Moduli Spaces and Grassmannians

Francesco Sorce

Università di Pisa
Dipartimento di Matematica

Abstract

In this document we introduce the concept of moduli spaces in algebraic geometry through the example of the Grassmannian scheme.

The first chapter introduces the basics of the functorial approach to algebraic geometry and its relation to moduli problems.

The second chapter is a quick overview of Grassmannians as defined set theoretically. We focus our attention on the Plücker embedding and prove that it identifies the Grassmannian with a projective variety.

In the third chapter we describe the reduced scheme structure on the Grassmannian and prove that it is a fine moduli space for the functor of quotients from $\mathcal{O}_T^n$ to a rank k vector bundle on T.

Contents

Introduction

The following type of *classification problem* occurs often in math:

> Consider some type of object and a notion of isomorphism which can be defined between them. We are interested in understanding the behaviour of isomorphism classes and how they relate to each other.

Finding a bijection between isomorphism classes and known objects is usually trivial[1], but for an answer to a classification problem to be satisfactory we usually require some information on *families* of isomorphism classes.

Miraculously, many such classification problems turn out to have a natural answer in the form of some geometric object. Usually the object can only be defined as the families themselves together with some geometric structure (this is the realm of the theory of stacks), but in more special circumstances one can find a more concrete space (usually a scheme) whose points represent isomorphism classes for our problem and whose geometric structure encodes information on the families. Such objects are called *moduli spaces* for the classification problem.

The best result we can hope for is finding a space which completely encodes how families behave[2], but this requirement is usually too strict. In this document we mostly deal with problems for which such a nice space exists: the Grassmannian and the Hilbert scheme.

Historical background

The history of moduli spaces begins with the article [6], where Riemann computes what we would now call the dimention of M_g, the moduli space of smooth projective algebraic curves of genus g, to be $3g-3$.

Although the argument given by Riemann can be made rigorous in modern language, he did not prove the existence of the space M_g itself. The first general construction of M_g as a space of some kind can be attributed to Teichmüller, which realized M_g as the quotient of the Teichmüller space T_g parametrizing complex structures up to isomorphism on a surface of genus g by the action of the group Γ_g of diffeomorphisms of the surface up to isotopy. The paper which establishes these ideas is [8].

Alexander Grothendieck introduced the functorial approach to analytic moduli theory and later on to algebraic geometry in general. Grothendieck was very interested

[1] for example, if the classes form a set they can be identified with a canonical set of the same cardinality.

[2] what will be formalized as a fine moduli space

in algebraic moduli theory and contributed to it greatly by introducing the Hilbert, Quot and Picard functors and showing their representability by schemes. However, Grothendieck did not end up publishing on M_g.

Among the first to study moduli spaces systematically was David Mumford. Inspired by invariant theory, Grothendieck's functorial approach and the existing constructions of moduli spaces like the one of principally polarized abelian varieties or the Chow varieties, Mumford developed Geometric Invariant Theory (commonly refered to as GIT), which can be described as a method to study and construct moduli spaces as quotients of algebraic groups. In the book [5] Mumford gives two constructions of M_g as a coarse moduli space.

Why category theory?

As we briefly mentioned, the modern approach to moduli problems if formalized via functors. It might not be clear why this is the most appropriate tool, and indeed it can seem more complicated than more concrete treatments in simple cases like the classification of lines through a point via projective space.

Nevertheless, the functorial approach has proven itself to be effective in many aspects, chief among them the formalization of the nebulous concept of "family" described above.

Following Grothendieck's ideas, a moduli problem is expressed as a contravariant functor

$$F : T \mapsto \{\text{families of objects over } T\}\big/\sim$$

where $\sim$ is the isomorphism relation imposed on families of objects.

Since we are mostly concerned about problems in algebraic geometry, and thus families over schemes, the functor is usually taken to be a presheaf on Sch/S for some base scheme S[3], i.e. $F : \mathrm{Sch}/S^{op} \to \mathrm{Set}$. To find the set of objects we want to classify up to isomorphism we can simply evaluate F on a point.

