Now that we have built our candidate for the inverse functor, all we need to show that that the two compositions are naturally isomorphic to the identity.

Proposition 1.37. The map $M \to X(D(M))$ which to an element $m \in M$ assigns the character which corresponds to the Hopf-Algebra homomorphism

$$\begin{array}{ccc} k[x^{\pm 1}] & \longrightarrow & k[M] \\ x & \longmapsto & t^m \end{array}$$

is a natural isomorphism.

Proof.
It is easy to check that $M \to X(D(M))$ is a group homomorphism.

inj. If $m_1 \neq m_2$ then $t^{m_1} \neq t^{m_2}$ and so the induced Hopf algebra homomorphisms are different.

onto Given lemma (1.35), we just need to show that the only group-like elements of $k[M]$ are the t^m for $m \in M$. Let us take any element $a = \sum_{m \in M} a_m t^m$ of $k[M]$ and impose that $\Delta(a) = a \otimes a$, then

$$\Delta(a) = \Delta\left(\sum_{m \in M} a_m t^m\right) = \sum_{m \in M} a_m \Delta(t^m) = \sum_{m \in M} a_m t^m \otimes t^m$$

$$a \otimes a = \left(\sum_{m \in M} a_m t^m\right) \otimes \left(\sum_{m' \in M} a_{m'} t^{m'}\right) = \left(\sum_{m, m' \in M} a_m a_{m'} t^m \otimes t^{m'}\right).$$

Since the $t^m \otimes t^{m'}$ form a basis of $k[M] \otimes k[M]$, if $m \neq m'$ then $a_m a_{m'} = 0$. Thus there exists at most one nonzero coefficient a_{m_0} and $a = a_{m_0} t^{m_0}$, but a must be invertible so $a_{m_0} \neq 0$. Also, again imposing the comultiplication condition, $a_{m_0}^2 = a_{m_0}$, which implies that $a_{m_0} = 1$ since it isn't 0.

□

Corollary 1.38. For $M = \mathbb{Z}^n$ we get $X(\mathbb{G}_m^n) \cong \mathbb{Z}^n$ and the characters are the ones we wrote above[a].

[a] $(a_1, \cdots, a_n) \mapsto a_1^{k_1} \cdots a_n^{k_n}$

Let us now consider the other composition:

Remark 1.39. There is a canonical map $\operatorname{Spec} A = G \to D(X(G))$.

Proof.
Let $\chi : G \to \mathbb{G}_m$ be a character of G. Upon composition with the inclusion $\mathbb{G}_m \subseteq \mathbb{A}^1$ we get a morphism in $\operatorname{Hom}(G, \mathbb{A}^1)$ and this set is canonically identified with A, so we get a map

$$\varphi : X(G) \to A.$$

This is a group homomorphism, which induces the desired map

$$\begin{array}{ccc} k[X(G)] & \longrightarrow & A \\ t^m & \longmapsto & \varphi(m) \end{array}.$$

$\square$

Lemma 1.40. Let G be an abstract group (no algebraic structure) and $\mathbb{K}$ be any field, if we take $\phi_i : G \to \mathbb{K}^*$ distinct group homomorphisms then the ϕ_i are linearly independent in[a] $\operatorname{Fun}(G, \mathbb{K})$

[a]not homomorphisms of any kind, just set theoretic functions. It is a $\mathbb{K}$-vector space by looking at the strucutre on the codomain.

Proof.
Let us assume by contradiction that we have a non-triavial relation $\sum a_i\phi_i = 0$ for some $a_i \in \mathbb{K}$ and let's assume that this relation has minimal length.

By definition, $\sum a_i\phi_i(gh) = \sum a_i\phi_i(g)\phi_i(h) = 0$ for all $g, h \in G$. Pick $\overline{g} \in G$ such that $\phi_1(\overline{g}) \neq \phi_2(\overline{g})$ (which we can do because $\phi_1 \neq \phi_2$). Setting $g = \overline{g}$ in the expression we get

$$\sum a_i\phi_i(\overline{g}h) = \sum \underbrace{a_i\phi_i(\overline{g})}_{\in\mathbb{K}}\phi_i(h) = 0$$

that is, $\sum a_i\phi_i(\overline{g})\phi_i = 0$ is an equality in $\operatorname{Fun}(G, \mathbb{K})$. Multiplying the initial relation by $\phi_1(\overline{g})$ we get

$$\sum a_i\phi_1(\overline{g})\phi_i = 0$$

subtracting the two functions we get

$$\sum a_i(\phi_1(\overline{g}) - \phi_i(\overline{g}))\phi_i = 0$$

which is a shorter (look at $i = 1$) non-trivial (look at $i = 2$) reation, which is a contradiction. $\square$

Proposition 1.41. If G is diagonalizable then the homomorphism $G \to D(X(G))$ is an isomorphism and $X(G)$ is finitely generated. Moreover, if $\operatorname{char} k = p \neq 0$ then $X(G)$ has no p-torsion.

Proof.
Take a diagonalizable group G and consider it as a closed subgroup $G \subseteq \mathbb{G}_m^n = \operatorname{Diag}_n$.

Since it is *closed* and $\mathbb{G}_m^n$ is affine, $G = \operatorname{Spec} A$ is also affine and we get a surjection[4] $k[\mathbb{Z}^n] \to A$.

Now note that we have $\mathbb{Z}^n \cong X(\mathbb{G}_m^n) \to X(G)$ and the surjection above factors

$$k[\mathbb{Z}^n] \to k[X(G)] \to A$$

since the composition is surjective, $k[X(G)] \to A$ is also surjective. To conclude the first part of the proof then, we just need to show that the map is also injective, but this follows from the lemma.

Now we concern ourselves with finite generation. Because of the isomorphism we just proved, the factorization

$$k[\mathbb{Z}^n] \to k[X(G)] \to A$$

now shows that $k[\mathbb{Z}^n] \to k[X(G)]$ is surjective because $k[\mathbb{Z}^n] \to A$ was. This lets us conclude that $\mathbb{Z}^n \to X(G)$ is surjective (and thus $X(G)$ is finitely generated) because otherwise $k[\mathbb{Z}^n] \to k[X(G)]$ wouldn't be.

Suppose now that $0 \neq p = \operatorname{char} k$. Let $\chi \in X(G)$ be a p-torsion character, i.e. $\chi^p = 1$, that is, $\chi(g)^p = 1$ for all $g \in G$. Because $x^p - 1 = (x-1)^p$ in characteristic p, $\chi(g) = 1$ for all $g \in G$, showing that $\chi = 1$ and thus the absence of p-torsion. □

Corollary 1.42. A connected subgroup of a torus is a torus.

Proof.
If $G \subseteq \mathbb{G}_m^n$, from the proposition we get that

$$G = D(X(G)) \cong \mathbb{G}_m^k \times \mu_{n_1} \times \cdots \times \mu_{n_r},$$

but if G is connected then all n_i must be 1 because otherwise that product would be disconnected. □

Having now verified both compositions we may formally state Cartier duality as a theorem now

Theorem 1.43 (Cartier duality)**.** The functor

$$D : (\text{fin.gen.AbGps}_{\text{no } p\text{-tors}})^{op} \to (\text{Diag.AlgGps}),$$

where $p = \operatorname{char} k$, is an equivalence of categories. The inverse functor is X.

Remark 1.44. If we allow group schemes the problem with p-torsion doesn't come up.

[4]the surjection corresponds to taking $k[\mathbb{Z}^n] \to k[\mathbb{Z}^n]/I$ where I is the ideal which defines G as $V(I) \subseteq \mathbb{G}_m^n$.

Image of map between tori is a torus

Proposition 1.45. Let $f : T_1 \to T_2$ be a homomorphism of tori, then the image is also a torus.

Proof.
Since $T_1 \to D(X(T_1))$ and $T_2 \to D(X(T_2))$ are isomorphisms and the appropriate diagrams commute, we have that f is induced by the corresponding homomorphism $M_2 \to M_1$ where $M_1 = X(T_1)$ and $M_2 = X(T_2)$.

Let $K = \ker(M_2 \to M_1)$ and note that $M_2 \twoheadrightarrow M_2/K \hookrightarrow M_1$. We claim that $L := \ker(k[M_2] \to k[M_1])$ is the ideal $I = (t^m - t^{m'} \mid \varphi(m) = \varphi(m'))$:

$\boxed{I \subseteq L}$ It suffices to note that the generators of I lie in L, indeed $t^m - t^{m'} \mapsto t^{\varphi(m)} - t^{\varphi(m')} = 0$.

$\boxed{L \subseteq I}$ Let $\sum_{m \in M_2} a_m t^m$ be a general element of L, then

$$\sum_{n \in M_1} \left(\sum_{m \in \varphi^{-1}(n)} a_m \right) t^n = 0 \overset{\text{lin.ind.}}{\Longrightarrow} \sum_{m \in \varphi^{-1}(n)} a_m = 0 \quad \forall n \in M_1$$

For a fixed n, if $a_{m_1}, a_{m_2} \neq 0$ for some $m_1, m_2 \in \varphi^{-1}(n)$ (if all are 0 ok, just one nonzero is impossible given that the whole sum is zero) we can write

$$\sum a_m t^m = \underbrace{a_{m_1}(t^{m_1} - t^{m_2})}_{\in I} + \underbrace{(a_{m_2} + a_{m_1})t^{m_2} + \sum_{m \neq m_1, m_2} a_m t^m}_{\text{removed term with } t^{m_1}}$$

iterating this process shows the other inclusion.

Thus we can factor $k[M_2] \to k[M_1]$ as $k[M_2] \twoheadrightarrow k[M_2]/I \hookrightarrow k[M_1]$. One can check that $k[M_2]/I = k[M_2/K]$. Since $M_2/K \hookrightarrow M_1$ and M_1 is a free abelian group, M_2/K is also free and thus

$$T_1 \twoheadrightarrow \underbrace{\operatorname{Spec} k[M_2/K]}_{\text{torus}} \hookrightarrow T_2$$

where to check injectivity we use $k[M_2] \to k[M_2/K]$ surjective and to check surjectivity, because subgroups are closed, it is enough to check for dominance and indeed $k[M_2/K] \to k[M_1]$ is injective. □

Remark 1.46. We could have just said that the image is a connected subgroup of a torus and thus is also a torus, but the proof given is more instructive.

1.3 1 parameter subgroups and lattices

We now define a *dual* notion to characters (we will make this precise shortly).

Definition 1.47 (1-parameter subgroup)**.** A **1-parameter subgroup** (or **cocharacter** or **1-ps**) of an algebraic group G is a homomorphism $\lambda : \mathbb{G}_m \to G$.

Exercise 1.48. A non-trivial quotient of $\mathbb{G}_m$ is isomorphic to $\mathbb{G}_m$. More generally, a non-trivial quotient of a torus is isomorphic to a torus.

Remark 1.49. If $\lambda : \mathbb{G}_m \to G$ is a homomorphism, the image is isomorphic to $\mathbb{G}_m / \ker \lambda$ by the first isomorphism theorem and because of the above exercise this quotient is again isomorphic to $\mathbb{G}_m$.

Remark 1.50. 1-paramenter subgroups of G form a group via

$$\lambda_1 \cdot \lambda_2 : \mathbb{G}_m \xrightarrow{(\lambda_1, \lambda_2)} G \times G \xrightarrow{m} G.$$

Remark 1.51. If G is abelian the group of 1-ps is abelian.

Proposition 1.52. If $(h_1, \cdots, h_n) \in \mathbb{Z}^n$, the morphism

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m^n \\ a & \longmapsto & (a^{h_1}, \cdots, a^{h_n}) \end{array}$$

is a 1-ps of $\mathbb{G}_m^n$. Moreover, all 1-ps of $\mathbb{G}_m^n$ are of this form. In particular, the group of 1-ps of $\mathbb{G}_m^n$ is isomorphic to $\mathbb{Z}^n$.

Proof.
If $\lambda : \mathbb{G}_m \to \mathbb{G}_m^n$ is a 1-ps then the compositions with the projections $\pi_i : \mathbb{G}_m^n \to \mathbb{G}_m$ yield characters of $\mathbb{G}_m$, so $\pi_i \circ \lambda(a) = a^{h_i}$ for some $h_i \in \mathbb{Z}$. ☐

Remark 1.53. In general, if $f : \mathbb{G}_m^{n_1} \to \mathbb{G}_m^{n_2}$ is a homomorphism then there are $k_1, \cdots, k_{n_1} \in \mathbb{Z}^{n_2}$ such that

$$f(a_1, \cdots, a_{n_1}) = a_1^{k_1} \cdots a_{n_1}^{k_{n_1}}$$

where $a^{(k_{1,h}, \cdots, k_{n_2,h})} = (a^{k_{1,h}}, \cdots, a^{k_{n_2,h}})$.

1.3.1 Character- and cocharacter- lattice

We have seen that the group of characters and cocharacters are free abelian groups of finite rank, let us formalize this by introducing lattices

Definition 1.54 (Lattice)**.** A **lattice** is a free abelian group of finite rank.

Definition 1.55 (Character lattice)**.** The **character lattice** of a torus T is the group of characters $M = X(T)$.

Definition 1.56 (Cocharacter lattice)**.** The **cocharacter lattice** of a torus T is the group of 1-parameter subgroups N.

Notation. If $m \in M$ we may write χ^m to mean the character m, similarly for $n \in N$ and λ^n. While this is technically redundant, it is useful when we identify M and N with the abstract $\mathbb{Z}^k$.

Proposition 1.57. The lattices M and N are dual.

Proof.
We have a symmetric $\mathbb{Z}$-bilinear pairing

$$\langle , \rangle : \begin{array}{ccc} M \times N & \longrightarrow & \mathbb{Z} \\ (\chi, \lambda) & \longmapsto & k \end{array}$$

where k is the unique integer such that $\chi \circ \lambda(a) = a^k$.

One can check that this becomes[5] the standard pairing $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by the dot product upon choosing an isomorphism $T \cong \mathbb{G}_m^n$. In particular this is a non-degenerate pairing. □

Remark 1.58. There is an isomorphism of groups

$$\begin{array}{ccc} N \otimes_{\mathbb{Z}} k^* & \longrightarrow & T \\ u \otimes t & \longmapsto & \lambda^u(t) \end{array}$$

this amounts to saying that $T \cong \mathbb{G}_m^{\operatorname{rnk} N}$.

Notation. From now on, the torus with cocharacter lattice N will be denoted by T_N. It's "the" torus because $T_N = D(N^*)$ where N^* is the dual of N in the sense we had above.

Remark 1.59. Fixing an isomorphism $T_N \cong \mathbb{G}_m^n$ is equivalent to fixing a $\mathbb{Z}$-basis of N (or M equivalently).

1.4 Actions and representations

Definition 1.60. An **action** of an algebraic group G on a variety X is a morphism $\mu : G \times X \to X$ that satisfies the diagrammatic axioms of an action:

$$\begin{tikzcd} G \times G \times X \arrow[r, "(id_G, \mu)"] \arrow[d, "(m, id_X)"'] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"'] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} k \times X \arrow[r, "\sim"] \arrow[d, "(e, id_X)"'] & X \\ G \times X \arrow[ur, "\mu"'] & \end{tikzcd}$$

[5] $\chi^{(k_1, \cdots, k_n)}(a_1, \cdots, a_n) = a_1^{k_1} \cdots a_n^{k_n}$ and $\lambda^{(k_1, \cdots, k_n)}(a) = (a^{k_1}, \cdots, a^{k_n})$, thus

$$\chi^{e_j}(\lambda^{e_i}(a)) = \chi^{e_j}(1, \cdots, \overset{\overset{i}{\downarrow}}{a}, \cdots, 1) = a^{\delta_{ij}}.$$

Example 1.61. The conjugation action

$$\begin{array}{ccc} \mathrm{GL}_n \times \mathcal{M}(n) & \longrightarrow & \mathcal{M}(n) \\ (A,B) & \longmapsto & ABA^{-1} \end{array}$$

is an action of the algebraic group GL_n on $\mathcal{M}(n)$.

Example 1.62. Multiplication

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ (a,z) & \longmapsto & a \cdot z \end{array}$$

is an action. This action *extends* to the projective line

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ (a,[x_0,x_1]) & \longmapsto & [x_0, ax_1] \end{array}$$

Definition 1.63. A (finite dimensional) **representation** of an algebraic group G is an algebraic group homomorphism $G \to \mathrm{GL}(V)$ for a (finite dimensional) k-vector space V.

Remark 1.64. A representation in this sense yields a set-theoretic linear action of G on $V \cong k^n$.

Fact 1.65. If $G = \operatorname{Spec} A$ is affine, so that A is a Hopf algebra, representations of G are in bijection with A-comodules, that is, k-vector spaces V equipped with the following data:

- $\rho : V \to V \otimes_k A$ a k-linear map
- $$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & A \otimes V \arrow[d, "id_A \otimes \rho"] \\ A \otimes V \arrow[r, "\Delta \otimes id_V"'] & A \otimes A \otimes V \end{tikzcd}$$
- $$\begin{tikzcd} k \otimes V & V \arrow[l, "\sim"'] \arrow[dl, "\rho"] \\ A \otimes V \arrow[u, "\varepsilon \otimes id_V"] & \end{tikzcd}$$

Moreover, subrepresentations correspond to subcomodules.

Idea.
From ρ you get $\begin{array}{ccc} G \times V & \longrightarrow & V \\ (g,v) & \longmapsto & \rho(v)(g) \end{array}$ and this is a linear action. $\square$

Proposition 1.66. Let $\alpha : T \to \mathrm{GL}(V)$ be a finite dimensional representation of the torus T. For $m \in M$ set

$$V_m = \{v \in V \mid \forall t \in T,\ \alpha(t)(v) = \chi^m(t) \cdot v\},$$

then

$$V = \bigoplus_{m \in M} V_m.$$

Sketch for $n = 1$.
Let $\rho : V \to V \otimes_k k[x^{\pm 1}]$ be the corresponding coaction to α.

- If $m \in M = \mathbb{Z}$ we have that (exercise)

$$V_m = \{v \in V \mid \rho(v) = v \otimes x^m\}.$$

- If $\rho(v) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes x^m$, then $f_m : V \to V$ is linear and $v = \sum_{m \in M} f_m(v)$.
- $f_m(v) \in V_m$
- $f_m \circ f_n = 0$ if $n \neq m$

This tells us that $\{f_n\}_{n \in \mathbb{Z}}$ is a family of orthogonal projectors, so $V = \bigoplus_{m \in M} V_m$. $\square$

Chapter 2

Affine toric varieties

2.1 Introduction

Definition 2.1 (Affine toric variety)**.** An **affine toric variety** is an irreducible affine variety X equipped with an open embedding of a torus T such that the translation action $T \times T \to T$ extends to an action of T on X.

Remark 2.2. The open torus is automatically dense in, and of the same dimension of, X.

Remark 2.3. The extension of the action is unique because if X and Y are irreducible affine and $f, g : X \to Y$ agree on a dense open subset then $f = g$.

Example 2.4. A torus is a toric variety.

Example 2.5. Affine space $\mathbb{A}^n$ is a toric variety, via the trivial embedding

$$\mathbb{G}_m^n = \{x_1 \cdots x_n \neq 0\} \subseteq \mathbb{A}^n.$$

Example 2.6. Let $C = V(x^3 - y^2) \subseteq \mathbb{A}^2$ with torus

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & C \\ t & \longmapsto & (t^2, t^3) \end{array}$$

and action

$$\begin{array}{ccc} \mathbb{G}_m \times C & \longrightarrow & C \\ (t,(x,y)) & \longmapsto & (t^2 x, t^3 y) \end{array}.$$

Notice that this affine toric variety is neither smooth nor normal[1].

Fact 2.7. A normal variety is smooth in codimension 1, that it, the singular locus has codimension at least 2. In particular a curve is normal iff they're smooth.

Example 2.8. Let $X = V(xy - z^2) \subseteq \mathbb{A}^3$ be the *quadric cone*. It can be shown that X is normal, but it is not smooth (not at the origin).

[1] $\operatorname{Spec} A$ irreducible affine variety is **normal** if all local rings are integrally closed in $\operatorname{Frac} A$. This is equivalent to A being integrally closed in $\operatorname{Frac} A$.

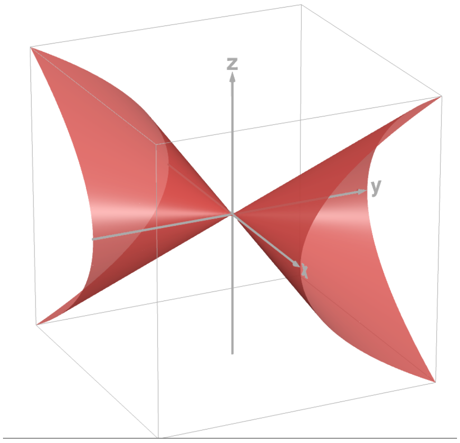


Figure 2.1: Quadric cone over the real numbers.

X is a toric variety with torus given by the image of[2]

$$\begin{array}{ccc} \mathbb{G}_m^2 & \longrightarrow & X \\ (s,t) & \longmapsto & (s^2, t^2, st) \end{array}$$

and action

$$\begin{array}{ccc} \mathbb{G}_m^2 \times X & \longrightarrow & X \\ ((s,t),(x,y,z)) & \longmapsto & (sx, st^2y, stz) \end{array}$$

Example 2.9. $X = V(xy - zw) \subseteq \mathbb{A}^4$ is a toric variety with torus

$$\begin{array}{ccc} \mathbb{G}_m^3 & \longrightarrow & X \\ (t_1,t_2,t_3) & \longmapsto & (t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \end{array}$$

and action

$$\begin{array}{ccc} \mathbb{G}_m^3 \times X & \longrightarrow & X \\ ((t_1,t_2,t_3),(x,y,z,w)) & \longmapsto & (t_1 x, t_2 y, t_3 z, t_1 t_2 t_3^{-1} w) \end{array}$$

2.2 Monoids

Definition 2.10 (Monoid)**.** A **monoid** is a set S with an operation $+$, which is commutative, associative and with a neutral element $0 \in S$.

