Intersection Theory on Moduli Spaces

Reading group organized by
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Typed by Sorce Francesco

Foreword

These notes contain the topics that were discussed during the reading group

Intersection Theory on Moduli Spaces

which met in Pisa during the second semester of the 2024/25 accademic year. The reading group was organized by Leonardini Pietro and Tarini Bernardo.

Each chapter corresponds to a lecture. The name of the speaker is shown at the start of the corresponding chapter.

Some information was added while typing the notes to make the delivery clearer, but these notes have no aspiration to be thorough. For further reading you may refer to the bibliography. Each chapter also mentions which sources are more relevant.

The goal of the reading group was to lay out the basic theory of algebraic stacks and intersection theory in order to combine the two in the context of moduli theory.

In the first two chapters we lay out the basic categorical notions involved and we define stacks, which are then upgraded to algebraic stacks in the third chapter with the motivation of moduli theory.

The fourth, fith and sixth chapters give a brief overview of intersection theory over schemes. We start by defining Chow rings and then deal with their functoriality. We then move on to vector bundles and how they interact with Chow rings, in particular looking at Chern classes. We conclude with a brief discussion of equivariant intersection theory.

In the last chapters we give a tour of the article [Vis98] by Angelo Vistoli where he computes the Chow ring of $\mathcal{M}_2$. This will give a practical application of the theory we developed in the previous parts.

Here is a list of all speakers, sorted by occurrence of their first lecture:

1. Marino Domenico
2. Sorce Francesco
3. Leonardini Pietro
4. Gallese Andrea
5. Minnocci Francesco
6. Tarini Bernardo

I want to conclude this introduction by thanking Pietro and Bernardo for organizing the group and having been available and supportive, especially with regards to these notes.

Contents

Part I

Stacks

Chapter 1

Grothendieck topologies

Speaker: Domenico Marino

The references for this chaper are [Vis07] and [Ols16].

1.1 Grothendieck topology and sheaves

Recall that a presheaf on X is a functor

$$F : \mathrm{Open}(X)^{op} \to \mathrm{Set}$$

and that a sheaf is a presheaf that respects certain gluing properties over open covers.

As we will see in the next chapters, it will be useful to try and define sheaves over more general categories, that is, we want to substitute $\mathrm{Open}(X)$ with some other category. The only topological notions which come up when we want to define sheaves are open covers of open subsets of X, so if we want to generalize sheaves we need define what "open covers" for objects $X \in \mathcal{C}$ should be.

Definition 1.1. Let $\mathcal{C}$ be a category. We define a **Grothendieck topology** on $\mathcal{C}$ to be the following data: for all $U \in \mathcal{C}$ we have a collection $\mathrm{Cov}(U)$ of sets of arrows $\{U_i \to U\}$ in $\mathcal{C}$, called **covers of** U, and these covers respect the following properties:

1. If $V \to U$ is an isomorphism in $\mathcal{C}$ then $\{V \to U\} \in \mathrm{Cov}(U)$.

2. If $\{U_i \to U\} \in \mathrm{Cov}(U)$ and $V \to U$ arrow in $\mathcal{C}$ then we can fix fiber products $U_i \times_U V \to V$ such that $\{U_i \times_U V \to V\} \in \mathrm{Cov}(V)$.

3. If $\{U_i \to U\} \in \mathrm{Cov}(U)$ and for all i we have $\{V_{ij} \to U_i\} \in \mathrm{Cov}(U_i)$ then $\{V_{ij} \to U\} \in \mathrm{Cov}(U)$, where the arrows $V_{ij} \to U$ are given by the composition $V_{ij} \to U_i \to U$.

A category with a fixed Grothendieck topology is called a **site**.

Remark 1.2. According to [GD67], what we have defined is called a *pretopology*, a full Grothendieck topology is a looser notion which consideres pretopologies equivalent

when they give the same sheaf theory. We are going to ignore this subtelty from now on. For a little more detail on this distinction you may also look at [Vis07].

Example 1.3. Let X be a topological space and let $\mathrm{Open}(X)$ be the category of its open subsets (the arrows are inclusions). The category $\mathrm{Open}(X)$ with the usual notion of open covers is a site:

1. If $V \to U$ is an isomorphism then we have an inverse $U \to V$, but the arrows are inclusions so $V = U$ and of course U is an open cover of U.

2. Let $\{U_i\}$ be an open cover of U and let V is another open set in X which is contained in U. Note that $V \times_U U_i = V \cap U_i$ because of how fibered products work in $\mathrm{Open}(X)$. We conclude the proof of this property by noting that indeed $\{V \cap U_i\}$ is an open cover of V if $V \subseteq U$.

3. Let $\{U_i\}$ be an open cover of U and for all i let $\{V_{ij}\}$ be an open cover of U_i, then it is clear that $\{V_{ij}\}$ (now with i variable) is an open cover of U.

Notation. If $\{U_i \to U\}$ is a cover of U with respect to some Grothendieck topology, we take U_{ij} to be some choice for $U_i \times_U U_j$ and similarly for U_{ijk}. From now on we will implicitly fix such fibered products.

We now move on to the general definition of presheaves, separated presheaves and sheaves:

Definition 1.4. Let $\mathcal{C}$ be a category. A **presheaf** on $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. If $f : V \to U$ is an arrow in $\mathcal{C}$ and $\xi \in F(U)$ for F presheaf, we define the **pullback of ξ along f** (also called the **restriction of ξ to V**) to be

$$F(f)(\xi) \in F(V).$$

The pullback of ξ may be written as $f^*\xi$ or $\xi|_V$ when no ambiguity may arise.

Suppose now that $\mathcal{C}$ is also a site, then a presheaf F on $\mathcal{C}$ is a

- **separated presheaf** if for all $U \in \mathcal{C}$, all $\{u_i : U_i \to U\} \in \mathrm{Cov}(U)$ and all pairs of elements $\xi, \eta \in F(U)$, we have that

$$(\forall i \quad \xi|_{U_i} = \eta|_{U_i}) \implies \xi = \eta.$$

- **sheaf** if for all $U \in \mathcal{C}$, all covers $\{U_i \to U\} \in \mathrm{Cov}(U)$ and all collections $\xi_i \in F(U_i)$ for each i such that $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$ for all i, j, there exists a unique $\xi \in F(U)$ such that $\xi|_{U_i} = \xi_i$.

Remark 1.5. One may define sheaves that take values in Groups, Rings and so on instead of just Sets. The next chapter will essentially define sheaves that take values in categories but we are getting ahead of ourselves.

Definition 1.6. If F, G are sheaves on $\mathcal{C}$, a **morphism between** them is a natural transformation of functors $\mathcal{C}^{op} \to \mathrm{Set}$.

Remark 1.7. A sheaf is in particular a separated presheaf.

Theorem 1.8. Let $\mathcal{C}$ be a site. The forgetful functor $\mathrm{Sh}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C})$ has a left adjoint called **sheafification**.

Proof.
A full proof is in [Ols16]. The main idea is to compose left adjoints $\mathrm{Pre}(\mathcal{C}) \to \mathrm{Sep}\,\mathrm{Pre}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$. For the first piece we identify two elements of $F(U)$ when they agree on a cover of U, for the second we add all gluings. $\square$

1.2 Examples of Grothendieck topologies

As we have already noted, $\mathrm{Open}(X)$ for some topological space X defines a site when considering open covers in the classical sense. The first new site which we define is a way the *global* version of this

Definition 1.9. Let $\{U_i \to U\}$ be a collection of functions to U. We say that these arrows are **jointly surjective** if the union of the images is U.

Example 1.10 (Global classical topology)**.** Let $\mathcal{C} = \mathrm{Top}$. For any topological space $X \in \mathrm{Top}$ we define $\mathrm{Cov}(X)$ to be the collection of jointly surjective sets of open immersions into X, that is, $\{j_i : U_i \to X\} \in \mathrm{Cov}(X)$ if $\bigcup_i \mathrm{Imm}\, j_i = X$ and all j_i are open immersions.

Remark 1.11. We can't define $\mathrm{Cov}(X) = \{\text{open covers of } X\}$ in the previous example because, for instance, the first property in the definition of Grothendieck topology fails.

We have a more algebro-geometric version of this example given by

Example 1.12 (Global Zariski topology)**.** Let $\mathcal{C} = \mathrm{Sch}\,/S$. If X is a scheme over S then we define $\mathrm{Cov}(X)$ to be the collection of jointly surjective sets of open embeddings of schemes over S.

So far these sites have not been too out of the ordinary, but now we start to get to more sofisticated, yet useful, examples

Definition 1.13. Let $f : X \to Y$ be a morphism of schemes. We say that f is **formally unramified** (resp. **formally smooth** / **formally étale**) if for all affine Y-schemes $Y' \to Y$ and all closed embeddings $Y_0' \to Y'$ defined by nilpotent ideals, then map
$$\mathrm{Hom}(Y', X) \to \mathrm{Hom}(Y_0', X)$$
is injective (resp. surjective / bijective).
Moreover, if f is also locally of finte presentation then we say that f is **unramified** / **smooth** / **étale** when the same conditions hold.

With these definitions we can define

Example 1.14 (Small smooth / étale topology)**.** Let $\widetilde{\mathcal{C}} = \operatorname{Sch}/S$ and let $\mathcal{C}$ be the full subcategory with objects given by arrows $X \to S$ which are smooth / étale. If $X \in \mathcal{C}$ then we can define $\operatorname{Cov}(X)$ to be jointly surjective sets of maps which are all smooth / étale.

Example 1.15 (Big étale topology)**.** Let $\mathcal{C} = \operatorname{Sch}/S$. If $X \in \mathcal{C}$ then we define $\operatorname{Cov}(X)$ to be jointly surjective sets of étale maps.

1.2.1 The fppf and fpqc topologies

The classical and étale topologies are already useful, but they are not the gold-standard for many applications.

For some motivation, consider the following problem: recall that if $f : X \to Y$ is a morphism of schemes and $\mathcal{P}$ is some property of this morphism, we say that the property is **stable under base change** if for all $Y' \to Y$, when we build a cartesian square

$$
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{\;f'\;} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;f\;} & Y
\end{array}
$$

then if f has property $\mathcal{P}$ it follows that f' also has property $\mathcal{P}$.

We want to explore the converse in some sense, that is: suppose we have $f : X \to Y$ and fix a cover $\{Y_i \to Y\}$. If all restrictions $X \times_Y Y_i \to Y_i$ have property $\mathcal{P}$, when is it the case that f itself has property $\mathcal{P}$?

The covers in the classical and étale topolgies make it "too easy" to have this "local to global" condition be true, to the point where it is not easy at all to check that the property holds locally. To have this local-to-global principle being useful, we may require finer topologies. The fppf and fpqc are the usual answers when such questions arise.

Definition 1.16 (fppf topology)**.** Let $\mathcal{C} = \operatorname{Sch}/S$. A cover for $X \in \mathcal{C}$ in the **fppf topology** (it stands for *faithfully flat and locally of finite presentation*) is a jointly surjective collection of flat maps which are locally of finite presentation.

Definition 1.17. A morphism $f : X \to Y$ is **fpqc** (it stands for *faithfully flat and quasi-compact*) if it is flat, surjective and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

Remark 1.18. Equivalent definitions of fpqc are given in [Vis07].

In order to define the fpqc topology we require the following proposition, which you can find in [Vis07].

Proposition 1.19. The property of being fpqc is stable under composition and base change. Moreover, if $f : X \to Y$ and $\{V_i \to Y\}$ is an open cover (for the classical topology) such that $f|_{f^{-1}(V_i)} : f^{-1}(V_I) \to V_i$ is fpqc for all i then f is fpqc.

Definition 1.20 (fpqc topology). Let $\mathcal{C} = \mathrm{Sch}\,/S$. A cover for $X \in \mathcal{C}$ in the **fpcq topology** is a collection of scheme morphisms $\{U_i \to X\}$ such that the induced map $\coprod U_i \to X$ is fpqc.

Remark 1.21. The topologies that we have defined have become gradually finer, that is

$$\mathrm{Zariski} \subseteq \text{étale} \subseteq \mathrm{fppf} \subseteq \mathrm{fpqc}$$

We conclude the chapter by stating that the fpqc and fppf topologies to give an answer to the local-to-global question we started with:

Proposition 1.22. Let $f : X \to Y$ be a morphism of schemes, $\{Y_i \to Y\}$ be an fpqc cover of Y and f_i be the pullbacks $f_i : X \times_Y Y_i \to Y_i$. It is the case that if all f_i have property $\mathcal{P}$ then f has $\mathcal{P}$ for $\mathcal{P}$ in the following (non-exhaustive) list

- separable
- surjective
- locally of finte type
- locally of finite presentation
- proper
- flat

- affine
- finite
- smooth
- unramified
- étale

Remark 1.23. An analogous result holds for fppf where the list consists of properties of the form "is an immersion" (for example open, closed and locally closed immersions).

Chapter 2

Fibered categories and Stacks

Speaker: Francesco Sorce

Everything in this chapter is talked about in greater detail in [Vis07].

2.1 Fibered categories

2.1.1 Definition

Definition 2.1. If we have the data of $\mathcal{F}$ and $\mathcal{C}$ two categories and $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ a functor we say that $\mathcal{F}$ is **over** $\mathcal{C}$.
In this context, if $p_{\mathcal{F}}(\xi) = U$ for some object ξ we say that ξ is over U.
Similarly if $p_{\mathcal{F}}(\xi \to \eta) = U \to V$ then $\xi \to \eta$ is over $U \to V$.

Notation. In the diagrams that follow, a normal or dashed arrow will be a morphism, while an allow with a tail (like $\mapsto$) represents applying the functor.

Definition 2.2. An arrow $\phi : \xi \to \eta$ in $\mathcal{F}$ over $\mathcal{C}$ is **cartesian** if for all $\phi : \zeta \to \eta$ and all $h : p_{\mathcal{F}}\zeta \to p_{\mathcal{F}}\xi$ which make the following diagram commute there exists a unique arrow $\zeta \to \xi$ in place of the dotted arrow:
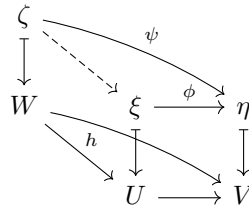
If $\xi \to \eta$ is cartesian and over $f : U \to V$ we say that ξ is the **pullback of** η **along** f. We may write $\xi = f^*\eta$ if we fix a choice of pullback.

Definition 2.3. Let us fix a category $\mathcal{C}$. The associated **arrow category** $\mathrm{Arr}(\mathcal{C})$ is the category whose objects are arrows in $\mathcal{C}$ and whose morphisms are commutative squares in $\mathcal{C}$.

Remark 2.4. We may view $\mathrm{Arr}(\mathcal{C})$ as a category over $\mathcal{C}$ by fixing the functor that to an arrow $X \to U$ assigns U.

Example 2.5. In order to better understand cartesian arrows (and to see where the name comes from) we determine which arrows in $\mathrm{Arr}(\mathcal{C})$ are cartesian.

If an arrow $(X \to U) \to (Y \to V)$ is cartesian, consider the following data:
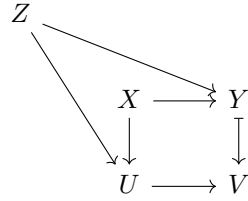
which we may equivalently rewrite
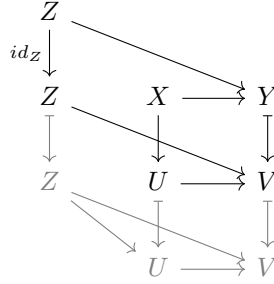
Since the arrow is cartesian, there is a unique way to complete the diagram above with maps $Z \to X$ and $Z \to U$. Since we ask for compatibility with the functor, $Z \to U$ is already known, so we have constructed $Z \to X$ in the first diagram. This makes us suspect that cartesian arrows in $\mathrm{Arr}(\mathcal{C})$ correspond to cartesian squares in $\mathcal{C}$, and this is true.

Indeed, suppose that $(X \to U) \to (Y \to V)$ is a cartesian square and let us consider the data
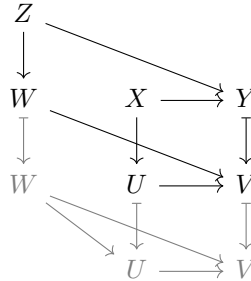
then we have a unique way to complete the diagram. The bottom map is determined by compatibility with the functor towards $\mathcal{C}$ and the top arrow can be chosen to be the one induced by the fact that we have a cartesian square an maps $Z \to Y$, $Z \to U$, where the second one is the composition $Z \to W \to U$.

