Stacks and intersection theory

Reading group

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A.A. 2024/25

Contents

Part I

Stacks

Chapter 1

Grothendieck topologies

Speaker: Domenico Marino

Chapter 2

Fibered categories and Stacks

Speaker: Francesco Sorce

Everything in this chapter is talked about in greater detail in [Vis07].

2.1 Fibered categories

2.1.1 Definition

Definition 2.1. If we have the data of $\mathcal{F}$ and $\mathcal{C}$ two categories and $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ a functor we say that $\mathcal{F}$ is **over** $\mathcal{C}$.
In this context, if $p_{\mathcal{F}}(\xi) = U$ for some object ξ we say that ξ is over U.
Similarly if $p_{\mathcal{F}}(\xi \to \eta) = U \to V$ then $\xi \to \eta$ is over $U \to V$.

Notation. In the diagrams that follow, a normal or dashed arrow will be a morphism, while an allow with a tail (like $\mapsto$) represents applying the functor.

Definition 2.2. An arrow $\phi : \xi \to \eta$ in $\mathcal{F}$ over $\mathcal{C}$ is **cartesian** if for all $\phi : \zeta \to \eta$ and all $h : p_{\mathcal{F}}\zeta \to p_{\mathcal{F}}\xi$ which make the following diagram commute there exists a unique arrow $\zeta \to \xi$ in place of the dotted arrow:

$$
\begin{tikzcd}
\zeta \arrow[d, mapsto] \arrow[dr, dashed] \arrow[drr, bend left, "\psi"] & & \\
W \arrow[dr, "h"] \arrow[drr, bend left] & \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
& U \arrow[r] & V
\end{tikzcd}
$$

If $\xi \to \eta$ is cartesian and over $f : U \to V$ we say that ξ is the **pullback of** η **along** f. We may write $\xi = f^*\eta$ if we fix a choice of pullback.

Definition 2.3. Let us fix a category $\mathcal{C}$. The associated **arrow category** $\mathrm{Arr}(\mathcal{C})$ is the category whose objects are arrows in $\mathcal{C}$ and whose morphisms are commutative squares in $\mathcal{C}$.

Remark 2.4. We may view $\mathrm{Arr}(\mathcal{C})$ as a category over $\mathcal{C}$ by fixing the functor that to an arrow $X \to U$ assigns U.

Example 2.5. In order to better understand cartesian arrows (and to see where the name comes from) we determine which arrows in $\mathrm{Arr}(\mathcal{C})$ are cartesian.

If an arrow $(X \to U) \to (Y \to V)$ is cartesian, consider the following data:

$$\begin{tikzcd}
Z \arrow[drr] \arrow[ddr] & & \\
& X \arrow[r] \arrow[d] & Y \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

which we may equivalently rewrite

$$\begin{tikzcd}
Z \arrow[d, "id_Z"'] \arrow[drr] & & \\
Z \arrow[d] \arrow[drr] & X \arrow[r] \arrow[d] & Y \arrow[d] \\
Z \arrow[dr] \arrow[drr] & U \arrow[r] \arrow[d] & V \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

Since the arrow is cartesian, there is a unique way to complete the diagram above with maps $Z \to X$ and $Z \to U$. Since we ask for compatibility with the functor, $Z \to U$ is already known, so we have constructed $Z \to X$ in the first diagram. This makes us suspect that cartesian arrows in $\mathrm{Arr}(\mathcal{C})$ correspond to cartesian squares in $\mathcal{C}$, and this is true.

Indeed, suppose that $(X \to U) \to (Y \to V)$ is a cartesian square and let us consider the data

$$\begin{tikzcd}
Z \arrow[d] \arrow[drr] & & \\
W \arrow[d] \arrow[drr] & X \arrow[r] \arrow[d] & Y \arrow[d] \\
W \arrow[dr] \arrow[drr] & U \arrow[r] \arrow[d] & V \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

then we have a unique way to complete the diagram. The bottom map is determined by compatibility with the functor towards $\mathcal{C}$ and the top arrow can be chosen to be the one induced by the fact that we have a cartesian square an maps $Z \to Y$, $Z \to U$, where the second one is the composition $Z \to W \to U$.

Fact 2.6. Cartesian arrows satisfy the following properties:

1. the composition of cartesian arrows is cartesian
2. if $\xi \to \zeta$ factors through η with $\eta \to \zeta$ being cartesian, then $\xi \to \zeta$ is cartesian if and only if $\xi \to \eta$ is
3. if ϕ is over an isomorphism then ϕ is cartesian if and only if it is an isomorphism

Definition 2.7. A category $\mathcal{F}$ over $\mathcal{C}$ is **fibered over** $\mathcal{C}$ if for all $f : U \to V$ in $\mathcal{C}$ and all $\eta \in \mathcal{F}$ over V there exists a cartesian arrow $\phi : \xi \to \eta$ over f. That is, given a "partial diagram" of the form

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

we can make it a square in such a way that the top side is a cartesian arrow.