Remark 2.11. The reference book [CLS11] calls these *semigroups*.

[2] this map is 2:1, to get the actual parametrization we need

$$\begin{array}{ccc} \mathbb{G}_m^2 & \longrightarrow & X \\ (s,t) & \longmapsto & (s, st^2, st) \end{array}$$

This is related to the fact that X is the quotient $\mathbb{A}^2/\mu_2$ by the action $-1(x,y) = (-x,-y)$.

Definition 2.12. If $A \subseteq S$ is a subset of a monoid, the **submonoid generated by** A **in** S is the smallest submonoid which contains A. Concretely it is

$$\langle A \rangle = \left\{ \sum_{a \in A} n_a a \mid n_a \in \mathbb{N},\ n_a = 0 \text{ for all but finitely many coeff.} \right\}$$

A monoid S is **finitely generated** if there exists a finite subset $A \subseteq S$ such that $S = \langle A \rangle$.

Remark 2.13. S is a finitely generated monoid if there exists a surjective monoid homomorphism

$$\mathbb{N}^n \twoheadrightarrow S.$$

Definition 2.14. S is an **affine monoid** if it is finitely generated and it is a submonoid of a lattice M.

Example 2.15. $\mathbb{N}^k \subseteq \mathbb{Z}^k$ is an affine monoid.

Example 2.16. $\mathbb{Z}/n\mathbb{Z}$ is a monoid but it is NOT affine because a lattice can't have a submonoid with torsion.

Example 2.17. $\langle (1,0), (1,1) \rangle \subseteq \mathbb{N} \oplus \mathbb{Z}/2\mathbb{Z}$ is also not affine because of torsion.

Definition 2.18 (Integrality)**.** A monoid S is **integral** (or **cancellative**) if $a + b = a + c \implies b = c$.

Fact 2.19. A monoid S is affine if and only if S is

- finitely generated,
- integral and
- torsion free.

Let us now define the left adjoint to the forgetful Ab $\to$ Mon:

Definition 2.20 (Associated group)**.** Let S be a monoid. There is an **associated abelian group** S^{gp}, which is the initial group with a morphism from S. Concretely

$$S^{gp} = \frac{\{(s, s') \mid s, s' \in S\}}{\sim}$$

where $(s_1, s_1') \sim (s_2, s_2')$ if there exists $s \in S$ such that[a]

$$s + s_1 + s_2' = s + s_2 + s_1'.$$

[a]think about localization on rings which are not domains.

Remark 2.21. S^{gp} is an abelian group and we have a map $S \to S^{gp}$ given by $s \mapsto [(s,0)]_\sim$.

Fact 2.22. Any morphism $S \to G$ for G abelian group factors uniquely through S^{gp}. More precisely

$$\mathrm{Hom}_{\mathrm{Mon}}(S,G) = \mathrm{Hom}_{\mathrm{Ab}}(S^{gp},G)$$

Remark 2.23. S is integral if and only if $S \to S^{gp}$ is injective, which happens if and only if S can be injected into an abelian group.

Definition 2.24. A monoid is **sharp** if the only invertible element is 0.

Definition 2.25. An element m of a sharp monoid S is **irreducible** if $m = m' + m''$ in S implies $m' = 0$ or $m'' = 0$.

Remark 2.26. If S is a sharp monoid, the irreducible elements generate the monoid.

Presentations of monoids

With monoids, the kernel is "sort of useless"

Example 2.27. Consider

$$\begin{array}{ccc} \mathbb{N}^2 & \longrightarrow & \mathbb{N} \\ (a,b) & \longmapsto & a+b \end{array}$$

this has trivial kernel (preimage of 0 is just $(0,0)$) but it is far from being injective.

Let $f : S \to S'$ be a surjective homomorphism. What we should look at instead of the kernel for the right analogue of the first isomorphism theorem is

$$E = \{(s,s') \in S \times S \mid f(s) = f(s')\}.$$

This set is an equivalence relation on $S \times S$, which is also a submonoid.

Definition 2.28 (Congruence relations)**.** A submonoid of $S \times S$ which defines an equivalence relation is called **congruence relation**.

Definition 2.29 (Coequalizer)**.** If $f, g : X \to Y$, the coequalizer is an object Z together with $h : Y \to Z$ such that $h \circ f = h \circ g : X \to Z$ and if W together with $h' : Y \to W$ is also such that $h' \circ f = c' \circ g$ then there exists a unique $Z \to W$ making everything commute.

$$\begin{tikzcd} & & W \\ X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[ur, "h'"] \arrow[r, "h"'] & Z \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

Fact 2.30. We can construct quotients of S by a congruence relation E on $S \times S$ by setting it to be the coequalizer of $E \subseteq S \times S \rightrightarrows S$, where the arrows are the two projections from $S \times S$ to S.
We call this object the **quotient of** S **by** E and denote it S/E.

Remark 2.31. If E is the relation constructed from $f : M \twoheadrightarrow M'$ homomorphism of abelian groups viewed as monoids then $E = \{(m, m') \in M \times M \mid f(m) = f(m')\} = \{(m, m') \mid m - m' \in \ker f\}$. It follows that $M' \cong M/\ker f$ is a coequalizer for $E \rightrightarrows M$, so our definition makes sense.

Definition 2.32 (presentation of a monoid)**.** The monoid associated to

$$\langle p_1, \cdots, p_r \mid a_1 = b_i,\ i \in \{1, \cdots, k\} \rangle,$$

where $a_i, b_i, \in \langle p_1, \cdots, p_r \rangle_{\mathbb{N}}$, is the quotient of $\mathbb{N}^r$ by the congruence relation generated by the (a_i, b_i) in $\mathbb{N}^r \times \mathbb{N}^r$.
A **presentation** of a monoid S is an isomorphism with a monoid constructed as above.

2.2.1 Monoid algebra

Since from abelian groups we costructed the group algebra and found connections to geometric objects, we want to generalize that construction to monoids.

Definition 2.33 (Monoid algebra)**.** For a monoid S, its **monoid algebra** $k[S]$ is the k-vector space which is freely generated by $\{t^s \mid s \in S\}$ and with multiplication induced by the operation on S.

Remark 2.34. In [CLS11] they write χ^s instead of t^s because they think of S inside $M = X(T)$ for some torus.

Remark 2.35. If S is actually a group then the monoid algebra and group algebras coincide.

Example 2.36. If $S = \mathbb{N}^n \subseteq \mathbb{Z}^n$ then $k[S] = k[x_1, \cdots, x_n]$.

Proposition 2.37. If S is a monoid with presentation

$$\langle p_1, \cdots, p_r \mid a_i = b_i,\ 1 \leq i \leq k \rangle,$$

then

$$k[S] = \frac{k[t_1, \cdots, t_r]}{(t^{a_i} - t^{b_i})}$$

where if $a_i = \sum a_{ij} p_j$ we set $t^{a_i} = \prod t_j^{a_{ij}}$.

Sketch.
Let R be the congruence relation on $\mathbb{N}^r$ generated by $\{(a_i, b_i)\}_{1 \leq i \leq k}$. Since $R \rightrightarrows \mathbb{N}^r \to S$ is a coequalizer and $S \mapsto k[S]$ is a left adjoint ($\mathrm{Hom}_{\mathrm{Mon}}(S, A) \cong \mathrm{Hom}_{k-\mathrm{Alg}}(k[S], A)$) it follows that

$$k[R] \overset{f}{\underset{g}{\rightrightarrows}} k[\mathbb{N}^r] \to k[S]$$

is a coequalizer in k-algebras, so $k[S] \cong k[\mathbb{N}^r]/I$ where $I = (f(x) - g(x) \mid x \in k[R])$. □

Example 2.38. Let $S = \langle (2, 0), (1, 1), (0, 2) \rangle \subseteq \mathbb{Z}^2$. This monoid can be seen to be isomorphic to

$$\langle p, q, r \mid p + q = 2r \rangle .$$

It follows that

$$k[S] \cong \frac{k[x, y, z]}{(xy - z^2)},$$

which is the coordinate ring of the quadric cone.

Example 2.39. Consider $S = \langle 2, 3 \rangle \subseteq \mathbb{N}$, which has presentation

$$\langle p, q \mid 3p = 2q \rangle .$$

It follows that

$$k[S] \cong \frac{k[x, y]}{(x^3 - y^2)},$$

the coordinate ring of the cusp curve.

2.3 Toric variety associated to a monoid

Inspired by the success of Cartier duality, we consider the analogous construction with affine monoids. Instead of diagonalizable algebraic groups we will get affine toric varieties:

Proposition 2.40. If S is an affine monoid then

1. $k[S]$ is a domain and a finitely generated k-algebra.
2. $\mathrm{Spec}\, k[S]$ is an affine toric variety, with torus $\mathrm{Spec}\, k[S^{gp}]$.

Proof.
Let us prove the two propositions

1. Since $S \subseteq M$, we have an obvious inclusion $k[S] \subseteq k[M]$ and $k[M]$ is a domain, so $k[S]$ also is. Since S is finitely generated, just take the formal variables associated to those generators and they will generate $k[S]$ as a k-algebra.
2. The inclusion $S \to M$ must factor through $S \to S^{gp} \to M$ by the universal property. Since M is free of finite rank, S^{gp} also is, thus $T = \mathrm{Spec}\, k[S^{gp}] = D(S^{gp})$ is a torus

(1.23) of dimension equal to the rank of S^{gp}. Moreover, $k[S^{gp}]$ is a localization of $k[S]$ in a single element: if $\{s_i\}_{1\leq i\leq k}$ are generators of S then[3]

$$k[S^{gp}] \cong k[S]_{\prod t^{s_i}} = k[S][t^{-s_1}, \cdots, t^{-s_k}]$$

and this isomorphism is induced by the natural map $k[S] \to k[S^{gp}]$. The induced morphism $\operatorname{Spec} k[S^{gp}] \to \operatorname{Spec} k[S]$ is then an open embedding (iso. on local rings).
The translation action of T on itself is the one given by

$$\begin{array}{ccc} k[S^{gp}] & \longrightarrow & k[S^{gp}] \otimes k[S^{gp}] \\ t^m & \longmapsto & t^m \otimes t^m \end{array},$$

which extends to an action on $\operatorname{Spec} k[S]$ by

$$\begin{array}{ccc} k[S] & \longrightarrow & k[S^{gp}] \otimes k[S] \\ t^m & \longmapsto & t^m \otimes t^m \end{array},$$

which makes sense because $S \subseteq S^{gp}$.

□

There is another construction to describe the toric variety associated to the monoid generated by a finite subset $A \subseteq M$ (recall that M is the character lattice of T for some torus).

Consider the morphism

$$\phi_A : \begin{array}{ccc} T_N & \longrightarrow & (\mathbb{A}^1)^A \\ x & \longmapsto & (\chi^a(x))_{a\in A} \end{array}$$

Remark 2.41. The image of ϕ_A is contained in the standard torus $\operatorname{Imm} \phi_A \subseteq (\mathbb{G}_m)^A \subseteq (\mathbb{A}^1)^A$. It follows that $\operatorname{Imm} \phi_A$ is also a torus because it is the image of a homomorphism between tori (1.45).

Let Y_A be the closure of $\operatorname{Imm} \phi_A$ in $(\mathbb{A}^1)^A$.

Proposition 2.42. Y_A is an affine toric variety, with torus given by the one associated to $\mathbb{Z}A \subseteq M$. More precisely, $Y_A \cong \operatorname{Spec} k[\mathbb{N}A]$.

Proof.
The morphism ϕ_A corresponds to the algebra homomorphism

$$\varphi_A : k[x_a \mid a \in A] \to k[M]$$

Note that

$$\overline{\operatorname{Imm} \phi_A} = V(\ker \varphi_A) = \operatorname{Spec} \frac{k[x_a \mid a \in A]}{\ker \varphi_A} = \operatorname{Spec} \operatorname{Imm} \varphi_A.$$

It is easy to see that $\operatorname{Imm} \varphi_A = k[\mathbb{N}A] \subseteq k[M]$. Since $\mathbb{N}A$ is an affine monoid we are done by (2.40) □

[3]exercise

Remark 2.43. The two constructions are the same upon choosing a finite set of generators A for S, letting us write $S = \mathbb{N}A$.

Definition 2.44 (Toric ideals)**.** The ideals of $k[\mathbb{N}^A]$ which give rise to toric varieties are called **Toric ideals**

Fact 2.45. Toric ideals are exactly the prime ideals which can be generated by binomials (differences of monic monomials).

We now want to show that this construction covers all affine toric varieties:

Remark 2.46. The torus T_N acts linearly on its own ring of regular functions $k[M]$ as follows: for $t \in T_N$ and $f \in k[M]$ ($f : T_N \to \mathbb{A}^1$) we define[4] $t \cdot f \in k[M]$ as

$$t \cdot f : \begin{array}{ccc} T_N & \longrightarrow & \mathbb{A}^1 \\ p & \longmapsto & f(t^{-1} \cdot p) \end{array}$$

where the product $t^{-1} \cdot p$ is the product of T_N as an algebraic group.

To be more precise, the action of T_N is induced by a comodule structure on $k[M]$, specifically

$$k[M] \xrightarrow{\Delta} k[M] \otimes k[M] \xrightarrow{S \otimes id} k[M] \otimes k[M].$$

Technically $k[M]$ is infinite dimensional, but every time we consider this action we will actually consider the restriction to a stable finite dimensional subspace.

Lemma 2.47. The only simultaneous eigenvectors of the action $T_N \curvearrowright k[M]$ given above are the characters.

Proof.
Note that $t \cdot \chi^m(p) = \chi^m(t^{-1} \cdot p) = \chi^m(t^{-1})\chi^m(p)$ on the torus, thus $t \cdot \chi^m = \chi^m(t^{-1})\chi^m$, that is, characters are simultaneous eigenvectors for this action of T_N.

Let us now prove that they are the only ones (up to scalars): if $\sum a_m \chi^m$ in $k[M]$ is a simultaneous eigenvector then

$$\alpha(t) \left(\sum a_m \chi^m \right) = t \cdot \left(\sum a_m \chi^m \right) = \sum \chi^m(t^{-1}) a_m \chi^m$$

for some function $\alpha : T_N \to k$, thus $a_m \alpha(t) = a_m \chi^m(t^{-1})$ for all m. If $a_{m_1} \neq 0 \neq a_{m_2}$ then $\chi^{m_1}(t^{-1}) = \alpha(t) = \chi^{m_2}(t^{-1})$, so $m_1 = m_2$ and thus the simultaneous eigenvector we chose must be of the form $a_m \chi^m$ for some $m \in M$. $\square$

Lemma 2.48. If $A \subseteq k[M]$ is a subspace which is stable under the action above then

$$A = \bigoplus_{t^m \in A} k t^m,$$

that is, A is generated by characters.

[4]the inverse in the definition is not needed since T_N is abelian, but it is put there for consistency with more general theory where it is needed to verify that the map given is indeed a left-action.

Proof.
Call $A' = \bigoplus_{t^m \in A} kt^m$. Clearly $A' \subseteq A$ so we just need the other inclusion. Pick $f \in A$ and write

$$f = \sum_{m \in B} c_m t^m$$

for $B \subseteq M$ finite and such that $c_m \neq 0$ for all $m \in B$. Note that

$$f \in A \cap \langle t^m \mid m \in B \rangle := V.$$

This intersection is a finite dimensional k-vector space which is stable under the T_N-action, so it is a finite dimensional representation of T_N. By proposition (1.66) it follows that V is generated by simultaneous eigenvectors of the action, which are the t^m by the lemma above. Writing what we have just said in symbols:

$$f \in V = \bigoplus_{\substack{m \in B \ s.t. \\ t^m \in A}} kt^m \subseteq \bigoplus_{t^m \in A} kt^m = A'.$$

□

Theorem 2.49. All affine T_N-toric varieties are isomorphic to one of the form $\operatorname{Spec} k[S]$ for some monoid $S \subseteq M = X(T_N)$.

Proof.
If $X = \operatorname{Spec} A$ is an affine toric variety, then $A \subseteq k[M]$ is stable for the action of T_N on $k[M]$. This is because $\cdot t^{-1} : T_N \to T_N$ extends to X by definition of toric variety. By the lemma above

$$A = \bigoplus_{t^m \in A} kt^m = k[S],$$

where $S = \{m \in M \mid t^m \in A\}$, which is a submonoid of M because A is an algebra.

Since A is finitely generated, there exist $f_1, \cdots, f_k$ such that $A = k[f_1, \cdots, f_k]$. By replacing each f_i with all the characters that you need to write it out, we can assume that the f_i are all of the form t^m.

It is now easy to check that the corresponding exponents generate S. □

2.4 Cones

It will turn out that (normal) affine toric varieties are described by cones lying in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is a lattice (it will be the cocharacter lattice of the resulting toric variety).

Definition 2.50. A **convex polyhedral cone** (from now on just **cone**) is a subset of $N_{\mathbb{R}}$ of the form

$$\sigma = \operatorname{Cone}(A) = \left\{ \sum_{n \in A} \lambda_n \cdot n \mid \lambda_n \geq 0 \right\} \subseteq N_{\mathbb{R}}$$

where $A \subseteq N_{\mathbb{R}}$ is a finite subset.

Remark 2.51. A cone σ is a convex subset of $N_{\mathbb{R}}$ and it is a "positive" cone, in the sense that if $v \in \sigma$ and $\lambda \in [0, +\infty) \subseteq \mathbb{R}$ then $\lambda v \in \sigma$.

Example 2.52. The positive quadrant

$$\left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0\right\} = \mathrm{Cone}((1,0),(0,1))$$

is a cone. $\mathrm{Cone}((1,0),(1,2))$ is also a cone, which is embedded differently.

Definition 2.53 (Orthant)**.** An **orthant** is a cone of the form $\mathrm{Cone}(e_1, \cdots, e_k) \subseteq \mathbb{R}^n$.

Example 2.54. $\mathrm{Cone}((1,0,0),(0,1,0),(1,0,1),(0,1,1)) \subseteq \mathbb{R}^3$ is a cone.

Example 2.55. A line in $\mathbb{R}^2$ is a cone, since it can be written $\mathrm{Cone}(v, -v)$. In general linear subspaces are cones.

Definition 2.56. A cone σ is **strongly** or **strictly convex** if it does not contain any positive dimensional subspace.

Definition 2.57. The **dimension** of σ, denoted $\dim \sigma$, is the dimension of the vector subspace of $N_{\mathbb{R}}$ spanned by σ.
A cone is **full-dimensional** if its dimension is the same as the rank of N.

2.4.1 General facts about cones

For references you can look at Fulton [Ful93] for most of these facts.

Proposition 2.58. A cone is closed in the respective $N_{\mathbb{R}}$.

Sketch.
Assume the following theorem by Carathéodory: *if* $v \in \mathrm{Cone}(A)$ *then there exists* $B \subseteq A$ *linearly independent such that* $v \in \mathrm{Cone}(B)$.

It follows that

$$\mathrm{Cone}(A) = \bigcup_{\substack{B \subseteq A \\ B \text{ lin. ind.}}} \mathrm{Cone}(B)$$

and this is a finite union of closed sets because $\mathrm{Cone}(B)$ can be identified with $\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-k}$ via a linear transformation for some k. $\square$

Definition 2.59. Two polytopes are said to be **combinatorially equivalent** if their poset of faces are isomorphic.

Is there any polytope which is combinatorially equivalent to one with rational vertices (i.e. vetices in $\mathbb{Q}^n$)? Surprisingly, no. In all dimensions above 8 there are some polytopes that contradict this (which is weird because one would think "I can just move the vertices a little").

For more details look up *non-realizable matroids.*

Hyperplanes and dual cone

Definition 2.60 (Hyperplane and closed half-space)**.** If $m \in M_\mathbb{R}$, we write

$$H_m = \{n \in N_\mathbb{R} \mid \langle m, n\rangle = 0\}$$

(the product is the one induced by $M \times N \to \mathbb{Z}$ upon tensoring with $\mathbb{R}$). Sets of this form are **hyperplanes** in $N_\mathbb{R}$.
We write H_m^+ for $\{n \in N_\mathbb{R} \mid \langle m, n\rangle \geq 0\}$ and call this a **closed half-space**.

Definition 2.61. H_m is a **supporting hyperplane** for a cone σ if $\sigma \subseteq H_m^+$. We call H_m^+ a **supporting half-space**.

Definition 2.62. The **dual cone** to a cone σ is

$$\sigma^\vee = \{m \in M_\mathbb{R} \mid \langle m, n\rangle \geq 0 \ \forall n \in \sigma\} \subseteq M_\mathbb{R}$$

Remark 2.63. By definition

$$\sigma^\vee = \bigcap_{\substack{m \in M_\mathbb{R} \ s.t. \\ H_m^+ \text{ supp. half-sp.}}} H_m^+,$$

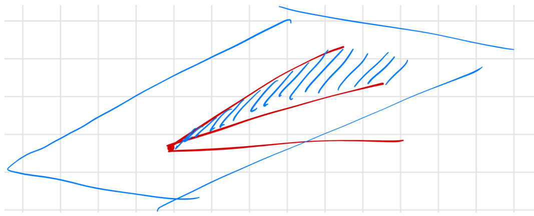
so H_m is supporting if and only if $m \in \sigma^\vee \setminus \{0\}$.

Fact 2.64. $\sigma^\vee$ is also a cone and $(\sigma^\vee)^\vee \cong \sigma$ under the identification $(N_\mathbb{R}^\vee)^\vee \cong N_\mathbb{R}$.

Fact 2.65. $m_1, \cdots, m_s$ generate $\sigma^\vee$ if and only if $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$. In particular, every cone is a finite intersection of half-spaces.

