Grassmannians as Fine Moduli Spaces

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Chapter 1

Moduli Spaces

The following type of *classification problem* occurs often in math:

> Consider some type of object and a notion of isomorphism which can be defined between them. We are interested in understanding the behaviour of isomorphism classes and how they relate to each other.

Finding a bijection between isomorphism classes and known objects is usually trivial[1], but for an answer to a classification problem to be satisfactory we usually require some information on *families* of isomorphism classes.

Miraculously, many such classification problems turn out to have a natural answer in the form of a space whose points are isomorphism classes and whose geometry encodes how families behave. Such an object is usually called a *moduli space*.
The best result we can hope for is finding a space which completely encodes how families behave[2], but this requirement is usually too strict. In this document we shall mostly deal with problems for which such a nice space exists.

The most fruitful formalization of family as intended above has been the functorial approach, the basics of which will be introduced in this chapter.

Most definitions given in this chapter follow [1] and [2].

1.1 Representable functors

In this section we introduce the basic concepts of the functorial approach and some of the required category theory.

1.1.1 Yoneda lemma and embedding

All categories considered in this document will be small. If $\mathcal{C}$ is a category, we shall write $X \in \mathcal{C}$ to mean "X is an object in $\mathcal{C}$".
If A and B are objects in a category (resp. categories / functors) then $\mathrm{Hom}\,(A, B)$ denotes the set of morphisms (resp. functors / natural transformations) from A to B.
The notation $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denotes the category of functors from $\mathcal{C}$ to $\mathcal{D}$, with morphisms being natural transformations.

[1] for example, if the classes form a set they can be identified with a canonical set of the same cardinality

[2] what will be formalized as a fine moduli space

Definition 1.1 (Presheaf)**.**
A contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is called a **presheaf** on $\mathcal{C}$. For any fixed category $\mathcal{C}$, the presheaves on $\mathcal{C}$ form a category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ with morphisms given by natural transformations.
If $T \in \mathcal{C}$ then we call the elements of $F(T)$ **families** over T.

Definition 1.2 (Hom-functor)**.**
Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We define the **Hom-functor** of X to be

$$h_X : \begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \mathrm{Hom}\,(T, X) \\ f : T \to S & \longmapsto & \circ f : \mathrm{Hom}\,(S, X) \to \mathrm{Hom}\,(T, X) \end{array}$$

Remark 1.3.
The Hom-functor is a presheaf.

Definition 1.4 (Moduli problem)**.**
Let S be a scheme. A presheaf on Sch/S is called a **moduli problem**.

Remark 1.5.
Usually we study moduli problems of the following form

$$\begin{array}{ccc} \mathrm{Sch}/S & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \{\text{families over } T\}\big/\sim \\ f : T \to R & \longmapsto & \text{pullback of families along } f \end{array}$$

where $\sim$ refers to the equivalence relation we are studying the families of classes of.

To proceed we will need the following

Lemma 1.6 (Yoneda Lemma)**.**
Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. For all contravariant functors $F : \mathcal{C}^{op} \to \mathrm{Set}$ the following sets are in a natural bijection

$$\mathrm{Hom}\,(h_X, F) \longleftrightarrow F(X).$$

Proof.
Given a natural trasformation ζ, we can take its image in $F(X)$ to be $\zeta_X(id_X)$.
On the other hand, for any given element $u \in F(X)$ we can define an arrow $h_X(T) \to F(T)$ for any $T \in \mathcal{C}$ by taking $f \mapsto F(f)(u)$. This collection of maps defines a natural transformation from h_X to F by commutativity of the following diagram for all $g : S \to T$

$$\begin{tikzcd}
\mathrm{Hom}\,(X,X) \arrow[dr, "\circ f"'] \arrow[drr, "\circ(f\circ g)"] \arrow[ddd, "F(\cdot)(u)"'] & & \\
& \mathrm{Hom}\,(T,X) \arrow[r, "\circ g"'] \arrow[d, "F(\cdot)(u)"'] & \mathrm{Hom}\,(S,X) \arrow[d, "F(\cdot)(u)"] \\
& F(T) \arrow[r, "F(g)"] & F(S) \\
F(X) \arrow[uur, "F(f)"] \arrow[uurr, "F(f\circ g)"'] & &
\end{tikzcd}$$

□

Definition 1.7 (Yoneda embedding)**.**
We define the **Yoneda embedding** of a category $\mathcal{C}$ to be the following functor

$$よ : \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \\ X & \longmapsto & h_X \\ f : X \to Y & \longmapsto & f\circ : h_X \to h_Y \end{array}$$

Proposition 1.8.
The functor よ *is fully faithful.*

Proof.
Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is fully faithful if for any two objects $A, B \in \mathcal{C}$ we have $\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$. In our case we want to check if

$$\mathrm{Hom}\,(X, Y) \cong \mathrm{Hom}\,(h_X, h_Y),$$

which is exactly the statement of the Yoneda lemma (1.6). □

Remark 1.9.
The Yoneda embedding is injective up to isomorphism on isomorphism classes of objects in $\mathcal{C}$.

Proof.
Suppose $\zeta : h_X \to h_Y$ is an isomorphism of functors, then $id_X \in h_X(X)$ corresponds to some $f = \zeta_X(id_X) : X \to Y$. Similarly $id_Y \in h_Y(Y)$ corresponds to some $g = \zeta_Y^{-1}(id_Y) : Y \to X$. By naturality of ζ we see can pull back id_X and f by g to see that

$$\begin{tikzcd} \mathrm{Hom}\,(X, X) \arrow[rr, "\zeta_X"] \arrow[dd, "\circ g"'] & & \mathrm{Hom}\,(X, Y) \arrow[dd, "\circ g"] \\ & \begin{tikzcd} id_X \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\ g \arrow[r, mapsto] & id_Y = f \circ g \end{tikzcd} & \\ \mathrm{Hom}\,(Y, X) \arrow[rr, "\zeta_Y"'] & & \mathrm{Hom}\,(Y, Y) \end{tikzcd}$$

and similarly we can show that $g \circ f = id_X$. □

Lemma 1.10 (Yoneda embedding is continuous)**.**
The Yoneda embedding preserves limits.

Proof.
Suppose X is the limit of the diagram $\{f_{ij} : X_j \to X_i\}$. If we apply the Yoneda embedding to the diagram we obtain

$$\left\{\circ f_{ij} : h_{X_j} \to h_{X_i}\right\}$$

Let F be any presheaf on $\mathcal{C}$ and suppose that we have morphisms $F \to h_{X_i}$ which make the diagram commute, then for all $T \in \mathcal{C}$ we have compatible and natural $F(T) \to \mathrm{Hom}\,(T, X_i)$. If $f \in F(T)$ then these arrows define several $f_i \in \mathrm{Hom}\,(T, X_i)$ which compose respecting the diagram. By the universal property of limits this defines uniquely a morphism $f_\ell \in \mathrm{Hom}\,(T, X)$ and we see that the assignment $f \mapsto f_\ell$ is the unique map from $F(T)$ to $\mathrm{Hom}\,(T, X)$ which makes the diagram in Set commute. Since all that we have done is natural in T we have effectively constructed a morphism $F \to h_X$ as we desired. □

1.1.2 Moduli spaces

Definition 1.11 (Representable functor)**.**
A presheaf F on $\mathcal{C}$ is **representable** if there exists a natural isomorphism $\zeta : F \to h_X$ for some $X \in \mathcal{C}$. In this case we say that the pair[3] (X, ζ) **represents** F.
If $a \in F(T)$ we call the map $\zeta_T(a) : T \to X$ the **classifying map** of the family a.

Definition 1.12 (Universal family)**.**
Instead of the pair (X, ζ) for $\zeta : F \to h_X$, we may consider the pair (X, ξ) where $\xi \in F(X)$ is the image of ζ^{-1} under the bijection predicated by the Yoneda lemma. The object ξ is called the **universal family** of X.

Remark 1.13.
The universal family is given by

$$\zeta_X^{-1}(id_X).$$

Definition 1.14 (Fine moduli space)**.**
Let F be a moduli functor. A scheme $X \in \mathrm{Sch}/S$ is a **fine moduli space** for F if X represents F.

Notation 1.15.
If U is a subscheme of T and $i : U \to T$ is the inclusion morphism, then if $\xi \in F(T)$ we will define its restriction to U to be

$$\xi\big|_U = F(i)(\xi).$$

Remark 1.16.
Because the Yoneda embedding is injective on isomorphism classes up to isomorphism (1.9), fine moduli spaces are unique up to isomorphism.

Example 1.17 (Projective space)**.**
Consider the functor

$$\mathbb{P}_n : \begin{array}{ccc} \mathrm{Sch}^{op} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \left\{ (\mathcal{L}, s_0, \cdots, s_n) \;\middle|\; \begin{array}{c} \mathcal{L} \text{ line bundle on } S,\ s_0, \cdots, s_n \in \mathcal{L}(S), \\ \forall x \in S,\ \langle (s_0)_x, \cdots, (s_n)_x \rangle_{\mathcal{O}_{S,x}} = \mathcal{L}_x \end{array} \right\} \Big/ \sim \\ f & \longmapsto & \text{pullback of sheaves and sections via } f \end{array}$$

where $(\mathcal{L}, (s_i)) \sim (\mathcal{L}', (s_i'))$ is there exists a sheaf isomorphism $\alpha : \mathcal{L} \to \mathcal{L}'$ such that $s_i = \alpha^* s_i'$ for all $i \in \{0, \cdots, n\}$.

It is a well know fact (Proposition 5.1.31 in [6]) that $\mathbb{P}_n(S) \cong \mathrm{Hom}\,(S, \mathbb{P}^n_{\mathbb{Z}})$ and that pullbacks behave as expected, thus $\mathbb{P}^n_{\mathbb{Z}}$ is a fine moduli space for $\mathbb{P}_n$.
From the statement of Proposition 5.1.31 in [6] it is also clear that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$ is a universal family.

Fine moduli spaces do not always exist. The simplest obstructions to having a fine moduli spaces are

- the functor is not a Zariski sheaf (see proposition (1.23))
- existence of non trivial automorphisms.

To get an idea of why the second condition is an obstruction consider the following

[3] usually we just say that X represents F

Proposition 1.18.
Let $F \in (\mathrm{Sch}/\mathbb{C})^{op} \to \mathrm{Set}$ be a moduli functor. If there exists an ***isotrivial family*** *$\mathcal{E} \in F(S)$ for $S \in \mathrm{Sch}/\mathbb{C}$ variety, i.e.*

- *for all $s, t \in S(\mathbb{C})$, the fiber $F(s)(\mathcal{E}) = \mathcal{E}_s = \mathcal{E}_t$ and*
- *the family $\mathcal{E}$ is not the pullback of an object $E \in F(\mathrm{Spec}\,\mathbb{C})$ along the structural morphism $S \to \mathrm{Spec}\,\mathbb{C}$,*

then there exists no fine moduli space for F.