Definition 2.8. Let $\mathcal{F}$ and $\mathcal{G}$ be fibered categories over $\mathcal{C}$. A **morphism of fibered categories over** $\mathcal{C}$ from $\mathcal{F}$ to $\mathcal{G}$ is a functor $F : \mathcal{F} \to \mathcal{G}$ such that $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$ which preserves cartesian arrows, i.e. if $\phi \in \mathcal{F}$ cartesian then $F(\phi)$ is cartesian in $\mathcal{G}$.

Example 2.9. Because of what we have already said, $\mathrm{Arr}(\mathcal{C})$ is fibered over $\mathcal{C}$ if and only if $\mathcal{C}$ admits fibered products.

2.1.2 Fibers

We now reach one of the main definitions which motivate fibered categories

Definition 2.10. Let $\mathcal{F}$ be a fibered over $\mathcal{C}$ and fix an object $U \in \mathcal{C}$. The **fiber of** $\mathcal{F}$ **over** U is the subcategory $\mathcal{F}(U)$ of $\mathcal{F}$ whose objects are the objects of $\mathcal{F}$ over U and whose morphisms are those $\underline{\text{over } id_U}$.

Example 2.11. Fix $U \in \mathcal{C}$. The fiber $\mathrm{Arr}(\mathcal{C})(U)$ is the comma category $\mathcal{C}/U$.

Remark 2.12. If $f : U \to V$ is a morphism in $\mathcal{C}$ and $\mathcal{F}$ is fibered over $\mathcal{C}$ then we may define a "restriction functor"

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U) \\ X & \longmapsto & f^*X \end{array}$$

upon choosing a unique pullback for each element. Such a choice is called a **cleavage**. For more details see [Vis07].

Remark 2.13. If $F : \mathcal{F} \to \mathcal{G}$ is a morphism of fibered categories over $\mathcal{C}$ and $U \in \mathcal{C}$ then it induces a functor $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

These definitions make $\mathcal{F}$ look like a presheaf $\mathcal{C}^{op} \to \mathrm{Cat}$

This is not quite correct because choosing a cleavage may lead to having two arrows that should be the same being naturally isomorphic instead. To be technically correct we would need to introduce pseudo-functors, but they behave like normal functors in basically every way.

Fact 2.14. There is a correspondence between categories fibered over $\mathcal{C}$ and pseudo-functors $\mathcal{C}^{op} \to \mathrm{Cat}$.

Proof (Idea).
To get the functor from the category fix a cleavage, then take an object to the fiber over it and a map to the pullback along it.

To get a category from the functor consider the category with objects (U, X) for $U \in \mathcal{C}$ and $X \in \mathcal{F}(U)$ and morphisms defined in the obvious way. The functor from this category to $\mathcal{C}$ is the projection on the first factor. The fact that this comes from a pseudo-functor automatically gives the existence of pullbacks. $\square$

Definition 2.15. A category is called a **set** if its objects form a set and the only morphisms are of the form id_X for some object X in the category.

Definition 2.16. A category is called a groupoid if every arrow in the category is an isomorphism.

Definition 2.17. Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$. We say that $\mathcal{F}$ is

- **fibered in sets** if $\mathcal{F}(U)$ is a set for all $U \in \mathcal{C}$
- **fibered in groupoids** if $\mathcal{F}(U)$ is a groupoid for all $U \in \mathcal{C}$.

Fact 2.18. A cateogry is fibered in sets if and only if the associated pseudo-functor is a functor in the usual sense.

Example 2.19. Let $\mathcal{C}$ be a (locally small) category and let us fix $X \in \mathcal{C}$. We have a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ which maps X to $h_X = \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$. This functor defines a category fibered over $\mathcal{C}$. The objects of this category are pairs $(Y, Y \to X)$ (which we will identify with $Y \to X$ itself) and a morphism from $Y \to X$ to $Z \to X$ is an arrow $Y \to Z$ in $\mathcal{C}$ such that $Y \to X = Y \to Z \to X$.

The fiber of this category over $Y \in \mathcal{C}$ is $\{Y \to X\} = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ but seen as a category. Since arrows in a fiber must be over the identify of the object of which they are the fiber, the only arrows in $\mathrm{Hom}(Y, X)$ are the identity of Y. This shows that the category we got from X is fibered in sets, which is what we expected because it came from a functor.

Schematically, we have the following embeddings and equivalences

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[dr, "\sim"'] & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \arrow[r, hook] \arrow[dr, "\sim"'] & \mathrm{Fibered}/\mathcal{C} \\
 & \text{Representable} \arrow[u, hook] & \text{Fibered in sets} \arrow[u, hook]
\end{tikzcd}$$

Proposition 2.20. Let $\mathcal{F}$ be a category over $\mathcal{C}$ (we do not assume fibered). $\mathcal{F}$ is fibered in groupoids over $\mathcal{C}$ if and only if the following conditions hold:

1. every arrow in $\mathcal{F}$ is cartesian
2. given a partial diagram $f : U \to V$ and $\eta \in \mathcal{F}$ over V, the exists some $\phi : \xi \to \eta$ over f.