Definition 2.66. A **face** of a cone σ is a subset of the form $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$. In this case we write $\tau \leq \sigma$.

Remark 2.67. If $\sigma = \operatorname{Cone}(A)$ then $\tau = \operatorname{Cone}(a \in A \mid a \in H_m)$. In particular τ is also a cone.



Definition 2.68. A face is **proper** if it is not σ itself.

Definition 2.69. The **relative interior** of a cone σ is

$$\text{Relint}(\sigma) = \sigma \setminus \bigcup_{\tau < \sigma} \tau,$$

that is, the topological interior of the cone as a subset of $\text{Span}_{\mathbb{R}}(\sigma)$.

Fact 2.70. The following are true:

- If $\tau_1, \tau_2 \leq \sigma$ then $\tau_1 \cap \tau_2 \leq \sigma$
- if $\tau' \leq \tau$ and $\tau \leq \sigma$ then $\tau' \leq \sigma$
- if $\tau \leq \sigma$ and $v, w \in \sigma$ are such that $v + w \in \tau$ then $v, w \in \tau$.

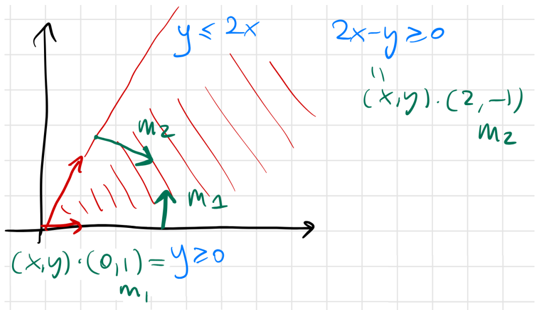
Definition 2.71. A **ray** (or **edge**) is a 1 dimensional face. A **facet** is a $\dim \sigma - 1$ dimensional face.

Fact 2.72. If σ is full-dimensional in $N_{\mathbb{R}}$ then in the representations like $\sigma = H^+_{m_1} \cap \cdots \cap H^+_{m_s}$ we can assume that $\sigma \cap H_{m_i}$ is a facet of σ for all i.

Remark 2.73. This is not the case if σ is not full-dimensional, for example for $\sigma = \text{Cone}((1,0)) \subseteq \mathbb{R}^2$ the only facet is $\{(0,0)\}$ but in order to write σ as the intersection of half-spaces we need some half-spaces with associated hyperplane being $\text{Span}((1,0))$ and so $\sigma \cap H$ for those hyperplanes is σ itself.

Fact 2.74. Every proper face is the intersection of all facets containing it.

Remark 2.75. If $N_{\mathbb{R}} \cong \mathbb{R}^n$ then we know that $M_{\mathbb{R}} \cong \mathbb{R}^n$ via the dual basis and we can think of one of the m_i that generate the dual cone as an "inward-pointing" normal vector to a facet of σ



Example 2.76. Let $\sigma = \text{Cone}((1,0),(1,2))$.
The half-planes that bound the cone are $y \geq 0$ and $2x - y \geq 0$, which correspond to $(0,1)$ and $(2,-1)$, which can be used to generate $\sigma^\vee = \text{Cone}((0,1),(-2,1))$.

Example 2.77. Take $\sigma = \text{Cone}((1,0)) \subseteq \mathbb{R}^2$, so $\sigma^\vee$ is $\text{Cone}((1,0),(0,1),(0,-1))$ which correspond to $x \geq 0$, $y \geq 0$ and $-y \geq 0$

Fact 2.78. The following are equivalent:

- σ is strictly convex,
- $\{0\}$ is a face of σ,
- $\sigma \cap (-\sigma) = \{0\}$,
- $\dim \sigma^\vee = \dim M_\mathbb{R}$.

Fact 2.79. Any cone σ contains a maximal linear subspace given by $\sigma \cap (-\sigma) = W$. Moreover, $\sigma / W \subseteq N_\mathbb{R}/W$ is strictly convex.

Definition 2.80. σ is **rational** if $\sigma = \text{Cone}(A)$ for $A \subseteq N$ (not $N_\mathbb{R}$ like before).

Fact 2.81. The dual and the faces of a rational cone are rational.

Fact 2.82. If $A \subseteq N$ then

$$\text{Cone}(A) \cap N_\mathbb{Q} = \left\{ \sum_{a \in A} q_a a \mid q_a \in \mathbb{Q} \right\}.$$

Definition 2.83. Let σ be a rational cone, its **minimal ray generators** are given as follows: if $\rho \leq \sigma$ is a ray (and thus rational), the minimal ray generator correspondint to it is the minimal generator of $\rho \cap N$ as a monoid, which is denoted u_ρ.

Fact 2.84. A strictly convex rational cone is "canonically" generated by its minimal ray generators:

$$\sigma = \mathrm{Cone}(u_\rho \mid \rho \text{ is a ray}).$$

Corollary 2.85. If σ is a rational full-dimensional cone then σ has minimal facet normals (minimal ray generators of the dual).

2.5 Affine toric varieties from cones

Notation. Let σ be a cone in $N_\mathbb{R}$. We write

$$S_\sigma = \sigma^\vee \cap M.$$

Remark 2.86. S_σ is a submonoid of M because if $m, m' \in \sigma^\vee \cap M$ then

$$\langle m + m', n\rangle = \langle m, n\rangle + \langle m' n\rangle \geq 0 + 0 = 0.$$

Lemma 2.87 (Gordan)**.** If σ is a rational polyhedral cone in $N_\mathbb{R}$, then $S_\sigma = \sigma^\vee \cap M$ is finitely generated.

Proof.
Write $\sigma^\vee = \mathrm{Cone}(T)$ with $T \subseteq M$ some finite subset. Consider

$$K = \left\{ \sum_{m \in T} a_m m \mid 0 \leq a_m < 1 \right\}.$$

Clearly K is bounded in $M_\mathbb{R}$, so $K \cap M$ is a finite set. We claim that $T \cup (K \cap M)$ generates S_σ as a monoid:

Let $w \in S_\sigma = \sigma^\vee \cap M$. We can write $w = \sum_{m \in T} \lambda_m m$ with $\lambda_m > 0$ real numbers. We can write $\lambda_m = \lfloor \lambda_m \rfloor + \{\lambda_m\}$ (floor and fractional part), so that

$$w = \underbrace{\sum_{m \in T} \lfloor \lambda_m \rfloor m}_{\in M} + \underbrace{\sum_{m \in T} \{\lambda_m\} m}_{\in K}.$$

But $\sum_{m \in T} \{\lambda_m\} m$ is also in M because it is $w - \sum_{m \in T} \lfloor \lambda_m \rfloor m$, so we have written w in the desired form. $\square$

Because of the correspondence between affine toric varieties and affine monoids that we built (2.40) we can give the following definition:

Definition 2.88. Let $\sigma \subseteq N_\mathbb{R}$ be a rational cone. Its affine toric variety is

$$U_\sigma = \mathrm{Spec}\, k[S_\sigma].$$

Remark 2.89. The torus of U_σ has character lattice $S_\sigma^{gp} \subseteq M$.

Remark 2.90. Why are we not taking Spec$[\sigma \cap M]$ (for σ cone in $M_\mathbb{R}$) instead? This is because the gluing process of affine pieces will be more natural if the cones are in $N_\mathbb{R}$

Proposition 2.91. The following are equivalent

1. $\dim U_\sigma = n = \dim N_\mathbb{R}$
2. the torus of U_σ is T_N
3. σ is strictly convex.

Proof.
First note that
$$\dim U_\sigma = \operatorname{rnk} S_\sigma^{gp} = \dim \operatorname{Cone}(S_\sigma) = \dim \sigma^\vee$$
From this, $\dim U_\sigma = n$ is equivalent to $\dim \sigma^\vee = n$ which we know is equivalent to σ being strongly convex.

For the other equivalence, we claim M/S_σ^{gp} is torsion free. This gives the desired equivalence because we get
$$\dim U_\sigma = n \iff \operatorname{rnk} S_\sigma^{gp} = \operatorname{rnk} M \overset{\text{claim}}{\iff} M = S_\sigma^{gp} \iff T_N \text{ is the torus in } U_\sigma.$$
We now prove that the claim holds. Let $m \in M$ and assume that $km \in S_\sigma^{gp}$ for some $k \in \mathbb{N}$. Then $km = m_1 - m_2$ for some $m_1, m_2 \in S_\sigma$ and so
$$M \ni m + m_2 = \frac{1}{k} m_1 + \frac{k-1}{k} m_2 \in \sigma^\vee$$
where the last inclusion holds by convexity. Thus $m = (m+m_2) - m_2$ implies $m \in S_\sigma^{gp}$ □

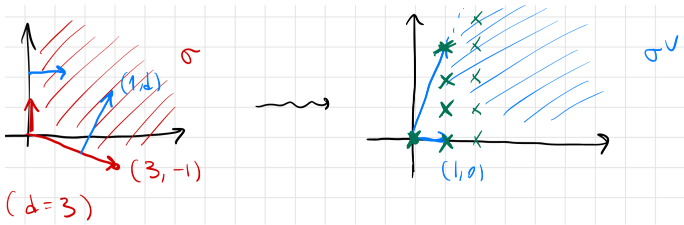
Because of this result, from now on a cone σ will be assumed to be strictly convex (i.e. $S_\sigma^{gp} = M$) and rational unless otherwise stated.

Example 2.92. Let $\sigma = \operatorname{Cone}(e_1) \subseteq \mathbb{R}^2$, then $\sigma^\vee = \operatorname{Cone}(e_1, e_2, -e_2)$

Example 2.93. If $\sigma = \operatorname{Cone}(e_1, \cdots, e_k) \subseteq \mathbb{R}^n$ is an orthant then
$$\sigma^\vee = \operatorname{Cone}(e_1, \cdots, e_k, \pm e_{k+1}, \cdots, \pm e_n).$$
It follows that $k[S_\sigma] = k[x_1, \cdots, x_k, x_{k+1}^{\pm 1}, \cdots, x_n^{\pm 1}]$ and so for an orthant
$$U_\sigma \cong \mathbb{A}^k \times \mathbb{G}_m^{n-k}.$$

Example 2.94. If $\sigma = \operatorname{Cone}(0) = \{0\}$ then $\sigma^\vee = M$ and so $U_\sigma = T_N$

Example 2.95 (Rational normal cone of degree d)**.** Let $d \in \mathbb{N} \setminus \{0\}$ and take $\sigma = \operatorname{Cone}(de_1 - e_2, e_2)$
It turns out that $S_\sigma = \langle (1, i) \mid 0 \le i \le d \rangle$ (not trivial yet). Let us study
$$U_\sigma = \operatorname{Spec} k[S_\sigma]$$
Setting $A = \{(1, i) \mid 0 \le i \le d\}$, we can see U_σ as Y_A, the closure of the image of
$$\begin{array}{ccc} \mathbb{G}_m^2 & \longrightarrow & \mathbb{A}^{d+1} \\ (s,t) & \longmapsto & (s, st^1, \cdots, st^d) \end{array}$$



Definition 2.96. The toric variety from the previous example is called the **rational normal cone of degree** d. It is the affine cone over the so called *rational curve of degree* d in $\mathbb{P}^d$.

Remark 2.97. It turns out that the ideal of the rational normal cone of degree d is $(x_i x_{j+1} - x_{i+1}x_j \mid 0 \le i < j \le d)$. Note that the generators are determinants of 2×2 matricies, specifically, all minors of

$$\begin{pmatrix} x_0 & \cdots & x_{d-1} \\ x_1 & \cdots & x_d \end{pmatrix}$$

Example 2.98. Consider $\sigma = \mathrm{Cone}(e_1, e_2, e_1+e_3, e_2+e_3)$. The equations that define this cone are $y \ge 0$, $z \ge 0$, $x \ge 0$ and $x+y-z \ge 0$, so $\sigma^\vee = \mathrm{Cone}(e_1, e_2, e_3, e_1+e_2-e_3)$.

You can check that $S_\sigma = \sigma^\vee \cap M \cong \langle p, q, r, s \mid p + q = r + s\rangle$, showing that

$$k[S_\sigma] \cong \frac{k[x,y,z,w]}{(xy - zw)}.$$

Remark 2.99. When σ is full-dimensional ($\sigma^\vee$ is strictly convex) it follows that S_σ is sharp and so (2.26) the irreducible elements of S_σ give a canonical generating set.

Definition 2.100. Let σ be a cone. If S_σ is sharp, the set

$$H = \{m \in S_\sigma \mid m \text{ irreducible}\}$$

is called the **Hilbert basis** of S_σ.

Fact 2.101. If σ is full dimensional (and so S_σ is sharp) then

- H is finite and generates S_σ
- H contains the minimal generators of the rays of $\sigma^\vee$
- every generating set of S_σ contains H

2.6 Normality and smoothness of affine toric varieties

2.6.1 Normality

Definition 2.102. If $X = \operatorname{Spec} A$ is an irreducible affine algebraic variety (A is a domain) then X is **normal** if $A \subseteq \operatorname{Frac} A$ is integrally closed.

Remark 2.103. X is normal if and only if all local rings of X are integrally closed in $\operatorname{Frac} A$. We are identifying the local rings with the subrings of $\operatorname{Frac} A$ below

$$A_{\mathfrak{m}_p} \cong \mathcal{O}_{X,p} = \left\{ f \in \operatorname{Frac} A \mid f = \frac{g}{h},\ h(p) \neq 0 \right\}.$$

Definition 2.104. An integral monoid S is **saturated** if for all $s \in S^{gp}$ such that there exists $k \in \mathbb{N} \setminus \{0\}$ such that $ks \in S$ we have $s \in S$.

Example 2.105. $S = \langle 2, 3 \rangle \subseteq \mathbb{N}$ is not saturated because $S^{gp} = \mathbb{Z}$ $(1 = 3 - 2)$ and $2 \cdot 1 = 2 \in S$ but $1 \notin S$.

Remark 2.106. In [CLS11] they say that $S \subseteq M$ is saturated if the condition holds for $m \in M$. The two definitions are not equivalent because $2\mathbb{N} \subseteq \mathbb{Z}$ is saturated for our definition but not theirs.

If $S^{gp} = M$ the two definitions are the same and this is always assumed in [CLS11] so nothing really changes but the true definition in monoid theory is the one we gave.

Proposition 2.107. For an affine toric variety X with torus T_N, the following are equivalent

1. X is normal
2. $X = \operatorname{Spec} k[S]$ for S saturated
3. There exists a strictly convex cone σ in $N_{\mathbb{R}}$ with $X \cong U_\sigma$

Proof.
Let us give the implications

$\boxed{1 \implies 2}$ Suppose X is normal and let $S \subseteq M$ be some monoid such that $S^{gp} = M$ and $X \cong \operatorname{Spec} k[S]$. Let $m \in S^{gp} = M$ and $k \in \mathbb{N} \setminus \{0\}$ be such that $km \in S$, then $t^{km} \in k[S]$ and $t^m \in k[M] \subseteq \operatorname{Frac}(k[S])$ is a root of the polynomial

$$y^k - t^{km} \in k[S][y].$$

Since $k[S]$ is integrally closed we get $t^m \in k[S]$ and so $m \in S$

$\boxed{2 \implies 3}$ Suppose S is saturated with $S^{gp} = M$. Let $A \subseteq S$ be a set of generators and take $\tau = \text{Cone}(A) \subseteq M_{\mathbb{R}}$. Define $\sigma = \tau^\vee$. This σ is strictly convex because τ is full dimensional by construction and clearly $S \subseteq \tau \cap M = \sigma^\vee \cap M$. We just need the other inclusion now. If $m \in \tau \cap M$ then $m \in M \subseteq M_{\mathbb{Q}}$ and so

$$m = \sum_{a \in A} q_a a$$

for some $q_a \in \mathbb{Q}$, $q_a \geq 0$. Upon taking the least common multiple of the denominators N we get a positive integer such that Nm is an integral linear combination of the elements of A, thus $Nm \in S$ and by saturatedness we have $m \in S$ as desired.

$\boxed{3 \implies 1}$ Let $\rho_1, \cdots, \rho_r$ be the rays of σ, then $\sigma^\vee = \bigcap_{i=1}^r \rho_i^\vee$ and so

$$k[S_\sigma] = \bigcap_{i=1}^r k[S_{\rho_i}] \subseteq k[M].$$

Since the intersection of integrally closed subrings is integrally closed we may suppose without loss of generality that $\sigma = \rho$ is a ray.

Let u_ρ be the minimal ray generator of ρ, then we can complete u_ρ to a $\mathbb{Z}$-basis of N: consider the exact sequence ($N' = \text{coker}(\langle u_\rho \rangle \subseteq N)$)

$$0 \to \langle u_\rho \rangle \to N \to N' \to 0$$

Note that N' is torsion free and thus free (finitely generated abelian group), so the sequence splits and we have $N \cong \langle u_\rho \rangle \oplus N'$.

We may therefore assume that $\rho = \text{Cone}(e_1)$, so that $\rho^\vee = \text{Cone}(e_1, \pm e_2, \cdots, \pm e_n)$, so $k[S_\rho] = k[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$ and this is integrally closed.

□

Remark 2.108. If S is integral but not saturated then it has a saturation S^{sat} given by $\{m \in S^{gp} \mid \exists k > 0,\ km \in S\}$. Note that

- $S \subseteq S^{sat} \subseteq S^{gp}$
- S^{sat} is finitely generated
- $(S^{sat})^{gp} = S^{gp}$

Moreover, the inclusion $k[S] \to k[S^{sat}]$ gives the "normalization" $\operatorname{Spec} k[S^{sat}] \to \operatorname{Spec} k[S]$

Example 2.109. Let $S = \langle 2, 3 \rangle \subseteq \mathbb{N}$ and note that $S^{sat} = \mathbb{N}$. Recall that $\langle 2, 3 \rangle = \langle p, q \mid 3p = 2q \rangle$, so

$$k[S] = \frac{k[x, y]}{(x^3 - y^2)}$$

and $C = \operatorname{Spec} k[S]$ is the cuspidal cubic in $\mathbb{A}^2$ (not normal variety). The normalizaion of this is

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & C \subseteq \mathbb{A}^2 \\ t & \longmapsto & (t^2, t^3) \end{array}$$

Example 2.110. Consider the monoid $S = \langle (2,0), (1,1), (0,2) \rangle \subseteq \mathbb{Z}^2$. We know that $\operatorname{Spec} k[S]$ is a normal variety, but the monoid does not "look" saturated. For example, $(0,1) \in \mathbb{Z}^2 \setminus S$ but $2(0,1) = (0,2) \in S$. The issue is that S^{gp} is smaller than $\mathbb{Z}^2$ and $(0,1) \notin S^{gp}$.

2.6.2 Smoothness

Remark 2.111. Recall that $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ in general. If $X \subseteq \mathbb{A}^n$ as $V(I)$ with $I = (f_1, \cdots, f_s)$ then $T_x X$ is defined by the linear equations $0 = d_x(f_i) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(x) x_j$ with $1 \le i \le n$.

Definition 2.112. An irreducible affine variety $X = \operatorname{Spec} A$ is **smooth** if $\dim T_x X = \dim X$ for all $x \in X$.

Fact 2.113 (Jacobian criterion)**.** An irreducible $X = V(f_1, \cdots, f_s) \subseteq \mathbb{A}^n$ of dimension d is smooth at $x \in X$ if and only if

$$\operatorname{rnk}\left(\frac{\partial f_i}{\partial x_j}(x)\right) = n - d.$$

We will see that an affine toric variety U_σ is smooth if and only if σ is a *smooth cone*:

Definition 2.114. A rational strongly convex cone $\sigma \subseteq N_\mathbb{R}$ is

- **smooth** (or **regular**) if the minimal ray generators of σ are part of a $\mathbb{Z}$-basis of N
- **simplicial** if the minimal ray generators are $\mathbb{R}$-linearly independent in $N_\mathbb{R}$.

Example 2.115. The cone $\mathbb{R}^k_{\ge 0} \subseteq \mathbb{R}^n$ is smooth. Moreover, all smooth cones are of this form up to the action of some element of $\mathrm{GL}(\mathbb{Z}, n)$.

Example 2.116. The cone $\sigma = \mathrm{Cone}((1,0),(1,2))$ is simplicial because $(1,0)$ and $(1,2)$ are linearly independent, but $(1,0)$ and $(1,2)$ cannot be part of a basis for σ because the element $(1,1)$ would never be reached despite being in the cone.

Example 2.117. The cone $\mathrm{Cone}((1,0,0),(0,1,0),(1,0,1),(0,1,1)) \subseteq \mathbb{R}^4$ is not simplicial because it has 4 minimal ray generators.

Remark 2.118. Points of $\operatorname{Spec} k[S]$ are in bijection with homomorphisms of monoids $S \to (k, \cdot)$:

$$\begin{aligned}\{\text{points of } \operatorname{Spec} k[S]\} \overset{NSS}{=} &\{\text{max. ideals of } k[S]\} = \\ = &\{\text{surjections of } k\text{-algebras } k[S] \to k\} = \\ = &\{\text{monoid homomorphisms } S \to (k, \cdot)\}\end{aligned}$$

where the last equality works because it amounts to choosing a value in k for each $s \in S$ (or equivalently $t^s \in k[S]$) which is compatible with the operations. The surjectivity works because $S \to (k, \cdot)$ being a homomorphism means that 0 goes to 1 and so the corresponding k-alegbra homomorphism has 1 in the image, making the map surjective.

