Stacks and intersection theory

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Contents

Part I

Stacks

Chapter 1

Grothendieck topologies

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The references for this chaper are [Vis07] and [Ols16].

1.1 Grothendieck topology and sheaves

Recall that a presheaf on X is a functor

$$F : \text{Open}(X)^{op} \to \text{Set}$$

and that a sheaf is a presheaf that respects certain gluing properties over open covers.

As we will see in the next chapters, it will be useful to try and define sheaves over more general categories, that is, we want to substitute $\text{Open}(X)$ with some other category. The only topological notions which come up when we want to define sheaves are open covers of open subsets of X, so if we want to generalize sheaves we need define what "open covers" for objects $X \in \mathcal{C}$ should be.

Definition 1.1. Let $\mathcal{C}$ be a category. We define a **Grothendieck topology** on $\mathcal{C}$ to be the following data: for all $U \in \mathcal{C}$ we have a collection $\text{Cov}(U)$ of sets of arrows $\{U_i \to U\}$ in $\mathcal{C}$, called **covers of** U, and these covers respect the following properties:

1. If $V \to U$ is an isomorphism in $\mathcal{C}$ then $\{V \to U\} \in \text{Cov}(U)$.

2. If $\{U_i \to U\} \in \text{Cov}(U)$ and $V \to U$ arrow in $\mathcal{C}$ then we can fix fiber products $U_i \times_U V \to V$ such that $\{U_i \times_U V \to V\} \in \text{Cov}(V)$.

3. If $\{U_i \to U\} \in \text{Cov}(U)$ and for all i we have $\{V_{ij} \to U_i\} \in \text{Cov}(U_i)$ then $\{V_{ij} \to U\} \in \text{Cov}(U)$, where the arrows $V_{ij} \to U$ are given by the composition $V_{ij} \to U_i \to U$.

A category with a fixed Grothendieck topology is called a **site**.

Remark 1.2. According to [GD67], what we have defined is called a *pretopology*, a full Grothendieck topology is a looser notion which consideres pretopologies equivalent

when they give the same sheaf theory. We are going to ignore this subtelty from now on. For a little more detail on this distinction you may also look at [Vis07].

Example 1.3. Let X be a topological space and let $\mathrm{Open}(X)$ be the category of its open subsets (the arrows are inclusions). The category $\mathrm{Open}(X)$ with the usual notion of open covers is a site:

1. If $V \to U$ is an isomorphism then we have an inverse $U \to V$, but the arrows are inclusions so $V = U$ and of course U is an open cover of U.

2. Let $\{U_i\}$ be an open cover of U and let V is another open set in X which is contained in U. Note that $V \times_U U_i = V \cap U_i$ because of how fibered products work in $\mathrm{Open}(X)$. We conclude the proof of this property by noting that indeed $\{V \cap U_i\}$ is an open cover of V if $V \subseteq U$.

3. Let $\{U_i\}$ be an open cover of U and for all i let $\{V_{ij}\}$ be an open cover of U_i, then it is clear that $\{V_{ij}\}$ (now with i variable) is an open cover of U.

Notation. If $\{U_i \to U\}$ is a cover of U with respect to some Grothendieck topology, we take U_{ij} to be some choice for $U_i \times_U U_j$ and similarly for U_{ijk}. From now on we will implicitly fix such fibered products.

We now move on to the general definition of presheaves, separated presheaves and sheaves:

Definition 1.4. Let $\mathcal{C}$ be a category. A **presheaf** on $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. If $f : V \to U$ is an arrow in $\mathcal{C}$ and $\xi \in F(U)$ for F presheaf, we define the **pullback of ξ along f** (also called the **restriction of ξ to V**) to be

$$F(f)(\xi) \in F(V).$$

The pullback of ξ may be written as $f^*\xi$ or $\xi|_V$ when no ambiguity may arise.

Suppose now that $\mathcal{C}$ is also a site, then a presheaf F on $\mathcal{C}$ is a

- **separated presheaf** if for all $U \in \mathcal{C}$, all $\{u_i : U_i \to U\} \in \mathrm{Cov}(U)$ and all pairs of elements $\xi, \eta \in F(U)$, we have that

$$(\forall i \quad \xi|_{U_i} = \eta|_{U_i}) \implies \xi = \eta.$$

- **sheaf** if for all $U \in \mathcal{C}$, all covers $\{U_i \to U\} \in \mathrm{Cov}(U)$ and all collections $\xi_i \in F(U_i)$ for each i such that $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$ for all i, j, there exists a unique $\xi \in F(U)$ such that $\xi|_{U_i} = \xi_i$.

Remark 1.5. One may define sheaves that take values in Groups, Rings and so on instead of just Sets. The next chapter will essentially define sheaves that take values in categories but we are getting ahead of ourselves.

Definition 1.6. If F, G are sheaves on $\mathcal{C}$, a **morphism between** them is a natural transformation of functors $\mathcal{C}^{op} \to \mathrm{Set}$.

Remark 1.7. A sheaf is in particular a separated presheaf.

Theorem 1.8. Let $\mathcal{C}$ be a site. The forgetful functor $\mathrm{Sh}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C})$ has a left adjoint called **sheafification**.

Proof.
A full proof is in [Ols16]. The main idea is to compose left adjoints $\mathrm{Pre}(\mathcal{C}) \to \mathrm{SepPre}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$. For the first piece we identify two elements of $F(U)$ when they agree on a cover of U, for the second we add all gluings. $\qquad\square$

1.2 Examples of Grothendieck topologies

As we have already noted, $\mathrm{Open}(X)$ for some topological space X defines a site when considering open covers in the classical sense. The first new site which we define is a way the *global* version of this

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Definition 1.9. Let $\{U_i \to U\}$ be a collection of functions to U. We say that these arrows are **jointly surjective** if the union of the images is U.

Example 1.10 (Global classical topology)**.** Let $\mathcal{C} = \mathrm{Top}$. For any topological space $X \in \mathrm{Top}$ we define $\mathrm{Cov}(X)$ to be the collection of jointly surjective sets of open immersions into X, that is, $\{j_i : U_i \to X\} \in \mathrm{Cov}(X)$ if $\bigcup_i \mathrm{Imm}\, j_i = X$ and all j_i are open immersions.

Remark 1.11. We can't define $\mathrm{Cov}(X) = \{$open covers of $X\}$ in the previous example because, for instance, the first property in the definition of Grothendieck topology fails.

We have a more algebro-geometric version of this example given by

Example 1.12 (Global Zariski topology)**.** Let $\mathcal{C} = \mathrm{Sch}/S$. If X is a scheme over S then we define $\mathrm{Cov}(X)$ to be the collection of jointly surjective sets of open embeddings of schemes over S.

So far these sites have not been too out of the ordinary, but now we start to get to more sofisticated, yet useful, examples

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Definition 1.13. Let $f : X \to Y$ be a morphism of schemes. We say that f is **formally unramified** (resp. **formally smooth** / **formally étale**) if for all affine Y-schemes $Y' \to Y$ and all closed embeddings $Y_0' \to Y'$ defined by nilpotent ideals, then map
$$\mathrm{Hom}(Y', X) \to \mathrm{Hom}(Y_0', X)$$
is injective (resp. surjective / bijective).
Moreover, if f is also locally of finte presentation then we say that f is **unramified** / **smooth** / **étale** when the same conditions hold.

With these definitions we can define

Example 1.14 (Small smooth / étale topology). Let $\widetilde{\mathcal{C}} = \mathrm{Sch}/S$ and let $\mathcal{C}$ be the full subcategory with objects given by arrows $X \to S$ which are smooth / étale. If $X \in \mathcal{C}$ then we can define $\mathrm{Cov}(X)$ to be jointly surjective sets of maps which are all smooth / étale.

Example 1.15 (Big étale topology). Let $\mathcal{C} = \mathrm{Sch}/S$. If $X \in \mathcal{C}$ then we define $\mathrm{Cov}(X)$ to be jointly surjective sets of étale maps.

