[TITOLO]

[Sottotitolo]

Francesco Sorce

Università di Pisa
Dipartimento di Matematica

Contents

Chapter 1

Moduli Spaces

Set theoretic issues: whenever I write that something is an element of a class, what I mean is that that object satisfies the proposition that defines the class.

1.1 Introduction to moduli problems

Definition 1.1 (Presheaf)**.**
A contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is called a **presheaf** on $\mathcal{C}$.

Definition 1.2 (Moduli problem)**.**
Let S be a scheme. A presheaf on Sch/S is called a **moduli problem**.

1.2 Fine and Coarse moduli spaces

1.3 Zariski sheaves and gluing of fine moduli spaces

Definition 1.3 (Equalizer)**.**
Let $\mathcal{C}$ be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \to C$. We say that the pair (A, h) is an **equalizer** of the diagram

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

if $h : A \to B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \to A$ which makes the diagram commute

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ Q \arrow[u, dashed] \arrow[ur, "q"'] & & \end{tikzcd}$$

Definition 1.4 (Zariski sheaf)**.**
A moduli problem $F \in (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a **Zariski sheaf** if for any S-scheme X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Proposition 1.5 (Representable moduli functors are Zariski sheaves)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem, then if there exists a fine moduli space M for F it must be the case that F is a Zariski sheaf.

Proof.
By composing with the natural isomorphism we may assume $F = h_M$. Let X be an S-scheme and $\{U_i \to X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\mathrm{Mor}(X, M) \longrightarrow \prod_k \mathrm{Mor}(U_k, M) \rightrightarrows \prod_{i,j} \mathrm{Mor}(U_i \cap U_j, M)$$

The arrows in this case correspond to restriction of morphisms, so the thesis is equivalent to the fact that restriction to a given set doesn't depend on the intermediate restrictions and that morphisms of schemes that coincide on double intersections glue to the union, both of which are true. $\square$

Definition 1.6 (Subfunctor)**.**
Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. A **subfunctor** of G is a pair (F, i) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $i : F \to G$ such that $i_X : F(X) \to G(X)$ is a monomorphism for all $X \in \mathcal{C}$.

Remark 1.7.
If $\mathcal{D} = \mathrm{Set}$ then we can express the same data equivalently as follows:
A functor $F : \mathcal{C} \to \mathrm{Set}$ is a subfunctor of $G : \mathcal{C} \to \mathrm{Set}$ if for all $X \in \mathcal{C}$ and for all $f \in Mor_{\mathcal{C}}(A, B)$

$$F(X) \subseteq G(X), \qquad \text{and} \qquad F(f) = G(f)|_{F(A)}.$$

In this case we write $F \subseteq G$.

Definition 1.8 (Fibered product of presheaves)**.**
Let $F, G, H : \mathcal{C}^{op} \to \mathrm{Set}$ be presheaves together with two natural transformations $\xi^1 : F \to H$ and $\xi^2 : G \to H$. We define their fibered product as follows:
If $X \in \mathcal{C}$ then

$$(F \times_H G)(X) = F(X) \times_{H(X)} G(X),$$

if $f : A \to B$ then[1]

$$(F \times_H G)(f) : \begin{array}{ccc} F(B) \times_{H(B)} G(B) & \longrightarrow & F(A) \times_{H(A)} G(A) \\ (b_1, b_2) & \longmapsto & (F(f)(b_1), G(f)(b_2)) \end{array}.$$

Definition 1.9 (Open subfunctor)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S-scheme T and any natural trasformation $h_T \to F$, the pullback $h_T \times_F G$ is representable by an open subscheme of T.

