Toric Varieties - Geometria Algebrica F

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Contents

Introduction

Le note sono in inglese per coerenza con la lingua in cui il corso è stato tenuto.

What is the course about?

The first part of the course deals with:

- Algebraic Tori, their actions and representations
- Affine toric varieties (with monoids) $\leftrightarrow$ cones in some $\mathbb{R}^n$
- Projective toric varieties $\leftrightarrow$ polytopes in some $\mathbb{R}^n$
- General toric varieties $\leftrightarrow$ fans in $\mathbb{R}^n$

We will then deal with (suject to change)

- Divisors/line bundles on toric varieties
- Cox ring of a toric variety
- Cohomology of divisors
- Toric morphisms and resolution of singularities
- and more!

Let us set some ground assumptions: we'll work over an algebraically closed field (and we will be lax about the characteristic of the field). The main reference for the course (Cox, Little, Schenk "Toric varieties") works over $\mathbb{C}$ but a lot of things work more generally.

Definition 0.1 (Toric variety)**.** An n-dimensional toric variety X is a (normal) k-variety equipped with an open immersion of an n-dimensional torus $T \subseteq X$, where $T \cong (k^*)^n$, and an action $T \times T \to T$ which extends to the whole of X [a].

[a] that is, it extends to a $T \times X \to X$

Remark 0.2. Normality is a standard assumption that we'll make at some point but some things work without it.

We'll see that the geometry of such an object is encoded in a combinatorial gadgets, reducing problems in algebraic geometry to problems in combinatorics, which is sometimes convenient.

Sometimes the opposite happens and results in combinatorics can be proven using algebraic geometry

Example 0.3 (McMullen's "g-conjecture")**.** The theorem is a characterization of the f-vectors of simple polytopes[1].

Definition 0.4 (f-vectors)**.** If P is a polytope, its ***f*-vector** is

$$(f_0(P), \cdots, f_d(P)), \quad \text{where } d = \dim P$$

and $f_i(P)$ is the number of i-dimensional faces. We may set $f_{-1}(P) = 1$.

It's reasonable to ask ourselves which f-vectors can appear. We may define the h-vector by setting

$$\sum_{i=0}^{d} f_i(t-1)^i = \sum_{i=0}^{d} h_i t^i, \quad \text{i.e. } h_i = \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} f_j, \; h_{-1} = 0$$

It is a theorem that the h-vector of a simple polytope is palindromic ($h_i = h_{d-i}$).

We obtain the ***g*-vector** by setting $g_i = h_i - h_{i-1}$. The conjecture was that

Theorem 0.5 (g-conjecture)**.** $f = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1}$ is the f-vector of a simple polytope if

1. $h_i = h_{d-1}$ for all $0 \leq i \leq \lfloor d/2 \rfloor$
2. $g_i \geq 0$ for all $0 \leq i \leq \lfloor d/2 \rfloor$
3. $(g_1, \cdots, g_{\lfloor d/2 \rfloor})$ is a "Macauly vector" if, when we write (uniquely)

$$g_i = \binom{n_i}{i} + \cdots + \binom{n_{r_i}}{r_i}$$

with $n_i > n_{i-1} > \cdots > n_{r_i}$ then

$$g_{i+1} \leq \binom{n_i + 1}{i+1} + \cdots + \binom{n_{r_i} + 1}{r_i + 1}$$

Stanley proved necessity using toric varieties. He proved that the g-vector of a simple polytope is the vector of dimensions for some cohomology ring of the associated toric variety.

Later McMullen found a completely combinatorial proof.

[1] for now, convex hull of a finite subset of $\mathbb{R}^n$

Part I

Basic theory of toric varieties

Chapter 1

Algebraic tori and their actions

1.1 Basic definitions

Definition 1.1 (Algebraic group)**.** An **algebraic group** G is a k-variety equipped with the the structure of a "group object" in the category of k-varieties, i.e. we have two morphisms and a *closed* point

$$m : G \times G \to G, \quad i : G \to G, \quad e \in G$$

that satisfy the usual group axioms "diagrammatically".

Example 1.2. Associativity can be expressed "diagrammatically" as

$$\begin{tikzcd} G \times G \times G \arrow[r, "(id_G, m)"] \arrow[d, "(m, id_G)"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

Definition 1.3 (Multiplicative group)**.** The **multiplicative group**, denoted $\mathbb{G}_m$, is the k-variety $\mathbb{A}^1 \setminus \{0\}$ equipped with the morphisms

$$m : \begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ (a, b) & \longmapsto & ab \end{array}$$

$$i : \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ a & \longmapsto & 1/a \end{array}$$

$$e = 1 \in \mathbb{A}^1 \setminus \{0\}$$

(we are identifying $\mathbb{G}_m = k^*$).

Remark 1.4. $\mathbb{G}_m$ is affine: $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and $\mathbb{A}^1 \setminus \{0\} = \mathbb{A}^1 \setminus V(x) = D(x)$, thus $D(x) = \operatorname{Spec}(k[x])_x = \operatorname{Spec}(k[x, x^{-1}]) = \operatorname{Spec} k[x^{\pm 1}]$.

