Moduli Spaces and Grassmannians

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Abstract

In this document we introduce the concept of moduli spaces in algebraic geometry through the example of the Grassmannian scheme.

The first chapter introduces the basics of the functorial approach to algebraic geometry and its relation to moduli problems.

The second chapter is a quick overview of Grassmannians as defined set theoretically. We focus our attention on the Plücker embedding and prove that it identifies the Grassmannian with a projective variety.

In the third chapter we describe the reduced scheme structure on the Grassmannian and prove that it is a fine moduli space for the functor of quotients from $\mathcal{O}_T^n$ to a rank k vector bundle on T.

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Introduction

The following type of *classification problem* occurs often in math:

> Consider some type of object and a notion of isomorphism which can be defined between them. We are interested in understanding the behaviour of isomorphism classes and how they relate to each other.

Finding a bijection between isomorphism classes and known objects is usually trivial[1], but for an answer to a classification problem to be satisfactory we usually require some information on *families* of isomorphism classes.

Miraculously, many such classification problems turn out to have a natural answer in the form of some geometric object. Usually the object can only be defined as the families themselves together with some geometric structure (this is the realm of the theory of stacks), but in more special circumstances one can find a more concrete space (usually a scheme) whose points represent isomorphism classes for our problem and whose geometric structure encodes information on the families. Such objects are called *moduli spaces* for the classification problem.

The best result we can hope for is finding a space which completely encodes how families behave[2], but this requirement is usually too strict. In this document we mostly deal with problems for which such a nice space exists: the Grassmannian and the Hilbert scheme.

Historical background

The history of moduli spaces begins with the article [6], where Riemann computes what we would now call the dimention of M_g, the moduli space of smooth projective algebraic curves of genus g, to be $3g-3$.

Although the argument given by Riemann can be made rigorous in modern language, he did not prove the existence of the space M_g itself. The first general construction of M_g as a space of some kind can be attributed to Teichmüller, which realized M_g as the quotient of the Teichmüller space T_g parametrizing complex structures up to isomorphism on a surface of genus g by the action of the group Γ_g of diffeomorphisms of the surface up to isotopy. The paper which establishes these ideas is [8].

Alexander Grothendieck introduced the functorial approach to analytic moduli theory and later on to algebraic geometry in general. Grothendieck was very interested

[1] for example, if the classes form a set they can be identified with a canonical set of the same cardinality.

[2] what will be formalized as a fine moduli space

in algebraic moduli theory and contributed to it greatly by introducing the Hilbert, Quot and Picard functors and showing their representability by schemes. However, Grothendieck did not end up publishing on M_g.

Among the first to study moduli spaces systematically was David Mumford. Inspired by invariant theory, Grothendieck's functorial approach and the existing constructions of moduli spaces like the one of principally polarized abelian varieties or the Chow varieties, Mumford developed Geometric Invariant Theory (commonly refered to as GIT), which can be described as a method to study and construct moduli spaces as quotients of algebraic groups. In the book [5] Mumford gives two constructions of M_g as a coarse moduli space.

Why category theory?

As we briefly mentioned, the modern approach to moduli problems if formalized via functors. It might not be clear why this is the most appropriate tool, and indeed it can seem more complicated than more concrete treatments in simple cases like the classification of lines through a point via projective space.

Nevertheless, the functorial approach has proven itself to be effective in many aspects, chief among them the formalization of the nebulous concept of "family" described above.

Following Grothendieck's ideas, a moduli problem is expressed as a contravariant functor

$$F : T \mapsto {\{\text{families of objects over } T\}}\big/{\sim}$$

where $\sim$ is the isomorphism relation imposed on families of objects.

Since we are mostly concerned about problems in algebraic geometry, and thus families over schemes, the functor is usually taken to be a presheaf on Sch/S for some base scheme S[3], i.e. $F : \mathrm{Sch}/S^{op} \to \mathrm{Set}$. To find the set of objects we want to classify up to isomorphism we can simply evaluate F on a point.

The functorial language allows for families to be pulled back via morphisms: if $f : S \to T$ is a morphism and $a \in F(T)$ is a family over T, then $F(f) : F(T) \to F(S)$ by contravariance and thus $F(f)(a) \doteqdot f^*a \in F(S)$ is a family over S.

There are several ways in which we can define a moduli space. The two most relevant are *fine* and *coarse* moduli spaces. A scheme M is a fine moduli space if we can recover the whole moduli functor from it[4]. M is a coarse moduli space if its $\mathbb{K}$-points are in bijection with $F(\operatorname{Spec}\mathbb{K})$ and if M is universal for this property.

In both cases we can interpret a family of objects over a scheme T as a morphism from T to M. Intuitively this is because a function from T to M is an assignment of an isomorphism class to each point of T, the added structure of a scheme morphism serves to define a "niceness" condition to the considered families. If M is a fine moduli space, then every family over T can be viewed as the pullback under a morphism $T \to M$ of a specific family $u \in F(M)$, called the *univeral family.*

[3] usually $\operatorname{Spec}\mathbb{K}$ for an algebraically closed field $\mathbb{K}$ or $\operatorname{Spec}\mathbb{Z}$.
[4] formally, when h_M and F are naturally isomorphic functors.

Chapter 1

Moduli Spaces

In this chapter we introduce the basic category theory used in the study of moduli spaces. After a quick review of the Yoneda embedding, we define representability of a functor and give the definition of fine and coarse moduli space. After that we give a quick overview of Zariski sheves and prove the representability criterion we will need in the third chapter.

Most definitions given in this chapter follow section 0.3 of [1].

1.1 Representable functors

In this section we introduce the basic concepts of the functorial approach and some of the required category theory.

1.1.1 Yoneda lemma and embedding

We use the following conventions:

- All categories considered in this document will be small.
- If $\mathcal{C}$ is a category, we shall write $X \in \mathcal{C}$ to mean "X is an object in $\mathcal{C}$".
- If A and B are objects in a category (respectively categories / functors) then $\mathrm{Hom}\,(A, B)$ denotes the set of morphisms (respectively functors / natural transformations) from A to B.

  In the first case, if $A, B \in \mathcal{C}$ we sometimes write $\mathrm{Hom}_{\,\mathcal{C}}(A, B)$ to emphasize the category in which we are considering the morphisms.
- The notation $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denotes the category of functors from $\mathcal{C}$ to $\mathcal{D}$, with morphisms being natural transformations.

Definition 1.1 (Presheaf)**.** A contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is called a **presheaf** on $\mathcal{C}$. If $T \in \mathcal{C}$ then we call the elements of $F(T)$ **families** over T.

Definition 1.2 (Presheaf category)**.** For any fixed category $\mathcal{C}$, the presheaves on $\mathcal{C}$ form a category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ with morphisms given by natural transformations.

Definition 1.3 (Hom-functor)**.** Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We define the **Hom-functor** of X to be

$$h_X : \begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \mathrm{Hom}\,(T, X) \\ f : T \to S & \longmapsto & h_X(f) : \mathrm{Hom}\,(S, X) \to \mathrm{Hom}\,(T, X) \end{array}$$

where, if $g : S \to X$ then $h_X(f)(g) = g \circ f : T \to X$.

Remark 1.4. The Hom-functor is a presheaf.

Definition 1.5 (Moduli problem)**.** Let S be a scheme. A presheaf on Sch/S is called a **moduli problem**.

Remark 1.6. Usually we study moduli problems of the following form

$$\begin{array}{ccc} \mathrm{Sch}/S & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \{\text{families over } T\}/\sim \\ f : T \to R & \longmapsto & \text{pullback of families along } f \end{array}$$

where $\sim$ refers to an equivalence relation imposed on the objects we are studying.

To proceed we will need the following

Lemma 1.7 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. For all contravariant functors $F : \mathcal{C}^{op} \to \mathrm{Set}$ the following sets are in a natural bijection*

$$\mathrm{Hom}\,(h_X, F) \longleftrightarrow F(X).$$

Proof. Given a natural trasformation ζ, we can take its image in $F(X)$ to be $\zeta_X(id_X)$. On the other hand, for any given element $u \in F(X)$ we can define an arrow $h_X(T) \to F(T)$ for any $T \in \mathcal{C}$ by taking $f \mapsto F(f)(u)$. This collection of maps defines a natural transformation from h_X to F by commutativity of the following diagram for all $g : S \to T$

$$\begin{tikzcd}
\mathrm{Hom}\,(X,X) \arrow[dr, "h_X(f)"'] \arrow[drr, "h_X(f\circ g)"] \arrow[ddd, "F(\cdot)(u)"'] & & \\
& \mathrm{Hom}\,(T,X) \arrow[r, "h_X(g)"'] \arrow[d, "F(\cdot)(u)"'] & \mathrm{Hom}\,(S,X) \arrow[d, "F(\cdot)(u)"] \\
& F(T) \arrow[r, "F(g)"] & F(S) \\
F(X) \arrow[ur, "F(f)"] \arrow[urr, "F(f\circ g)"'] & &
\end{tikzcd}$$

□

Definition 1.8 (Yoneda embedding)**.** We define the **Yoneda embedding** of a category $\mathcal{C}$ to be the following functor

$$h_\bullet : \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \\ X & \longmapsto & h_X \\ f : X \to Y & \longmapsto & h_f : h_X \to h_Y \end{array}$$

where if $g : T \to X$ then $h_f(g) = f \circ g : T \to Y$.

Remark 1.9. If $X = Y$ then $h_{id_X} = id_{h_X}$. Indeed if $f : T \to X$ then

$$(h_{id_X})_T(f) = id_X \circ f = f = (id_{h_X})_T(f).$$

Proposition 1.10. *The functor $h_\bullet$ is fully faithful.*

Proof. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is fully faithful if for any two objects $A, B \in \mathcal{C}$ we have $\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$. In our case we want to check if

$$\mathrm{Hom}\,(X, Y) \cong \mathrm{Hom}\,(h_X, h_Y),$$

which is exactly the statement of the Yoneda lemma (1.7). □

Remark 1.11. The Yoneda embedding is injective up to isomorphism on isomorphism classes of objects in $\mathcal{C}$.

Proof. If $\varphi : h_A \to h_B$ is an isomorphism with inverse φ', by the Yoneda lemma they coorespond to maps $f : A \to B$ and $f' : B \to A$, which must be inverses because the Yoneda embedding is fully faithful.

