Toric Varieties - Geometria Algebrica F

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Contents

Introduction

Syllabus

The first part of the course deals with:

- Algebraic Tori, their actions and representations
- Affine toric varieties (with monoids) $\leftrightarrow$ cones in some $\mathbb{R}^n$
- Projective toric varieties $\leftrightarrow$ polytopes in some $\mathbb{R}^n$
- General toric varieties $\leftrightarrow$ fans in $\mathbb{R}^n$

We will then deal with (suject to change)

- Divisors/line bundles on toric varieties
- Cox ring of a toric variety
- Cohomology of divisors
- Toric morphisms and resolution of singularities
- and more...?

The main reference for this course —"Toric varieties" by Cox, Little, Schenck [CLS11] —is available in the same folder as this PDF.

What is the course about?

We will work over an algebraically closed field (and we will be lax about the characteristic of the field). In [CLS11] the authors work over $\mathbb{C}$ but many results hold more generally.

The main goal of the course is understanding toric varieties:

Definition 0.1 (Toric variety). An n-dimensional toric variety X is a (normal) k-variety equipped with an open immersion of an n-dimensional torus $T \subseteq X$, where $T \cong (k^*)^n$, and an action $T \times T \to T$ which extends to the whole of X [a].

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[a] that is, it extends to a $T \times X \to X$

Remark 0.2. Normality is a standard assumption that we'll make at some point but some things work without it.

We'll see that the geometry of such an object is encoded in a combinatorial gadget, converting problems in algebraic geometry to problems in combinatorics, which is sometimes convenient.

The opposite reduction is also possible and has been used historically. One of the main examples of a combinatorial problem being solved via the geometry of toric varieties is

Example 0.3 (McMullen's "g-conjecture")**.** The then conjecture, and now theorem, is a characterization of the f-vectors of simple polytopes[1].

Definition 0.4 (f-vectors)**.** If P is a polytope, its f**-vector** is

$$(f_0(P), \cdots, f_d(P)), \quad \text{where } d = \dim P$$

and $f_i(P)$ is the number of i-dimensional faces. We may set $f_{-1}(P) = 1$.

It's reasonable to ask ourselves which f-vectors can appear. We may define the h**-vector** by setting

$$\sum_{i=0}^{d} f_i(t-1)^i = \sum_{i=0}^{d} h_i t^i, \quad \text{i.e. } h_i = \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} f_j, \ h_{-1} = 0.$$

It was a known theorem that the h-vector of a simple polytope is palindromic (i.e. $h_i = h_{d-i}$). From the h-vector we obtain the g**-vector** by setting $g_i = h_i - h_{i-1}$.

The conjecture was that

Theorem 0.5 (g-conjecture)**.** $f = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1}$ is the f-vector of a simple polytope if

1. $h_i = h_{d-1}$ for all $0 \le i \le \lfloor d/2 \rfloor$

2. $g_i \ge 0$ for all $0 \le i \le \lfloor d/2 \rfloor$

3. $(g_1, \cdots, g_{\lfloor d/2 \rfloor})$ is a "Macauly vector" if, when we write (uniquely)

$$g_i = \binom{n_i}{i} + \cdots + \binom{n_{r_i}}{r_i}$$

with $n_i > n_{i-1} > \cdots > n_{r_i}$ then

$$g_{i+1} \le \binom{n_i + 1}{i + 1} + \cdots + \binom{n_{r_i} + 1}{r_i + 1}$$

Stanley proved necessity using toric varieties. He proved that the g-vector of a simple polytope is the vector of dimensions for some cohomology ring of the associated toric variety.

Later McMullen found a completely combinatorial proof but for some time the only proof of this combinatorial fact passed through the geometry of toric varieties.

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[1] for now a simple polytope is the convex hull of a finite subset of $\mathbb{R}^n$

Part I

Basic theory of toric varieties

Chapter 1

Algebraic tori and their actions

1.1 Basic definitions

> **Definition 1.1** (Algebraic group)**.** An **algebraic group** G is a k-variety equipped with the the structure of a "group object" in the category of k-varieties, i.e. we have two morphisms and a *closed* point
>
> $$m : G \times G \to G, \quad i : G \to G, \quad e \in G$$
>
> that satisfy the usual group axioms "diagrammatically".

Example 1.2. Associativity can be expressed "diagrammatically" as

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{(id_G, m)} & G \times G \\
{\scriptstyle (m, id_G)}\downarrow & & \downarrow {\scriptstyle m} \\
G \times G & \xrightarrow{\quad m \quad} & G
\end{array}
$$

Remark 1.3. If $G = \operatorname{Spec} A$ is an affine variety, a structure of algebraic group is equivalent to a structure of **Hopf algebra** on A:

$$
\begin{aligned}
m : G \times G \to G & \quad\longleftrightarrow\quad \Delta : A \to A \otimes_k A \\
i : G \to G & \quad\longleftrightarrow\quad S : A \to A \\
e : \operatorname{Spec} k \to G & \quad\longleftrightarrow\quad \varepsilon : A \to k
\end{aligned}
$$

and the homomorphisms Δ, S, ε satisfy the diagrammatic group axioms with the arrows reversed.

Remark 1.4. If G and H are algebraic groups, $G \times H$ is also naturally an algebraic group. For example

$$
m_{G \times H} : \quad
\begin{array}{ccc}
(G \times H) \times (G \times H) & \longrightarrow & G \times H \\
((g_1, h_1), (g_2, h_2)) & \longmapsto & (m_G(g_1, g_2), m_H(h_1, h_2))
\end{array} .
$$

Definition 1.5 (Homomorphism between Algebraic groups)**.** If G, H are algebraic groups over k then a homomorphism $f : G \to H$ is a morphism of k-varieties such that

$$
\begin{array}{ccc}
G \times G & \xrightarrow{(f,f)} & H \times H \\
{\scriptstyle m}\big\downarrow & & {\scriptstyle m}\big\downarrow \\
G & \xrightarrow{\quad f \quad} & H
\end{array}
$$

Remark 1.6. If G and H are affine, the axioms of homomorphism dualize to what a homomorphism of Hopf algebras should be.

Remark 1.7. All algebraic subgroups of an algebraic group are closed subvarieties.

The first example of algebraic group we present is the multiplicative group

Definition 1.8 (Multiplicative group)**.** The **multiplicative group**, denoted $\mathbb{G}_m$, is the k-variety $\mathbb{A}^1 \setminus \{0\}$ equipped with the morphisms

$$
m : \begin{array}{ccc}
\mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\
(a, b) & \longmapsto & ab
\end{array}
$$

$$
i : \begin{array}{ccc}
\mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\
a & \longmapsto & 1/a
\end{array}
$$

$$
e = 1 \in \mathbb{A}^1 \setminus \{0\}
$$

(we are identifying $\mathbb{G}_m = k^*$).

Remark 1.9. $\mathbb{G}_m$ is affine, indeed $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and $\mathbb{A}^1 \setminus \{0\} = \mathbb{A}^1 \setminus V(x) = D(x)$, thus

$$
D(x) = \operatorname{Spec}(k[x])_x = \operatorname{Spec}(k[x, x^{-1}]) = \operatorname{Spec} k[x^{\pm 1}].
$$

If you are uncomfortable with "x^{-1}" appearing you may simply think of this coordinate ring as

$$
\frac{k[x, y]}{(xy - 1)}.
$$

Remark 1.10. The multiplication $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ can be described as the morphism corresponding to the k-algebra homomorphism

$$
\begin{array}{ccc}
k[x^{\pm 1}] & \longrightarrow & k[y^{\pm 1}] \otimes_k k[z^{\pm 1}] \\
x & \longmapsto & y \otimes z
\end{array} \quad .
$$

Similarly, the inverse corresponds to

$$
\begin{array}{ccc}
k[x^{\pm 1}] & \longrightarrow & k[y^{\pm 1}] \\
x & \longmapsto & y^{-1}
\end{array}
$$

and the neutral element corresponds to[1]

$$k[x^{\pm 1}] \longrightarrow k$$
$$x \longmapsto 1$$

Definition 1.11 (Algebraic tori)**.** The **standard n-dimensional algebraic torus over** k is $\mathbb{G}_m^n$. An **algebraic torus**[a] is an algebraic group T which is isomorphic to $\mathbb{G}_m^n$ for some n.

---

[a]we may simply say "torus" if no confusion can occur.

Remark 1.12. If $k = \mathbb{C}$ then $\mathbb{G}_m^n = (\mathbb{C}^*)^n$, which is homotopy equivalent to $(S^1)^n$. This $(S^1)^n$ is the "maximal compact subgroup" and is the reason why these groups are called tori in the first place.

1.2   Cartier duality

In this section we will define an equivalence of categories between finitely generated abelian groups[2] and a specific type of algebraic groups. Under this correspondence, tori will be "dual" to finitely generated free abelian groups.

1.2.1   Group algebra and Cartier dual

The first step is transforming general (finitely generated) abelian groups into (finite type reduced) algebras over k, the way we do this is via the

Definition 1.13 (Associated group algebra)**.** If M is a finitely generated abelian group, the k**-group algebra of** M, denoted by $k[M]$, is the k-vector space spanned formally by the basis $\{t^m \mid m \in M\}$ together with the multiplication induced by $t^m t^{m'} = t^{m+m'}$.

Example 1.14. If $M = \mathbb{Z}^n$ then

$$k[\mathbb{Z}^n] = k[t^{(1,0,\cdots,0)}, t^{(-1,0,\cdots,0)}, \cdots, t^{(0,\cdots,0,-1)}] = k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}],$$

which is the coordinate ring of $(\mathbb{G}_m)^n$.

---

[1]recall that a k-point e of the variety G can be seen as a morphism $\operatorname{Spec} k \to G$ with set-theoretic image e.
[2]with no p-torsion if $p = \operatorname{char} k \neq 0$

Fact 1.15. These formulas give $k[M]$ a Hopf algebra structure for all finitely generated abelian groups M

$$\Delta: \begin{array}{ccc} k[M] & \longrightarrow & k[M] \otimes_k [M] \\ t^m & \longmapsto & t^m \otimes t^m \end{array}$$

$$S: \begin{array}{ccc} k[M] & \longrightarrow & k[M] \\ t^m & \longmapsto & t^{-m} \end{array}$$

$$\varepsilon: \begin{array}{ccc} k[M] & \longrightarrow & k \\ t^m & \longmapsto & 1 \end{array}$$

Remark 1.16. If we see $\mathbb{G}_m^n$ as $\operatorname{Spec} k[\mathbb{Z}^n]$ then the usual algebraic group structure is the one induced by the maps we just mentioned.