Fact 2.6. Cartesian arrows satisfy the following properties:

1. the composition of cartesian arrows is cartesian

2. if $\xi \to \zeta$ factors through η with $\eta \to \zeta$ being cartesian, then $\xi \to \zeta$ is cartesian if and only if $\xi \to \eta$ is

3. if ϕ is over an isomorphism then ϕ is cartesian if and only if it is an isomorphism

Definition 2.7. A category $\mathcal{F}$ over $\mathcal{C}$ is **fibered over** $\mathcal{C}$ if for all $f : U \to V$ in $\mathcal{C}$ and all $\eta \in \mathcal{F}$ over V there exists a cartesian arrow $\phi : \xi \to \eta$ over f. That is, given a "partial diagram" of the form

$$
\begin{array}{c}
\eta \\
\big\downarrow \\
U \xrightarrow{\ f\ } V
\end{array}
$$

we can make it a square in such a way that the top side is a cartesian arrow.

Definition 2.8. Let $\mathcal{F}$ and $\mathcal{G}$ be fibered categories over $\mathcal{C}$. A **morphism of fibered categories over** $\mathcal{C}$ from $\mathcal{F}$ to $\mathcal{G}$ is a functor $F : \mathcal{F} \to \mathcal{G}$ such that $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$ which preserves cartesian arrows, i.e. if $\phi \in \mathcal{F}$ cartesian then $F(\phi)$ is cartesian in $\mathcal{G}$.

Example 2.9. Because of what we have already said, $\mathrm{Arr}(\mathcal{C})$ is fibered over $\mathcal{C}$ if and only if $\mathcal{C}$ admits fibered products.

2.1.2   Fibers

We now reach one of the main definitions which motivate fibered categories

Definition 2.10. Let $\mathcal{F}$ be a fibered over $\mathcal{C}$ and fix an object $U \in \mathcal{C}$. The **fiber of** $\mathcal{F}$ **over** U is the subcategory $\mathcal{F}(U)$ of $\mathcal{F}$ whose objects are the objects of $\mathcal{F}$ over U and whose morphisms are those over id_U.

Example 2.11. Fix $U \in \mathcal{C}$. The fiber $\mathrm{Arr}(\mathcal{C})(U)$ is the comma category $\mathcal{C}/U$.

Remark 2.12. If $f : U \to V$ is a morphism in $\mathcal{C}$ and $\mathcal{F}$ is fibered over $\mathcal{C}$ then we may define a "restriction functor"

$$
\begin{array}{ccc}
\mathcal{F}(V) & \longrightarrow & \mathcal{F}(U) \\
X & \longmapsto & f^*X
\end{array}
$$

upon choosing a unique pullback for each element. Such a choice is called a **cleavage**. For more details see [Vis07].

Remark 2.13. If $F : \mathcal{F} \to \mathcal{G}$ is a morphism of fibered categories over $\mathcal{C}$ and $U \in \mathcal{C}$ then it induces a functor $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

These definitions make $\mathcal{F}$ look like a presheaf $\mathcal{C}^{op} \to \mathrm{Cat}$

This is not quite correct because choosing a cleavage may lead to having two arrows that should be the same being naturally isomorphic instead. To be technically correct we would need to introduce pseudo-functors, but they behave like normal functors in basically every way.

Fact 2.14. There is a correspondence between categories fibered over $\mathcal{C}$ and pseudo-functors $\mathcal{C}^{op} \to \mathrm{Cat}$.

Proof (Idea).
To get the functor from the category fix a cleavage, then take an object to the fiber over it and a map to the pullback along it.

To get a category from the functor consider the category with objects (U, X) for $U \in \mathcal{C}$ and $X \in \mathcal{F}(U)$ and morphisms defined in the obvious way. The functor from this category to $\mathcal{C}$ is the projection on the first factor. The fact that this comes from a pseudo-functor automatically gives the existence of pullbacks. $\square$

Definition 2.15. A category is called a **set** if its objects form a set and the only morphisms are of the form id_X for some object X in the category.

Definition 2.16. A category is called a **groupoid** if every arrow in the category is an isomorphism.

Definition 2.17. Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$. We say that $\mathcal{F}$ is

- **fibered in sets** if $\mathcal{F}(U)$ is a set for all $U \in \mathcal{C}$

- **fibered in groupoids** if $\mathcal{F}(U)$ is a groupoid for all $U \in \mathcal{C}$.

Fact 2.18. A cateogry is fibered in sets if and only if the associated pseudo-functor is a functor in the usual sense.

Example 2.19. Let $\mathcal{C}$ be a (locally small) category and let us fix $X \in \mathcal{C}$. We have a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ which maps X to $h_X = \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$. This functor defines a category fibered over $\mathcal{C}$. The objects of this category are pairs $(Y, Y \to X)$ (which we will identify with $Y \to X$ itself) and a morphism from $Y \to X$ to $Z \to X$ is an arrow $Y \to Z$ in $\mathcal{C}$ such that $Y \to X = Y \to Z \to X$.

The fiber of this category over $Y \in \mathcal{C}$ is $\{Y \to X\} = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ but seen as a category. Since arrows in a fiber must be over the identify of the object of which they

are the fiber, the only arrows in $\mathrm{Hom}(Y,X)$ are the identity of Y. This shows that the category we got from X is fibered in sets, which is what we expected because it came from a functor.

Schematically, we have the following embeddings and equivalences

$$\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \longrightarrow \mathrm{Fibered}/\mathcal{C}$$

Representable   Fibered in sets

Proposition 2.20. Let $\mathcal{F}$ be a category over $\mathcal{C}$ (we do not assume fibered). $\mathcal{F}$ is fibered in groupoids over $\mathcal{C}$ if and only if the following conditions hold:

1. every arrow in $\mathcal{F}$ is cartesian

2. given a partial diagram $f : U \to V$ and $\eta \in \mathcal{F}$ over V, the exists some $\phi : \xi \to \eta$ over f.

Sketch.
If $\mathcal{F}$ is fibered in groupoids, 2. holds by fiberedness and 1. is a consequence of point 2 in fact 2.6.

Viceversa, if 1. and 2. hold then $\mathcal{F}$ is fibered. Since all arrows in a fiber are over an isomorphism (the identity of the object we are taking the fiber), again by fact 2.6 we see that all arrows must be isomorphisms because they are cartesian. $\square$

2.2 Stacks

Now that we have explored the formalism of fibered categories, let us see how it interacts with the notion of Grothendieck topology we dealt with in the previous chapter.

2.2.1 Motivating example

Let us consider the arrow category associated to the category of topological spaces Arr(Top). We may interpret objects in this category to be *spaces lying above a base space*. In what follows it may be useful to imagine objects in this category to be things like covering maps or fiber bundles, though of course the notion of "continuous maps" is much more general.

Taking this as our point of view, two concepts we may want to explore are the following:

- Suppose we have two objects over the same base space and an open cover for that base. Under what conditions can we construct a map between them locally?

- Suppose we have an open cover and that for each element of the cover we have a space over it. Under what conditions can we glue these pieces to get an object over the entire base space?

Studying these questions will eventually lead us to what a prestack and a stack should be.

Recall that we have a standard Grothendieck topology on Top for which

$$\text{Cov}(U) = \{\{U_i \to U\} \mid \text{jointly surjective open immersions}\}.$$

Notation. We will write U_{ij} instead of $U_i \cap U_j$ and similarly U_{ijk} for triple intersections.

Gluing maps

Suppose we fix $U \in \text{Top}$ and an open cover[1] $\{U_i \to U\}$. We also fix two objects $\xi : X \to U$ and $\eta : Y \to U$. The answer to our first question can be summerized in the following proposition:

Proposition 2.21. Let U, $\{U_i\}$, ξ, η be as above. Suppose we have maps $f_i : \xi^{-1}(U_i) \to \eta^{-1}(U_i)$ for all i such that

$$f_i|_{\xi^{-1}(U_{ij})} = f_j|_{\eta^{-1}(U_{ij})},$$

then there exists a unique $f : X \to Y$ which is compatible with the maps to U and such that $f|_{\xi^{-1}(U_i)} = f_i$ for all i.

Notice that we may restate the proposition as

Proposition 2.22. The functor

$$\underline{\text{Hom}}_U(X,Y) : \begin{array}{ccc} (\text{Top}/U)^{op} & \longrightarrow & \text{Set} \\ Z \to U & \longmapsto & \text{Hom}_Z(X \times_U Z, Y \times_U Z) \end{array}$$

is a sheaf with respect to the Grothendieck topology induced by Top on Top $/U$.

To see the equivalence, note that $X \times_U U_i = \xi^{-1}(U_i)$ and $Y \times_U U_i = \eta^{-1}(U_i)$, so when you evaluate the functor on the open cover of U, the fact that it is a sheaf corresponds exactly to the gluing property.

Gluing spaces

We now look at what it means to glue spaces using a cover:

Proposition 2.23. Let us fix a topological space U together with an open cover $\{U_i\}$. For each i let $u_i : X_i \to U_i$ be an element of Arr(Top). We also fix isomorphisms $\phi_{ij} : u_j^{-1}(U_{ij}) \to u_i^{-1}(U_{ij})$ for each pair of indices in such a way so that

$$\phi_{ik}|_{u_k^{-1}(U_{ijk})} = \phi_{ij}|_{u_j^{-1}(U_{ijk})} \circ \phi_{jk}|_{u_i^{-1}(U_{ijk})}.$$

It follows that there exists $u : X \to U$ continuous, together with homeomorphisms $\phi_i : u^{-1}(U_i) \to X_i$ such that $\phi_{ij} = \phi_i|_{u^{-1}(U_{ij})} \circ (\phi_j|_{u^{-1}(U_{ij})})^{-1}$.

---

[1]in the standard sense. Everything works the same with jointly surjective open immersions but the notation gets bothersome rather quickly.

Remark 2.24. The notation is rather tedious, but the geometric image is actually rather simple: the ϕ_{ij} serve as a guide to tell us which points should be identified and the condition $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ (usually called a **cocycle condition**) ensures that these identifications define an equivalence realtion on $\coprod X_i$.
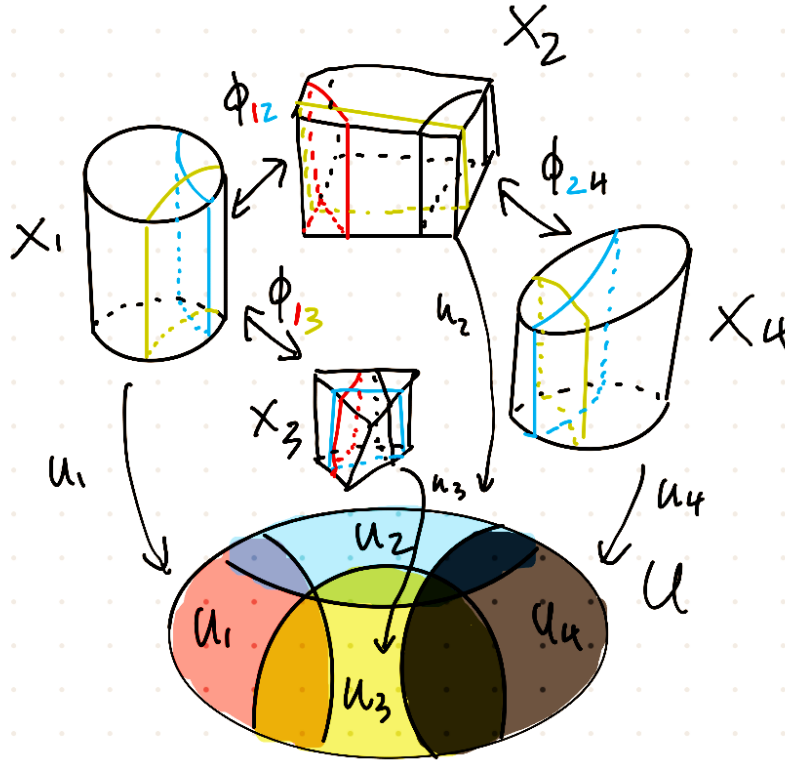
2.2.2 Descent data

We are now ready to move onto stacks. The main idea is to generalize the type of *local data* that we used to glue before.

For the rest of the chapter $\mathcal{C}$ is a site with a fixed Grothendieck topology and $\mathcal{F}$ is fibered category over $\mathcal{C}$.

Definition 2.25. Fix $U \in \mathcal{C}$ and a cover $\{U_i \to U\} \in \mathrm{Cov}(U)$. An **object with descent data** is a pair $(\{\xi_i\}, \{\phi_{ij}\})$ where $\xi_i \in \mathcal{F}(U_i)$ for all i and $\phi_{ij} : \mathrm{pr}_2^*\xi_j \to \mathrm{pr}_1^*\xi_i$ are isomorphisms which live in $\mathcal{F}(U_{ij})$ such that

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk}.$$

The ϕ_{ij} are called **transition isomorphisms**.

If you are (very reasonably) getting confused with the pullbacks and indices, it

may (or may not) be useful to admire *the hypercube*

$$
\begin{array}{c}
\text{pr}_1^*\xi_j \xleftarrow{\phi_{jk}} \text{pr}_2^*\xi_k \longrightarrow \xi_k \\
\end{array}
$$
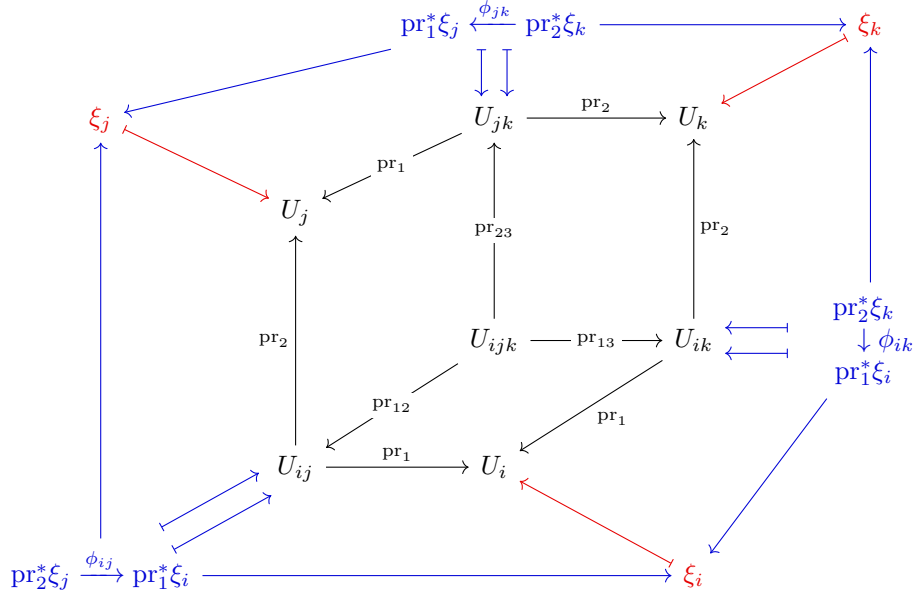
Notation. Let $\mathcal{C}$ be a site, $\mathcal{F}$ fibered over $\mathcal{C}$, $U \in \mathcal{C}$ and $\{U_i \to U\} \in \text{Cov}(U)$. Objects with descent data over $\{U_i \to U\}$ form a category[2], which we denote using

$$\mathcal{F}(\{U_i \to U\}).$$

Remark 2.26. Fix $U \in \mathcal{C}$, $\{u_i : U_i \to U\} \in \text{Cov}(U)$ and $\xi \in \mathcal{F}(U)$. From ξ we can obtain an object with descent data by restriction. To be more precise, we can define

$$\xi_i = u_i^*\xi, \qquad \phi_{ij} : \text{pr}_2^* u_j^*\xi \to \text{pr}_1^* u_i^*\xi$$

where ϕ_{ij} is the canonical isomorphism induced by the fact that both objects amount to the pullback of ξ along $U_{ij} \to U$. The resulting object with descent data is $(\{\xi_i\}, \{\phi_{ij}\})$.

What we have described is actually a functor from $\mathcal{F}(U)$ to $\mathcal{F}(\{U_i \to U\})$.

Definition 2.27. An object with descent data is called **effective** if it is in the essential image of the functor above.

We are now finally ready to define prestacks and stacks:

---

[2]the morphisms are collections of maps and commutative diagrams which are compatible with the indices.

Definition 2.28. Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$ site. $\mathcal{F}$ is a

- **prestack** if for all $U \in \mathcal{C}$ and all $\{U_i \to U\} \in \mathrm{Cov}(U)$, the functor

$$\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$$

  is *fully faithful*.