1.2.1   The fppf and fpqc topologies

The classical and étale topologies are already useful, but they are not the gold-standard for many applications.

For some motivation, consider the following problem: recall that if $f : X \to Y$ is a morphism of schemes and $\mathcal{P}$ is some property of this morphism, we say that the property is **stable under base change** if for all $Y' \to Y$, when we build a cartesian square

$$
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

then if f has property $\mathcal{P}$ it follows that f' also has property $\mathcal{P}$.

We want to explore the converse in some sense, that is: suppose we have $f : X \to Y$ and fix a cover $\{Y_i \to Y\}$. If all restrictions $X \times_Y Y_i \to Y_i$ have property $\mathcal{P}$, when is it the case that f itself has property $\mathcal{P}$?

The covers in the classical and étale topolgies make it "too easy" to have this "local to global" condition be true, to the point where it is not easy at all to check that the property holds locally. To have this local-to-global principle being useful, we may require finer topologies. The fppf and fpqc are the usual answers when such questions arise.

Definition 1.16 (fppf topology). Let $\mathcal{C} = \mathrm{Sch}/S$. A cover for $X \in \mathcal{C}$ in the **fppf topology** (it stands for *faithfully flat and locally of finite presentation*) is a jointly surjective collection of flat maps which are locally of finite presentation.

Definition 1.17. A morphism $f : X \to Y$ is **fpqc** (it stands for *faithully flat and quasi-compact*) if it is flat, surjective and every quasi-compati open subset of Y is the image of a quasi-compact open subset of X.

Remark 1.18. Equivalent definitions of fpqc are given in [Vis07].

In order to define the fpqc topology we require the following proposition, which you can find in [Vis07].

Proposition 1.19. The property of being fpqc is stable under composition and base change. Moreover, if $f : X \to Y$ and $\{V_i \to Y\}$ is an open cover (for the classical topology) such that $f|_{f^{-1}(V_i)} : f^{-1}(V_I) \to V_i$ is fpqc for all i then f is fpqc.

Definition 1.20 (fpqc topology)**.** Let $\mathcal{C} = \mathrm{Sch}/S$. A cover for $X \in \mathcal{C}$ in the **fpcq topology** is a collection of scheme morphisms $\{U_i \to X\}$ such that the induced map $\coprod U_i \to X$ is fpqc.

Remark 1.21. The topologies that we have defined have become gradually finer, that is

$$\mathrm{Zariski} \subseteq \text{étale} \subseteq \mathrm{fppf} \subseteq \mathrm{fpqc}$$

We conclude the chapter by stating that the fpqc and fppf topologies to give an answer to the local-to-global question we started with:

Proposition 1.22. Let $f : X \to Y$ be a morphism of schemes, $\{Y_i \to Y\}$ be an fpqc cover of Y and f_i be the pullbacks $f_i : X \times_Y Y_i \to Y_i$. It is the case that if all f_i have property $\mathcal{P}$ then f has $\mathcal{P}$ for $\mathcal{P}$ in the following (non-exhaustive) list

- separable
- surjective
- locally of finte type
- locally of finite presentation
- proper
- flat

- affine
- finite
- smooth
- unramified
- étale

Remark 1.23. An analogous result holds for fppf where the list consists of properties of the form "is an immersion" (for example open, closed and locally closed immersions).

Chapter 2

Fibered categories and Stacks

Speaker: Francesco Sorce

Everything in this chapter is talked about in greater detail in [Vis07].
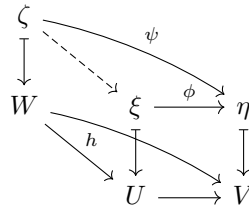
2.1 Fibered categories

2.1.1 Definition

> **Definition 2.1.** If we have the data of $\mathcal{F}$ and $\mathcal{C}$ two categories and $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ a functor we say that $\mathcal{F}$ is **over** $\mathcal{C}$.
> In this context, if $p_{\mathcal{F}}(\xi) = U$ for some object ξ we say that ξ is over U.
> Similarly if $p_{\mathcal{F}}(\xi \to \eta) = U \to V$ then $\xi \to \eta$ is over $U \to V$.

Notation. In the diagrams that follow, a normal or dashed arrow will be a morphism, while an allow with a tail (like $\mapsto$) represents applying the functor.

> **Definition 2.2.** An arrow $\phi : \xi \to \eta$ in $\mathcal{F}$ over $\mathcal{C}$ is **cartesian** if for all $\phi : \zeta \to \eta$ and all $h : p_{\mathcal{F}}\zeta \to p_{\mathcal{F}}\xi$ which make the following diagram commute there exists a unique arrow $\zeta \to \xi$ in place of the dotted arrow:
>
>
>
> If $\xi \to \eta$ is cartesian and over $f : U \to V$ we say that ξ is the **pullback of** η **along** f. We may write $\xi = f^*\eta$ if we fix a choice of pullback.

Definition 2.3. Let us fix a category $\mathcal{C}$. The associated **arrow category** $\text{Arr}(\mathcal{C})$ is the category whose objects are arrows in $\mathcal{C}$ and whose morphisms are commutative squares in $\mathcal{C}$.
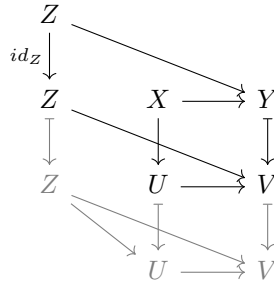
Remark 2.4. We may view $\text{Arr}(\mathcal{C})$ as a category over $\mathcal{C}$ by fixing the functor that to an arrow $X \to U$ assigns U.

Example 2.5. In order to better understand cartesian arrows (and to see where the name comes from) we determine which arrows in $\text{Arr}(\mathcal{C})$ are cartesian.

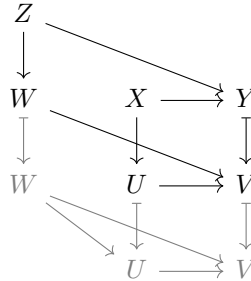If an arrow $(X \to U) \to (Y \to V)$ is cartesian, consider the following data:

$$
\begin{array}{ccc}
Z & & \\
& \searrow & \\
& X \longrightarrow Y & \\
& \downarrow \quad\quad \downarrow & \\
& U \longrightarrow V &
\end{array}
$$
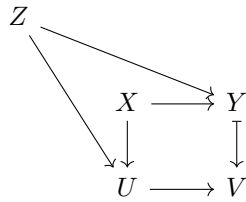
which we may equivalently rewrite

$$
\begin{array}{ccc}
Z & & \\
id_Z \downarrow & \searrow & \\
Z & X \longrightarrow Y & \\
\downarrow & \downarrow \quad\quad \downarrow & \\
Z & U \longrightarrow V & \\
& \downarrow \quad\quad \downarrow & \\
& U \longrightarrow V &
\end{array}
$$

Since the arrow is cartesian, there is a unique way to complete the diagram above with maps $Z \to X$ and $Z \to U$. Since we ask for compatibility with the functor, $Z \to U$ is already known, so we have constructed $Z \to X$ in the first diagram. This makes us suspect that cartesian arrows in $\text{Arr}(\mathcal{C})$ correspond to cartesian squares in $\mathcal{C}$, and this is true.

Indeed, suppose that $(X \to U) \to (Y \to V)$ is a cartesian square and let us consider the data

$$
\begin{array}{ccc}
Z & & \\
\downarrow & \searrow & \\
W & X \longrightarrow Y & \\
\downarrow & \downarrow \quad\quad \downarrow & \\
W & U \longrightarrow V & \\
& \downarrow \quad\quad \downarrow & \\
& U \longrightarrow V &
\end{array}
$$

then we have a unique way to complete the diagram. The bottom map is determined by compatibility with the functor towards $\mathcal{C}$ and the top arrow can be chosen to be the one induced by the fact that we have a cartesian square an maps $Z \to Y$, $Z \to U$, where the second one is the composition $Z \to W \to U$.