Remark 1.17. If M is finitely generated then $k[M]$ is of finite type over k. It turns out that it is also reduced when M has no elements of order divided by the characteristic of k.

Definition 1.18 (Cartier dual). If M is a finitely generated abelian group, $D(M) := \operatorname{Spec} k[M]$ is the **cartier dual** of M.

Let us compute the cartier dual of another type of finitely generated abelian group:

Example 1.19. If $M = \mathbb{Z}/n\mathbb{Z}$ then the group algebra is

$$k[\mathbb{Z}/n\mathbb{Z}] = \frac{k[t]}{(t^n - 1)}.$$

$\operatorname{Spec} k[\mathbb{Z}/n\mathbb{Z}]$ then is the closed subvariety (and subgroup) of $\mathbb{G}_m$ described by the equation $t^n = 1$, i.e. the group of the n-th roots of unity μ_n

Definition 1.20 (Group of n-th roots of unity). $\mu_n = D(\mathbb{Z}/n\mathbb{Z})$.

Remark 1.21. If $n = p = \operatorname{char} k$ then $(t^p - 1) = (t - 1)^p$, so μ_p would be a point. To get any interesting geometric information in this case you need to allow nilpotens, stumbling into the teorritory of group schemes.

Since we know the structure theorem for finitely generated abelian groups, let us consider the following

Exercise 1.22. $D(M \oplus N) = D(M) \times D(N)$.

Solution (Sketch).
It is enough to note that $k[M \oplus N] = k[M] \otimes k[N]$ and this follows from the fact that

$$t^{(m,n)} = t^{(m,0)}t^{(0,n)}.$$

$\square$

It follows that

Proposition 1.23. For a general finitely generated abelian group
$$M = \mathbb{Z}^n \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$$
the Cartier dual is
$$D(M) \cong \mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_k}.$$

Since we hope to find an equivalence of categories, let us try to understand another way in which we can view these types of algebraic groups.

Remark 1.24. GL_n is an algebraic group: It is a variety when seen as[3] $\mathbb{A}^{n^2} \setminus V(\det)$ and it can be checked that matrix multiplication and inversion are morphisms of k-varieties.

Definition 1.25 (Diagonizable group)**.** An algebraic group is called **diagonalizable** if it is isomorphic to a (closed) subgroup of $\mathrm{Diag}_n \subseteq \mathrm{GL}_n$ for some n

Remark 1.26. $\mathrm{Diag}_n \cong \mathbb{G}_m^n$ and the isomorphism is given by ignoring the entries which aren't on the diagonal.

Remark 1.27. $D(M)$ is diagonalizable, because
$$D(M) \cong \mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_k} \subseteq \mathbb{G}_m^{n+k} \cong \mathrm{Diag}_{n+k}.$$

Proposition 1.28. If $\varphi : M \to N$ is a group homomorphism
$$k[\varphi] : \begin{array}{ccc} k[M] & \longrightarrow & k[N] \\ t^m & \longmapsto & t^{\varphi(m)} \end{array}$$
is a k-algebra homomorphism and so $D(\varphi) = \mathrm{Spec}(k[\varphi]) : D(N) \to D(M)$ is a morphism of k-varieties.
This is actually a homomorphism of algebraic groups and the association is functorial.

Cartier duality is that statement that
$$D : (\text{fin.gen.AbGps}_{\text{no } p\text{-tors}})^{op} \to (\text{Diag.AlgGps}),$$
where $p = \mathrm{char}\,k$, is an equivalence of categories. To prove this fact we will build an inverse functor

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[3]the determinant is a homogeneous polynomial of degree n

1.2.2 Character group

To find the "inverse" functor, we want to build a finitely generated abelian group from an algebraic group. The construction that will end up being what we want is the *group of characters*

Definition 1.29 (Character). A **character** of an algebraic group G is a homomorphism $\chi : G \to \mathbb{G}_m$. We denote the set of all characters $X(G)$.

Remark 1.30. The characters of an algebraic group G form an abelian group via:

$$\chi_1 : G \to \mathbb{G}_m, \quad \chi_2 : G \to \mathbb{G}_m \quad \leadsto \quad \chi_1 \cdot \chi_2 : G \xrightarrow{(\chi_1, \chi_2)} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m.$$

From now on $X(G)$ will always also have the group structure.

Example 1.31. If $G = \mathbb{G}_m$ then for $k \in \mathbb{Z}$

$$
\begin{array}{ccc}
\mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\
a & \longmapsto & a^k
\end{array}
$$

is a character, which corresponds to

$$
\begin{array}{ccc}
k[x^{\pm 1}] & \longrightarrow & k[x^{\pm 1}] \\
x & \longmapsto & x^k
\end{array}
$$

Example 1.32. If $G = \mathbb{G}_m^n$ and $(k_1, \cdots, k_n) \in \mathbb{Z}^n$ then

$$
\begin{array}{ccc}
\mathbb{G}_m^n & \longrightarrow & \mathbb{G}_m \\
(a_1, \cdots, a_n) & \longmapsto & a_1^{k_1} \cdots a_n^{k_n}
\end{array}
\cdot
$$

We will see that these are all the characters on the torus.

Example 1.33. If $G = \mathrm{GL}_n$ the determinant is a character

$$
\begin{array}{ccc}
\mathrm{GL}_n & \longrightarrow & \mathbb{G}_m \\
M & \longmapsto & \det M
\end{array}
$$

Definition 1.34 (Group-like elements). A **group-like element** in a Hopf algebra A is an $a \in A$ such that a is invertible and $\Delta(a) = a \otimes a$.

Lemma 1.35. If $G = \operatorname{Spec} A$ is an affine algebraic group, characters of G correspond to group-like elements of A.

Proof.
A character $\chi : \operatorname{Spec} A \to \mathbb{G}_m$ corresponds to a homomorphism of Hopf algebras $k[x^{\pm 1}] \to A$ which sends x to some $a \in A$. The homomorphism is uniquely determined by a so we just need to check which elements of A can be the image of x. Since x has an inverse, $a \in A^*$ and $\Delta(a) = a \otimes a$ because $\Delta(x) = x \otimes x$. On the other hand, an element which satisfies those properties does yield a Hopf-algebra homomorphism, so we are done. $\qquad\square$

1.2.3 Proof of Cartier duality

Remark 1.36. Constructing the character group extends to a functor

$$X : (\text{AlgGps}) \to (\text{AbGps})$$

via pullback, i.e. the map $f : G \to H$ becomes

$$X(f) \quad \begin{matrix} X(H) & \longrightarrow & X(G) \\ \chi & \longmapsto & \chi \circ f \end{matrix}$$

Now that we have built our candidate for the inverse functor, all we need to show that that the two compositions are naturally isomorphic to the identity.

Proposition 1.37. The map $M \to X(D(M))$ which to an element $m \in M$ assigns the character which corresponds to the Hopf-Algebra homomorphism

$$\begin{matrix} k[x^{\pm 1}] & \longrightarrow & k[M] \\ x & \longmapsto & t^m \end{matrix}$$

is a natural isomorphism.

Proof.
It is easy to check that $M \to X(D(M))$ is a group homomorphism.

inj. If $m_1 \neq m_2$ then $t^{m_1} \neq t^{m_2}$ and so the induced Hopf algebra homomorphisms are different.

onto Given lemma (1.35), we just need to show that the only group-like elements of $k[M]$ are the t^m for $m \in M$. Let us take any element $a = \sum_{m \in M} a_m t^m$ of $k[M]$ and impose that $\Delta(a) = a \otimes a$, then

$$\Delta(a) = \Delta\left(\sum_{m \in M} a_m t^m\right) = \sum_{m \in M} a_m \Delta(t^m) = \sum_{m \in M} a_m t^m \otimes t^m$$

$$a \otimes a = \left(\sum_{m \in M} a_m t^m\right) \otimes \left(\sum_{m' \in M} a_{m'} t^{m'}\right) = \left(\sum_{m,m' \in M} a_m a_{m'} t^m \otimes t^{m'}\right).$$

Since the $t^m \otimes t^{m'}$ form a basis of $k[M] \otimes k[M]$, if $m \neq m'$ then $a_m a_{m'} = 0$. Thus there exists at most one nonzero coefficient a_{m_0} and $a = a_{m_0} t^{m_0}$, but a must be invertible so $a_{m_0} \neq 0$. Also, again imposing the comultiplication condition, $a_{m_0}^2 = a_{m_0}$, which implies that $a_{m_0} = 1$ since it isn't 0.

$\square$

Corollary 1.38. For $M = \mathbb{Z}^n$ we get $X(\mathbb{G}_m^n) \cong \mathbb{Z}^n$ and the characters are the ones we wrote above[a].

---
[a] $(a_1, \cdots, a_n) \mapsto a_1^{k_1} \cdots a_n^{k_n}$

Let us now consider the other composition:

Remark 1.39. There is a canonical map $\operatorname{Spec} A = G \to D(X(G))$.

Proof.
Let $\chi : G \to \mathbb{G}_m$ be a character of G. Upon composition with the inclusion $\mathbb{G}_m \subseteq \mathbb{A}^1$ we get a morphism in $\operatorname{Hom}(G, \mathbb{A}^1)$ and this set is canonically identified with A, so we get a map
$$\varphi : X(G) \to A.$$
This is a group homomorphism, which induces the desired map
$$\begin{array}{ccc} k[X(G)] & \longrightarrow & A \\ t^m & \longmapsto & \varphi(m) \end{array}.$$

$\square$

Lemma 1.40. Let G be an abstract group (no algebraic structure) and $\mathbb{K}$ be any field, if we take $\phi_i : G \to \mathbb{K}^*$ distinct group homomorphisms then the ϕ_i are linearly independent in[a] $\operatorname{Fun}(G, \mathbb{K})$

---
[a]not homomorphisms of any kind, just set theoretic functions. It is a $\mathbb{K}$-vector space by looking at the strucutre on the codomain.

Proof.
Let us assume by contradiction that we have a non-triavial relation $\sum a_i \phi_i = 0$ for some $a_i \in \mathbb{K}$ and let's assume that this relation has minimal length.