- **stack** if for all $U \in \mathcal{C}$ and all $\{U_i \to U\} \in \mathrm{Cov}(U)$, the functor

$$\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$$

  is an *equivalence*.

Remark 2.29. The two conditions can be intuitively understood as:

- A prestack is a fibered category where if two objects are the same locally (taking into account gluing) then they are isomorphic.

- A stack is a prestack where all descent data glues to an object.

Another way in which we can intepret the prestack condition is given in the following

Proposition 2.30. Let $\mathcal{F}$ be fibered over $\mathcal{C}$ site. $\mathcal{F}$ is a prestack if and only if for all $U \in \mathcal{C}$ and all $\xi, \eta \in \mathcal{F}(U)$, the functor

$$\underline{\mathrm{Hom}}_U(\xi, \eta) : \begin{array}{ccc} (\mathcal{C}/U)^{op} & \longrightarrow & \mathrm{Set} \\ f : Z \to U & \longmapsto & \mathrm{Hom}_Z(f^*\xi, f^*\eta) \end{array}$$

is a sheaf on $\mathcal{C}/U$.

Sketch.
Fix $U \in \mathcal{C}$ and $\{U_i \to U\} \in \mathrm{Cov}(U)$. The functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ being fully faithful means that for all $\xi, \eta \in \mathcal{F}(U)$ we have

$$\mathrm{Hom}_U(\xi, \eta) \cong \mathrm{Hom}_{\{U_i \to U\}}((\{\xi_i\}, \{\alpha_{ij}\}), (\{\eta_i\}, \{\beta_{ij}\}))$$

where $(\{\xi_i\}, \{\alpha_{ij}\})$ is the image of ξ and similarly for the other object and η.

Concretely, this means that maps $\xi \to \eta$ can be defined uniquely on an open cover, which is exactly what it means for $\underline{\mathrm{Hom}}_U(\xi, \eta)$ to be a sheaf. $\square$

Corollary 2.31. $\mathrm{Arr}(\mathrm{Top})$ is a stack.

We close the chapter by noting that when $\mathcal{F}/\mathcal{C}$ is fibered in sets, we find the notions of separated presheaf and sheaf over $\mathcal{C}$ respectively, meaning that we have, in some very precise sense, defined *sheaves in categories*. This concept will be trated in more detail in the next chapter.

Proposition 2.32. Let $\mathcal{C}$ be a site and consider a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. Identify F with the fibered category it defines. Then

- F is a prestack if and only if it is a separated presheaf,

- F is a stack if and only if F is a sheaf.

Example 2.33. The fibered category associated to an object $X \in \mathcal{C}$ is a stack because it corresponds to the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$.

Chapter 3

Algebraic Stacks

Speaker: Pietro Leonardini

3.1 Motivation: moduli problems

3.1.1 The dream

Let us recall what an elliptic curve is:

Definition 3.1 (Elliptic curve)**.** An **elliptic curve** over k is a 1-dimensional scheme $C \to \operatorname{Spec} k$ over k which is geometrically integral, proper, smooth and of genus[a] 1 with a fixed k-rational point $e : \operatorname{Spec} k \to C$.

---
[a]the genus is $\dim_k H^1(C, \mathcal{O}_C)$

Since the relative approach is central in the theory of schemes, we may want to define an elliptic curve over some base scheme S. The most sensible approach is to say that an elliptic curve over S should be a "family of elliptic curves parametrized by S". Formally:

Definition 3.2 (Elliptic curve over S)**.** An **elliptic curve** over S is a proper, smooth, flat[a] morphism of schemes $p : E \to S$ with a fixed section $e : S \to E$ of p such that for all geometric points $\operatorname{Spec} \Omega \to S$, the fiber is an elliptic curve over Ω:

$$
\begin{array}{ccc}
E \times_S \operatorname{Spec} \Omega & \longrightarrow & E \\
\downarrow & & \downarrow{\scriptstyle p} \\
\operatorname{Spec} \Omega & \longrightarrow & S
\end{array}
$$

---
[a]Flatness corresponds in a vague sense to the property that the fibers should "deform continuously". This notion can be made more precise, for example by looking at Hilbert polynomials among other things, but this vague intuition will be enough for our purposes.

When studying elliptic curves it would be useful to find some space $M_{1,1}$ such that for every family $E \to S$ we have a unique $\phi : S \to M_{1,1}$ such that

$$
\begin{array}{ccc}
E & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\ \phi\ } & M_{1,1}
\end{array}
$$

This ϕ should be thought of naively as the map that to the point $P \in S(\Omega)$ assigns the point of $M_{1,1}(\Omega)$ that corresponds to the isomorphism class of the elliptic curve given by the fiber of $p : E \to S$ over P.

In other words, we would be very happy if all families of elliptic curves could be described as the pullback of some *universal family* $\mathcal{E}$ over $M_{1,1}$ along some morphism $S \to M_{1,1}$. Intuitively, the universal family should be the one defined by setting the fiber over a point of $M_{1,1}$ to be the elliptic curve corresponding to that point.

A useful way to formulate this problem is to define the following functor[1]

$$
\begin{array}{ccccc}
 & \mathrm{Sch}^{op} & \longrightarrow & & \mathrm{Set} \\
F : & S & \longmapsto & \{E \to S \mid \text{elliptic curves}\}\,/\text{iso.} \\
 & f : T \to S & \longmapsto & (E_1 \to S) \mapsto (f^*E_1 \to T)
\end{array}
$$

We can re-formulate the requirements on $M_{1,1}$ by saying that $M_{1,1}$ should **represent** this functor, i.e. there is a natural isomorphism of functors

$$
h_{M_{1,1}} \cong F
$$

where $h_{M_{1,1}}$ is the contravariant Hom-functor associated to $M_{1,1}$.

We recall the following simple but foundational result

Theorem 3.3 (Yoneda lemma). There is a natural correspondence

$$
\mathrm{Hom}(h_X, F) \leftrightarrow F(X)
$$

Sketch.
Suppose we have a natural transformation $\phi : h_X \to F$, then we can find an element of $F(X)$ by taking $\phi_X(id_X) = \xi$.

Let us now fix an element $\xi \in F(X)$. Let T be a scheme. We may construct a map $\mathrm{Hom}(T, X) \to F(T)$ by sending $f : T \to X$ to $F(f)(\xi)$. If $g : T \to S$ is a morphism then $\eta : f \mapsto g \circ f$ is mapped to $F(g) : F(S) \to F(T)$. It is easy to check that this defines a natural transformation $h_X \to F$ and that the two constructions are inverses of each other. $\square$

Corollary 3.4. The functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ that sends X to h_X is fully faithful.

---

[1] we technically have no guarantee that families over a scheme should form a set. This issue will be solved automatically later when we consider a fibered category over Sch instead of a functor.

3.1.2 A good attempt

Let us try to find some way of parametrizing isomorphism classes of elliptic curves. The main idea comes from the following fact:

Fact 3.5. If char $k \neq 2, 3$ then every elliptic curve is isomorphic over $\overline{k}$ to some

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

for some $\lambda \in k \setminus \{0, 1\}$.

Definition 3.6 (j-invariant). Let

$$j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}$$

Fact 3.7. The map

$$\phi : \begin{array}{ccc} \overline{k} \setminus \{0,1\} & \longrightarrow & \overline{k} \\ \lambda & \longmapsto & j(E_\lambda) \end{array}$$

is surjective and $6 : 1$ except in $j = 0$ and $j = 1728$, where it is $2 : 1$ and $3 : 1$ respectively.

Fact 3.8. Two elliptic curves are isomorphic over $\overline{k}$ if and only if they have the same j-invariant.

Theorem 3.9. Let $E \to S$ be an elliptic curve, then there exists a Zariski affine open cover of S given by $\{U_i = \operatorname{Spec} A_i\}$ such that for all i

$$E_i := \operatorname{Spec}\left(\frac{A_i[x,y]}{(y^2 - x^3 - Ax - B)}\right) \longrightarrow E$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\operatorname{Spec} A_i \longrightarrow S$$

for some $A, B \in A_i$.

Remark 3.10. With $E \to S$ elliptic curve and a cover like above, the j-invariants of $E_i \to \operatorname{Spec} A_i$ are sections of the structure sheaf of $\operatorname{Spec} A_i$ which coincide on the intersection, so they glue to a section $j \in \Gamma(S, \mathcal{O}_S)$.

From this discussion it follows that the best candidate for $M_{1,1}$ is $\mathbb{A}^1$ where the curve over $j \in \mathbb{A}^1$ is the one with that j-invariant. Unfortunately $\mathbb{A}^1$ does not work.

Example 3.11. Suppose $\mathbb{A}^1$ represents the functor F from before. Consider then the two families

$$
\begin{array}{ccc}
\mathrm{Spec}\left(\frac{k[x,y,t^{\pm}]}{(y^2-x^3+t)}\right) & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, k[t,t^{-1}] & \longrightarrow & \mathbb{A}^1
\end{array}
$$

and

$$
\mathrm{Spec}\left(\frac{k[x,y]}{(y^2-x^3+1)}\right) \times \mathrm{Spec}\left(k[t^{\pm}]\right)
$$

$$
=
$$

$$
\begin{array}{ccc}
\mathrm{Spec}\left(\frac{k[x,y,t^{\pm}]}{(y^2-x^3+1)}\right) & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, k[t,t^{-1}] & \longrightarrow & \mathbb{A}^1
\end{array}
$$

If $\mathbb{A}^1$ represented the functor, these two families should be isomorphic since for every specific choice $t \neq 0$ we have that $y^2 = x^3 - t$ and $y^2 = x^3 - 1$ have the same j-invariant and so both families induce the same (constant) map $\mathrm{Spec}\, k[t^{\pm}] \to \mathbb{A}^1$.

However, the two families are NOT isomorphic by the following argument: It is possible to prove via the j-invariants that two Weierstrass equations yield isomorphic elliptic curves over $\mathrm{Spec}\, k[t^{\pm}]$ if and only if there exists some $u \in k[t^{\pm}]$ such that substituting x with $u^2 x$ and y with $u^3 y$ transforms one equation into the other up to factoring out powers of u. Therefore we require for there to exist some u such that

$$
u^6 y^2 - u^6 x^3 + t = u^k(y^2 - x^3 + 1) \iff t = u^6
$$

and this is impossible.

Remark 3.12. A slogan to keep in mind is

The presence of non-trivial automorphisms prevent the moduli problem from having a fine moduli space.

Example 3.13. If we consider the same families from before but now over the base $\mathrm{Spec}\, \frac{k[t^{\pm 1},w]}{(t-w^6)}$ instead of $\mathrm{Spec}\, k[t^{\pm 1}]$, they now become isomorphic, that is

$$
\frac{k[x,y,t^{\pm},w]}{(y^2 - x^3 + 1, w^6 - t)} \cong \frac{k[x,y,t^{\pm},w]}{(y^2 - x^3 + t, w^6 - t)}.
$$

This is because now we can choose $u = w$ with the notation from before.

The idea behind this observation is that if we allow ourselves to look at finer topologies[2] on Sch we may be able to solve our representability issue. We are starting to see how viewing F as a functor is not enough; we should view it as some kind of sheaf.

3.1.3 Our dreams crumble?

So, what now? We have two options:

---

[2]In this case, the natural map $\mathrm{Spec}\, \frac{k[t^{\pm 1},w]}{(t-w^6)} \to \mathrm{Spec}\, k[t^{\pm 1}]$ is an étale cover.

- Rigidify the problem hoping that we get something representable

- Enlarge the category among which we search for the moduli space (Stacky approach)

Notation. From now on we identify the scheme X with the functor $h_X : \mathrm{Sch}^{op} \to \mathrm{Set}$ and we identify it with the appropriate fibered category over Sch.

Remark 3.14. $X(k) = \mathrm{Hom}(\mathrm{Spec}\, k, X)$.

The main idea:

Instead of the functor F from before **we consider the category $\mathcal{M}_{1,1}$ fibered in groupoids** over Sch defined the same way.

Remark 3.15. The issue with families forming sets is now irrelevant. To be more precise, the groupoid associated to a scheme S is the one whose objects are elliptic curves over S and whose morphisms are isomorphisms of families (making this category a groupoid by construction).

Theorem 3.16. The fibered category $\mathcal{M}_{1,1}$ is a stack for the fpqc topology[a].

---

[a]and thus for the coarser topologies like Zariski or Étale too.

Proof.
Olsson [Ols16], chapter 13 □

Remark 3.17. A morphism $\phi : X \to \mathcal{M}_{1,1}$ of stacks corresponds by (a version of) the Yoneda lemma to an element of the fiber $\mathcal{M}_{1,1}(X)$.

Said another way, there is a correspondence between families of curves over X (the category $\mathcal{M}_{1,1}(X)$) and morphisms $X \to \mathcal{M}_{1,1}$.

From a categorical point of view, $\mathcal{M}_{1,1}$ does exactly what we want. Now we try to understand in which ways it can be thought of as a space.

3.2 Algebraic stacks

If we want to think about stacks in a geometric way, we may want to impose some "good behaviour" conditions. For example, one thing we may require is that if two "normal" objects (for example schemes) live inside our stack, their intersection should still be a "normal" object. The key definition to formalize this idea is

Definition 3.18 (Representable morphism). Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks, then it is said to be **representable** if for all schemes T and morphisms $T \to \mathcal{Y}$ there exists an algebraic space[a] V such that

$$
\begin{array}{ccc}
V & \longrightarrow & T \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y}
\end{array}
$$

is cartesian.

We say that a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ has a property $\mathcal{P}$ for $\mathcal{P}$ a property of morphisms of schemes/algebraic spaces which is stable under base change if the morphism is representable and for all $T \to \mathcal{Y}$, the induced arrow $V \to T$ as above has the property $\mathcal{P}$.

––––––––––––––––––––

[a]An **algebraic space** is a gluing of affine schemes with respect to the étale topology instead of the Zariski topology.

Remark 3.19. The properties "begin smooth", "begin surjective" and "begin étale" are stable under base change, so it makes sense to talk about smooth/surjective/étale morphisms of stacks.

Definition 3.20 (Algebraic stack). A stack over a scheme S, i.e. a morphism of stacks $\mathcal{X} \to S$, is said to be **algebraic** if

1. the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and

2. there exists a smooth and surjective morphism $\pi : X \to \mathcal{X}$ for X scheme. This scheme is called an **atlas** for the algebraic stack.

If $\pi : X \to \mathcal{X}$ is étale then $\mathcal{X} \to S$ is called a **Deligne-Mumford stack**.

Remark 3.21. Property 1. looks very mysterious but the following is an equivalent and, perhaps clearer, formulation: for all T, U schemes, the fibered product $T \times_{\mathcal{X}} U$ exists and is an algebraic space. We may think of this as saying that "intersecting schemes in a stack gives an algebraic space".

Thinking about this set-theoretically may help in understanding the equivalence.

Remark 3.22. Property 2. can be thought of as a "finiteness" condition. The case of Deligne-Mumford stacks makes this even clearer because étale morphisms can be thought of as unramified covers, so we are saying that our stack admits a cover with total space being a scheme.

3.3   Quotient stacks

In this section we will explore quotient stacks and they essentially allow us to generalize in algebraic geometry the concept of taking the quotient of a space under the action of a group. There is a different (but related) machinery for the same general goal called *Geometric Invariant Theory* (GIT) but the aspired results are different. In

GIT one is concerned with *when quotients exist* in some category (usually schemes), while quotient stacks allow us to *define something that behaves like a quotient* without worrying about these existence questions but forcing us to generalize our notion of "space".

Like any stack, quotient stacks are categories. Let us define the objects:

Definition 3.23 (*G*-fibration)**.** Let X be a scheme, G a group scheme. A *G*-**fibration** over X is a scheme ξ with an action $\mu : G \times \xi \to \xi$ and a G-invariant morphism $\pi : \xi \to X$, that is, we have a commutative diagram

$$
\begin{array}{ccc}
G \times \xi & \xrightarrow{\mu} & \xi \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi} \\
\xi & \xrightarrow{\pi} & X
\end{array}
$$

Example 3.24. $\xi = G \times X$ with the obvious projection $\xi \to X$ is a G-fibration. This is called the **trivial** G-fibration.

Definition 3.25 (Principal G-bundle)**.** A **principal G-bundle** with respect to the fpqc (or fppf, étale etc) topology is a G-fibration which is locally trivial, i.e. there exists a cover $\{U_i \to X\}$ such that $\xi|_{U_i} \to U_i$ is isomorphic to $G \times U_i \to U_i$.