□

Lemma 1.12 (Yoneda embedding is continuous)**.** *The Yoneda embedding preserves limits.*

Proof. Suppose X is the limit of the diagram $\{f_{ij} : X_j \to X_i\}$. If we apply the Yoneda embedding to the diagram we obtain

$$\left\{h_{f_{ij}} : h_{X_j} \to h_{X_i}\right\}$$

Let F be any presheaf on $\mathcal{C}$ and suppose that we have morphisms $F \to h_{X_i}$ which make the diagrams commute, then for all $T \in \mathcal{C}$ we have compatible and natural $F(T) \to \mathrm{Hom}\,(T, X_i)$. If $f \in F(T)$ then these arrows define several $f_i \in \mathrm{Hom}\,(T, X_i)$ which compose with the f_{ij} respecting the diagram. By the universal property of limits this defines uniquely a morphism $f_\ell \in \mathrm{Hom}\,(T, X)$ and we see that the assignment $f \mapsto f_\ell$ is the unique map from $F(T)$ to $\mathrm{Hom}\,(T, X)$ which makes the diagram in Set commute. Since all that we have done is natural in T, we have effectively constructed a morphism $F \to h_X$ as we desired. □

1.1.2 Moduli spaces

Definition 1.13 (Representable functor)**.** A presheaf F on $\mathcal{C}$ is **representable** if there exists a natural isomorphism $\zeta : F \to h_X$ for some $X \in \mathcal{C}$. In this case we say that the pair[1] (X, ζ) **represents** F. If $a \in F(T)$ we call $\zeta_T(a) : T \to X$ the **classifying map** of the family a.

Definition 1.14 (Universal family)**.** Instead of the pair (X, ζ) for $\zeta : F \to h_X$, we may consider the pair (X, ξ) where $\xi \in F(X)$ is the image of ζ^{-1} under the bijection predicated by the Yoneda lemma. The object ξ is called the **universal family** of X.

Remark 1.15. The universal family is given by

$$\zeta_X^{-1}(id_X).$$

Definition 1.16 (Fine moduli space)**.** Let F be a moduli functor. A scheme $X \in \mathrm{Sch}/S$ is a **fine moduli space** for F if X represents F.

[1] usually we just say that X represents F

Notation 1.17. If U is a subscheme of T and $i : U \to T$ is the inclusion morphism, then if $\xi \in F(T)$ we will define its restriction to U to be

$$\xi|_U = F(i)(\xi).$$

Remark 1.18. Because the Yoneda embedding is injective on isomorphism classes up to isomorphism (1.11), fine moduli spaces are unique up to isomorphism.

Example 1.19 (Projective space)**.** Consider the functor

$$\mathbb{P}_n : \begin{array}{ccc} \mathrm{Sch}^{op} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \left\{ (\mathcal{L}, s_0, \cdots, s_n) \mid \begin{array}{c} \mathcal{L} \text{ line bundle on } S,\ s_0, \cdots, s_n \in \mathcal{L}(S), \\ \forall x \in S,\ \langle (s_0)_x, \cdots, (s_n)_x \rangle_{\mathcal{O}_{S,x}} = \mathcal{L}_x \end{array} \right\} \Big/ \sim \\ f & \longmapsto & \text{pullback of sheaves and sections via } f \end{array}$$

where $(\mathcal{L}, (s_i)) \sim (\mathcal{L}', (s_i'))$ is there exists a sheaf isomorphism $\alpha : \mathcal{L} \to \mathcal{L}'$ such that $s_i = \alpha^* s_i'$ for all $i \in \{0, \cdots, n\}$.

It is a well know fact (Proposition 5.1.31 in [4]) that $\mathbb{P}_n(S) \cong \mathrm{Hom}\,(S, \mathbb{P}^n_{\mathbb{Z}})$ and that pullbacks behave as expected, thus $\mathbb{P}^n_{\mathbb{Z}}$ is a fine moduli space for $\mathbb{P}_n$. From the statement of Proposition 5.1.31 in [4] it is also clear that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$ is a universal family.

Fine moduli spaces do not always exist. The simplest obstructions to having a fine moduli spaces are

- the functor is not a Zariski sheaf (see proposition (1.26))
- existence of non trivial automorphisms.

To get an idea of why the second condition we cite the following

Proposition 1.20. *Let $F \in (\mathrm{Sch}/\mathbb{C})^{op} \to \mathrm{Set}$ be a moduli functor. If there exists a variety $S \in \mathrm{Sch}/\mathbb{C}$ such that $\mathcal{E} \in F(S)$ is an* ***isotrivial family****, i.e.*

- *for all $s, t \in S(\mathbb{C})$, the fiber $F(s)(\mathcal{E}) = \mathcal{E}_s = \mathcal{E}_t$ and*
- *the family $\mathcal{E}$ is not the pullback of an object $E \in F(\mathrm{Spec}\,\mathbb{C})$ along the structural morphism $S \to \mathrm{Spec}\,\mathbb{C}$,*

then there exists no fine moduli space for F.

Proof. This is Proposition 0.3.21 in [1]. □

A weaker notion of moduli space is that of coarse moduli space:

Definition 1.21 (Coarse moduli space)**.** Let F be a moduli problem. A pair (X, ζ) for $X \in \mathrm{Sch}/S$ and $\zeta : F \to h_X$ natural transformation is a **coarse moduli space** for F if

- $\zeta_{\mathrm{Spec}\,\mathbb{K}} : F(\mathrm{Spec}\,\mathbb{K}) \to \mathrm{Hom}\,(\mathrm{Spec}\,\mathbb{K}, X)$ is a bijection for all algebraically closed fields $\mathbb{K}$
- for any scheme Y and $\eta : F \to h_Y$ natural transformation there exists a unique morphism $\alpha : X \to Y$ such that $\eta = h_\alpha \circ \zeta$.

Remark 1.22. A fine moduli space is also a coarse moduli space.

Proof. The first condition is trivially verified. For the second condition, if (Y, η) is defined as above and (X, u) is the fine moduli space with universal family u then we can take $\alpha = \eta_X(u)$. □

1.2 Zariski sheaves and gluing of fine moduli spaces

One approach to show representability of a moduli problem is emulating the gluing properties of sheaves. Indeed it is possible to show that representable functors are sheaves of some kind and this realization will eventually lead us to the representability criterion we will use later in this document.

1.2.1 Zariski sheaves

First, let us formalize a way in which a functor can be a sheaf. First we recall the definition of equalizer:

Definition 1.23 (Equalizer)**.** Let $\mathcal{C}$ be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \to C$. We say that the pair (A, h) is an **equalizer** of the diagram

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

if $h : A \to B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \to A$ which makes the diagram commute

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ Q \arrow[u, dashed] \arrow[ur, "q"'] & & \end{tikzcd}$$

Definition 1.24 (Zariski sheaf)**.** A moduli problem $F \in (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a **Zariski sheaf** if for any S-scheme X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Remark 1.25. Using the Yoneda lemma (1.7) we may equivalently require the following diagram to be an equalizer

$$\mathrm{Hom}\left(h_X, F\right) \longrightarrow \prod_k \mathrm{Hom}\left(h_{U_k}, F\right) \rightrightarrows \prod_{i,j} \mathrm{Hom}\left(h_{U_i \cap U_j}, F\right)$$

Proposition 1.26 (Representable moduli functors are Zariski sheaves)**.** *Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem, then if there exists a fine moduli space M for F it must be the case that F is a Zariski sheaf.*

Proof. Up to composing with the natural isomorphism, we may assume $F = h_M$. Let X be an S-scheme and $\{U_i \to X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\mathrm{Hom}\left(X, M\right) \longrightarrow \prod_k \mathrm{Hom}\left(U_k, M\right) \rightrightarrows \prod_{i,j} \mathrm{Hom}\left(U_i \cap U_j, M\right)$$

The arrows in this case correspond to restriction of morphisms, so what we need to verify is

- $\mathrm{res}^{U_i}_{U_i\cap U_j} \circ \mathrm{res}^X_{U_i} = \mathrm{res}^{U_j}_{U_i\cap U_j} \circ \mathrm{res}^X_{U_j}$
- a collection of maps $\{f_i : U_i \to M\}$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ glues uniquely to a map $f : X \to M$

both of which are well known properties of maps and scheme morphisms respectively.

□

1.2.2 Open cover of a moduli problem

Let us now define the analogue of an open cover for functors

Definition 1.27 (Subfunctor)**.** A functor $F : \mathcal{C} \to \mathrm{Set}$ is a **subfunctor** of $G : \mathcal{C} \to \mathrm{Set}$ if for all $X \in \mathcal{C}$ and for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$

$$F(X) \subseteq G(X), \qquad \text{and} \qquad F(f) = G(f)|_{F(A)}.$$

In this case we write $F \subseteq G$.

Definition 1.28 (Fibered product of presheaves)**.** Let $F, G, H : \mathcal{C}^{op} \to \mathrm{Set}$ be presheaves together with two natural transformations $\eta : F \to H$ and $\zeta : G \to H$. We define their fibered product as the following functor

$$F \times_H G : \begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathrm{Set} \\ X & \longmapsto & F(X) \times_{H(X)} G(X) \\ f : A \to B & \longmapsto & (b_1, b_2) \mapsto (F(f)(b_1), G(f)(b_2)) \end{array}$$

where the fibered product $F(X) \times_{H(X)} G(X)$ in defined through the maps η_X and ζ_X. The map $(F \times_H G)(f)$ is well defined because if $(b_1, b_2) \in F(B) \times_{H(B)} G(B)$ then $\eta_B(b_1) = \zeta_B(b_2)$, thus

$$\eta_A(F(f)(b_1)) = H(f)(\eta_B(b_1)) = H(f)(\zeta_B(b_2)) = \zeta_A(G(f)(b_2)).$$

Definition 1.29 (Open subfunctor)**.** Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S-scheme T and any natural transformation $h_T \to F$, there exists an open subscheme $U \subseteq T$ such that

$$\begin{tikzcd} U \arrow[r, "h_\bullet"] \arrow[d, phantom, "\cap|"] & h_U \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & G \arrow[d] \\ T \arrow[r, "h_\bullet"] & h_T \arrow[r] & F \end{tikzcd}$$

i.e. U represents $h_T \times_F G$ and the map from h_U to h_T is given by the inclusion.

Because of the Yoneda lemma, giving a natural transformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows

Definition 1.30 (Open subfunctor v.2)**.** A subfunctor $G \subseteq F$ is open if for any S-scheme T and any family $\xi \in F(T)$ there exists an open subscheme $\iota : U \hookrightarrow T$ such that the following diagram is natural in R for all $R \in \mathrm{Sch}/S$, commutes and is cartesian[2]

$$\begin{tikzcd} \mathrm{Hom}\,(R, U) \arrow[r, "G\circ h_\iota(\cdot)(\xi)"] \arrow[d, hook, "h_\iota"'] \arrow[dr, phantom, "\ulcorner"] & G(R) \arrow[d, hook, "\subseteq"] \\ \mathrm{Hom}\,(R, T) \arrow[r, "F(\cdot)(\xi)"'] & F(R) \end{tikzcd}$$

[2]for any map $f : R \to U$ there exists a $g : R \to U$ such that $f = \iota \circ g$ if and only if $F(f)(\xi) \in G(R)$.