By definition, $\sum a_i \phi_i(gh) = \sum a_i \phi_i(g) \phi_i(h) = 0$ for all $g, h \in G$. Pick $\overline{g} \in G$ such that $\phi_1(\overline{g}) \neq \phi_2(\overline{g})$ (which we can do because $\phi_1 \neq \phi_2$). Setting $g = \overline{g}$ in the expression we get
$$\sum a_i \phi_i(\overline{g}h) = \sum \underbrace{a_i \phi_i(\overline{g})}_{\in \mathbb{K}} \phi_i(h) = 0$$
that is, $\sum a_i \phi_i(\overline{g}) \phi_i = 0$ is an equality in $\operatorname{Fun}(G, \mathbb{K})$. Multiplying the initial relation by $\phi_1(\overline{g})$ we get
$$\sum a_i \phi_1(\overline{g}) \phi_i = 0$$
subtracting the two functions we get
$$\sum a_i (\phi_1(\overline{g}) - \phi_i(\overline{g})) \phi_i = 0$$
which is a shorter (look at $i = 1$) non-trivial (look at $i = 2$) reation, which is a contradiction. $\square$

Proposition 1.41. If G is diagonalizable then the homomorphism $G \to D(X(G))$ is an isomorphism and $X(G)$ is finitely generated. Moreover, if $\operatorname{char} k = p \neq 0$ then $X(G)$ has no p-torsion.

Proof.
Take a diagonalizable group G and consider it as a closed subgroup $G \subseteq \mathbb{G}_m^n = \operatorname{Diag}_n$.

Since it is *closed* and $\mathbb{G}_m^n$ is affine, $G = \operatorname{Spec} A$ is also affine and we get a surjection[4] $k[\mathbb{Z}^n] \to A$.

Now note that we have $\mathbb{Z}^n \cong X(\mathbb{G}_m^n) \to X(G)$ and the surjection above factors

$$k[\mathbb{Z}^n] \to k[X(G)] \to A$$

since the composition is surjective, $k[X(G)] \to A$ is also surjective. To conclude the first part of the proof then, we just need to show that the map is also injective, but this follows from the lemma.

Now we concern ourselves with finite generation. Because of the isomorphism we just proved, the factorization

$$k[\mathbb{Z}^n] \to k[X(G)] \to A$$

now shows that $k[\mathbb{Z}^n] \to k[X(G)]$ is surjective because $k[\mathbb{Z}^n] \to A$ was. This lets us conclude that $\mathbb{Z}^n \to X(G)$ is surjective (and thus $X(G)$ is finitely generated) because otherwise $k[\mathbb{Z}^n] \to k[X(G)]$ wouldn't be.

Suppose now that $0 \neq p = \operatorname{char} k$. Let $\chi \in X(G)$ be a p-torsion character, i.e. $\chi^p = 1$, that is, $\chi(g)^p = 1$ for all $g \in G$. Because $x^p - 1 = (x - 1)^p$ in characteristic p, $\chi(g) = 1$ for all $g \in G$, showing that $\chi = 1$ and thus the absence of p-torsion. $\qquad\square$

Corollary 1.42. A connected subgroup of a torus is a torus.

Proof.
If $G \subseteq \mathbb{G}_m^n$, from the proposition we get that

$$G = D(X(G)) \cong \mathbb{G}_m^k \times \mu_{n_1} \times \cdots \times \mu_{n_r},$$

but if G is connected then all n_i must be 1 because otherwise that product would be disconnected. $\qquad\square$

Having now verified both compositions we may formally state Cartier duality as a theorem now

Theorem 1.43 (Cartier duality). The functor

$$D : (\text{fin.gen.AbGps}_{\text{no } p\text{-tors}})^{op} \to (\text{Diag.AlgGps}),$$

where $p = \operatorname{char} k$, is an equivalence of categories. The inverse functor is X.

Remark 1.44. If we allow group schemes the problem with p-torsion doesn't come up.

---

[4]the surjection corresponds to taking $k[\mathbb{Z}^n] \to k[\mathbb{Z}^n]/I$ where I is the ideal which defines G as $V(I) \subseteq \mathbb{G}_m^n$.

Image of map between tori is a torus

> **Proposition 1.45.** Let $f : T_1 \to T_2$ be a homomorphism of tori, then the image is also a torus.

Proof.
Since $T_1 \to D(X(T_1))$ and $T_2 \to D(X(T_2))$ are isomorphisms and the appropriate diagrams commute, we have that f is induced by the corresponding homomorphism $M_2 \to M_1$ where $M_1 = X(T_1)$ and $M_2 = X(T_2)$.

Let $K = \ker(M_2 \to M_1)$ and note that $M_2 \twoheadrightarrow M_2/K \hookrightarrow M_1$. We claim that $L := \ker(k[M_2] \to k[M_1])$ is the ideal $I = (t^m - t^{m'} \mid \varphi(m) = \varphi(m'))$:

$\boxed{I \subseteq L}$ It suffices to note that the generators of I lie in L, indeed $t^m - t^{m'} \mapsto t^{\varphi(m)} - t^{\varphi(m')} = 0$.

$\boxed{L \subseteq I}$ Let $\sum_{m \in M_2} a_m t^m$ be a general element of L, then

$$\sum_{n \in M_1} \left(\sum_{m \in \varphi^{-1}(n)} a_m \right) t^n = 0 \stackrel{\text{lin.ind.}}{\Longrightarrow} \sum_{m \in \varphi^{-1}(n)} a_m = 0 \quad \forall n \in M_1$$

For a fixed n, if $a_{m_1}, a_{m_2} \neq 0$ for some $m_1, m_2 \in \varphi^{-1}(n)$ (if all are 0 ok, just one nonzero is impossible given that the whole sum is zero) we can write

$$\sum a_m t^m = \underbrace{a_{m_1}(t^{m_1} - t^{m_2})}_{\in I} + \underbrace{(a_{m_2} + a_{m_1})t^{m_2} + \sum_{m \neq m_1, m_2} a_m t^m}_{\text{removed term with } t^{m_1}}$$

iterating this process shows the other inclusion.

Thus we can factor $k[M_2] \to k[M_1]$ as $k[M_2] \twoheadrightarrow k[M_2]/I \hookrightarrow k[M_1]$. One can check that $k[M_2]/I = k[M_2/K]$. Since $M_2/K \hookrightarrow M_1$ and M_1 is a free abelian group, M_2/K is also free and thus

$$T_1 \twoheadrightarrow \underbrace{\operatorname{Spec} k[M_2/K]}_{\text{torus}} \hookrightarrow T_2$$

where to check injectivity we use $k[M_2] \to k[M_2/K]$ surjective and to check surjectivity, because subgroups are closed, it is enough to check for dominance and indeed $k[M_2/K] \to k[M_1]$ is injective. $\square$

Remark 1.46. We could have just said that the image is a connected subgroup of a torus and thus is also a torus, but the proof given is more instructive.

1.3 1 parameter subgroups and lattices

We now define a *dual* notion to characters (we will make this precise shortly).

> **Definition 1.47** (1-parameter subgroup). A **1-parameter subgroup** (or **cocharacter** or **1-ps**) of an algebraic group G is a homomorphism $\lambda : \mathbb{G}_m \to G$.

Exercise 1.48. A non-trivial quotient of $\mathbb{G}_m$ is isomorphic to $\mathbb{G}_m$. More generally, a non-trivial quotient of a torus is isomorphic to a torus.

Remark 1.49. If $\lambda : \mathbb{G}_m \to G$ is a homomorphism, the image is isomorphic to $\mathbb{G}_m / \ker \lambda$ by the first isomorphism theorem and because of the above exercise this quotient is again isomorphic to $\mathbb{G}_m$.

Remark 1.50. 1-paramenter subgroups of G form a group via

$$\lambda_1 \cdot \lambda_2 : \mathbb{G}_m \xrightarrow{(\lambda_1, \lambda_2)} G \times G \xrightarrow{m} G.$$

Remark 1.51. If G is abelian the group of 1-ps is abelian.

Proposition 1.52. If $(h_1, \cdots, h_n) \in \mathbb{Z}^n$, the morphism

$$
\begin{array}{ccc}
\mathbb{G}_m & \longrightarrow & \mathbb{G}_m^n \\
a & \longmapsto & (a^{h_1}, \cdots, a^{h_n})
\end{array}
$$

is a 1-ps of $\mathbb{G}_m^n$. Moreover, all 1-ps of $\mathbb{G}_m^n$ are of this form. In particular, the group of 1-ps of $\mathbb{G}_m^n$ is isomorphic to $\mathbb{Z}^n$.

Proof.
If $\lambda : \mathbb{G}_m \to \mathbb{G}_m^n$ is a 1-ps then the compositions with the projections $\pi_i : \mathbb{G}_m^n \to \mathbb{G}_m$ yield characters of $\mathbb{G}_m$, so $\pi_i \circ \lambda(a) = a^{h_i}$ for some $h_i \in \mathbb{Z}$. $\qquad \square$

Remark 1.53. In general, if $f : \mathbb{G}_m^{n_1} \to \mathbb{G}_m^{n_2}$ is a homomorphism then there are $k_1, \cdots, k_{n_1} \in \mathbb{Z}^{n_2}$ such that

$$f(a_1, \cdots, a_{n_1}) = a_1^{k_1} \cdots a_{n_1}^{k_{n_1}}$$

where $a^{(k_{1,h}, \cdots, k_{n_2,h})} = (a^{k_{1,h}}, \cdots, a^{k_{n_2,h}})$.

1.3.1 Character- and cocharacter- lattice

We have seen that the group of characters and cocharacters are free abelian groups of finite rank, let us formalize this by introducing lattices

---

Definition 1.54 (Lattice). A **lattice** is a free abelian group of finite rank.

Definition 1.55 (Character lattice). The **character lattice** of a torus T is the group of characters $M = X(T)$.

Definition 1.56 (Cocharacter lattice). The **cocharacter lattice** of a torus T is the group of 1-parameter subgroups N.

Notation. If $m \in M$ we may write χ^m to mean the character m, similarly for $n \in N$ and λ^n. While this is technically redundant, it is useful when we identify M and N with the abstract $\mathbb{Z}^k$.

Proposition 1.57. The lattices M and N are dual.

Proof.
We have a symmetric $\mathbb{Z}$-bilinear pairing

$$\langle,\rangle: \begin{array}{ccc} M \times N & \longrightarrow & \mathbb{Z} \\ (\chi,\lambda) & \longmapsto & k \end{array}$$

where k is the unique integer such that $\chi \circ \lambda(a) = a^k$.

One can check that this becomes[5] the standard pairing $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by the dot product upon choosing an isomorphism $T \cong \mathbb{G}_m^n$. In particular this is a non-degenerate pairing. $\qquad\square$

Remark 1.58. There is an isomorphism of groups

$$\begin{array}{ccc} N \otimes_{\mathbb{Z}} k^* & \longrightarrow & T \\ u \otimes t & \longmapsto & \lambda^u(t) \end{array}$$

this amounts to saying that $T \cong \mathbb{G}_m^{\mathrm{rnk}\, N}$.

Notation. From now on, the torus with cocharacter lattice N will be denoted by T_N. It's "the" torus because $T_N = D(N^*)$ where N^* is the dual of N in the sense we had above.