Proof.
This is Proposition 0.3.21 in [1]. [ISERT LATER MAYBE] □

A weaker notion of moduli space is that of coarse moduli space:

Definition 1.19 (Coarse moduli space)**.**
Let F be a moduli problem. A pair (X, ζ) for $X \in \mathrm{Sch}/S$ and $\zeta : F \to h_X$ natural transformation is a **coarse moduli space** for F if

- $\zeta_{\mathrm{Spec}\,\mathbb{K}} : F(\mathrm{Spec}\,\mathbb{K}) \to \mathrm{Hom}\,(\mathrm{Spec}\,\mathbb{K}, X)$ is a bijection for all algebraically closed fields $\mathbb{K}$
- for any scheme Y and $\eta : F \to h_Y$ natural transformation there exists a unique morphism $\alpha : X \to Y$ such that $\eta = h_\alpha \circ \zeta$.

Remark 1.20.
A fine moduli space is also a coarse moduli space.

Proof.
The first condition is trivially verified. For the second condition, if (Y, η) is defined as above and (X, u) is the fine moduli space with universal family u then we can take $\alpha = \eta_X(u)$. □

1.2 Zariski sheaves and gluing of fine moduli spaces

One approach to show representability of a moduli problem is emulating the gluing properties of sheaves. Indeed it is possible to show that representable functors are sheaves of some kind and this realization will eventually lead us to the representability criterion we will use later in this document.

1.2.1 Zariski sheaves

First, let us formalize a way in which a functor can be a sheaf. First we recall the definition of equalizer:

Definition 1.21 (Equalizer)**.**
Let $\mathcal{C}$ be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \to C$. We say that the pair (A, h) is an **equalizer** of the diagram

$$B \overset{f}{\underset{g}{\rightrightarrows}} C$$

if $h : A \to B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \to A$ which makes the diagram commute

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ Q \arrow[u, dashed] \arrow[ur, "q"'] & & \end{tikzcd}$$

Definition 1.22 (Zariski sheaf)**.**
A moduli problem $F \in (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a **Zariski sheaf** if for any S-scheme X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Proposition 1.23 (Representable moduli functors are Zariski sheaves)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem, then if there exists a fine moduli space M for F it must be the case that F is a Zariski sheaf.

Proof.
Up to composing with the natural isomorphism, we may assume $F = h_M$. Let X be an S-scheme and $\{U_i \to X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\mathrm{Hom}\,(X, M) \longrightarrow \prod_k \mathrm{Hom}\,(U_k, M) \rightrightarrows \prod_{i,j} \mathrm{Hom}\,(U_i \cap U_j, M)$$

The arrows in this case correspond to restriction of morphisms, so what we need to verify is

- $\mathrm{res}^{U_i}_{U_i \cap U_j} \circ \mathrm{res}^X_{U_i} = \mathrm{res}^{U_j}_{U_i \cap U_j} \circ \mathrm{res}^X_{U_j}$
- a collection of maps $\{f_i : U_i \to M\}$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ glues uniquely to a map $f : X \to M$

both of which are true. □

1.2.2 Open cover of a moduli problem

Let us now define the analogue of an open cover for functors

Definition 1.24 (Subfunctor)**.**
A functor $F : \mathcal{C} \to \mathrm{Set}$ is a **subfunctor** of $G : \mathcal{C} \to \mathrm{Set}$ if for all $X \in \mathcal{C}$ and for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$

$$F(X) \subseteq G(X), \qquad \text{and} \qquad F(f) = G(f)|_{F(A)}.$$

In this case we write $F \subseteq G$.

Definition 1.25 (Fibered product of presheaves)**.**
Let $F, G, H : \mathcal{C}^{op} \to \mathrm{Set}$ be presheaves together with two natural transformations $\eta : F \to H$ and $\zeta : G \to H$. We define their fibered product as the following functor

$$\begin{array}{rccc} & \mathcal{C}^{op} & \longrightarrow & \mathrm{Set} \\ F \times_H G : & X & \longmapsto & F(X) \times_{H(X)} G(X) \\ & f : A \to B & \longmapsto & (b_1, b_2) \mapsto (F(f)(b_1), G(f)(b_2)) \end{array}$$

where the fibered product $F(X) \times_{H(X)} G(X)$ in defined through the maps η_X and ζ_X. The map $(F \times_H G)(f)$ is well defined because if $(b_1, b_2) \in F(B) \times_{H(B)} G(B)$ then $\eta_B(b_1) = \zeta_B(b_2)$, thus

$$\eta_A(F(f)(b_1)) = H(f)(\eta_B(b_1)) = H(f)(\zeta_B(b_2)) = \zeta_A(G(f)(b_2)).$$

Definition 1.26 (Open subfunctor)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S-scheme T and any natural transformation $h_T \to F$, there exists an open subscheme U of T such that U represents $h_T \times_F G$, i.e.

$$\begin{tikzcd} U \arrow[r, mapsto] \arrow[d, phantom, "\cap\mid"] & h_U \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & G \arrow[d] \\ T \arrow[r, mapsto] & h_T \arrow[r] & F \end{tikzcd}$$

Remark 1.27.
By the Yoneda lemma, giving a natural transformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows:

A subfunctor $G \subseteq F$ is open if for any S-scheme T and any family $\xi \in F(T)$ there exists an open subscheme $U \subseteq T$ such that the following diagram is natural in R, commutes and for any map $f : R \to U$ there exists a $g : R \to U$ such that $f = \subseteq \circ g$ if and only if $F(f)(\xi) \in G(R)$[4]

$$\begin{tikzcd} \mathrm{Hom}\,(R, U) \arrow[r, "G(\subseteq\circ\ \cdot\ )(\xi)"] \arrow[d, hook, "\subseteq\circ"'] & G(R) \arrow[d, hook, "\subseteq"] \\ \mathrm{Hom}\,(R, T) \arrow[r, "F(\cdot)(\xi)"'] & F(R) \end{tikzcd}$$

Definition 1.28 (Open cover of a functor)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. A collection of open subfunctors $\{F_i \to F\}$ is an **open cover** of F if for any S-scheme T and any natural transformation $h_T \to F$, there exists an open cover $\{U_i \to T\}$ of T such that U_i represents $h_T \times_F F_i$ for all i.

Remark 1.29.
Like above, we can rephrase the definition as follows:
A collections of open subfunctors $F_i \subseteq F$ form an open cover of F if for any S-scheme T and any family $\xi \in F(T)$, there exists an open cover $\{U_i \to T\}$ of T such that $\xi|_{U_i} \in F_i(U_i)$ for all i.

1.2.3 Representability criterion

Finally, we come to the main result of this chapter

Theorem 1.30 (Representability by open cover)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a Zariski sheaf and let $\{F_i \to F\}$ be an open cover of it by representable subfunctors, then F is representable.

Proof.
For this proof we will mainly follow the version presented in [5] (Theorem 8.9, page 212).
Let us fix X_i schemes and $\xi_i \in F_i(X_i)$ such that (X_i, ξ_i) is a fine moduli space for F_i. For all S-schemes T we have

$$(F_i \times_F F_j)(T) = F_i(T) \times_{F(T)} F_j(T) = F_i(T) \cap F_j(T) \subseteq F(T),$$

[4] the "only if" is trivially true by commutativity but for the "if" we are using the fact that $h_U \cong h_T \times_F G$.

thus $F_i \times_F F_j = F_j \times_F F_i \doteqdot F_{i,j}$. We define $F_{i,j,k}$ analogously.

Since F_j is an open subfunctor of F, there exists an open subscheme $U_{ij} \subseteq X_i$ which represents $h_{X_i} \times_F F_j \cong F_i \times_F F_j = F_{i,j}$. We can define $U_{ji} \subseteq X_j$ similarly and since they are both moduli spaces for $F_{i,j}$ they are isomorphic. Let $\varphi_{ji} : U_{ij} \to U_{ji}$ be the isomorphism given by $\varphi_{ji} = \alpha_{U_{ij}}(id_{U_{ij}})$ for α natural isomorphism which makes the following diagram commute

$$\begin{array}{ccccc} h_{U_{ij}} & \cong & h_{X_i} \times_F F_j & \cong & F_{i,j} \\ {\scriptstyle\alpha}\big\downarrow & & & & \| \\ h_{U_{ji}} & \cong & F_i \times_F h_{X_j} & \cong & F_{i,j} \end{array}$$

Note that if T is an S-scheme and $f \in h_{U_{ij}}(T)$ then

$$h_{\varphi_{ji}}(f) = \alpha_{U_{ij}}(id_{U_{ij}}) \circ f = \alpha_T(id_{U_{ij}} \circ f) = \alpha_T(f),$$

so α is the image of φ_{ji} under the Yoneda embedding.

We want to show that the X_i can be glued along the U_{ij} using the isomorphisms φ_{ji}. First we need to show that $\varphi_{ji}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and then we have to verify the cocycle condition $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$.

The first condition follows immediately from the fact that F_k is an open subfunctor and our construction of the φ_{ji}.

Since the Yoneda embedding preserves limits (1.10) it preserves fibered products, so we see that the following diagram commutes

$$\begin{array}{ccccccc} h_{U_{ij} \cap U_{ik}} & \xrightarrow{\doteqdot} & h_{U_{ij}} \times_{h_{X_i}} h_{U_{ik}} & \cong & F_{i,j} \times_{F_i} F_{i,k} & \cong & F_{i,j,k} \\ {\scriptstyle\varphi_{ji}}\big\downarrow & & & & & & \| \\ h_{U_{ji} \cap U_{jk}} & \xrightarrow{\doteqdot} & h_{U_{ji}} \times_{h_{X_j}} h_{U_{jk}} & \cong & F_{i,j} \times_{F_j} F_{j,k} & \cong & F_{i,j,k} \end{array}$$

therefore, to prove that $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ it is enough to verify the trivial equality

$$id_{F_{i,j,k}} \circ id_{F_{i,j,k}} = id_{F_{i,j,k}}.$$

Let X to be the scheme obtained by gluing the X_i as indicated above.
Note that $\xi_i = \varphi_{ji}^* \xi_j$, so if we look at these families as elements of $F(X)$ we see that $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$. Since F is a Zariski sheaf, the ξ_i can be glued to a family $\xi \in F(X)$.

To finish the proof we need to verify that (X, ξ) is a fine moduli space for F:
Let T be an S-scheme and and let us consider a family $\zeta \in F(T)$. Since $\{F_i \to F\}$ is an open cover of F, there exists an open cover $\{V_i \to T\}$ of T such that $\zeta|_{V_i} \in F_i(V_i) \cong \operatorname{Hom}(V_i, X_i)$. Since F is a sheaf and $\zeta_i|_{V_i \cap V_j} = \zeta_j|_{V_i \cap V_j}$, the morphisms $V_i \to X_i$ corresponding to the ζ_i glue to a morphism $f : T \to X$ such that $f^* \xi = \zeta$ (by construction). $\square$

Remark 1.31.
Adopting the notation of the proof we see that if we have a candidate fine moduli space X, an open cover $\{X_i \to X\}$ such that X_i represents F_i and we can verify that $\varphi_{ji} : X_i \cap X_j \to X_i \cap X_j$ is the identity for all i, j, then X represents F.