The functorial language allows for families to be pulled back via morphisms: if $f : S \to T$ is a morphism and $a \in F(T)$ is a family over T, then $F(f) : F(T) \to F(S)$ by contravariance and thus $F(f)(a) \doteqdot f^*a \in F(S)$ is a family over S.

There are several ways in which we can define a moduli space. The two most relevant are *fine* and *coarse* moduli spaces. A scheme M is a fine moduli space if we can recover the whole moduli functor from it[4]. M is a coarse moduli space if its $\mathbb{K}$-points are in bijection with $F(\operatorname{Spec}\mathbb{K})$ and if M is universal for this property.

In both cases we can interpret a family of objects over a scheme T as a morphism from T to M. Intuitively this is because a function from T to M is an assignment of an isomorphism class to each point of T, the added structure of a scheme morphism serves to define a "niceness" condition to the considered families. If M is a fine moduli space, then every family over T can be viewed as the pullback under a morphism $T \to M$ of a specific family $u \in F(M)$, called the *univeral family.*

[3]usually $\operatorname{Spec}\mathbb{K}$ for an algebraically closed field $\mathbb{K}$ or $\operatorname{Spec}\mathbb{Z}$.
[4]formally, when h_M and F are naturally isomorphic functors.

Chapter 1

Moduli Spaces

In this chapter we introduce the basic category theory used in the study of moduli spaces. After a quick review of the Yoneda embedding, we define representability of a functor and give the definition of fine and coarse moduli space. After that we give a quick overview of Zariski sheaves and prove representability results that we will need in the third chapter.

We adopt the following conventions:

- All categories considered in this document will be small.
- If $\mathcal{C}$ is a category, we shall write $X \in \mathcal{C}$ to mean "X is an object in $\mathcal{C}$".
- If $A, B \in \mathcal{C}$, we denote the set of morphisms from A to B with $\mathrm{Hom}\,(A, B)$ or $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for specificity.
- If A and B are R-modules we write $\mathrm{Hom}_{R}(A, B)$ instead of $\mathrm{Hom}_{R\text{-Mod}}(A, B)$.

Most definitions given in this chapter follow section 0.3 of [1].

1.1 Yoneda lemma

Definition 1.1 (Presheaf)**.** A contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is called a **presheaf** on $\mathcal{C}$. If $T \in \mathcal{C}$ then we call the elements of $F(T)$ **families** over T.

Definition 1.2 (Presheaf category)**.** For any fixed category $\mathcal{C}$, the presheaves on $\mathcal{C}$ form a category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ with morphisms given by natural transformations.

Definition 1.3 (Hom-functor)**.** Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. We define the **Hom-functor** of X to be

$$h_X : \begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \mathrm{Hom}\,(T, X) \\ f : T \to S & \longmapsto & h_X(f) : \begin{array}{ccc} \mathrm{Hom}\,(S, X) & \longrightarrow & \mathrm{Hom}\,(T, X) \\ g & \longmapsto & g \circ f \end{array} \end{array}$$

Remark 1.4. The Hom-functor is a presheaf.

Lemma 1.5 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. If F is a presheaf on $\mathcal{C}$ then the following sets are in a natural bijection*

$$\mathrm{Hom}\,(h_X, F) \longleftrightarrow F(X).$$

Proof. Given a natural trasformation ζ, we can take its image in $F(X)$ to be $\zeta_X(id_X)$. On the other hand, for any given element $u \in F(X)$ we can define an arrow $h_X(T) \to F(T)$ for any $T \in \mathcal{C}$ by taking $f \mapsto F(f)(u)$. This collection of maps defines a natural transformation from h_X to F because for all $g : S \to T$ and for all $f \in h_X(T)$

$$F(g)(F(f)(u)) = (F(g) \circ F(f))(u) = F(f \circ g)(u) = F(h_X(g)(f))(u).$$

To conclude it is enough to check that the two assignments are inverses:

$$F(f)(\zeta_X(id_X)) = \zeta_T(h_X(f)(id_X)) = \zeta_T(f), \quad F(id_X)(u) = u.$$