Proposition 3.26. Let X be a scheme and G a linear algebraic group acting on X via $\mu : X \times G \to X$. Assume that the action is free[a], then X/G exists as an algebraic space.

Moreover $\rho : X \to X/G$ is a principal G-bundle with respect to the étale topology and the following is cartesian

$$
\begin{array}{ccc}
X \times G & \xrightarrow{\pi_1} & X \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \rho} \\
X & \longrightarrow & X/G
\end{array}
$$

---

[a]for all schemes $\mu_T : X(T) \times G(T) \to X(T)$ is a free action of $G(T)$

Remark 3.27. If we fix $f : U \to X/G$ and take ξ to be the pullback of $X \to X/G$ along $U \to X/G$

$$
\begin{array}{ccc}
\xi & \xrightarrow{\alpha} & X \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \rho} \\
U & \xrightarrow{f} & X/G
\end{array}
$$

then

1. $\pi : \xi \to U$ is a G-bundle and

2. $\alpha : \xi \to X$ is G-equivariant.

Definition 3.28. Let X be a scheme and G a smooth linear algebraic group acting on X, the **quotient stack** $[X/G]$ is the fibered category over schemes

$$U \mapsto \{(\pi : \xi \to U, \alpha : \xi \to X) \mid \pi \text{ principal } G\text{-bundle}, \alpha \ G\text{-equivariant}\} / \sim$$

If $X = \operatorname{Spec} k$, $[\operatorname{Spec} k / G] = \mathbb{B}G$ is called the **classifying space** of G.

Let's pause for a moment to understand why this definition makes sense. As defined, the quotient stack is the moduli space for the problem of classifying principal G-bundles with an equivariant map to X. So maps to $[X/G]$ correspond to this data.

On the other hand, what should maps to a quotient of X by G entail? As in remark (3.27) by pullback of $X \to X/G$ via such a map we should get a G-bundle and a G-equivariant map to X.

Putting these facts together, $[X/G]$ behaves like a categorical quotient should.

Theorem 3.29. $[X/G]$ is an algebraic stack

Partial idea.

Let us convince ourselves that $[X/G]$ should be an algebraic stack with atlas X:

We want a morphism $\pi : X \to [X/G]$ (which intituvely should be the quotient map). Such a morphism corresponds via the Yoneda lemma to a family of principal G-bundles over X, which we can choose to be

$$\begin{array}{ccc} X \times G & \xrightarrow{\pi_1} & X \\ {\scriptstyle \mu} \downarrow & & \\ X & & \end{array}$$

If we fix $\varphi : U \to [X/G]$, this corresponds to a family

$$\begin{array}{ccc} \xi & \xrightarrow{\alpha} & X \\ {\scriptstyle \eta} \downarrow & & \\ U & & \end{array}$$

It turns out that $\xi = X \times_{[X/G]} U$ so we would need to show that $\eta : \xi \to U$ is smooth and surjective (not obvious but provable).

$\square$

3.3.1 Back to elliptic curves

Let us consider the scheme[3] $U = \operatorname{Spec} k[A, B, 1/\Delta]$ where $\Delta = -16(4A^3 + 27B^2)$, and let $G = \mathbb{G}_m = k[u^{\pm 1}]$. G actis on U by looking at equivalent Weierstrass forms via

---

[3]$U \subseteq \mathbb{A}^2$ is the possible pairs (A, B) of coefficients in a Weierstrass form for an elliptic curve. Localizing at the discriminant enforces that for any point $(A, B) \in U$, the curve $y^2 = x^3 + Ax + B$ is an elliptic curve.

the change of variables
$$x \mapsto u^2 x, y \mapsto u^3 y.$$
Concretely, the action maps A to $u^{-4}A$, B to $u^{-6}B$ and Δ to $u^{-12}\Delta$.

Fact 3.30. $\mathcal{M}_{1,1} \cong [U/G]$.

Part II

Intersection theory

Chapter 4

Definition of Chow ring

Speaker: Andrea Gallese

The main reference for this chapter is [EH16].

4.1 Algebrizing intersection

Think back to our roots: algebraic geometry is the study of solutions to systems of polynomial equations, which we can think of as intersecting some geometric objects. It seems reasonable then that we ought to study carefully what it means for varieties to intersect.

One way in which we may approach this question is by asking the question "can we make intersection into an algebraic operation?". The answer is yes and the algebraic objects which come out from this idea are the Chow rings.

In this chapter we take k to be an alegbraically closed field. All schemes will be smooth and of finite type[1] over k.

4.1.1 Chow group

> **Definition 4.1.** Let X be a scheme as above. We define its **algebraic cycles** to be
> $$Z(X) = \mathbb{Z}\left[\text{integral closed subschemes of } X\right] = \bigoplus_k Z_k(X).$$
> where $Z_k(X)$ is the same thing for subschemes of dimension k.

Remark 4.2. $Z(X) = Z(X^{red})$.

Remark 4.3. We may also grade $Z(X)$ with respect to codimension instead of dimension, which we will do eventually.

---

[1]note that this implies Noetherian

Definition 4.4. Given $Y \subseteq X$ closed subscheme with irreducible components $Y_1, \cdots, Y_r$, we define the **algebraic cycle associated to** Y to be

$$\langle Y \rangle = \sum_{i=1}^{r} \ell_i [Y_i]$$

where[a] $\ell_i = \text{length}(\mathcal{O}_{Y,y_i})$ for $y_i \in Y_i$ the generic point.

---
[a]length of the module over itself. In particular, if $\mathcal{O}_{Y,y_i}$ is reduced then $\ell_i = 1$.

Remark 4.5. Compare this with the notion of the degree of a quasicoherent sheaf on $\mathbb{P}^n$. In that case we have that if $\text{Supp} \, \mathcal{F} = Y = \bigcup Y_i$ like above then

$$\deg \mathcal{F} = \sum_{\dim Y_i = \dim Y} \ell_i \deg Y_i.$$

Of course this is a different notion but the similarity is not accidental. When we later define the product in the Chow ring we will recover the notion of degree by considering the intersection with an hyperplane, which is what the original "naive" definition of degree used to be before the formalization of Hilbert polynomials came about.

Example 4.6. Consider $V(xy) \subseteq \mathbb{A}^2$. The irreducible components are $V(x)$ and $V(y)$ with generic points (x) and (y) respectively. Note that $\left(\frac{k[x,y]}{(x)} \right)_{(x)} = k(y)$, so it is reduced and the length is 1, same idea for the other irreducible component.

Consider now $V(x^2) \subseteq \mathbb{A}^2$. The only irreducible component is $V(x)$ with generic point (x) but now $\left(\frac{k[x,y]}{(x^2)} \right)_{(x)}$ is not reduced because $x \neq 0$ and $x^2 = 0$. This is reflected in the fact that this ring has length 2, which can be seen by noting that

$$(0) \subsetneq (x) \subsetneq (1)$$

is a composition series. The fact that the length is 2 recovers the fact that we started with $V(x^2)$ instead of $V(x)$.

The group of cycles is way too large for any practical purpose, so we want to define when two cycles should be thought of as the same. We give the following definition in hope that it models "continuous deformations" in some sense:

Definition 4.7. We define the subgroup $\text{Rat}(X)$ of $Z(X)$ to be the one which is generated as follows

$$\text{Rat}(X) = \langle \langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle \rangle$$

where Φ varies among integral subvarieties of $X \times \mathbb{P}^1$ and t_0, t_1 are rational points of the projective line, i.e. $t_0, t_1 \in \mathbb{P}^1(k)$.

We can now define our main character

Definition 4.8. Let X be a smooth scheme of finite type over k. We define its **Chow group** as the quotient

$$CH(X) = \frac{Z(X)}{\mathrm{Rat}(X)}.$$

If two cycles give the same element in $CH(X)$ we say that they are **rationally equivalent** and we may write this with $\sim$.

Remark 4.9. The Chow group is graded by dimension and by codimension.

Example 4.10. The cycles $[y = 0] + [x = 0]$ and $[xy = 1]$ in $\mathbb{A}^2$ are rationally equivalent. Consider

$$\Phi = V(xy - t) \subseteq \mathbb{P}^1 \times \mathbb{A}^2$$

where t is the coordinate[2] of $\mathbb{P}^1$. Let $t_0 = 0$ and $t_1 = 1$, which are clearly rational points of $\mathbb{P}^1$, then we have

$$[y = 0] + [x = 0] = \langle V(xy) \rangle = \langle \Phi \cap \{0\} \times \mathbb{A}^2 \rangle$$

and similarly for the other cycle. Thus $[y = 0] + [x = 0] \sim [xy = 1]$.
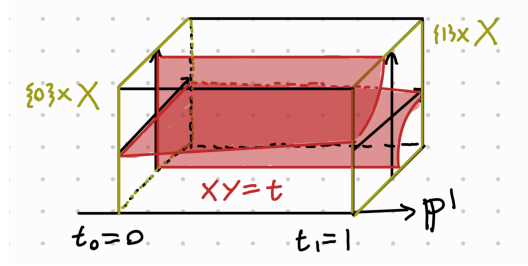
Remark 4.11. It may be useful to think of rational equivalence as some type of homotopy equivalence. This would make the Chow ring something akin to the cohomology ring of the varieties. These analogies are beatiful and deep, but we shall not persue them further.

Example 4.12. In $\mathbb{A}^2$ all points are rationally equivalent. To find the relation just multiply $\mathbb{P}^1$ with $\mathbb{A}^2$ and consider the line which connects the two points at different slices like in the picture

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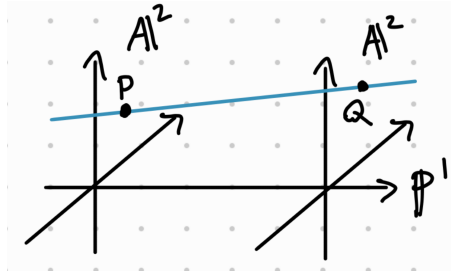
[2]to be more precise, if the coordinates are $[s : t]$ we are taking $\Phi = V(sxy - t)$

Example 4.13. In $\mathbb{A}^2$, all lines passing through the origin are rationally equivalent. Recall that GL_2 acts on $\mathbb{A}^2$ and that $GL_2 \subseteq \mathbb{A}^4$. Moreover, the orbit of some line L under the action of GL_2 consits of lines. This suggests a way to construct a rational equivalence between $[L]$ and $[gL]$ for $g \in GL_2$.

The idea is to take the line $g(t)$ in $\mathbb{A}^4$ which connects $id \in GL_2$ to $g \in GL_2$ and then consider Φ to be the variety where each slice corresponds to $g(t)L$.

4.1.2 Chow ring

Now that we have our objects that we want to intersect, we want to count the "number of intersections" somehow.

If we think about this, we find a problem when trying to define what the intersection ought to be when tangencies are involved. For example, if we intersect a line with a circle we always get two intersection points except when the line is tangent to the circle, in which case the two intersection points *collapse* to a single point. We want to count that as an intersection of multiplicity 2, not a single intersection.
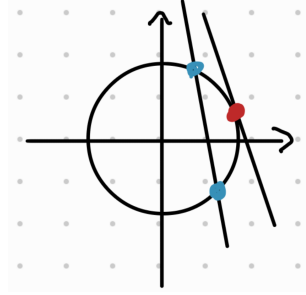
Let us define the "well behaved" cases of intersection first:

Definition 4.14. Let A, B be integral subvarieties of X. We say that A and B are **transverse** at $p \in A \cap B$ if $T_p A + T_p B = T_p X$, i.e.

$$\operatorname{codim} T_p A + \operatorname{codim} T_p B = \operatorname{codim}(T_p A \cap T_p B).$$

We say that A and B are **generically transverse** if they are transverse at all generic points of the intersection.

We now cite a lemma which tells us that we may reduce "ill behaved" cases to "well behaved" ones

---

Lemma 4.15 (Moving lemma)**.** Let X be a smooth quasi-projective variety, then the following holds:

- For all $\alpha, \beta \in CH(X)$ there exist $A, B \in Z(X)$ such that $\alpha = [A]$, $\beta = [B]$ and A, B are generically transverse.

- The class $[A \cap B]$ depends only on $[A]$ and $[B]$, not the representatives A, B.

With this lemma we can now put a product on $CH(X)$ which formalizes intersection, making it a ring.

Theorem 4.16. Let X be smooth and quasi-projective, then there exists a unique product on $CH(X)$ such that for all A, B subvarieties of X, if they are generically transverse then $[A] \cdot [B] = [A \cap B]$.

This makes $CH(X)$ into a graded (by codimension), associative, commutative and unital ring.

Remark 4.17. The identity of the Chow ring of X is the **fundemental class**, i.e. $[X]$. This has degree (codimension) 0 as we would expect. Intuitively, this corresponds to the simple fact that

$$A \subseteq X \implies A \cap X = A.$$

4.1.3 Non-triviality

The first non-triviality result concerns codimention 0 cycles, that is, irreducible components of the ambient variety.

Theorem 4.18. If X is irreducible and $\dim X = n$ then

$$CH_n(X) = CH^0(X) \cong \mathbb{Z}$$

and it is generated by $[X]$. If X has irreducible components given by $X_1, \cdots, X_r$, then $[X_1], \cdots, [X_r]$ generate a free abelian subgroup of $CH(X)$ which is isomorphic to $\mathbb{Z}^r$.

Sketch.

If $\Phi \subseteq X \times \mathbb{P}^1$ gives some rational equivalence then $\Phi \subseteq X_i \times \mathbb{P}^1$ by irreducibility. So we may only consider the irreducible case. If $\Phi \cap (\{t_0\} \times X) = X$ then $\Phi = X \times \mathbb{P}^1$, again by irreduciblity. $\square$

When considering codimension 1 we get back the theory of divisors and the class group, as one would expect

Proposition 4.19. Suppose X is irreducible of dimension n, then[a]

$$CH_{n-1}(X) = CH^1(X) = \mathrm{Cl}(X) \cong \mathrm{Pic}(X)$$

The divisor $D = \sum n_i Y_i$ corresponds to the cycle $\sum n_i [Y_i]$.

[a]Recall that X is smooth, so in particular it is locally factorial and thus $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$.

Rough sketch.

The correspondence is clear at the level of $Z(X)$ and $\mathrm{Div}(X)$. What we need to see is that the two equivalence relations translate into each other. If $f \in k(X)$ then we

may take $\Phi = V(f(x) - t)$ to see that $[\{f = 0\}] = [\{f = \infty\}]$, so $\operatorname{div}(f)$ does map to something in $\operatorname{Rat}(X)$. To conclude we would need to show that $\operatorname{Rat}(X)$ is generated by the relations induced by principals divisors in this way. $\qquad \square$

4.2 Mayer-Vietoris and Excision

Given the resemblance we noted before with cohomology rings, the following theorems seem rather natural (though we will not provide proofs):

Theorem 4.20 (Mayer-Vietoris)**.** Let $X_1, X_2 \subseteq X$ closed subvarieties, then we have an exact sequence

$$CH(X_1 \cap X_2) \to CH(X_1) \oplus CH(X_2) \to CH(X_1 \cup X_2) \to 0.$$

Theorem 4.21 (Excision)**.** Let $Y \subseteq X$ be a closed subvariety, then we have an exact sequence
$$CH(Y) \to CH(X) \to CH(X \setminus Y) \to 0$$

Remark 4.22. The exact sequence above may also be called *the localization sequence.*

4.2.1 Issue with Chow rings of affine space

Having now defined Chow rings, we want to compute some of them explicitly. We might attempt this first with affine space, but we start to run into some issues.

Example 4.23. If we think about the Chow ring of spaces like $\mathbb{A}^2$ we start to run into some problems. For example, consider $[C]^2$ for C some circle. To compute this intersection we may move a little one copy of the circle as in the picture
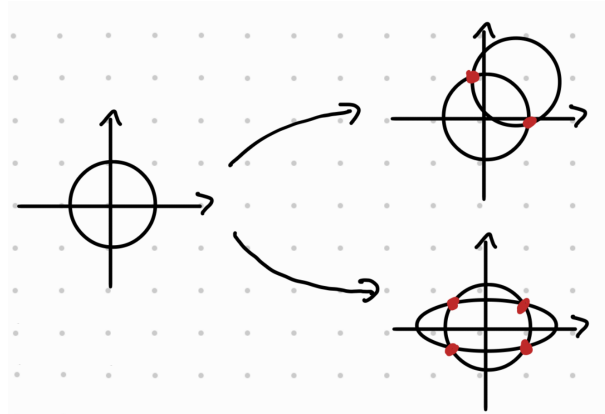
The issue is that in the bottom case the four points of intersection are still in the affine plane, but in the top case they are at infinity, yielding some undesidered consequences which we will make more explicit in the next result.