Sketch.
If $\mathcal{F}$ is fibered in groupoids, 2. holds by fiberedness and 1. is a consequence of point 2 in fact 2.6.

Viceversa, if 1. and 2. hold then $\mathcal{F}$ is fibered. Since all arrows in a fiber are over an isomorphism (the identity of the object we are taking the fiber), again by fact 2.6 we see that all arrows must be isomorphisms because they are cartesian. $\square$

2.2 Stacks

Now that we have explored the formalism of fibered categories, let us see how it interacts with the notion of Grothendieck topology we dealt with in the previous chapter.

2.2.1 Motivating example

Let us consider the arrow category associated to the category of topological spaces Arr(Top). We may interpret objects in this category to be *spaces lying above a base space*. In what follows it may be useful to imagine objects in this category to be things like covering maps or fiber bundles, though of course the notion of "continuous maps" is much more general.

Taking this as our point of view, two concepts we may want to explore are the following:

- Suppose we have two objects over the same base space and an open cover for that base. Under what conditions can we construct a map between them locally?
- Suppose we have an open cover and that for each element of the cover we have a space over it. Under what conditions can we glue these pieces to get an object over the entire base space?

Studying these questions will eventually lead us to what a prestack and a stack should be.

Recall that we have a standard Grothendieck topology on Top for which

$$\mathrm{Cov}(U) = \{\{U_i \to U\} \mid \text{jointly surjective open immersions}\}.$$

Notation. We will write U_{ij} instead of $U_i \cap U_j$ and similarly U_{ijk} for triple intersections.

Gluing maps

Suppose we fix $U \in$ Top and an open cover[1] $\{U_i \to U\}$. We also fix two objects $\xi : X \to U$ and $\eta : Y \to U$. The answer to our first question can be summerized in the following proposition:

Proposition 2.21. Let U, $\{U_i\}$, ξ, η be as above. Suppose we have maps $f_i : \xi^{-1}(U_i) \to \eta^{-1}(U_i)$ for all i such that

$$f_i|_{\xi^{-1}(U_{ij})} = f_j|_{\eta^{-1}(U_{ij})},$$

then there exists a unique $f : X \to Y$ which is compatible with the maps to U and such that $f|_{\xi^{-1}(U_i)} = f_i$ for all i.

Notice that we may restate the proposition as

Proposition 2.22. The functor

$$\underline{\mathrm{Hom}}_U(X, Y) : \begin{array}{ccc} (\mathrm{Top}/U)^{op} & \longrightarrow & \mathrm{Set} \\ Z \to U & \longmapsto & \mathrm{Hom}_Z(X \times_U Z, Y \times_U Z) \end{array}$$

is a sheaf with respect to the Grothendieck topology induced by Top on Top$/U$.

To see the equivalence, note that $X \times_U U_i = \xi^{-1}(U_i)$ and $Y \times_U U_i = \eta^{-1}(U_i)$, so when you evaluate the functor on the open cover of U, the fact that it is a sheaf corresponds exactly to the gluing property.

Gluing spaces

We now look at what it means to glue spaces using a cover:

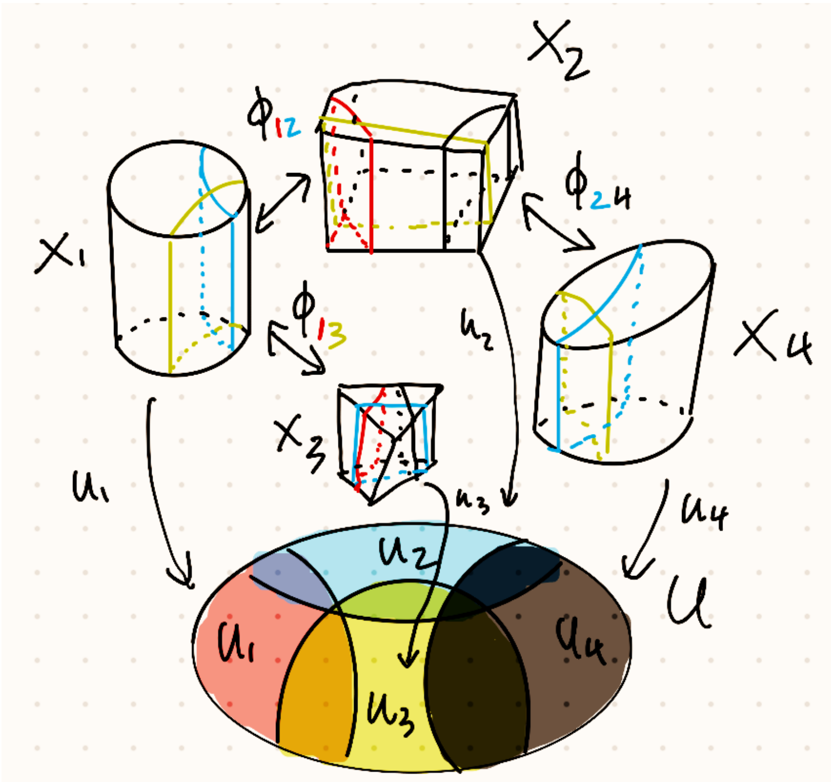
Proposition 2.23. Let us fix a topological space U together with an open cover $\{U_i\}$. For each i let $u_i : X_i \to U_i$ be an element of Arr(Top). We also fix isomorphisms $\phi_{ij} : u_j^{-1}(U_{ij}) \to u_i^{-1}(U_{ij})$ for each pair of indices in such a way so that