Remark 1.5. All affine algebraic groups can be described dually as spectra of **Hopf algebras**

Example 1.6. $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ can be described as the map corresponding to

$$\begin{array}{ccc} k[x^{\pm 1}] & \longrightarrow & k[y^{\pm 1}] \otimes_k k[z^{\pm 1}] \\ x & \longmapsto & y \otimes z \end{array}$$

the inverse corresponds to

$$\begin{array}{ccc} k[x^{\pm 1}] & \longrightarrow & k[y^{\pm 1}] \\ x & \longmapsto & y^{-1} \end{array}$$

and the neutral element corresponds to[1]

$$\begin{array}{ccc} k[x^{\pm 1}] & \longrightarrow & k \\ x & \longmapsto & 1 \end{array}$$

Remark 1.7. In general, if $G = \operatorname{Spec} A$ is an affine variety, a structure of algebraic group is equivalent to a structure of Hopf algebra on A:

$$\begin{array}{rcl} m : G \times G \to G & \longleftrightarrow & \Delta : A \to A \otimes_k A \\ i : G \to G & \longleftrightarrow & S : A \to A \\ e : \operatorname{Spec} k \to G & \longleftrightarrow & \varepsilon : A \to k \end{array}$$

Remark 1.8. If G and H are algebraic groups, $G \times H$ is also naturally an alegbraic group.

Definition 1.9 (Algebraic tori)**.** The **standard n-dimensional algebraic torus over** k is $\mathbb{G}_m^n$. An **algebraic torus** is an algebraic groups T which is isomorphic to $\mathbb{G}_m^n$ for some n.
We may omit the adjective "algebraic" when appropriate.

Remark 1.10. If $k = \mathbb{C}$ then $\mathbb{G}_m^n = (\mathbb{C}^*)^n$, which is homotopy equivalent to $(S^1)^n$. This $(S^1)^n$ is the "maximal compact subgroup".

1.2 Cartier duality

In some sense which we will make precise, tori are "dual" to finitely generated torsion-free (and thus free) abelian groups.

Definition 1.11 (Associated group algebra)**.** If M is a finitely generated abelian group, the **k-group algebra of** M, denoted by $k[M]$, is the freely generated k-vector space with formal basis $\{t^m \mid m \in M\}$ and multiplication induced by $t^m t^{m'} = t^{m+m'}$.

[1] recall that a k-point e of the variety G can be seen as a morphism $\operatorname{Spec} k \to G$ with set-theoretic image e.

Example 1.12. If $M = \mathbb{Z}^n$ then

$$k[\mathbb{Z}^n] = k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}],$$

which is the coordinate ring of $(\mathbb{G}_m)^n$.

Moreover, the group structure of $\mathbb{G}_m^n$ is given by

$$\Delta : \begin{array}{ccc} k[\mathbb{Z}^n] & \longrightarrow & k[\mathbb{Z}^n] \otimes_k [\mathbb{Z}^n] \\ t^m & \longmapsto & t^m \otimes t^m \end{array}$$

$$S : \begin{array}{ccc} k[\mathbb{Z}^n] & \longrightarrow & k[\mathbb{Z}^n] \\ t^m & \longmapsto & t^{-m} \end{array}$$

$$\varepsilon : \begin{array}{ccc} k[\mathbb{Z}^n] & \longrightarrow & k \\ t^m & \longmapsto & 1 \end{array}$$

Fact 1.13. These formulas give a Hopf algebra structure on $k[M]$ for all abelian groups M

$$\Delta : \begin{array}{ccc} k[M] & \longrightarrow & k[M] \otimes_k [M] \\ t^m & \longmapsto & t^m \otimes t^m \end{array}$$

$$S : \begin{array}{ccc} k[M] & \longrightarrow & k[M] \\ t^m & \longmapsto & t^{-m} \end{array}$$

$$\varepsilon : \begin{array}{ccc} k[M] & \longrightarrow & k \\ t^m & \longmapsto & 1 \end{array}$$

Remark 1.14. $k[M]$ is finitely generated and reduced, so there is a (classical) affine variety $D(M) := \operatorname{Spec} k[M]$ which inherits the structure of an algebraic group.

Definition 1.15 (Cartier dual)**.** If M is a finitely generated abelian group, $D(M)$ is the **cartier dual** of M.

Example 1.16. If $M = \mathbb{Z}/n\mathbb{Z}$ then the group algebra is

$$k[\mathbb{Z}/n\mathbb{Z}] = \frac{k[t]}{(t^n - 1)}.$$

$\operatorname{Spec} k[\mathbb{Z}/n\mathbb{Z}]$ then is the closed subvariety (and subgroup) of $\mathbb{G}_m$ described by the equation $t^n = 1$, i.e. the group of the n-th roots of unity μ_n

Definition 1.17 (Group of n-th roots of unity)**.** $\mu_n = D(\mathbb{Z}/n\mathbb{Z})$.

Remark 1.18. If $n = p = \operatorname{char} k$ then $(t^p - 1) = (t - 1)^p$, so μ_p would be a point. To get any interesting geometric information in this case you need to allow nilpotens and you end up with a group scheme.

Exercise 1.19. $D(M \oplus N) = D(M) \times D(N)$.

For a general finitely generated abelian group

$$M = \mathbb{Z}^n \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$$

we get

$$D(M) \cong \mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_k}.$$

Remark 1.20. GL_n is an algebraic group, indeed $\mathrm{GL}_n \subseteq \mathbb{A}^{n^2}$ and we can give it the structure of a variety by seeing it as the principal open subset associated to the determinant (seen as a regular function on $\mathbb{A}^{n^2}$). Matrix multiplication and inversion can be checked to be morphisms.

Definition 1.21 (Diagonizable group)**.** An algebraic group is called **diagonalizable** if it is isomorphic to a (closed) subgroup of $\mathrm{Diag}_n \subseteq \mathrm{GL}_n$ for some n

Remark 1.22. $\mathrm{Diag}_n \cong \mathbb{G}_m^n$

Fact 1.23. We have an equivalence of categories

$$D : (fin.gen.AbGps) \to (Diagonalizable.AlgGroups)$$