Remark 1.10.
By the Yoneda lemma, giving a natural transformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows:
A subfunctor $G \subseteq F$ is open if for any S-scheme T and any family $\xi \in F(T)$ there

[1] the map is well defined because $\xi^1_A(F(f)(b_1)) = H(f)(\xi^1_B(b_1)) = H(f)(\xi^2_B(b_2)) = \xi^2_A(G(f)(b_2))$.

exists an open subscheme $U \subseteq T$ such that the following diagram is natural in R and commutes

$$\begin{tikzcd} \mathrm{Mor}(R,U) \arrow[r, "G(\subseteq\circ\ \cdot\ )(\xi)"] \arrow[d, hook, "\subseteq\circ"'] & G(R) \arrow[d, hook, "\subseteq"] \\ \mathrm{Mor}(R,T) \arrow[r, "F(\cdot)(\xi)"'] & F(R) \end{tikzcd}$$

and a map $f \in \mathrm{Mor}(R,T)$ factors as $R \xrightarrow{g} U \subseteq T$ <u>if and only if</u> $F(f)(\xi) \in G(R)$[2].

Definition 1.11 (Open cover of a functor)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. A collection of open subfunctors $\{F_i\}$ is an **open cover** of F if for any S-scheme T and any natural transformation $h_T \to F$, the open subschemes U_i that represent the pullbacks $h_T \times_F F_i$ form an open cover of T.

Remark 1.12.
Like above, we can rephrase the definition as follows:
A collections of open subfunctors $F_i \subseteq F$ form an open cover of F if for any S-scheme T and any family $\xi \in F(T)$, there exists an open cover $\{U_i\}$ of T such that $\xi|_{U_i} \in F_i(U_i)$ for all i.

The definitions above let us state the following criterion for representability:

Proposition 1.13 (Representability by open cover)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem which is a Zariski sheaf and let $\{F_i\}$ be an open cover of it by representable subfunctors, then F is representable.

Proof.
Let X_i be the fine moduli space for F_i and let $\xi_i \in F_i(X_i)$ be their universal families. Since F_j is an open subfunctor of F, there exists an open subscheme $U_{ij} \subseteq X_i$ such that the diagram is a cartesian square

$$\begin{tikzcd} h_{U_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & h_{X_i} \arrow[d] \\ F_j \arrow[r] & F \end{tikzcd}$$

Evaluating the diagram on U_{ij} yields

$$\begin{tikzcd} \mathrm{Mor}(U_{ij},U_{ij}) \arrow[r] \arrow[d] & \mathrm{Mor}(U_{ij},X_i) \arrow[d] \\ F_j(U_{ij}) \arrow[r] & F(U_{ij}) \end{tikzcd}$$

The top left set contains $id_{U_{ij}}$, which corresponds to the universal family in

$$F_j(U_{ij}) \times_{F(U_{ij})} \mathrm{Mor}(U_{ij}, X_i),$$

given by $(\xi|_{U_{ij}}, \iota_i)$, where $\iota_i : U_{ij} \to X_i$ is the inclusion and $\xi_j|_{U_{ij}} = F_j(f_i)(\xi_j)$ for

$$\begin{tikzcd} h_{U_{ij}}(U_{ij}) \arrow[d, phantom, sloped, "\in"] & \cong & F_j(U_{ij}) \times_{F(U_{ij})} h_{X_i}(U_{ij}) \arrow[r, "pr_1"] & F_j(U_{ij}) & \cong & \mathrm{Mor}(U_{ij}, X_j) \arrow[d, phantom, sloped, "\in"] \\ id_{U_{ij}} \arrow[rrrrr, mapsto] & & & & & f_i \end{tikzcd}$$

[2]the "only if" is trivially true by commutativity but for the "if" we are using the fact that $h_U \cong h_T \times_F G$.

The images of these two elements in $F(U_{ij})$ are $\xi_j|_{U_{ij}}$ (since $F_j \subseteq F$) and $\xi_i|_{U_{ij}} = \iota_i^* \xi_i$, so by commutativity

$$\xi_j|_{U_{ij}} = \xi_i|_{U_{ij}}.$$

We now evaluate the cartesian square that defines U_{ji} on U_{ij}:

$$\begin{array}{ccc} \mathrm{Mor}(U_{ij}, U_{ji}) & \longrightarrow & \mathrm{Mor}(U_{ij}, X_j) \\ \downarrow & & \downarrow \\ F_i(U_{ij}) & \longrightarrow & F(U_{ij}) \end{array}$$