$$\phi_{ik}|_{u_k^{-1}(U_{ijk})} = \phi_{ij}|_{u_j^{-1}(U_{ijk})} \circ \phi_{jk}|_{u_i^{-1}(U_{ijk})}.$$

It follows that there exists $u : X \to U$ continuous, together with homeomorphisms $\phi_i : u^{-1}(U_i) \to X_i$ such that $\phi_{ij} = \phi_i|_{u^{-1}(U_{ij})} \circ (\phi_j|_{u^{-1}(U_{ij})})^{-1}$.

[1] in the standard sense. Everything works the same with jointly surjective open immersions but the notation gets bothersome rather quickly.

Remark 2.24. The notation is rather tedious, but the geometric image is actually rather simple: the ϕ_{ij} serve as a guide to tell us which points should be identified and the condition $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ (usually called a **cocycle condition**) ensures that these identifications define an equivalence realtion on $\coprod X_i$.



2.2.2 Descent data

We are now ready to move onto stacks. The main idea is to generalize the type of *local data* that we used to glue before.

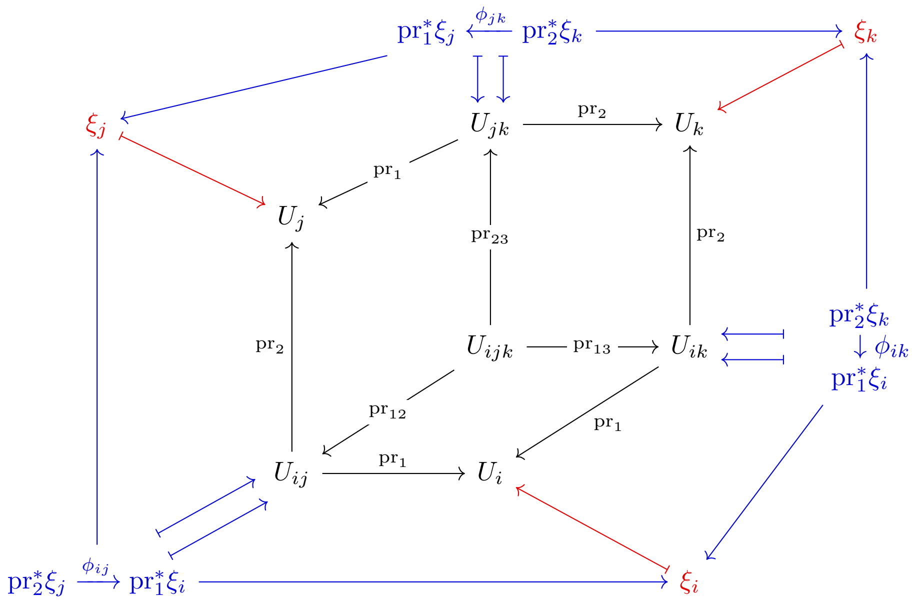
For the rest of the chapter $\mathcal{C}$ is a site with a fixed Grothendieck topology and $\mathcal{F}$ is fibered category over $\mathcal{C}$.

Definition 2.25. Fix $U \in \mathcal{C}$ and a cover $\{U_i \to U\} \in \mathrm{Cov}(U)$. An **object with descent data** is a pair $(\{\xi_i\}, \{\phi_{ij}\})$ where $\xi_i \in \mathcal{F}(U_i)$ for all i and $\phi_{ij} : \mathrm{pr}_2^*\xi_j \to \mathrm{pr}_1^*\xi_i$ are isomorphisms which live in $\mathcal{F}(U_{ij})$ such that

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk}.$$

The ϕ_{ij} are called **transition isomorphisms**.