Definition 1.31 (Open cover of a functor)**.** Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. A collection of open subfunctors $\{F_i \to F\}$ is an **open cover** of F if for any S-scheme T and any natural transformation $h_T \to F$, the open subschemes U_i of T determined by the F_i form an open cover of T.

Like before we can rephrase this definition in terms of families

Definition 1.32 (Open cover of a functor v.2)**.** Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. A collection of open subfunctors $F_i \subseteq F$ forms an open cover of F if for any S-scheme T and any family $\xi \in F(T)$, the subschemes $U_i \subseteq T$ determined by the openness of the F_i form a cover of T.

Remark 1.33. If $\{F_i \to F\}$ is an open cover of the functor F then for any S-scheme T and any family $\xi \in F(T)$ there exists an open cover $\{U_i \to T\}$ of T such that $\xi|_{U_i} \in F_i(U_i)$ for all i.

1.2.3 Representability criterion

Finally, we come to the main results of this chapter

Proposition 1.34. *Let F and G be Zariski sheaves, $\{F_i \to F\}$ and $\{G_i \to G\}$ be open covers and $f_i : F_i \to G_i$ be natural transformations such that $f_i|_{F_i \cap F_j} = f_j|_{F_i \cap F_j}$. Then there exists a natural transformation $f : F \to G$ which restricts to f_i on F_i.*

Moreover, if each f_i is an isomorphism then f too is an isomorphism.

Proof. Let T be a scheme and $\zeta : h_T \to F$ a natural transformation. Let $\{\iota_i : U_i \to T\}$ be the open cover induced by $\{F_i \to F\}$ through ζ by the definition of open subfunctor cover.

$$\begin{array}{ccccc} h_{U_i} & \xrightarrow{\eta_i} & F_i & \xrightarrow{f_i} & G_i \\ {\scriptstyle h_{\iota_i}}\downarrow & \ulcorner & \cap\!\!| & & \cap\!\!| \\ h_T & \xrightarrow[\zeta]{} & F & & G \end{array}$$

where η_i is the map $\zeta \circ h_{\iota_i}$ with its codomain restricted. This map is well defined because the square is cartesian. Let $g_i = f_i \circ \eta_i$ and note that

$$g_i|_{h_{U_i \cap U_j}} = f_i|_{F_i \cap F_j} \circ \eta_i|_{h_{U_i \cap U_j}} = f_j|_{F_i \cap F_j} \circ \eta_j|_{h_{U_i \cap U_j}} = g_j|_{h_{U_i \cap U_j}}.$$

Because G is a Zariski sheaf, there exists $\zeta' : h_T \to G$ such that $\zeta' \circ h_{\iota_i} = g_i$. Because of the Yoneda lemma (1.7), we have constructed a map $F(T) \to G(T)$ which is functorial in T, i.e. $f : F \to G$. By construction it is also clear that $f|_{F_i} = f_i$.

Suppose now that the f_i are isomorphisms. Let f be the morphism $F \to G$ obtained as above and let $g : G \to F$ be the morphism obtained the same way but from the morphisms $f_i^{-1} : G_i \to F_i$. It is easy to see that the restrictions of $f \circ g$ (respectively $g \circ f$) to the G_i (respectively F_i) and the double intersections yield the identity, so the compositions themselves must glue to the identity of F and G. $\square$

Theorem 1.35 (Representability by open cover)**.** *Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a Zariski sheaf and let $\{F_i \to F\}$ be an open cover of it by representable subfunctors, then F is representable.*

Sketch. We fix schemes X_i and families $\xi_i \in F_i(X_i)$ such that (X_i, ξ_i) is a fine moduli space for F_i. For all S-schemes T we have

$$(F_i \times_F F_j)(T) = F_i(T) \times_{F(T)} F_j(T) = F_i(T) \cap F_j(T) \subseteq F(T),$$

thus $F_i \times_F F_j = F_j \times_F F_i \doteqdot F_{i,j}$.

Using the openness of F_j we find $U_{ij} \subseteq X_i$ which represents $h_{X_i} \times_F F_j \cong F_{i,j}$. By uniqueness of moduli spaces we see that there exists an isomorphism $\varphi_{ji} : U_{ij} \to U_{ji}$, which we can choose in such a way as to translate to the identity transformation $F_{i,j} = F_{j,i}$.

Our choice for the maps φ_{ji} makes the cocycle condition $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ hold. We can thus glue the X_i to a scheme X. Since $\xi_i|_{U_ij} = \xi_j|_{U_{ij}}$ by construction of φ_{ji}, we find a family $\xi \in F(X)$ by the sheaf property of F. It follows easily that (X, ξ) represents F.

□

Remark 1.36. Adopting the notation of the proof we see that if we have a candidate fine moduli space X, an open cover $\{X_i \to X\}$ such that X_i represents F_i and we can verify that $\varphi_{ji} : X_i \cap X_j \to X_i \cap X_j$ is the identity for all i, j, then X represents F.

Chapter 2

Grassmannians as projective varieties

In this chapter we introduce Grassmannians from the point of view of classical algebraic geometry. We are interested in Grassmannians in the context of classification problems because their definition leads us to suspect that they are a moduli space for certain families of vector spaces. In the next chapter we will indeed find that they are fine moduli spaces for a functor that formalizes *families of k-vector subspaces of* $\mathbb{K}^n$.

We first define Grassmannians set-theoretically, then we will find a bijection between this set and Zariski-closed subset of some projective space. This bijection will allow us to endow the Grassmannian with the structure of a projective algebraic variety.

2.1 First definitions and conventions

Definition 2.1 (Grassmannian)**.** Let $k \leq n$ be a pair of positive integers. We define the (n, k)**-Grassmannian**, denoted[1] $\mathrm{Gr}(k, n, \mathbb{K})$, as the set of $(n-k)$-dimensional[2] $\mathbb{K}$-vector subspaces of $\mathbb{K}^n$.

Remark 2.2. We may equivalently define $\mathrm{Gr}(k, n)$ to be the following set:

$$\left\{\ker \varphi \mid \varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k),\ \operatorname{rnk} \varphi = k\right\}.$$

Lemma 2.3. *Let $\varphi, \psi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ be linear maps of full rank. The following conditions are equivalent:*

1. $\ker \varphi = \ker \psi$,
2. *there exists $\theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta \circ \psi$.*

Proof. The implication 2. $\implies$ 1. is a straight forward computation, the other can be derived by completing a basis of H to a basis $\mathcal{B}$ of $\mathbb{K}^n$ and defining θ to be the change of basis between the images of $\mathcal{B}$ under φ and ψ. $\square$

[1] the field will be omitted when clear from context

[2] The most natural choice might be to take the k-dimentional subspaces, but for later convenience we adopt this convention. It is also worth noting that if we fix a basis, the map $H \mapsto H^{\perp}$ gives a bijection between $\mathrm{Gr}(k, n)$ and $\mathrm{Gr}(n-k, n)$.

Corollary 2.4. *We may redefine Grassmannians in terms of linear maps as follows:*

$$\mathrm{Gr}(k,n) = {\{\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \varphi \text{ surjective.}\}}\big/{\sim}$$

where $\varphi \sim \psi$ *if and only if* $\exists \theta \in \mathrm{GL}(\mathbb{K}^k)$ *such that* $\varphi = \theta \circ \psi$.

We conclude this introductory section with some notation and conventions.

Definition 2.5 (Multiindicies)**.** We define a (k,n)**-multiindex** as an element of $\{1, \cdots, n\}^k$. Our notation for a multiindex I will usually be $I = (i_1, \cdots, i_k)$.
We denote the set of **ordered** (k,n)**-multiindicies** by

$$\omega(k,n) = \left\{(i_1, \cdots, i_k) \in \{1, \cdots, n\}^k \mid i_1 < \cdots < i_k\right\}.$$

If $I \in \omega(k,n)$, we write[3]

- $\widehat{I}$ for the element of $\omega(n-k,n)$ such that $I \cup \widehat{I} = \{1, \cdots, n\}$ and
- σ_I for the permutation that sends $\widehat{I} * I$ to $(1, \cdots, n)$.

If A is a $k \times n$ matrix and I is a (k,n)-multiindex, we denote the **I-minor of** A by A_I, i.e.

$$A_I = \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{pmatrix}.$$

If B is an $\alpha \times \beta$ matrix, $i \in \{1, \cdots, \alpha\}$ and $j \in \{1, \cdots, \beta\}$ we denote the $(\alpha-1)\times(\beta-1)$ matrix obtained from B by deleting the i-th row and the j-th column with $B_{\times i, \times j}$.

Remark 2.6. If $I = (i_1, \cdots, i_k)$ is a (k,n)-multiindex and $\mathcal{B} = \{e_1, \cdots, e_n\}$ is a basis of $\mathbb{K}^n$ we define

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Note that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\} = \{e_I \mid I \in \omega(k,n)\}$$

forms a basis for $\bigwedge^k \mathbb{K}^n$, which we call the **basis induced by** $\mathcal{B}$ and denote with $\wedge^k \mathcal{B}$ or simply $\mathcal{B}$ with a slight abuse of notation.

Notation 2.7. Whenever a basis $\mathcal{B}$ of $\mathbb{K}^\ell$ is fixed, we will identify $\bigwedge^\ell \mathbb{K}^\ell$ with $\mathbb{K}$ by sending only element of $\wedge^\ell \mathcal{B}$ to $1 \in \mathbb{K}$. This isomorphism is denoted $\eta_{\mathcal{B}} : \bigwedge^\ell \mathbb{K}^\ell \to \mathbb{K}$. If $\mathcal{D}$ is a basis of $\mathbb{K}^m$ then we define $\eta^{\mathcal{B}}_{\mathcal{D}} = \eta_{\mathcal{D}}^{-1} \circ \eta_{\mathcal{B}}$.

Remark 2.8 (Matrix form for the Grassmannian)*.* If we fix bases $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and $e_1, \cdots, e_k$ of $\mathbb{K}^k$, then we can identify $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ with the set of $k \times n$ matrices with coefficients in $\mathbb{K}$. As a consequence of this we find yet another form for $\mathrm{Gr}(k,n)$:

$$\mathrm{Gr}(k,n) = {\{A \in \mathcal{M}(k,n) \mid \mathrm{rnk}\, A = k\}}\big/{\sim},$$

where $A \sim B \iff \exists P \in \mathrm{GL}(k)$ *s.t.* $A = PB$.

[3] $\cup$ and $*$ denote the union of the underlying sets and concatenation respectively.