Lemma 2.119. The action of T_N on $\operatorname{Spec} k[S]$ has a fixed point if and only if S is sharp. In this case there is exactly one fixed point, which corresponds to $S \to (k, \cdot)$ given by $s \mapsto \begin{cases} 0 & \text{if } s \neq 0 \\ 1 & \text{if } s = 0 \end{cases}$

Proof.
If $p \in \operatorname{Spec} k[S]$ corresponds to $\gamma : S \to (k, \cdot)$ and we fix $a \in T_N$, let us compute $a \cdot p$: recall that the action is described by

$$\begin{array}{ccc} k[S] & \longrightarrow & k[M] \otimes k[S] \\ t^s & \longmapsto & t^s \otimes t^s \end{array}$$

and so it maps $(a, p) \in T_N \times X$ to the point which corresponds to $k[S] \to k$ given by

$$k[S] \longrightarrow k[M] \otimes k[S] \longrightarrow k \otimes k = k$$

$$t^s \longmapsto t^s \otimes t^s \longmapsto \chi^s(a)\gamma(s)$$

so the homomorphism $\gamma' : S \to (k, \cdot)$ which corresponds to $a \cdot p$ is given by $\gamma'(s) = \chi^s(a)\gamma(s)$.

The point is fixed if $\chi^s(a)\gamma(s) = \gamma(s)$ for all $a \in T_N$, $\in S$. For $s = 0$ $\gamma(s) = 1$ ok because it has to be a homomorphism, for $s \neq 0$ this implies that $\gamma(s) = 0$ in k (because $\exists a \in T_N$ such that $\chi^s(a) \neq 1$), so the only possible γ is the one in the statement, which is a homomorphism if and only if S is sharp. ☐

Remark 2.120. The point in the statement of the lemma above can be thought of as the "most singular point of X".

Remark 2.121. A toric variety U_σ has a fixed point for the action of the torus if and only if σ is full-dimensional.

Remark 2.122. The maximal ideal of $k[S]$ corresponding to the the torus fixed point (when S is sharp) is $(t^m \mid m \in S \setminus \{0\})$.

Example 2.123. In $k[\mathbb{N}^n] = k[x_1, \cdots, x_n]$ this ideal would be $(x_1, \cdots, x_n)$.

Proposition 2.124. If σ is a strongly convex cone of maximal dimension and $p_\sigma \in U_\sigma$ is the torus fixed point then

$$\dim_k T_{p_\sigma} U_\sigma = |H|$$

where H is the Hilbert basis of S_σ.

Proof.
The maximal ideal which corresponds to p_σ is $\mathfrak{m} = (t^m \mid m \in \sigma^\vee \cap M \setminus \{0\})$ and as a k-vector space we have

$$\mathfrak{m} = \bigoplus_{\substack{m \neq 0, \\ m \in S_\sigma}} kt^m = \bigoplus_{m \in H \setminus \{0\}} kt^m \oplus \underbrace{\bigoplus_{\substack{m \text{ reducible} \\ m \neq 0}} kt^m}_{\mathfrak{m}^2}$$

so $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = |H|$. Since $\mathfrak{m}/\mathfrak{m}^2 \cong \dfrac{\mathfrak{m}A_\mathfrak{m}}{\mathfrak{m}^2 A_\mathfrak{m}} = \mathfrak{m}_{p_\sigma}/\mathfrak{m}_{p_\sigma}^2$ we are done. $\square$

Theorem 2.125. We have that U_σ is smooth $\iff$ σ is a smooth cone.

Proof.
We give the two implications

$\boxed{\Longleftarrow}$ If σ is smooth then we can assume up to an integral change of basis that $\sigma = \mathrm{Cone}(e_1, \cdots, e_k) \subseteq \mathbb{R}^n$ and $\sigma^\vee = \mathrm{Cone}(e_1, \cdots, e_k, \pm e_{k+1}, \cdots, \pm e_n)$, so $U_\sigma = \mathbb{A}^n \times \mathbb{G}_m^{n-k}$, which is a smooth variety.

$\boxed{\Longrightarrow}$ We first consider the full dimensional case and then massage the general case into it:

$\boxed{\text{full-dim}}$ Note that $\sigma^\vee$ is strictly convex. Let $p_\sigma \in U_\sigma$ be the torus fixed point. Smoothness at p_σ implies that

$$n = \dim U_\sigma = \dim T_{p_\sigma} U_\sigma = |H| \geq |\text{rays of } \sigma^\vee| \geq n$$

where the first inequality comes from the fact that H contains the minimal ray generators, while the second comes from $\sigma^\vee$ being full-dimensional[5]. It follows that $\sigma^\vee$ has n rays.

Since $M = S_\sigma^{gp}$, the n minimal ray generators of $\sigma^\vee$ must be a $\mathbb{Z}$-basis of M by a rank argument. Thus $\sigma^\vee$ is smooth and so σ itself is smooth.

$\boxed{\text{general}}$ Consider the saturated (so we also consider elements of N that lie in σ after taking some multiple) $\mathbb{Z}$-span $N_1 \subseteq N$ of $\sigma \cap N$. We can now write $N = N_1 \oplus N_2$ because we constructed $N_2 = N/N_1$ in a way that makes it torsion-free.

We can now think of σ as a cone in $(N_1)_\mathbb{R}$ also, not just $N_\mathbb{R}$. These give two monoid algebras

$$k[S_{\sigma,N}] = k[\sigma^\vee \cap M], \quad k[S_{\sigma,N_1}] = k[\sigma^\vee \cap M_1]$$

where $M_1 = (N_1)^\vee$.

It turns out that[6] (exercise) $S_{\sigma,N} \cong S_{\sigma,N_1} \oplus M_2$, and so $k[S_{\sigma,N}] \cong k[S_{\sigma,N_1}] \otimes k[M_2]$, so

$$U_{\sigma,N} \cong U_{\sigma,N_1} \times T_{N_2}.$$

Now if U_σ is smooth, it follows that U_{σ,N_1} is smooth (exercise, look at dimensions of tangents in products). Now U_{σ,N_1} is like in the above case, so $\sigma \subseteq N_1$ is smooth, meaning that it must be smooth in N also.

$\square$

2.7 Faces correspond to affine open subsets

Consider σ a strictly convex rational cone. Let $\tau \leq \sigma$ be a face. We will now see that U_τ can naturally be identified with a principal open subset of U_σ.

[5] we always assume σ strictly convex.

[6] for example, take $\sigma = \mathrm{Cone}(e_1) \subseteq \mathbb{R}^2$. $N_1 = \mathbb{Z} \times \{0\} \subseteq \mathbb{Z}^2$, M_2 is the perpendicular ****************

Recall that $\tau \leq \sigma$ means that there exists some $m \in \sigma^\vee$ such that

$$\tau = \sigma \cap H_m.$$

Proposition 2.126. If $\tau \leq \sigma$ is cut out by the hyperplane H_m we have that $k[S_\tau] = k[\tau^\vee \cap M]$ is naturally identified with $k[S_\sigma]_{t^m}$.

Proof.
If $\tau \leq \sigma$ then S_σ is a submonoid of S_τ and $\langle m, n \rangle = 0$ for all $n \in \tau$ means that $\pm m \in S_\tau$. This implies that $S_\sigma + \mathbb{N}(-m)$ is a submonoid of S_τ. If we check that this inclusion is an equality we are done because localizing at t^m is the same as adding t^{-m} to the generators.

Take $m' \in S_\tau$. Note that $\langle m', n \rangle \geq 0$ for all $n \in \tau$. Let $\sigma = \mathrm{Cone}(S)$ with $S \subseteq N$ finite and consider

$$C = \max \{ |\langle m', s \rangle| \mid s \in S \} \in \mathbb{N}.$$

If we show that $m' + Cm \in S_\sigma$ then we are done. To check this note that if $u \in \sigma$ then

$$\langle m' + Cm, u \rangle = \langle m', u \rangle + C \langle m, u \rangle .$$

If $u \in \tau$ then $\langle m, u \rangle = 0$ and $\langle m', u \rangle \geq 0$ since $m' \in S_\tau$ and we are done. Otherwise $\langle m, n \rangle \geq 1$ and therefore, for $u = s \in S$ minimal ray generator, we have

$$\langle m', s \rangle + C \langle m, s \rangle \geq \langle m', s \rangle + C \geq 0$$

where the last inequality comes from the definition of C. $\square$

Remark 2.127. If σ and σ' are cones in $N_\mathbb{R}$ and $\sigma \cap \sigma' = \tau$ is a face of both, we have a diagram

$$\begin{tikzcd} U_\sigma & & U_{\sigma'} \\ & U_\tau \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd}$$

We will be able to glue U_σ and $U_{\sigma'}$ along U_τ to get a, possibly non-affine, toric variety.

Chapter 3

Projective toric varieties

3.1 Introduction

Definition 3.1. A **projective toric variety** is an irreducible, normal projective variety X equipped with an open embedding $T \subseteq X$ of an algebraic torus such that the translation action of T extends to X.

Remark 3.2. Projective space $\mathbb{P}^n$ is a projective toric variety with torus given by

$$\mathbb{P}^n \setminus V(x_0 \cdots x_n).$$

This is the same torus that we get on all the affine charts.

The translation action extends as follows:

$$\begin{array}{ccc} \mathbb{G}_m^n \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ ((a_1, \cdots, a_n), [x_0, \cdots, x_n]) & \longmapsto & [x_0, a_1 x_1, \cdots, a_n x_n] \end{array}$$

The character lattice of this torus $T_{\mathbb{P}^n}$ can be thought of as follows: recall that we have

$$\mathbb{A}^{n+1} \setminus \{0\} \xrightarrow{\pi} \frac{\mathbb{A}^{n+1} \setminus \{0\}}{\mathbb{G}_m} \cong \mathbb{P}^n$$

and this induces

$$0 \to \mathbb{G}_m \to \mathbb{G}_m^{n+1} \to T_{\mathbb{P}^n} \to 0$$

where the first inclusion is via matricies of the form λI. Dually we get a short exact sequence of the character lattices

$$0 \to M_{\mathbb{P}^n} \to \mathbb{Z}^{n+1} \to \mathbb{Z} \to 0$$

so we may write

$$M_{\mathbb{P}^n} = \left\{ (a_0, \cdots, a_n) \in \mathbb{Z}^{n+1} \mid \sum a_i = 0 \right\} \subseteq \mathbb{Z}^{n+1}.$$

Now, given a finte subset $A \subseteq M$ (let us write $A = \{a_1, \cdots, a_s\}$) we can consider

$$\varphi_A : \begin{array}{ccc} T_N & \longrightarrow & \mathbb{G}_m^s \\ t & \longmapsto & (\chi^{a_1}(t), \cdots, \chi^{a_s}(t)) \end{array}$$

and then the composition

$$\psi_A : T_N \xrightarrow{\varphi_A} \mathbb{G}_m^s \hookrightarrow \mathbb{A}^2 \setminus \{0\} \twoheadrightarrow \mathbb{P}^{s-1}.$$

The closure of the image of ψ_A inside $\mathbb{P}^{s-1}$ is the **projective toric variety** X_A **associated to** A

Proposition 3.3. X_A as above is a projective toric variety. $\dim X_A = \dim \mathrm{Affspan}_{\mathbb{R}} A$ where the last notation means *the affine subspace generated by* A in $M_{\mathbb{R}}$.

Proof.
Let T be the image of $T_N \to \mathbb{G}_m^s \to T_{\mathbb{P}^{s-1}}$, which is still a torus by (1.45). Note that X_A is the closure of T in $\mathbb{P}^{s-1}$.

If $t \in T$, $t \cdot T = T \subseteq X_A$ and $\overline{t \cdot T} = t \cdot \overline{T} = t \cdot X_A$, so $t \cdot X_A \subseteq X_A$, but the same holds for t^{-1}, thus the action extends.

$\dim X_A = \dim T = \mathrm{rnk}_{\mathbb{Z}} M'$ where $M' = X(T)$. We can compute M':

$$\begin{tikzcd} T_N \arrow[r, two heads] \arrow[dr] & T \arrow[d, hook] \\ & T_{\mathbb{P}^{s-1}} \end{tikzcd}$$

yields dually (apply $X(\cdot)$ functor)

$$\begin{tikzcd} M & M' \arrow[l, hook'] \\ & M_{\mathbb{P}^{s-1}} \arrow[ul] \arrow[u, two heads] \end{tikzcd}$$

so M' is the image of $M_{\mathbb{P}^{s-1}} \to M$, which is induced by the map $\mathbb{Z}^s \to M$ which sends e_i to a_i, so the image is exactly

$$\left\{ \sum k_i a_i \mid \sum k_i = 0 \right\} = \langle a_i - a_j \mid i \neq j \rangle \subseteq M.$$

Upon tensoring this with $\mathbb{R}$ we get the vector subspace of $M_{\mathbb{R}}$ associated to the affine subspace generated by A. $\square$

Remark 3.4. One may expect $Y_A \subseteq \mathbb{A}^s$ to be related to the affine cone over X_A. The two are the same if and only if $I(Y_A)$ is homogeneous iff exists $n \in N$ and k positive such that ************** (i.e. A is contained in an affine hyperplane of $M_{\mathbb{R}}$).

Remark 3.5. The toric variety $X_A \subseteq \mathbb{P}^{s-1}$ is covered by affine toric varieties, given by the intersections $X_A \cap U_i$. The $X_A \cap U_i$ are indeed affine and they are toric because they all contain T. In fact $X_A \cap U_i = \overline{T}^{U_i}$.

Proposition 3.6. The monoid of $X_A \cap U_i$ is the submonoid A_i of M generated by $a_j - a_i$ for $j \neq i$.

Proof.
It suffices to show that $X_A \cap U_i$ is the closure of the image of $T_N \to U_i \to \mathbb{A}^{s-1}$. If $t \in T_N$ then the maps go

$$t \mapsto [\chi^{a_1}(t), \cdots, \chi^{a_s}(t)] \mapsto (\chi^{a_1 - a_i}(t), \cdots, \chi^{a_s - a_i}(t))$$

and this is exactly what we want. $\square$

Remark 3.7. A_i^{gp} is exactly the character lattice of T that we found the proof before.

Example 3.8 (Rational normal curve)**.** Let $A \subseteq \mathbb{Z}^2$ be the subset given by $A = \{(0,d), (1, d-1), \cdots, (d, 0)\}$. The affine toric variety Y_A is what we called *rational normal cone of degree d*.

The projective toric variety X_A is called the **rational normal curve of degree** d in $\mathbb{P}^d$ and Y_A is its affine cone in $\mathbb{A}^{d+1}$.

Example 3.9. Let $A = \{e_1, e_2, e_3, e_1 + e_2 - e_3\}$. The affine toric variety is

$$Y_A = \operatorname{Spec} \frac{k[x,y,z,w]}{(xy - zw)} \subseteq \mathbb{A}^4$$

The projective toric variety X_A is the one in $\mathbb{P}^3$ given by the same equation $xy = zw$. This is actually isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ ([y_0, y_1], [z_0, z_1]) & \longmapsto & [y_0 z_0, y_1 z_1, y_0 z_1, y_1 z_0] \end{array}$$

3.2 Polytopes

We have seen that affine toric varieties are described by cones. For projective toric varieties we have a similar correspondence with polytopes.

Definition 3.10. A **polytope** in $M_{\mathbb{R}}$ is the convex hull of a finite subset $A \subseteq M_{\mathbb{R}}$, i.e.

$$P = \operatorname{Conv}(A) = \left\{ \sum_{a \in A} \lambda_a a \mid \lambda_a \geq 0,\ \sum \lambda_a = 1 \right\}.$$

Given such a P we can construct a cone

$$\operatorname{Cone}(A \times \{1\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

We can recover the polytope by slicing the cone at height 1.

This corresponcence is sometimes useful to prove things about polytopes by reducing to the case of cones.

Definition 3.11. The **dimension** of a polytope P is the dimension of the smalled affine subspace of $M_{\mathbb{R}}$ which contains P.

Definition 3.12. Let $u \in N_{\mathbb{R}}$ and $b \in \mathbb{R}$. They determine an **affine hyperplane**

$$H_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle = b\} \subseteq M_{\mathbb{R}}$$

and a **closed half-space**

$$H^+_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq b\} \subseteq M_{\mathbb{R}}$$

Definition 3.13. A subset $Q \subseteq P$ is a **face** if there exist $n \in N_{\mathbb{R}}$ and $b \in \mathbb{R}$ such that $P \subseteq H^+_{u,b}$ (in this case we say that $H_{u,b}$ is a **supporting hyperplane**) and $Q = P \cap H_{u,b}$.

Remark 3.14. Faces of a polytope are polytopes. Moreover, if $P = \text{Conv}(A)$ then $Q = \text{Conv}(A \cap H_{u,b})$ for $H_{u,b}$ supporting hyperplane which defines Q.

Definition 3.15. Faces of dimension 0 are called **vertices**, those of dimension 1 are **edges** and those of codimension 1 are **facets**.

Fact 3.16. If P is a polytope then

- $P = \text{Conv}(\text{vertices of } P)$
- If $P = \text{Conv}(A)$ and $v \in P$ is a vertex then $v \in A$
- if $Q \leq P$ then

$$\{\text{faces of } Q\} = \{\text{faces of } P \text{ contained in } Q\}$$

- if $Q < P$ (proper face) then

$$Q = \bigcap_{\substack{F \text{ facet of } P\\ Q \leq F}} F$$

- a polytope is a finite intersection of closed half-spaces
- any finite intersection of closed half-spaces which is bounded is a polytope

Fact 3.17. When P is full-dimensional, each facet F has a *unique* supporting hyperplane.

Notation. If F is a facet of P full-dimensional we use H^+_F to denote the associated supporting hyperplane and we denote by $u_F \in N_{\mathbb{R}}$, $a_F \in \mathbb{R}$ the pair such that

$$H^+_F = H_{u_F, -a_F}.$$

The sign of a_F is that way just for convention, it will make some computations easier later on. Note that the pair (u_F, a_F) is not unique but it become unique up to positive scaling.

Definition 3.18. A polytope P is a **lattice polytope** if there exists $A \subseteq M$ finite such that $P = \mathrm{Conv}(A)$.

Remark 3.19. This is equivalent to saying that all vertices of P lie in M.

Fact 3.20. The following propositions hold

- Faces of lattice polytopes are lattice polytopes
- in the description of P as $P = \bigcup_{i=1}^{s} H^{+}_{u_i,s_i}$ we can assume that the u_i are also points in the lattice N
- If P is a full-dimensional lattice polytope we have a presentation

$$P = \bigcup_{F \text{ facet of } P} H_F^{+}$$

  and we can assume that u_F is the minimal ray generator of $\mathrm{Cone}(u_F)$.
- The presentation above for a given P is unique and the pairs (u_F, a_F) chosen as above (u_F minimal ray generator) are unique.

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \ \forall F \text{ facet of } P\}\,.$$

Example 3.21. The standard n-dimensional simplex $\Delta_n = \mathrm{Conv}(0, e_1, \cdots, e_n)$ is a polytope of dimension n. It has exactly $n+1$ vertices.

3.3 Toric varieties from polytopes

Now the idea is, given a lattice polytope P, which we assume to be full-dimensional, is to take $X_{P\cap M}$

Remark 3.22. If M is a lattice and P is a lattice polytope in $M_{\mathbb{R}}$, $P \cap M$ is a finite set.

This works, but if we want the combinatorics of P to reflect the geometry of $X_{P\cap M}$ correctly, we need P to have "enough" lattice points.

There are two notions that are related to this issue: *normality* and *very ampleness.* We will only discuss the second one.

3.3.1 Very ampleness

Definition 3.23. A lattice polytope is **very ample** if for all vertices v of P, the monoid

$$\langle P \cap M - v \rangle = \langle m - v \mid m \in P \cap M \rangle$$

is saturated.

Remark 3.24. The idea of taking the difference with v translates to making v the origin

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Definition 3.25. If P, Q are subsets of $M_{\mathbb{R}}$, their **Minkowski sum** is

$$P + Q = \{p + q \mid p \in P,\ q \in Q\}$$

Remark 3.26. If $P = \mathrm{Conv}(A)$ and $Q = \mathrm{Conv}(B)$ then $P + Q = \mathrm{Conv}(A + B)$.

Notation. If $k > 0$ and $P = \mathrm{Conv}(A)$, then we set kP to be the polytope defined by $\mathrm{Cone}(\{ka \mid a \in A\})$. If $k \in \mathbb{N}$, this also coincides with the iterated Minkowski sum

$$\underbrace{P + \cdots + P}_{k \text{ times}} = \{m_1 + m_2 + \cdots + m_k \mid m_i \in P\}\,.$$

Remark 3.27. If P is defined by $\{\langle m, n_i \rangle \geq b_i \mid i \in \{1, \cdots, s\}\}$ then

$$kP = \{\langle m, n_i \rangle \geq kb_i\ \forall i\}$$

Fact 3.28. Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope with $\operatorname{rnk} M \geq 2$. Then kP is very ample for all $k \geq n - 1$.

Remark 3.29. If $\operatorname{rnk} M = 1$ we have no issue in finding a very ample multiple.

3.3.2 The projective variety

Let P be a full-dimensional lattice polytope. The associated projective variety is

$$X_P = X_{(kP) \cap M}$$

for $k \in \mathbb{N}$ such that kP is very ample.

Remark 3.30. This will yield a well defined abstract variety, though the embedding in the ambient projective spaces change with respect to k.