If you are (very reasonably) getting confused with the pullbacks and indices, it may (or may not) be useful to admire *the hypercube*



Notation. Let $\mathcal{C}$ be a site, $\mathcal{F}$ fibered over $\mathcal{C}$, $U \in \mathcal{C}$ and $\{U_i \to U\} \in \mathrm{Cov}(U)$. Objects with descent data over $\{U_i \to U\}$ form a category[2], which we denote using

$$\mathcal{F}(\{U_i \to U\}).$$

Remark 2.26. Fix $U \in \mathcal{C}$, $\{u_i : U_i \to U\} \in \mathrm{Cov}(U)$ and $\xi \in \mathcal{F}(U)$. From ξ we can obtain an object with descent data by restriction. To be more precise, we can define

$$\xi_i = u_i^*\xi, \qquad \phi_{ij} : \mathrm{pr}_2^* u_j^*\xi \to \mathrm{pr}_1^* u_i^*\xi$$

where ϕ_{ij} is the canonical isomorphism induced by the fact that both objects amount to the pullback of ξ along $U_{ij} \to U$. The resulting object with descent data is $(\{\xi_i\}, \{\phi_{ij}\})$.

What we have described is actually a functor from $\mathcal{F}(U)$ to $\mathcal{F}(\{U_i \to U\})$.

Definition 2.27. An object with descent data is called **effective** if it is in the essential image of the functor above.

We are now finally ready to define prestacks and stacks:

[2]the morphisms are collections of maps and commutative diagrams which are compatible with the indices.

Definition 2.28. Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$ site. $\mathcal{F}$ is a

- **prestack** if for all $U \in \mathcal{C}$ and all $\{U_i \to U\} \in \text{Cov}(U)$, the functor
$$\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$$
is *fully faithful.*
- **stack** if for all $U \in \mathcal{C}$ and all $\{U_i \to U\} \in \text{Cov}(U)$, the functor
$$\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$$
is an *equivalence.*

Remark 2.29. The two conditions can be intuitively understood as:

- A prestack is a fibered category where if two objects are the same locally (taking into account gluing) then they are isomorphic.
- A stack is a prestack where all descent data glues to an object.

Another way in which we can intepret the prestack condition is given in the following

Proposition 2.30. Let $\mathcal{F}$ be fibered over $\mathcal{C}$ site. $\mathcal{F}$ is a prestack if and only if for all $U \in \mathcal{C}$ and all $\xi, \eta \in \mathcal{F}(U)$, the functor

$$\underline{\text{Hom}}_U(\xi, \eta) : \begin{array}{ccc} (\mathcal{C}/U)^{op} & \longrightarrow & \text{Set} \\ f : Z \to U & \longmapsto & \text{Hom}_Z(f^*\xi, f^*\eta) \end{array}$$

is a sheaf on $\mathcal{C}/U$.

Sketch.
Fix $U \in \mathcal{C}$ and $\{U_i \to U\} \in \text{Cov}(U)$. The functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ being fully faithful means that for all $\xi, \eta \in \mathcal{F}(U)$ we have

$$\text{Hom}_U(\xi, \eta) \cong \text{Hom}_{\{U_i \to U\}}((\{\xi_i\}, \{\alpha_{ij}\}), (\{\eta_i\}, \{\beta_{ij}\}))$$

where $(\{\xi_i\}, \{\alpha_{ij}\})$ is the image of ξ and similarly for the other object and η.

Concretely, this means that maps $\xi \to \eta$ can be defined uniquely on an open cover, which is exactly what it means for $\underline{\text{Hom}}_U(\xi, \eta)$ to be a sheaf. $\square$

Corollary 2.31. Arr(Top) is a stack.

We close the chapter by noting that when $\mathcal{F}/\mathcal{C}$ is fibered in sets, we find the notions of separated presheaf and sheaf over $\mathcal{C}$ respectively, meaning that we have, in some very precise sense, defined *sheaves in categories.* This concept will be trated in more detail in the next chapter.

Proposition 2.32. Let $\mathcal{C}$ be a site and consider a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. Identify F with the fibered category it defines. Then

- F is a prestack if and only if it is a separated presheaf,
- F is a stack if and only if F is a sheaf.

Example 2.33. The fibered category associated to an object $X \in \mathcal{C}$ is a stack because it corresponds to the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$.

Chapter 3

Algebraic Stacks

Speaker: Pietro Leonardini

3.1 Motivation: moduli problems

3.1.1 The dream

Let us recall what an elliptic curve is:

Definition 3.1 (Elliptic curve)**.** An **elliptic curve** over k is a 1-dimensional scheme $C \to \operatorname{Spec} k$ over k which is geometrically integral, proper, smooth and of genus[a] 1 with a fixed k-rational point $e : \operatorname{Spec} k \to C$.