2.2 The Plücker embedding

In this section we define an injection from the Grassmannian to a projective space. The idea behind this map is to take appropriate wedge products in such a way as to transform the several vectors defining a vector subspace into a single vector and then to projectivize. Our approach differs slightly from the usual one[4] because we consider equivalence classes of maps rather than equivalence classes of bases.

Definition 2.9 (Plücker map)**.** Let $k \leq n$ be a pair of positive integers. We define the **Plücker map** as[5]

$$\wedge^k : \begin{array}{ccc} \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \\ \varphi & \longmapsto & \wedge^k \varphi \end{array},$$

where $(\wedge^k \varphi)(v_1 \wedge \cdots \wedge v_k) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_k)$.

Remark 2.10. If $\mathcal{B} = \{v_1, \cdots, v_k\}$ is a basis of $\mathbb{K}^k$, $\mathcal{C}\mathrm{an} = \{e_1, \cdots, e_k\}$ is the canonical basis and $[\cdot]_{\mathcal{B}} : \mathbb{K}^k \to \mathbb{K}^k$ is the isomorphism which sends v_i to e_i then

$$\wedge^k(\varphi)(v_1 \wedge \cdots \wedge v_k) = \det \left([\varphi(v_1)]_{\mathcal{B}} | \cdots | [\varphi(v_k)]_{\mathcal{B}}\right) e_1 \wedge \cdots \wedge e_k.$$

Remark 2.11. The codomain of the Plücker map is isomorphic to $\bigwedge^k \mathbb{K}^n$, indeed

$$\operatorname{Hom}_{\mathbb{K}} \left(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k \right) \cong \left(\bigwedge^k \mathbb{K}^n \right)^\vee \cong \bigwedge^k \mathbb{K}^n.$$

If $\mathcal{B} = \{e_1, \cdots, e_n\}$ is a basis of $\mathbb{K}^n$ and $\mathcal{D} = \{e_1, \cdots, e_k\}$ is a basis of $\mathbb{K}^k$ then we can write one such isomorphism concretely as

$$\zeta_{\mathcal{B},\mathcal{D}} : \begin{array}{ccc} \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ \psi & \longmapsto & \displaystyle\sum_{I \in \omega(k,n)} \eta_{\mathcal{D}}(\psi(e_I)) e_I \end{array}.$$

When the bases are fixed we simply write ζ.

Notation 2.12. If bases are fixed we define $\phi \doteqdot \zeta \circ \wedge^k$.

Remark 2.13 (Matrix form of the Plücker map). If we fix bases $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and $e_1, \cdots, e_k$ of $\mathbb{K}^k$ then, up to identifying $\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ with $\mathcal{M}(k, n)$, we have

$$\phi : \begin{array}{ccc} \mathcal{M}(k,n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \sum_{I \in \omega(k,n)} \det A_I e_I \end{array}$$

Proposition 2.14. *The image of the Plücker map is a cone.*

Proof. We have $\lambda \wedge^k \varphi = \wedge^k(\alpha \circ \varphi)$ for any $\alpha \in \mathrm{GL}(\mathbb{K}^k)$ with determinant λ. $\square$

Lemma 2.15. *If* $\varphi \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ *then* $\operatorname{rnk} \varphi < k$ *if and only if* $\wedge^k(\varphi) = 0$.

Proof. $\wedge^k(\varphi)$ is the zero map if an only if the set $\{\varphi(v_1), \cdots, \varphi(v_k)\}$ is linearly dependent for any choice of $v_1, \cdots, v_k$, i.e. φ is not of full rank. $\square$

[4] briefly illustrated in [2], pages 79 and 80

[5] the map $\wedge^k \varphi$ is well defined because if we view it as a map $\wedge^k \varphi : (\mathbb{K}^n)^k \to \bigwedge^k \mathbb{K}^k$ then it is multilinear and alternating.

Lemma 2.16. *Let $\varphi : \mathbb{K}^n \to \mathbb{K}^k$ be a full rank linear map, then*

$$\ker\varphi = \left\{ z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \right\}.$$

Proof. The inclusion $\subseteq$ is trivial. If $\varphi(z) \neq 0$ we can find $k-1$ vectors of the desired form by completing $\varphi(z)$ to a basis $\varphi(z), v_2, \cdots, v_k$ of $\mathbb{K}^k$ and then taking w_i to be any element of $\varphi^{-1}(v_i)$. The set is not empty by surjectivity of φ. $\square$

Proposition 2.17 (Injectivity of the Plücker map up to scalars)**.** *Let $\sim$ be the equivalence relation defined in corollary (2.4), then for any two full rank linear maps $\varphi, \psi : \mathbb{K}^n \to \mathbb{K}^k$*

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{K}^* \ s.t.\ \wedge^k(\varphi) = \lambda \wedge^k(\psi).$$

Proof. We prove both implications:

$\boxed{\Longrightarrow}$ If $\varphi = \theta \circ \psi$ for $\theta \in \mathrm{GL}(\mathbb{K}^k)$ then

$$\wedge^k(\varphi) = \wedge^k(\theta \circ \psi) = (\det\theta) \wedge^k(\psi).$$

$\boxed{\Longleftarrow}$ From lemma (2.3) we see that it is enough to prove that $\ker\varphi = \ker\psi$. We conclude by applying lemma (2.16) as follows:

$$\begin{aligned}\ker\varphi &= \left\{ z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \right\} = \\ &= \left\{ z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \lambda \wedge^k(\psi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \right\} = \\ &= \left\{ z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \wedge^k(\psi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \right\} = \ker\psi.\end{aligned}$$

$\square$

Remark 2.18. Because of proposition (2.17) and lemma (2.15), there exists a unique h such that the diagram commutes

$$\begin{tikzcd} \left\{ \varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \mathrm{rnk}\,\varphi = k \right\} \arrow[r, "{[\wedge^k]}"] \arrow[d, "\pi_\sim"] & \mathbb{P}(\mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)) \\ \mathrm{Gr}(k,n) \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Moreover, such an h must be injective by proposition (2.17).

Definition 2.19 (Plücker embedding)**.** Let us fix a basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and a basis $e_1, \cdots, e_k$ of $\mathbb{K}^k$. We define the **Plücker embedding** as follows[6]

$$\mathrm{Pl} : \begin{array}{ccc} \mathrm{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [\varphi] & \longmapsto & \left[\displaystyle\sum_{1 \le i_1 < \cdots < i_k \le n} \det(\varphi(e_{i_1}) \mid \cdots \mid \varphi(e_{i_k})) e_{i_1} \wedge \cdots \wedge e_{i_k} \right] \end{array}$$

The entries of the homogeneous $\binom{n}{k}$-tuple associated to $[\varphi] \in \mathrm{Gr}(k,n)$ are called the **Plücker coordinates** of $[\varphi]$.

[6]we can omit the basis in which we calulated the determinats because we will soon see that the resulting point in $\mathbb{P}(\bigwedge^k \mathbb{K}^n)$ does not depend on this choice.

Remark 2.20 (Well defined and injective). If we fix bases for $\mathbb{K}^n$ and $\mathbb{K}^k$ and ζ is the isomorphism $\mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \to \bigwedge^k \mathbb{K}^n$ discussed during remark (2.11), we see that the following diagram commutes

$$\begin{tikzcd} \mathbb{P}(\mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)) \arrow[r, "\mathbb{P}(\zeta)"] & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ \mathrm{Gr}(k,n) \arrow[u, "h"] \arrow[ur, "\mathrm{Pl}"'] & \end{tikzcd}$$

This proves that the Plücker embedding is well defined and injective.

Remark 2.21. $\mathrm{Pl} \circ \pi_{\sim} = \mathbb{P}(\zeta \circ \wedge^k) = \mathbb{P}(\phi)$.

Remark 2.22. The Plücker embedding depends on the choice of basis for $\mathbb{K}^n$ but not on the one for $\mathbb{K}^k$.

Changing the basis of $\mathbb{K}^k$ simply multiplies all Plücker coordinates by the same nonzero scalar (the determinant of the change of basis), which does not change the point they describe in $\mathbb{P}(\bigwedge^k \mathbb{K}^k)$.

The dependence on the basis of $\mathbb{K}^n$ is inevitable because $\mathrm{GL}(\mathbb{K}^n)$ acts transitively on $\mathrm{Gr}(k,n)$ viewed as the set of $(n-k)$-dimentional subspaces of $\mathbb{K}^n$.

Remark 2.23 (Matrix form of the Plücker embedding). If we fix a basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and identify $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ with $\mathcal{M}(k,n)$ then

$$\mathrm{Pl}: \begin{array}{ccc} \mathrm{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [A]_{\sim} & \longmapsto & \left[\sum_{I \in \omega(k,n)} \det A_I e_I\right]_{\mathbb{K}^*} \end{array}$$

2.3 The image of the Plücker embedding is closed

Thus far we have identified $\mathrm{Gr}(k,n)$ with a subset of some projective space. We seek to show that this subset is closed in the Zariski topology.

2.3.1 Some linear algebra results

Definition 2.24 (Divisibility). We say that $\omega \in \bigwedge^k \mathbb{K}^n$ is **divisible** by $v \in \mathbb{K}^n$ if there exists $\varepsilon \in \bigwedge^{k-1} \mathbb{K}^n$ such that $\omega = \varepsilon \wedge v$.

Lemma 2.25. *Let $\omega \in \bigwedge^k \mathbb{K}^n$. For any given nonzero vector v, ω is divisible by v if and only if $\omega \wedge v = 0$.*

Proof. If $\omega = \varepsilon \wedge v$ then $\omega \wedge v = \varepsilon \wedge v \wedge v = 0$. If $\omega \wedge v = 0$ then by writing ω in a base containing v we can see that the simple multivectors with nonzero coefficients must contain v as a factor, so can factor out v by multilinearity and get a decomposition of the form $\omega = \varepsilon \wedge v$. □

Corollary 2.26 (Total decomposibility criterion). *Let $\omega \in \bigwedge^k \mathbb{K}^n$ and define*

$$D_\omega = \{v \in \mathbb{K}^n \mid \omega \wedge v = 0\}.$$

If $\dim D_\omega \geq k$ then $\omega = \lambda v_1 \wedge \cdots \wedge v_k$ for any set of linearly independent vectors $\{v_1, \cdots, v_k\}$ in D_ω and some scalar λ. Moreover $\lambda \neq 0$ if and only if $\dim D_\omega = k$.