Proposition 4.24. The Chow ring of affine space is $CH(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n] \cong \mathbb{Z}$, concentrated in degree 0.

Proof.
Let Y be any positive-codimension cycle and move it[3] so that it does not pass through 0. We define[4]

$$\Phi = \left\{ (t, ty) \mid y \in Y, \ t \in \mathbb{A}^1 \right\} = V(\{f(z/t) \mid f \in I(Y)\}) \subseteq \mathbb{P}^1 \times \mathbb{A}^n.$$

Let us consider the slices of Φ at $t_0 = 1$ and $t_1 = \infty$:

$\boxed{t = 1}$ The ideal is $I(Y)$ itself ($f(z/1) = f(z)$) and we get back Y.

$\boxed{t = \infty}$ Since Y does not go through 0, there is some $f \in I(Y)$ such that $f(0) \neq 0$. If we take the slice at ∞ we get that $f(z/\infty) = f(0) \neq 0$ is in the ideal which defines $\Phi \cap \{\infty\} \times \mathbb{A}^n$, so the slice is $\emptyset$.

We have just shown that Φ defines an equivalence between Y and $\emptyset$, meaning that all cycles in positive codimension are 0 in the cohomology ring. $\square$

Remark 4.25. The geometric intuition is that we are scaling the set Y by a factor of t. Since we chose Y so that it does not pass through the origin, if we "take the limit" for $t \to \infty$ we are scaling all coordinates to be so large that the figure tY disappears to infinity.

Corollary 4.26. If $U \subseteq \mathbb{A}^n$ open, $CH(U) \cong \mathbb{Z}$ with generator $[U]$.

Idea.
Use excision. $\square$

The bottom line is that "good intersection theory" requires some properness condition.

4.3 Functoriality and proper push-forward

We would like our construction of Chow rings to be functorial, that is, if we consider Chow rings for objects we want to have homomorphisms which are induced by maps between those objects. Unfortunaltely this is rather tricky. The morphisms between schemes have to be proper to have a nice pushforward between the Chow rings.

Remark 4.27. If $f : X \to Y$ is proper and $A \subseteq X$ subvariety then $f(A)^{red} \subseteq Y$ is a subvariety.

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[3]translated copies of the same cycle are clearly rationally equivalent.
[4]by $f(z/t)$ what we really mean is: if $t = x_1/x_0$ is the coordinate of $\mathbb{P}^1$, take f and homogenize it with respect to x_1 to get $H_{x_1}(f)$. By the notation $f(z/t)$ we mean $(H_{x_1}(f)(x_0 z))/(x_1^{\deg f})$.

Theorem 4.28. If $f : X \to Y$ proper, we can get $f_* : CH(X) \to CH(Y)$. On generators it behaves as follows

$$f_*[A] = \begin{cases} 0 & \text{if } \dim f(A) < \dim A \\ d[f(A)] & \text{if } \dim f(A) = \dim A \text{ and } d = \deg(A \to f(A)) \end{cases}$$

Theorem 4.29. f_* defines a dimension-preserving homomorphism of the Chow-groups.
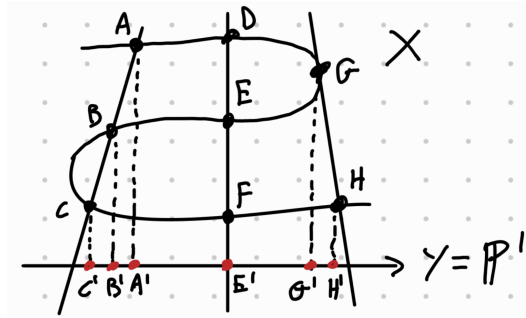
Figure 4.1: The cycle $A+B+C \sim D+E+F \sim 2G+H$, gets mapped to $A'+B'+C' \sim 3E' \sim 2G' + H'$. In all cases multiplicities are preserved.

Proposition 4.30. If $X \to \operatorname{Spec} k$ is proper, then we have a unique map deg : $CH(X) = CH(pt) \cong \mathbb{Z}$ which preserves dimension (so $[Y] \mapsto 0$ if $\dim Y > 0$ and $[pt] \mapsto 1$).

4.4 Projective space

Since properness seems to play a big role in this theory, we compute the Chow ring of projective space.

Definition 4.31. Let $X \subseteq \mathbb{P}^n$ be an irreducible closed subvariety of dimension k. The **degree** of X is the number of intersection points, counted with multiplicities, it has when intersected with k hyperplanes in general position.

Remark 4.32. The following is an equivalent, but perhaps more reassuring, definition of degree: let

$$P(t) = \frac{h(t)}{(1-t)^k}, \quad h(t) \in \mathbb{Z}[t], \ h(1) \neq 0$$

be the Poincaré-Hilbert series of $X \subseteq \mathbb{P}^n$, then the degree of X is $h(1)$.

Definition 4.33. Let X be a scheme of finite type of k. We call a finite collection of irreducible locally closed subschemes X_i an **affine stratification** if

- each X_i is isomorphic to some $\mathbb{A}^{k_i}$,

- X is the disjoint union of the X_i,

- the closure of each X_i us given by a union of some X_j.

Theorem 4.34. We have that as graded rings

$$CH(\mathbb{P}^n) \cong \frac{\mathbb{Z}[t]}{(t^{n+1})}$$

where t is the class of some linear hyperplane. Moreover, if $X \subseteq \mathbb{P}^n$ irreducible of codimension k and degree d then $[X] = dt^k$.

Sketch.
The proof consists of the following steps

- Consider the chain of inclusions

$$\{pt\} = \mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \cdots \subseteq \mathbb{P}^n$$

given by fixing the last free coordinate to be 0 to get from one term to the previous one. Consider the differences $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1}$, which are all isomorphic to affine space. It turns out that the U_i form an affine stratification of $\mathbb{P}^n$.

- Prove that $CH(\mathbb{P}^n)$ is generated by the classes of $[\overline{U_i}]$. The idea to prove the result for general affine stratifications. Proceed by induction on the number of strata. Let U_0 be a minimal strata and note that beacuse of that $\overline{U_0} = U_0$. By excision we have:

$$\mathbb{Z} = CH(U_0) = CH(\overline{U_0}) \to CH(X) \to CH(X \setminus \overline{U_0}) \to 0$$

and the last bit is a scheme with affine stratification that has one less strata, therefore the Chow ring of the last term is generated by the $[\overline{U_i}]$ for $i \neq 0$ by inductive hypothesis.

- Intersecting k hyperplanes gives a $(n-k)$-plane, so $t^k = [(n-k)\text{-plane}]$.

- Let $M \subseteq \mathbb{P}^n$ be an k-plane, we have a surjective[5] map

$$CH^k(\mathbb{P}^n) \xrightarrow[\text{deg}]{\cdot [M]} CH^n(\mathbb{P}^n) \xrightarrow{=} CH(pt)$$

and the full composition is the degree map, so this is also injective ($[X]$ is in the kernel when it has empty intersection with general M and this cannot happen if X has codimension k). Thus $CH^k(\mathbb{P}^n) \cong \mathbb{Z}$ and this map is actually an isomorphism of abelian groups. Moreover, if $[X] \in CH^k(\mathbb{P}^n)$ then[6]

$$[X][M] = \deg X[pt] = \deg X t^n = (\deg X t^k)t^{n-k} = (\deg X t^k)[M]$$

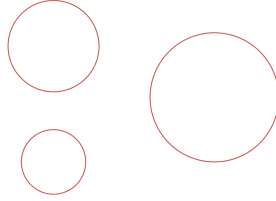[5]$(n-k)$-plane $\cap$ k-plane $= [pt]$, generator of $CH^n(\mathbb{P}^n)$
[6]by definition, the degree of an $(n-k)$-subvariety is the number of intersections with a general k-plane

and so $[X] = \deg X t^k$.

$\square$

To get some practice with this ring we conclude with the following exercise

Exercise 4.35. In $\mathbb{A}^2$ draw three circles C_1, C_2, C_3.

How many circles can be drawn which are tangent to C_1, C_2 and C_3?

Solution.
A circle in the plane is given by an equation of the form $(x - A)^2 + (y - B)^2 = C$. If we homogenize we see that the two points at infinity $[1 : \pm i : 0]$ always lie on such a homogenized circle. We may therefore define a circle in $\mathbb{P}^2$ to be a conic which passes through these two points.

Recall that conics in $\mathbb{P}^2$ are parametrized by $\mathbb{P}^5$ by looking at the coefficients in their general equation

$$ax^2 + bxy + cy^2 + dxz + ezy + fz^2 = 0.$$

By imposing the fact that the circles should go through two distinct points we get that circles are parametrized by a $\mathbb{P}^3$. More precisely, they correspond to the intersection of the two hypersurfaces $a + bi - c = 0$ and $a - bi - c = 0$ in $\mathbb{P}^5$.

What does it mean for a circle to be tangent to another fixed circle C? Let $Z_C = \{\text{circles tangent to } C\} \subseteq \mathbb{P}^3$ and consider the pairs

$$\Phi = \{(r, D) \mid r \in C, \ D \text{ circle tangent to } C \text{ at } r\}.$$

We have two projections $\Phi \to C$ and $\Phi \to Z_C$. Note that the fibers of $\Phi \to C$ are isomorphic to $\mathbb{P}^1$: up to translation, suppose that $r = 0$. If D is some circle which is tangent to C at 0 then all circles which are tangent to C at that same point are re-scaled versions of D.
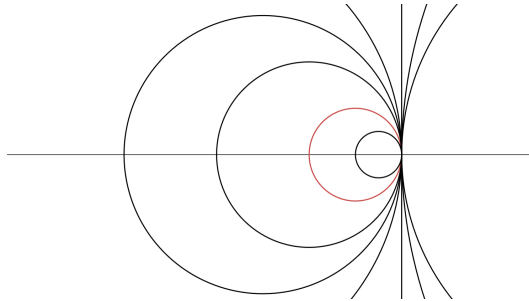
So Φ should be a surface of some kind (fiber bundle over a circle with fibers given by $\mathbb{P}^1$). This surface should have degree 2 because tangency between a circle

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$D : (x - a)^2 + (y - b)^2 = c$ and a fixed circle $C : (x - a_0)^2 + (y - b_0)^2 = c_0$ is checked by imposing that the following system of equations has a double solution

$$\begin{cases} (x - a)^2 + (y - b)^2 = c \\ (x - a_0)^2 + (y - b_0)^2 = c_0 \end{cases}$$

and the way we impose that is by setting the discriminant of a resulting quadratic equation to be 0. The fact that the degree should be two comes from the fact that the discriminant has degree two in the coefficients of the equation.

Moreover, Z_C is birational to Φ (from a circle in Z_C we can get the point of tangency it has with C).

So $Z_{C_1} \subseteq \mathbb{P}^3$ is a surface of degree 2. In the Chow group $CH(\mathbb{P}^3)$ we thus have $2t = [Z_{C_1}]$ and so

$$[Z_{C_1} \cap Z_{C_2} \cap Z_{C_3}] = [Z_{C_1}][Z_{C_2}][Z_{C_3}] = (2t)^3 = 8t^3.$$

$8t^3 \in CH(\mathbb{P}^3)$ is the class of eight points, so *8 is the number we were looking for*[7].

$\square$

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[7]when the three circles are in general enough position

Chapter 5

Pullback and Chern classes

Speaker: Francesco Minnocci

The main references for this chapter are [Ful84] and [EH16].

5.1 Pullback

Let $f : X \to Y$ be a morphism. In the last chapter we managed to construct a pushforward when f is proper. Our first goal in this chapter is to construct a pullback, at least in some cases.

To be more concrete, we want a map $Z(Y) \to Z(X)$ which preserves rational equivalence (so it induces a map at the level of Chow groups) and the intersection product.

5.1.1 Divisors and rational equivalence

Definition 5.1. Let $V \subseteq X$ be a subvariety with generic point ξ and fix $f \in k(X)^*$. Note that $f = f_1/f_2$ with $f_1, f_2 \in \mathcal{O}_{X,\xi}$[a]. We define the **order** of f along V to be

$$\operatorname{ord}_V f := \operatorname{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/(f_1)) - \operatorname{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/(f_2)).$$

[a]this amounts to saying that, locally, a rational function is the ratio of regular functions.

Definition 5.2. Let $f \in k(X)^*$, we define its **associated divisor** to be

$$\operatorname{div} f = \sum_V \operatorname{ord}_V(f)[V].$$

Proposition 5.3. A cycle $\alpha \in Z^k(X)$ is rationally equivalent to 0 if and only if there exists a finite number of varieties W_i together with inclusions[a] $\iota_i : W_i \to X$ and rational functions $f_i \in k(W_i)^*$ such that $\alpha = \sum (\iota_i)_*[\mathrm{div}(f_i)]$.

---

[a]recall that closed immersions are proper morphisms.

5.1.2 Regular immersions

Definition 5.4. A **regular immersion** of codimension r is a locally closed immersion given locally by a quotient of an ideal which is generated by a regular sequence of length r.

Notation. In this chapter, when we say flat we also mean that all fibers should the same dimension. This is automatic when the codomain is connected because of the properties of flat morphisms.

Fact 5.5. If Y is smooth then we can factor any morphism $f : X \to Y$ as

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
& \Gamma_f \searrow & \uparrow \mathrm{pr}_2 \\
& & X \times Y
\end{array}
$$

where Γ_f is the graph-morphism, which is a regular immersion, and pr_2 is the projection on the second factor, which is flat.

This shows that, for Y smooth, we may construct pullbacks by only considering morphisms which are regular immersions or flat.

Theorem 5.6. If i is a regular immersion then i^* exists and resprects the functorial properties

Remark 5.7. If pullbacks exist for X smooth, and $\Delta : X \to X \times X$ is a regular immersion then, if $V, W \subseteq X$ are subvarieties, we want

$$[V][W] = \Delta^*([V \times W]),$$

that is, we want the product $[V][W]$ to correspond to "the points of X which under the diagonal map to pairs in $V \times W$", i.e. the intersection.

5.1.3 Flat pullback

Theorem 5.8. If $f : Y \to X$ is flat of codimension n and $V \subseteq X$ is a subvariety, the function
$$f^*([V]) = [f^{-1}(V)]$$
is a homomorphism $Z(Y) \to Z(X)$ of degree n (if we grade cycles by dimension).

Proposition 5.9. Suppose we have a cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
f' \downarrow & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

with f proper and g flat, then for all $\alpha \in Z(X)$

$$f'_* g'^*(\alpha) = g^* f_*(\alpha).$$

Proposition 5.10. Let $f : X \to Y$ be a flat morphism and $\alpha \in Z^k(Y)$ be an element of $\mathrm{Rat}(Y)$, then $f^*\alpha \sim 0$.

Proof.
Without loss of generality, we may assume $\alpha = [\Phi(0)] - [\Phi(\infty)]$ where $\Phi(t) = \Phi \cap \{t\} \times Y$ for $\Phi \subseteq \mathbb{P}^1 \times Y$ closed and $t \in \mathbb{P}^1(k)$. Consider the base change diagram

where $W = ((id, f))^{-1}(\Phi)$ and p, q, g, h are the appropriate projections.

$$
\begin{aligned}
f^*(\alpha) =& f^*([\Phi(0)] - [\Phi(\infty)]) = f^* q_*([g^{-1}(0)] - [g^{-1}(\infty)]) = \\
=& p_*((id, f))^*([g^{-1}(0)] - [g^{-1}(\infty)]) = \\
=& p_*([h^{-1}(0)] - [h^{-1}(\infty)]).
\end{aligned}
$$

If $W = \bigcup W_i$ with W_i irreducible, we may write $[W] = \sum m_i[W_i]$ for appropriate $m_i \in \mathbb{Z}$. Note that if $h_i = h|_{W_i}$ then

$$[h^{-1}(t)] = \sum m_i h_i^{-1}(t).$$

Since

$$[h_i^{-1}(0)] - [h_i^{-1}(\infty)] = [\text{div}(h_i)]$$

this shows that

$$f^*(\alpha) = \sum m_i p_*[\text{div}(h_i)] \sim 0.$$

$\square$

Corollary 5.11. We have a well defined degree-preserving group homomorphism $f^* : CH^\bullet(Y) \to CH^\bullet(X)$.

Remark 5.12. f^* is also a ring homomorphism.

Remark 5.13. The pullback $f^* : CH(Y) \to CH(X)$ induces a $CH(Y)$-module structure on $CH(X)$.