Fact 2.6. Cartesian arrows satisfy the following properties:

1. the composition of cartesian arrows is cartesian

2. if $\xi \to \zeta$ factors through η with $\eta \to \zeta$ being cartesian, then $\xi \to \zeta$ is cartesian if and only if $\xi \to \eta$ is

3. if ϕ is over an isomorphism then ϕ is cartesian if and only if it is an isomorphism

Definition 2.7. A category $\mathcal{F}$ over $\mathcal{C}$ is **fibered over** $\mathcal{C}$ if for all $f : U \to V$ in $\mathcal{C}$ and all $\eta \in \mathcal{F}$ over V there exists a cartesian arrow $\phi : \xi \to \eta$ over f. That is, given a "partial diagram" of the form

$$
\begin{array}{c}
\eta \\
\Big\downarrow \\
U \xrightarrow{\ f\ } V
\end{array}
$$

we can make it a square in such a way that the top side is a cartesian arrow.

Definition 2.8. Let $\mathcal{F}$ and $\mathcal{G}$ be fibered categories over $\mathcal{C}$. A **morphism of fibered categories over** $\mathcal{C}$ from $\mathcal{F}$ to $\mathcal{G}$ is a functor $F : \mathcal{F} \to \mathcal{G}$ such that $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$ which preserves cartesian arrows, i.e. if $\phi \in \mathcal{F}$ cartesian then $F(\phi)$ is cartesian in $\mathcal{G}$.

Example 2.9. Because of what we have already said, $\mathrm{Arr}(\mathcal{C})$ is fibered over $\mathcal{C}$ if and only if $\mathcal{C}$ admits fibered products.

2.1.2 Fibers

We now reach one of the main definitions which motivate fibered categories

Definition 2.10. Let $\mathcal{F}$ be a fibered over $\mathcal{C}$ and fix an object $U \in \mathcal{C}$. The **fiber of** $\mathcal{F}$ **over** U is the subcategory $\mathcal{F}(U)$ of $\mathcal{F}$ whose objects are the objects of $\mathcal{F}$ over U and whose morphisms are those over id_U.

Example 2.11. Fix $U \in \mathcal{C}$. The fiber $\mathrm{Arr}(\mathcal{C})(U)$ is the comma category $\mathcal{C}/U$.

Remark 2.12. If $f : U \to V$ is a morphism in $\mathcal{C}$ and $\mathcal{F}$ is fibered over $\mathcal{C}$ then we may define a "restriction functor"

$$
\begin{array}{ccc}
\mathcal{F}(V) & \longrightarrow & \mathcal{F}(U) \\
X & \longmapsto & f^*X
\end{array}
$$

upon choosing a unique pullback for each element. Such a choice is called a **cleavage**. For more details see [Vis07].

Remark 2.13. If $F : \mathcal{F} \to \mathcal{G}$ is a morphism of fibered categories over $\mathcal{C}$ and $U \in \mathcal{C}$ then it induces a functor $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

These definitions make $\mathcal{F}$ look like a presheaf $\mathcal{C}^{op} \to \mathrm{Cat}$

This is not quite correct because choosing a cleavage may lead to having two arrows that should be the same being naturally isomorphic instead. To be technically correct we would need to introduce pseudo-functors, but they behave like normal functors in basically every way.

Fact 2.14. There is a correspondence between categories fibered over $\mathcal{C}$ and pseudo-functors $\mathcal{C}^{op} \to \mathrm{Cat}$.

Proof (Idea).
To get the functor from the category fix a cleavage, then take an object to the fiber over it and a map to the pullback along it.

To get a category from the functor consider the category with objects (U, X) for $U \in \mathcal{C}$ and $X \in \mathcal{F}(U)$ and morphisms defined in the obvious way. The functor from this category to $\mathcal{C}$ is the projection on the first factor. The fact that this comes from a pseudo-functor automatically gives the existence of pullbacks. $\qquad\square$

Definition 2.15. A category is called a **set** if its objects form a set and the only morphisms are of the form id_X for some object X in the category.

Definition 2.16. A category is called a groupoid if every arrow in the category is an isomorphism.

Definition 2.17. Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$. We say that $\mathcal{F}$ is

- **fibered in sets** if $\mathcal{F}(U)$ is a set for all $U \in \mathcal{C}$

- **fibered in groupoids** if $\mathcal{F}(U)$ is a groupoid for all $U \in \mathcal{C}$.

Fact 2.18. A cateogry is fibered in sets if and only if the associated pseudo-functor is a functor in the usual sense.

Example 2.19. Let $\mathcal{C}$ be a (locally small) category and let us fix $X \in \mathcal{C}$. We have a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ which maps X to $h_X = \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$. This functor defines a category fibered over $\mathcal{C}$. The objects of this category are pairs $(Y, Y \to X)$ (which we will identify with $Y \to X$ itself) and a morphism from $Y \to X$ to $Z \to X$ is an arrow $Y \to Z$ in $\mathcal{C}$ such that $Y \to X = Y \to Z \to X$.

The fiber of this category over $Y \in \mathcal{C}$ is $\{Y \to X\} = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ but seen as a category. Since arrows in a fiber must be over the identify of the object of which they

are the fiber, the only arrows in $\mathrm{Hom}(Y, X)$ are the identity of Y. This shows that the category we got from X is fibered in sets, which is what we expected because it came from a functor.

Schematically, we have the following embeddings and equivalences

$$\mathcal{C} \lhook\joinrel\longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \lhook\joinrel\longrightarrow \mathrm{Fibered}/\mathcal{C}$$

$$\text{Representable} \qquad \text{Fibered in sets}$$

Proposition 2.20. Let $\mathcal{F}$ be a category over $\mathcal{C}$ (we do not assume fibered). $\mathcal{F}$ is fibered in groupoids over $\mathcal{C}$ if and only if the following conditions hold:

1. every arrow in $\mathcal{F}$ is cartesian

2. given a partial diagram $f : U \to V$ and $\eta \in \mathcal{F}$ over V, the exists some $\phi : \xi \to \eta$ over f.

Sketch.
If $\mathcal{F}$ is fibered in groupoids, 2. holds by fiberedness and 1. is a consequence of point 2 in fact 2.6.

Viceversa, if 1. and 2. hold then $\mathcal{F}$ is fibered. Since all arrows in a fiber are over an isomorphism (the identity of the object we are taking the fiber), again by fact 2.6 we see that all arrows must be isomorphisms because they are cartesian. $\qquad\square$

2.2 Stacks

Now that we have explored the formalism of fibered categories, let us see how it interacts with the notion of Grothendieck topology we dealt with in the previous chapter.

2.2.1 Motivating example

Let us consider the arrow category associated to the category of topological spaces Arr(Top). We may interpret objects in this category to be *spaces lying above a base space*. In what follows it may be useful to imagine objects in this category to be things like covering maps or fiber bundles, though of course the notion of "continuous maps" is much more general.

Taking this as our point of view, two concepts we may want to explore are the following:

- Suppose we have two objects over the same base space and an open cover for that base. Under what conditions can we construct a map between them locally?

- Suppose we have an open cover and that for each element of the cover we have a space over it. Under what conditions can we glue these pieces to get an object over the entire base space?

Studying these questions will eventually lead us to what a prestack and a stack should be.

Recall that we have a standard Grothendieck topology on Top for which

$$\mathrm{Cov}(U) = \{\{U_i \to U\} \mid \text{jointly surjective open immersions}\}.$$

Notation. We will write U_{ij} instead of $U_i \cap U_j$ and similarly U_{ijk} for triple intersections.