Proof. For the first part of the result we may just iterate the above lemma. If $\lambda = 0$ then $D_\omega = \mathbb{K}^n$, so its dimention is not k. If the dimention is greater than k then we may subtract two total decompositions differing only by one vector and use linear independence to check that the coefficients must have been zero. □

Proposition 2.27. *There is a canonical isomorphism between* $\mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^n \mathbb{K}^n)$ *and* $\bigwedge^{n-k}\mathbb{K}^n$ *given by*

$$\Xi : \begin{array}{ccc} \bigwedge^{n-k}\mathbb{K}^n & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^n \mathbb{K}^n) \\ \omega & \longmapsto & \omega \wedge \cdot \end{array}.$$

For any basis $\mathcal{B} = \{e_1, \cdots, e_n\}$ *of* $\mathbb{K}^n$ *the map*

$$\Gamma_{\mathcal{B}} : \begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^n \mathbb{K}^n) & \longrightarrow & \bigwedge^{n-k}\mathbb{K}^n \\ \psi & \longmapsto & \sum_{I\in\omega(n-k,n)} \mathrm{sgn}\,\sigma_I \eta_{\mathcal{B}}(\psi(e_{\hat{I}}))e_I \end{array}$$

is the inverse of Ξ.

Proof. The map is clearly base independent and linear. Let $\wedge^k\mathcal{B}$ be the basis induced on $\bigwedge^k \mathbb{K}^n$ by $\mathcal{B}$. Concluding from here is simply a matter of computing $\Gamma_{\mathcal{B}}(\Xi(\omega))$ by writing ω in terms of its coordinates in $\wedge^k\mathcal{B}$ and verifying that $\Xi(\Gamma_{\mathcal{B}}(\psi))$ and ψ agree on $\wedge^k\mathcal{B}$.

□

Corollary 2.28. *Let* $\psi \in \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$. *If* $\mathcal{B} = \{e_1, \cdots, e_n\}$ *and* $\mathcal{B}' = \{e'_1, \cdots, e'_n\}$ *are two bases for* $\mathbb{K}^n$ *and* $\mathcal{D} = \{e_1, \cdots, e_k\}$ *and* $\mathcal{D}' = \{e'_1, \cdots, e'_k\}$ *are bases for* $\mathbb{K}^k$, *there exists* $\mu \in \mathbb{K}\setminus\{0\}$ *such that*

$$\sum_{I\in\omega(n-k,n)} \mathrm{sgn}\,\sigma_I \eta_{\mathcal{D}}(\psi(e_{\hat{I}}))e_I = \mu \sum_{I\in\omega(n-k,n)} \mathrm{sgn}\,\sigma_I \eta_{\mathcal{D}'}(\psi(e'_{\hat{I}}))e'_I.$$

Proof. Note that

$$\sum_{I\in\omega(n-k,n)} \mathrm{sgn}\,\sigma_I \eta_{\mathcal{D}}(\psi(e_{\hat{I}}))e_I = \Xi^{-1}(\eta^{\mathcal{D}}_{\mathcal{B}} \circ \psi)$$

and similarly the other expression is $\Xi^{-1}(\eta^{\mathcal{D}'}_{\mathcal{B}'} \circ \psi)$. It is therefore enough to show that $\eta^{\mathcal{B}}_{\mathcal{D}} = \mu\eta^{\mathcal{B}'}_{\mathcal{D}'}$ for some $\mu \in \mathbb{K}\setminus\{0\}$, which is true because $\dim_{\mathbb{K}} \mathrm{Hom}_{\mathbb{K}}(\bigwedge^n \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) = 1$ and both $\eta^{\mathcal{B}}_{\mathcal{D}}$ and $\eta^{\mathcal{B}'}_{\mathcal{D}'}$ are not the zero map. □

2.3.2 Rank condition for the image

Lemma 2.29. *Fix bases* $\mathcal{B} = \{e_1, \cdots, e_n\}$ *of* $\mathbb{K}^n$ *and* $\mathcal{D} = \{e_1, \cdots, e_k\}$ *of* $\mathbb{K}^k$. *A multilinear alternating form* $\psi \in \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ *is in the image of the Plücker map* $\wedge^k$ *if and only if there exists* $\lambda \in \mathbb{K}$ *and linearly independent vectors* $z_1, \cdots, z_{n-k}$ *such that*

$$\sum_{I\in\omega(n-k,n)} \mathrm{sgn}\,\sigma_I \eta_{\mathcal{D}}(\psi(e_{\hat{I}}))e_I = \lambda z_{(1,\cdots,n-k)}.$$

Proof. We show both implications

$\boxed{\Longrightarrow}$ If $\psi = \wedge^k\varphi$, the equality follows by choosing $z_1, \cdots, z_{n-k}$ to be a basis of $\ker\varphi$. Completing this set to a basis of $\mathbb{K}^n$ and using corollary (2.28) gives the result after a simple calculation.

$\boxed{\Longleftarrow}$ Let $\mathcal{Z} = \{z_1, \cdots, z_n\}$ be a basis of $\mathbb{K}^n$ which extends the given $z_1, \cdots, z_{n-k}$. We can take φ to be

$$\varphi(z_i) = \begin{cases} 0 & \text{if } 1 \le i \le n-k \\ \left(\mu\lambda\mathrm{sgn}\,\sigma_{(1,\cdots,n-k)}\right)e_1 & \text{if } i = n-k+1 \\ e_{i-n+k} & \text{if } i > n-k+1 \end{cases}$$

where $\mu \in \mathbb{K}\setminus\{0\}$ is such that $\eta^{\mathcal{B}}_{\mathcal{D}} = \mu\eta^{\mathcal{Z}}_{\mathcal{D}}$.

□

Definition 2.30. Let $\mathcal{B} = \{e_1, \cdots, e_n\}$ and $\mathcal{D} = \{e_1, \cdots, e_k\}$ be bases of $\mathbb{K}^n$ and $\mathbb{K}^k$ respectively. For any $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ we define $\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ to be

$$\Phi_{\mathcal{B},\mathcal{D}}(\psi) : \begin{array}{ccc} \mathbb{K}^n & \longrightarrow & \bigwedge^{n-k+1} \mathbb{K}^n \\ v & \longmapsto & \displaystyle\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(e_{\widehat{I}}))e_I \wedge v \end{array} .$$

Remark 2.31. The rank of $\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ does not depend on the choice of basis. Indeed if we change basis, by corollary (2.28) we see that

$$\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(e_{\widehat{I}}))e_I \wedge v = \mu \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}'}(\psi(e'_{\widehat{I}}))e'_I \wedge v,$$

so $\ker \Phi_{\mathcal{B},\mathcal{D}}(\psi)$ does not depend on the basis and thus neither do nullity or rank.

For this reason we will write propositions which only concern the rank of $\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ omitting the bases.

Remark 2.32. $\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ is linear in ψ.

Proposition 2.33. *An alternating multilinear map $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ is in the image of the Plücker map $\wedge^k$ if and only if $\Phi(\psi)$ has rank at most k.*

Proof. For the $\implies$ arrow, choose a basis $\mathcal{Z} = \{z_1, \cdots, z_n\}$ for $\mathbb{K}^n$ which extends a basis for $\ker \varphi$. Because of how we proved lemma (2.29), we see that if $v \in \ker \varphi$ then $\Phi_{\mathcal{Z},\mathcal{D}}(\wedge^k \varphi)(v) = \lambda z_{(1,\cdots,n-k)} \wedge v$, which is zero by linear dependence. Thus the nullity of $\Phi(\wedge^k \varphi)$ is at least $\dim \ker \varphi = n - k$.

Given $n-k$ linearly independent vectors in $\ker \Phi(\psi)$, by the total decomposibility criterion (2.26) there exists $\lambda \in \mathbb{K}$ such that

$$\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(z_{\widehat{I}}))z_I = \lambda z_1 \wedge \cdots \wedge z_{n-k}.$$

This concludes by lemma (2.29). □

Definition 2.34. Let $\mathcal{B} = \{e_1, \cdots, e_n\}$ and $\mathcal{D} = \{e_1, \cdots, e_k\}$ be bases of $\mathbb{K}^n$ and $\mathbb{K}^k$ respectively. Let $\zeta_{\mathcal{B},\mathcal{D}} : \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \to \bigwedge^k \mathbb{K}^n$ be the isomorphism discussed during remark (2.11). We define

$$\widetilde{\Phi}_{\mathcal{B},\mathcal{D}} : \begin{array}{ccc} \bigwedge^k \mathbb{K}^n & \longrightarrow & \operatorname{Hom}_{\mathbb{K}}\left(\mathbb{K}^n, \bigwedge^{n-k+1} \mathbb{K}^n\right) \\ \omega & \longmapsto & \Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega)) \end{array}$$

Proposition 2.35. *The rank of $\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}(\omega)$ does not depend on the choice of basis.*

Proof. If $\omega = \sum_{I \in \omega(k,n)} p_I e_I$ then an easy calculation shows that v is an element of $\ker \Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega))$ if and only if for all $I \in \omega(k,n)$ either $p_I = 0$ or $e_{\widehat{I}} \wedge v = 0$, which is the same as saying $\omega \wedge v = 0$. The last condition is base independent so $\ker \Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega))$ must be also. □

Remark 2.36. $\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}$ is linear.

Let us define a matrix with coefficients in[7] $\mathbb{K}[z_I \mid I \in \omega(k,n)]$ which represents $\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}$:

Let $B^I \in \mathcal{M}\left(\binom{n}{n-k+1}, n, \mathbb{K}\right)$ be the matrix which represents $\Phi_{\mathcal{B},\mathcal{D}}(\zeta_{\mathcal{B},\mathcal{D}}^{-1}(e_I))$ in the bases induced by $\mathcal{B}$ and $\mathcal{D}$. By linearity

$$\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}\left(\sum_{I\in\omega(k,n)} a_I e_I\right)(v) = \sum_{I\in\omega(k,n)} a_I \Phi_{\mathcal{B},\mathcal{D}}(\zeta_{\mathcal{B},\mathcal{D}}^{-1}(e_I))(v) = \sum_{I\in\omega(k,n)} a_I B^I v.$$

We define the matrix which represents $\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}$ to be

$$M_{\mathcal{B},\mathcal{D}} = \sum_{I,\omega(k,n)} B^I z_I = \left(\sum_{I\in\omega(k,n)} (B^I)_{i,j} z_I\right)_{i,j}.$$

Remark 2.37. The rank of $\Phi_{\mathcal{B},\mathcal{D}}(\sum_{I\in\omega(k,n)} p_I e_I)$ is exactly the rank of $M|_{z_I=p_I}$.

The previous remark together with proposition (2.33) tells us that

$$\begin{aligned}\operatorname{Imm}(\zeta_{\mathcal{B},\mathcal{D}} \circ \wedge^k) = &\left\{\sum_{I\in\omega(k,n)} p_I e_I \mid \operatorname{rnk}\ M|_{z_I=p_I} < k+1\right\} = \\ =&V(\{\det m \mid m \text{ is a } (k+1)\times(k+1) \text{ minor of } M\}),\end{aligned}$$

which is evidently a Zariski-closed subset of $\bigwedge^k \mathbb{K}^n$.