Remark 1.59. Fixing an isomorphism $T_N \cong \mathbb{G}_m^n$ is equivalent to fixing a $\mathbb{Z}$-basis of N (or M equivalently).

1.4 Actions and representations

Definition 1.60. An **action** of an algebraic group G on a variety X is a morphism $\mu: G \times X \to X$ that satisfies the diagrammatic axioms of an action:

$$
\begin{array}{ccc}
G \times G \times X & \xrightarrow{(id_G,\mu)} & G \times X \\
{\scriptstyle (m,id_X)}\downarrow & & \downarrow{\scriptstyle \mu} \\
G \times X & \xrightarrow{\mu} & X
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Spec}\, k \times X & \xrightarrow{\sim} & X \\
{\scriptstyle (e,id_X)}\downarrow & \nearrow{\scriptstyle \mu} & \\
G \times X & &
\end{array}
$$

---

[5] $\chi^{(k_1,\cdots,k_n)}(a_1,\cdots,a_n) = a_1^{k_1}\cdots a_n^{k_n}$ and $\lambda^{(k_1,\cdots,k_n)}(a) = (a^{k_1},\cdots,a^{k_n})$, thus

$$\chi^{e_j}(\lambda^{e_i}(a)) = \chi^{e_j}(1,\cdots,\overset{\overset{i}{\downarrow}}{a},\cdots,1) = a^{\delta_{ij}}.$$

Example 1.61. The conjugation action

$$\begin{aligned} \mathrm{GL}_n \times \mathcal{M}(n) &\longrightarrow \mathcal{M}(n) \\ (A, B) &\longmapsto ABA^{-1} \end{aligned}$$

is an action of the algebraic group GL_n on $\mathcal{M}(n)$.

Example 1.62. Multiplication

$$\begin{aligned} \mathbb{G}_m &\longrightarrow \mathbb{A}^1 \\ (a, z) &\longmapsto a \cdot z \end{aligned}$$

is an action. This action *extends* to the projective line

$$\begin{aligned} \mathbb{G}_m \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ (a, [x_0, x_1]) &\longmapsto [x_0, ax_1] \end{aligned}$$

Definition 1.63. A (finite dimensional) **representation** of an algebraic group G is an algebraic group homomorphism $G \to \mathrm{GL}(V)$ for a (finite dimensional) k-vector space V.

Remark 1.64. A representation in this sense yields a set-theoretic linear action of G on $V \cong k^n$.

Fact 1.65. If $G = \mathrm{Spec}\, A$ is affine, so that A is a Hopf algebra, representations of G are in bijection with A-comodules, that is, k-vector spaces V equipped with the following data:

- $\rho : V \to V \otimes_k A$ a k-linear map

-
$$\begin{array}{ccc} V & \xrightarrow{\ \rho\ } & A \otimes V \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle id_A \otimes \rho} \\ A \otimes V & \xrightarrow[\Delta \otimes id_V]{} & A \otimes A \otimes V \end{array}$$

-
$$\begin{array}{ccc} k \otimes V & \xleftarrow{\ \sim\ } & V \\ {\scriptstyle \varepsilon \otimes id_V}\uparrow & \swarrow{\scriptstyle \rho} & \\ A \otimes V & & \end{array}$$

Moreover, subrepresentations correspond to subcomodules.

Idea.

From ρ you get $\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto \rho(v)(g) \end{aligned}$ and this is a linear action. $\qquad\square$

Proposition 1.66. Let $\alpha : T \to \mathrm{GL}(V)$ be a finite dimensional representation of the torus T. For $m \in M$ set

$$V_m = \{v \in V \mid \forall t \in T, \ \alpha(t)(v) = \chi^m(t) \cdot v\},$$

then

$$V = \bigoplus_{m \in M} V_m.$$

Sketch for $n = 1$.
Let $\rho : V \to V \otimes_k k[x^{\pm 1}]$ be the corresponding coaction to α.

- If $m \in M = \mathbb{Z}$ we have that (exercise)

$$V_m = \{v \in V \mid \rho(v) = v \otimes x^m\}.$$

- If $\rho(v) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes x^m$, then $f_m : V \to V$ is linear and $v = \sum_{m \in M} f_m(v)$.

- $f_m(v) \in V_m$

- $f_m \circ f_n = 0$ if $n \neq m$

This tells us that $\{f_n\}_{n \in \mathbb{Z}}$ is a family of orthogonal projectors, so $V = \bigoplus_{m \in M} V_m$.
$\square$

Chapter 2

Affine toric varieties

2.1 Introduction

> **Definition 2.1** (Affine toric variety). An **affine toric variety** is an irreducible affine variety X equipped with an open embedding of a torus T such that the translation action $T \times T \to T$ extends to an action of T on X.

Remark 2.2. The open torus is automatically dense in, and of the same dimension of, X.

Remark 2.3. The extension of the action is unique because if X and Y are irreducible affine and $f, g : X \to Y$ agree on a dense open subset then $f = g$.

Example 2.4. A torus is a toric variety.

Example 2.5. Affine space $\mathbb{A}^n$ is a toric variety, via the trivial embedding

$$\mathbb{G}_m^n = \{x_1 \cdots x_n \neq 0\} \subseteq \mathbb{A}^n.$$

Example 2.6. Let $C = V(x^3 - y^2) \subseteq \mathbb{A}^2$ with torus

$$
\begin{array}{ccc}
\mathbb{G}_m & \longrightarrow & C \\
t & \longmapsto & (t^2, t^3)
\end{array}
$$

and action

$$
\begin{array}{ccc}
\mathbb{G}_m \times C & \longrightarrow & C \\
(t, (x, y)) & \longmapsto & (t^2 x, t^3 y)
\end{array} .
$$

Notice that this affine toric variety is neither smooth nor normal[1].

> **Fact 2.7.** A normal variety is smooth in codimension 1, that it, the singular locus has codimension at least 2. In particular a curve is normal iff they're smooth.

Example 2.8. Let $X = V(xy - z^2) \subseteq \mathbb{A}^3$ be the *quadric cone.* It can be shown that X is normal, but it is not smooth (not at the origin).

---

[1]Spec A irreducible affine variety is **normal** if all local rings are integrally closed in Frac A. This is equivalent to A being integrally closed in Frac A.
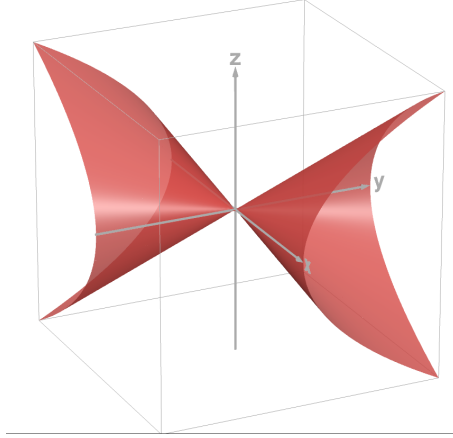
Figure 2.1: Quadric cone over the real numbers.

X is a toric variety with torus given by the image of[2]

$$
\begin{array}{ccc}
\mathbb{G}_m^2 & \longrightarrow & X \\
(s,t) & \longmapsto & (s^2, t^2, st)
\end{array}
$$

and action

$$
\begin{array}{ccc}
\mathbb{G}_m^2 \times X & \longrightarrow & X \\
((s,t),(x,y,z)) & \longmapsto & (sx, st^2 y, stz)
\end{array}
$$

Example 2.9. $X = V(xy - zw) \subseteq \mathbb{A}^4$ is a toric variety with torus

$$
\begin{array}{ccc}
\mathbb{G}_m^3 & \longrightarrow & X \\
(t_1, t_2, t_3) & \longmapsto & (t_1, t_2, t_3, t_1 t_2 t_3^{-1})
\end{array}
$$

and action

$$
\begin{array}{ccc}
\mathbb{G}_m^3 \times X & \longrightarrow & X \\
((t_1, t_2, t_3),(x,y,z,w)) & \longmapsto & (t_1 x, t_2 y, t_3 z, t_1 t_2 t_3^{-1} w)
\end{array}
$$

2.2 Monoids

Definition 2.10 (Monoid). A **monoid** is a set S with an operation $+$, which is commutative, associative and with a neutral element $0 \in S$.

Remark 2.11. The reference book [CLS11] calls these *semigroups*.

---

[2]this map is 2:1, to get the actual parametrization we need

$$
\begin{array}{ccc}
\mathbb{G}_m^2 & \longrightarrow & X \\
(s,t) & \longmapsto & (s, st^2, st)
\end{array}
$$

This is related to the fact that X is the quotient $\mathbb{A}^2/\mu_2$ by the action $-1(x,y) = (-x,-y)$.

Definition 2.12. If $A \subseteq S$ is a subset of a monoid, the **submonoid generated by** A **in** S is the smallest submonoid which contains A. Concretely it is

$$\langle A \rangle = \left\{ \sum_{a \in A} n_a a \mid n_a \in \mathbb{N}, \ n_a = 0 \text{ for all but finitely many coeff.} \right\}$$

A monoid S is **finitely generated** if there exists a finite subset $A \subseteq S$ such that $S = \langle A \rangle$.

Remark 2.13. S is a finitely generated monoid if there exists a surjective monoid homomorphism

$$\mathbb{N}^n \twoheadrightarrow S.$$

Definition 2.14. S is an **affine monoid** if it is finitely generated and it is a submonoid of a lattice M.

Example 2.15. $\mathbb{N}^k \subseteq \mathbb{Z}^k$ is an affine monoid.

Example 2.16. $\mathbb{Z}/n\mathbb{Z}$ is a monoid but it is NOT affine because a lattice can't have a submonoid with torsion.

Example 2.17. $\langle (1,0), (1,1) \rangle \subseteq \mathbb{N} \oplus \mathbb{Z}/2\mathbb{Z}$ is also not affine because of torsion.

Definition 2.18 (Integrality)**.** A monoid S is **integral** (or **cancellative**) if $a + b = a + c \implies b = c$.

Fact 2.19. A monoid S is affine if and only if S is

- finitely generated,

- integral and

- torsion free.

Let us now define the left adjoint to the forgetful $\mathrm{Ab} \to \mathrm{Mon}$:

Definition 2.20 (Associated group)**.** Let S be a monoid. There is an **associated abelian group** S^{gp}, which is the initial group with a morphism from S. Concretely

$$S^{gp} = \frac{\{(s, s') \mid s, s' \in S\}}{\sim}$$

where $(s_1, s'_1) \sim (s_2, s'_2)$ if there exists $s \in S$ such that[a]

$$s + s_1 + s'_2 = s + s_2 + s'_1.$$

---
[a]think about localization on rings which are not domains.