Gluing maps

Suppose we fix $U \in \mathrm{Top}$ and an open cover[1] $\{U_i \to U\}$. We also fix two objects $\xi : X \to U$ and $\eta : Y \to U$. The answer to our first question can be summerized in the following proposition:

Proposition 2.21. Let U, $\{U_i\}$, ξ, η be as above. Suppose we have maps $f_i : \xi^{-1}(U_i) \to \eta^{-1}(U_i)$ for all i such that

$$f_i|_{\xi^{-1}(U_{ij})} = f_j|_{\eta^{-1}(U_{ij})},$$

then there exists a unique $f : X \to Y$ which is compatible with the maps to U and such that $f|_{\xi^{-1}(U_i)} = f_i$ for all i.

Notice that we may restate the proposition as

Proposition 2.22. The functor

$$\underline{\mathrm{Hom}}_U(X,Y) : \begin{array}{ccc} (\mathrm{Top}/U)^{op} & \longrightarrow & \mathrm{Set} \\ Z \to U & \longmapsto & \mathrm{Hom}_Z(X \times_U Z, Y \times_U Z) \end{array}$$

is a sheaf with respect to the Grothendieck topology induced by Top on Top$/U$.

To see the equivalence, note that $X \times_U U_i = \xi^{-1}(U_i)$ and $Y \times_U U_i = \eta^{-1}(U_i)$, so when you evaluate the functor on the open cover of U, the fact that it is a sheaf corresponds exactly to the gluing property.

Gluing spaces

We now look at what it means to glue spaces using a cover:

Proposition 2.23. Let us fix a topological space U together with an open cover $\{U_i\}$. For each i let $u_i : X_i \to U_i$ be an element of Arr(Top). We also fix isomorphisms $\phi_{ij} : u_j^{-1}(U_{ij}) \to u_i^{-1}(U_{ij})$ for each pair of indices in such a way so that

$$\phi_{ik}|_{u_k^{-1}(U_{ijk})} = \phi_{ij}|_{u_j^{-1}(U_{ijk})} \circ \phi_{jk}|_{u_i^{-1}(U_{ijk})}.$$

It follows that there exists $u : X \to U$ continuous, together with homeomorphisms $\phi_i : u^{-1}(U_i) \to X_i$ such that $\phi_{ij} = \phi_i|_{u^{-1}(U_{ij})} \circ (\phi_j|_{u^{-1}(U_{ij})})^{-1}$.

---

[1] in the standard sense. Everything works the same with jointly surjective open immersions but the notation gets bothersome rather quickly.

Remark 2.24. The notation is rather tedious, but the geometric image is actually rather simple: the ϕ_{ij} serve as a guide to tell us which points should be identified and the condition $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ (usually called a **cocycle condition**) ensures that these identifications define an equivalence realtion on $\coprod X_i$.
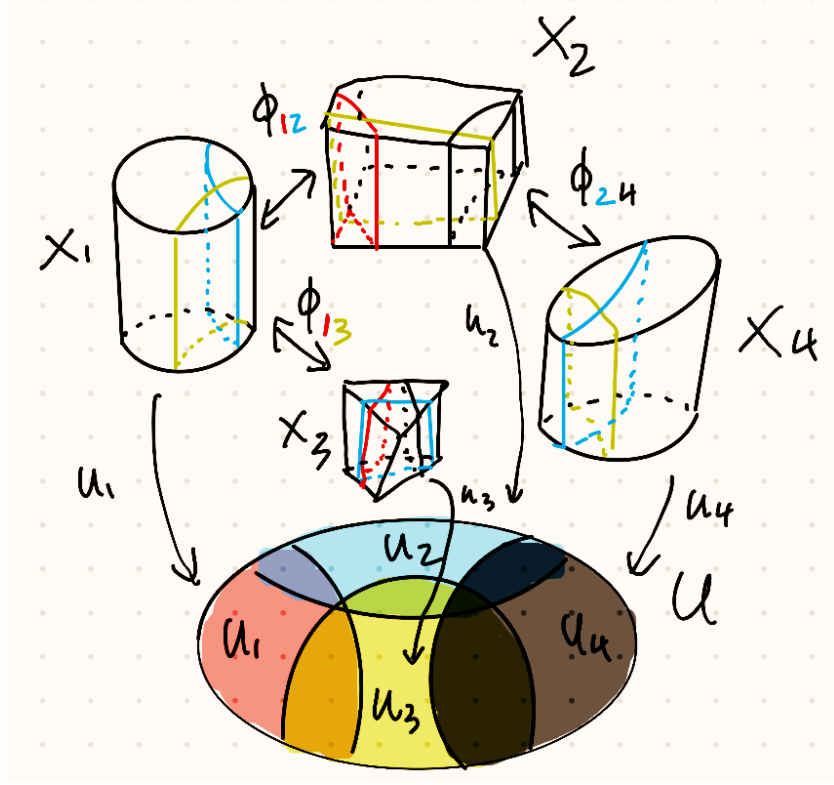
2.2.2 Descent data

We are now ready to move onto stacks. The main idea is to generalize the type of *local data* that we used to glue before.

For the rest of the chapter $\mathcal{C}$ is a site with a fixed Grothendieck topology and $\mathcal{F}$ is fibered category over $\mathcal{C}$.

Definition 2.25. Fix $U \in \mathcal{C}$ and a cover $\{U_i \to U\} \in \text{Cov}(U)$. An **object with descent data** is a pair $(\{\xi_i\}, \{\phi_{ij}\})$ where $\xi_i \in \mathcal{F}(U_i)$ for all i and $\phi_{ij} : \text{pr}_2^* \xi_j \to \text{pr}_1^* \xi_i$ are isomorphisms which live in $\mathcal{F}(U_{ij})$ such that

$$\text{pr}_{13}^* \phi_{ik} = \text{pr}_{12}^* \phi_{ij} \circ \text{pr}_{23}^* \phi_{jk}.$$

The ϕ_{ij} are called **transition isomorphisms**.

If you are (very reasonably) getting confused with the pullbacks and indices, it

may (or may not) be useful to admire *the hypercube*

$$
\begin{array}{c}
\text{pr}_1^*\xi_j \xleftarrow{\phi_{jk}} \text{pr}_2^*\xi_k \longrightarrow \xi_k
\end{array}
$$
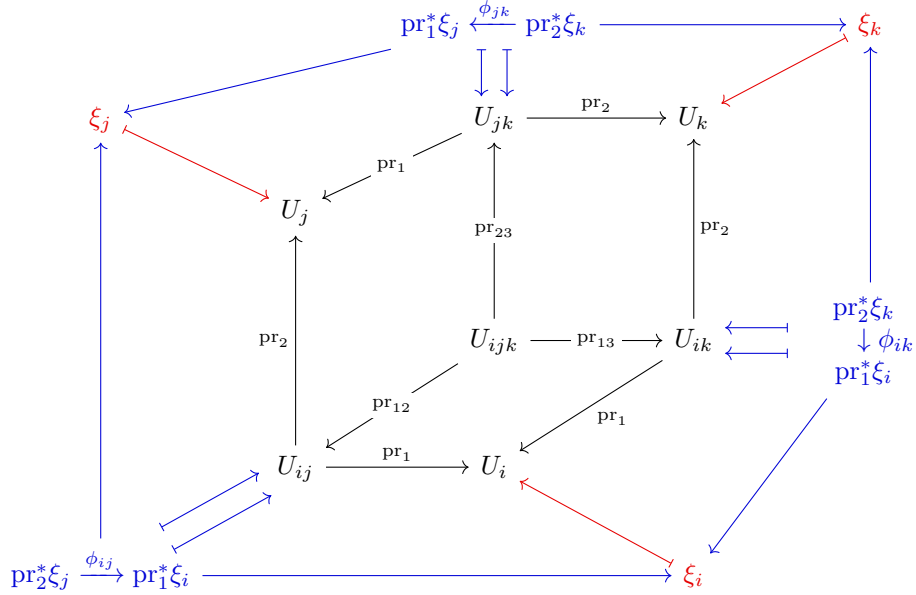
Notation. Let $\mathcal{C}$ be a site, $\mathcal{F}$ fibered over $\mathcal{C}$, $U \in \mathcal{C}$ and $\{U_i \to U\} \in \text{Cov}(U)$. Objects with descent data over $\{U_i \to U\}$ form a category[2], which we denote using

$$
\mathcal{F}(\{U_i \to U\}).
$$

Remark 2.26. Fix $U \in \mathcal{C}$, $\{u_i : U_i \to U\} \in \text{Cov}(U)$ and $\xi \in \mathcal{F}(U)$. From ξ we can obtain an object with descent data by restriction. To be more precise, we can define

$$
\xi_i = u_i^*\xi, \qquad \phi_{ij} : \text{pr}_2^* u_j^*\xi \to \text{pr}_1^* u_i^*\xi
$$

where ϕ_{ij} is the canonical isomorphism induced by the fact that both objects amount to the pullback of ξ along $U_{ij} \to U$. The resulting object with descent data is $(\{\xi_i\}, \{\phi_{ij}\})$.