It follows trivially that the projectivization[8] of this set (i.e. the image of Pl) is closed in $\mathbb{P}(\bigwedge^k \mathbb{K}^n)$, so we found a bijection between $\mathrm{Gr}(k,n)$ and a projective variety, which we can use to endow $\mathrm{Gr}(k,n)$ with the structure of one.

Remark 2.38. The determinants we used to show that the image of the Plücker embedding is closed do not generate the ideal of that variety. The most well known set of generators for that ideal are the **Plücker relations** (Theorem 2.4.3 in [2], page 80).

[7] what we will later call the Bracket ring

[8] recall (2.14) that $\operatorname{Imm} \wedge^k$ is a cone.

Chapter 3

Representability of the Grassmannian functor

In this chapter we assume a fixed basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and $e_1, \cdots, e_k$ of $\mathbb{K}^k$. We also identify $\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ with $\mathcal{M}(k,n)$.

To differentiate the scheme morphisms we define in this chapter from the morphisms of varieties defined previously we use a superscript s for the latter, i.e.

$$\phi^s : \begin{array}{ccc} \mathcal{M}(k,n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \sum_{I\in\omega(k,n)} \det A_I e_I \end{array}$$

$$\mathrm{Pl}^s : \begin{array}{ccc} \mathrm{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [A]_\sim & \longmapsto & \left[\sum_{I\in\omega(k,n)} \det A_I e_I\right]_{\mathbb{K}^*} \end{array}$$

Notation 3.1. Let I be an ideal of the ring A and J be a homogeneous ideal of the graded ring B. We adopt the following notation

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subseteq \mathfrak{p}\}, \qquad V_+(J) = \{\mathfrak{p} \in \operatorname{Proj} B \mid I \subseteq \mathfrak{p}\}.$$

If the sets above are considered as closed subschemes we take the reduced structure.

3.1 Grassmannians as projective schemes

To connect Grassmannians to the world of representable functors we describe them scheme-theoretically by emulating the construction from the previous chapter using rings and ring homomorphisms.

Definition 3.2 (Bracket ring)**.** We define the **bracket ring** (see page 79 of [2]) as the ring of polynomial functions on $\bigwedge^k \mathbb{K}^n$, i.e.

$$\mathcal{B}_{k,n} \doteqdot \frac{\mathbb{K}[z_I \mid I \in \{1, \cdots, n\}^k]}{(\{z_I - \operatorname{sgn}(\sigma) z_{\sigma(I)}\}_{\sigma\in S_k})} \cong \mathbb{K}[z_I \mid I \in \omega(k,n)].$$

Definition 3.3 (Ring of generic matrices)**.** Let $\mathbb{K}[X_{k,n}] \doteqdot \mathbb{K}[x_{1,1}, \cdots, x_{k,n}]$ denote the polynomial ring with $k \cdot n$ variables. We define the **generic matrix** by

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \cdots & x_{k,n} \end{pmatrix}$$

and by the same token we use X_I to denote the generic $k \times k$ minor determined by the multiindex I and $\det X_I$ to write the formal determinant of this minor.

Remark 3.4. The ring $\mathbb{K}[X_{k,n}]$ is the coordinate ring of $\mathcal{M}(k,n)$.

Remark 3.5. The familiar $\mathcal{M}(k,n)$ and $\bigwedge^k \mathbb{K}^n$ can be identified with the $\mathbb{K}$-points of the affine schemes $\operatorname{Spec} \mathbb{K}[X_{k,n}]$ and $\operatorname{Spec} \mathcal{B}_{k,n}$ respectively (Example 2.3.32 of [4]).

Definition 3.6 (Plücker ring homomorphism)**.** We define[1] the **Plücker ring homomorphism** or simply **Plücker homomorphism** as

$$\phi^{\#} : \begin{array}{ccc} \mathcal{B}_{k,n} & \longrightarrow & \mathbb{K}[X_{k,n}] \\ z_I & \longmapsto & \det X_I \end{array}$$

For brevity we will denote $\operatorname{Spec} \phi^{\#}$ by ϕ.

Remark 3.7. If we interpret $\mathcal{B}_{k,n}$ and $\mathbb{K}[X_{k,n}]$ as the coordinate rings of the affine spaces $\bigwedge^k \mathbb{K}^n$ and $\mathcal{M}(k,n)$ respectively, then it is clear by construction that the map induced between them from $\phi^{\#}$ is $A \mapsto \sum_{I \in \omega(k,n)} \det A_I e_I$, which we already know as ϕ^s.

Proposition 3.8. $\ker \phi^{\#}$ *is a homogeneous prime ideal which does not contain all of the* z_I*.*

Proof. We prove the three properties

- $\ker \phi^{\#}$ is prime because $\mathbb{K}[X_{k,n}]$ is an integral domain.
- $\ker \phi^{\#}$ is homogeneous because if $\sum_d f_d \in \ker \phi^{\#}$ for f_d homogeneous of degree d then $0 = \sum_d \phi^{\#}(f_d)$, but $\phi^{\#}(f_d)$ is homogeneous of degree kd, so for every d it must be the case that $f_d \in \ker \phi^{\#}$.
- $(z_I) \nsubseteq \ker \phi^{\#}$ because $\deg \phi^{\#}(z_I) = \deg(\det X_I) = k > 0$ for all z_I.

□

Proposition 3.9. *Let* $t : \mathrm{Var}/\mathbb{K} \to \mathrm{Sch}/\mathbb{K}$ *be the fully faithfull functor defined as in Proposition 2.6 of [3]. Then* $V_+(\ker \varphi^{\#}) \cong t(\operatorname{Imm} \mathrm{Pl}^s)$*.*

Proof. Because t is fully faithfull, we only need to show that $V_+(\ker(\varphi^{\#}))(\mathbb{K}) \cong \operatorname{Imm} \mathrm{Pl}^s$. This is equivalent to

$$\operatorname{Imm} \phi^s \cong V(\ker \phi^{\#})(\mathbb{K}) = \overline{\operatorname{Imm} \phi|_{\mathcal{M}(k,n)}} \overset{(3.7)}{=} \overline{\operatorname{Imm} \phi^s},$$

which is true because[2] $\operatorname{Imm} \phi^s = \overline{\operatorname{Imm} \phi^s}$ □

From now on $\mathrm{Gr}(k,n)$ will denote $V_+(\ker \phi^{\#})$, while $\mathrm{Gr}(k,n)(\mathbb{K})$ will denote what we used to write as $\mathrm{Gr}(k,n)$.

[1]This definition is inspired by that of ϕ at page 79 of [2].
[2]this is the main result of section 2.3

3.1.1 Standard affine cover of the Grassmannian scheme

Recall that projective space admits a standard affine cover given by the loci where one indeterminate does not vanish. In our case we see that

$$\operatorname{Proj} \mathcal{B}_{k,n} = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left((\mathcal{B}_{k,n})^0_{z_I} \right) = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left(\mathbb{K}\left[\frac{z_J}{z_I} \mid J \in \omega(k,n) \right] \right),$$

where the subscript denotes localization with multiplicative part $\{1, z_I, z_I^2, \cdots\}$ and the superscript 0 denotes the fact that we are only considering terms of degree 0 in this ring (this is the notation used in [4]).

This open affine cover of $\operatorname{Proj} \mathcal{B}_{k,n}$ induces an open cover on $\operatorname{Gr}(k,n)$ as follows:

$$\operatorname{Gr}(k,n) = V_+(\ker \phi^{\#}) = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left(\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}} \right)^0_{z_I} \right).$$

Notation 3.10. Let us fix $I \in \omega(k,n)$, then we denote the restriction of $\phi^{\#}$ as

$$\begin{array}{rccc} \phi^{\#}_I : & \mathbb{K}\left[\frac{z_J}{z_I} \mid J \in \omega(k,n) \right] & \longrightarrow & \mathbb{K}[X_{k,n}]^0_{\det X_I} \\ & \frac{z_J}{z_I} & \longmapsto & \frac{\det X_J}{\det X_I} \end{array}$$

Remark 3.11. The image of $\phi^{\#}_I$ is

$$\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n) \right],$$

thus by the first isomorphism theorem we have

$$\frac{(\mathcal{B}_{k,n})^0_{z_I}}{\ker \phi^{\#}_I} \cong \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n) \right].$$

For brevity we will usually shorten the notation to $\mathbb{K}\left[\frac{\det X_J}{\det X_I} \right]$.

Remark 3.12. The following equality holds by the properties of localization

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}} \right)_{z_I} = \frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}},$$

thus

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}} \right)^0_{z_I} = \left(\frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}} \right)^0 = \frac{(\mathcal{B}_{k,n})^0_{z_I}}{\ker \phi^{\#}_I} \cong \mathbb{K}\left[\frac{\det X_J}{\det X_I} \right]$$

Putting what we have said together, we have shown that up to some canonical identifications

$$\operatorname{Gr}(k,n) = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I} \right] \right) \doteqdot \bigcup_{I \in \omega(k,n)} \operatorname{Gr}_I(k,n).$$

Proposition 3.13. *$\operatorname{Gr}_I(k,n)$ is isomorphic to $\mathbb{A}^{k(n-k)}_{\mathbb{K}}$ as a scheme.*

Proof. Since they are both affine schemes, it is enough to show that their coordinate rings are isomorphic. To simplify the notation we will set $w_J = \frac{\det X_J}{\det X_I}$ and if S is a multiindex we will write S^i_j for the multiindex where the i-th entry is substituted by $j \in \{1, \cdots, n\}$.

Without loss of generality we may assume that $I = (1, \cdots, k)$. An analogous argument will work for any choice of multiindex. First we will prove that

$$\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right] = \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J = I^j_{\ell_j},\ j \in \{1, \cdots, k\},\ \ell_j \notin I\right],$$

then we will show that the RHS (which we will denote R for brevity) is isomorphic to $\mathbb{K}[Y_{k,n-k}] = \mathbb{K}[y_{1,1}, \cdots, y_{k,n-k}]$.