Chapter 2

Classical Grassmannians

In this chapter we introduce Grassmannians from the point of view of classical algebraic geometry. We are interested in Grassmannians in the context of classification problems because their definition leads us to suspect that they are a moduli space for certain families of vector spaces. In the next chapter we will indeed find that they are fine moduli spaces for a functor that formalizes *families of k-vector subspaces of* $\mathbb{K}^n$.

We first define Grassmannians set-theoretically, then we will find a bijection between this set and Zariski-closed subset of some projective space. This bijection will allow us to endow the Grassmannian with the structure of a projective algebraic variety.

2.1 First definitions and conventions

Definition 2.1 (Grassmannian)**.**
Let $k \leq n$ be a pair of positive integers. We define the (n,k)**-Grassmannian**, denoted[1] $\mathrm{Gr}(k,n,\mathbb{K})$, as the set of $(n-k)$-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^n$.

Remark 2.2.
We may equivalently define $\mathrm{Gr}(k,n)$ to be the following set:

$$\left\{\ker\varphi \mid \varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n,\mathbb{K}^k),\ \mathrm{rnk}\,\varphi = k\right\}.$$

Proof.
If $H \in \mathrm{Gr}(k,n)$, let $z_1,\cdots,z_n$ be a basis of $\mathbb{K}^n$ such that $H = \mathrm{Span}\,(z_1,\cdots,z_{n-k})$ and let $e_1,\cdots,e_k$ be any basis of $\mathbb{K}^k$. We can view H as the kernel of the (rank k) linear map given by

$$\varphi(z_i) = \begin{cases} 0 & \text{if } i \leq n-k \\ e_{i-n+k} & \text{otherwise} \end{cases}$$

On the other hand, if φ is a rank k linear map then, by the rank-nullity theorem, its kernel is an $n-k$ dimensional subspace of $\mathbb{K}^n$. $\square$

Lemma 2.3.
Let $\varphi,\psi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n,\mathbb{K}^k)$ be linear maps of full rank. The following conditions are equivalent:

1. $\ker\varphi = \ker\psi$,

[1] the field will be omitted when clear from context

2. *there exists $\theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta \circ \psi$.*

Proof.
Let us prove both implications:

$\boxed{2. \implies 1.}$ $\ker \varphi = \ker(\theta \circ \psi) = \psi^{-1}(\ker \theta) = \psi^{-1}(\{0\}) = \ker \psi$.

$\boxed{1. \implies 2.}$ Let $z_1, \cdots, z_n$ be a basis of $\mathbb{K}^n$ such that $\ker \varphi = \ker \psi = \mathrm{Span}\,(z_1, \cdots, z_{n-k})$. By construction $\varphi(z_{n-k+1}), \cdots, \varphi(z_n)$ and $\psi(z_{n-k+1}), \cdots, \psi(z_n)$ are bases of $\mathbb{K}^k$. Let θ be the change of basis on $\mathbb{K}^k$ determined by $\theta(\psi(z_i)) = \varphi(z_i)$ for all $n - k < i \leq n$. By construction θ is nonsingular and φ agrees with $\theta \circ \psi$ on a basis of $\mathbb{K}^n$.

$\square$

Corollary 2.4.
We may redefine Grassmannians in terms of linear maps as follows:

$$\mathrm{Gr}(k,n) = {\left\{\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \varphi \textit{ surjective.}\right\}}\big/{\sim}$$

where $\varphi \sim \psi$ if and only if $\exists \theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta \circ \psi$.

We conclude this introductory section with some notation and conventions.

Definition 2.5 (Multiindicies)**.**
We define a (k,n)-**multiindex** as an element of $\{1, \cdots, n\}^k$. Our notation for a multiindex I will usually be $I = (i_1, \cdots, i_k)$.
We denote the set of **ordered** (k,n)-**multiindicies** with

$$\omega(k,n) = \left\{(i_1, \cdots, i_k) \in \{1, \cdots, n\}^k \mid i_1 < \cdots < i_k\right\}.$$

If $I \in \omega(k,n)$, we denote by $\widehat{I}$ the element of $\omega(n-k,n)$ whose elements are those of $\{1, \cdots, n\}$ missing from I.
For a given multiindex $I \in \omega(k,n)$, define σ_I to be the permutation that sends $\widehat{I} \cup I$ to $\{1, \cdots, n\}$, where $\cup$ denotes concatenation.
If A is a $k \times n$ matrix and I is a (k,n)-multiindex, we denote the ***I*-minor of** A by A_I, i.e.

$$A_I = \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{pmatrix}.$$

If B is an $\alpha \times \beta$ matrix, $i \in \{1, \cdots, \alpha\}$ and $j \in \{1, \cdots, \beta\}$ we denote the $(\alpha-1)\times(\beta-1)$ matrix obtained from B by deleting the i-th row and the j-th column with $B_{\times i, \times j}$.

Remark 2.6.
If $I = (i_1, \cdots, i_k)$ is a (k,n)-multiindex and $e_1, \cdots, e_n$ is a basis of $\mathbb{K}^n$ we define

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Note that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\} = \{e_I \mid I \in \omega(k,n)\}$$

forms a basis for $\bigwedge^k \mathbb{K}^n$.

Notation 2.7.
Whenever a basis $\mathcal{B}$ of $\mathbb{K}^\ell$ is fixed, we will identify $\bigwedge^\ell \mathbb{K}^\ell$ with $\mathbb{K}$ by sending the wedge of the ordered basis to $1 \in \mathbb{K}$. This isomorphism is denoted $\eta_{\mathcal{B}} : \bigwedge^\ell \mathbb{K}^\ell \to \mathbb{K}$. If $\mathcal{D}$ is a basis of $\mathbb{K}^m$ then we define $\eta^{\mathcal{B}}_{\mathcal{D}} = \eta^{-1}_{\mathcal{D}} \circ \eta_{\mathcal{B}}$.

2.2 The Plücker embedding

In this section we define an injection from the Grassmannian to a projective space. The idea behind this map is to take appropriate wedge products in such a way as to trasform the several vectors defining a vector subspace into a single vector and then to projectivize.

Definition 2.8 (Plücker map)**.**
Let $k \le n$ be a pair of positive integers. We define the **Plücker map** as[2]

$$\wedge^k : \begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \\ \varphi & \longmapsto & \wedge^k \varphi \end{array},$$

where $(\wedge^k \varphi)(v_1 \wedge \cdots \wedge v_k) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_k)$.

Remark 2.9.
If $\mathcal{B} = \{v_1, \cdots, v_k\}$ is a basis of $\mathbb{K}^k$, $\mathcal{C}\mathrm{an} = \{e_1, \cdots, e_k\}$ is the canonical basis and $[\cdot]_{\mathcal{B}} : \mathbb{K}^k \to \mathbb{K}^k$ is the isomorphism which sends v_i to e_i then

$$\wedge^k(\varphi)(v_1 \wedge \cdots \wedge v_k) = \det\left([\varphi(v_1)]_{\mathcal{B}} | \cdots | [\varphi(v_k)]_{\mathcal{B}}\right) e_1 \wedge \cdots \wedge e_k,$$

which we will often identify with the coefficient, i.e. its image under $\eta^{\mathcal{B}}_{\mathcal{C}\mathrm{an}}$.

In the interest of readability, when $\mathcal{B}$ is fixed we will write $\wedge^k(\varphi)(v_1 \wedge \cdots \wedge v_k)$ or $\varphi(v_1) \wedge \cdots \wedge \varphi(v_k)$ to mean $\det\left([\varphi(v_1)]_{\mathcal{B}} | \cdots | [\varphi(v_k)]_{\mathcal{B}}\right)$. This gives the first notation two interpretations but which is meant will be clear from context.

Remark 2.10.
The codomain of the Plücker map is isomorphic to $\bigwedge^k \mathbb{K}^n$, indeed

$$\mathrm{Hom}_{\mathbb{K}}\left(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k\right) \cong \left(\bigwedge^k \mathbb{K}^n\right)^{\vee} \cong \bigwedge^k \mathbb{K}^n.$$

The isomorphism depends on the choice of basis for $\mathbb{K}^n$ and $\mathbb{K}^k$:
If $\mathcal{B} = \{e_1, \cdots, e_n\}$ is a basis of $\mathbb{K}^n$ and $\mathcal{D} = \{e_1, \cdots, e_k\}$ is a basis of $\mathbb{K}^k$ then the isomorphism takes on the following form

$$\zeta_{\mathcal{B},\mathcal{D}} : \begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ \psi & \longmapsto & \displaystyle\sum_{I \in \omega(k,n)} \eta_{\mathcal{D}}(\psi(e_I)) e_I \end{array}.$$

Proposition 2.11.
The image of the Plücker map is a cone.

Proof.
For any $\lambda \in \mathbb{K}^*$ and any map $\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ note that

$$\lambda \wedge^k (\varphi) = \wedge^k(\alpha \circ \varphi)$$

for any automorphism α of $\mathbb{K}^k$ with determinant λ. We can construct one such α by fixing a basis of $\mathbb{K}^k$ and defining α to be the map corresponding to the matrix

$$\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

□

[2]the map $\wedge^k \varphi$ is well defined because if we view it as a map $\wedge^k \varphi : (\mathbb{K}^n)^k \to \bigwedge^k \mathbb{K}^k$ then it is multilinear and alternating.

Lemma 2.12.
If $\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ then $\operatorname{rnk}\varphi < k$ if and only if $\wedge^k(\varphi) = 0$.

Proof.
$\wedge^k(\varphi)$ is the zero map if an only if the set $\{\varphi(v_1), \cdots, \varphi(v_k)\}$ is linearly dependent for any choice of $v_1, \cdots, v_k$, i.e. φ is not of full rank. □

Lemma 2.13.
Let $\varphi : \mathbb{K}^n \to \mathbb{K}^k$ be a full rank linear map, then

$$\ker\varphi = \left\{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0\right\}.$$

Proof.
If $\varphi(z) = 0$ then for any $w_2, \cdots, w_k \in \mathbb{K}^k$ we see that

$$\wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \wedge \varphi(w_2) \wedge \cdots \wedge \varphi(w_k) = 0.$$

Suppose now that $\varphi(z) \neq 0$ and let $v_2, \cdots, v_k$ be such that $\{\varphi(z), v_2, \cdots, v_k\}$ forms a basis for $\mathbb{K}^k$. Since φ is surjective, there exist $w_2, \cdots, w_k$ such that $\varphi(w_i) = v_i$ for all $2 \leq i \leq k$. By construction

$$\wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = \varphi(z) \wedge v_2 \wedge \cdots \wedge v_k \neq 0.$$

□

Proposition 2.14 (Injectivity of the Plücker map up to scalars)**.**
Let $\sim$ be the equivalence relation defined in corollary (2.4), then for any two full rank linear maps $\varphi, \psi : \mathbb{K}^n \to \mathbb{K}^k$

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{K}^* \ s.t.\ \wedge^k(\varphi) = \lambda \wedge^k(\psi).$$

Proof.
We prove both implications:

$\boxed{\Longrightarrow}$ If $\varphi = \theta \circ \psi$ for $\theta \in \mathrm{GL}(\mathbb{K}^k)$ then it follows easily from known properties of the determinant that

$$\wedge^k(\varphi) = \wedge^k(\theta \circ \psi) = (\det\theta) \wedge^k(\psi).$$

$\boxed{\Longleftarrow}$ From lemma (2.3) we see that it is enough to prove that $\ker\varphi = \ker\psi$. We conclude by applying lemma (2.13) as follows:

$$\begin{aligned}\ker\varphi &= \left\{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0\right\} = \\ &= \left\{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \lambda \wedge^k(\psi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0\right\} = \\ &= \left\{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \wedge^k(\psi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0\right\} = \ker\psi.\end{aligned}$$

□

Remark 2.15.
Because of proposition (2.14) and lemma (2.12) there exists a unique h such that the diagram commutes

$$\begin{tikzcd} \left\{\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \operatorname{rnk}\varphi = k\right\} \arrow[r, "{[\wedge^k]}"] \arrow[d, "\pi_\sim"] & \mathbb{P}(\mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)) \\ \mathrm{Gr}(k, n) \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Moreover, such an h must be injective by proposition (2.14).