What we have described is actually a functor from $\mathcal{F}(U)$ to $\mathcal{F}(\{U_i \to U\})$.

Definition 2.27. An object with descent data is called **effective** if it is in the essential image of the functor above.

We are now finally ready to define prestacks and stacks:

---

[2]the morphisms are collections of maps and commutative diagrams which are compatible with the indices.

Definition 2.28. Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$ site. $\mathcal{F}$ is a

- **prestack** if for all $U \in \mathcal{C}$ and all $\{U_i \to U\} \in \mathrm{Cov}(U)$, the functor

$$\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$$

  is *fully faithful*.

- **stack** if for all $U \in \mathcal{C}$ and all $\{U_i \to U\} \in \mathrm{Cov}(U)$, the functor

$$\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$$

  is an *equivalence*.

Remark 2.29. The two conditions can be intuitively understood as:

- A prestack is a fibered category where if two objects are the same locally (taking into account gluing) then they are isomorphic.

- A stack is a prestack where all descent data glues to an object.

Another way in which we can intepret the prestack condition is given in the following

Proposition 2.30. Let $\mathcal{F}$ be fibered over $\mathcal{C}$ site. $\mathcal{F}$ is a prestack if and only if for all $U \in \mathcal{C}$ and all $\xi, \eta \in \mathcal{F}(U)$, the functor

$$\underline{\mathrm{Hom}}_U(\xi, \eta) : \begin{array}{ccc} (\mathcal{C}/U)^{op} & \longrightarrow & \mathrm{Set} \\ f : Z \to U & \longmapsto & \mathrm{Hom}_Z(f^*\xi, f^*\eta) \end{array}$$

is a sheaf on $\mathcal{C}/U$.

Sketch.
Fix $U \in \mathcal{C}$ and $\{U_i \to U\} \in \mathrm{Cov}(U)$. The functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ being fully faithful means that for all $\xi, \eta \in \mathcal{F}(U)$ we have

$$\mathrm{Hom}_U(\xi, \eta) \cong \mathrm{Hom}_{\{U_i \to U\}}((\{\xi_i\}, \{\alpha_{ij}\}), (\{\eta_i\}, \{\beta_{ij}\}))$$

where $(\{\xi_i\}, \{\alpha_{ij}\})$ is the image of ξ and similarly for the other object and η.

Concretely, this means that maps $\xi \to \eta$ can be defined uniquely on an open cover, which is exactly what it means for $\underline{\mathrm{Hom}}_U(\xi, \eta)$ to be a sheaf. $\square$

Corollary 2.31. $\mathrm{Arr}(\mathrm{Top})$ is a stack.

We close the chapter by noting that when $\mathcal{F}/\mathcal{C}$ is fibered in sets, we find the notions of separated presheaf and sheaf over $\mathcal{C}$ respectively, meaning that we have, in some very precise sense, defined *sheaves in categories*. This concept will be trated in more detail in the next chapter.

Proposition 2.32. Let $\mathcal{C}$ be a site and consider a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. Identify F with the fibered category it defines. Then

- F is a prestack if and only if it is a separated presheaf,

- F is a stack if and only if F is a sheaf.

Example 2.33. The fibered category associated to an object $X \in \mathcal{C}$ is a stack because it corresponds to the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$.

Chapter 3

Algebraic Stacks

Speaker: Pietro Leonardini

3.1 Motivation: moduli problems

3.1.1 The dream

Let us recall what an elliptic curve is:

Definition 3.1 (Elliptic curve)**.** An **elliptic curve** over k is a 1-dimensional scheme $C \to \operatorname{Spec} k$ over k which is geometrically integral, proper, smooth and of genus[a] 1 with a fixed k-rational point $e : \operatorname{Spec} k \to C$.

---
[a]the genus is $\dim_k H^1(C, \mathcal{O}_C)$

Since the relative approach is central in the theory of schemes, we may want to define an elliptic curve over some base scheme S. The most sensible approach is to say that an elliptic curve over S should be a "family of elliptic curves parametrized by S". Formally:

Definition 3.2 (Elliptic curve over S)**.** An **elliptic curve** over S is a proper, smooth, flat morphism of schemes $p : E \to S$ with a fixed section $e : S \to E$ of p such that for all $\operatorname{Spec} \Omega$ geometric point, the pullback is an elliptic curve over Ω:

$$
\begin{array}{ccc}
E \times_S \operatorname{Spec} \Omega & \longrightarrow & E \\
\downarrow & & \downarrow \\
\operatorname{Spec} \Omega & \longrightarrow & S
\end{array}
$$

When studying elliptic curves it would be useful to find some space $M_{1,1}$ such that

for every family $E \to S$ we have a unique $\phi : S \to M_{1,1}$ such that

$$
\begin{array}{ccc}
E & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\ \phi\ } & M_{1,1}
\end{array}
$$

that is, we would be very happy if all families of elliptic curves could be described as the pullback of some *universal family* $\mathcal{E}$ over $M_{1,1}$ along some morphism $S \to M_{1,1}$. Intuitively, the universal family should be the one defined by the fact that the fiber over a point of $M_{1,1}$ is the elliptic curve corresponding to that point. The morphism should be the one that to each point of S assigns some elliptic curve, i.e. a point of $M_{1,1}$ (of course this naive view is too simple when dealing with schemes but it motivates why we look for such a map).

A useful way to formulate this problem is to define the following functor[1]

$$
F : \begin{array}{ccc}
\mathrm{Sch}^{op} & \longrightarrow & \mathrm{Set} \\
S & \longmapsto & \{E \to S \mid \text{elliptic curves}\} \,/\text{iso.} \\
f : T \to S & \longmapsto & (E_1 \to S) \mapsto (f^* E_1 \to T)
\end{array}
$$

We can re-formulate the requirements on $M_{1,1}$ by saying that $M_{1,1}$ should **represent** this functor, i.e. there is a natural isomorphism of functors

$$
h_{M_{1,1}} \cong F
$$

where $h_{M_{1,1}}$ is the contravariant Hom-functor associated to $M_{1,1}$.

We recall the following simple but foundational result

Theorem 3.3 (Yoneda lemma)**.** There is a natural correspondence

$$
\mathrm{Hom}(h_X, F) \leftrightarrow F(X)
$$

---

sketch.
Suppose we have a natural transformation $\phi : h_X \to F$, then we can find an element of $F(X)$ by taking $\phi_X(id_X) = \xi$.

Let us now fix an element $\xi \in F(X)$. Let T be a scheme. We may construct a map $\mathrm{Hom}(T, X) \to F(T)$ by sending $f : T \to X$ to $F(f)(\xi)$. If $g : T \to S$ is a morphism then $\eta : f \mapsto g \circ f$ is mapped to $F(g) : F(S) \to F(T)$. It is easy to check that this defines a natural transformation $h_X \to F$ and that the two constructions are inverses of each other. $\qquad\square$

Corollary 3.4. The functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ that sends X to h_X is fully faithful.