- Let us consider the formal matrix

$$M = \begin{pmatrix} 1 & & & w_{I^1_{k+1}} & \cdots & w_{I^1_n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & w_{I^k_{k+1}} & \cdots & w_{I^k_n} \end{pmatrix}.$$

We can see that $M = (X_I)^{-1} X$, where $(X_I)^{-1}$ is the formal inverse of X_I, which exists because $\det X_I$ is invertible. More precisely

$$(X_I)^{-1} = \frac{1}{\det X_I} \mathrm{Adj}\,(X_I)$$

where $\mathrm{Adj}\,(X_I)$ is the adjugate matrix of X_I. The equality $M = (X_I)^{-1} X$ holds for the first k columns by definition of inverse, for the other columns we can see that they agree on every entry:

$$\begin{aligned} \frac{1}{\det X_I} (\mathrm{Adj}\,(X_I) X)_{i,j} =& \frac{1}{\det X_I} \sum_{\ell=1}^{k} \left((-1)^{i+\ell} \det\,(X_I)_{\times \ell, \times i}\right) x_{\ell,j} = \\ =& \frac{1}{\det X_I} \det X_{I^i_j} = \\ =& w_{I^i_j}. \end{aligned}$$

We have thus proven that for any (k,n)-multiindex J we have

$$\det M_J = \det(X_I^{-1} X)_J = \frac{1}{\det X_I} \det X_J = w_J.$$

Since $\det M_J$ is a polynomial expression in the ring R by definition of M and the w_J are generators for $\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right]$, we have shown the nontrivial inclusion and thus equality.

- Let us consider the following ring homomorphism

$$\chi : \begin{array}{ccc} \mathbb{K}[Y_{k,n-k}] & \longrightarrow & R \\ y_{i,j} & \longmapsto & w_{I^i_j} \end{array}.$$

It is surjective by construction, so we just need to show that it is injective to find the desired isomorphism.

Suppose by contradiction that there exists a nonzero polynomial $p \in \mathbb{K}[Y_{k,n-k}]$ which maps to 0. If $\overline{\mathbb{K}}$ is an algebraic closure[3] of $\mathbb{K}$ we can consider the lift

$$\widetilde{\chi}: \begin{array}{ccc} \overline{\mathbb{K}}[Y_{k,n-k}] & \longrightarrow & \widetilde{R} = \overline{\mathbb{K}}[w_{I^i_j}] \\ y_{i,j} & \longmapsto & w_{I^i_j} \end{array}$$

Note that if $\chi(p) = 0$ then $\widetilde{\chi}(p) = 0$ because $R \subseteq \widetilde{R}$ and $\widetilde{\chi}|_{\mathbb{K}[Y_{k,n-k}]} = \chi$. Consider now any matrix of the form

$$A = \left(I_k \mid \widetilde{A}\right) = (a_{i,j})_{i,j}$$

where I_k is the $k \times k$ identity matrix and $\widetilde{A} \in \mathcal{M}(k, n-k, \overline{\mathbb{K}})$. From what we have said above it follows that $\det A_{I^i_j} = a_{i,j}$, so

$$p(\widetilde{A}) = p\left(\left(\det A_{I^i_j}\right)_{\substack{i\in\{1,\cdots,k\},\\ j\in\{k+1,\cdots,n\}}}\right) = \widetilde{\chi}(p)(A) = 0.$$

We have shown that p has infinitely many roots in $\overline{K}$, so if we fix the value of $k(n-k)-1$ indeterminates the resulting polynomial is the 0 polynomial. If we reiterate this reasoning we eventually prove that $p = 0$ in $\overline{\mathbb{K}}[Y_{k,n-k}]$, but $0 \in \mathbb{K}[Y_{k,n-k}] \subseteq \overline{\mathbb{K}}[Y_{k,n-k}]$, so p is the zero polynomial in the original ring, contradicting our hypothesis.

□

Remark 3.14. Since $\mathrm{Gr}_I(k,n)$ is affine, the scheme $\mathrm{Gr}_I(k,n) \cap \mathrm{Gr}_J(k,n) = \mathrm{Gr}_I(k,n)_{z_J}$ is affine for any choice of multiindices.

3.2 Grassmannian moduli functor

Let us consider the following functor

$$\mathbb{G}(k,n): \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q\}/_{\sim} \\ f : S \to T & \longmapsto & (\alpha : \mathcal{O}_T^n \to Q) \mapsto (f^*\alpha : \mathcal{O}_S^n \to f^*Q) \end{array}$$

where Q is a locally free sheaf of rank k on T and two surjections $\alpha : \mathcal{O}_T^n \twoheadrightarrow Q$, $\beta : \mathcal{O}_T^n \twoheadrightarrow V$ are equivalent if and only if there exist an isomorphism of sheaves $\theta : Q \to V$ such that the diagram commutes

$$\begin{tikzcd} \mathcal{O}_T^n \arrow[r, two heads, "\alpha"] \arrow[dr, two heads, "\beta"'] & Q \arrow[d, "\theta"] \\ & V \end{tikzcd}$$

We have functoriality because of the composition properties of pullbacks.

In this this section we prove that the Grassmann scheme represents this functor.

[3]we can take any field extension $\mathbb{K} \subseteq \mathbb{F}$ where $\mathbb{F}$ is an infinite field.

3.2.1 Open subfunctor cover of the Grassmannian

Notation 3.15. For any multiindex $I \in \omega(k,n)$ and any scheme T we define the following morphism of sheaves

$$s_I^T : \begin{array}{ccc} \mathcal{O}_T^k & \longrightarrow & \mathcal{O}_T^n \\ e_j & \longmapsto & e_{i_j} \end{array}.$$

If there is no ambiguity we write s_I.

Definition 3.16 (Principal subfunctor of the Grassmannian)**.** Fixed a multiindex $I \in \omega(k,n)$ we define the following functor

$$\mathbb{G}_I(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \left\{\mathcal{O}_T^n \overset{\alpha}{\twoheadrightarrow} Q \mid \alpha \circ s_I \text{ surjective}\right\}\big/\sim \\ f & \longmapsto & \alpha \mapsto f^*\alpha \end{array}$$

where the equivalence relation is the same as the one defined for $\mathbb{G}(k,n)$.

Remark 3.17. The functor $\mathbb{G}_I(k,n)$ is well defined.

Proof. First we observe that $\mathbb{G}_I(k,n)(T)$ is well defined because if $\psi = \theta \circ \alpha$ with θ isomorphism of sheaves then on each stalk we have

$$\psi_x \circ (s_I)_x = \theta_x \circ \varphi_x \circ (s_I)_x,$$

which is surjective if and only if $\varphi_x \circ (s_I)_x$ is surjective.

Consider now a morphism $f : S \to T$, then

$$f^*\alpha \circ s_I^S = f^*\alpha \circ f^* s_I^T = f^*(\alpha \circ s_I^T)$$

is surjective if and only if it is surjective on all stalks, i.e. if and only if for all $s \in S$ we have that the following map is surjective

$$f^*(\alpha \circ s_I^T)_s = (\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} \mathcal{O}_{S,s},$$

which is true because the tensor product is right-exact. □

We recall a basic fact about sheaves

Lemma 3.18. *The support of a finite type quasicoherent sheaf $\mathcal{F}$ on a scheme X is a closed subset*[4].

Proposition 3.19. *The $\mathbb{G}_I(k,n)$ are open subfunctors of $\mathbb{G}(k,n)$.*

Proof. The inclusion $\mathbb{G}_I(k,n)(T) \subseteq \mathbb{G}(k,n)(T)$ is apparent, so we just need to show that if we fix a quotient $[\alpha : \mathcal{O}_T^n \twoheadrightarrow Q]$ in $\mathbb{G}(k,n)(T)$ then we can find an open subscheme of T which represents $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$.

Let us fix a representative α for the given quotient. The locus where $\alpha \circ s_I : \mathcal{O}_T^k \to Q$ is surjective is the complement of the support of its cokernel sheaf $\mathcal{K}$, i.e.

$$(\alpha \circ s_I)_x \text{ surjective} \iff x \notin \operatorname{Supp} \mathcal{K}.$$

[4]For more detail see Section 01B4 in [7]

Note that by the definition of $\sim$ and properties of isomorphisms of sheaves, the first condition does not depend of the choice of representative for $[\alpha]$, so $\operatorname{Supp}\mathcal{K}$ only depends on $[\alpha]$. By lemma[5] (3.18) the set $U_I = T \setminus \operatorname{Supp}\mathcal{K}$ is open.

We now want to show that U_I represents the functor $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$, that is we want to show that if $f : S \to T$ is a morphism of $\mathbb{K}$-schemes then f factors through U_I if and only if $f^*\alpha : \mathcal{O}_S^n \to f^*Q \in \mathrm{Gr}_I(S)$.

Note that $f(s) \in U_I$ if and only if $(\alpha \circ s_I^T)_{f(s)}$ is surjective which, by Nakayama's lemma applied to the cokernels, is equivalent to the surjectivity of

$$(\alpha \circ s_I^T)|_{f(s)} : k(f(s))^k \to Q_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} k(f(s)).$$

Observe that

$$\begin{aligned} f^*(\alpha \circ s_I^T)|_s =& f^*(\alpha \circ s_I^T)_s \otimes_{\mathcal{O}_{S,s}} id_{k(s)} = \\ =& ((\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{\mathcal{O}_{S,s}}) \otimes_{\mathcal{O}_{S,s}} id_{k(s)} = \\ =& (\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{k(s)} = \\ =& (\alpha \circ s_I^T)|_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{k(s)} \end{aligned}$$

where the last equality is simply emphasizing how the restriction of scalars by $f_s^\#$ acts. From the last line we see that

$$f^*(\alpha \circ s_I^T)|_s \text{ is surjective} \quad \Longleftrightarrow \quad (\alpha \circ s_I^T)|_{f(s)} \text{ is surjective.}$$

By Nakayama's lemma we can again consider equivalently $f^*(\alpha \circ s_I^T)_s = (f^*\alpha)_s \circ (s_I^S)_s$.

We have thus shown that $f(s) \in U_I$ if and only if $(f^*\alpha)_s \circ (s_I^S)_s$ is surjective, i.e. f factors through U_I if and only if $(f^*\alpha) \circ s_I^S$ is surjective, i.e. $f^*\alpha \in \mathbb{G}_I(k,n)(S)$. $\square$

Proposition 3.20. *The collection $\{\mathbb{G}_I(k,n)\}$ is a Zariski open subfunctor cover of $\mathbb{G}(k,n)$.*

Proof. For any $\mathbb{K}$-scheme S and any quotient $[\alpha] \in \mathrm{Gr}(k,n)(S)$ (without loss of generality we choose a representative α) we need to show that for any $s \in S$ there exists a multiindex I such that $s \in U_I$ defined as in the previous proposition.

We are therefore looking for a multiindex I such that $(\alpha \circ s_I)_s$ is surjective. By Nakayama's lemma this is equivalent to showing that there exists an I such that

$$k(s)^k \xrightarrow{s_I} k(s)^n \xrightarrow{\alpha_s} Q_s \otimes_{\mathcal{O}_{S,s}} k(s),$$

which is trivially true since $\operatorname{rnk} \alpha_s = k$. $\square$

3.2.2 Representability of the Grassmannian functor

Lemma 3.21. *Let $\alpha : \mathcal{O}^n \twoheadrightarrow Q$ and $\beta : \mathcal{O}^n \twoheadrightarrow Q'$ be surjective sheaf morphisms. If there exists an isomorphism $\theta : Q \to Q'$ such that $\beta = \theta \circ \alpha$ then θ is unique.*

Proof. First, observe that if $\alpha = \beta$ then by surjectivity and commutativity $\theta = id_Q$.