Given what we have already said, $(\xi_i|_{U_{ij}}, f_i) \in F_i(U_{ij}) \times_{F(U_{ij})} h_{X_j}(U_{ij})$, so it defines a morphism $\varphi_{ij} : U_{ij} \to U_{ji}$. We observe that φ_{ij} and φ_{ji} are inverses of each other. Indeed if we consider $\varphi_{ji} \circ \varphi_{ij}$ as an element of the top left set in the diagram

$$\begin{array}{ccc} \mathrm{Mor}(U_{ij}, U_{ij}) & \longrightarrow & \mathrm{Mor}(U_{ij}, X_i) \\ \downarrow & & \downarrow \\ F_j(U_{ij}) & \longrightarrow & F(U_{ij}) \end{array}$$

we notice that the two projections are given by

$$\iota_i \circ \varphi_{ji} \circ \varphi_{ij} = f_j \circ \varphi_{ij} = f_i$$

and

$$F_j(f_i \circ \varphi_{ji} \circ \varphi_{ij})(\xi_j) = F_j(f_j \circ \varphi_{ij})(\xi_j) = F_j(f_i)(\xi_j) = \xi_j|_{U_{ij}},$$

which are the same projections as $id_{U_{ij}}$, so $\varphi_{ji} \circ \varphi_{ij} = id_{U_{ij}}$. A symmetric argument yields $\varphi_{ij} \circ \varphi_{ji} = id_{U_{ji}}$. Note that with our notation $\varphi_{ii} = id_{U_{ii}} = id_{X_i}$.
We also remark that

$$F(\varphi_{ij})(\xi_j) = F_j(\iota_j \circ \varphi_{ij})(\xi_j) = F_j(f_i)(\xi_j) = \xi_j|_{U_{ij}} = \xi_i|_{U_{ij}}.$$

We now want to show that the X_i can be glued along the U_{ij} using the isomorphisms φ_{ij}.

Intersection We want to check that

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}.$$

Given the symmetry of the indicies and having already proven that $\varphi_{ji} = \varphi_{ij}^{-1}$, we just need to show inclusion. By the equivalent definition of open subfunctor, we know that the morphism $\varphi_{ij} : U_{ij} \cap U_{ik} \to U_{ji}$ factors through $U_{ji} \cap U_{jk}$ if and only if $\varphi_{ij}^* \xi_j|_{U_{ji} \cap U_{jk}} \in F(U_{ij} \cap U_{ik})$, which is true because

$$\varphi_{ij}^* \xi_j|_{U_{ji} \cap U_{jk}} = \xi_i|_{U_{ij} \cap U_{ik}}.$$

Cocycle cond. We now want to verify that

$$\varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}} = \varphi_{ik}|_{U_{ik} \cap U_{ij}},$$

but this simply follows from the fact that both maps pullback $\xi_k|_{U_{ki} \cap U_{kj}}$ to $\xi_i|_{U_{ij} \cap U_{ik}}$.

We can thus define X to be the scheme obtained by gluing the X_i along the U_{ij}. Moreover, since F is a Zariski sheaf, the ξ_i must glue to a family $\xi \in F(X)$[3].

We now only need to verify that (X, ξ) is a fine moduli space for F:
Let T be an S-scheme and and let us consider a family $\zeta \in F(T)$. Since $\{F_i\}$ is an open cover of F, there exists an open cover $\{V_i\}$ of T such that $\zeta_{U_i} \in F_i(V_i) \cong \mathrm{Mor}(V_i, X_i)$. Since F is a sheaf $\zeta_i|_{V_i \cap V_j} = \zeta_j|_{V_i \cap V_j}$, so the morphisms $V_i \to X_i$ corresponding to the ζ_i glue to a morphism $f : T \to X$ such that $f^*\xi = \zeta$ (by construction). $\square$

[3]because $\xi_i|_{U_ij} = \xi_j|_{U_ij}$.