Definition 2.16 (Plücker embedding)**.**
Let us fix a basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and a basis $e_1, \cdots, e_k$ of $\mathbb{K}^k$. We define the **Plücker embedding** as follows

$$\mathrm{Pl}: \begin{array}{ccc} \mathrm{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [\varphi] & \longmapsto & \left[\displaystyle\sum_{1 \le i_1 < \cdots < i_k \le n} \det(\varphi(e_{i_1}) \mid \cdots \mid \varphi(e_{i_k})) e_{i_1} \wedge \cdots \wedge e_{i_k} \right] \end{array}$$

The entries of the homogeneous $\binom{n}{k}$-tuple associated to $[\varphi] \in \mathrm{Gr}(k,n)$ are called the **Plücker coordinates** of $[\varphi]$.

Remark 2.17 (Well defined and injective)*.*
If we fix bases for $\mathbb{K}^n$ and $\mathbb{K}^k$ and ζ is the isomorphism $\mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \to \bigwedge^k \mathbb{K}^n$ discussed during remark (2.10), we see that the following diagram commutes

$$\begin{tikzcd} \mathbb{P}(\mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)) \arrow[r, "\mathbb{P}(\zeta)"] & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ \mathrm{Gr}(k,n) \arrow[u, "h"] \arrow[ur, "\mathrm{Pl}"'] & \end{tikzcd}$$

This proves that the Plücker embedding is well defined and injective.

Remark 2.18*.*
If we define $\phi = \zeta \circ \wedge^k$ then we see that

$$\mathrm{Pl} \circ \pi_{\sim} = \mathbb{P}(\phi).$$

This identity will guide our definition of the Plücker embedding in the next chapter.

Remark 2.19*.*
The Plücker embedding depends on the choice of basis for $\mathbb{K}^n$ but not on the one for $\mathbb{K}^k$, since the effect of changing the basis of $\mathbb{K}^k$ is that of multiplying all Plücker coordinates by the same nonzero scalar (the determinant of the change of basis), which doesn't change the point they describe in $\mathbb{P}(\bigwedge^k \mathbb{K}^k)$.
The dependence on the basis of $\mathbb{K}^n$ is inevitable because $\mathrm{GL}(\mathbb{K}^n)$ acts transitively on $\mathrm{Gr}(k,n)$ viewed as the set of $(n-k)$-dimentional subspaces of $\mathbb{K}^n$.

2.3 The image of the Plücker embedding is closed

Thus far we have identified $\mathrm{Gr}(k,n)$ with a subset of some projective space. We seek to show that it is a closed subset in the Zariski topology. Our approach mostly adapts parts of [7].

First we need some linear algebra results

Definition 2.20 (Divisibility)**.**
We say that $\omega \in \bigwedge^k \mathbb{K}^n$ is **divisible** by $v \in \mathbb{K}^n$ if there exists $\varepsilon \in \bigwedge^{k-1} \mathbb{K}^n$ such that $\omega = \varepsilon \wedge v$.

Lemma 2.21.
Let $\omega \in \bigwedge^k \mathbb{K}^n$. For any given nonzero vector v, ω is divisible by v if and only if $\omega \wedge v = 0$.

Proof.
If $\omega = \varepsilon \wedge v$ then $\omega \wedge v = \varepsilon \wedge v \wedge v = 0$.
Suppose now that $\omega \wedge v = 0$. Let $v_1, \cdots, v_n$ be a basis of $\mathbb{K}^n$ such that $v_1 = v$. If we write

$$\omega = \sum_{I \in \omega(k,n)} p_I v_I$$

then we see that for any given multiindex I, either $p_I = 0$ or $v_I \wedge v = 0$. Since $v_1, \cdots, v_n$ is a basis, $v_I \wedge v_1 = 0$ if and only if $1 \in I$, i.e. $v_I = v \wedge v_{(i_2, \cdots, i_k)}$, therefore

$$\omega = v \wedge \underbrace{\left(\sum_{2 \le i_2 < \cdots i_k \le n} p_{(1,i_2,\cdots,i_k)} e_{(i_2,\cdots,i_k)} \right)}_{\doteqdot (-1)^{k-1}\varepsilon} = \varepsilon \wedge v.$$

$\square$

Corollary 2.22 (Total decomposibility criterion)**.**
Let $\omega \in \bigwedge^k \mathbb{K}^n$. If $\dim \{v \in \mathbb{K}^n \mid \omega \wedge v = 0\} \ge k$ then $\omega = \lambda v_1 \wedge \cdots \wedge v_k$ for any set of linearly independent vectors $\{v_1, \cdots, v_k\}$ in $\{v \in \mathbb{K}^n \mid \omega \wedge v = 0\}$ and some scalar λ. Moreover $\lambda \neq 0$ if and only if $\dim \{v \in \mathbb{K}^n \mid \omega \wedge v = 0\} = k$.

Proof.
The set $\{v \in \mathbb{K}^n \mid \omega \wedge v = 0\}$ is clearly a subspace of $\mathbb{K}^n$. Let $\{v_1, \cdots, v_k\}$ be linearly independent vectors of this space. By iterating the above lemma we see that

$$\omega = \lambda \wedge v_1 \wedge \cdots \wedge v_k$$

for some $\lambda \in \bigwedge^0 \mathbb{K}^n = \mathbb{K}$.
If $\lambda = 0$ then clearly $\{v \in \mathbb{K}^n \mid \omega \wedge v = 0\} = \mathbb{K}^n$. If v_{k+1} is such that $\omega \wedge v_{k+1} = 0$ and $\{v_1, \cdots, v_{k+1}\}$ is linearly independent then, proceding as above,

$$\lambda v_1 \wedge \cdots \wedge v_k = \omega = \mu v_1 \wedge \cdots \wedge v_{k-1} \wedge v_{k+1}.$$

By multilinearity

$$0 = v_1 \wedge \cdots \wedge v_{k-1} \wedge (\lambda v_k - \mu v_{k+1}),$$

i.e. $\lambda v_k - \mu v_{k+1} \in \operatorname{Span}(v_1, \cdots, v_{k-1})$. By linear independence $\lambda v_k - \mu v_{k+1} = 0$ and thus, again by linear independence, $\lambda = \mu = 0$. $\square$

Proposition 2.23.
There is a canonical isomorphism between $\operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^n \mathbb{K}^n)$ and $\bigwedge^{n-k} \mathbb{K}^n$ given by

$$\Xi : \begin{array}{ccc} \bigwedge^{n-k} \mathbb{K}^n & \longrightarrow & \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^n \mathbb{K}^n) \\ \omega & \longmapsto & \omega \wedge \cdot \end{array}.$$

For any basis $\mathcal{B} = \{e_1, \cdots, e_n\}$ of $\mathbb{K}^n$ the map

$$\Gamma_{\mathcal{B}} : \begin{array}{ccc} \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^n \mathbb{K}^n) & \longrightarrow & \bigwedge^{n-k} \mathbb{K}^n \\ \psi & \longmapsto & \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{B}}(\psi(e_{\widehat{I}})) e_I \end{array}$$

is the inverse of Ξ.

Proof.
The map is clearly base independent and linear.
For all e_J

$$\begin{aligned}\Xi(\Gamma_{\mathcal{B}}(\psi))(e_J) = &\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I \eta_{\mathcal{B}}(\psi(e_{\widehat{I}}))e_I \wedge e_J = \\ &= \operatorname{sgn}\sigma_{\widehat{J}}\eta_{\mathcal{B}}(\psi(e_J))e_{\widehat{J}} \wedge e_J = \\ &= \eta_{\mathcal{B}}(\psi(e_J))e_{(1,\cdots,n)} = \\ &= \psi(e_J),\end{aligned}$$

so $\Xi(\Gamma_{\mathcal{B}}(\psi))$ agrees with ψ on a basis.
If $\omega = \sum_{J\in\omega(n-k,n)} p_J e_J$ then

$$\begin{aligned}\operatorname{sgn}\sigma_I\eta_{\mathcal{B}}(\omega \wedge e_{\widehat{I}}) = &\sum_{J\in\omega(n-k,n)} p_J \operatorname{sgn}\sigma_I\eta_{\mathcal{B}}(e_J \wedge e_{\widehat{I}}) = \\ &= p_I\eta_{\mathcal{B}}(\operatorname{sgn}\sigma_I e_I \wedge e_{\widehat{I}}) = \\ &= p_I\eta_{\mathcal{B}}(e_{(1,\cdots,n)}) = \\ &= p_I,\end{aligned}$$

thus

$$\Gamma_{\mathcal{B}}(\Xi(\omega)) = \sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I\eta_{\mathcal{B}}(\omega\wedge e_{\widehat{I}})e_I = \sum_{I\in\omega(n-k,n)} p_I e_I = \omega.$$

□

Corollary 2.24.
Let $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$. If $\mathcal{B} = \{e_1,\cdots,e_n\}$ and $\mathcal{B}' = \{e'_1,\cdots,e'_n\}$ are two bases for $\mathbb{K}^n$ and $\mathcal{D} = \{e_1,\cdots,e_k\}$ and $\mathcal{D}' = \{e'_1,\cdots,e'_k\}$ are bases for $\mathbb{K}^k$, there exists $\mu \in \mathbb{K}\setminus\{0\}$ such that

$$\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I\eta_{\mathcal{D}}(\psi(e_{\widehat{I}}))e_I = \mu \sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I\eta_{\mathcal{D}'}(\psi(e'_{\widehat{I}}))e'_I.$$

Proof.
Note that

$$\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I\eta_{\mathcal{D}}(\psi(e_{\widehat{I}}))e_I = \Xi^{-1}(\eta^{\mathcal{D}}_{\mathcal{B}} \circ \psi)$$

and similarly the other expression is $\Xi^{-1}(\eta^{\mathcal{D}'}_{\mathcal{B}'} \circ \psi)$.
It is therefore enough to show that $\eta^{\mathcal{B}}_{\mathcal{D}} = \mu\eta^{\mathcal{B}'}_{\mathcal{D}'}$ for some $\mu \in \mathbb{K}\setminus\{0\}$, which is true because $\dim_{\mathbb{K}} \operatorname{Hom}_{\mathbb{K}}(\bigwedge^n \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) = 1$ and both $\eta^{\mathcal{B}}_{\mathcal{D}}$ and $\eta^{\mathcal{B}'}_{\mathcal{D}'}$ are not the zero map.
□

Lemma 2.25.
Fix bases $\mathcal{B} = \{e_1,\cdots,e_n\}$ of $\mathbb{K}^n$ and $\mathcal{D} = \{e_1,\cdots,e_k\}$ of $\mathbb{K}^k$. A multilinear alternating form $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ is in the image of the Plücker map $\wedge^k$ if and only if there exists $\lambda \in \mathbb{K}$ and linearly independent vectors $z_1,\cdots,z_{n-k}$ such that

$$\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I\eta_{\mathcal{D}}(\psi(e_{\widehat{I}}))e_I = \lambda z_{(1,\cdots,n-k)}.$$

Proof.
For simplicity we omit the $\eta_{\mathcal{D}}$.