Remark 2.21. S^{gp} is an abelian group and we have a map $S \to S^{gp}$ given by $s \mapsto [(s, 0)]_{\sim}$.

Fact 2.22. Any morphism $S \to G$ for G abelian group factors uniquely through S^{gp}. More precisely

$$\mathrm{Hom}_{\mathrm{Mon}}(S, G) = \mathrm{Hom}_{\mathrm{Ab}}(S^{gp}, G)$$

Remark 2.23. S is integral if and only if $S \to S^{gp}$ is injective, which happens if and only if S can be injected into an abelian group.

Definition 2.24. A monoid is **sharp** if the only invertible element is 0.

Definition 2.25. An element m of a sharp monoid S is **irreducible** if $m = m' + m''$ in S implies $m' = 0$ or $m'' = 0$.

Remark 2.26. If S is a sharp monoid, the irreducible elements generate the monoid.

Presentations of monoids

With monoids, the kernel is "sort of useless"

Example 2.27. Consider

$$\begin{array}{ccc} \mathbb{N}^2 & \longrightarrow & \mathbb{N} \\ (a, b) & \longmapsto & a + b \end{array}$$

this has trivial kernel (preimage of 0 is just $(0, 0)$) but it is far from being injective.

Let $f : S \to S'$ be a surjective homomorphism. What we should look at instead of the kernel for the right analogue of the first isomorphism theorem is

$$E = \{(s, s') \in S \times S \mid f(s) = f(s')\}.$$

This set is an equivalence relation on $S \times S$, which is also a submonoid.

Definition 2.28 (Congruence relations)**.** A submonoid of $S \times S$ which defines an equivalence relation is called **congruence relation**.

Definition 2.29 (Coequalizer)**.** If $f, g : X \to Y$, the coequalizer is an object Z together with $h : Y \to Z$ such that $h \circ f = h \circ g : X \to Z$ and if W together with $h' : Y \to W$ is also such that $h' \circ f = c' \circ g$ then there exists a unique $Z \to W$ making everything commute.

Fact 2.30. We can construct quotients of S by a congruence relation E on $S \times S$ by setting it to be the coequalizer of $E \subseteq S \times S \rightrightarrows S$, where the arrows are the two projections from $S \times S$ to S.

We call this object the **quotient of S by E** and denote it S/E.

Remark 2.31. If E is the relation constructed from $f : M \twoheadrightarrow M'$ homomorphism of abelian groups viewed as monoids then $E = \{(m, m') \in M \times M \mid f(m) = f(m')\} = \{(m, m') \mid m - m' \in \ker f\}$. It follows that $M' \cong M/\ker f$ is a coequalizer for $E \rightrightarrows M$, so our definition makes sense.

Definition 2.32 (presentation of a monoid)**.** The monoid associated to

$$\langle p_1, \cdots, p_r \mid a_1 = b_i, \ i \in \{1, \cdots, k\} \rangle,$$

where $a_i, b_i, \in \langle p_1, \cdots, p_r \rangle_{\mathbb{N}}$, is the quotient of $\mathbb{N}^r$ by the congruence relation generated by the (a_i, b_i) in $\mathbb{N}^r \times \mathbb{N}^r$.

A **presentation** of a monoid S is an isomorphism with a monoid constructed as above.

2.2.1 Monoid algebra

Since from abelian groups we costructed the group algebra and found connections to geometric objects, we want to generalize that construction to monoids.

Definition 2.33 (Monoid algebra)**.** For a monoid S, its **monoid algebra** $k[S]$ is the k-vector space which is freely generated by $\{t^s \mid s \in S\}$ and with multiplication induced by the operation on S.

Remark 2.34. In [CLS11] they write χ^s instead of t^s because they think of S inside $M = X(T)$ for some torus.

Remark 2.35. If S is actually a group then the monoid algebra and group algebras coincide.

Example 2.36. If $S = \mathbb{N}^n \subseteq \mathbb{Z}^n$ then $k[S] = k[x_1, \cdots, x_n]$.

Proposition 2.37. If S is a monoid with presentation

$$\langle p_1, \cdots, p_r \mid a_i = b_i, \ 1 \leq i \leq k \rangle,$$

then

$$k[S] = \frac{k[t_1, \cdots, t_r]}{(t^{a_i} - t^{b_i})}$$

where if $a_i = \sum a_{ij} p_j$ we set $t^{a_i} = \prod t_j^{a_{ij}}$.

Sketch.

Let R be the congruence relation on $\mathbb{N}^r$ generated by $\{(a_i, b_i)\}_{1 \le i \le k}$. Since $R \rightrightarrows \mathbb{N}^r \to S$ is a coequalizer and $S \mapsto k[S]$ is a left adjoint $(\mathrm{Hom}_{\mathrm{Mon}}(S, A) \cong \mathrm{Hom}_{k-\mathrm{Alg}}(k[S], A))$ it follows that

$$k[R] \underset{g}{\overset{f}{\rightrightarrows}} k[\mathbb{N}^r] \to k[S]$$

is a coequalizer in k-algebras, so $k[S] \cong k[\mathbb{N}^r]/I$ where $I = (f(x) - g(x) \mid x \in k[R])$.
$\square$

Example 2.38. Let $S = \langle (2,0), (1,1), (0,2) \rangle \subseteq \mathbb{Z}^2$. This monoid can be seen to be isomorphic to

$$\langle p, q, r \mid p + q = 2r \rangle.$$

It follows that

$$k[S] \cong \frac{k[x,y,z]}{(xy - z^2)},$$

which is the coordinate ring of the quadric cone.

Example 2.39. Consider $S = \langle 2, 3 \rangle \subseteq \mathbb{N}$, which has presentation

$$\langle p, q \mid 3p = 2q \rangle.$$

It follows that

$$k[S] \cong \frac{k[x,y]}{(x^3 - y^2)},$$

the coordinate ring of the cusp curve.

2.3 Toric variety associated to a monoid

Inspired by the success of Cartier duality, we consider the analogous construction with affine monoids. Instead of diagonalizable algebraic groups we will get affine toric varieties:

Proposition 2.40. If S is an affine monoid then

1. $k[S]$ is a domain and a finitely generated k-algebra.

2. $\mathrm{Spec}\, k[S]$ is an affine toric variety, with torus $\mathrm{Spec}\, k[S^{gp}]$.

Proof.

Let us prove the two propositions

1. Since $S \subseteq M$, we have an obvious inclusion $k[S] \subseteq k[M]$ and $k[M]$ is a domain, so $k[S]$ also is. Since S is finitely generated, just take the formal variables associated to those generators and they will generate $k[S]$ as a k-algebra.

2. The inclusion $S \to M$ must factor through $S \to S^{gp} \to M$ by the universal property. Since M is free of finite rank, S^{gp} also is, thus $T = \mathrm{Spec}\, k[S^{gp}] = D(S^{gp})$ is a torus

(1.23) of dimension equal to the rank of S^{gp}. Moreover, $k[S^{gp}]$ is a localization of $k[S]$ in a single element: if $\{s_i\}_{1 \le i \le k}$ are generators of S then[3]

$$k[S^{gp}] \cong k[S]_{\prod t^{s_i}} = k[S][t^{-s_1}, \cdots, t^{-s_k}]$$

and this isomorphism is induced by the natural map $k[S] \to k[S^{gp}]$. The induced morphism $\operatorname{Spec} k[S^{gp}] \to \operatorname{Spec} k[S]$ is then an open embedding (iso. on local rings).

The translation action of T on itself is the one given by

$$\begin{array}{ccc} k[S^{gp}] & \longrightarrow & k[S^{gp}] \otimes k[S^{gp}] \\ t^m & \longmapsto & t^m \otimes t^m \end{array},$$

which extends to an action on $\operatorname{Spec} k[S]$ by

$$\begin{array}{ccc} k[S] & \longrightarrow & k[S^{gp}] \otimes k[S] \\ t^m & \longmapsto & t^m \otimes t^m \end{array},$$

which makes sense because $S \subseteq S^{gp}$.

$\square$

There is another construction to describe the toric variety associated to the monoid generated by a finite subset $A \subseteq M$ (recall that M is the character lattice of T for some torus).

Consider the morphism

$$\phi_A : \begin{array}{ccc} T_N & \longrightarrow & (\mathbb{A}^1)^A \\ x & \longmapsto & (\chi^a(x))_{a \in A} \end{array}$$

Remark 2.41. The image of ϕ_A is contained in the standard torus $\operatorname{Imm} \phi_A \subseteq (\mathbb{G}_m)^A \subseteq (\mathbb{A}^1)^A$. It follows that $\operatorname{Imm} \phi_A$ is also a torus because it is the image of a homomorphism between tori (1.45).

Let Y_A be the closure of $\operatorname{Imm} \phi_A$ in $(\mathbb{A}^1)^A$.

Proposition 2.42. Y_A is an affine toric variety, with torus given by the one associated to $\mathbb{Z}A \subseteq M$. More precisely, $Y_A \cong \operatorname{Spec} k[\mathbb{N}A]$.

Proof.
The morphism ϕ_A corresponds to the algebra homomorphism

$$\varphi_A : k[x_a \mid a \in A] \to k[M]$$

Note that

$$\overline{\operatorname{Imm} \phi_A} = V(\ker \varphi_A) = \operatorname{Spec} \frac{k[x_a \mid a \in A]}{\ker \varphi_A} = \operatorname{Spec} \operatorname{Imm} \varphi_A.$$

It is easy to see that $\operatorname{Imm} \varphi_A = k[\mathbb{N}A] \subseteq k[M]$. Since $\mathbb{N}A$ is an affine monoid we are done by (2.40) $\square$

---

[3]exercise

Remark 2.43. The two constructions are the same upon choosing a finite set of generators A for S, letting us write $S = \mathbb{N}A$.

> **Definition 2.44** (Toric ideals). The ideals of $k[\mathbb{N}^A]$ which give rise to toric varieties are called **Toric ideals**

> **Fact 2.45.** Toric ideals are exactly the prime ideals which can be generated by binomials (differences of monic monomials).

We now want to show that this construction covers all affine toric varieties:

Remark 2.46. The torus T_N acts linearly on its own ring of regular functions $k[M]$ as follows: for $t \in T_N$ and $f \in k[M]$ ($f : T_N \to \mathbb{A}^1$) we define[4] $t \cdot f \in k[M]$ as

$$t \cdot f : \begin{array}{ccc} T_N & \longrightarrow & \mathbb{A}^1 \\ p & \longmapsto & f(t^{-1} \cdot p) \end{array}$$

where the product $t^{-1} \cdot p$ is the product of T_N as an algebraic group.