---

[1] we technically have no guarantee that families over a scheme should form a set. This issue will be solved automatically later when we consider a fibered category over Sch instead of a functor.

3.1.2 A good attempt

Let us try to find some way of parametrizing isomorphism classes of elliptic curves. The main idea comes from the following fact:

Fact 3.5. If char $k \neq 2, 3$ then every elliptic curve is isomorphic over $\overline{k}$ to some

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

for some $\lambda \in k \setminus \{0, 1\}$.

Definition 3.6 (*j*-invariant). Let

$$j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

Fact 3.7. The map

$$\phi : \begin{array}{ccc} \overline{k} \setminus \{0, 1\} & \longrightarrow & \overline{k} \\ \lambda & \longmapsto & j(E_\lambda) \end{array}$$

is surjective and $6 : 1$ except in $j = 0$ and $j = 1728$, where it is $2 : 1$ and $3 : 1$ respectively.

Fact 3.8. Two elliptic curves are isomorphic over $\overline{k}$ if and only if they have the same *j*-invariant.

Theorem 3.9. Let $E \to S$ be an elliptic curve, then there exists a Zariski affine open over of S given by $\{U_i = \operatorname{Spec} A_i\}$ such that for all i

$$\begin{array}{ccc} E_i := \operatorname{Spec}\left(\dfrac{A_i[x, y]}{(y^2 - x^3 - Ax - B)} \right) & \longrightarrow & E \\ \downarrow & & \downarrow \\ \operatorname{Spec} A_i & \longrightarrow & S \end{array}$$

for some $A, B \in A_i$.

Remark 3.10. With $E \to S$ elliptic curve and a cover like above, the *j*-invariants of $E_i \to \operatorname{Spec} A_i$ are sections of the structure sheaf of $\operatorname{Spec} A_i$ which coincide on the intersection, so they glue to a section $j \in \Gamma(S, \mathcal{O}_S)$.

From this discussion it follows that the best candidate for $M_{1,1}$ is $\mathbb{A}^1$ where the curve over $j \in \mathbb{A}^1$ is the one with that *j*-invariant. Unfortunately $\mathbb{A}^1$ does not work.

Example 3.11. Suppose $\mathbb{A}^1$ represents the functor F from before. Consider then the two families

$$
\begin{array}{ccc}
\operatorname{Spec}\left(\frac{k[x,y,t^{\pm}]}{(y^2-x^3+t)}\right) & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
\operatorname{Spec} k[t,t^{-1}] & \longrightarrow & \mathbb{A}^1
\end{array}
$$

and

$$
\operatorname{Spec}\left(\frac{k[x,y]}{(y^2-x^3+1)}\right) \times \operatorname{Spec}\left(k[t^{\pm 1}]\right)
$$

$$
=
$$

$$
\begin{array}{ccc}
\operatorname{Spec}\left(\frac{k[x,y,t^{\pm}]}{(y^2-x^3+1)}\right) & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
\operatorname{Spec} k[t,t^{-1}] & \longrightarrow & \mathbb{A}^1
\end{array}
$$

If $\mathbb{A}^1$ represented the functor, these two families should be isomorphic (both are pullbacks along $\operatorname{Spec} k[t,t^{-1}] \to \mathbb{A}^1$ of the universal family), but they are not:

it is possible to prove via the j-invariants that two Weierstrass equations yield isomorphic curves if and only if there exists some $u \in k[t^{\pm 1}]$ such that substituting x with $u^2 x$ and y with $u^3 y$ transforms one equation into the other up to factoring out powers of u. Therefore we require for there to exist some u such that

$$
u^6 y^2 - u^6 x^3 + t = u^k(y^2 - x^3 + 1) \Longleftrightarrow t = u^6
$$

and this is impossible.

Remark 3.12. A slogan to keep in mind is

The presence of non-trivial automorphisms prevent the moduli problem from having a fine moduli space.

Example 3.13. If we consider the same families from before but now over the base $\operatorname{Spec} \frac{k[t^{\pm 1}, w]}{(t-w^6)}$ instead of $\operatorname{Spec} k[t^{\pm 1}]$, they now become isomorphic, that is

$$
\frac{k[x,y,t^{\pm},w]}{(y^2-x^3+1, w^6-t)} \cong \frac{k[x,y,t^{\pm},w]}{(y^2-x^3+t, w^6-t)}.
$$

This is because now we can choose $u = w$ with the notation from before.

The idea behind this observation is that if we allow ourselves to look at finer topologies on Sch we may be able to solve our representability issue. We are starting to see how viewing F as a functor is not enough; we should view it as some kind of sheaf.

3.1.3   Our dreams crumble?

So, what now? We have two options:

- Rigidify the problem (GIT approach)

- Enlarge the category among which we search for the moduli space (Stacky approach)

Notation. From now on we identify the scheme X with the functor $h_X : \mathrm{Sch}^{op} \to \mathrm{Set}$ and we identify it with the appropriate fibered category over Sch.

Remark 3.14. $X(k) = \mathrm{Hom}(\mathrm{Spec}\, k, X)$.

The main idea:

Instead of the functor F from before **we consider the category $\mathcal{M}_{1,1}$ fibered in groupoids** over Sch defined the same way.

Remark 3.15. The issue with families forming sets is now irrelevant. To be more precise, the groupoid associated to a scheme S is the one whose objects are elliptic curves over S and whose morphisms are isomorphisms of families (making this category a groupoid by construction).

Theorem 3.16. The fibered category $\mathcal{M}_{1,1}$ is a stack for the fpqc topology.

Proof.
Olsson [Ols16], chapter 13 $\hfill\square$

Remark 3.17. A morphism $\phi : X \to \mathcal{M}_{1,1}$ of stacks corresponds by (a version of) the Yoneda lemma to an element of the fiber $\mathcal{M}_{1,1}(X)$.

Said another way, there is a correspondence between families of curves over X (the category $\mathcal{M}_{1,1}(X)$) and morphisms $X \to \mathcal{M}_{1,1}$.

From a categorical point of view, $\mathcal{M}_{1,1}$ does exactly what we want. Now we try to understand in which ways it can be thought of as a space.

3.2 Algebraic stacks

If we want to think about stacks in a geometric way, we may want to impose some "good behaviour" conditions. For example, one thing we may require is that if two "normal" objects (for example schemes) live inside our stack, their intersection should still be a "normal" object. The key definition to formalize this idea is

---

Definition 3.18 (Representable morphism)**.** Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks, then it is said to be **representable** if for all schemes T and morphisms $T \to \mathcal{Y}$ there exists an algebraic space V such that

$$
\begin{array}{ccc}
V & \longrightarrow & T \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y}
\end{array}
$$

is cartesian.
We say that a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ has a property $\mathcal{P}$ for $\mathcal{P}$ a property of morphisms of schemes/algebraic spaces which is stable under base change if the morphism is representable and for all $T \to \mathcal{Y}$, the induced arrow $V \to T$ as above has the property $\mathcal{P}$.

Remark 3.19. The properties "begin smooth", "begin surjective" and "begin étale" are stable under base change.

Definition 3.20 (Algebraic stack)**.** A stack over a scheme S, i.e. a morphism of stacks $\mathcal{X} \to S$ is said to be **algebraic** if

1. the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and

2. there exists a smooth and surjective morphism $\pi : X \to \mathcal{X}$ for X scheme. This scheme is called an **atlas** for the algebraic stack.

If $\pi : X \to \mathcal{X}$ is étale then $\mathcal{X} \to S$ is called a **Deligne-Mumford stack**.

Remark 3.21. Property 1. looks very mysterious but the following is an equivalent and, perhaps clearer, formulation: for all T, U schemes, the fibered product $T \times_{\mathcal{X}} U$ exists and is an algebraic space. We may think of this as saying that "intersecting schemes in a stack gives an algebraic space".

Thinking about this set-theoretically may help in understanding the equivalence.