□

Definition 1.6 (Yoneda embedding)**.** We define the **Yoneda embedding** of a category $\mathcal{C}$ to be the following functor

$$h_\bullet : \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \\ X & \longmapsto & h_X \\ f : X \to Y & \longmapsto & h_f : h_X \to h_Y \end{array}$$

where if $g : T \to X$ then $h_f(g) = f \circ g : T \to Y$.

Proposition 1.7. *The functor $h_\bullet$ is fully faithful.*

Proof. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is fully faithful if for any two objects $A, B \in \mathcal{C}$ we have $\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$. In our case we want to verify that

$$\mathrm{Hom}\,(X, Y) \cong \mathrm{Hom}\,(h_X, h_Y),$$

which is exactly the statement of the Yoneda lemma (1.5) for $F = h_Y$. □

Proposition 1.8. *The Yoneda embedding is injective up to isomorphism on isomorphism classes of objects in $\mathcal{C}$.*

Proof. A natural isomorphism $\zeta : h_A \to h_B$ and its inverse ζ' correspond to maps $f : A \to B$ and $f' : B \to A$ via the Yoneda lemma. Note that

$$h_\bullet(f \circ f') = h_{f \circ f'} = h_f \circ h_{f'} = h_B(\cdot)(f) \circ h_A(\cdot)(f') \overset{\text{Yoneda}}{=} \zeta \circ \zeta' = id_{h_B},$$

thus, because $h_\bullet$ if fully faithful, we see that $f \circ f' = id_B$. An analogous argument works for $f' \circ f$. □

Lemma 1.9. *The Yoneda embedding preserves limits.*

Proof. Suppose X is the limit of the diagram $\{f_{ij} : X_j \to X_i\}$. If we apply the Yoneda embedding to the diagram we obtain

$$\left\{h_{f_{ij}} : h_{X_j} \to h_{X_i}\right\}$$

Let F be any presheaf on $\mathcal{C}$ and suppose that we have morphisms $F \to h_{X_i}$ which make the diagrams commute, then for all $T \in \mathcal{C}$ we have compatible and natural $F(T) \to \mathrm{Hom}\,(T, X_i)$. If $f \in F(T)$ then these arrows define several $f_i \in \mathrm{Hom}\,(T, X_i)$ which compose with the f_{ij} respecting the diagram. By the universal property of limits this defines uniquely a morphism $f_\ell \in \mathrm{Hom}\,(T, X)$ and we see that the assignment $f \mapsto f_\ell$ is the unique map from $F(T)$ to $\mathrm{Hom}\,(T, X)$ which makes the diagram in Set commute. Since all that we have done is natural in T, we have effectively constructed a morphism $F \to h_X$ as we desired. □

1.2 Moduli problems

Definition 1.10 (Representable functor)**.** A presheaf F on $\mathcal{C}$ is **representable** if there exists a natural isomorphism $\zeta : F \to h_X$ for some $X \in \mathcal{C}$. In this case we say that the pair[1] (X, ζ) **represents** F. If $a \in F(T)$ we call $\zeta_T(a) : T \to X$ the **classifying map** of the family a.

Definition 1.11 (Universal family)**.** Given a functor F and an object $X \in \mathcal{C}$ that represents it via the isomorphism $\zeta : F \to h_X$, the **universal family** of X is

$$\zeta_X^{-1}(id_X) \in F(X).$$

Remark 1.12*.* The universal family is the element of $F(X)$ which corresponds to ζ^{-1} under the Yoneda lemma (1.5).

We now specify our study to the category of schemes:

Definition 1.13 (Moduli problem)**.** Let S be a scheme. A presheaf on Sch/S is called a **moduli problem** or **moduli functor**.