[a]the genus is $\dim_k H^1(C, \mathcal{O}_C)$

Since the relative approach is central in the theory of schemes, we may want to define an elliptic curve over some base scheme S. The most sensible approach is to say that an elliptic curve over S should be a "family of elliptic curves parametrized by S". Formally:

Definition 3.2 (Elliptic curve over S)**.** An **elliptic curve** over S is a proper, smooth, flat morphism of schemes $p : E \to S$ with a fixed section $e : S \to E$ of p such that for all $\operatorname{Spec}\Omega$ geometric point, the pullback is an elliptic curve over Ω:

$$\begin{tikzcd} E \times_S \operatorname{Spec}\Omega \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec}\Omega \arrow[r] & S \end{tikzcd}$$

When studying elliptic curves it would be useful to find some space $M_{1,1}$ such that for every family $E \to S$ we have a unique $\phi : S \to M_{1,1}$ such that

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ S \arrow[r, "\phi"'] & M_{1,1} \end{tikzcd}$$

that is, we would be very happy if all families of elliptic curves could be described as the pullback of some *universal family* $\mathcal{E}$ over $M_{1,1}$ along some morphism $S \to M_{1,1}$. Intuitively, the universal family should be the one defined by the fact that the fiber over a point of $M_{1,1}$ is the elliptic curve corresponding to that point. The morphism should be the one that to each point of S assigns some elliptic curve, i.e. a point of $M_{1,1}$ (of course this naive view is too simple when dealing with schemes but it motivates why we look for such a map).

A useful way to formulate this problem is to define the following functor[1]

$$F : \begin{array}{ccc} \mathrm{Sch}^{op} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \{E \to S \mid \text{elliptic curves}\}/\text{iso.} \\ f : T \to S & \longmapsto & (E_1 \to S) \mapsto (f^*E_1 \to T) \end{array}$$

We can re-formulate the requirements on $M_{1,1}$ by saying that $M_{1,1}$ should **represent** this functor, i.e. there is a natural isomorphism of functors

$$h_{M_{1,1}} \cong F$$

where $h_{M_{1,1}}$ is the contravariant Hom-functor associated to $M_{1,1}$.

We recall the following simple but foundational result

Theorem 3.3 (Yoneda lemma)**.** There is a natural correspondence

$$\mathrm{Hom}(h_X, F) \leftrightarrow F(X)$$

sketch.
Suppose we have a natural transformation $\phi : h_X \to F$, then we can find an element of $F(X)$ by taking $\phi_X(id_X) = \xi$.

Let us now fix an element $\xi \in F(X)$. Let T be a scheme. We may construct a map $\mathrm{Hom}(T, X) \to F(T)$ by sending $f : T \to X$ to $F(f)(\xi)$. If $g : T \to S$ is a morphism then $\eta : f \mapsto g \circ f$ is mapped to $F(g) : F(S) \to F(T)$. It is easy to check that this defines a natural transformation $h_X \to F$ and that the two constructions are inverses of each other. $\square$

Corollary 3.4. The functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ that sends X to h_X is fully faithful.

[1] we technically have no guarantee that families over a scheme should form a set. This issue will be solved automatically later when we consider a fibered category over Sch instead of a functor.

3.1.2 A good attempt

Let us try to find some way of parametrizing isomorphism classes of elliptic curves. The main idea comes from the following fact:

Fact 3.5. If $\operatorname{char} k \neq 2, 3$ then every elliptic curve is isomorphic over $\overline{k}$ to some

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

for some $\lambda \in k \setminus \{0, 1\}$.

Definition 3.6 (j-invariant)**.** Let

$$j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}$$

Fact 3.7. The map

$$\phi : \begin{array}{ccc} \overline{k} \setminus \{0,1\} & \longrightarrow & \overline{k} \\ \lambda & \longmapsto & j(E_\lambda) \end{array}$$

is surjective and $6:1$ except in $j = 0$ and $j = 1728$, where it is $2:1$ and $3:1$ respectively.

Fact 3.8. Two elliptic curves are isomorphic over $\overline{k}$ if and only if they have the same j-invariant.

Theorem 3.9. Let $E \to S$ be an elliptic curve, then there exists a Zariski affine open over of S given by $\{U_i = \operatorname{Spec} A_i\}$ such that for all i

$$\begin{tikzcd} E_i := \operatorname{Spec}\left(\dfrac{A_i[x,y]}{(y^2 - x^3 - Ax - B)}\right) \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec} A_i \arrow[r] & S \end{tikzcd}$$

for some $A, B \in A_i$.

Remark 3.10. With $E \to S$ elliptic curve and a cover like above, the j-invariants of $E_i \to \operatorname{Spec} A_i$ are sections of the structure sheaf of $\operatorname{Spec} A_i$ which coincide on the intersection, so they glue to a section $j \in \Gamma(S, \mathcal{O}_S)$.

From this discussion it follows that the best candidate for $M_{1,1}$ is $\mathbb{A}^1$ where the curve over $j \in \mathbb{A}^1$ is the one with that j-invariant. Unfortunately $\mathbb{A}^1$ does not work.