Remark 3.22. Property 2. can be thought of as "finiteness" condition. The case of Deligne-Mumford stacks makes this even clearer because étale morphisms can be thought of as unramified covers, so we are saying that our stack admits a cover with total space being a scheme.

3.3 Quotient stacks

Let us study a specific type of algebraic stacks: quotients.

Definition 3.23 (G-fibration)**.** Let X be a scheme, G a group scheme. A **G-fibration** over X is a scheme ξ with an action $\mu : G \times \xi \to \xi$ and a G-invariant morphism $\pi : \xi \to X$, that is, we have a commutative diagram

$$
\begin{array}{ccc}
G \times \xi & \xrightarrow{\ \mu\ } & \xi \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi} \\
\xi & \xrightarrow{\ \pi\ } & X
\end{array}
$$

Example 3.24. $\xi = G \times X$ with the obvious projection $\xi \to X$ is a G-fibration. This is called the **trivial** G-fibration.

Definition 3.25 (Principal G-bundle)**.** A **principal G-bundle** with respect to the fpqc (or fppf, étale etc) topology is a G-fibration which is locally trivial, i.e. there exists a cover $\{U_i \to X\}$ such that $\xi|_{U_i} \to U_i$ is isomorphic to $G \times U_i \to U_i$.

Proposition 3.26. Let X be a scheme and G a linear algebraic group acting on X via $\mu : X \times G \to X$. Assume that the action is free[a], then X/G exists as an algebraic space.

Moreover $\rho : X \to X/G$ is a principal G-bundle with respect to the étale topology and the following is cartesian

$$
\begin{array}{ccc}
X \times G & \xrightarrow{\pi_1} & X \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \rho} \\
X & \longrightarrow & X/G
\end{array}
$$

---

[a]for all schemes $\mu_T : X(T) \times G(T) \to X(T)$ is a free action of $G(T)$

Remark 3.27. If we fix $f : U \to X/G$ and take ξ to be the pullback of $X \to X/G$ along $U \to X/G$

$$
\begin{array}{ccc}
\xi & \xrightarrow{\alpha} & X \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \rho} \\
U & \xrightarrow{f} & X/G
\end{array}
$$

then

1. $\pi : \xi \to U$ is a G-bundle and

2. $\alpha : \xi \to X$ is G-equivariant.

Definition 3.28. Let X be a scheme and G a smooth linear algebraic group acting on X, the **quotient stack** $[X/G]$ is the fibered category over schemes

$$U \mapsto \{(\pi : \xi \to U, \alpha : \xi \to X) \mid \pi \text{ principal } G\text{-bundle}, \alpha \ G\text{-equivariant}\} / \sim$$

If $X = \operatorname{Spec} k$, $[\operatorname{Spec} k/G] = \mathbb{B}G$ is called the **classifying space** of G.

Theorem 3.29. $[X/G]$ is an algebraic stack

Let us quickly convince ourselves that $[X/G]$ should be an algebraic stack by looking at what the atlas should be:

The obvious candidate for an atlas is $\pi : X \to [X/G]$ (think "quotient map is a cover"). This morphism corresponds via the Yoneda lemma to a family of principal G-bundles over X, which is

$$
\begin{array}{ccc}
X \times G & \xrightarrow{\pi_1} & X \\
{\scriptstyle \mu}\downarrow & & \\
X & &
\end{array}
$$

If we fix $\varphi : U \to [X/G]$, this corresponds to a family

$$
\begin{array}{ccc}
\xi & \xrightarrow{\alpha} & X \\
\eta \downarrow & & \\
U & &
\end{array}
$$

with $\eta : \xi \to U$ smooth and surjective (not obvious but provable)

3.3.1 Back to elliptic curves

Let us consider the scheme[2] $U = \operatorname{Spec} k[A, B, 1/\Delta]$ where $\Delta = -16(4A^3 + 27B^2)$, and let $G = \mathbb{G}_m = k[u^{\pm 1}]$. G actis on U by looking at equivalent Weierstrass forms via the change of variables

$$
x \mapsto u^2 x, y \mapsto u^3 y.
$$

Concretely, the action maps A to $u^{-4}A$, B to $u^{-6}B$ and Δ to $u^{-12}\Delta$.

Fact 3.30. $\mathcal{M}_{1,1} \cong [U/G]$.

---

[2]$U \subseteq \mathbb{A}^2$ is the possible pairs (A, B) of coefficients in a Weierstrass form for an elliptic curve. Localizing at the discriminant enforces that for any point $(A, B) \in U$, the curve $y^2 = x^3 + Ax + B$ is an elliptic curve.

Part II

Intersection theory

Chapter 4

Intersection theory 1

Speaker: Andrea Gallese

The main reference for this chapter is [EH16].

4.1 Motivation

Think back to our roots: algebraic geometry is the study of solutions to systems of polynomial equations, which we can think of as intersecting some geometric objects.

In this chapter, we will consider schemes of finite type over algebraically closed fields. Eventually we will impose smoothness.

Definition 4.1. Let X be a scheme as above. We define the **algebraic cycles** to be
$$Z(X) = \mathbb{Z}\,[\text{integral closed subschemes of } X] = \bigoplus_k Z_k(X).$$

where $Z_k(X)$ is the same thing for subschemes of dimension k.

Remark 4.2. $Z(X) = Z(X^{red})$.

Definition 4.3. Given $Y \subseteq X$ closed subscheme, if $Y_1, \cdots, Y_r$ are its irreducible components the we define its **associated algebraic cycle** to be

$$\langle Y \rangle = \sum \ell_i [Y_i]$$

where[a] $\ell_i = \text{length}(\mathcal{O}_{X,y_i})$ for $y_i \in Y_i$ the generic point.

---
[a]length of the module over itself

Definition 4.4. Let us consider the subgroup of $Z(X)$ given by

$$\text{Rat}(X) = \langle Z \cap (t_0 \times X) - Z \cap (t_1 \times X) \rangle$$

where is some integral subvariety $Z \subseteq \mathbb{P}^1 \times X$ and $t_0, t_1 \in \mathbb{P}^1(k)$.
We define the **Chow group** as the quotient

$$CH(X) = \frac{Z(X)}{\text{Rat}(X)}$$

Example 4.5. $Z = \{xy = t\} \subseteq \mathbb{P}^1 \times X$, and if we slice Z at $t = 0$ and $t = 1$ we get $\{xy = 1\}$ and $\{xy = 0\}$, so these two give equivalent cycles.

Remark 4.6. The Chow group is graded by dimension and by codimension.

Example 4.7. In $\mathbb{A}^2$ all points are rationally equivalent:
multiply by $\mathbb{P}^1$ and take the line connecting the two points

Example 4.8. We have an action $\text{GL}_2 \curvearrowright \mathbb{A}^2$ and $\text{GL}_2 \subseteq \mathbb{A}^4$. If L is a line and $g \in \text{GL}_2(k)$ then $[L] \sim [gL]$.
The idea is to take the line in $\mathbb{A}^4$ which connects $id \in \text{GL}_2$ to $g \in \text{GL}_2$.

Now that we have our objects that we want to intersect, we want to count the "number of intersections" somehow.

4.1.1 Trasversality

Definition 4.9. Let A, B be integral subvarieties of X. We say that A and B are **transverse** at $p \in A \cap B$ if $T_p A + T_p B = T_p X$, i.e. $\text{codim}\, T_p A + \text{codim}\, T_p B = \text{codim}(T_p A \cap T_p B)$.
A and B are **generically transverse** if they are transverse at the generic points of the intersection.

Lemma 4.10 (Moving lemma). The following propositions hold when X is a smooth quasi-projective variety:

- For all $\alpha, \beta \in CH(X)$ there exist $A, B \in Z(X)$ such that $\alpha = [A]$, $\beta = [B]$ and A, B are generically transverse

- The class $[A \cap B]$ does not depend on the representatives A, B from the previous point.