Proposition 5.14 (Projection formula). If f is a flat and proper morphism between smooth varieties then f_* is a homomorphism of $CH(Y)$-modules, i.e.

$$f_*(f^*(\beta) \cdot \alpha) = \beta \cdot f_*(\alpha)$$

Remark 5.15. Plugging in $\alpha = 1_{CH(X)} = [X]$ we have that $f_* f^*(\beta) = (\deg f)\beta$.

5.2 Chern classes

We now move on to one of the first applications of the Chow ring: giving vector bundles their associated Chern classes.

5.2.1 Brief review of vector bundles

Definition 5.16. $\mathcal{E}$ is a **vector bundle** if it is a locally free sheaf of $\mathcal{O}_X$-modules of some finite rank r.

Remark 5.17. Vector bundles of rank 1 are called **line bundles**.

Definition 5.18. The **dual**[a] of a vector bundle $\mathcal{E}$ is $\mathcal{E}^\vee = \text{Hom}(\mathcal{E}, \mathcal{O}_X)$.

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[a]also called **inverse** in the case of line bundles.

Definition 5.19. An **algebraic vector bundle** is a morphism $p : E \to X$ of schemes such that for all $x \in X$ there exists U neighborhood of x such that $f^{-1}(U) \cong \mathbb{A}^r \times U$ and the transition maps are linear fiber-wise.

Remark 5.20. We have a correspondence between the two notions given by sending a locally free sheaf $\mathcal{E}$ to

$$\underline{\mathrm{Spec}}_{\mathcal{O}_X}(\underline{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{E}^\vee))$$

and sending an algebraic vector bundle to its sheaf of local sections.

We can also construct a projective version of bundles by taking

$$\mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_{\mathcal{O}_X}(\underline{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{E}^\vee)).$$

5.2.2 First Chern class of a line bundle

Recall that we are taking X to be smooth and noetherian, so in particular also irreducible and has a generic point ξ. It follows that non-empty open subsets of X are dense.

Definition 5.21. Let L be a line bundle on X, then a **rational section** of L is a germ $f \in L_\xi$.

Remark 5.22. Since L is locally free of rank 1, $L_\xi \cong \mathcal{O}_{X,\xi} = k(X)$ as $\mathcal{O}_{X,\xi}$-modules. It follows that a nonzero rational section σ always exists because $k(X)^* \neq \emptyset$.

Proposition 5.23. A nonzero rational section determines a divisor on X. Moreover, all divisors obtained this way are rationally equivalent.

Proof.
Let $\{U_i\}$ be a trivializing open cover for L and let us fix isomorphisms $\varphi_i : L_{U_i} \cong \mathcal{O}_{U_i}$. Set $f_i = (\varphi_i)_\xi(\sigma)$. This allows us to construct divisors $\mathrm{div}(f_i)$ on each U_i. On the intersections U_{ij} both of the divisors we got separately on U_i and U_j agree because the automorphism of $\mathcal{O}(U_{ij})$ induced by the equality $L_{U_i}|_{U_{ij}} = L_{U_j}|_{U_{ij}}$ must be multiplication by an invertible element of $g_{ij} \in \mathcal{O}(U_{ij})^*$ (that is, $f_j|_{U_{ij}} = g_{ij}f_j|_{U_{ij}}$) and therefore

$$\mathrm{div}(f_j)\big|_{U_{ij}} = \underline{\mathrm{div}(g_{ij})}^0 + \mathrm{div}(f_i)\big|_{U_{ij}}.$$

Changing the isomorphisms $L_{U_i} \cong \mathcal{O}_{U_i}$ does not change the associated divisor for similar reasons.

Let us call this divisor $\mathrm{div}(\sigma)$. Since $L_\xi \cong k(X)$, it follows that two different nonzero rational sections σ and τ must determine a rational function $f = \sigma/\tau \in k(X)^*$. It follows that

$$\mathrm{div}(\sigma) = \mathrm{div}(f) + \mathrm{div}(\tau),$$

that is, $\mathrm{div}(\sigma) \sim \mathrm{div}(\tau)$. $\qquad\square$

Definition 5.24. We define $c_1(L)$ to be the class in $CH^1(X)$ of any divisor associated to a nonzero rational section of L.

Example 5.25. Let $X = \mathbb{P}^n$, $L = \mathcal{O}(d)$ with $d \geq 0$. Then $c_1(L) = [H] = d[h]$ where h is a hyperplane (because H is a degree d hypersurface). This comes from the fact that global sections of $\mathcal{O}(d)$ are homogeneous polynomials of degree d.

Proposition 5.26. If X is smooth, $c_1 : \mathrm{Pic}(X) \to CH^1(X)$ is an isomorphism. In particular, if σ, σ' are rational sections of L and L' respectively, the rational section $\sigma \otimes \sigma'$ of $L \otimes L'$ behaves as follows:

$$\mathrm{div}(f \otimes f') = \mathrm{div}(f) + \mathrm{div}(f').$$

5.2.3 Axiomatic definition of Chern classes

There is another definition of c_1 which works for all vector bundles (and yields as many classes as the rank of the bundle itself).

Theorem 5.27. If $\mathcal{E}$ is a rank r vector bundle on X then $CH^\bullet(\mathbb{P}(\mathcal{E}))$ is a free $CH^\bullet(X)$-module of rank r.
Moreover, if $L = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and $h = c_1(L) \in CH^1(\mathbb{P}(\mathcal{E}))$, then $\{1, \cdots, h^{r-1}\}$ is a basis of $CH^\bullet(\mathbb{P}(\mathcal{E}))$ and the only relation is of the form

$$h^r + c_1(\mathcal{E})h^{r-1} + \cdots + c_r(\mathcal{E}) = 0$$

for some unique $c_i(\mathcal{E}) \in CH^i(X)$.

Remark 5.28. This is a generalization of the fact that $CH(\mathbb{P}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ for x hyperplane.

Example 5.29. Let $\mathcal{E} = L$ be a line bundle and $\pi : \mathbb{P}(L) \to X$ be its projectivization. Note that[1] $\mathcal{O}_{\mathbb{P}(L)}(1) = \pi^* L^\vee$. Let $h = c_1(\pi^* L^\vee) = -\pi^* c_1(L) = -c_1(L)[\mathbb{P}(L)]$ be the generator of the Chow group. With the new definition of c_1 we got in the previous theorem (which we now denote $\widetilde{c}_1$ to avoid confusion) we have

$$h^1 + \widetilde{c}_1(L)h^0 = 0 \implies h = -\widetilde{c}_1(L)[\mathbb{P}(L)]$$

so indeed the two definitions of c_1 coincide for line bundles.

Definition 5.30. We define the **total Chern class** of $\mathcal{E}$ to be

$$c(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E})$$

---

[1] $\pi^* L$ is the tautological bundle of $\mathbb{P}(\mathcal{E})$

Proposition 5.31. The following hold

- If f is flat, $f^*(c(\mathcal{E})) = c(f^*(\mathcal{E}))$

- If we have an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ of vector bundles then

$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}),$$

i.e.

$$c_k(\mathcal{F}) = \sum_{i+j=k} c_i(\mathcal{E})c_j(\mathcal{G})$$

Moreover, in the case of a split exact sequence we have $c(\mathcal{E} \oplus \mathcal{G}) = c(\mathcal{E})c(\mathcal{G})$.

5.2.4 Splitting principle

Theorem 5.32 (Splitting principle). If $E \to X$ vector bundle of rank n then there exists $\varphi : Y \to X$ smooth such that

1. $\varphi^* : CH(Y) \to CH(X)$ is injective

2. $\varphi^* E \to Y$ has a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots, \subseteq E_n = \varphi^* E$$

with E_i/E_{i+1} locally free of rank 1.

Corollary 5.33. We have that

$$\varphi^* c(E) = c(\varphi^* E) = \prod_{i=0}^{n-1} c(E_i/E_{i+1})$$

Example 5.34. If $E = \bigoplus_{i=1}^{r} L_i$ and $\alpha_i = c_1(L_i)$ then

$$c(E) = \prod_{i=1}^{r}(1 + \alpha_i), \quad c(E^\vee) = \prod_{i=1}^{r}(1 - \alpha_i), \quad c_k(E^\vee) = (-1)^k c_k(E)$$

Definition 5.35. If $\mathcal{E}$ is a vector bundle of rank r, $\det \mathcal{E} = \bigwedge^r \mathcal{E}$.

Proposition 5.36. $c_1(E) = c_1(\det E)$.

Idea.
If $E = \bigoplus L_i$ then $c_1(E) = \sum c_1(L_i) = c_1(\bigotimes L_i) = c_1(\det(\bigotimes L_i))$. The general case follows from this combined with the splitting principle. □

Chapter 6

Chow rings of vector bundles and equivariant setting

Speaker: Bernardo Tarini

6.1 Chow ring of vector bundles

Recall that we have a correspondence

$$\{\text{algebraic vector bundles of rank } n\} \quad \longleftrightarrow \quad \{\text{locally free sheaves of rank } n\}$$
$$E \to X \quad\quad \longmapsto \quad\quad \text{sheaf of sections of } E$$
$$\underline{\mathrm{Spec}}_{\mathcal{O}_X}(\mathrm{Sym}\,\mathcal{E}^{\vee}) \quad\quad \longleftarrow\!\shortmid \quad\quad \mathcal{E}$$

6.1.1 Chow ring of a vector bundle is isomorphic to the Chow ring of the base

Recall the principal of Noetherian induction:

Lemma 6.1. Let X be a Noetherian topological space, and let P be a property of closed subset of X. Assume that for any closed subset Y of X, if P holds for every proper closed subset of Y, then P holds for Y[a]. Then P holds for X.

---
[a]In particular, P must hold for the empty set.

Theorem 6.2. Let $\pi : E \to X$ be a vector bundle. The pullback map $\pi^* : CH^{\bullet}(X) \to CH^{\bullet}(E)$ is an isomorphism and the inverse is provided by taking the bullback via the zero-section $s : X \to E$.

Proof.
Injectivity of π^* follows by noting that

$$\pi \circ s = id_X \implies s^* \circ \pi^* = (\pi \circ s)^* = id_{CH^{\bullet}(X)}.$$

Now we move on to surjectivity. First let us reduce to the case where $E \cong X \times \mathbb{A}^n$ is the trivial bundle: Let $\operatorname{Spec} R = U \subseteq X$ be a dense affine trivializing open subset. Note that $CH^\bullet(\emptyset) = CH^\bullet(E_{|_\emptyset})$, so by Noetherian induction we may assume that surjectivity hold for all proper closed subsets of X, in particular also for $Z = X \setminus U$ because $\dim Z < \dim X$ guarantees properness. Consider now the commutative diagram

$$CH^\bullet(\mathbb{A}^n \times U)$$

$$\|$$

$$
\begin{array}{ccccccc}
CH^\bullet(E_{|_Z}) & \xrightarrow{i_*} & CH^\bullet(E) & \longrightarrow & CH^\bullet(E_{|_U}) & \longrightarrow & 0 \\
{\scriptstyle \pi^*}\big\uparrow & & {\scriptstyle \pi^*}\big\uparrow & & {\scriptstyle \pi^*}\big\uparrow & & \\
CH^\bullet(Z) & \longrightarrow & CH^\bullet(X) & \longrightarrow & CH^\bullet(U) & \longrightarrow & 0
\end{array}
$$

Because of what we said the left-most vertical arrow is surjective and the rows are exact, so if the arrow on the right is surjective, the one in the middle also is.

We may therefore suppose $E = X \times \mathbb{A}^n$. We may now view this bundle as a series of consecutive line bundles

$$U \times \mathbb{A}^n \to U \times \mathbb{A}^{n-1} \to \cdots \to U \times \mathbb{A}^1 \to U$$

so without loss of generality we may actually consider $E = U \times \mathbb{A}^1$. Without loss of generality we also consider U to be irreducible (WHY???????)

Take $\alpha = [V] \in CH^\bullet(U \times \mathbb{A}^1)$ for V irreducible. Because of the diagram

$$
\begin{array}{ccc}
CH^\bullet(E_{|_{\overline{\pi(V)}}}) & \xrightarrow{i_*} & CH^\bullet(U \times \mathbb{A}^1) \\
\big\uparrow{\scriptstyle \pi^*} & & \big\uparrow{\scriptstyle \pi^*} \\
CH^\bullet(\overline{\pi(V)}) & \xrightarrow{i_*} & CH^\bullet(U)
\end{array}
$$

we may (by noetherian induction ??????) suppose that $U = \overline{\pi(V)}$. Therefore $\dim U \leq \dim V \leq \dim U + 1$. If $\dim V = \dim U + 1$ then by irreducibility $V = U \times \mathbb{A}^1$ and so $[V] = \pi^*[U]$.

Let us now suppose $\dim V = \dim U$. Consider the following diagram

$$
\begin{array}{ccc}
V_\xi & \longrightarrow & V \\
\text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\
\mathbb{A}^1_{k(\xi)} \quad = \quad E_\xi & \longrightarrow & \mathbb{A}^1 \times U \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow \\
U_\xi & \longrightarrow & U
\end{array}
$$

where ξ is the generic point of U. Note that $V_\xi = V(p(x))$ for $p(x) \in \operatorname{Frac}(R)[x]$. But if this happens at the generic fiber we can take a open set U' where the denominators of p are inverted and find that $V \cap U' = V(p(x))$.

Now consider the localization sequence

$$CH^\bullet(E_{|_{U \setminus U'}}) \to CH^\bullet(E) \to CH^\bullet(E_{|_{U'}}) \to 0$$

Since $[V]$ goes to 0 we have that $[V] = \sum n_i[V_i]$ where $V_i \subseteq E_{|_{U \setminus U'}}$, which has a dimension smaller than that of $E_{|_U}$, so we are done by Noetherian induction. $\qquad\square$

Corollary 6.3. $CH^\bullet(\mathbb{A}^n) \cong CH^\bullet(\operatorname{Spec} k) \cong \mathbb{Z}$

Recall the projection formula: if $f : V \to X$ is a proper map between smooth schemes over k then $f_* : CH^\bullet(Y) \to CH^\bullet(X)$ is an homomorphism of $CH^\bullet(X)$-modules, that is

$$f_*(f^*(\alpha)\beta) = (f_*\beta)\alpha.$$

6.1.2   Self intersection formula

Definition 6.4 (Normal bundle). Let $i : X \to Y$ be a regular immersion of smooth schemes of codimension r. If $\mathcal{I} \subseteq \mathcal{O}_Y$ is the ideal which defines X then $\mathcal{I}/\mathcal{I}^2$ is locally free sheaf of rank r. The normal bundle is

$$\mathcal{N}_{X/Y} = \underline{\operatorname{Spec}}_{\mathcal{O}_X}\left(\operatorname{Sym}\mathcal{I}/\mathcal{I}^2\right)$$

Theorem 6.5 (Self-intersection formula). Let $i : X \to Y$ be a regular immersion of smooth schemes of codimension r. Then $i^*i_*(\alpha) = \alpha c_r(\mathcal{N}_{X/Y})$.

Corollary 6.6. Let $E \to X$ be a vector bundle and let $s : X \to E$ be the 0-section. Set $E_0 = s(X) \subseteq E$. Then

$$CH^\bullet(E \setminus E_0) \cong \frac{CH^\bullet(X)}{(c_r(E))}.$$

Proof.
By the self-intersection formula, $s^*s_*(\alpha) = \alpha c_r(\mathcal{N}_{X/E})$. Note also that $\mathcal{N}_{X/E} \cong E$ because $X \subseteq E$ is viewed as the zero-section (OK INTUITIVELY, BUT WHY FORMALLY ??????).

Combining these facts with the localization exact sequence and the isomorphism between $CH^\bullet(X)$ and $CH^\bullet(E)$ we get

$$CH^\bullet(X) \xrightarrow{\ s_*\ } CH^\bullet(E) \longrightarrow CH^\bullet(E \setminus E_0) \longrightarrow 0$$

$$\begin{array}{c} {\scriptstyle \cdot c_r(E)} \searrow \quad \| \wr\, s^* \\ CH^\bullet(X) \end{array}$$

which shows that

$$CH^\bullet(E \setminus E_0) \cong \frac{CH^\bullet(X)}{(c_r(E))}.$$

$\square$

6.1.3 Geometric interpretation of Chern classes

Theorem 6.7. If s is a global section of $E \to X$ such that codim $Z(s) = \text{rnk } E$ then $[Z(s)] = c_r(E)$.

Remark 6.8. In the case of the first Chern class of codim $Z(s) = \text{rnk } E = 1$ we get back the first definition of c_1 we used for line bundles.