Recall that $X_A \subseteq \mathbb{P}^{s-1}$ is covered by affine toric varieties: via (3.6) we have (for $A = \{a_1, \cdots, a_s\}$)

$$X_A = \bigcup_{i=1}^{s} X_A \cap U_i$$

Lemma 3.31. If $A = P \cap M$ then

$$X_A = \bigcup_{a_i \text{ vertex of } P} X_A \cap U_i$$

Proof.
Let $\{a_j\}_{j\in J}$ be the vertices of P. Fix $a_i \in A \setminus \{a_j\}_{j\in J}$. We want to find $j \in J$ such that $X_A \cap U_i \subseteq X_A \cap U_j$. Note that (exercise)

$$P \cap M_{\mathbb{Q}} = \left\{ \sum_{j\in J} r_j a_j \mid r_j \in \mathbb{Q}_{\geq 0},\ \sim r_j = 1 \right\},$$

so we can write

$$a_i = \sum_{j\in J} r_j a_j.$$

If we clear the denominators we get

$$ka_i = \sum k_j a_j, \quad k, k_j \in \mathbb{N},\ k \neq 0,\ \sum k_j = k.$$

From this we get

$$\sum_{j\in J} k_j(a_j - a_i) = 0.$$

Let $j_0 \in J$ be such that $k_{j_0} \neq 0$. It follows that

$$k_{j_0}(a_i - a_{j_0}) = \sum_{j\in J\setminus\{j_0\}} k_j(a_j - a_i)$$

so $a_i - a_{j_0} \in S_i = \langle a_k - a_i \mid k \neq i \rangle$ and S_i is the monoid which corresponds to $X_A \cap U_i$. Note that $a_{j_0} - a_i \in S_i$ by definition, so also having $k_{j_0}(a_i - a_{j_0}) \in S_i$ means that $a_{j_0} - a_i$ is invertible in S_i.

Note that $k[S_i]_{t^{a_j - a_i}}$ is the coordinate ring of $X_A \cap U_i \cap U_j$, but for j_0

$$k[S_i]_{t^{a_{j_0} - a_i}} = k[S_i]$$

so $X_A \cap U_i \cap U_{j_0} = X_A \cap U_i$, that is, $X_A \cap U_i \subseteq X_A \cap X_{j_0}$. $\square$

Theorem 3.32. Assume P is a very ample full-dimensional lattice polytope. Then

- if $a_i \in P \cap M$ is a vertex, then $X_{P\cap M} \cap U_i \cong U_{\sigma_i} = \operatorname{Spec} k[\sigma_i^\vee \cap M]$ where $\sigma_i \subseteq N_{\mathbb{R}}$ is the strongly convex cone which is dual to $C_i = \operatorname{Cone}(P \cap M - a_i)$. Moreover $\dim \sigma_i = n$
- The torus of $X_{P\cap M}$ is T_N.

Proof.
Since a_i is a vertex and P is full-dimensional, C_i is strongly convex and full-dimensional.

Now S_i (monoid that corresponds to $X_A \cap U_i$) is a submonoid $S_i \subseteq C_i \cap M = \sigma_i^\vee \cap M$ by construction.

Since P is very ample, S_i is saturated and as in a proof which we have seen ($2 \implies 3$ from (2.107)) it follows that we have equality.

The fact that the torus is T_N follows from the fact that the σ_i are strictly convex and that the torus of X_A is the same as the torus of $X_A \cap U_i$ for any i. □

The cones σ_i assemble into the **normal fan** of the polytope P: if we write

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_F \rangle \geq -a_F \ \forall F \text{ facet}\}$$

and fix a vertex $v \in P$, at v we have a cone

$$C_v = \operatorname{Cone}(P \cap M - v)$$

and $\sigma_v = C_v^\vee$ as in the proof. There is a bijection

$$\begin{array}{ccc} \{Q \leq P,\ v \in Q\} & \longleftrightarrow & \{\tau \leq C_v\} \\ Q & \longmapsto & Q_v = \operatorname{Cone}(Q \cap M - v) \\ Q_\tau = (\tau + v) \cap P & \longleftarrow & \tau \end{array}$$

This bijection preserves inclusions, intersection, dimension etc.

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In particular facets of C_v correspond to facets of P containing v, so

$$C_v = \{m \in M_\mathbb{R} \mid \langle m, u_F \rangle \geq 0 \ \forall \text{facet containing } v\}.$$

So $\sigma_v = C_v^\vee = \operatorname{Cone}(u_F \mid v \in F)$.

We can extend this association *vertices* $\to$ *cones* to all faces of P as follows:

$$Q \leq P \mapsto \sigma_Q = \operatorname{Cone}(u_F \mid Q \subseteq F)$$

Example 3.33. If $F \leq P$ is a facet, σ_F is the ray generated by u_F. If $Q = P$ then $\sigma_P = \operatorname{Cone}(\emptyset) = \{0\}$.

Definition 3.34. The cones $\{\sigma_Q \mid Q \leq P\}$ give the **normal fan** of P, denoted Σ_P.

Definition 3.35. A **fan** Σ in $N_\mathbb{R}$ is a finite collection of strongly convex cones such that

1. for all $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of both
2. if $\sigma \in \Sigma$ and $\tau \leq \sigma$ then $\tau \in \Sigma$.

Example 3.36. PICTURE

Proposition 3.37. If $\tau \leq \sigma$ there is a dual face $\tau^* \leq \sigma^\vee$ defined as $\sigma^\vee \cap ((\operatorname{Span}_\mathbb{R} \tau)^\perp)$. This construction gives an inclusion-reversing bijection between faces of σ and faces of $\sigma^\vee$.

Example 3.38. DRAWING FROM LECTURES

Remark 3.39. For all $u \in N_{\mathbb{R}} \setminus \{0\}$ there exists a unique $b \in \mathbb{R}$ such that $H_{u,b}^+ \supseteq P$ and $H_{u,b} \cap P \neq \emptyset$.

Theorem 3.40. The normal fan of a polytope P is a fan.

Sketch.
We have the following steps:

1. Note that

$$\sigma_Q = \{u \in N_{\mathbb{R}} \mid \exists b \in \mathbb{R} \ s.t. \ H_{u,b} \text{ is supporting and } Q \subseteq H_{u,b} \cap P\},$$

indeed

$\boxed{\subseteq}$ take $u \in \sigma_Q$, then $u = \sum_{Q \subseteq F} \lambda_F u_F$ for $\lambda_F \geq 0$. Let $b_0 = \sum_{F \text{ facet}, Q \subseteq F} -\lambda_F a_F \in \mathbb{R}$. By construction[1], $P \subseteq H_{u,b_0}^+$ and $Q \subseteq H_{u,b_0} \cap P$ because $Q = \bigcap_{Q \subseteq F} F$

$\boxed{\supseteq}$ Assume that $b \in \mathbb{R}$ is such that $H_{u,b}$ is supporting and $Q \subseteq H_{u,b} \cap P$. Let v be a vertex of Q (which is also a vertex of P). From $P \subseteq H_{u,b}^+$ and $P \in H_{u,b}$ it follows that $C_v \subseteq H_{u,0}^+$, i.e. $u \in (C_v)^\vee = \sigma_v = \mathrm{Cone}(u_F \mid v \in F)$, thus $u = \sum_{v \in F} \lambda_F u_F$ with some $\lambda_F \geq 0$. We have to show that if $Q \not\subseteq F$ then $\lambda_F = 0$: fix F_0 such that $Q \not\subseteq F_0$ and $p \in Q \setminus F_0$. $p, v \in Q \subseteq H_{u,b}$, so

$$b = \langle p, u \rangle = \sum \lambda_F \langle p, u_F \rangle$$

but also

$$b = \langle v, u \rangle = \sum \lambda_F \langle v, u_F \rangle = -\sum_{v \in F} \lambda_F a_F$$

so $\sum_{v \in F} \lambda_F \langle p, u_F \rangle = -\sum_{v \in F} \lambda_F a_F$, but $\langle p, u_F \rangle \geq -a_F$ for all F, so we get equality everywhere $\lambda_F \neq 0$. Since $p \notin F_0$ we have $\langle p, u_{F_0} \rangle > -a_{F_0}$, so $\lambda_{F_0} = 0$.

2. If $Q \leq P$ and $F \leq P$ facet then $u_F \in \sigma_Q$ if and only if $Q \subseteq F$
3. if $Q \subseteq Q'$ then $\sigma_{Q'} \leq \sigma_Q$ and all faces of σ_Q are of this form[2].
4. $\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$ where Q'' is the smallest face of P which contains both Q and Q'.

□

Remark 3.41. σ_Q is strictly convex because each σ_Q is a face of some σ_v and σ_v is strictly convex because P is full-dimensional.

Remark 3.42. The σ_v are the **maximal cones** of Σ_P since any other σ_Q is a face of some σ_v.

[1] $\langle m, u \rangle = \sum \lambda_F \langle m, u_F \rangle \geq -\sum \lambda_F a_F = b_0$
[2] for this you need duality of faces for a cone σ (3.37).

Definition 3.43. A fan Σ in $N_{\mathbb{R}}$ is called **complete** if

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} = N_R$$

Proposition 3.44. If P is a full-dimensional lattice polytope then Σ_P is complete.

Proof.
Fix $u \in N_{\mathbb{R}} \setminus \{0\}$ and set $b = \min\{\langle v, u\rangle \mid v \text{ vertex of } P\}$. Then $P \subseteq H^+_{u,b}$ and there exists v_0 vertex such that $\langle v_0, u\rangle = b$, that is, $v_0 \in H_{u,b_0}$. From what we have seen, this implies that $u \in \sigma_{v_0} \subseteq |\Sigma_P|$. ☐

Remark 3.45. The normal fan of P is invariant with respect to dilations and translations by integral vectors, that is,

$$\Sigma_P = \sigma_{kP+m}$$

for any $k \in \mathbb{N}$ and $m \in M$.

Together with the next proposition, this implies that the projective toric varieties $X_{kP}, X_P,\ X_{P+m}$ are all abstractly isomorphic. The only difference is the embedding in projective space.

Proposition 3.46. If P is a very ample full-dimensional lattice polytope. Let $v \neq w$ be vertices of P and let Q be the smallest face of P which contains both. Then[a]

$$X_{P\cap M} \cap U_v \cap U_w \cong U_{\sigma_Q} = \operatorname{Spec} k[\sigma_Q^\vee \cap M].$$

[a] $\sigma_Q = \sigma_v \cap \sigma_w$ so intersections at the level of cones in the fan describe how the affine patches of the toric variety are glued together.

Proof.
We have inclusions

$$\begin{array}{ccccc}
U_{\sigma_v} & & & & U_{\sigma_w} \\
\parallel & & & & \parallel \\
X_{P\cap M} \cap U_v & & & & X_{P\cap M} \cap U_w \\
& \supseteq & & \subseteq & \\
& & X_{P\cap M} \cap U_v \cap U_w & &
\end{array}$$

and we identify the double intersection both with $(U_{\sigma_v})_{t^{w-v}} \subseteq U_{\sigma_v}$ and $(U_{\sigma_w})_{t^{v-w}} \subseteq U_{\sigma_w}$.

We need to show that, for instance, $(U_{\sigma_v})_{t^{w-v}}$ can be identified with U_{σ_Q}. Note that $w - v \in C_v = \sigma_v^\vee$ so $\tau := H_{w-v} \cap \sigma_v \leq \sigma_v$. We saw that $(U_{\sigma_v})_{t^{w-v}} \cong U_\tau$ (3.32).

Let us check that $\tau = \sigma_Q$. We know that $\sigma_Q = \sigma_v \cap \sigma_w$ from the proof that the normal fan is a fan (3.40), i.e. we want $H_{w-v} \cap \sigma_v = \sigma_w \cap \sigma_v$.

$\boxed{\subseteq}$ If $n \in H_{w-v} \cap \sigma_v \setminus \{0\}$ then there exists a unique $b \in \mathbb{R}$ such that $H_{u,b}$ is supporting for P and $u \in \sigma_v$ implies $v \in H_{u,b}$ (proposition from little ago). Also $u \in H_{w-v}$, that is, ${}^{w,u} = \langle v, u\rangle$, so putting the two facts together $\langle w, u\rangle = b$, that is, $u \in \sigma_w$.

$\boxed{\supseteq}$ If $u \in \sigma_v \cap \sigma_w \setminus \{0\}$ and $b \in \mathbb{R}$ such that $H_{u,b}$ supporting then $u \in \sigma_v$ implies $v \in H_{u,b}$ and so .... $\langle w - v, u\rangle = 0$ which implies $u \in H_{w-v}$.

□

Remark 3.47. What we are sayng is that the toric variety depends only on the fan in some sense, not the polytope.

Remark 3.48. This shows that for a full-dimensional lattice polytope P, $X_P = X_{(kP)\cap M}$ where kP is very ample as an abstract variety / scheme only depends on the normal fan Σ_P and can be constructed directly from it.

Example 3.49. Let $P = \Delta_n = \mathrm{Cone}(0, e_1, \cdots, e_n) \subseteq \mathbb{R}^n$. Let $A = \Delta_n \cap \mathbb{Z}^n = \{0, e_1, \cdots, e_n\}$.

$$\phi_A : \begin{array}{ccc} \mathbb{G}_m^n & \longrightarrow & \mathbb{P}^n \\ (a_1, \cdots, a_n) & \longmapsto & [1, a_1, a_2, \cdots, a_n] \end{array}$$

and this is exactly an embedding of the torus of $\mathbb{P}^n$, which is dense, so $X_{\Delta_n} = \mathbb{P}^n$.

Let us now try $k\Delta_n$. Then $X_{k\Delta_n}$ is still isomorphic to $\mathbb{P}^n$ but it is embedded in $\mathbb{P}^{\binom{n+k}{k}-1}$ via the Veronese embedding. For example, for $n = k = 2$ we have $2\Delta_2 \cap \mathbb{Z}^2 = \{(0,0), (1,0), (2,0), (0,1), (0,2), (1,1)\}$ and

$$\phi_A : \begin{array}{ccc} \mathbb{G}_m^2 & \longrightarrow & \mathbb{P}^5 \\ (a, b) & \longmapsto & [1, a, a^2, b, b^2, ab] \end{array}$$

This extends to

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ [x_0, x_1, x_2] & \longmapsto & [x_0^2, x_0x_1, x_1^2, x_0x_2, x_2^2, x_1x_2] \end{array}$$

which is the Veronese embedding.

Example 3.50. Consider the trapezoids given by the convex hull of

$$(0,1), (0,0), (0,b), (0,a)$$

and let $X_{a,b}$ be the associated toric variety. If $b - a = b' - a'$ then $X_{a,b} \cong x_{a',b'}$ because the fan doesn't change (even though we it's not necessarily the case that we get between such isomorphic polytope by scaling and translating).

This toric variety is called Hirzebruch surface H_r where $r = b - a \in \mathbb{N}$. Another description for it is $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-r))$.

Proposition 3.51. If P is a full-dimensional lattice polytope, then X_P is normal (because the affine pieces are of the form U_{σ_v} for σ_v stictly convex) and X_P is smooth if and only if Σ_P is smooth fan (i.e. all cones in Σ_P are smooth).

Proof.
It follows from previous results and locality of the two properties. □

Chapter 4

General normal toric varieties

Recall that a scheme is **separated** if the image of the diagonal is closed.

Fact 4.1. All quasi-projective varieties are separated.

Definition 4.2. An **(abstract) variety over** k is an integral separated scheme of finite type over k.

Definition 4.3. A **toric variety** is a variety X over k with dense open torus $T_N \subseteq X$ such that the translation action of T_N on itself extends to X.

4.1 Toric varieties from fans

Given a fan Σ in $N_{\mathbb{R}}$ we have affine toric varieties U_σ for each $\sigma \in \Sigma$, which we are going to glue together as follows:

Recall that if $\tau \leq \sigma$ then $\tau = H_m \cap \sigma$ and (2.126)

$$k[S_\tau] \cong k[S_\sigma]_{t^m}$$

and so

$$U_\tau \cong (U_\sigma)_{t^m}.$$

Lemma 4.4. If $\tau = \sigma_1 \cap \sigma_2$ and it is a face of both then there exists $m \in (\sigma_1^\vee) \cap (-\sigma_2)^\vee \cap M$ such that

$$\sigma_1 \cap H_m = \sigma_2 \cap H_m = \tau.$$

This is called the **separating hyperplane**.

By the lemma, we can identify U_τ with both $(U_{\sigma_1})_{t^m}$ and $(U_{\sigma_2})_{t^{-m}}$, so we can use this isomorphism $g_{\sigma_1,\sigma_2} : (U_{\sigma_1})_{t^m} \to (U_{\sigma_2})_{t^{-m}}$ to glue U_{σ_1} and U_{σ_2} along U_τ.

It is possible to check (exercise) that the compatibilities are satisfied (descent data stuff). It is useful in the verification to consider the following diagram (showing its commutativity) for $\sigma, \sigma', \sigma'' \in \Sigma$:

$$\begin{tikzcd}
 & U_{\sigma\cap\sigma'} \arrow[r] \arrow[dr] & U_{\sigma} \\
U_{\sigma\cap\sigma'\cap\sigma''} \arrow[ur] \arrow[r] \arrow[dr] & U_{\sigma\cap\sigma''} \arrow[ur] \arrow[dr] & U_{\sigma'} \\
 & U_{\sigma'\cap\sigma''} \arrow[r] \arrow[ur] & U_{\sigma''}
\end{tikzcd}$$

We denote the resulting variety by X_Σ.

Theorem 4.5. X_Σ is a toric variety.

Proof.
The torus of X_Σ is $U_\sigma \cong T_N$ for $\sigma = \{0\}$, which is contained in any other U_σ as a dense open. So it is a dense open in X_Σ as well. The actions $T_N \times U_\sigma \to U_\sigma$ are compatible with the gluing data so they glue to a global action $T_N \times X_\Sigma \to X_\Sigma$ which extends the torus action.

Let us now check that X_Σ is separated. It is enough to show that for all $\sigma_1, \sigma_2 \in \Sigma$ with intersection τ then the "diagonal" $\Delta : U_\tau \to U_{\sigma_1} \times U_{\sigma_2}$ has closed image. This is because the image of the actual diagonal is the union of these images and so it will be a finite union of closed subsets of $X_\Sigma \times X_\Sigma$. This is now an algebraic question because that morphism is closed when the map

$$\begin{array}{ccc} k[S_{\sigma_1}] \otimes k[S_{\sigma_2}] & \longrightarrow & k[S_\tau] \\ t^m \otimes t^n & \longmapsto & t^{m+n} \end{array}$$

is surjective. This is the case because $S_\tau = S_{\sigma_1} + S_{\sigma_2}$ as submonoids of M, indeed

$\boxed{\subseteq}$ Recall that $S_\tau = S_{\sigma_1} + \mathbb{N}(-m) \subseteq S_{\sigma_1} + S_{\sigma_2}$ for H_m separating hyperplane. The inclusion of $-m$ in S_{σ_2} follows because $m \in (-\sigma_2)^\vee \implies -m \in \sigma_2^\vee$.

$\boxed{\supseteq}$ $\sigma_1^\vee + \sigma_2^\vee \subseteq (\sigma_1 \cap \sigma_2)^\vee = \tau^\vee$ and now intersect with M.

$\square$

We will see later that every toric variety is of this form.

4.1.1 Examples

Example 4.6. The fan of $\mathbb{P}^2$ is the normal fan of the the simplex Δ_2:

$$\Sigma_{\Delta_2} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_1 \cap \sigma_2, \sigma_1 \cap \sigma_3, \sigma_2 \cap \sigma_3, \{0\}\}$$

where $\sigma_1 = \mathrm{Cone}((1,0),(0,1))$, $\sigma_2 = \mathrm{Cone}((0,1),(-1,-1))$ and $\sigma_3 = \mathrm{Cone}((1,0),(-1,-1))$.

Note that $\det\left(\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}\right) = -1$ is invertible in $\mathbb{Z}$, so $(1,0)$, $(-1,-1)$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ and σ_3 is smooth. A similar remark holds for the other cones.

Note that $\sigma_1^\vee \cap M = \langle e_1, e_2 \rangle$ so $U_{\sigma_1} \cong \operatorname{Spec} k[x,y]$. Similarly $U_{\sigma_2} = \operatorname{Spec} k[x^{-1}, x^{-1}y]$ and $U_{\sigma_3} = \operatorname{Spec} k[y^{-1}, xy^{-1}]$. Abstractly $U_{\sigma_1} \cong U_{\sigma_2} \cong U_{\sigma_3} \cong \mathbb{A}^2$ but the notation shows us the transition functions.

If in $\mathbb{P}^2$ we have $[x_0, x_1, x_2]$ we are saying $x = x_1/x_0$ and $y = x_2/x_0$. Indeed $x_0/x_1 = x^{-1}$, $x_2/x_1 = x^{-1}y$ etc.

Example 4.7. The fan of $\mathbb{P}^n$ is the one in $\mathbb{R}^n$ given by the cones generated by proper (possibly empty) subsets of

$$\{e_1, \cdots, e_n, -e_1 - \cdots - e_n\}.$$

Example 4.8. Affine and projective toric varieties are of this form. For U_σ take $U_\sigma = \{\text{faces of } \sigma\}$ and in the projective case we take the normal fan.

Remark 4.9. All toric varieties of dimension 1 are $\mathbb{G}_m$, $\mathbb{A}^1$ and $\mathbb{P}^1$, given by the possible fans in $\mathbb{R}$: $\{\{0\}\}$, $\{\mathrm{Cone}(1), \{0\}\}$ and $\{\mathrm{Cone}(1), \mathrm{Cone}(-1), \{0\}\}$.

Example 4.10. Consider the fan $\Sigma = \{\tau_1, \tau_2, \{0\}\}$ with $\tau_1 = \mathrm{Cone}((1,0))$ and $\tau_2 = \mathrm{Cone}((0,1))$ in $\mathbb{R}^2$.

X_Σ is obtained by gluing together $U_{\tau_1} = \mathbb{A}^1 \times \mathbb{G}_m$ and $U_{\tau_2} = \mathbb{G}_m \times \mathbb{A}^1$ along $\mathbb{G}_m \times \mathbb{G}_m$. This results in $\mathbb{A}^2 \setminus \{0\}$, which we know to be neither affine nor projective.

Remark 4.11. We will see that there is a bijection between torus orbits on X_Σ and cones in Σ, so deliting a cone σ (and all other cones which contain it as a face) from the fan corresponds to removing the corresponding orbit.