Example 3.11. Suppose $\mathbb{A}^1$ represents the functor F from before. Consider then the two families

$$\begin{array}{ccc} \operatorname{Spec}\left(\frac{k[x,y,t^{\pm}]}{(y^2-x^3+t)}\right) & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[t,t^{-1}] & \longrightarrow & \mathbb{A}^1 \end{array}$$

and

$$\operatorname{Spec}\left(\frac{k[x,y]}{(y^2-x^3+1)}\right) \times \operatorname{Spec}\left(k[t^{\pm 1}]\right)$$

$$=$$

$$\begin{array}{ccc} \operatorname{Spec}\left(\frac{k[x,y,t^{\pm}]}{(y^2-x^3+1)}\right) & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[t,t^{-1}] & \longrightarrow & \mathbb{A}^1 \end{array}$$

If $\mathbb{A}^1$ represented the functor, these two families should be isomorphic (both are pullbacks along $\operatorname{Spec} k[t,t^{-1}] \to \mathbb{A}^1$ of the universal family), but they are not:

it is possible to prove via the j-invariants that two Weierstrass equations yield isomorphic curves if and only if there exists some $u \in k[t^{\pm 1}]$ such that substituting x with $u^2 x$ and y with $u^3 y$ transforms one equation into the other up to factoring out powers of u. Therefore we require for there to exist some u such that

$$u^6 y^2 - u^6 x^3 + t = u^k (y^2 - x^3 + 1) \iff t = u^6$$

and this is impossible.

Remark 3.12. A slogan to keep in mind is

The presence of non-trivial automorphisms prevent the moduli problem from having a fine moduli space.

Example 3.13. If we consider the same families from before but now over the base $\operatorname{Spec} \frac{k[t^{\pm 1},w]}{(t-w^6)}$ instead of $\operatorname{Spec} k[t^{\pm 1}]$, they now become isomorphic, that is

$$\frac{k[x,y,t^{\pm},w]}{(y^2-x^3+1, w^6-t)} \cong \frac{k[x,y,t^{\pm},w]}{(y^2-x^3+t, w^6-t)}.$$

This is because now we can choose $u = w$ with the notation from before.

The idea behind this observation is that if we allow ourselves to look at finer topologies on Sch we may be able to solve our representability issue. We are starting to see how viewing F as a functor is not enough; we should view it as some kind of sheaf.

3.1.3 Our dreams crumble?

So, what now? We have two options:

- Rigidify the problem (GIT approach)
- Enlarge the category among which we search for the moduli space (Stacky approach)

Notation. From now on we identify the scheme X with the functor $h_X : \mathrm{Sch}^{op} \to \mathrm{Set}$ and we identify it with the appropriate fibered category over Sch.

Remark 3.14. $X(k) = \mathrm{Hom}(\mathrm{Spec}\, k, X)$.

The main idea:

Instead of the functor F from before **we consider the category $\mathcal{M}_{1,1}$ fibered in groupoids** over Sch defined the same way.

Remark 3.15. The issue with families forming sets is now irrelevant. To be more precise, the groupoid associated to a scheme S is the one whose objects are elliptic curves over S and whose morphisms are isomorphisms of families (making this category a groupoid by construction).

Theorem 3.16. The fibered category $\mathcal{M}_{1,1}$ is a stack for the fpqc topology.

Proof.
Olsson [Ols16], chapter 13 □

Remark 3.17. A morphism $\phi : X \to \mathcal{M}_{1,1}$ of stacks corresponds by (a version of) the Yoneda lemma to an element of the fiber $\mathcal{M}_{1,1}(X)$.

Said another way, there is a correspondence between families of curves over X (the category $\mathcal{M}_{1,1}(X)$) and morphisms $X \to \mathcal{M}_{1,1}$.

From a categorical point of view, $\mathcal{M}_{1,1}$ does exactly what we want. Now we try to understand in which ways it can be thought of as a space.

3.2 Algebraic stacks

If we want to think about stacks in a geometric way, we may want to impose some "good behaviour" conditions. For example, one thing we may require is that if two "normal" objects (for example schemes) live inside our stack, their intersection should still be a "normal" object. The key definition to formalize this idea is

Definition 3.18 (Representable morphism)**.** Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks, then it is said to be **representable** if for all schemes T and morphisms $T \to \mathcal{Y}$ there exists an algebraic space V such that

$$\begin{tikzcd} V \arrow[r] \arrow[d] & T \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is cartesian.
We say that a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ has a property $\mathcal{P}$ for $\mathcal{P}$ a property of morphisms of schemes/algebraic spaces which is stable under base change if the morphism is representable and for all $T \to \mathcal{Y}$, the induced arrow $V \to T$ as above has the property $\mathcal{P}$.

Remark 3.19. The properties "begin smooth", "begin surjective" and "begin étale" are stable under base change.

Definition 3.20 (Algebraic stack)**.** A stack over a scheme S, i.e. a morphism of stacks $\mathcal{X} \to S$ is said to be **algebraic** if

1. the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and
2. there exists a smooth and surjective morphism $\pi : X \to \mathcal{X}$ for X scheme. This scheme is called an **atlas** for the algebraic stack.

If $\pi : X \to \mathcal{X}$ is étale then $\mathcal{X} \to S$ is called a **Deligne-Mumford stack**.

Remark 3.21. Property 1. looks very mysterious but the following is an equivalent and, perhaps clearer, formulation: for all T, U schemes, the fibered product $T \times_{\mathcal{X}} U$ exists and is an algebraic space. We may think of this as saying that "intersecting schemes in a stack gives an algebraic space".