Let $\theta, \eta : Q \to Q'$ be isomorphisms such that $\beta = \theta \circ \alpha$ and $\beta = \eta \circ \alpha$. Then $\theta^{-1} \circ \eta : Q \to Q$ is an isomorphism such that $\theta^{-1} \circ \eta \circ \alpha = \theta^{-1} \circ \beta = \alpha$, so $\theta^{-1} \circ \eta = id_Q$ and thus $\theta = \eta$. $\square$

[5] $\mathcal{K}$ is of finite type because locally it is given by quotients of finite rank free modules.

Proposition 3.22. *The Grassmannian functor $\mathbb{G}(k,n)$ is a Zariski sheaf.*

Proof. Consider a $\mathbb{K}$-scheme T and an open cover $\{U_i \to T\}$. Consider now quotients $\alpha_i : \mathcal{O}^n_{U_i} \twoheadrightarrow Q_i$ such that

$$\alpha_i|_{U_i \cap U_j} \sim \alpha_j|_{U_i \cap U_j}.$$

Because of lemma (3.21), the isomorphism giving the equivalence above is unique. Let $\varphi_{ji} : Q_i|_{U_i\cap U_j} \to Q_j|_{U_i \cap U_j}$ be this isomorphism. Because of the uniqueness $\varphi_{ii} = id_{Q_i}$ and $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$, so have the data to glue the Q_i to a locally free sheaf of rank k over T, which we denote by Q.

Up to isomorphism, let us consider $\alpha_i : \mathcal{O}^n_{U_i} \twoheadrightarrow Q|_{U_i}$ for all i. If we fix any open set $V \subseteq T$ we see that, if $s \in \mathcal{O}^n_T(V)$ is a section, we can define $\alpha_V(s)$ by gluing the $\alpha_i(s|_{U_i})$, which we can do by construction of Q and the choice of representative for the α_i. By construction $\alpha_{U_i} = \alpha_i$ and it is in fact the only such morphism, so we have verified the gluing property of sheaves for $\mathbb{G}(k,n)$. □

Proposition 3.23. *The affine scheme $\mathrm{Gr}_I(k,n)$ represents the functor $\mathbb{G}_I(k,n)$.*

Proof. First we prove that for any $\mathbb{K}$-scheme $\mathrm{Hom}_{\mathsf{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathbb{G}_I(T)$, then we need to check that, given a map $f : S \to T$, the pullbacks behave well.

- From our work in the first section of this chapter we can see that

$$\mathrm{Hom}_{\mathsf{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right).$$

Let us now consider the following maps

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) & \longleftrightarrow & \left\{\alpha : \mathcal{O}^n_T \to \mathcal{O}^k_T \mid \alpha \circ s_I = id_{\mathcal{O}^k_T}\right\} \\ \varphi & \longmapsto & \eta(\varphi) \\ \rho(\alpha) : \frac{\det X_J}{\det X_I} \mapsto \frac{d(\alpha,J)}{d(\alpha,I)} & \longleftarrow & \alpha \end{array}$$

where $d(\alpha, L)$ is the determinant of the L minor of the matrix associated to α_T in the canonical basis and $\eta(\varphi)$ is defined on an open subset V of T by[6]

$$\eta(\varphi)_V(e_j) = \sum_{r=1}^{k} (\mathrm{res}^T_V \circ \varphi)\left(\frac{\det X_{I^{i_r}_j}}{\det X_I}\right) e_r = (\mathrm{res}^T_V \circ \varphi)\left(X_I^{-1} X\right) e_j.$$

The maps are well defined because $\alpha \circ s_I = id_{\mathcal{O}^k_T} \implies d(\alpha, I) = 1$ and

$$\frac{\det X_{I^{i_r}_{i_s}}}{\det X_I} = \delta_{r,s} \implies \eta(\varphi) \circ s_I = id_{\mathcal{O}^k_T}.$$

An argument analogous to the one presented in the proof of proposition (3.13) tells us that η and ρ are inverses.

Observe now that

$$\begin{array}{ccc} \left\{\alpha : \mathcal{O}^n_T \to \mathcal{O}^k_T \mid \alpha \circ s_I = id_{\mathcal{O}^k_T}\right\} & \longleftrightarrow & \{\alpha : \mathcal{O}^n_T \twoheadrightarrow Q \mid \alpha \circ s_I \text{ isomorphism}\}/\sim \\ \alpha & \longmapsto & [\alpha] \\ (\beta \circ s_I)^{-1} \circ \beta & \longleftarrow & [\beta] \end{array}$$

[6] $\mathrm{res}^T_V : \mathcal{O}_T(T) \to \mathcal{O}_T(V)$ denotes the restriction map given by the structure of sheaf.

is a bijection. Indeed $(\beta \circ s_I)^{-1} \circ \beta \sim \beta$ by definition of $\sim$ and if $\theta : Q \to Q'$ is any isomorphism of sheaves then

$$(\theta \circ \alpha \circ s_I)^{-1} \circ \theta \circ \alpha = (\alpha \circ s_I)^{-1} \circ \theta^{-1} \circ \theta \circ \alpha = id_{\mathcal{O}_T^k} \circ \alpha = \alpha.$$

Finally, we see that

$$\{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q \mid \alpha \circ s_I \text{ isomorphism}\}_{\diagup\sim} = \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q \mid \alpha \circ s_I \text{ surjective}\}_{\diagup\sim}$$

because on all stalks $\alpha \circ s_I$ is an endomorphism of finitely generated modules.

- Let $f : S \to T$ be a morphism of $\mathbb{K}$-schemes. Recall that

$$\begin{array}{ccc} \mathbb{G}_I(k,n)(T) & \longrightarrow & \mathbb{G}_I(k,n)(S) \\ [\alpha] & \longmapsto & [f^*\alpha] \end{array}.$$

Under the bijection presented, imposing naturality gives

$$\begin{array}{ccc} \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} & \longrightarrow & \left\{\beta : \mathcal{O}_S^n \to \mathcal{O}_S^k \mid \beta \circ s_I = id_{\mathcal{O}_S^k}\right\} \\ \alpha & \longmapsto & f^*\alpha \end{array}$$

since $f^*\alpha \circ s_I^S = f^*(\alpha \circ s_I^T) = f^*(id_{\mathcal{O}_T^k}) = id_{\mathcal{O}_S^k}$.

If we impose naturality again we get

$$\begin{array}{ccc} \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) & \longrightarrow & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_S(S)\right) \\ \varphi & \longmapsto & \rho(f^*\eta(\varphi)) \end{array}$$

We claim that $\rho(f^*(\eta(\varphi))) = f^\#(T) \circ \varphi$. Since η is the inverse of ρ, it is enough to prove that $f^*(\eta(\varphi)) = \eta(f^\#(T) \circ \varphi)$. The equality holds because for all $s \in S$, if $t = f(s) \in T$, then both maps are represented by the matrix

$$f_s^\# \left(\left(\varphi(X_I^{-1}X)\right)_t\right).$$

We conclude by recalling that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) & \xrightarrow{\mathrm{Spec}} & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) \\ {\scriptstyle h_{\mathrm{Gr}_I(k,n)}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], f^\#(T)\right)} \\ \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(S, \mathrm{Gr}_I(k,n)) & \xrightarrow{\mathrm{Spec}} & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_S(S)\right) \end{array}$$

□

Theorem 3.24. *The Grassmann scheme* $\mathrm{Gr}(k,n)$ *is a fine moduli space for the Grassmann functor* $\mathbb{G}(k,n)$.

Proof. We know that $\{\mathbb{G}_I(k,n) \to \mathbb{G}(k,n)\}$ is an open cover (3.20), that $\mathbb{G}(k,n)$ is a Zariski sheaf (3.22) and that $h_{\mathrm{Gr}_I(k,n)} \cong \mathbb{G}_I(k,n)$ (3.23). If we can show that these isomorphisms restrict well to the double intersection we have the desired result by proposition (1.34).

Let T be a scheme and let us consider a morphism

$$f \in \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}\left(T, \mathrm{Gr}_I(k,n) \cap \mathrm{Gr}_J(k,n)\right) = \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}\left(T, \mathrm{Gr}(k,n) \cap D_+(z_I z_J)\right).$$

If we apply a well known adjunction result for morphisms towards an affine scheme we get

$$f^{\#}(T) \in \operatorname{Hom}_{\mathbb{K}\text{-alg}}\left(\left(\mathbb{K}\left[\det X_L\right]_{\det X_I \det X_J}\right)^0, \mathcal{O}_T(T)\right).$$

By the universal property of localization, we may identify this set with

$$\left\{\beta \in \operatorname{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_L}{\det X_I}\right], \mathcal{O}_T(T)\right) \mid \beta\left(\frac{\det X_J}{\det X_I}\right) \neq 0\right\}.$$

Applying the functor η defined during the proof of proposition (3.23), which we will denote η^I to emphasize which determinant we consider at the denominator, we obtain

$$\eta^I(f^{\#}(T)) \in \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k},\ \alpha \circ s_J \text{ isomorphism}\right\}.$$

Observe that we can identify this set with

$$\{\alpha : \mathcal{O}_T^n \to Q \mid \alpha \circ s_I \text{ and } \alpha \circ s_J \text{ surjective}\}/\sim\; = (\mathbb{G}_I(k,n) \times_{\mathbb{G}(k,n)} \mathbb{G}_J(k,n))(T),$$

so to conclude the proof we just need to verify that $\eta^I(f^{\#}(T)) \sim \eta^J(f^{\#}(T))$ in $\mathbb{G}(k,n)$. Note that $\eta^I_V(f^{\#}(T))(e_j) = f^{\#}(V)(X_I^{-1}X)e_j$, so if the matrix $X_J^{-1}X_I$ can be described only using elements in $\mathbb{K}\left[\det X_L\right]^0_{\det X_I \det X_J}$, we can define $\theta_V(e_j)$ to be $f^{\#}(V)(X_J^{-1}X_I)e_j$ and see that $\theta \circ \eta^I = \eta^J$. Indeed, the map is well defined because

$$X_J^{-1}X = \left(\frac{\det X_{J_j^{j_i}}}{\det X_J}\right)_{i\in\{1,\cdots,k\},j\in\{1,\cdots,n\}} \implies X_J^{-1}X_I = \left(\frac{\det X_{J_{i_\ell}^{j_r}}}{\det X_J}\right)_{r,\ell\in\{1,\cdots,k\}}.$$

An analogous procedure for $X_I^{-1}X_J$ would define an inverse of θ. $\square$

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