Example 4.12. Consider $\Sigma = \{\sigma_1, \sigma_2\}$ with $\sigma_1 = \mathrm{Cone}((0,1),(1,1))$ and $\sigma_2 = \mathrm{Cone}((1,0),(1,1))$. It turns out that X_Σ in this case is $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$. Recall that $\mathrm{Bl}_{(0,0)}\mathbb{A}^2 = V(x_0y - x_1x) \subseteq \mathbb{P}^1 \times \mathbb{A}^2$. If $x_0 \neq 0$ and we name $t = x_1/x_0$ then we get that $\mathrm{Bl}_{(0,0)}\mathbb{A}^2 \cap U_0 \times \mathbb{A}^2 = \mathbb{A}^3$ looks like $V(y - tx)$, which is isomorphic to $\mathbb{A}^2 = \operatorname{Spec} k[x,t]$.

The X_Σ is obtained by gluing two copies of $U_{\sigma_1} \cong \mathbb{A}^2$ and $U_{\sigma_2} \cong \mathbb{A}^2$. It is possible to check that the gluing conditions look like the ones we implied while looking at the affine charts of $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$: $\sigma_1^\vee = \mathrm{Cone}(e_1, e_2 - e_1)$, $\sigma_2^\vee = \mathrm{Cone}(e_2, e_1 - e_2)$, so $U_{\sigma_1} = \operatorname{Spec} k[x, yx^{-1}]$, $U_{\sigma_2} = \operatorname{Spec} k[y, xy^{-1}]$ and now if we say $y = xt$ then we get the conditions from before.

Remark 4.13. More generally, the fan generated by $\{e_1, \cdots, e_n, e_1 + \cdots + e_n\}$ gives $\mathrm{Bl}_0\mathbb{A}^n$.

Definition 4.14. If Σ' and Σ are fans in $N_\mathbb{R}$, Σ' is a **refinement** of Σ if for all $\sigma' \in \Sigma'$ there exists $\sigma \in \Sigma$ such that $\sigma' \subseteq \sigma$.

Remark 4.15. The previous example was a special case of the following result: if Σ' is a refinement of Σ there is an induced "toric morphism" $X_{\Sigma'} \to X_\Sigma$ which is always proper and birational.

4.2 Orbit-cone correspondence

As we mentioned, there is a correspondence between torus orbits in X_Σ and cones in Σ. This allows us to reconstruct the fan Σ starting from X_Σ.

The way to detect cones of Σ from the T_N-action is by loocking at limits $\lim_{t\to 0}\lambda^n(t)$ of 1-parameter subgroups $\lambda^n : \mathbb{G}_m \to T_N$. This statement doesn't make sense as stated but we are trying to emulate limits like for 1-ps in differential geometry. If $k = \mathbb{C}$ the limit is the actual limit in the euclidia topology.

Definition 4.16. Let $\lambda^n : \mathbb{G}_m \to T_N \subseteq X_\Sigma$ be a 1-ps. $\lim_{t\to 0} \lambda^n(t)$ is defined to be $\widetilde{\lambda^n}(0)$ if λ^n extends to a morphism $\widetilde{\lambda^n} : \mathbb{A}^1 \to X_\Sigma$ (which is uniquely determined if it exists by separatedness of X_Σ).

Example 4.17. The 1-ps $\mathbb{G}_m \to \mathbb{G}_m \subseteq \mathbb{A}^1$ given by $\lambda^n(t) = t$ has $\lim_{t\to 0} \lambda^n(t) = 0$. The one given by $\lambda^n(t) = t^{-1}$ does not extend and so has no limit.

Remark 4.18. The codomain of the extension matters. The map $t \mapsto t^{-1}$ seen as a morphism $\mathbb{G}_m \to \mathbb{P}^1$ DOES extend to $\mathbb{A}^1 \to \mathbb{P}^1$ and the value at 0 would be the point at infinity.

For u varying in N, the possible limits $\lim_{t\to 0} \lambda^u(t) \in X_\Sigma$ are finitely many, one for each cone in Σ. It will be the case that the limit is γ_σ for σ cone exactly when $u \in \mathrm{Relint}(\sigma) \subseteq N_\mathbb{R}$.

Example 4.19. In $\mathbb{P}^2$ consider the cocharacter $u = (a, b) \in N = \mathbb{Z}^2$ and the relative 1-parameter subgroup

$$\lambda^{(a,b)} : \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{P}^2 \\ t & \longmapsto & [1, t^a, t^b] \end{array}$$

What is the limit

$$\lim_{t\to 0}[1, t^a, t^b] = ?$$

- If $a, b > 0$ then $[1, t^a, t^b] \to [1, 0, 0]$.
- If $a < 0$ and $b > a$ then $[1, t^a, t^b] = [t^{-a}, 1, t^{b-a}]$ so in that case the limit is $[0, 1, 0]$.
- If $b < 0$ and $a > b$ then the limit is $[0, 0, 1]$.
- If $a = 0$ and $b > 0$ then $[1, 1, t^b] \to [1, 1, 0]$.
- If $a = b$ and $b < 0$ then $[0, 1, 1]$.
- If $b = 0$ and $a > 0$ then $[1, 0, 1]$.
- Finally, for $a = b = 0$ we get $[1, 1, 1]$.

Definition 4.20. Given a fan Σ, the limit points γ_σ are defined as follows:
$\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$ is defined by the monoid homomorphism[a]

$$\gamma_\sigma : \begin{array}{ccc} S_\sigma & \longrightarrow & (k, \cdot) \\ m & \longmapsto & \begin{cases} 1 & \text{if } m \in \sigma^\vee \cap M \cap \sigma^\perp \\ 0 & \text{otherwise} \end{cases} \end{array}$$

[a]the intersection with $\sigma^\perp$ is relevant only if σ is not full-dimensional.

Remark 4.21. The map γ_σ above is a homomorphism

Proof.
$\sigma^\perp \cap \sigma^\vee$ is a face of $\sigma^\vee$, so if $m, m' \in \sigma^\vee \cap M$, having $m + m' \in \sigma^\vee \cap M \cap \sigma^\perp$ implies $m + m' \in \sigma^\perp$ ******** ☐

Remark 4.22. ******* and the torus-fixed point p_σ of U_σ **** that we analyzed before.

Example 4.23. If $\sigma = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}^2$ (Cone(e_1)) then $S_\sigma = \mathbb{N} \oplus \mathbb{Z}$ ($\sigma^\vee \cap \mathbb{Z}^2$) then

$$\gamma_\sigma : \begin{array}{ccc} \mathbb{N} \oplus \mathbb{Z} & \longrightarrow & k \\ (n,m) & \longmapsto & \begin{cases} 1 & \text{if } n = 0 \\ 0 \end{cases} \end{array}$$

$U_\sigma = \mathbb{A}^1 \times \mathbb{G}_m$ and $\gamma_\sigma \leftrightarrow (0,1)$, when there is a torus factor, i.e. σ not full-dimensional, and $U_\sigma \cong U_{\sigma,N_1} \times T_{N_2}$ where N_1 is the saturated $\mathbb{Z}$-span of $\sigma \cap N$

$\gamma_\sigma = (p_{\sigma,N_1}, e)$ where the first is the torus-fixed point of U_{σ,N_1} and e is the neutral element of T_{N_2}.

Remark 4.24. If $\tau \leq \sigma$ then $U_\tau \subseteq U_\sigma$ as a principal open, so γ_τ is also a point of U_σ, corresponding to the monoid homomorphism

$$\begin{array}{ccc} S_\sigma & \longrightarrow & k \\ m & \longmapsto & \begin{cases} 1 & \text{if } m \in \sigma^\vee \cap \tau^\perp \cap M \\ 0 & \text{otherwise} \end{cases} \end{array}$$

Remark 4.25. The different γ_σ are distinct as points of X_Σ. The idea is to prove that if $\tau < \sigma$ then[1] $\gamma_\sigma \notin U_\tau$, because in that case $\gamma_\sigma = \gamma_{\sigma'}$ but $\sigma \cap \sigma'$ would be a proper face of at least one of σ or σ' if they were different cones, contradiction.

The idea now is to show that the orbits of the torus action are precisely the orbits of these γ_σ, which we write $\mathcal{O}(\sigma) = T_N \cdot \gamma_\sigma$.

Lemma 4.26. The limit $\lim_{t\to 0} \lambda^u(t)$ exists in U_σ if and only if for all $m \in S_\sigma$, $\lim_{t\to 0} \chi^m \lambda^u(t)$ exists in $\mathbb{A}^1$.

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\lambda^u"] \arrow[rrr, bend right] & T_N \arrow[r, phantom, "\subseteq"] & U_\sigma \arrow[r, "\chi^m"] & \mathbb{A}^1 \end{tikzcd}$$

Proof.
We give the two implications

$\boxed{\Longrightarrow}$ If $\mathbb{G}_m \to U_\sigma$ exentds to $\mathbb{A}^1$ then the composite $\mathbb{G}_m \to U_\sigma \to \mathbb{A}^1$ will also extend by composing the extension with $\chi^m : U_\sigma \to \mathbb{A}^1$.

$\boxed{\Longleftarrow}$ If $A = \{a_1, \cdots, a_s\} \subseteq M$ is a finite set of generators for S_σ then $k[x_1, \cdots, x_s] \twoheadrightarrow k[S_\sigma]$ and this induces a closed embedding $U_\sigma \hookrightarrow \mathbb{A}^s$. By assumption, $\mathbb{G}_m \to U_\sigma \to \mathbb{A}^s$ extends to $\mathbb{A}^1 \to \mathbb{A}^s$ (it does in all coordinates). Since U_σ is closed, the extension will factor through U_σ (you can take the closure Z of the images of $\mathbb{G}_m$ and $\mathbb{A}^1$ in $\mathbb{A}^s$, which are the same because $\mathbb{G}_m$ is dense in $\mathbb{A}^1$, and then $Z \subseteq U_\sigma$ because U_σ is closed and Z is the closure of the image of $\mathbb{G}_m$ which is contained in the image of U_σ, showing the desired factorization).

□

[1] use the fact that the image of the embedding of $U_\tau \hookrightarrow U_\sigma$ is given by the homomorphisms $S_\sigma \to k$ such that $\gamma(m) \in k^*$, where $m \in M$ is such that $\tau = \sigma \cap H_m$.

Remark 4.27. We can also say that, when the limit exists, the limit point in U_σ corresponds to the homomorphism

$$\begin{array}{ccc} S_\sigma & \longrightarrow & k \\ m & \longmapsto & \lim_{t\to 0}\chi^m\lambda^u(t) \end{array},$$

indeed, using the embedding $U_\sigma \subseteq \mathbb{A}^s$ as in the proof, points of U_σ become points of $\mathbb{A}^s$ (homomorphisms $\mathbb{N}^s \to k$ obtained by precomposing with the presentation of S_σ given by fixing generators) and the limit point is now the one with coordinated given by that formula for $m = a_i$ with $1 \le i \le s$. Since $a_1, \cdots, a_s$ generate S_σ, the homomorphisms agree on generators of the domain.

Proposition 4.28. The limit $\lim_{t\to 0}\lambda^u(t)$ exists in U_σ if and only if $u \in \sigma$ in $N_\mathbb{R}$ and if $u \in \mathrm{Relint}(\sigma)$ then the limit is γ_σ.

Proof.
By the lemma (4.26), the limit exists in U_σ if and only if $\lim_{t\to 0}\chi^m\lambda^u(t)$ exists in $\mathbb{A}^1$ for all $m \in S_\sigma$. Let us write $t^{\langle m,n\rangle} = \chi^m\lambda^u(t)$. We have that the limit exists if and only if for all $m \in S_\sigma$ we have $\langle m, u\rangle \ge 0$, that is, $u \in (\sigma^\vee)^\vee = \sigma$.

Thanks to the previous remark, we can say that the limit point will correspond to the homomorphism

$$\begin{array}{ccc} S_\sigma & \longrightarrow & k \\ m & \longmapsto & \lim_{t\to 0} t^{\langle m,u\rangle} \end{array}$$

Now, if $u \in \mathrm{Relint}(\sigma)$ then (exercise)

$$\begin{cases} \langle m,u\rangle > 0 & \text{if } m \in S_\sigma \setminus \sigma^\perp \\ \langle m,u\rangle = 0 & \text{if } m \in S_\sigma \cap \sigma^\perp \end{cases}$$

and this gives exactly γ_σ as a limit point[2]. □

We will now describe the orbits $\mathcal{O}(\sigma)$ of the torus action on X_Σ and their closures $V(\sigma)$ starting from the fan Σ and then embed them in X_Σ.

For $\sigma \in \Sigma$, let $N_\sigma \subseteq N$ be the saturated sublattice of N generated by $\sigma \cap N$. We have that

$$N(\sigma) = N/N_\sigma$$

is also a lattice and its dual can be canonically identified with $M(\sigma) = \sigma^\perp \cap M$ via the non-degenerate pairing $M(\sigma) \times N(\sigma) \to \mathbb{Z}$ induced by $M \times N \to \Sigma$.

Let $\mathcal{O}(\sigma)$ be the torus corresponding to these lattices, $\mathcal{O}(\sigma) = \mathrm{Spec}\, k[M(\sigma)]$. Note that $\dim_\mathbb{R}(N_\sigma)_\mathbb{R} = \dim\sigma$, so $\dim\mathcal{O}(\sigma) = n - \dim\sigma$, where $n = \mathrm{rnk}\, N$.

Also $M(\sigma) \subseteq M$ gives a surjection of tori $T_N \twoheadrightarrow \mathcal{O}(\sigma)$, which gives an action of T_N on $\mathcal{O}(\sigma)$.

To define $V(\sigma)$ we consider the "star" of σ in Σ:

Definition 4.29. Given a fan Σ and a cone σ in the fan, the **star** of σ is

$$\mathrm{Star}(\sigma) = \{\tau \in \Sigma \mid \sigma \le \tau\}.$$

[2] the idea is that $\lim_{t\to 0} t^a$ for $a > 0$ is 0, while $\lim_{t\to 0} t^0 = \lim_{t\to 0} 1 = 1$.

Remark 4.30. the images of the cones in $\mathrm{Star}(\sigma)$ in the quotient $N(\sigma) = N/S_\sigma$ form a fan, which we still denote $\mathrm{Star}(\sigma)$.

PICTURE

Let $V(\sigma) = X_{\mathrm{Star}(\sigma)}$, the toric variety given by this fan in $N(\sigma)_{\mathbb{R}}$. This is an $\mathcal{O}(\sigma)$-toric variety (i.e., $\mathcal{O}(\sigma)$ is the torus for this variety).

By construction, $V(\tau) = \bigcup_{\tau \le \sigma} U_\sigma(\tau)$ where

$$U_\sigma(\tau) = \operatorname{Spec} k[\overline{\sigma}^\vee \cap M(\tau)]$$

where $\overline{\sigma} \in \mathrm{Star}(\tau)$ is the quotient σ/N_τ.

We can embed $V(\tau)$ in X_σ as an orbit closure: we can construct the embedding locally as follows:

fix σ such that $\tau \le \sigma$. We have a closed embedding $U_\sigma(\tau) \hookrightarrow U_\sigma$ corresponding the homomorphism $k[\sigma^\vee \cap M] \to k[\sigma^\vee \cap M \cap \tau^\perp]$ given by sending t^m to t^m if $m \in \tau^\perp$ or to 0 otherwise. Equivalently this amounts to extending $\gamma : \sigma^\vee \cap M \cap \tau^\perp \to k$ to

$$\widetilde{\gamma} : \begin{array}{ccc} \sigma^\vee \cap M & \longrightarrow & k \\ m & \longmapsto & \begin{cases} \gamma(m) & \text{if } m \in \tau^\perp \\ 0 & \text{overwise} \end{cases} \end{array}$$

This makes sense because $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$.

These embeddings are compatible: if $\tau \le \sigma \le \sigma'$ then

$$\begin{tikzcd} U_\sigma(\tau) \arrow[r, hook, "closed"] \arrow[d, hook, "open"'] & U_\sigma \arrow[d, hook, "open"] \\ U_{\sigma'}(\tau) \arrow[r, hook, "closed"'] & U_{\sigma'} \end{tikzcd}$$

commutes (check on the algebras).

So these maps glue to a closed embedding $V(\tau) \to \bigcup_{\tau \le \sigma} U_\sigma \subseteq X_\Sigma$, that is, we now only know that the first of the two immersions is closed. To finish we just need to show that if $V(\tau) \cap U_{\sigma'} \neq \emptyset$ then $\tau \le \sigma'$.

Lemma 4.31. If τ, σ are cones such that $\tau \subseteq \sigma$ then τ is a face of σ if and only if for all $v, w \in \sigma$ we have $v + w \in \tau \implies v, w \in \tau$.

Corollary 4.32. If τ, τ' are faces of a cone σ such that $\tau \subseteq \tau'$ then $\tau \le \tau'$.

Proposition 4.33. If $V(\tau) \cap U_{\sigma'} \neq \emptyset$ then $\tau \le \sigma'$.

Proof.

Assume $\gamma \in V(\tau) \cap U_{\sigma'}$. From what we have seen, there exists $\sigma \in \Sigma$ such that $\tau \le \sigma$ and $\gamma \in U_\sigma$, so

$$\gamma \in V(\tau) \cap U_\sigma \cap U_{\sigma'} = U_\sigma(\tau) \cap U_{\sigma'} = \operatorname{Spec} k[\sigma^\vee \cap \tau^\perp \cap M] \cap U_{\sigma'}.$$

The inclusion $U_\sigma(\tau) \subseteq U_\sigma$ at the level of points corresponds to extension by 0 (if we view the points as monoid homomorphisms to $(k, \cdot)$).

Since $U_\sigma(\tau) \subseteq U_\sigma$, $U_\sigma(\tau) \cap U_{\sigma'} = U_\sigma(\tau) \cap U_{\sigma\cap\sigma'}$ and $U_{\sigma\cap\sigma'} \subseteq U_\sigma$ corresponds to the points $\alpha : \sigma^\vee \cap M \to k$ such that $\alpha(m) \in k^*$ for $m \in M$ such that $H_m \cap \sigma = \sigma \cap \sigma'$.

Since $\gamma \in U_\sigma(\tau) \cap U_{\sigma\cap\sigma'}$ it must be the case that $m \in \tau^\perp$ because outside of $\tau^\perp$ we extend by 0 and $\alpha(m)$ has to be invertible. This means that $\langle m, n\rangle = 0$ for $n \in \tau$ and since $H_m \cap \sigma = \sigma \cap \sigma'$ we have $\tau \subseteq \sigma \cap \sigma'$. By corollary (4.32), we have $\tau \leq \sigma \cap \sigma'$ and since $\sigma \cap \sigma' \leq \sigma'$ we have $\tau \leq \sigma'$. □

Remark 4.34. These maps are T_N-equivariant, where the action of T_N on $\mathcal{O}(\sigma)$ and $V(\sigma)$ are induced by the surjection $T_N \twoheadrightarrow \mathcal{O}(\sigma)$ corresponding to $M(\sigma) \subseteq M$.

Proof.
If $\tau \leq \sigma$ we need to check that the diagram commutes

$$\begin{tikzcd}
T_N \times U_\sigma(\tau) \arrow[r] \arrow[d] & \mathcal{O}(\sigma) \times U_\sigma(\tau) \arrow[r] & U_\sigma(\tau) \arrow[d] \\
T_N \times U_\sigma \arrow[rr] & & U_\sigma
\end{tikzcd}$$

and this is true after looking at the coordinate rings (exercise). □

Remark 4.35. This implies that $\mathcal{O}(\sigma) \subseteq X_\Sigma$ is an orbit for the torus action and that $V(\sigma)$ is its closure (because it is closed and $\mathcal{O}(\sigma)$ is its dense torus when seen as a toric variety).

If $\tau \leq \tau'$ then we have a closed embedding $V(\tau') \hookrightarrow V(\tau)$ making the diagram commute

$$\begin{tikzcd}
V(\tau') \arrow[rr, hook] \arrow[dr, hook] & & V(\tau) \arrow[dl, hook'] \\
& X_\Sigma &
\end{tikzcd}$$

This can be described locally as: for $\tau' \leq \sigma$ we have $U_\sigma(\tau') \hookrightarrow U_\sigma(\tau)$ closed immersion described alegbraically by

$$\begin{array}{ccc}
k[\sigma^\vee \cap \tau^\perp \cap M] & \longrightarrow & k[\sigma^\vee \cap (\tau')^\perp \cap M] \\
t^m & \longmapsto & \begin{cases} t^m & \text{if } m \in (\tau')^\perp \\ 0 & \text{otherwise} \end{cases}
\end{array}$$

This is compatible with the embedding in U_σ.

Remark 4.36. This embedding $V(\tau') \subseteq V(\tau)$ can also be seen as the embedding $V(\tau') \subseteq X_{\mathrm{Star}(\tau)}$.

This gives a map

$$\begin{array}{ccc}
\{\text{cones in } \Sigma\} & \longrightarrow & \{\text{torus orbits in } X_\Sigma\} \\
\sigma & \longmapsto & \mathcal{O}(\sigma)
\end{array}$$

(or equivalently we may take {orbit closures} and assign $\sigma \mapsto V(\sigma)$).

Remark 4.37. The map is injective because of the fact (exercise) that $\gamma_\sigma \in \mathcal{O}(\sigma)$ and $\gamma_\sigma \notin \mathcal{O}(\sigma')$ for $\sigma' \neq \sigma$.

This also shows that $\mathcal{O}(\sigma) = T_N \cdot \gamma_\sigma$.

Recall that $\dim \mathcal{O}(\sigma) = \dim T_N - \dim \sigma$.

Proposition 4.38. We have

1. $U_\sigma = \bigcup_{\tau \leq \sigma} \mathcal{O}(\tau)$, so in particular every torus orbit is one of the $\mathcal{O}(\sigma)$.
2. $\mathcal{O}(\sigma) = V(\sigma) \setminus \bigcup_{\sigma < \tau} V(\tau)$.
3. $V(\tau) = \bigcup_{\tau \leq \sigma} \mathcal{O}(\sigma)$.