Remark 6.9. To describe $c_{r-i}(E)$ we need to find $s_1, \cdots, s_{i+1}$ global sections of E such that codim $Z(s_1, \cdots, s_{i+1}) = r - i$ where $Z(s_1, \cdots, s_{i+1}) = Z(s_1 \wedge \cdots \wedge s_{i+1})$. If this equality holds then

$$[Z(s_1, \cdots, s_{i+1})] = c_{r-i}(E)$$

6.2 Equivariant setting

We want to define a suitable intersection theory for the category whose objects are smooth G-varieties and whose maps are G-equivariant maps. We take G to be a smooth linear reductive algebraic group. Also $k = \bar{k}$.

If X is a G-variety we want to define

$$CH^\bullet_G(X) = CH^\bullet([X/G])$$

We'd like that when X/G is a scheme then $CH^\bullet_G(X) = CH^\bullet(X/G)$.

Unfortunately, quotients of schemes do not always exist. However, in the previous section we showed that $CH^\bullet(X) = CH^\bullet(E)$ for any vector bundle on X, so if we can find a suitable G-bundle on X we might still be in luck. Unfotunately, this idea too cannot work quite yet, since the quotient can never be a scheme (for example, the zeros have too many automorphisms), but if we remove from the bundle a G-invariant closed subset Z of "high enough codimension" and the quotient exists in that case then we may as well use that quotient to compute the Chow ring up to the degree which corresponds to the codimension of that Z we removed. We are therefore give the following definition, which relies on what are called *approximation spaces*.

Definition 6.10. Let X be a G-variety. Let V be a G-representation such that there exists $U \subseteq V$ open which is G-invariant and such that $\text{codim}(V \setminus U) > i$ and U/G is a scheme. We define[a]

$$CH^i_G(X) = CH^i((X \times U)/G)$$

We also set[b]

$$CH^i(BG) := CH^i_G(\text{Spec } k) = CH^i(U/G).$$

---
[a]note that U/G existing implies that $(X \times U)/G$ exists where the action is the diagonal action.
[b]Even though it is just notation at this stage, we may think about BG as some variety that has that Chow ring. This can be formalized and BG is called the **classifying space** of G.

If this well defined? Suppose we have two approximation spaces $U = V \setminus Z$ and $U' = V' \setminus Z'$. Let $\widetilde{U} = (V \oplus V') \setminus (X \times V' \cup V \times Z')$ and consider the diagram of

equivariant inclusions

$$V \times V' \setminus Z \times V' \longleftarrow \widetilde{U} \longrightarrow V \times V' \setminus V \times Z'$$

with maps down to U and U'.

This passes to the quotients

$$\frac{V \times V' \setminus Z \times V'}{G} \longleftarrow \widetilde{U}/G \longrightarrow \frac{V \times V' \setminus V \times Z'}{G}$$

with maps down to U/G and U'/G.

Note that the maps in this diagram are actually vector bundles so the Chow groups of everthing involved here are isomorphic.

Remark 6.11. Quotient maps are fppf, so all properties of maps we may care about can be checked locally.

6.2.1 Examples of Chow rings of classifying spaces

We may compute equivariant Chow rings via approximations spaces like in the definition:

Example 6.12 (Chow ring of classifying space of tori)**.** Consider the standard scaling action $\mathbb{G}_m \curvearrowright \mathbb{A}^{r+1} \setminus \{0\}$ and note that it has a quotient $\mathbb{P}^r = \mathbb{A}^{r+1} \setminus \{0\}/\mathbb{G}_m$.

Consider now $T = \mathbb{G}_m^n$ and note that it acts on $U = \prod_{i=1}^n (\mathbb{A}^{r+1} \setminus \{0\})$ with quotient $U/T = (\mathbb{P}^r)^n$. We can use these as approximation spaces. Note that

$$CH^\bullet((\mathbb{P}^r)^n) \cong \mathbb{Z}[h_1, \cdots, h_n]/(\{h_i^{r+1}\}_{1 \leq i \leq n})$$

with $h_i = c_1(X_i)$ for X_i the 1-dimensional representation induced by the character $\mathrm{pr}_i : T = \mathbb{G}_m^n \to \mathbb{G}_m = \mathrm{GL}_1$.

The idea will be to take a projective limit

$$CH^\bullet(B\mathbb{G}_m^n) = \varprojlim_r \frac{\mathbb{Z}[h_1, \cdots, h_n]}{(h_i^{r+1})}$$

The following lemmas may also be useful in this type of computations if we do not wish to go through approximation spaces:

Lemma 6.13. Suppose X is a G-variety where G acts transitively (that is, $G \times X \to X$ should be an epimorphism of fppf sheaves). Pick $x \in X(k)$ and set $S = \mathrm{stab}(x) \leq G$, then
$$CH_G^\bullet(X) = CH^\bullet(BS).$$

Idea.
Use the isomorphism
$$(X \times U)/G \cong U/S$$

$\square$

Lemma 6.14. If $H \cong \mathbb{A}_k^n$ is an algebraic group, the left action on itself is by affine transformations and G is a linear algebraic group which acts on H by affine transformations then
$$CH_{H \rtimes G}^\bullet(X) \cong CH_G^\bullet(X)$$
for any $H \rtimes G$-variety X.

Idea.
Consider

$$
\frac{X \times U \times H}{H \rtimes G} \quad \cong \quad \frac{X \times U \times \frac{H \rtimes G}{G}}{H \rtimes G} \quad \cong \quad \frac{X \times U}{G}
$$

$$\downarrow$$

$$\frac{X \times U}{H \rtimes G}$$

and note that the vertical map is an affine bundle. Like in the case of vector bundles, the pullback via an affine bundle gives an isomorphism of Chow rings. $\qquad \square$

Example 6.15 (Chow ring of $B\,\mathrm{GL}_n$). There is an isomorphism

$$CH^\bullet B\,\mathrm{GL}_n \cong \mathbb{Z}[c_1, \cdots, c_n]$$

where the c_i are the i-th Chern classes of the natural representation of GL_n.

Proof.
We proceed by induction on n. We already know the result is true for $n = 1$ because $\mathrm{GL}_1 = \mathbb{G}_m$. Let V be the natural representation of $G = GL_n$[1]. On $V \setminus \{0\}$ we have that G acts transitively. We can view V as a vector bundle over a point and so $V \setminus \{0\}$ is that bundle where we removed the zero section. Proceeding in a similar way to before

$$CH_G^\bullet(\mathrm{Spec}\,k) \longrightarrow CH_G^\bullet(V) \longrightarrow CH_G^\bullet(V \setminus \{0\}) \longrightarrow 0$$

$$\searrow_{\cdot c_n(V)} \qquad \| \wr$$

$$CH_G^\bullet(\mathrm{Spec}\,k)$$

Note that where $S = \mathrm{stab}(e_1)$ can be interpreted as $S = H \rtimes \mathrm{GL}_{n-1}$ by writing matricies in S in the form

$$
\begin{pmatrix}
1 & * & \cdots & * \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix}.
$$

Note also that by the first lemma $CH_G^\bullet(V \setminus \{0\}) \cong CH_S^\bullet(\mathrm{Spec}\,k)$ and by the second lemma $CH_S^\bullet(\mathrm{Spec}\,k) \cong CH_S^\bullet(\mathrm{Spec}\,k) \cong CH_{\mathrm{GL}_{n-1}}^\bullet(\mathrm{Spec}\,k)$.

By inductive hypothesis, we have shown that the $c_1, \cdots, c_n$ generate $CH_G^\bullet(V) = CH^\bullet(B\,\mathrm{GL}_n)$ (the first $n-1$ generate the quotient $CH^\bullet(B\,\mathrm{GL}_{n-1})$ and by the exact sequence if we also add c_n we get everything).

---

[1]that is, $V = k^n$ and the action of $g \in G$ is the usual linear transformation associated to that matrix

To show algebraic independence note that we have a torus $T \subseteq \mathrm{GL}_n$ so $f : BT \to B\,\mathrm{GL}_n$ and the GL_n representation V becomes the representation $\bigoplus \chi_i$ where the χ_i are given by the characters.

Note that $f^*(c_i) \in CH^\bullet(BT)$ is the i-th symmetric polynomial in the ξ_r^i, which are the chern classes of characters.

Lemma 6.16. The symmetric functions σ_i are algebraically independent into $\mathbb{Z}[\xi_i, \cdots, \xi_r] \cong CH^\bullet(BT)$.

So $CH^\bullet(B\,\mathrm{GL}_n) = \mathbb{Z}[c_1, \cdots, c_n]$ where $c_i = c_i(V)$. $\qquad \square$

Part III

Chow ring of M2

Chapter 7

Reduction to a quotient of the Chow ring of projective space

Speakers: Pietro Leonardini

In this and the following chapters our aim is to review the article [Vis98].
Recall that $\mathcal{M}_g$ is the stack such that $S \to \mathcal{M}_g$ corresponds to

$$
\begin{array}{ccc}
C_k & \longrightarrow & C \\
\downarrow & & \downarrow{\scriptstyle\rho} \\
\operatorname{Spec} k & \longrightarrow & S
\end{array}
$$

with ρ proper and smooth morphism and $C_k \to \operatorname{Spec} k$ is a smooth curve of genus g.

Recall that if S is a base scheme, G affine smooth group scheme, X scheme, $G \curvearrowright X$ then $[X/G]$ is the stack which parametrizes (T, P, π) with

- $T \in \operatorname{Sch}/S$

- P is a G_T principal bundle

- $\pi : P \to X \times_S T$ is G_T-equivariant.

Fact 7.1. $\mathcal{M}_2$ is a quotient stack, i.e. it can be written in the form

$$
\mathcal{M}_2 = [X/G]
$$

Recall that if G is a g-dimensional algebraic group acting on X, and V is an ℓ-dimensional representation of G such that $U \subseteq V$ and G acts freely on U[1] and $\operatorname{codim}(V \setminus U) > n - j$ then we get $X \times U \to (X \times U)/G$ and ****** A is CH *****

$$
A_i^G(X) := A_{i+\ell-g}((X \times U)/G)
$$

for $i > j$ by definition (what we called approximation spaces).

---

[1] so that $U \to U/G$ equists

7.1 main theorem

Theorem 7.2. Consider $\mathcal{M}_2 \to \operatorname{Spec} k$. If char $k \neq 2, 3$ then

$$A^\bullet(\mathcal{M}_2) = \frac{\mathbb{Z}[\lambda_1, \lambda_2]}{(10\lambda_1, 2\lambda_1^2 - 24\lambda_2)}$$

On $\mathcal{M}_2$ there is a natural vector bundle $\mathcal{E}$ called the Hodge bundle.

7.1.1 Vector bundles on stacks

If $\mathcal{X}$ is an algebraic stack we can consider the category $(\operatorname{Sch}/\mathcal{X})$ with objects (T, t) with $t : T \to \mathcal{X}$ and T a scheme, and $(f, f^b) : (T, t) \to (T', t')$ is a commuting triangle

and an isomorphism of functors $f^b : t \to t' \circ f$.

We can make this category into a site by considering (T, t), t smooth ?

$$\left\{ (T_i, t_i) \xrightarrow{(f, f^b)} (T, t) \right\}$$

with f_i étale.

Definition 7.3. A sheaf of $\mathcal{O}_\mathcal{X}$-modules on this site is $\left\{ F_{(T,t)}, \phi_{(f,f^b)} \right\}$ with $F_{(T,t)}$ sheaf for all $t : T \to \mathcal{X}$ and $\phi : f^* F_{(T,t)} \to F_{(T',t')}$

If I have a sheaf F on X together with an isomorphism $\sigma : \operatorname{pr}_1^* F \to p^* F$ (the map is the action $p : X \times G \to X$) then a sheaf on $[X/G]$ is like (F, σ) together with some conditions

The Hodge bundle is defined as follows: given a curve $\pi : C \to S$ we consider the relative dualizing sheaf $\omega_{C/S}$. This yields a sheaf $\pi_*(\omega_{C/S})$ on S.

Proposition 7.4. $\pi_*(\omega_{C/S})$ is locally free of rank 2.

Example 7.5. A sheaf on $BG = [\operatorname{Spec} k/G]$ is defined as follows: if $\rho : \operatorname{Spec} k \times G \to \operatorname{Spec} k$ is the action and V is a sheaf on $\operatorname{Spec} k$ (so a vector space) with an isomorphism $\sigma : \operatorname{pr}_1^* V \to \rho^* V$, that is an action of G on V, so sheaves on BG are representations of G.

Consider the stack $\mathcal{Y}$ where the objects are $(\pi : C \to S, \alpha)$ with π like in the definition of $\mathcal{M}_2$ and $\alpha : \mathcal{O}_S^2 \to \pi_* \omega$ an isomorphism.

(Not all $\pi : C \to S$ from $\mathcal{M}_2$ may appear here, only those for which $\pi_* \omega$ is trivial. This won't really be a problem because we will forget this trivializations soon)

Proposition 7.6. $\mathcal{Y}$ is a stack , actually it is an algebraic space because the automorphisms are trivial.

Proposition 7.7. We have an action of GL_2 on $\mathcal{Y}$ given by $A \cdot (\pi, \alpha) = \pi, \alpha \circ A^{-1}$.

In particular $\mathcal{M}_2 = [\mathcal{Y}/\mathrm{GL}_2]$.

Proposition 7.8. Consider

$$\mathbb{A}^7 = \{\text{space of sixtic forms in 2 variables}\}$$

that is, elements of the form

$$\phi(x_0, x_1) = a_0 x_0^6 + a_1 x_0^5 x_1 + \cdots + a_7 x_1^6.$$

Note that $\mathrm{GL}_2 \curvearrowright \mathbb{A}^7$ by $A\phi(x) = \det A^2 \phi(A^{-1}x)$. Consider $X \subseteq \mathbb{A}^2$ the open ste of forms with distinct roots.
Then $\mathcal{M}_2 = [\mathcal{Y}/\mathrm{GL}_2] \cong [X/\mathrm{GL}_2]$.

$$[\mathbb{A}^7/\mathrm{GL}_2]$$

$$[\mathcal{Y}/\mathrm{GL}_2] \quad \cong \quad [X/\mathrm{GL}_2] \longrightarrow B\,\mathrm{GL}_2$$

We have an exact sequence

$$CH_{\mathrm{GL}_2}^\bullet(\mathbb{A}^7 \setminus X) \longrightarrow CH_{\mathrm{GL}_2}^\bullet(\mathbb{A}^7) \longrightarrow CH_{\mathrm{GL}_2}^\bullet(X) \longrightarrow 0$$

$$\| \wr$$

$$CH^\bullet(B\,\mathrm{GL}_2) \quad = \quad CH_{\mathrm{GL}_2}^\bullet(\mathrm{Spec}\,k)$$

By what we computed last chapter, $CH^\bullet(B\,\mathrm{GL}_2) = \mathbb{Z}[\lambda_1, \lambda_2]$ with $\lambda_1 = c_1(\mathcal{E})$ and $\lambda_2 = c_2(\mathcal{E})$ where $\mathcal{E}$ is the Hodge bundle (because the natural representation of GL_2 corresponded via the isomorphism with $\mathcal{Y}$ to the automorphisms of $\pi_*\omega$ when it is trivial, so the automorphisms of $\mathcal{E}$).

(possibly to put earlier) $\mathrm{GL}_2 \curvearrowright \mathbb{A}^2 = F$ linear forms in two variables via

$$A\phi(x) = \phi(A^{-1}x)$$

So GL_2 acts on $\mathrm{Sym}^6 F = E = \mathbb{A}^7$ with $Af(x) = f(A^{-1}x)$.

$$\mathbb{P}^6 \quad = \quad \mathbb{P}(E)$$
$$\cap$$
$$Z \quad = \quad \mathbb{P}(X) \longleftarrow X$$

$X \to Z$ is a $\mathbb{G}_m$-bundle, that is, a line bundle via $D^{\otimes 2} \otimes \mathcal{O}(-1)$ where D is the determinant bundle of the standard representation of GL_2. (D comes from the det A^2 in $A\phi(x) = \det A^2 \phi(A^{-1}x)$ for the action of GL_2 on X).

Thanks to the map $A^\bullet_{\mathrm{GL}_2}(Z) \to A^\bullet_{\mathrm{GL}_2}(X) = A^\bullet_{\mathrm{GL}_2}(M \setminus Z_0) \to 0$ where M is the total space of the line bundle associated to $X \to Z$ ($M = X \cup Z_0$ for Z_0 the zero section)

$CH^\bullet_{\mathrm{GL}_2}(X) = CH^\bullet_{\mathrm{GL}_2}(Z)/c_1(M)$
if $t = c_1(\mathcal{O}(1))$ then $c_1(M) = 2\lambda_1 - t$.