Example 1.14 (Moduli problem of smooth curves of fixed genus)**.** A *family of smooth curves of genus* g over a scheme S is a smooth and proper scheme morphism $C \to S$ such that for all $s \in S$ the fiber C_s is a connected, smooth and proper curve of genus g. The moduli problem of smooth curves of genus g is the functor

$$\begin{array}{ccc} \mathrm{Sch}/\mathbb{C}^{op} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \{\text{families of smooth curves of genus } g \text{ over } S\}/\!\sim \\ T \to S & \longmapsto & (C \to S) \mapsto (C \times_S T \to T) \end{array}$$

where two families $C \to S$ and $C' \to S$ are equivalent if there exists an isomorphism between C and C' which is compatible with the structure maps over S.

Definition 1.15 (Fine moduli space)**.** Let F be a moduli functor. A scheme $X \in \mathrm{Sch}/S$ is a **fine moduli space** for F if X represents F.

Remark 1.16*.* Because the Yoneda embedding is injective on isomorphism classes up to isomorphism (1.8), fine moduli spaces are unique up to isomorphism.

Example 1.17 (Projective space)**.** Consider the functor

$$\mathbb{P}_n : \begin{array}{ccc} \mathrm{Sch}^{op} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \left\{ (\mathcal{L}, s_0, \cdots, s_n) \mid \begin{array}{c} \mathcal{L} \text{ line bundle on } S,\ s_0, \cdots, s_n \in \mathcal{L}(S), \\ \forall x \in S,\ \langle (s_0)_x, \cdots, (s_n)_x \rangle_{\mathcal{O}_{S,x}} = \mathcal{L}_x \end{array} \right\} /\!\sim \\ f & \longmapsto & \text{pullback of sheaves and sections via } f \end{array}$$

where $(\mathcal{L}, (s_i)) \sim (\mathcal{L}', (s_i'))$ is there exists a sheaf isomorphism $\alpha : \mathcal{L} \to \mathcal{L}'$ such that $s_i = \alpha^* s_i'$ for all $i \in \{0, \cdots, n\}$.

It is a well know fact (Proposition 5.1.31 in [4]) that $\mathbb{P}_n(S) \cong \mathrm{Hom}\,(S, \mathbb{P}^n_{\mathbb{Z}})$ and that pullbacks behave as expected, thus $\mathbb{P}^n_{\mathbb{Z}}$ is a fine moduli space for $\mathbb{P}_n$. From the statement of Proposition 5.1.31 in [4] it is also clear that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$ is a universal family.

Fine moduli spaces do not always exist. The simplest obstructions to having a fine moduli spaces are

[1] usually we just say that X represents F

- the functor is not a Zariski sheaf (see proposition (1.25))
- existence of non trivial automorphisms.

To get an idea for why the second condition is an obstruction we cite the following

Proposition 1.18. *Let $F \in (\mathrm{Sch}/\mathbb{C})^{op} \to \mathrm{Set}$ be a moduli functor. If there exists a variety $S \in \mathrm{Sch}/\mathbb{C}$ such that $\mathcal{E} \in F(S)$ is an **isotrivial family**, i.e.*

- *for all $s, t \in S(\mathbb{C})$, the fiber $F(s)(\mathcal{E}) = \mathcal{E}_s = \mathcal{E}_t$ and*
- *the family $\mathcal{E}$ is not the pullback of an object $E \in F(\operatorname{Spec}\mathbb{C})$ along the structural morphism $S \to \operatorname{Spec}\mathbb{C}$,*

then there exists no fine moduli space for F.

Proof. This is Proposition 0.3.21 in [1]. □

Remark 1.19. This proposition can be used to show that the moduli functor of smooth curves of fixed genus is not representable in general.