$\boxed{\Longrightarrow}$ Suppose that $\psi = \wedge^k(\varphi)$ and let $\{z_1, \cdots, z_n\}$ be a basis of $\mathbb{K}^n$ such that $z_1, \cdots, z_{n-k}$ are linearly independent vectors in $\ker \varphi$, then

$$\begin{aligned} \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \wedge^k \varphi(e_{\widehat{I}}) e_I &\overset{(2.24)}{=} \\ &= \mu \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \wedge^k \varphi(z_{\widehat{I}}) z_I = \\ &= \left(\mu \operatorname{sgn} \sigma_{(1,\cdots,n-k)} \wedge^k \varphi(z_{(n-k+1,\cdots,n)})\right) z_{(1,\cdots,n-k)}. \end{aligned}$$

$\boxed{\Longleftarrow}$ Let $z_1, \cdots, z_n$ be a basis of $\mathbb{K}^n$ which extends $\{z_1, \cdots, z_{n-k}\}$ and define $\widetilde{\varphi}$ by

$$\widetilde{\varphi}(z_i) = \begin{cases} 0 & \text{if } 1 \le i \le n-k \\ e_1 & \text{if } i = n-k+1 \\ e_{i-n+k} & \text{if } i > n-k+1 \end{cases}$$

Let $\alpha = \wedge^k \widetilde{\varphi}(z_{(n-k+1,\cdots,n)})$ and consider the following chain of equalities

$$\begin{aligned} \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \psi(e_{\widehat{I}}) e_I =& \lambda z_{(1,\cdots,n-k)} = \\ &= \frac{\lambda}{\alpha} \wedge^k \widetilde{\varphi}(z_{(n-k+1,\cdots,n)}) z_{(1,\cdots,n-k)} = \\ &= \left(\frac{\lambda}{\alpha} \operatorname{sgn} \sigma_{(1,\cdots,n-k)}\right) \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \wedge^k \widetilde{\varphi}(z_{\widehat{I}}) z_I = \\ &= \left(\frac{\lambda}{\alpha} \operatorname{sgn} \sigma_{(1,\cdots,n-k)}\right) \mu \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \wedge^k \widetilde{\varphi}(e_{\widehat{I}}) e_I \end{aligned}$$

where the third equality follows from the construction of $\widetilde{\varphi}$ and the last is (2.24). If we now define φ by

$$\varphi(z_i) = \begin{cases} 0 & \text{if } 1 \le i \le n-k \\ \mu \left(\frac{\lambda}{\alpha} \operatorname{sgn} \sigma_{(1,\cdots,n-k)}\right) e_1 & \text{if } i = n-k+1 \\ e_{i-n+k} & \text{if } i > n-k+1 \end{cases}$$

with the same reasoning we see that

$$\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \psi(e_{\widehat{I}}) e_I = \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \wedge^k \varphi(e_{\widehat{I}}) e_I.$$

By linear independence, this shows that for all $J \in \omega(k,n)$ we have

$$\psi(e_J) = \wedge^k \varphi(e_J),$$

so ψ and $\wedge^k(\varphi)$ agree on a basis of $\bigwedge^k \mathbb{K}^n$ and are thus the same map.

□

Definition 2.26.
Let $\mathcal{B} = \{e_1, \cdots, e_n\}$ and $\mathcal{D} = \{e_1, \cdots, e_k\}$ be bases of $\mathbb{K}^n$ and $\mathbb{K}^k$ respectively. For any $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ we define $\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ to be

$$\Phi_{\mathcal{B},\mathcal{D}}(\psi) : \begin{array}{ccc} \mathbb{K}^n & \longrightarrow & \bigwedge^{n-k+1} \mathbb{K}^n \\ v & \longmapsto & \displaystyle\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(e_{\widehat{I}})) e_I \wedge v \end{array} .$$

Remark 2.27.
The rank of $\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ does not depend on the choice of basis. Indeed if we change basis, by corollary (2.24) we see that

$$\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I \eta_{\mathcal{D}}(\psi(e_{\widehat{I}}))e_I \wedge v = \mu \sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I \eta_{\mathcal{D}'}(\psi(e'_{\widehat{I}}))e'_I \wedge v,$$

so $\Phi_{\mathcal{B},\mathcal{D}}(\psi)(v) = 0$ holds for a pair of bases $(\mathcal{B},\mathcal{D})$ if and only if it holds for all such pairs.
Because of this we will write propositions which concern only the rank of $\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ omitting the bases.

Remark 2.28.
$\Phi_{\mathcal{B},\mathcal{D}}(\psi)$ is linear in ψ.

Proposition 2.29.
An alternating multilinear map $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ is in the image of the Plücker map $\wedge^k$ if and only if $\Phi(\psi)$ has rank at most k.

Proof.
Let $\mathcal{D}$ be any basis of $\mathbb{K}^k$. We prove both implications

$\boxed{\Longrightarrow}$ Suppose that $\psi = \wedge^k(\varphi)$ and let $\mathcal{Z} = \{z_1, \cdots, z_n\}$ be a basis of $\mathbb{K}^n$ such that the first $n-k$ vectors are a basis of $\ker\varphi$. Note that

$$\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I \eta_{\mathcal{D}}(\psi(z_{\widehat{I}}))z_I = \operatorname{sgn}\sigma_{(1,\cdots,n-k)}\eta_{\mathcal{D}}(\wedge^k\varphi(z_{(n-k+1,\cdots,n)}))z_{(1,\cdots,n-k)}.$$

If $v \in \ker\varphi$ then $z_{(1,\cdots,n-k)} \wedge v = 0$ and therefore by the above equality $v \in \ker\Phi_{\mathcal{Z},\mathcal{D}}(\psi)$. This means that $\Phi(\psi)$ has a nullity of at least $n-k$ (i.e. rank at most k).

$\boxed{\Longleftarrow}$ Suppose that $\{z_1 \cdots, z_{n-k}\}$ are linearly independent elements of $\ker\Phi(\psi)$. By the total decomposability criterion (2.22) there exists $\lambda \in \mathbb{K}$ such that

$$\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I \eta_{\mathcal{D}}(\psi(z_{\widehat{I}}))z_I = \lambda z_1 \wedge \cdots \wedge z_{n-k}.$$

This concludes by lemma (2.25).

□

Definition 2.30.
Let $\mathcal{B} = \{e_1, \cdots, e_n\}$ and $\mathcal{D} = \{e_1, \cdots, e_k\}$ be bases of $\mathbb{K}^n$ and $\mathbb{K}^k$ respectively. Let $\zeta_{\mathcal{B},\mathcal{D}} : \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \to \bigwedge^k \mathbb{K}^n$ be the isomorphism discussed during remark (2.10). We define

$$\widetilde{\Phi}_{\mathcal{B},\mathcal{D}} : \begin{array}{ccc} \bigwedge^k \mathbb{K}^n & \longrightarrow & \operatorname{Hom}_{\mathbb{K}}\left(\mathbb{K}^n, \bigwedge^{n-k+1}\mathbb{K}^n\right) \\ \omega & \longmapsto & \Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega)) \end{array}$$

Remark 2.31.
Let $\omega = \sum_{I\in\omega(k,n)} p_I e_I$.

$$\begin{aligned}\Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega))(v) &= \sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I \eta_{\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega)(e_{\widehat{I}}))e_I \wedge v = \\ &= \sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I p_{\widehat{I}} e_I \wedge v,\end{aligned}$$

so $v \in \ker \Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega))$ if and only if for all $I \in \omega(k,n)$ either $p_I = 0$ or $e_{\widehat{I}} \wedge v = 0$, which is the same as saying $\omega \wedge v = 0$ if and only if $\omega = 0$ or $v = 0$. This is a base independent condition, so the rank of $\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}(\omega)$ does not depend on the choice of basis.

Remark 2.32.
$\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}$ is linear.

Let us define a matrix with coefficients in[3] $\mathbb{K}[z_I \mid I \in \omega(k,n)]$ which represents $\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}$:
Let $B^I \in \mathcal{M}\left(\binom{n}{n-k+1}, n, \mathbb{K}\right)$ be the matrix which represents $\Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(e_I))$ in the bases induced by $\mathcal{B}$ and $\mathcal{D}$. By linearity

$$\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}\left(\sum_{I\in\omega(k,n)} a_I e_I\right)(v) = \sum_{I\in\omega(k,n)} a_I \Phi_{\mathcal{B},\mathcal{D}}(\zeta^{-1}_{\mathcal{B},\mathcal{D}}(e_I))(v) = \sum_{I\in\omega(k,n)} a_I B^I v.$$

We define the matrix which represents $\widetilde{\Phi}_{\mathcal{B},\mathcal{D}}$ to be

$$M_{\mathcal{B},\mathcal{D}} = \sum_{I,\omega(k,n)} B^I z_I = \left(\sum_{I\in\omega(k,n)} (B^I)_{i,j} z_I\right)_{i,j}.$$

Remark 2.33.
The rank of $\Phi_{\mathcal{B},\mathcal{D}}(\sum_{I\in\omega(k,n)} p_I e_I)$ is exactly the rank of $M|_{z_I=p_I}$.

The previous remark together with proposition (2.29) tells us that

$$\begin{aligned}\operatorname{Imm}(\zeta_{\mathcal{B},\mathcal{D}} \circ \wedge^k) = &\left\{\sum_{I\in\omega(k,n)} p_I e_I \mid \operatorname{rnk}\ M|_{z_I=p_I} < n-k+1\right\} = \\ =&V(\{\det m \mid m \text{ is a } (n-k+1)\times(n-k+1) \text{ minor of } M\}),\end{aligned}$$

which is evidently a Zariski-closed subset of $\bigwedge^k \mathbb{K}^n$.
It follows trivially that the projectivization[4] of this set (i.e. the image of Pl) is closed in $\mathbb{P}(\bigwedge^k \mathbb{K}^n)$, so we found a bijection between $\operatorname{Gr}(k,n)$ and a projective variety, which we can use to endow $\operatorname{Gr}(k,n)$ with the structure of one.

Remark 2.34.
The determinants we used to show that the image of the Plücker embedding is closed do not generate the ideal of that variety. The most well known set of generators for that ideal are the **Plücker relations**. For their construction [BUT NOT THE PROOF THAT THEY ARE GENERATORS, STILL HAVE TO CHOOSE A REFERENCE] see [7].