Proof.
We prove the three propositions

1. Pick $\gamma \in U_\sigma$, which we see as a homomorphism $\gamma : \sigma^\vee \cap M \to k$. Look at $\gamma^{-1}(k^*) \subseteq \sigma^\vee \cap M$ and note that the cone generated by this submonoid is a face of $\sigma^\vee$ (follows from the fact that if $m, m' \in \sigma^\vee \cap M$ are such that $m + m' \in \gamma^{-1}(k^*)$ then $\gamma(m)\gamma(m') \in k^* \implies \gamma(m), \gamma(m') \in k^*$, that is, $m, m' \in \gamma^{-1}(k^*)$). By taking the dual (as a face, not as a cone) of this face we get $\gamma^{-1}(k^*) = \sigma^\vee \cap \tau^\perp \cap M$ for some $\tau \leq \sigma$. This means exactly that $\gamma \in \mathcal{O}(\tau) = \operatorname{Spec} k[\tau^\perp \cap M]$.

2. Changing notation so that $N = N(\sigma)$, we reduce to proving that
$$T_N = X_\Sigma \setminus \bigcup_{\tau \neq 0} V(\tau)$$
Intersecting with U_σ for some $\sigma \in \Sigma$ this becomes
$$T_N = U_\sigma \setminus \bigcup_{\tau \neq 0} V(\tau) \cap U_\sigma = U_\sigma \setminus \bigcup_{\tau \neq 0} U_\sigma(\tau)$$
and thus follows from 1. when applied to U_σ:
$$U_\sigma = \coprod_{\tau \leq \sigma} \mathcal{O}(\tau) = T_N \sqcup \coprod_{0 \neq \tau \leq \sigma} \mathcal{O}(\tau) \subseteq T_N \sqcup \bigcup_{0 \neq \tau \leq \sigma} U_\sigma(\tau).$$

3. Follows from 2. by induction on $n - \dim \tau$.

$\square$

Example 4.39. Let Σ be the fan of $\mathbb{A}^2$, that is
$$\Sigma = \{\sigma, \tau_1, \tau_2, \{0\}\}$$
for
$$\sigma = \operatorname{Cone}(e_1, e_2) \quad \text{and} \quad \tau_i = \operatorname{Cone}(e_i).$$
We get

$\mathbb{A}^2 = U_\sigma = \mathcal{O}(\sigma) \sqcup \mathcal{O}(\tau_1) \sqcup \mathcal{O}(\tau_2) \sqcup \mathcal{O}(0) = \{(0,0)\} \sqcup \{(0,y) \mid y \neq 0\} \sqcup \{(x,0) \mid x \neq 0\} \sqcup \mathbb{G}_m^2.$

Indeed: $N(\tau_2) = N/N_{\tau_2} = \mathbb{Z}$ and
$$V(\tau_2) = X_{\operatorname{Star}(\tau_2)} = \mathbb{A}^1 = \operatorname{Spec} k[t] \hookrightarrow U_\sigma = \operatorname{Spec} k[x,y]$$
let $\sigma' = \tau_1/N_{\tau_2}$
$(\sigma')^\vee \tau_2^\perp \cap M = \langle e_1 \rangle \cap (\sigma')^\vee \cong \mathbb{N}$. So $V(\tau_2) \subseteq \mathbb{A}^2$ is the x-axis.......

In the projective case, i.e. for X_P with P full-dimensional lattice polytope, for a face $Q \leq P$ there is a cone $\sigma_Q \in \Sigma_P$, so the orbit closures on X_P are exactly the $V(\sigma_Q)$ for faces $Q \leq P$. This correspondence becomes more "visual".

For example, note that $Q \mapsto V(\sigma_Q) \subseteq X_P$ preserves dimensions.

Proposition 4.40. $V(\sigma_Q) \cong X_Q$ as toric varieties.

Example 4.41. $\mathbb{P}^2$ comes from Δ_2 for example. The edges of Δ_2 correspond to the coordinate hyperplanes in $\mathbb{P}^2$, the intersection points are the three origins ($[1,0,0]$, $[0,1,0]$, $[0,0,1]$). The interior corresponds to the torus.

Remark 4.42. If $k = \mathbb{C}$ there is a continuous (for the euclidian topology) function (called moment/momentum map)

$$\mathbb{CP}^2 \to \Delta_2$$

which is a "degenerate torus fibration", i.e. the fibers of points in the interior of Δ_2 the fiber is a torus, over an edge the fibers are circles and over the vertices they are points.

In the case of $\mathbb{CP}^1$ the fibers over points in $(0,1)$ are circles and over the extremes $\{0,1\}$ you get points.

4.3 Rough sketch that every toric variety comes from a fan

Theorem 4.43 (Sumihiro)**.** If $T_N \curvearrowright X$ with X normal and separated then there exists $\{U_i\}$ open affine inveriant cover of X.

Every such U_i is an affine normal T_N-toric variety $U_i \cong U_{\sigma_i}$ for $\sigma_i \subseteq N_{\mathbb{R}}$ strongly convex rational cone. Then one shows

- $U_i \cap U_j$ can be identified with $U_{\sigma_i \cap \sigma_j}$
- $\sigma_i \cap \sigma_j$ is a face of both (this follows from the general fact that if $\tau \subseteq \sigma$ and $U_\tau \to U_\sigma$ is an open embedding (induced by $\sigma^\vee \to \tau^\vee$) then it follows that $\tau \leq \sigma$).
- $X \cong X_\Sigma$ where Σ is the fan consisting of the σ_i and their faces.

4.4 Properness of toric varieties

4.4.1 Properness and valuative criterion

Note that in algebraic geometry, classical compactness is almost always verified because Zariski open sets are very big. Properness is the correct analogue.

Definition 4.44. An abstract variety X is **proper** if it is separated and for all Y abstract variety the projection $X \times Y \to Y$ is closed.

Example 4.45. $\mathbb{P}^n$ is proper, $Z \subseteq \mathbb{P}^n$ closed is also proper.

Example 4.46. $\mathbb{A}^n$ is not proper. For example $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is not closed ($V(xy-1)$ goes to $\mathbb{G}_m$, which is not closed in $\mathbb{A}^1$).

Remark 4.47. Sometimes proper is called complete. The idea is that X does not have any "punctures", i.e. limits of points on curves always exist.

Remark 4.48. X is both proper and affine only if X composed of a finite amount of points.

Definition 4.49. A morphism $f : X \to Y$ of varieties is **proper** if it is universally closed[a], i.e. f is closed and for all $Z \to Y$ morphism the projection $X \times_Y Z \to Z$ is closed.

[a] in general you also impose separated of finite type but these conditions are automatic from our definition of abstract varieties

Fact 4.50 (Valuative criterion for properness)**.** If X, Y are varieties (finite type schemes over k) and $f : X \to Y$ is a morphism, then f is proper if and only if for all R DVR over k with fraction field $\mathbb{K}$, in any diagram

$$\begin{tikzcd} \operatorname{Spec} \mathbb{K} \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a unique dotted arrow that makes everything commute.

Example 4.51. Let $R = k[[t]]$. The fraction field is $\mathbb{K} = k[[t]][t^{-1}] = k((x))$. $\operatorname{Spec} R$ has two elements, the ideal (0) and the maximal ideal (t).

$\operatorname{Spec} \mathbb{K} \to \operatorname{Spec} R$ in an embedding of the generic point in $\operatorname{Spec} R$ (because $K = R_{(0)}$). The generic point is like a "fuzzy neighborhood of a curve" or "a piece of a curve". A diagram like before means that we have some piece of a curve in X and we have a curve plus an actual point in Y, then the criterion says that f is proper when we can find a unique "limit point" of the piece of curve in X which commutes with the projections to Y.

If we take $Y = \operatorname{Spec} k$, we see that properness of X corresponds to some kind of existence and uniqueness of limits for curves in X.

Remark 4.52. You could refrase the valuative criterion using smooth projective curves instead of $\operatorname{Spec} R$ and a non-empty open $U \subseteq C$ instead of $\operatorname{Spec} \mathbb{K}$, but this form is much more convenient.

Example 4.53. Let's check that $\mathbb{P}^n \to \operatorname{Spec} k$ is proper using the valuative criterion. Let R be a DVR with uniformizing parameter π and fraction field $\mathbb{K}$. Suppose we have a diagram

$$\begin{tikzcd} \operatorname{Spec} \mathbb{K} \arrow[r] \arrow[d, hook] & \mathbb{P}^n \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

The morphism $\operatorname{Spec} \mathbb{K} \to \mathbb{P}^n$ corresponds to a closed point $\mathbb{P}^n(\mathbb{K})$, that is, $[x_0, \cdots, x_n]$ with $x_i \in \mathbb{K}$ and (say) $x_0 \neq 0$. Since R is a DVR, the x_i have a valuation $v(x_i)$. Let $k = \min\{v(x_i) \mid 0 \leq i \leq n,\ x_i \neq 0\}$. Note that

$$[x_0, \cdots, x_n] = [\pi^{-k}x_0, \cdots, \pi^{-k}x_n]$$

and now by construction the valuation of the coordinates in the second form lie in R. Moreover, there is a j such that $v(\pi^{-k}x_j) = 0$, that is, $\pi^{-k}x_j \in R^*$, so the morphism $\operatorname{Spec} R \to \mathbb{A}^n \cong U_j \subseteq \mathbb{P}^n$ given by

$$\begin{array}{ccc} k[y_1, \cdots, y_n] & \longrightarrow & R \\ y_i & \longmapsto & \frac{\pi^{-k}x_i}{\pi^{-k}x_j} \end{array}$$

provides a lift, which is also unique, showing the hypothesis of the valuative criterion.

Using the valuative criterion, we will show that X_Σ is proper if and only if Σ is a complete fan.

4.4.2 Toric morphisms

Let N, N' be two lattices with fans Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$.

Given a group homomorphism $\varphi : N \to N'$ which is compatible with the fans, i.e. for all $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma'$ such that $\varphi_\mathbb{R}(\sigma) \subseteq \sigma'$ for $\varphi_\mathbb{R} = \varphi \otimes id_\mathbb{R} : N_\mathbb{R} \to N'_\mathbb{R}$.

In this situation we can construct an induced morphism $\varphi_* : X_\Sigma \to X_{\Sigma'}$.

If $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ such that $\varphi_\mathbb{R}(\sigma) \subseteq \sigma'$ then we have $\varphi_\mathbb{R}|_\sigma : \sigma \to \sigma'$ and this defines a morphism of monoids $(\sigma')^\vee \cap M' \to \sigma^\vee \cap M$, which gives a morphism $U_\sigma \to U'_\sigma$.

These morphisms are compatible (exercise) so they glue to a global morphism $X_\Sigma \to X_{\Sigma'}$.

Definition 4.54. A morphism defined as above is called a **toric morphism**.

Remark 4.55. Any toric morphism $f = \varphi_* : X_\Sigma \to X_{\Sigma'}$ is[3] $(T_N \to T_{N'})$-equivariant, that is, we have a commutative diagram

$$\begin{array}{ccc} T_N \times X_\Sigma & \longrightarrow & T_{N'} \times X_{\Sigma'} \\ \downarrow & & \downarrow \\ X_\Sigma & \longrightarrow & X_{\Sigma'} \end{array}$$

This is because it commutes at the level of

$$\begin{array}{ccc} T_N \times T_N & \longrightarrow & T_{N'} \times T_{N'} \\ \downarrow & & \downarrow \\ T_N & \longrightarrow & T_{N'} \end{array}$$

which are dense (and all schemes are separated here).

[3] the map $T_N \to T_{N'}$ is defined by starting from $\varphi : N \to N'$, getting $\varphi^\vee : M' \to M$ and then taking the Cartier duals $T_N = D(M) \to D(M') = T_{N'}$

Example 4.56. Consider $\varphi = id_{\mathbb{Z}^2} : \mathbb{Z}^2 \to \mathbb{Z}^2$, Σ the fan generated by $\sigma_1 = \operatorname{Cone}((0,1),(1,1))$, $\sigma_2 = \operatorname{Cone}((1,0),(1,1))$ and Σ' the fan generated by $\operatorname{Cone}((0,1),(1,0))$.

Recall that $X_\Sigma = \operatorname{Bl}_{(0,0)} \mathbb{A}^2$ and $X_{\Sigma'} = \mathbb{A}^2$. Note that φ is compatible with the fans and the induced toric morphism $\varphi_* : \operatorname{Bl}_{(0,0)} \mathbb{A}^2 \to \mathbb{A}^2$ is the blow-up map.

Theorem 4.57. A toric morphism φ_* is proper if and only if $\varphi_{\mathbb{R}}^{-1}(|\Sigma'|) = |\Sigma|$.

Proof.
Note that the inclusion $|\Sigma| \subseteq \varphi_{\mathbb{R}}^{-1}(|\Sigma'|)$ is true by definition.

We use the valuative criterion for both arrows:

$\boxed{\Longrightarrow}$ Assume φ_* is proper and by contradiction let $u \in N_{\mathbb{R}}$ be such that $\varphi_{\mathbb{R}}(u) \in |\Sigma'|$ but $u \notin |\Sigma|$. Restating the first part of the assumption, there exists $\sigma' \in \Sigma'$ such that $\varphi(u) \in \sigma'$. Recall that $\lim_{t\to 0} \lambda^u(t)$ exists in U_σ if and only if $u \in \sigma$ (4.26), so by assumption $\lambda^{\varphi(u)} : \mathbb{G}_m \to T_{N'} \subseteq U_{\sigma'}$ has a limit as $t \to 0$, but $\lambda^u : \mathbb{G}_m \to T_N \subseteq X_\Sigma$ does not have a limit.

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & T_N \arrow[r] \arrow[d] & X_\Sigma \arrow[d] \\
 & T_{N'} \arrow[r] & X_{\Sigma'}
\end{tikzcd}$$

Note that $\varphi_* \circ \lambda^u = \lambda^{\varphi(u)}$ so

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{K} \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d] & X_\Sigma \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \mathbb{A}^1 \arrow[r, "\widetilde{\lambda^{\varphi(u)}}"'] \arrow[ur, "\nexists"] & X_{\Sigma'}
\end{tikzcd}$$

where $R = \mathcal{O}_{\mathbb{A}^1,0}$ and $\widetilde{\lambda^{\varphi(u)}}$ is the extension of $\lambda^{\varphi(u)}$. Since the blue arrow cannot exist, the lift from the DVR also can't (details to check), contraddicting properness.

$\boxed{\Longleftarrow}$ Suppose we have $\varphi_{\mathbb{R}}^{-1}(|\Sigma'|) = |\Sigma|$. Let us consider a diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{K} \arrow[r] \arrow[d] & X_\Sigma \arrow[d, "\varphi_*"] & \supseteq & T_N \arrow[d, "\varphi_*"] \\
\operatorname{Spec} R \arrow[r] & X_{\Sigma'} & \supseteq & T_{N'}
\end{tikzcd}$$

we take the following (non trivial) fact as a given: since X_Σ is irreducible, we allow ourselves to check the lifting property for diagrams where $\operatorname{Spec} \mathbb{K}$ maps into T_N.

Now now that $\operatorname{Spec} R \to X_{\Sigma'}$ will factor through some $U_{\sigma'}$ for some $\sigma' \in \Sigma'$. We want to find $\sigma \in \Sigma$ and a lifting

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{K} \arrow[r] \arrow[d] & T_N & \subseteq & U_\sigma \arrow[dll, "\varphi_*"] \\
\operatorname{Spec} R \arrow[r] \arrow[urrr, dashed] & U_{\sigma'} & &
\end{tikzcd}$$

Passing to the algebras this means the following:

$$\begin{tikzcd}
\mathbb{K} & k[M] \arrow[l] & k[S_\sigma] \arrow[l, phantom, "\supseteq"] \arrow[dll, dashed] \\
R \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] & k[S_{\sigma'}] \arrow[l] \arrow[u] \arrow[ur]
\end{tikzcd}$$

the homomorphism $k[M] \to \mathbb{K}$ is encoded by a group homomorphism $\alpha : M \to (\mathbb{K}^*, \cdot)$.

If we compose $v \circ \alpha : M \to \mathbb{K}^* \to \mathbb{Z}$ we get an element of $(M^\vee)^\vee = N$ and a lift exists only when $v(\alpha(m)) \geq 0$ for all $m \in S_\sigma$, that is, when $v \circ \alpha \in (\sigma^\vee)^\vee = \sigma$. So the existence of a lift reduces to showing that something belongs to σ.

Now, $\alpha \circ \varphi^\vee : M' \to \mathbb{K}^*$ corresponds to $\operatorname{Spec} \mathbb{K} \to T_N \to T_{N'}$ and we know that this lifts to $\operatorname{Spec} R$ so $v \circ \alpha \circ \varphi^\vee$ is non-negative on $S_{\sigma'}$. So $\varphi(v \circ \alpha) \in (\sigma'^\vee)^\vee = \sigma'$.

Now, by assumption, there exists a cone $\sigma \in \Sigma$ such that $v \circ \alpha \in \sigma$ (and $\varphi_\mathbb{R}(\sigma) \subseteq \sigma'$) so $v \circ \alpha \geq 0$ on S_σ and thus we have a factorization of $k[S_\sigma] \to k[M] \xrightarrow{\alpha} \mathbb{K}$ through R.

Uniqueness of the lift follows from separatedness.

□

Remark 4.58. If $N' = \{0\}$ then $X_{\Sigma'} = \operatorname{Spec} k$, so if Σ is complete then X_Σ is proper.

4.5 More on toric morphisms

There are two important types of toric morphisms: refinements and changes of lattice.

Definition 4.59. A fan Σ' is a **refinement** (ir **subdivision**) of Σ if for all $\sigma' \in \Sigma'$ there exists some $\sigma \in \Sigma$ with $\sigma' \subseteq \sigma$ (this says that $id_N : N \to N$ is compatible with the fans) and $|\Sigma| = |\Sigma'|$.

Theorem 4.60. Refinements induce proper and birational toric morphisms.

Remark 4.61. Refinements can be used to reduce singularities.

Fact 4.62. Blowups at torus-invariant closed subvarieties can also be described by refinements.

Definition 4.63. Let Σ be a fan in $N_\mathbb{R}$ and consider a finite-index[a] sublattice $N' \subseteq N$. Note that $N_\mathbb{R} = N'_\mathbb{R}$, so Σ is also a fan in $N'_\mathbb{R}$. The inclusion $N' \hookrightarrow N$ is compatible with Σ.

[a] N/N' finite. For example $n_1\mathbb{Z} \times \cdots, n_n\mathbb{Z} \subseteq \mathbb{Z}^n$ with $n_i \neq 0$.

$$0 \to N' \to N \to Q \to 0$$

with Q torsion

$$0 \to \underbrace{\mathrm{Hom}(Q,\mathbb{Z})}_{=0} \to \underbrace{\mathrm{Hom}(N,\mathbb{Z})}_{=M} \to \underbrace{\mathrm{Hom}(N',\mathbb{Z})}_{=M'} \to \mathrm{Ext}^1(Q,\mathbb{Z}) \to 0$$

so we have $M \hookrightarrow M'$, which yields $T_{N'} = D(M') \twoheadrightarrow D(M) = T_N$.

$G = \ker(T_{N'} \to T_N) \subseteq T_{N'}$ is a finite group ($\operatorname{char} k = 0$). Now $T_{N'}$ acts on $X_{\Sigma,N'}$ and therefore G also acts on $X_{\Sigma,N'}$.

Fact 4.64. We have an isomorphism[a] $X_{\Sigma,N'}/G \cong X_{\Sigma,N}$

[a]there is a way to construct quotients of schemes for for actions of finite groups. In the affine case you take the spectrum of the subring of invariants of the ring of regular functions.

Recall that a toric morphism $\varphi_* : X_\Sigma \to X_{\Sigma'}$ restricts to a homomorphism of algebraic groups $T_N \to T_{N'}$.

Theorem 4.65. A morphism $f : X_\Sigma \to X_{\Sigma'}$ is toric if and only if when we restrict it to T_N we get a homomorphism of algebraic groups $f|_{T_N} : T_N \to T_{N'}$.

Proof.
The homomorphism $f|_{T_N} : T_N \to T_{N'}$ gives a homomorphism $\varphi : N \to N'$ by functoriality. This homomorphism yields $T_N \to T_{N'}$ back by the usual construction (φ gives $\varphi^\vee : M' \to M$ which gives $k[M'] \to k[M]$ and so $T_N \to T_{N'}$). To conclude we just need to check compatibility with the fans because then $\varphi_* : X_\Sigma \to X_{\Sigma'}$ will be well defined and on the torus it gives $f|_{T_N}$ back, which by separatedness shows that $f = \varphi_*$.

Note that $f : X_\Sigma \to X_{\Sigma'}$ is $(T_N \to T_{N'})$-equivariant (commutes on tori and extend). The equivariance implies that the image of a T_N-orbit in X_Σ is contained in a $T_{N'}$-orbit of $X_{\Sigma'}$.

The idea is to use the orbit-cone correspondence to show compatibility.

Pick $\sigma \in \Sigma$ and consider $\mathcal{O}(\sigma) \subseteq X_\Sigma$. We just noted that there must exist $\mathcal{O}(\sigma') \subseteq X_{\Sigma'}$ orbit such that $f(\mathcal{O}(\sigma)) \subseteq \mathcal{O}(\sigma')$.