To be more precise, the action of T_N is induced by a comodule structure on $k[M]$, specifically

$$k[M] \xrightarrow{\Delta} k[M] \otimes k[M] \xrightarrow{S \otimes id} k[M] \otimes k[M].$$

Technically $k[M]$ is infinite dimensional, but every time we consider this action we will actually consider the restriction to a stable finite dimensional subspace.

> **Lemma 2.47.** The only simultaneous eigenvectors of the action $T_N \curvearrowright k[M]$ given above are the characters.

Proof.
Note that $t \cdot \chi^m(p) = \chi^m(t^{-1} \cdot p) = \chi^m(t^{-1})\chi^m(p)$ on the torus, thus $t \cdot \chi^m = \chi^m(t^{-1})\chi^m$, that is, characters are simultaneous eigenvectors for this action of T_N.

Let us now prove that they are the only ones (up to scalars): if $\sum a_m \chi^m$ in $k[M]$ is a simultaneous eigenvector then

$$\alpha(t) \left(\sum a_m \chi^m \right) = t \cdot \left(\sum a_m \chi^m \right) = \sum \chi^m(t^{-1}) a_m \chi^m$$

for some function $\alpha : T_N \to k$, thus $a_m \alpha(t) = a_m \chi^m(t^{-1})$ for all m. If $a_{m_1} \neq 0 \neq a_{m_2}$ then $\chi^{m_1}(t^{-1}) = \alpha(t) = \chi^{m_2}(t^{-1})$, so $m_1 = m_2$ and thus the simultaneous eigenvector we chose must be of the form $a_m \chi^m$ for some $m \in M$. $\qquad\square$

> **Lemma 2.48.** If $A \subseteq k[M]$ is a subspace which is stable under the action above then
> $$A = \bigoplus_{t^m \in A} kt^m,$$
> that is, A is generated by characters.

---

[4]the inverse in the definition is not needed since T_N is abelian, but it is put there for consistency with more general theory where it is needed to verify that the map given is indeed a left-action.

Proof.

Call $A' = \bigoplus_{t^m \in A} kt^m$. Clearly $A' \subseteq A$ so we just need the other inclusion. Pick $f \in A$ and write

$$f = \sum_{m \in B} c_m t^m$$

for $B \subseteq M$ finite and such that $c_m \neq 0$ for all $m \in B$. Note that

$$f \in A \cap \langle t^m \mid m \in B \rangle := V.$$

This intersection is a finite dimensional k-vector space which is stable under the T_N-action, so it is a finite dimensional representation of T_N. By proposition (1.66) it follows that V is generated by simultaneous eigenvectors of the action, which are the t^m by the lemma above. Writing what we have just said in symbols:

$$f \in V = \bigoplus_{\substack{m \in B \ s.t. \\ t^m \in A}} kt^m \subseteq \bigoplus_{t^m \in A} kt^m = A'.$$

$\square$

Theorem 2.49. All affine T_N-toric varieties are isomorphic to one of the form $\operatorname{Spec} k[S]$ for some monoid $S \subseteq M = X(T_N)$.

Proof.

If $X = \operatorname{Spec} A$ is an affine toric variety, then $A \subseteq k[M]$ is stable for the action of T_N on $k[M]$. This is because $\cdot t^{-1} : T_N \to T_N$ extends to X by definition of toric variety. By the lemma above

$$A = \bigoplus_{t^m \in A} kt^m = k[S],$$

where $S = \{m \in M \mid t^m \in A\}$, which is a submonoid of M because A is an algebra.

Since A is finitely generated, there exist $f_1, \cdots, f_k$ such that $A = k[f_1, \cdots, f_k]$. By replacing each f_i with all the characters that you need to write it out, we can assume that the f_i are all of the form t^m.

It is now easy to check that the corresponding exponents generate S. $\square$

2.4 Cones

It will turn out that (normal) affine toric varieties are described by cones lying in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is a lattice (it will be the cocharacter lattice of the resulting toric variety).

Definition 2.50. A **convex polyhedral cone** (from now on just **cone**) is a subset of $N_{\mathbb{R}}$ of the form

$$\sigma = \operatorname{Cone}(A) = \left\{ \sum_{n \in A} \lambda_n \cdot n \mid \lambda_n \geq 0 \right\} \subseteq N_{\mathbb{R}}$$

where $A \subseteq N_{\mathbb{R}}$ is a finite subset.

Remark 2.51. A cone σ is a convex subset of $N_\mathbb{R}$ and it is a "positive" cone, in the sense that if $v \in \sigma$ and $\lambda \in [0, +\infty) \subseteq \mathbb{R}$ then $\lambda v \in \sigma$.

Example 2.52. The positive quadrant

$$\left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, \ y \geq 0\right\} = \mathrm{Cone}((1,0),(0,1))$$

is a cone. $\mathrm{Cone}((1,0),(1,2))$ is also a cone, which is embedded differently.

Definition 2.53 (Orthant). An **orthant** is a cone of the form $\mathrm{Cone}(e_1, \cdots, e_k) \subseteq \mathbb{R}^n$.

Example 2.54. $\mathrm{Cone}((1,0,0),(0,1,0),(1,0,1),(0,1,1)) \subseteq \mathbb{R}^3$ is a cone.

Example 2.55. A line in $\mathbb{R}^2$ is a cone, since it can be written $\mathrm{Cone}(v,-v)$. In general linear subspaces are cones.

Definition 2.56. A cone σ is **strongly** or **strictly convex** if it does not contain any positive dimensional subspace.

Definition 2.57. The **dimension** of σ, denoted $\dim \sigma$, is the dimension of the vector subspace of $N_\mathbb{R}$ spanned by σ.
A cone is **full-dimensional** if its dimension is the same as the rank of N.

2.4.1 General facts about cones

For references you can look at Fulton [Ful93] for most of these facts.

Proposition 2.58. A cone is closed in the respective $N_\mathbb{R}$.

Sketch.
Assume the following theorem by Carathéodory: *if* $v \in \mathrm{Cone}(A)$ *then there exists* $B \subseteq A$ *linearly independent such that* $v \in \mathrm{Cone}(B)$.

It follows that
$$\mathrm{Cone}(A) = \bigcup_{\substack{B \subseteq A \\ B \text{ lin. ind.}}} \mathrm{Cone}(B)$$

and this is a finite union of closed sets because $\mathrm{Cone}(B)$ can be identified with $\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-k}$ via a linear transformation for some k. $\qquad\square$

Definition 2.59. Two polytopes are said to be **combinatorially equivalent** if their poset of faces are isomorphic.

Is there any polytope which is combinatorially equivalent to one with rational vertices (i.e. vetices in $\mathbb{Q}^n$)? Surprisingly, no. In all dimensions above 8 there are some polytopes that contradict this (which is weird because one would think "I can just move the vertices a little").

For more details look up *non-realizable matroids*.

Hyperplanes and dual cone

Definition 2.60 (Hyperplane and closed half-space)**.** If $m \in M_\mathbb{R}$, we write

$$H_m = \{n \in N_\mathbb{R} \mid \langle m, n \rangle = 0\}$$

(the product is the one induced by $M \times N \to \mathbb{Z}$ upon tensoring with $\mathbb{R}$). Sets of this form are **hyperplanes** in $N_\mathbb{R}$.
We write H_m^+ for $\{n \in N_\mathbb{R} \mid \langle m, n \rangle \geq 0\}$ and call this a **closed half-space**.

Definition 2.61. H_m is a **supporting hyperplane** for a cone σ if $\sigma \subseteq H_m^+$. We call H_m^+ a **supporting half-space**.

Definition 2.62. The **dual cone** to a cone σ is

$$\sigma^\vee = \{m \in M_\mathbb{R} \mid \langle m, n \rangle \geq 0 \ \forall n \in \sigma\} \subseteq M_\mathbb{R}$$

Remark 2.63. By definition

$$\sigma^\vee = \bigcap_{\substack{m \in M_\mathbb{R} \ s.t. \\ H_m^+ \ \text{supp. half-sp.}}} H_m^+,$$

so H_m is supporting if and only if $m \in \sigma^\vee \setminus \{0\}$.

Fact 2.64. $\sigma^\vee$ is also a cone and $(\sigma^\vee)^\vee \cong \sigma$ under the identification $(N_\mathbb{R}^\vee)^\vee \cong N_\mathbb{R}$.

Fact 2.65. $m_1, \cdots, m_s$ generate $\sigma^\vee$ if and only if $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$. In particular, every cone is a finite intersection of half-spaces.

Definition 2.66. A **face** of a cone σ is a subset of the form $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$. In this case we write $\tau \leq \sigma$.

Remark 2.67. If $\sigma = \text{Cone}(A)$ then $\tau = \text{Cone}(a \in A \mid a \in H_m)$. In particular τ is also a cone.
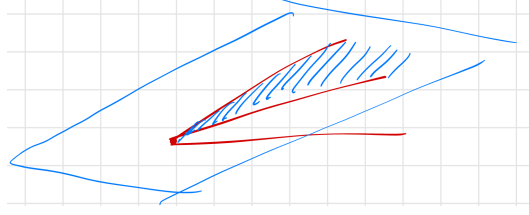
Definition 2.68. A face is **proper** if it is not σ itself.

Fact 2.69. The following are true:

- If $\tau_1, \tau_2 \leq \sigma$ then $\tau_1 \cap \tau_2 \leq \sigma$

- if $\tau' \leq \tau$ and $\tau \leq \sigma$ then $\tau' \leq \sigma$

- if $\tau \leq \sigma$ and $v, w \in \sigma$ are such that $v + w \in \tau$ then $v, w \in \tau$.

Definition 2.70. A **ray** (or **edge**) is a 1 dimensional face. A **facet** is a $\dim \sigma - 1$ dimensional face.

Fact 2.71. If σ is full-dimensional in $N_\mathbb{R}$ then in the representations like $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$ we can assume that $\sigma \cap H_{m_i}$ is a facet of σ for all i.

Remark 2.72. This is not the case if σ is not full-dimensional, for example for $\sigma = \mathrm{Cone}((1,0)) \subseteq \mathbb{R}^2$ the only facet is $\{(0,0)\}$ but in order to write σ as the intersection of half-spaces we need some half-spaces with associated hyperplane being $\mathrm{Span}((1,0))$ and so $\sigma \cap H$ for those hyperplanes is σ itself.

Fact 2.73. Every proper face is the intersection of all facets containing it.