[3] what we will later call (k,n)-braket ring
[4] recall (2.11) that $\operatorname{Imm} \wedge^k$ is a cone.

Chapter 3

Representability of the Grassmannian functor

In this chapter, unless otherwise specified, we have fixed a basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and a basis $e_1, \cdots, e_k$ of $\mathbb{K}^k$. In case of ambiguity we will refer to these bases as *canonical*.

Having fixed a base, we will identify $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ with the set of $k \times n$ matrices with coefficients in $\mathbb{K}$, which we will denote $\mathcal{M}(k, n)$. As a consequence of this we find yet another form for $\mathrm{Gr}(k, n)$:

$$\mathrm{Gr}(k, n) = {\{A \in \mathcal{M}(k, n) \mid \mathrm{rnk}\, A = k\}}\big/{\sim},$$

where $A \sim B \iff \exists P \in \mathrm{GL}(k)\ s.t.\ A = PB$.

We may rewrite the maps from the previous chapter as follows:

$$\phi^s : \begin{array}{ccc} \mathcal{M}(k, n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \sum_{I \in \omega(k,n)} \det A_I e_I \end{array}$$

$$\mathrm{Pl}^s : \begin{array}{ccc} \mathrm{Gr}(k, n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [A]_\sim & \longmapsto & \left[\sum_{I \in \omega(k,n)} \det A_I e_I\right]_{\mathbb{K}^*} \end{array}$$

where we use the superscript s to distinguish these maps with the ones we will define for schemes.

3.1 Grassmannians as projective schemes

To connect Grassmannians to the world of representable functors we shall redefine them scheme-theoretically by emulating the construction from the previous chapter using rings and ring homomorphisms.

Definition 3.1 (Bracket ring)**.**
We define the **bracket ring** (see page 79 of [4]) as the ring of polynomial functions on $\bigwedge^k \mathbb{K}^n$, i.e.

$$\mathcal{B}_{k,n} \doteqdot \frac{\mathbb{K}[z_I \mid I \in \{1, \cdots, n\}^k]}{(\{z_I - \mathrm{sgn}\,(\sigma) z_{\sigma(I)}\}_{\sigma \in S_k})} \cong \mathbb{K}[z_I \mid I \in \omega(k, n)].$$

Definition 3.2 (Ring of generic matrices)**.**
Let $\mathbb{K}[X_{k,n}] \doteqdot \mathbb{K}[x_{1,1}, \cdots, x_{k,n}]$ denote the polynomial ring with $k \cdot n$ variables. We define the **generic matrix** by

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \cdots & x_{k,n} \end{pmatrix}$$

and by the same token we denote by X_I the generic $k \times k$ minor determined by the multiindex I and by $\det X_I$ the formal determinant of this minor.

Remark 3.3.
The ring $\mathbb{K}[X_{k,n}]$ is the coordinate ring of $\mathcal{M}(k,n)$.

Remark 3.4.
The familiar $\mathcal{M}(k,n)$ and $\bigwedge^k \mathbb{K}^n$ can be identified with the $\mathbb{K}$-points of the affine schemes $\operatorname{Spec} \mathbb{K}[X_{k,n}]$ and $\operatorname{Spec} \mathcal{B}_{k,n}$ respectively (Example 2.3.32 of [6]).

Definition 3.5 (Plücker ring homomorphism)**.**
We define the **Plücker ring homomorphism** or simply **Plücker homomorphism** as

$$\phi^{\#} : \begin{array}{ccc} \mathcal{B}_{k,n} & \longrightarrow & \mathbb{K}[X_{k,n}] \\ z_I & \longmapsto & \det X_I \end{array}$$

For brevity we will denote $\operatorname{Spec} \phi^{\#}$ by ϕ.
This definition is inspired by that of ϕ at page 79 of [4].

Proposition 3.6.
The kernel of the Plücker homomorphism is an homogeneous prime ideal which does not contain $(\{z_I\}_{I \in \omega(k,n)})$.

Proof.
Since $\mathbb{K}[X_{k,n}]$ is an integral domain, $\ker \phi^{\#}$ is prime.
By definition of homogeneous ideal, we want to show that if $f = \sum f_d$ for d homogeneous and $f \in \ker \phi^{\#}$ then $f_d \in \ker \phi^{\#}$ for all d.
Looking at the definition of $\phi^{\#}$ we see that $\phi^{\#}(f_d)$ is a homogeneous polynomial of degree kd, in particular if $d \neq h$ then $\deg \phi^{\#}(f_d) \neq \deg \phi^{\#}(f_h)$. Since

$$0 = \phi^{\#}(f) = \sum \phi^{\#}(f_d)$$

this proves that $\phi^{\#}(f_d) = 0$ for all d.
Finally, observe that $\deg \phi^{\#}(z_I) = \deg(\det X_I) = k > 0$, so $z_I \notin \ker \phi^{\#}$. $\square$

Proposition 3.7.
The induced map $\phi|_{\operatorname{Spec}(\mathbb{K}[X_{k,n}])(\mathbb{K})} : \operatorname{Spec}(\mathbb{K}[X_{k,n}])(\mathbb{K}) \to \operatorname{Spec}(\mathcal{B}_{k,n})(\mathbb{K})$ *is equal to* $\phi^s : \mathcal{M}(k,n) \to \bigwedge^k \mathbb{K}^n$ *under the identification mentioned above, i.e. for all matrices* $A \in \mathcal{M}(k,n)$ *with entries* $a_{i,j}$ *we have*

$$(\phi^{\#})^{-1}((x_{i,j} - a_{i,j})) = (z_I - \det A_I).$$

Proof.
First we observe that for any multiindex I

$$\det X_I - \det A_I \in (x_{i,j} - a_{i,j}),$$

thus $(z_I - \det A_I) \subseteq (\phi^\#)^{-1}((x_{i,j} - a_{i,j}))$.
Since $(z_I - \det A_I)$ is a $\mathbb{K}$-point, it is in particular a maximal ideal of the Bracket ring, thus we have the desired equality if $1 \notin (\phi^\#)^{-1}((x_{i,j} - a_{i,j}))$, which is the case because otherwise $(x_{i,j} - a_{i,j})$ would not be proper. $\square$

Proposition 3.8.
The $\mathbb{K}$-points of $V_+(\ker(\phi^\#))$ correspond to $\operatorname{Imm}\operatorname{Pl}^s$.

Proof.
First we note that

$$V_+(\ker\phi^\#) = \operatorname{Proj}\frac{\mathcal{B}_{k,n}}{\ker\phi^\#} \subseteq \operatorname{Proj}\mathcal{B}_{k,n}.$$

Since ϕ becomes ϕ^s on $\mathbb{K}$-points we see that

$$Z(\ker\phi^\#) = \overline{\operatorname{Imm}\ (\operatorname{Spec}\phi^\#)\,|_{\operatorname{Spec}\,(\mathbb{K}[X_{k,n}])(\mathbb{K})}} = \overline{\operatorname{Imm}\phi^s} \overset{\substack{\text{previous}\\ \text{chapter}}}{=} \operatorname{Imm}\phi^s.$$

It follows from Corollary 2.3.44 in [6] that the $\mathbb{K}$-points of $V_+(\ker\phi^\#)$ correspond to

$$Z_+(\ker\phi^\#) = \mathbb{P}(Z(\ker\phi^\#)) = \mathbb{P}(\operatorname{Imm}\phi^s) = \operatorname{Imm}\operatorname{Pl}^s.$$

$\square$

This result allows us to redefine the Grassmannian as a projective scheme. We can obtain the Plücker embedding of the classical Grassmannian back by looking at $\mathbb{K}$-points.

From now on $\operatorname{Gr}(k,n)$ will denote $V_+(\ker\phi^\#)$, while $\operatorname{Gr}(k,n)(\mathbb{K})$ will denote what we used to write as $\operatorname{Gr}(k,n)$.

3.1.1 Standard affine cover of the Grassmannian scheme

Recall that projective space admits a standard affine cover given by the locus of where one indeterminate does not vanish. In our case we see that

$$\operatorname{Proj}\mathcal{B}_{k,n} = \bigcup_{I\in\omega(k,n)} \operatorname{Spec}\left((\mathcal{B}_{k,n})^0_{z_I}\right) = \bigcup_{I\in\omega(k,n)} \operatorname{Spec}\left(\mathbb{K}\left[\frac{z_J}{z_I} \mid J\in\omega(k,n)\right]\right).$$

where the subscript denotes localization with multiplicative part $\{1, z_I, z_I^2, \cdots\}$ and the superscript 0 denotes the fact that we are only considering terms of degree 0 in this ring (this is the notation used in [6]).

This open affine cover of $\operatorname{Proj}\mathcal{B}_{k,n}$ induces an open cover on $\operatorname{Gr}(k,n)$ as follows:

$$\operatorname{Gr}(k,n) = V_+(\ker\phi^\#) = \bigcup_{I\in\omega(k,n)} \operatorname{Spec}\left(\left(\frac{\mathcal{B}_{k,n}}{\ker\phi^\#}\right)^0_{z_I}\right).$$

Notation 3.9.
Let us fix $I \in \omega(k,n)$, then we denote the localization of $\phi^\#$ as

$$\phi_I^\# : \begin{array}{ccc} \mathbb{K}\left[\frac{z_J}{z_I} \mid J\in\omega(k,n)\right] & \longrightarrow & \mathbb{K}[X]^0_{\det X_I} \\ \frac{z_J}{z_I} & \longmapsto & \frac{\det X_J}{\det X_I} \end{array}$$

Remark 3.10.
The image of $\phi_I^{\#}$ is

$$\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right],$$

thus by the first isomorphism theorem we have

$$\frac{(\mathcal{B}_{k,n})_{z_I}^0}{\ker \phi_I^{\#}} \cong \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right].$$

Remark 3.11.
The following equality holds by the properties of localization

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}}\right)_{z_I} = \frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}},$$

thus

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}}\right)_{z_I}^0 = \left(\frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}}\right)^0 = \frac{(\mathcal{B}_{k,n})_{z_I}^0}{\ker \phi_I^{\#}} \cong \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right]$$

Putting what we have said together, we have shown that up to some canonical identifications

$$\mathrm{Gr}(k,n) = \bigcup_{I\in\omega(k,n)} \mathrm{Spec}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right]\right) \doteqdot \bigcup_{I\in\omega(k,n)} \mathrm{Gr}_I(k,n).$$

Proposition 3.12.
$\mathrm{Gr}_I(k,n)$ *is isomorphic to* $\mathbb{A}_{\mathbb{K}}^{k(n-k)}$ *as a scheme.*

Proof.
Since they are both affine schemes, it is enough to show that their coordinate rings are isomorphic.
To simplify the notation we will set $w_J = \frac{\det X_J}{\det X_I}$ and if S is a multiindex we will write S_j^i for the multiindex where the i-th entry is substituted by $j \in \{1, \cdots, n\}$.

Without loss of generality we may assume that $I = (1, \cdots, k)$. An analogous argument will work for any choice of multiindex.
First we will prove that

$$\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right] = \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J = I_{\ell_j}^j,\ j \in \{1, \cdots, k\},\ \ell_j \notin I\right],$$

then we will show that the RHS (which we will denote R for brevity) is isomorphic to $\mathbb{K}[Y_{k,n-k}] = \mathbb{K}[y_{1,1}, \cdots, y_{k,n-k}]$.