Let us show that $f|_{U_\sigma}$ has image in $U_{\sigma'}$. Recall that

$$U_\sigma = \coprod_{\tau \leq \sigma} \mathcal{O}(\tau)$$

If $\tau \leq \sigma$ there will exist some $\tau' \in \Sigma'$ with $f(\underline{\mathcal{O}(\tau)}) \subseteq \mathcal{O}(\tau')$. The factorization we want happens if τ' is a face of σ'. Recall that $\overline{\mathcal{O}(\tau)} = V(\tau) = \coprod_{\tau \leq \widetilde{\sigma}} \mathcal{O}(\widetilde{\sigma}) \supseteq \mathcal{O}(\sigma)$. So

$$f(\mathcal{O}(\sigma)) \subseteq f(\overline{\mathcal{O}(\tau)}) \subseteq \overline{\mathcal{O}(\tau')} = V(\tau')$$

Since $V(\tau') = \coprod_{\tau' \leq \widetilde{\sigma}'} \mathcal{O}(\widetilde{\sigma}')$ this shows that $\tau' \leq \sigma'$ because intersecting orbits must be the same.

Having now reduced to the affine case $f|_{U_\sigma} : U_\sigma \to U_{\sigma'}$, now we can show that $\varphi_{\mathbb{R}}(\sigma) \subseteq \sigma'$. It is enough to show that $\varphi(\sigma \cap N) \subseteq \sigma' \cap N'$.

Pick $u \in \sigma \cap N$, so that $\lim_{t\to 0} \lambda^u(t)$ exists in U_σ. Note that $f \circ \lambda^u = \lambda^{\varphi(u)}$ and thanks to the equivariance, the image of the limit of $\lambda^u(t)$ via f will be a limit for $\lambda^{\varphi(u)}(t)$ in $U_{\sigma'}$. Therefore $\varphi(u) \in \sigma'$. □

Another interesting kind of toric morphisms are "locally trivial fibrations".

Example 4.66. The fan for $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the four quadrants of $\mathbb{R}^2$.

Example 4.67. See [CLS11] for details. Consider the trapezoid $\mathrm{Conv}((0,0),(0,1),(a,0),(b,1))$. Set $r = b - a$. The associated toric variety is the Hirzebruch surface. The normal fan of this shape is a kind of funky version of the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ above.

You can take the projection $\mathbb{Z}^2 \to \mathbb{Z}$ which is compatible with the fans of the Hirzebruch surface and of $\mathbb{P}^1$ respectively. It turns out that locally $H_r \to \mathbb{P}^1$ looks like $U \times \mathbb{P}^1$ but globally it is not a product.

The fan of the fibers look like vertical sections.

Chapter 5

Divisors on toric varieties

5.1 Class group

Let Σ be a fan in $N_{\mathbb{R}}$.

Notation. Set $\Sigma(k) := \{\sigma \in \Sigma \mid \dim \sigma = k\}$.

For each $\rho \in \Sigma(1)$, by orbit-cone correspondence (4.38), we get a prime divisor $D_\rho = V(\rho) = \overline{\mathcal{O}(\rho)}$.

Let $v_\rho : k(X_\Sigma)^* \to \mathbb{Z}$ be the valuation of the local ring of D_ρ.

Recall that each ρ has a minimal ray generator $u_\rho \in \rho \cap N$. Note that for $m \in M$,

$$\chi^m : T_N \to \mathbb{G}_m$$

is a regular function defined on an open subset of X_Σ, so it yields a rational function on X_Σ.

Proposition 5.1. $v_\rho(\chi^m) = \langle m, u_\rho \rangle$.

Proof.
Extend u_ρ to a $\mathbb{Z}$-basis of N, say $u_\rho, e_2, \cdots, e_n$. Recall that $U_\rho \cong \mathbb{A}^1 \times \mathbb{G}_m^{n-1}$ and $D_\rho \cap U_\rho$ under the isomorphism is described by $x_1 = 0$, so

$$\mathcal{O}_{X_\Sigma, D_\rho} \cong \mathcal{O}_{U_\rho, D_\rho \cap U_\rho} \cong \mathcal{O}_{\mathbb{A}^1 \times \mathbb{G}_m^{n-1}, \{x_1 = 0\}} = k[x_1, x_2^{\pm}, \cdots, x_n^{\pm}]_{(x_1)}.$$

Thus $v_\rho(f)$ is the unique integer such that $f = x_1^{v_\rho(f)} g/h$ where $g, h \notin (x_1)$ for any $f \in k(X_\Sigma)^*$.

Using the dual basis to the $u_\rho, e_2, \cdots, e_n$ of N we can write $m = \sum \langle m, e_i \rangle e_i^\vee$ (we set $e_1 = u_\rho$), so

$$\chi^m = \prod \chi^{\langle m, e_i \rangle e_i^\vee} = x_1^{\langle m, u_\rho \rangle} \cdot x_2^{\langle m, e_2 \rangle} \cdots x_n^{\langle m, e_n \rangle}.$$

□

Proposition 5.2. We have that

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

Proof.
The irreducible components of $X_\Sigma \backslash T_N$ are exactly the D_ρ by the orbit-cone correspondence. Moreover, χ^m is defined and not zero on T_N so it follows that $\mathrm{Supp}(\mathrm{div}(\chi^m)) \subseteq X_\Sigma \setminus T_N$. By the previous computation we are done. ☐

Remark 5.3. Note that D_ρ are torus invariant and so every linear combination $\sum a_\rho D_\rho$ is torus-invariant (with the obvious induced action).

These are actually ALL the torus-invariant divisors (easy from the orbit-cone correspondence).

Notation. We use $\mathrm{Div}_{T_N}(X_\Sigma)$ to denote the subgroup of $\mathrm{Div}(X_\Sigma)$ given by the torus-invariant divisors.

Proposition 5.4. There is an exact sequence

$$M \to \mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0$$

where the first map is $m \mapsto \mathrm{div}(\chi^m)$. In particular every divisor on X_Σ is linearly equivalent to a torus-invariant one.
Moreover, the first map is injective if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$.

Proof.
We start from the localization sequence induced by the inclusion $T_N \subseteq X_\Sigma$ (we are using the fact that the D_ρ generate the invariant divisors)

$$\mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(T_N) \to 0$$

Since $k[x_1^\pm, \cdots, x_n^\pm]$ is a UFD we have that $\mathrm{Cl}(T_N) = 0$.

The composite $M \to \mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma)$ is clearly 0 because the images of the first map are principal.

Suppose $D \in \mathrm{Div}_{T_N}(X_\Sigma)$ is such that $D = \mathrm{div}(f) = \sum a_\rho D_\rho$. Then $\mathrm{div}(f)|_{T_N} = 0$, so $f \in \mathcal{O}^*_{T_N}$, that is, $f = c\chi^m$. So $\mathrm{div}(f) = \mathrm{div}(c\chi^m) = \mathrm{div}(\chi^m)$, proving exactness.

Now, $\mathrm{div}(\chi^m) = 0$ in $\mathrm{Div}_{T_N}(X_\Sigma)$ means that $\langle m, u_\rho \rangle = 0$ for all $\rho \in \Sigma(1)$. This is equivalent to $m = 0$ exactly when the $\{u_\rho\}$ span $N_\mathbb{R}$. ☐

Remark 5.5. The condition "$\{u_\rho\}$ spans $N_\mathbb{R}$" corresponds geometrically to the fact that X_Σ has no "torus factors", that is, X_Σ is not equivariantly isomorphic to some $\mathbb{G}_m^k \times X_{\Sigma'}$.

Proposition 5.6. The following are equivalent

1. X_Σ has a torus factor
2. there exists a non-constant morphism $X_\Sigma \to \mathbb{G}_m$
3. The $\{u_\rho \mid \rho \in \Sigma(1)\}$ do not span $N_\mathbb{R}$.

Proof.
We give the implications

$\boxed{1. \implies 2.}$ If $X_\Sigma \cong \mathbb{G}_m^k \times X_{\Sigma'}$ we can take any projection on one of the first k factors.

$\boxed{2. \implies 3.}$ If $f : X_\Sigma \to \mathbb{G}_m$ is non-constant then the restriction $f|_{T_N} : T_N \to \mathbb{G}_m$ is a non-constant morphism, i.e. a non-constant invertible element of $\mathcal{O}_{T_N}(T_N)$, so $f|_{T_N} = c\chi^m$ for $c \in k^*$ and $m \in M \setminus \{0\}$.

By multiplying by c^{-1} we can assume that $c = 1$, so $f : X_\Sigma \to \mathbb{G}_m$ is now a toric morphism, since it restricts to a homomorphism on the tori. These are induced by a non-constant homomorphism $\varphi : N \to \mathbb{Z}$ which is compatible with the fans (4.65). The fan of $\mathbb{G}_m$ is the origin inside $\mathbb{R} = \mathbb{Z}_\mathbb{R}$, so $\varphi(|\Sigma|) = 0$ and so $\{u_\rho \mid \rho \in \Sigma(1)\}$ does not span $N_\mathbb{R}$, otherwise $\varphi : N \to \mathbb{Z}$ would have to be the 0 map.

$\boxed{3. \implies 1.}$ Basically already seen. You get a proper sublattice of N by taking the $\mathbb{Z}$-span of $\{u_\rho\}$ and now so complete to a basis.

□

We can now compute $\mathrm{Cl}(X_\Sigma)$ algorithmically:

1. Fix a basis $e_1, \cdots, e_n$ for M
2. Fix minimal ray generators $u_1, \cdots, u_r$ which form a basis for $\mathrm{Div}_{T_N}(X_\Sigma)$ ******
3. If $\{\rho_1, \cdots,\}$
4. $$\begin{array}{ccc} M & \longrightarrow & \mathrm{Div}_{T_N}(X_\Sigma) \\ \wr\Vert & & \wr\Vert \\ \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^r \end{array}$$
   where the second isomorphism is using the basis D_ρ. This map corresponds to the matrix
   $$\begin{pmatrix} \langle e_1, u_1\rangle & \cdots & \langle e_n, u_1\rangle \\ \vdots & \ddots & \vdots \\ \langle e_1, u_r\rangle & \cdots & \langle e_n, u_r\rangle \end{pmatrix}$$
   and you compute the cokernel of this matrix using the smith normal form.

Remark 5.7. Using this description, you can show that $\mathrm{Cl}(X_\Sigma \times X_{\Sigma'}) = \mathrm{Cl}(X_\Sigma) \oplus \mathrm{Cl}(X_{\Sigma'})$, which is NOT true for general normal varieties.

5.2 Cartier divisors on toric varieties

To get $\mathrm{Pic}(X_\Sigma)$ we want to consider $\mathrm{CDiv}_{T_N}(X_\Sigma) \subseteq \mathrm{Div}_{T_N}(X_\Sigma)$. Note that $M \to \mathrm{Div}_{T_N}(X_\Sigma)$ has image contained in $\mathrm{CDiv}_{T_N}(X_\Sigma)$.

Proposition 5.8. We have an exact sequence

$$M \to \mathrm{CDiv}_{T_N}(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0$$

This is also exact on the left exactly when X_Σ has no torus factors

5.2.1 Torus invariant Cartier divisors: affine case

What is $\mathrm{CDiv}_{T_N}(X_\Sigma)$? For Weil divisors we had $\mathrm{Div}_{T_N}(X_\Sigma) = \bigoplus \mathbb{Z}D_\rho$.

Proposition 5.9. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex cone. Then every T_N-invariant Cartier divisor on U_σ is the divisor of a character, so $\mathrm{Pic}(U_\sigma) = 0$.

Proof.
Let $R = k[S_\sigma]$. Suppose that $D = \sum a_\rho D_\rho$ is T_N-invariant and Cartier. Suppose $a_\rho \neq 0$ for the ρ which appear in the sum.

Suppose D is effective. Note that the D_ρ intersect non-trivially: $\mathcal{O}(\sigma) \subseteq \overline{\mathcal{O}(\rho)} = D_\rho$, so $\mathcal{O}(\sigma) \subseteq \bigcap D_\rho$. Pick any point $p \in \mathcal{O}(\sigma)$. Since D is Cartier, there exists $U \subseteq U_\sigma$ open such that $p \in U$ and $D|_U = \mathrm{div}(f)|_U$ for some $f \in k(U_\sigma)^*$. We can assume $U = (U_\sigma)_h = \mathrm{Spec}\, R_h$ for some $h \in R$ because principal open sets form a basis. Since D is effective, $\mathrm{div}(f)|_U \geq 0$, so $f \in R_h$. If $f = g/h^k$, since $h \in R_h^*$ and multiplying by h doesn't change $\mathrm{div}(f)|_U$, we can just assume $f = g \in R$.

Consider now[1] $I = \{g \in k(U_\sigma) \mid g = 0 \text{ or } \mathrm{div} g \geq D\}$. Since $D \geq 0$ we have $\mathrm{div} g \geq 0$ so I is an ideal of R. This ideal is T_N-invariant because D is. Thherefore I is a subrepresentation of R under the action of T_N, meaning that

$$I = \bigoplus_{\mathrm{div}(t^m) \geq D} k t^m$$

Note that $f \in I$ because

$$\mathrm{div} f = \sum v_\rho(f) D_\rho + \underbrace{\sum_{E \neq D_\rho} v_E(f) E}_{\geq 0 \text{ because } f \in R} \geq \sum v_\rho(f) D_\rho + 0 = D$$

where the last equality holds because of our choice for U.

Thus $f = \sum a_i t^{m_i}$ with $\mathrm{div}(t^{m_i}) \geq D$ for all i. On U,

$$\mathrm{div}(t^{m_i})|_U \geq D|_U = \mathrm{div}(f)|_U \implies \mathrm{div}(t^{m_i}/f)|_U \geq 0 \implies t^{m_i}/f \in \mathcal{O}(U)$$

So we can write

$$1 = \sum a_i t^{m_i}/f$$

and the fact that $t^{m_i}/f \in \mathcal{O}(U)$ we can evaluate that equality at $p \in U$ to get

$$1 = \sum a_i t^{m_i}(p)/f(p)$$

so $t^{m_i}/f(p) \neq 0$ for some i. Let $V \subseteq U$ be an open set where t^{m_i}/f is never 0 and $p \in V$. We get

$$\mathrm{div}(t^{m_i}/f)|_V = 0 \implies \mathrm{div}(t^{m_i})|_V = \mathrm{div}(f)|_V = D|_V$$

Since V intersects every D_ρ non-trivially we get that $\mathrm{div}(t^{m_i}) = D$ on U_σ.

Consider now a general invariant Cartier divisor D. Since σ is strongly convex, there exists some $m \in M$ such that $\langle m, u_\rho \rangle > 0$ for all $\rho \in \sigma(1)$ (because the origin is a face of σ). This implies that $D' = D + \mathrm{div}(t^{km})$ for k big enough we get an effective invariant Cartier divisor. By the previous case $D' = \mathrm{div}(t^{m_i})$ and so $D = \mathrm{div}(t^{m_i - km})$. ∎

[1] this set is $\Gamma(\mathcal{O}(-D))$

Example 5.10. If $\sigma = \text{Cone}((d,-1),(0,1)) \subseteq \mathbb{R}^2$ we have that $\text{Cl}(U_\sigma) \cong \mathbb{Z}/d\mathbb{Z}$ but $\text{Pic}(U_\sigma) = 0$, so the generator of the class group corresponds to a Weil divisor which is not Cartier.

If we take out the torus fixed point $\gamma_\sigma \in U_\sigma$, $X = U_\sigma \setminus \{\gamma_\sigma\}$ (we remove the maximal cone). Now X is a smooth toric variety and in this case $\text{Pic}(X) = \text{Cl}(X)$. We left the rays untouched so $\text{Cl}(X) = \text{Cl}(U_\sigma)$, so we suddenly got nontrivial line bundles by removing a point!

Remark 5.11. You can show that if Σ contains a cone of dimension equal to that of the lattice then $\text{Pic}(X_\Sigma)$ is torsion free.

Sketch.
Suppose $\text{Pic}(X_\Sigma)$ has torsion and take a representative in $\text{CDiv}_{T_N(X_\Sigma)}$, that is, take an invariant Cartier Divisor D such that $kD = \text{div}(t^m)$. We have to show that $D = \text{div}(t^{m'})$.

If σ is a cone of maximal dimension then $\{u_\rho \mid \rho \in \sigma(1)\}$ spans $N_\mathbb{R}$. $D|_{U_\sigma}$ is the divisor of a character $t^{m'}$ by (5.9).

Since $\text{div}(t^m)|_{U_\sigma} = \text{div}(t^{km'})|_{U_\sigma}$ then $m = km'$ and so we get $D = \text{div}(t^{m'})$ on X_Σ because the rays of σ span (details to fill in). $\square$

Proposition 5.12. X_Σ is smooth if and only if $\text{Pic}(X_\Sigma) = \text{Cl}(X_\Sigma)$

Proof.
The first implication is always true. Suppose $\text{Pic}(X_\Sigma) = \text{Cl}(X_\Sigma)$. We want to show that $\sigma \in \Sigma$ is smooth.

Since $\text{Cl}(X_\Sigma) \to \text{Cl}(U_\sigma)$ is surjective, every divisor on U_σ is Cartier because the restriction of a Cartier divisor is Cartier and $\text{Cl}(X_\Sigma) = \text{Pic}(X_\Sigma)$. Since $\text{Pic}(U_\sigma) = 0$ it follows that $\text{Cl}(U_\sigma) = 0$, thus $M \to \text{Div}_{T_N}(U_\sigma)$ is surjective.

If $\sigma(1) = \{\rho_1, \cdots, \rho_r\}$ then the map $M \to \text{Div}_{T_N}(U_\sigma)$ in coordinate is

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{Z}^r \\ m & \longmapsto & (\langle m, u_{\rho_i} \rangle) \end{array}$$

This map is dual to

$$\phi : \begin{array}{ccc} \mathbb{Z}^r & \longrightarrow & N \\ e_i & \longmapsto & u_{\rho_i} \end{array}$$

Since $\phi^\vee$ is surjective it follows that (using a little homological algebra) ϕ is injective and that $\operatorname{coker} \phi$ is torsion-free. This implies that $\{u_{\rho_i}\}$ can be completed to a $\mathbb{Z}$-basis of N and so σ is a smooth cone. $\square$

Remark 5.13. One can also show that X_Σ is simplicial (every cone in Σ is simplicial, that is, the minimal ray generators are $\mathbb{R}$-linearly independent) if and only if every Weil divisor is $\mathbb{Q}$-Cartier, that is, it has a positive multiple which is Cartier, that is, the index $[\text{Cl}(X_\Sigma) : \text{Pic}(X_\Sigma)]$ is finite.

5.2.2 Torus invariant Cartier divisors

Recall that if $D \in \text{Div}(X)$ is Cartier then there exists an open cover $\{U_i\}$ of X and $f_i \in k(X)$ for all i such that $D|_{U_i} = \text{div}(f_i)|_{U_i}$. We may call $\{(U_i, f_i)\}$ a *Cartier data* for D.

Proposition 5.14. Let $D \in \mathrm{Div}_{T_N}(X_\Sigma)$ and let us write $D = \sum a_\rho D_\rho$. The following are equivalent

1. D is Cartier
2. D is principal on U_σ for all $\sigma \in \Sigma$.
3. For all $\sigma \in \Sigma$ there exists $m_\sigma \in M$ such that[a] $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$.
4. for all[b] $\sigma \in \Sigma_{max}$ there exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$.

Moreover, if D is Cartier, then the m_σ in proposition 4 are unique in the quotient $M/M(\sigma)$ for $M(\sigma) = \sigma^\perp \cap M$.
In particular, if $\tau \leq \sigma$ then $m_\tau \equiv m_\sigma$ modulo $M(\tau)$.

[a] i.e. $\sum a_\rho D_\rho \cap U_\sigma = D|_{U_\sigma} = \mathrm{div}(t^{-m_\sigma})|_{U_\sigma}$ and $D_\rho \cap U_\sigma \neq \emptyset$ if and only if $\rho \leq \sigma$.
[b] maximal cones in Σ by inclusion. They may be of different dimension.

Proof.
The only new part for the equivalences is 4.. 3. clearly implies 4.. The fact that 4 implies the others follows from the fact that if $\sigma \leq \sigma'$ then $m_{\sigma'}$ is a valid choice for m_σ in proposition 3.

Now we prove the uniqueness. If m_σ and $m_{\sigma'}$ are two elements that satisfy the condition, then $\langle m_\sigma, u_\rho \rangle = -a_\rho = \langle m_{\sigma'}, u_\rho \rangle$ for all $\rho \in \sigma(1)$. Thus

$$\langle m_\sigma - m_{\sigma'}, u_\rho \rangle = 0 \implies \langle m_\sigma - m_{\sigma'}, u \rangle = 0 \ \forall u \in \sigma.$$

□

Notation. We call $\{m_\sigma\}_{\sigma\in\Sigma}$ the *Cartier data* for the divisor D.

Remark 5.15. The minus sign in $\langle m_\sigma, u_\rho \rangle = -a_\rho$ is a convention. It matches the convention used when writing the presentations for polytopes $\langle m, u_F \rangle \geq -a_F$.

This choice boils down to the fact that the $\mathcal{O}_X$-module associated to a divisor D is defined as

$$\mathcal{O}_X(D)(U) = \left\{ f \in k(X)^* \mid (\mathrm{div} f + D)|_U \geq 0 \right\} \cup \{0\}.$$

Note that if $\tau \leq \sigma$ then we have $\sigma^\perp \cap M \subseteq \tau^\perp \cap M$ and this induces a map $M/M(\sigma) \to M/M(\tau)$. This maps $[m_\sigma]$ to $[m_\tau]$.

The groups $\{M/M(\sigma)\}_{\sigma\in\Sigma}$ form an inverse system ordered by Σ and the face-relation. It turns out that

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \cong \varprojlim_{\sigma\in\Sigma} M/M(\sigma)$$

This just means that Cartier divisors correspond exactly to compatible collections of $\{[m_\sigma]\}_{\sigma\in\Sigma}$.

Bibliography

[CLS11] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Soc., 2011.

[Ful93] W. Fulton. *Introduction to Toric Varieties*. Annals of mathematics studies. Princeton University Press, 1993.