Remark 2.74. If $N_\mathbb{R} \cong \mathbb{R}^n$ then we know that $M_\mathbb{R} \cong \mathbb{R}^n$ via the dual basis and we can think of one of the m_i that generate the dual cone as an "inward-pointing" normal vector to a facet of σ

Example 2.75. Let $\sigma = \mathrm{Cone}((1,0),(1,2))$.
The half-planes that bound the cone are $y \geq 0$ and $2x - y \geq 0$, which correspond to $(0,1)$ and $(2,-1)$, which can be used to generate $\sigma^\vee = \mathrm{Cone}((0,1),(-2,1))$.

Example 2.76. Take $\sigma = \mathrm{Cone}((1,0)) \subseteq \mathbb{R}^2$, so $\sigma^\vee$ is $\mathrm{Cone}((1,0),(0,1),(0,-1))$ which correspond to $x \geq 0$, $y \geq 0$ and $-y \geq 0$
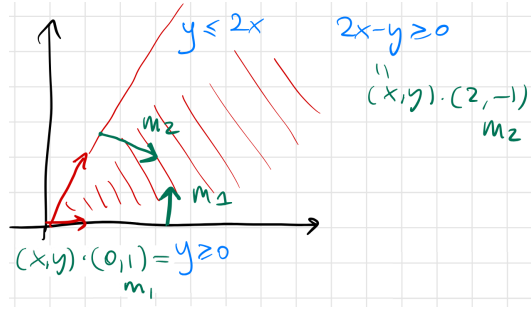
Fact 2.77. The following are equivalent:

- σ is strictly convex,
- $\{0\}$ is a face of σ,
- $\sigma \cap (-\sigma) = \{0\}$,
- $\dim \sigma^\vee = \dim M_\mathbb{R}$.

Fact 2.78. Any cone σ contains a maximal linear subspace given by $\sigma \cap (-\sigma) = W$. Moreover, $\sigma/W \subseteq N_\mathbb{R}/W$ is strictly convex.

Definition 2.79. σ is **rational** if $\sigma = \mathrm{Cone}(A)$ for $A \subseteq N$ (not $N_\mathbb{R}$ like before).

Fact 2.80. The dual and the faces of a rational cone are rational.

Fact 2.81. If $A \subseteq N$ then

$$\mathrm{Cone}(A) \cap N_\mathbb{Q} = \left\{ \sum_{a \in A} q_a a \mid q_a \in \mathbb{Q} \right\}.$$

Definition 2.82. Let σ be a rational cone, its **minimal ray generators** are given as follows: if $\rho \leq \sigma$ is a ray (and thus rational), the minimal ray generator correspondint to it is the minimal generator of $\rho \cap N$ as a monoid, which is denoted u_ρ.

Fact 2.83. A strictly convex rational cone is "canonically" generated by its minimal ray generators:

$$\sigma = \mathrm{Cone}(u_\rho \mid \rho \text{ is a ray}).$$

Corollary 2.84. If σ is a rational full-dimensional cone then σ has minimal facet normals (minimal ray generators of the dual).

2.5 Affine toric varieties from cones

Notation. Let σ be a cone in $N_{\mathbb{R}}$. We write

$$S_\sigma = \sigma^\vee \cap M.$$

Remark 2.85. S_σ is a submonoid of M because if $m, m' \in \sigma^\vee \cap M$ then

$$\langle m + m', n \rangle = \langle m, n \rangle + \langle m'n \rangle \geq 0 + 0 = 0.$$

Lemma 2.86 (Gordan). If σ is a rational polyhedral cone in $N_{\mathbb{R}}$, then $S_\sigma = \sigma^\vee \cap M$ is finitely generated.

Proof.
Write $\sigma^\vee = \text{Cone}(T)$ with $T \subseteq M$ some finite subset. Consider

$$K = \left\{ \sum_{m \in T} a_m m \mid 0 \leq a_m < 1 \right\}.$$

Clearly K is bounded in $M_{\mathbb{R}}$, so $K \cap M$ is a finite set. We claim that $T \cup (K \cap M)$ generates S_σ as a monoid:

Let $w \in S_\sigma = \sigma^\vee \cap M$. We can write $w = \sum_{m \in T} \lambda_m m$ with $\lambda_m > 0$ real numbers. We can write $\lambda_m = \lfloor \lambda_m \rfloor + \{\lambda_m\}$ (floor and fractional part), so that

$$w = \underbrace{\sum_{m \in T} \lfloor \lambda_m \rfloor m}_{\in M} + \underbrace{\sum_{m \in T} \{\lambda_m\} m}_{\in K}.$$

But $\sum_{m \in T} \{\lambda_m\} m$ is also in M because it is $w - \sum_{m \in T} \lfloor \lambda_m \rfloor m$, so we have written w in the desired form. $\square$

Because of the correspondence between affine toric varieties and affine monoids that we built (2.40) we can give the following definition:

Definition 2.87. Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational cone. Its affine toric variety is

$$U_\sigma = \text{Spec}\, k[S_\sigma].$$

Remark 2.88. The torus of U_σ has character lattice $S_\sigma^{gp} \subseteq M$.

Remark 2.89. Why are we not taking $\text{Spec}[\sigma \cap M]$ (for σ cone in $M_{\mathbb{R}}$) instead? This is because the gluing process of affine pieces will be more natural if the cones are in $N_{\mathbb{R}}$

> **Proposition 2.90.** The following are equivalent
>
> 1. $\dim U_\sigma = n = \dim N_\mathbb{R}$
>
> 2. the torus of U_σ is T_N
>
> 3. σ is strictly convex.

Proof.
First note that
$$\dim U_\sigma = \operatorname{rnk} S_\sigma^{gp} = \dim \operatorname{Cone}(S_\sigma) = \dim \sigma^\vee$$
From this, $\dim U_\sigma = n$ is equivalent to $\dim \sigma^\vee = n$ which we know is equivalent to σ being strongly convex.

For the other equivalence, we claim M/S_σ^{gp} is torsion free. This gives the desired equivalence because we get

$$\dim U_\sigma = n \iff \operatorname{rnk} S_\sigma^{gp} = \operatorname{rnk} M \overset{\text{claim}}{\iff} M = S_\sigma^{gp} \iff T_N \text{ is the torus in } U_\sigma.$$

We now prove that the claim holds. Let $m \in M$ and assume that $km \in S_\sigma^{gp}$ for some $k \in \mathbb{N}$. Then $km = m_1 - m_2$ for some $m_1, m_2 \in S_\sigma$ and so

$$M \ni m + m_2 = \frac{1}{k} m_1 + \frac{k-1}{k} m_2 \in \sigma^\vee$$

where the last inclusion holds by convexity. Thus $m = (m+m_2) - m_2$ implies $m \in S_\sigma^{gp}$
$\square$

Because of this result, from now on a cone σ will be assumed to be strictly convex (i.e. $S_\sigma^{gp} = M$) and rational unless otherwise stated.

Example 2.91. Let $\sigma = \operatorname{Cone}(e_1) \subseteq \mathbb{R}^2$, then $\sigma^\vee = \operatorname{Cone}(e_1, e_2, -e_2)$

Example 2.92. If $\sigma = \operatorname{Cone}(e_1, \cdots, e_k) \subseteq \mathbb{R}^n$ is an orthant then
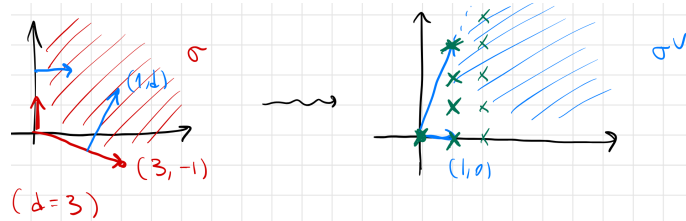
$$\sigma^\vee = \operatorname{Cone}(e_1, \cdots, e_k, \pm e_{k+1}, \cdots, \pm e_n).$$

It follows that $k[S_\sigma] = k[x_1, \cdots, x_k, x_{k+1}^{\pm 1}, \cdots, x_n^{\pm 1}]$ and so for an orthant

$$U_\sigma \cong \mathbb{A}^k \times \mathbb{G}_m^{n-k}.$$

Example 2.93. If $\sigma = \operatorname{Cone}(0) = \{0\}$ then $\sigma^\vee = M$ and so $U_\sigma = T_N$

Example 2.94 (Rational normal cone of degree d)**.** Let $d \in \mathbb{N} \setminus \{0\}$ and take $\sigma = \operatorname{Cone}(de_1 - e_2, e_2)$

It turns out that $S_\sigma = \langle (1, i) \mid 0 \le i \le d \rangle$ (not trivial yet). Let us study

$$U_\sigma = \operatorname{Spec} k[S_\sigma]$$

Setting $A = \{(1, i) \mid 0 \le i \le d\}$, we can see U_σ as Y_A, the closure of the image of

$$\begin{array}{ccc} \mathbb{G}_m^2 & \longrightarrow & \mathbb{A}^{d+1} \\ (s, t) & \longmapsto & (s, st^1, \cdots, st^d) \end{array}$$

Definition 2.95. The toric variety from the previous example is called the **rational normal cone of degree** d. It is the affine cone over the so called *rational curve of degree d* in $\mathbb{P}^d$.

Remark 2.96. It turns out that the ideal of the rational normal cone of degree d is $(x_i x_{j+1} - x_{i+1} x_j \mid 0 \le i < j \le d)$. Note that the generators are determinants of 2×2 matricies, specifically, all minors of

$$\begin{pmatrix} x_0 & \cdots & x_{d-1} \\ x_1 & \cdots & x_d \end{pmatrix}$$

Example 2.97. Consider $\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$. The equations that define this cone are $y \ge 0$, $z \ge 0$, $x \ge 0$ and $x + y - z \ge 0$, so $\sigma^\vee = \operatorname{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3)$.
You can check that $S_\sigma = \sigma^\vee \cap M \cong \langle p, q, r, s \mid p + q = r + s \rangle$, showing that

$$k[S_\sigma] \cong \frac{k[x, y, z, w]}{(xy - zw)}.$$

Remark 2.98. When σ is full-dimensional ($\sigma^\vee$ is strictly convex) it follows that S_σ is sharp and so (2.26) the irreducible elements of S_σ give a canonical generating set.

Definition 2.99. Let σ be a cone. If S_σ is sharp, the set

$$H = \{m \in S_\sigma \mid m \text{ irreducible}\}$$

is called the **Hilbert basis** of S_σ.

Fact 2.100. If σ is full dimensional (and so S_σ is sharp) then

- H is finite and generates S_σ

- H contains the minimal generators of the rays of $\sigma^\vee$

- every generating set of S_σ contains H

2.6 Normality and smoothness of affine toric varieties

2.6.1 Normality

Definition 2.101. If $X = \operatorname{Spec} A$ is an irreducible affine algebraic variety (A is a domain) then X is **normal** if $A \subseteq \operatorname{Frac} A$ is integrally closed.