- Let us consider the formal matrix

$$M = \begin{pmatrix} 1 & & & w_{I_{k+1}^1} & \cdots & w_{I_n^1} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & w_{I_{k+1}^k} & \cdots & w_{I_n^k} \end{pmatrix}$$

We can see that $M = (X_I)^{-1} X$, where $(X_I)^{-1}$ is the formal inverse of X_I, which exists because $\det X_I$ is invertible. More precisely

$$(X_I)^{-1} = \frac{1}{\det X_I} \mathrm{Adj}\,(X_I)$$

where $\mathrm{Adj}\,(X_I)$ is the adjugate matrix to X_I. The equality $M = (X_I)^{-1} X$ holds for the first k columns by definition of inverse, for the other columns we can see that they agree on every entry:

$$\begin{aligned}\frac{1}{\det X_I}(\mathrm{Adj}\,(X_I)X)_{i,j} =& \frac{1}{\det X_I} \sum_{\ell=1}^{k} \left((-1)^{i+\ell} \det\,(X_I)_{\times\ell,\times i}\right) x_{\ell,j} = \\ =& \frac{1}{\det X_I} \det X_{I_j^i} = \\ =& w_{I_j^i}\end{aligned}$$

We have thus proven that for any (k,n)-multiindex J we have

$$\det M_J = \det(X_I^{-1}X)_J = \frac{1}{\det X_I} \det X_J = w_J.$$

Since $\det M_J$ is a polynomial expression in the ring R by definition of M and the w_J are generators for $\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right]$, we have shown the nontrivial inclusion and thus equality.

- Let us consider the following ring homomorphism

$$\chi : \begin{array}{ccc} \mathbb{K}[Y_{k,n-k}] & \longrightarrow & R \\ y_{i,j} & \longmapsto & w_{I_j^i} \end{array}.$$

It is obviously surjective so we just need to show that it is injective to find the desired isomorphism.
Suppose by contradiction that there exists a nonzero polynomial $p \in \mathbb{K}[Y_{k,n-k}]$ which maps to 0. If $\overline{\mathbb{K}}$ is an algebraic closure[1] of $\mathbb{K}$ we can consider the lift $\widetilde{\chi} : \overline{\mathbb{K}}[Y_{k,n-k}] \to \widetilde{R} = \overline{\mathbb{K}}[w_{I_j^i}]$. Observe that if p maps to 0 according to the original map, then it would also map to 0 according to this extension. Consider now any matrix of the form

$$A = \left(I_k \mid \widetilde{A}\right) = (a_{i,j})_{i,j}$$

where I_k is the $k \times k$ identity matrix and $\widetilde{A} \in \mathcal{M}(k, n-k, \overline{\mathbb{K}})$. From what we have said above it follows that $\det A_{I_j^i} = a_{i,j}$, so

$$p(\widetilde{A}) = p\left(\left(\det A_{I_j^i}\right)_{\substack{i\in\{1,\cdots,k\},\\ j\in\{k+1,\cdots,n\}}}\right) = \widetilde{\chi}(p)(A) = 0.$$

We have shown that p has infinitely many roots, so if we fix the value of $k(n-k)-1$ indeterminates the resulting polynomial is the 0 polynomial. If we reiterate this reasoning we eventually prove that $p = 0$ in $\overline{\mathbb{K}}[Y_{k,n-k}]$, but $0 \in \mathbb{K}[Y_{k,n-k}] \subseteq \overline{\mathbb{K}}[Y_{k,n-k}]$, so p is the zero polynomial in the original ring, contradicting our hypothesis.

□

3.2 Grassmannian moduli functor

Let us consider the following functor

$$\mathbb{G}(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q\}/\sim \\ f : S \to T & \longmapsto & (\alpha : \mathcal{O}_T^n \to Q) \mapsto (f^*\alpha : \mathcal{O}_S^n \to f^*Q) \end{array}$$

[1] we can take any field extension $\mathbb{K} \subseteq \mathbb{F}$ where $\mathbb{F}$ is an infinite field.

where Q is a locally free sheaf of rank k on T and two surjections $\alpha : \mathcal{O}_T^n \twoheadrightarrow Q$, $\beta : \mathcal{O}_T^n \twoheadrightarrow V$ are equivalent if and only if there exist an isomorphism of sheaves $\theta : Q \to V$ such that the diagram commutes

$$\begin{tikzcd} \mathcal{O}_T^n \arrow[r, two heads, "\alpha"] \arrow[dr, "\beta"'] & Q \arrow[d, "\theta"] \\ & V \end{tikzcd}$$

We have functoriality because of the composition properties of pullbacks.

In this this section we will prove that the Grassmann scheme represents this functor.

3.2.1 Open subfunctor cover of the Grassmannian

Notation 3.13.
For any multiindex I and any scheme T we define the following morphism of sheaves

$$s_I^T : \begin{array}{ccc} \mathcal{O}_T^k & \longrightarrow & \mathcal{O}_T^n \\ e_j & \longmapsto & e_{i_j} \end{array} .$$

If there is no ambiguity on the scheme we will write s_I.

Definition 3.14 (Principal subfunctor of the Grassmannian)**.**
Fixed a multiindex $I \in \omega(k, n)$ we define the following association

$$\mathbb{G}_I(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \left\{ \mathcal{O}_T^n \overset{\alpha}{\twoheadrightarrow} Q \mid \alpha \circ s_I \text{ surjective} \right\}\big/\sim \\ f & \longmapsto & \alpha \mapsto f^*\alpha \end{array}$$

where the equivalence relation is the same as the one defined for $\mathbb{G}(k, n)$.

Remark 3.15.
The association $\mathbb{G}_I(k, n)$ is functor.

Proof.
First we observe that $\mathbb{G}_I(k, n)(T)$ is well defined because if $\psi = \theta \circ \alpha$ with θ isomorphism of sheaves then on each stalk we have

$$\psi_x \circ (s_I)_x = \theta_x \circ \varphi_x \circ (s_I)_x,$$

which is surjective if and only if $\varphi_x \circ (s_I)_x$ is surjective.

Consider now a map $f : S \to T$, then

$$f^*\alpha \circ s_I^S = f^*\alpha \circ f^* s_I^T = f^*(\alpha \circ s_I^T)$$

is surjective if and only if it is surjective on all stalks, i.e. if and only if for all $s \in S$ we have that the following map is surjective

$$f^*(\alpha \circ s_I^T)_s = (\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} \mathcal{O}_{S,s}$$

which is true because the tensor product is right-exact. $\square$

Lemma 3.16.
The support of a finite type quasicoherent sheaf $\mathcal{F}$ on a scheme X is a closed subset[2]*.*

[2] This statement is presented as Exercise 4.D. in [8]

Proof.
Since the support is a local notion and we can take an open affine cover of any scheme, we may assume $X = \operatorname{Spec} A$.
From the theory of quasicoherent sheaves on affine schemes we know that there exists a finitely generated A-module M such that $\mathcal{F} = \widetilde{M}$. Let $m_1, \cdots, m_k$ be generators for M, then

$$\operatorname{Supp} M = \bigcup_{i=1}^{k} \operatorname{Supp} m_i.$$

Since we have written $\operatorname{Supp} M$ as a finite union of sets, it is enough to show that $\operatorname{Supp} m$ is closed for all $m \in M$, which is true because

$$\operatorname{Supp} m = V(0 :_A m),$$

indeed

$$0 = m_{\mathfrak{p}} \iff \exists s \in A \setminus \mathfrak{p} \; s.t. \; sm = 0 \iff \exists t \in (0 :_A m) \setminus \mathfrak{p},$$

thus $\mathfrak{p} \in \operatorname{Supp} m$ if and only if $0 :_A m \subseteq \mathfrak{p}$, i.e. $\mathfrak{p} \in V(0 :_A m)$. □

Proposition 3.17.
The $\mathbb{G}_I(k,n)$ are open subfunctors of $\mathbb{G}(k,n)$.

Proof.
We will follow the approach showcased in [3].
The inclusion $\mathbb{G}_I(k,n)(T) \subseteq \mathbb{G}(k,n)(T)$ is apparent so we just need to show that if we fix a quotient $[\alpha : \mathcal{O}_T^n \twoheadrightarrow Q]$ in $\mathbb{G}(k,n)(T)$ then we can find an open subscheme of T which represents $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$.

Let us fix a representative α for the given quotient and define $\alpha \circ s_I : \mathcal{O}_T^k \to Q$. The locus where this map is surjective is the complement of the support of its cokernel sheaf $\mathcal{K}$, i.e. $\alpha \circ s_I$ is surjective on $\mathcal{O}_{T,x}^k$ if and only if $x \notin \operatorname{Supp} \mathcal{K}$. By lemma (3.16) the set U_I where $\alpha \circ s_I$ is surjective is an open subset of T ($\mathcal{K}$ is of finite type because locally it is the sheaf associated to the quotient of finite rank free modules).
We now want to show that U_I represents the functor $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$, that is we want to show that if $f : S \to T$ is a morphism of $\mathbb{K}$-schemes then f factors through U_I if and only if $f^*\alpha : \mathcal{O}_S^n \to f^*Q \in \operatorname{Gr}_I(S)$.

Note that $f(s) \in U_I$ if and only if $(\alpha \circ s_I^T)_{f(s)}$ is surjective which, by Nakayama's lemma applied to the cokernels, is equivalent to the surjectivity of

$$(\alpha \circ s_I^T)|_{f(s)} : k(f(s))^k \to Q_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} k(f(s)).$$

This is a linear map of vector spaces, so we can consider the pullback by the field extension $f_s^\# : k(f(s)) \to k(s)$ by keeping the same matrix. This operation preserves surjectivity and yields[3]

$$f^*(\alpha \circ s_I^T)|_s : k(s)^k \to Q_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} k(s).$$

By Nakayama's lemma we can again consider equivalently $f^*(\alpha \circ s_I^T)_s = (f^*\alpha)_s \circ (s_I^S)_s$. We have thus shown that $f(s) \in U_I$ if and only if $(f^*\alpha)_s \circ (s_I^S)_s$ is surjective, therefore f factors through U_I if and only if $(f^*\alpha) \circ s_I^S$ is surjective, i.e. $f^*\alpha \in \mathbb{G}_I(k,n)(S)$. □

[3] $f^*(\alpha \circ s_I^T)|_s = f^*(\alpha \circ s_I^T)_s \otimes_{\mathcal{O}_{S,s}} id_{k(s)} = ((\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{\mathcal{O}_{S,s}}) \otimes_{\mathcal{O}_{S,s}} id_{k(s)} = (\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{k(s)} = (\alpha \circ s_I^T)|_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{k(s)}$ where the last equality is simply emphasizing how the restriction of scalars $f_s^\# : k(f(s)) \to k(s)$ acts.

Proposition 3.18.
The collection $\{\mathbb{G}_I(k,n)\}$ is a Zariski open subfunctor cover of $\mathbb{G}(k,n)$.