PICTURE OF TANGENT LINE MOVING TO A TRANSVERSE ONE

Theorem 4.11. Let X be smooth and quasi-projective, then there exists a unique product on $CH(X)$ such that for all A, B subvarieties of X, if they are generically transverse then $[A] \cdot [B] = [A \cap B]$.
This makes $CH(X)$ into a graded (by codimension), associative, commutative ring.

Remark 4.12. The identity of the Chow group of X is the **fundemental class**, the class of the whole $[X]$.

Example 4.13. Example of the two circles

Theorem 4.14. If X is irreducible and $\dim X = n$ then $CH_n(X) \cong \mathbb{Z}$ and it is generated as a group by $[X]$.
If X has irreducible components given by $X_1, \cdots, X_r$, then $[X_1], \cdots, [X_r]$ generate a free abelian subgroup of $CH(X)$ which is isomorphic to $\mathbb{Z}^r$.

Sketch.
If $Z \subseteq X \times \mathbb{P}^1$ gives some rational equivalence then $Z \subseteq X_i \times \mathbb{P}^1$ by irreducibility. So we may only consider the irreducible case.

If $Z \cap (t_0 \times X) = X_i$ then $Z = X \times \mathbb{P}^1$ by irreduciblity. $\qquad\square$

4.2 Divisors

Proposition 4.15. Suppose X is irreducible of dimension n, then

$$CH_{n-1}(X) \cong \mathrm{Pic}(X)$$

the correspondence is that the divisor $D = \sum n_i Y_i$ corresponds to $\sum n_i [Y_i]$.

Proof (Well defined).
If $f \in k(X)$ then we may take $Z = \{f(x) = t\}$ to see that $[\{f = 0\}] = [\{f = \infty\}]$, so $\mathrm{div}(f)$ does map to 0.

Also, $\mathrm{Rat}(X)$ is generated by the relations induced by principals divisors in this way. $\qquad\square$

Proposition 4.16. The Chow ring of affine space is $CH(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n] \cong \mathbb{Z}$.

Proof.
Let Y be any positive codimension cycle not passing through 0 (just move it a little if necessary). We define

$$Z = \{(t, tz) \mid z \in Y\} = V(\{f(z/t) \mid f \in I(Y)\})$$

such an f exists because Y does not go through 0 and so $f(z/\infty) = f(0)$, thus Z defines an equivalence between Y for $t = 1$ and $\emptyset$ for $t = \infty$. $\qquad\square$

4.3 Mayer-Vietoris and Excision

Theorem 4.17 (Mayer-Vietoris)**.** Let $X_1, X_2 \subseteq X$ closed subvarieties, then we have an exact sequence

$$CH(X_1 \cap X_2) \to CH(X_1) \oplus CH(X_2) \to CH(X_1 \cup X_2) \to 0.$$

Theorem 4.18 (Excision)**.** Let $Y \subseteq X$ be a closed subvariety, then we have an exact sequence
$$CH(Y) \to CH(X) \to CH(X \setminus Y) \to 0$$

Corollary 4.19. If $U \subseteq \mathbb{A}^n$ open, $CH(U) \cong \mathbb{Z}$ as an abelian group and it is generated by $[U]$.

4.4 Functoriality / proper push-forward

Remark 4.20. If $f : X \to Y$ is proper and $A \subseteq X$ subvariety then $f(A)^{red} \subseteq Y$ is a subvariety.

Theorem 4.21. If $f : X \to Y$ proper, we can get $f_* : CH(X) \to CH(Y)$. On generators it behaves as follows

$$f_*[A] = \begin{cases} 0 & \text{if } \dim f(A) < \dim A \\ d[f(A)] & \text{if } \dim f(A) = \dim A \text{ and } d = \deg(A \to f(A)) \end{cases}$$

Theorem 4.22. f_* defines a dimension-preserving morphism of the Chow-groups

PICTURE OF $A + B + C$ being equivalent to $D + E + F$, equivalent to $G + H$ MAPS TO STUFF WORKING OUT

Proposition 4.23. If $X \to \operatorname{Spec} k$ is proper, then we have a unique map deg : $CH(X) = CH(pt) \cong \mathbb{Z}$ which preserves dimension (so $[Y] \mapsto 0$ if $\dim Y > 0$ and $[pt] \mapsto 1$).

4.5   Projective space

Theorem 4.24. We have that

$$CH(\mathbb{P}^n) \cong \frac{\mathbb{Z}[t]}{(t^{n+1})}$$

where t is the class of some linear hyperplane. Moreover, if $X \subseteq \mathbb{P}^n$ irreducible of codimension k and degree[a] d then $[X] = dt^k$.

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[a]Hilbert polynomial nonesense

Proof.

Lemma 4.25 (Affine stratification)**.** Consider the chain of inclusions

$$\{pt\} = \mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \cdots \subseteq \mathbb{P}^n$$

and consider $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1}$ (affine, we are choosing an affine chart).
It follows that $CH(\mathbb{P}^n)$ is generated by the classes of $[\overline{U_i}]$.

Sketch.
Induction on numeber of strata + excision

$$Z = CH(\overline{U_0}) = CH(U_0) \to CH(\mathbb{P}^n) \to CH(\mathbb{P}^n \setminus \{pt\})$$

and the last bit is generated by $[\overline{U_i}]$ by inductive hypothesis. $\qquad\square$

Lemma 4.26. Intersecting k hyperplanes gives a $(n-k)$-plane, so $t^k = [(n-k)\text{-plane}]$.

Lemma 4.27. Let $M \subseteq \mathbb{P}^n$ be a k-plane, we have a surjective[a] map

$$CH^k(\mathbb{P}^n) \xrightarrow{\cdot [M]} CH^n(\mathbb{P}^n) \cong \mathbb{Z}$$

and the full composition is the degree of $- \cdot [M]$ (DRAW DIAGRAM LATER)

---
[a]$(n-k)$-plane $\cap$ k-plane $= [pt]$, generator of $CH^n(\mathbb{P}^n)$

$\square$

Exercise 4.28. In $\mathbb{A}^2$ draw three circles C_1, C_2, C_3. How many circles can I draw which are tangent to C_1, C_2, C_3? Turns out it's 8.

Solution.

A circle in the plane is $(x - A)^2 + (y - B)^2 = C$. If we homogenize we get two points at infinity $[1 : \pm i : 0]$. We define a circle in $\mathbb{P}^2$ to be a conic which passes through these two points.

Recall that conics in $\mathbb{P}^2$ are parametrized by $\mathbb{P}^5$ by looking at the coefficients. By imposing the fact that the circles should go through two distinct points we get that circles are parametrized by a $\mathbb{P}^3$.

What does it mean for a circle to be tangent to C fixed circle? Let $Z_C = \{\text{circles tangent to } C\} \subseteq \mathbb{P}^3$. Let

$$\Phi = \{(r, D) \mid r \in C, \ D \in \text{circle tangent to } C \text{ at } r\}$$

We have two projections $\Phi \to C$ and $\Phi \to Z_D$. Note that the fibers of $\Phi \to C$ are $\mathbb{P}^1$ (homotety of the same circle getting bigger and bigger). So Φ should be a surface of some kind. It should have degree 2 because tangency is checked by imposing that the discriminant of the quadratic equation obtained by substituting the equations of the two circles into each other is a quadratic.

Moreover, Z_D is birational to Φ (from a circle in Z_D we can get the point of tangency it has with C, giving the equivalence).

So $Z_{C_1} \subseteq \mathbb{P}^3$ is a surface of degree 2. In the Chow group $CH(\mathbb{P}^3)$ we have $2t = [Z_{C_1}]$ and so

$$[Z_{C_1} \cap Z_{C_2} \cap Z_{C_3}] = [Z_{C_1}][Z_{C_2}][Z_{C_3}] = (2t)^3 = 8t^3$$

and $8t^3 \in CH(\mathbb{P}^3)$ is the class of eight points. $\qquad\square$

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