Remark 2.102. X is normal if and only if all local rings of X are integrally closed in $\operatorname{Frac} A$. We are identifying the local rings with the subrings of $\operatorname{Frac} A$ below

$$A_{\mathfrak{m}_p} \cong \mathcal{O}_{X,p} = \left\{ f \in \operatorname{Frac} A \mid f = \frac{g}{h}, \ h(p) \neq 0 \right\}.$$

Definition 2.103. An integral monoid S is **saturated** if for all $s \in S^{gp}$ such that there exists $k \in \mathbb{N} \setminus \{0\}$ such that $ks \in S$ we have $s \in S$.

Example 2.104. $S = \langle 2, 3 \rangle \subseteq \mathbb{N}$ is not saturated because $S^{gp} = \mathbb{Z}$ ($1 = 3 - 2$) and $2 \cdot 1 = 2 \in S$ but $1 \notin S$.

Remark 2.105. In [CLS11] they say that $S \subseteq M$ is saturated if the condition holds for $m \in M$. The two definitions are not equivalent because $2\mathbb{N} \subseteq \mathbb{Z}$ is saturated for our definition but not theirs.

If $S^{gp} = M$ the two definitions are the same and this is always assumed in [CLS11] so nothing really changes but the true definition in monoid theory is the one we gave.

Proposition 2.106. For an affine toric variety X with torus T_N, the following are equivalent

1. X is normal

2. $X = \operatorname{Spec} k[S]$ for S saturated

3. There exists a strictly convex cone σ in $N_{\mathbb{R}}$ with $X \cong U_{\sigma}$

Proof.
Let us give the implications

$\boxed{1 \implies 2}$ Suppose X is normal and let $S \subseteq M$ be some monoid such that $S^{gp} = M$ and $X \cong \operatorname{Spec} k[S]$. Let $m \in S^{gp} = M$ and $k \in \mathbb{N} \setminus \{0\}$ be such that $km \in S$, then $t^{km} \in k[S]$ and $t^m \in k[M] \subseteq \operatorname{Frac}(k[S])$ is a root of the polynomial

$$y^k - t^{km} \in k[S][y].$$

Since $k[S]$ is integrally closed we get $t^m \in k[S]$ and so $m \in S$

Suppose S is saturated with $S^{gp} = M$. Let $A \subseteq S$ be a set of generators and take $\tau = \mathrm{Cone}(A) \subseteq M_{\mathbb{R}}$. Define $\sigma = \tau^{\vee}$. This σ is strictly convex because τ is full dimensional by construction and clearly $S \subseteq \tau \cap M = \sigma^{\vee} \cap M$. We just need the other inclusion now. If $m \in \tau \cap M$ then $m \in M \subseteq M_{\mathbb{Q}}$ and so

$$m = \sum_{a \in A} q_a a$$

for some $q_a \in \mathbb{Q}$, $q_a \geq 0$. Upon taking the least common multiple of the denominators N we get a positive integer such that Nm is an integral linear combination of the elements of A, thus $Nm \in S$ and by saturatedness we have $m \in S$ as desired.

Let $\rho_1, \cdots, \rho_r$ be the rays of σ, then $\sigma^{\vee} = \bigcap_{i=1}^{r} \rho_i^{\vee}$ and so

$$k[S_\sigma] = \bigcap_{i=1}^{r} k[S_{\rho_i}] \subseteq k[M].$$

Since the intersection of integrally closed subrings is integrally closed we may suppose without loss of generality that $\sigma = \rho$ is a ray.

Let u_ρ be the minimal ray generator of ρ, then we can complete u_ρ to a $\mathbb{Z}$-basis of N: consider the exact sequence ($N' = \mathrm{coker}(\langle u_\rho \rangle \subseteq N)$)

$$0 \to \langle u_\rho \rangle \to N \to N' \to 0$$

Note that N' is torsion free and thus free (finitely generated abelian group), so the sequence splits and we have $N \cong \langle u_\rho \rangle \oplus N'$.

We may therefore assume that $\rho = \mathrm{Cone}(e_1)$, so that $\rho^{\vee} = \mathrm{Cone}(e_1, \pm e_2, \cdots, \pm e_n)$, so $k[S_\rho] = k[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$ and this is integrally closed.

$\square$

Remark 2.107. If S is integral but not saturated then it has a saturation S^{sat} given by $\{m \in S^{gp} \mid \exists k > 0, \ km \in S\}$. Note that

- $S \subseteq S^{sat} \subseteq S^{gp}$

- S^{sat} is finitely generated

- $(S^{sat})^{gp} = S^{gp}$

Moreover, the inclusion $k[S] \to k[S^{sat}]$ gives the "normalization" $\mathrm{Spec}\, k[S^{sat}] \to \mathrm{Spec}\, k[S]$

Example 2.108. Let $S = \langle 2, 3 \rangle \subseteq \mathbb{N}$ and note that $S^{sat} = \mathbb{N}$. Recall that $\langle 2, 3 \rangle = \langle p, q \mid 3p = 2q \rangle$, so

$$k[S] = \frac{k[x, y]}{(x^3 - y^2)}$$

and $C = \mathrm{Spec}\, k[S]$ is the cuspidal cubic in $\mathbb{A}^2$ (not normal variety). The normalizaion of this is

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & C \subseteq \mathbb{A}^2 \\ t & \longmapsto & (t^2, t^3) \end{array}$$

Example 2.109. Consider the monoid $S = \langle (2,0), (1,1), (0,2) \rangle \subseteq \mathbb{Z}^2$. We know that $\mathrm{Spec}\, k[S]$ is a normal variety, but the monoid does not "look" saturated. For example, $(0,1) \in \mathbb{Z}^2 \setminus S$ but $2(0,1) = (0,2) \in S$. The issue is that S^{gp} is smaller than $\mathbb{Z}^2$ and $(0,1) \notin S^{gp}$.

2.6.2 Smoothness

Remark 2.110. Recall that $T_x X = (\mathfrak{m}_x / \mathfrak{m}_x^2)^{\vee}$ in general. If $X \subseteq \mathbb{A}^n$ as $V(I)$ with $I = (f_1, \cdots, f_s)$ then $T_x X$ is defined by the linear equations $0 = d_x(f_i) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(x) x_j$ with $1 \le i \le n$.

Definition 2.111. An irreducible affine variety $X = \operatorname{Spec} A$ is **smooth** if $\dim T_x X = \dim X$ for all $x \in X$.

Fact 2.112 (Jacobian criterion). An irreducible $X = V(f_1, \cdots, f_s) \subseteq \mathbb{A}^n$ of dimension d is smooth at $x \in X$ if and only if

$$\operatorname{rnk}\left(\frac{\partial f_i}{\partial x_j}(x)\right) = n - d.$$

We will see that an affine toric variety U_σ is smooth if and only if σ is a *smooth cone*:

Definition 2.113. A rational strongly convex cone $\sigma \subseteq N_{\mathbb{R}}$ is

- **smooth** (or **regular**) if the minimal ray generators of σ are part of a $\mathbb{Z}$-basis of N

- **simplicial** if the minimal ray generators are $\mathbb{R}$-linearly independent in $N_{\mathbb{R}}$.

Example 2.114. The cone $\mathbb{R}_{\ge 0}^k \subseteq \mathbb{R}^n$ is smooth. Moreover, all smooth cones are of this form up to the action of some element of $\operatorname{GL}(\mathbb{Z}, n)$.

Example 2.115. The cone $\sigma = \operatorname{Cone}((1,0),(1,2))$ is simplicial because $(1,0)$ and $(1,2)$ are linearly independent, but $(1,0)$ and $(1,2)$ cannot be part of a basis for σ because the element $(1,1)$ would never be reached despite being in the cone.

Example 2.116. The cone $\operatorname{Cone}((1,0,0),(0,1,0),(1,0,1),(0,1,1)) \subseteq \mathbb{R}^4$ is not simplicial because it has 4 minimal ray generators.

Remark 2.117. Points of $\operatorname{Spec} k[S]$ are in bijection with homomorphisms of monoids $S \to (k, \cdot)$:

$$\{\text{points of } \operatorname{Spec} k[S]\} \overset{NSS}{=} \{\text{max. ideals of } k[S]\} =$$
$$= \{\text{surjections of } k\text{-algebras } k[S] \to k\} =$$
$$= \{\text{monoid homomorphisms } S \to (k, \cdot)\}$$

where the last equality works because it amounts to choosing a value in k for each $s \in S$ (or equivalently $t^s \in k[S]$) which is compatible with the operations. The surjectivity works because $S \to (k, \cdot)$ being a homomorphism means that 0 goes to 1 and so the corresponding k-alegbra homomorphism has 1 in the image, making the map surjective.

Lemma 2.118. The action of T_N on $\operatorname{Spec} k[S]$ has a fixed point if and only if S is sharp. In this case there is exactly one fixed point, which corresponds to $S \to (k, \cdot)$ given by $s \mapsto \begin{cases} 0 & \text{if } s \neq 0 \\ 1 & \text{if } s = 0 \end{cases}$

Proof.

If $p \in \operatorname{Spec} k[S]$ corresponds to $\gamma : S \to (k, \cdot)$ and we fix $a \in T_N$, let us compute $a \cdot p$: recall that the action is described by

$$
\begin{array}{ccc}
k[S] & \longrightarrow & k[M] \otimes k[S] \\
t^s & \longmapsto & t^s \otimes t^s
\end{array}
$$

and so it maps $(a, p) \in T_N \times X$ to the point which corresponds to $k[S] \to k$ given by

$$
k[S] \longrightarrow k[M] \otimes k[S] \longrightarrow k \otimes k = k
$$

$$
t^s \longmapsto t^s \otimes t^s \longmapsto \chi^s(a)\gamma(s)
$$

so the homomorphism $\gamma' : S \to (k, \cdot)$ which corresponds to $a \cdot p$ is given by $\gamma'(s) = \chi^s(a)\gamma(s)$.

The point is fixed if $\chi^s(a)\gamma(s) = \gamma(s)$ for all $a \in T_N$, $\in S$. For $s = 0$ $\gamma(s) = 1$ ok because it has to be a homomorphism, for $s \neq 0$ this implies that $\gamma(s) = 0$ in k (because $\exists a \in T_N$ such that $\chi^s(a) \neq 1$), so the only possible γ is the one in the statement, which is a homomorphism if and only if S is sharp. $\qquad \square$

Remark 2.119. The point in the statement of the lemma above can be thought of as the "most singular point of X".

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