Proof.
For any $\mathbb{K}$-scheme S and any quotient $[\alpha] \in \mathrm{Gr}(k,n)(S)$ (without loss of generality we choose a representative α) we need to show that for any $s \in S$ there exists a multiindex I such that $s \in U_I$ defined as in the previous proposition.
We are therefore looking for a multiindex I such that $(\alpha \circ s_I)_s$ is surjective. By Nakayama's lemma this is equivalent to showing that there exists and I such that

$$k(s)^k \xrightarrow{s_I} k(s)^n \xrightarrow{\alpha_s} Q_s \otimes_{\mathcal{O}_{S,s}} k(s),$$

which is trivially true since $\operatorname{rnk} \alpha_s = k$. $\square$

3.2.2 Representability of the Grassmannian functor

Proposition 3.19.
The Grassmannian functor $\mathbb{G}(k,n)$ is a Zariski sheaf.

Proof.
Consider a $\mathbb{K}$-scheme T and an open cover $\{U_i\}$. Consider now quotients $\alpha_i : \mathcal{O}_{U_i}^n \twoheadrightarrow Q_i$ such that

$$\alpha_i|_{U_i \cap U_j} \sim \alpha_j|_{U_i \cap U_j}.$$

By definition of $\sim$ there exist isomorphisms of sheaves $\varphi_{ji} : Q_i|_{U_i \cap U_j} \to Q_j|_{U_i \cap U_j}$. If we define $\varphi_{ii} = id_{Q_i}$ and fix the isomorphisms in such a way that $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ we have the data to glue the Q_i to a locally free sheaf of rank k over T, which we denote by Q. Now, up to isomorphism let us consider $\alpha_i : \mathcal{O}_{U_i}^n \twoheadrightarrow Q|_{U_i}$ for all i. If we fix any open set $V \subseteq T$ we see that if $s \in \mathcal{O}_T^n(V)$ is a section, we can define $\alpha_V(s)$ by gluing the $\alpha_i(s|_{U_i})$, which we can do by construction of Q and the choice of representative for the α_i. By construction $\alpha_{U_i} = \alpha_i$ and it is in fact the only such morphism, so we have verified the gluing property of sheaves for $\mathbb{G}(k,n)$. $\square$

Proposition 3.20.
The affine scheme $\mathrm{Gr}_I(k,n)$ represents the functor $\mathbb{G}_I(k,n)$.

Proof.
First we prove that for any $\mathbb{K}$-scheme $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathbb{G}_I(T)$, then we need to check that given a map $f : S \to T$ the pullbacks behave well.

- From our work in the first section of this chapter we can see that

  $$\mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right).$$

  Let us now consider the following maps

  $$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) & \longleftrightarrow & \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} \\ \varphi & \longmapsto & \eta(\varphi) \\ \rho(\alpha) : \frac{\det X_J}{\det X_I} \mapsto \frac{d(\alpha,J)}{d(\alpha,I)} & \longleftarrow & \alpha \end{array}$$

where $d(\alpha, L)$ is the determinant of the L minor of the matrix associated to α_T in the canonical basis and $\eta(\varphi)$ is defined on an open subset V of T as follows[4]:

$$\eta(\varphi)_V(e_j) = \sum_{i=1}^{k} (\text{res}_V^T \circ \varphi) \left(\frac{\det X_{I_j^i}}{\det X_I} \right) e_i.$$

An argument similar to the one presented in the proof of proposition (3.12) tells us that η and ρ are inverses.

Observe now that

$$\begin{array}{ccc} \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} & \longleftrightarrow & \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q \mid \alpha \circ s_I \text{ isomorphism}\}/\sim \\ \alpha & \longmapsto & [\alpha] \\ (\beta \circ s_I)^{-1} \circ \beta & \longleftarrow & [\beta] \end{array}$$

is a bijection. Indeed $(\beta \circ s_I)^{-1} \circ \beta \sim \beta$ by definition of $\sim$ and if $\theta : Q \to Q'$ is any isomorphism of sheaves then

$$(\theta \circ \alpha \circ s_I)^{-1} \circ \theta \circ \alpha = (\alpha \circ s_I)^{-1} \circ \theta^{-1} \circ \theta \circ \alpha = id_{\mathcal{O}_T^k} \circ \alpha = \alpha.$$

Finally, we see that

$$\left\{\alpha : \mathcal{O}_T^n \twoheadrightarrow \mathcal{O}_T^k \mid \alpha \circ s_I \text{ isomorphism}\right\}/\sim = \left\{\alpha : \mathcal{O}_T^n \twoheadrightarrow \mathcal{O}_T^k \mid \alpha \circ s_I \text{ surjective}\right\}/\sim$$

because on all stalks $\alpha \circ s_I$ is an endomorphism of finitely generated modules.

- Let $f : S \to T$ be a morphism of $\mathbb{K}$-schemes. Recall that

$$\begin{array}{ccc} \mathbb{G}_I(k,n)(T) & \longrightarrow & \mathbb{G}_I(k,n)(S) \\ [\alpha] & \longmapsto & [f^*\alpha] \end{array}.$$

Under the bijection presented, imposing naturality gives

$$\begin{array}{ccc} \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} & \longrightarrow & \left\{\beta : \mathcal{O}_S^n \to \mathcal{O}_S^k \mid \beta \circ s_I = id_{\mathcal{O}_S^k}\right\} \\ \alpha & \longmapsto & f^*\alpha \end{array}$$

since $f^*\alpha \circ s_I^S = f^*(\alpha \circ s_I^T) = f^*(id_{\mathcal{O}_T^k}) = id_{\mathcal{O}_S^k}$.

If we impose naturality again we get

$$\begin{array}{ccc} \text{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) & \longrightarrow & \text{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_S(S)\right) \\ \varphi & \longmapsto & \rho(f^*\eta(\varphi)) \end{array}$$

We claim that $\rho(f^*(\eta(\varphi))) = f^\#(T) \circ \varphi$. Since η is bijective and the inverse of ρ, it is enough to prove that $f^*(\eta(\varphi)) = \eta(f^\#(T) \circ \varphi)$. Since they are both sheaf morphisms on S it is enough to show that they induce the same morphism on all stalks. Let us fix $s \in S$ and $t = f(s) \in T$, then

$$\begin{aligned} f^*(\eta(\varphi))_s((e_j)_s) =& \eta(\varphi)_t((e_j)_t) \otimes_{\mathcal{O}_{T,t}} \mathcal{O}_{S,s} = \\ =& \sum_{i=1}^{k} \left(\varphi \left(\frac{\det X_{I_j^i}}{\det X_I} \right) \right)_t (e_i)_t \otimes_{\mathcal{O}_{T,t}} \mathcal{O}_{S,s} \overset{\text{[NOT SURE IF THIS WORKS]}}{=} \\ =& \sum_{i=1}^{k} f_s^\# \left(\left(\varphi \left(\frac{\det X_{I_j^i}}{\det X_I} \right) \right)_t \right) (e_i)_s = \\ =& \sum_{i=1}^{k} \left(f^\#(T) \circ \varphi \left(\frac{\det X_{I_j^i}}{\det X_I} \right) \right)_s (e_i)_s = \eta(f^\#(T) \circ \varphi)_s. \end{aligned}$$

[4] $\text{res}_V^T : \mathcal{O}_T(T) \to \mathcal{O}_T(V)$ denotes the restriction map given by the structure of sheaf.

We conclude by recalling that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \arrow[r, "\mathrm{Spec}"] \arrow[d, "\circ f"] & \mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) \arrow[d, "f^{\#}(T)\circ"'] \\
\mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}(S, \mathrm{Gr}_I(k,n)) \arrow[r, "\mathrm{Spec}"] & \mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_S(S)\right)
\end{tikzcd}$$

□

Theorem 3.21.
The Grassmann scheme $\mathrm{Gr}(k,n)$ *is a fine moduli space for the Grassmann functor* $\mathbb{G}(k,n)$.

Proof.
We know that $\mathbb{G}(k,n)$ is representable because of theorem (1.30), indeed we have shown that $\{\mathbb{G}_I(k,n)\}$ is an open subfunctor cover of $\mathbb{G}(k,n)$ (3.18) and that each $\mathbb{G}_I(k,n)$ is representable (3.20). Recall that in the proof of theorem (1.30) we constructed the fine moduli space of the functor by gluing fine moduli spaces of each open subfunctor in the cover. Since $\mathrm{Gr}(k,n) = \bigcup_{I\in\omega(k,n)} \mathrm{Gr}_I(k,n)$ and each $\mathrm{Gr}_I(k,n)$ represents $\mathbb{G}_I(k,n)$, the theorem follows if we can show that the gluing maps found in the proof of theorem (1.30) correspond to the identity of $\mathrm{Gr}_I(k,n)\cap\mathrm{Gr}_J(k,n)$ for any pair of ordered multiindicies.
Recall that the morphisms we are trying to identify correspond to the identity of $\mathbb{G}_I(k,n)\times_{\mathbb{G}(k,n)}\mathbb{G}_J(k,n)$. Note that

$$(\mathbb{G}_I(k,n)\times_{\mathbb{G}(k,n)}\mathbb{G}_J(k,n))(T) = {\{\alpha : \mathcal{O}_T^n \to Q \mid \alpha\circ s_I \text{ and } \alpha\circ s_J \text{ surjective}\}}\big/{\sim}$$

can be identified as we did while proving (3.20) with

$$\left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha\circ s_I = id_{\mathcal{O}_T^k},\ \alpha\circ s_J \text{ isomorphism}\right\},$$

which corresponds to

$$\left\{\beta \in \mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_L}{\det X_I}\right], \mathcal{O}_T(T)\right) \mid \beta\left(\frac{\det X_J}{\det X_I}\right) \neq 0\right\},$$

which in turn can be identified by the universal property of localization with

$$\mathrm{Hom}_{\mathbb{K}\text{-alg}}\left(\left(\mathbb{K}\left[\det X_L\right]_{\det X_I \det X_J}\right)^0, \mathcal{O}_T(T)\right).$$

Applying Spec this corresponds to

$$\mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}\left(T, \mathrm{Gr}(k,n)\cap D_+(z_I z_J)\right) = \mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}\left(T, \mathrm{Gr}_I(k,n)\cap\mathrm{Gr}_J(k,n)\right).$$

By functoriality of all the maps involved, we have shown that $\mathrm{Gr}_I(k,n)\cap\mathrm{Gr}_J(k,n)$ represents $\mathbb{G}_I(k,n)\times_{\mathbb{G}(k,n)}\mathbb{G}_J(k,n)$.
Following the isomorphisms involved we see that the identity of $\mathbb{G}_I(k,n)\times_{\mathbb{G}(k,n)}\mathbb{G}_J(k,n)$ translates to the identity of $h_{\mathrm{Gr}_I(k,n)\cap\mathrm{Gr}_J(k,n)}$, so the gluing maps are the image under the identity functor of the identity map in $h_{\mathrm{Gr}_I(k,n)\cap\mathrm{Gr}_J(k,n)}(\mathrm{Gr}_I(k,n)\cap\mathrm{Gr}_J(k,n))$ i.e. they are the identity on the intersections. □

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