So $CH^\bullet_{\mathrm{GL}_2}(\mathbb{P}^6) = CH^\bullet(B\,\mathrm{GL}_2)[t]/(t^7 + c_1(E)t^6 + \cdots + c_6(E)) = \frac{\mathbb{Z}[\lambda_1, \lambda_2, t]}{(p(t, \lambda_1, \lambda_2))}$ where

$$p(\lambda_1, \lambda_2, t) = (t+6\ell_1)(t+6\ell_2)(t+5\ell_1+\ell_2)(t+\ell_1+5\ell_2)(t+4\ell_1+2\ell_2)(t+2\ell_1+4\ell_2)(t+3\ell_1+3\ell_2)$$

where ℓ_1 and ℓ_2 are the Chern roots

$$t^2 + \lambda_1 t + \lambda_2 = (t+\ell_1)(t+\ell_2)$$

indeed

$$c_1(F) = -\lambda_1, \; c_2(F) = \lambda_2$$
$$F^\vee \cong L_1 \oplus L_2$$

$\mathrm{Sym}^2 F = \{x \otimes x, x \otimes y, y \otimes y\}$ so

$$(t + \ell_1)(t + \ell_2) \rightsquigarrow (t + 2\ell_1, t + \ell_1 + \ell_2, t + 2\ell_2)$$

Recall, we wanted $CH(X)$ and we know $CH(X) = CH(Z)/(\lambda_1 - t)$. By the exact sequence $CH(\mathbb{P}^6) \to CH(Z) \to CH(X) \to 0$ we can find some relations on $CH(Z)$ by knowing the relations on $CH(\mathbb{P}^6)$

Since C is hyperelliptic ($\pi : C \to S, \alpha$) ω_π is generated by global sections, $f : C \to \mathbb{P}^1_S$ double cover, $f^*\mathcal{O}(1) = \omega_\pi = \mathcal{O}_C(1)$.

$f_*\mathcal{O}_C$, $f^\sharp : \mathcal{O}_{\mathbb{P}^1_S} \hookrightarrow f_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1_S} \oplus L$ (it splits). This is because in general (looking locally at a double cover)

$$A \to B = A^2 = \frac{A[\theta]}{(\theta^2 - a)} \xrightarrow{tr} A$$

sending $b \in B$ seen as $b : B \to B$ by multiplication as an A-linear map on A^2. If $\{1, \theta\}$ generators of $B = A^2$, $tr(1) = 2$, $tr(\theta) = tr \left[\begin{smallmatrix} 0 & a \\ 1 & 0 \end{smallmatrix}\right]$ and so we get a split.

It turns out that $L = \mathcal{O}(-3)$ by Riemann-Hurwitz. ($L^{\otimes -2} = \mathcal{O}(\text{branch locus}) = \mathcal{O}(6)$)

So we have an isomorphism $\alpha : \omega_\pi \cong f^*\mathcal{O}(1)$, which corresponds to $f_*\omega = f_*f^*\mathcal{O}(1)$, so a section of $f_*\omega(-1)$ and by duality it becomes

$$f_*\omega(-1) = \omega(-1) \oplus (\omega(-1) \otimes L^{-1})$$

$L \to \omega(-1)$, $L \otimes L \to \mathcal{O}$, $\omega(-1)^{\otimes 2} \to \mathcal{O}$

7.2 Summary

char $k \neq 2, 3, 5$.

$\mathcal{M}_2 \cong [Y/\operatorname{GL}_2]$ where Y is algebraic space with

$$Y = \left\{ (\pi : C \to S, \alpha : \pi_* \omega_\pi \xrightarrow{\sim} \mathcal{O}_S^2) \right\}$$

and action $(M \in \operatorname{GL}_2)$

$$M \cdot (\pi, \alpha) = (\pi, \alpha \circ M^{-1}).$$

We have $Y \cong X$ with

$$X = \{\text{degree 6 forms in 2 variables with distinct roots}\} \overset{open}{\subseteq} \mathbb{A}^7 = \operatorname{Sym}^6 F$$

where $F = \{\text{linear forms in 2 variables}\}$. The isomorphism in GL_2-invariant and the action of GL_2 on $\mathbb{A}^7$ is given by

$$M \cdot \phi((x, y)) = \det(M)^2 \phi(M^{-1}(x, y)).$$

where V is the natural representation of GL_2. Thus

$$c_1(\mathcal{E}) := \lambda_1 = c_1(V), c_2(\mathcal{E}) := \lambda_2 = c_2(V)$$

Now we consider the quotient by the scaling action

$$
\begin{array}{ccc}
X & \subseteq & \mathbb{A}^7 \\
\downarrow & & \downarrow \\
Z & \subseteq & \mathbb{P}^6
\end{array}
$$

and so the twist in $M \cdot \phi(x) = \det M^2 \phi(M^{-1} x)$ becomes irrelevant.

$$CH_{\operatorname{GL}_2}(X) \cong \frac{CH_{\operatorname{GL}_2}(Z)}{c_1((\det \mathcal{E})^{\otimes 2} \otimes \mathcal{O}(-1))} = \frac{CH_{\operatorname{GL}_2}(Z)}{2\lambda_1 - t}$$

We also know that

$$CH_{\operatorname{GL}_2}(Z) = \frac{CH_{\operatorname{GL}_2}(\mathbb{P}^6)}{p_i(\lambda_1, \lambda_2, t)}$$

for some relations p_i, so if we can find those we are done.

Chapter 8

Finding the missing relations

Speakers: Bernardo Tarini

8.1 Find the relations

Consider $\mathbb{P}^6 \to \operatorname{Spec} k$ as a projective bundle. We get a projective bundle

$$\frac{\mathbb{P}^6 \times U}{\operatorname{GL}_2} \to \frac{U}{\operatorname{GL}_2}$$

and

$$CH_{\operatorname{GL}_2}(\mathbb{P}^6) = \frac{CH_{\operatorname{GL}_2}(\operatorname{Spec} k)[t]}{(\sum t^{7-i} c_i(\operatorname{Sym}^6 F))} = \frac{\mathbb{Z}[\lambda_1, \lambda_2][t]}{(\sum t^{7-i} c_i(\operatorname{Sym}^6 F))}$$

Set $\beta = \sum t^{7-i} c_i(\operatorname{Sym}^6 F)$. One can check that

$$\beta = (t^2 - 6\lambda_1 t + 8\lambda_1^2 + 4\lambda_2)(t - 3\lambda_1).$$

Now, let $i : \mathbb{P}^6 \setminus Z \to \mathbb{P}^6$ be the closed immersion. We want to compute the ideal $i_*(CH_{\operatorname{GL}_2}(\mathbb{P}^6 \setminus Z)) \subseteq CH_{\operatorname{GL}_2}(\mathbb{P}^6)$.

By definition, since Z are the square free forms, we have for $r = 1, 2$ or 3 the map

$$\pi_r : \quad \begin{array}{ccc} \mathbb{P}^r \times \mathbb{P}^{6-2r} = \mathbb{P}(\operatorname{Sym}^r F) \times \mathbb{P}(\operatorname{Sym}^{6-2r} F) & \longrightarrow & \mathbb{P}^6 \\ (f, g) & \longmapsto & f^2 g \end{array}$$

π_r has image contained in $\mathbb{P}^6 \setminus Z$ by construction (for $r = 1$ the image IS $\mathbb{P}^6 \setminus Z$).

Lemma 8.1. Let $\pi_1 \oplus \pi_2 \oplus \pi_3 : \coprod_{r=1}^3 \mathbb{P}^r \times \mathbb{P}^{6-2r} \to \mathbb{P}^6 \setminus Z$. Then for all $V \subseteq \mathbb{P}^6 \setminus Z$ irreducible closed there exists $\widetilde{V} \subseteq \coprod_{r=1}^3 \mathbb{P}^r \times \mathbb{P}^{6-2r}$ that maps birationally to V via the map.
Moreover, if V is GL_2-invariant then $\widetilde{V}$ is GL_2-invariant.[a]

---

[a]This is basically the definition of a Chow envelope, proper map such that all irred in codomain we have irred in domain that maps birationally to it.

Proof.

We can view a subvariety of $\mathbb{P}^6 \setminus Z$ ad a map $\operatorname{Spec} L \to \mathbb{P}^6 \setminus Z$ (where L is some extension of k). Let f be the form of degree 6 that comes from $\operatorname{Spec} L$ via this map (it has L coefficients). Let us write f as $f = u^2 v$ up to considering a finite extension of L (here the characteristic plays a role, we want $\operatorname{char} k = 0$ or bigger than 6). Having now written $f = u^2 v$ we have that (u, v) maps to f via our map. If we specify that u should have maximal degree then we also get the GL_2-invariance because the fiber is invariant and we have a unique possible u with maximal degree. $\qquad \square$

Remark 8.2. Let $W \subseteq U \times S$ be a G-invariant irreducible closed subset. If ξ is the generic point the lemma amounts to saying that we have surjectivity on the Chow rings.

Let $\xi_1 = p_1^* \mathcal{O}_{\mathbb{P}^r}(1), \xi_2 = p_2^* \mathcal{O}_{\mathbb{P}^{6-2r}}(1)$.

$$\pi_r^*(\mathcal{O}(1)) = p_1^*(\mathcal{O}(1))^{\otimes 2} \otimes p_2^*(\mathcal{O}(1))$$

So If $t \in CH_{\operatorname{GL}_2}(\mathbb{P}^6)$ we have $\pi_r^* t = 2\xi_1 + \xi_2$.

We know that $\mathbb{P}^r \to \operatorname{Spec} k$ makes $CH(\mathbb{P}^r)$ be generated by powers of ξ_1. We also have $CH(\mathbb{P}^r \times \mathbb{P}^{6-2r})$ is generated as $CH(\mathbb{P}^r)$-module by ξ_2, but by the formula $\pi_r^* t - 2\xi_1 = \xi_2$ we can ignore ξ_2 when looking at $CH(\mathbb{P}^r \times \mathbb{P}^{6-2r})$ as a $CH(\mathbb{P}^6)$-module via π_r.

By the projection formula, $(\pi_r)_*$ has image the ideal generated by $\left\{ (\pi_r)_* \xi_1^i \mid i = 0, \cdots, r \right\}$.

Consider

$$
\begin{array}{ccc}
(\mathbb{P}^1)^r \times (\mathbb{P}^1)^{6-2r} & \dashrightarrow{\delta_r} & (\mathbb{P}^1)^6 \\
\downarrow{\rho_r} & & \downarrow{\rho} \\
\mathbb{P}^r \times \mathbb{P}^{6-2r} & \xrightarrow{\pi_r} & \mathbb{P}^6
\end{array}
$$

where the vertical arrows correspond to multiplying the 1-forms and the dashed one simply duplicates the first r entries.

Notation. Let $x_i = p_i^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1))) \in CH_{\operatorname{GL}_2}(\mathbb{P}^1)^6$.

Remark 8.3. These inherit relations $x_i^2 - \lambda_1 x_i + \lambda_2 = 0$ from the definition (look at the bundles).

Also, $\rho^* t = x_1 + \cdots + x_6$.

If $t_k = \sum_{i=1}^6 x_i^k$ then

$$t_k = a_k + b_k t \quad \text{with } a_k, b_k \in \mathbb{Z}[\lambda_1, \lambda_2]$$

(we are identifying t with $\rho^* t$ and using the relation to remove higher powers of the x_i). We have therefore shown that $t_k \in \operatorname{Imm}(\rho^*)$.

Moreover, if $s_k = k! \sum_{1 \le i_1 < \cdots < i_k \le 6} x_{i_1} \cdots x_{i_k}$ then $s_k \in \rho^*(CH_{\operatorname{GL}_2}(\mathbb{P}^6))$. This is because $s_1 = t$ and the others follow by induction using the "Newton formulas"

$$s_k = \sum (-1)^{k-1} \frac{(k-1)!}{(k-i)!} s_{k-i} t_i$$

Example 8.4. $s_1 = t$, $s_2 = 6\lambda_2 - \lambda_1 t + t^2$.

Now, note that ρ and ρ_r are finite (obviously proper and finite fibers). By miracle flatness they are flat (because the schemes are smooth). Therefore we have a well defined degree. $\deg \rho = 6!$ (permute the factors) and $\deg \rho_r = r!(6 - 2r)!$.

By proper pushforward this means that

$$\rho_* s_k = 6! s_k, \quad \rho_* t_k = 6! t_k$$

and similarly for ρ_r. This allows us to see ξ_1^s as $(\rho_r)_*(\xi_1^s/(r!(6-2r)!))$.

Since $CH_{\mathrm{GL}_2}(\mathbb{P}^6)$ is torsion free, we can find the pushforward $(\pi_r)_*(\xi_1^s) \in CH_{\mathrm{GL}_2}(\mathbb{P}^6) \otimes$ $\mathbb{Q}$ without losing relations.

Remark 8.5. $\rho_*(x_{i_1} \cdots x_{i_k}) = \frac{6!}{\binom{6}{k}k!} s_k = (6-k)! s_k.$

Fact 8.6. If $L \to S$ vector bundle of rank 2 with S smooth and $\Delta : \mathbb{P}(L) \to$ $\mathbb{P}(L) \times_S \mathbb{P}(L)$ is the diagonal, $[\Delta(\mathbb{P}(L))] = x_1 + x_2 + c_1(L)$ where $x_i = p_i^*(c_1(\mathcal{O}(1)))$.

Sketch.
$P = \mathbb{P}(L)$ We have the Belimon sequence

$$0 \to p_1^* \Omega_{P/S}(1) \otimes p_2^* \mathcal{O}(-1) \to \mathcal{O}_{P \times_S P} \to \mathcal{O}_\Delta \to 0$$

and the Euler sequence

$$0 \to \Omega_{P/S} \to \pi^* L^\vee \otimes \mathcal{O}_P(-1) \to \mathcal{O}_P \to 0$$

$\square$

In our case we get

$$[\delta_r((\mathbb{P}^1)^r \times \mathbb{P}^{6-2r})] = (x_1 + x_2 - \lambda_1)(x_3 + x_4 - \lambda_1) \cdots (x_{2r-1} + x_{2r} - \lambda_1)$$

Remark 8.7. $\delta_r^* x_{2i-1} = \delta_r^* x_{2i}$ for $i = 1, \cdots, r$ and so $\rho_r^*(\xi_1) = x_1 + x_3 + \cdots + x_{2r-1}$.

By projection formula

$$(\delta_r)_*(\rho_r^*(\xi_1)) = (x_1 + \cdots + x_{2r-1})[\delta_r((\mathbb{P}^1)^r \times (\mathbb{P}^1)^{6-2r})] = (x_1 + \cdots + x_{2r-1})(x_1 + x_2 - \lambda_1) \cdots (x_{2r-1} + x_{2r} - \lambda_1)$$

so

$$(\delta_r)_*(\rho_r^*(\xi_1^s)) = (x_1 + \cdots + x_{2r-1})^s (x_1 + x_2 - \lambda_1) \cdots (x_{2r-1} + x_{2r} - \lambda_1)$$

Now we push forward via ρ and this will yield $(\pi_r)_*(\xi_1^s)$ by commutativity.

Example 8.8.

$$(\pi_1)_*(\xi_1^0) = (\pi_1)_*(1) = \frac{1}{4!} \rho_*(x_1^0(x_1 + x_2 - \lambda_1)) = \frac{1}{4!}(2s_1(6-1)! - s_0\lambda_1 6!) = \frac{2 \cdot 5! t - \lambda_1 6!}{4!}$$

Now, if we view this inside $CH_{\mathrm{GL}_2}(Z)$ we also have the relation $2\lambda_1 = t$ so in conclusion we have

$$\frac{1}{4!}(4\lambda_1 5! - \lambda_1 6!) = -10\lambda_1.$$

Example 8.9.

$$(\pi_1)_*(\xi_1) = \frac{1}{4!}\rho_*(x_1(x_1 + x_2 - \lambda_1)) = \frac{\rho_*(\lambda_1 x_1 - \lambda_2 + x_1 x_2 - \lambda_1 x_1)}{4!} = \frac{\rho_*(-\lambda_2 + x_1 x_2)}{4!} = \frac{-6!\lambda_2 + (6-2)! s_2}{4!} =$$

now again substitute $2\lambda_1 = t$ and we get

$$(\pi_1)_*(\xi_1) = -24\lambda_2 + 2\lambda_1^2$$

It turns out that all other relations come from these two.

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