Chapter 2

Grassmanians

2.1 Set-theoretic definition

Definition 2.1 (Grassmannian)**.**
Let $k \leq n$ be a pair of positive integers. We define the (n,k)**-Grassmannian**, denoted[1] $\mathrm{Gr}(k,n,\mathbb{K})$, as the set of $(n-k)$-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^n$.

Remark 2.2 (Definition via quotients).
We may equivalently define $\mathrm{Gr}(k,n)$ to be the following set:

$$\left\{\ker\varphi \mid \varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n,\mathbb{K}^k),\ \mathrm{rnk}\,\varphi = k\right\}.$$

Lemma 2.3.
Let $\varphi,\psi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n,\mathbb{K}^k)$ be linear maps of full rank. The following conditions are equivalent:

1. $\ker\varphi = \ker\psi$*,*
2. *there exists $\theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta\circ\psi$.*

Proof.
We shall prove the two implications:

$\boxed{2. \implies 1.}$ $\ker(\theta\circ\psi) = \psi^{-1}(\ker\theta) = \psi^{-1}(\{0\}) = \ker\psi$.

$\boxed{1. \implies 2.}$ Let $z_1,\cdots,z_{n-k}$ be a basis of $\ker\varphi = \ker\psi$ and let $z_1,\cdots,z_{n-k},v_1,\cdots,v_k$ be a completion of it to a basis of $\mathbb{K}^n$. By construction $\varphi(v_1),\cdots,\varphi(v_k)$ and $\psi(v_1),\cdots,\psi(v_k)$ are bases of $\mathbb{K}^k$. Let θ be the linear automorphism of $\mathbb{K}^k$ determined by $\theta(\psi(v_i)) = \varphi(v_i)$ for all i. By construction θ is nonsingular and φ agrees with $\theta\circ\psi$ on a basis of $\mathbb{K}^n$.

□

Corollary 2.4.
We may identify Grassmannians in terms of linear maps as follows:

$$\mathrm{Gr}(k,n) = {\left\{\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n,\mathbb{K}^k) \mid \varphi \text{ surjective.}\right\}}\big/{\sim}$$

where $\varphi \sim \psi$ if and only if $\exists\theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta\circ\psi$.

[1] we shall often omit the field when clear from context

2.1.1 The Plücker embedding

To make the study of Grassmanians easier, we want to identify $\mathrm{Gr}(k,n)$ with a subset of some projective space.

Definition 2.5 (Plücker map)**.**
Let $k \leq n$ be a pair of positive integers. We define the **Plücker map** as follows:

$$\phi : \begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \\ \varphi & \longmapsto & \wedge^k \varphi \end{array},$$

where $(\wedge^k \varphi)(v_1, \cdots, v_k) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_k)$.

Remark 2.6.
$\mathrm{rnk}\,\varphi < k$ if and only if $\phi(\varphi) = 0$.

Proof.
$\phi(\varphi)$ is the zero map if an only if $\varphi(v_1), \cdots, \varphi(v_k)$ are always linearly dependent, i.e. if and only if φ is not of full rank. □

Proposition 2.7.
Let $\sim$ be the equivalence relation defined in corollary (2.4), then for linear full rank maps $\varphi, \psi : \mathbb{K}^n \to \mathbb{K}^k$

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{K}^* \ s.t.\ \phi(\varphi) = \lambda \phi(\psi).$$

Proof.

□

2.2 Definition as a projective scheme

In order to write $\mathbb{K}$-algebra morphisms that correspond to what we've done geometrically, we shall switch to the language of matricies.

Definition 2.8 (Multiindicies)**.**
We define a (k,n)**-multiindex** as an element of $\{1, \cdots, n\}^k$. Our notation for a multiindex I will usually be $I = (i_1, \cdots, i_k)$.
If A is a $k \times n$ matrix and I is a (k,n)-multiindex, we denote the I-minor by A_I, i.e.

$$A_I = \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{pmatrix}.$$

2.3 Moduli functor

2.3.1 Affine cover

2.3.2 Representability of the Grassmannian functor