Thinking about this set-theoretically may help in understanding the equivalence.

Remark 3.22. Property 2. can be thought of as "finiteness" condition. The case of Deligne-Mumford stacks makes this even clearer because étale morphisms can be thought of as unramified covers, so we are saying that our stack admits a cover with total space being a scheme.

3.3 Quotient stacks

Let us study a specific type of algebraic stacks: quotients.

Definition 3.23 (G-fibration)**.** Let X be a scheme, G a group scheme. A **G-fibration** over X is a scheme ξ with an action $\mu : G \times \xi \to \xi$ and a G-invariant morphism $\pi : \xi \to X$, that is, we have a commutative diagram

$$\begin{tikzcd} G \times \xi \arrow[r, "\mu"] \arrow[d, "\pi_2"'] & \xi \arrow[d, "\pi"] \\ \xi \arrow[r, "\pi"'] & X \end{tikzcd}$$

Example 3.24. $\xi = G \times X$ with the obvious projection $\xi \to X$ is a G-fibration. This is called the **trivial** G-fibration.

Definition 3.25 (Principal G-bundle)**.** A **principal G-bundle** with respect to the fpqc (or fppf, étale etc) topology is a G-fibration which is locally trivial, i.e. there exists a cover $\{U_i \to X\}$ such that $\xi|_{U_i} \to U_i$ is isomorphic to $G \times U_i \to U_i$.

Proposition 3.26. Let X be a scheme and G a linear algebraic group acting on X via $\mu : X \times G \to X$. Assume that the action is free[a], then X/G exists as an algebraic space.
Moreover $\rho : X \to X/G$ is a principal G-bundle with respect to the étale topology and the following is cartesian

$$\begin{tikzcd} X \times G \arrow[r, "\pi_1"] \arrow[d, "\mu"'] & X \arrow[d, "\rho"] \\ X \arrow[r] & X/G \end{tikzcd}$$

[a]for all schemes $\mu_T : X(T) \times G(T) \to X(T)$ is a free action of $G(T)$

Remark 3.27. If we fix $f : U \to X/G$ and take ξ to be the pullback of $X \to X/G$ along $U \to X/G$

$$\begin{tikzcd} \xi \arrow[r, "\alpha"] \arrow[d, "\pi"'] & X \arrow[d, "\rho"] \\ U \arrow[r, "f"'] & X/G \end{tikzcd}$$

then

1. $\pi : \xi \to U$ is a G-bundle and
2. $\alpha : \xi \to X$ is G-equivariant.

Definition 3.28. Let X be a scheme and G a smooth linear algebraic group acting on X, the **quotient stack** $[X/G]$ is the fibered category over schemes

$$U \mapsto \{(\pi : \xi \to U, \alpha : \xi \to X) \mid \pi \text{ principal } G\text{-bundle, } \alpha \ G\text{-equivariant}\} / \sim$$

If $X = \operatorname{Spec} k$, $[\operatorname{Spec} k/G] = \mathbb{B}G$ is called the **classifying space** of G.

Theorem 3.29. $[X/G]$ is an algebraic stack

Let us quickly convince ourselves that $[X/G]$ should be an algebraic stack by looking at what the atlas should be:

The obvious candidate for an atlas is $\pi : X \to [X/G]$ (think "quotient map is a cover"). This morphism corresponds via the Yoneda lemma to a family of principal G-bundles over X, which is

$$\begin{tikzcd} X \times G \arrow[r, "\pi_1"] \arrow[d, "\mu"'] & X \\ X & \end{tikzcd}$$

If we fix $\varphi : U \to [X/G]$, this corresponds to a family

$$\begin{tikzcd} \xi \arrow[r, "\alpha"] \arrow[d, "\eta"'] & X \\ U & \end{tikzcd}$$

with $\eta : \xi \to U$ smooth and surjective (not obvious but provable)

3.3.1 Back to elliptic curves

Let us consider the scheme[2] $U = \operatorname{Spec} k[A, B, 1/\Delta]$ where $\Delta = -16(4A^3 + 27B^2)$, and let $G = \mathbb{G}_m = k[u^{\pm 1}]$. G actis on U by looking at equivalent Weierstrass forms via the change of variables

$$x \mapsto u^2 x, y \mapsto u^3 y.$$

Concretely, the action maps A to $u^{-4}A$, B to $u^{-6}B$ and Δ to $u^{-12}\Delta$.

Fact 3.30. $\mathcal{M}_{1,1} \cong [U/G]$.

[2] $U \subseteq \mathbb{A}^2$ is the possible pairs (A, B) of coefficients in a Weierstrass form for an elliptic curve. Localizing at the discriminant enforces that for any point $(A, B) \in U$, the curve $y^2 = x^3 + Ax + B$ is an elliptic curve.

Part II

Intersection theory

Chapter 4

???

Bibliography

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