A weaker notion of moduli space is that of coarse moduli space:

Definition 1.20 (Coarse moduli space)**.** Let F be a moduli problem. A pair (X, ζ) for $X \in \mathrm{Sch}/S$ and $\zeta : F \to h_X$ natural transformation is a **coarse moduli space** for F if

- $\zeta_{\operatorname{Spec}\mathbb{K}} : F(\operatorname{Spec}\mathbb{K}) \to \operatorname{Hom}(\operatorname{Spec}\mathbb{K}, X)$ is a bijection for all algebraically closed fields $\mathbb{K}$
- for any scheme Y and $\eta : F \to h_Y$ natural transformation there exists a unique morphism $\alpha : X \to Y$ such that $\eta = h_\alpha \circ \zeta$.

Proposition 1.21. *A fine moduli space is also a coarse moduli space.*

Proof. The first condition is trivially verified. For the second condition, if (Y, η) is defined as above and (X, u) is the fine moduli space with universal family u then we can take $\alpha = \eta_X(u)$. □

1.3 Zariski sheaves and gluing of fine moduli spaces

One approach to show representability of a moduli problem is emulating the gluing properties of sheaves. Indeed it is possible to show that representable functors are sheaves of some kind. This realization will lead to some results that aid in showing representability.

1.3.1 Zariski sheaves

First, let us formalize a way in which a functor can be a sheaf. First we recall the definition of equalizer:

Definition 1.22 (Equalizer)**.** Let $\mathcal{C}$ be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \to C$. We say that the the diagram

$$A \xrightarrow{h} B \overset{f}{\underset{g}{\rightrightarrows}} C$$

is an **equalizer** if $h : A \to B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \to A$ which makes the diagram commute

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ Q \arrow[u, dashed] \arrow[ur, "q"'] & & \end{tikzcd}$$

Definition 1.23 (Zariski sheaf)**.** A moduli problem $F \in (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a **Zariski sheaf** if for any S-scheme X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Remark 1.24. Using the Yoneda lemma (1.5), we may equivalently consider

$$\mathrm{Hom}\,(h_X, F) \longrightarrow \prod_k \mathrm{Hom}\,(h_{U_k}, F) \rightrightarrows \prod_{i,j} \mathrm{Hom}\,(h_{U_i \cap U_j}, F)$$

Proposition 1.25 (Representable moduli functors are Zariski sheaves)**.** *Let* $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ *be a moduli problem, then if there exists a fine moduli space* M *for* F *it must be the case that* F *is a Zariski sheaf.*

Proof. Up to composing with the natural isomorphism, we may assume $F = h_M$. Let X be an S-scheme and $\{U_i \to X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\mathrm{Hom}\,(U, M) \xrightarrow{Res} \prod_i \mathrm{Hom}\,(U_i, M) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} \prod_{i,j} \mathrm{Hom}\,(U_i \cap U_j, M)$$

The arrows correspond to restriction of morphisms, so what we need to verify is that

- $\mathrm{res}^{U_i}_{U_i \cap U_j} \circ \mathrm{res}^X_{U_i} = \mathrm{res}^{U_j}_{U_i \cap U_j} \circ \mathrm{res}^X_{U_j}$ and that
- a collection of maps $\{f_i : U_i \to M\}$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ glues uniquely to a map $f : X \to M$.

Both propositions are well known properties of scheme morphisms. □

1.3.2 Open cover of a moduli problem

Definition 1.26 (Subfunctor)**.** A functor $G : \mathcal{C} \to \mathrm{Set}$ is a **subfunctor** of $F : \mathcal{C} \to \mathrm{Set}$ if for all $X, A, B \in \mathcal{C}$ and for all $f \in \mathrm{Hom}\,(A, B)$

$$G(X) \subseteq F(X), \qquad \text{and} \qquad G(f) = F(f)|_{G(A)}.$$

In this case we write $G \subseteq F$.

Remark 1.27. If F and G are presheaves and $f : A \to B$ then $G(f) = F(f)|_{G(B)}$.

Definition 1.28 (Fibered product of presheaves)**.** Let $F, G, H : \mathcal{C}^{op} \to \mathrm{Set}$ be presheaves together with two natural transformations $\eta : F \to H$ and $\zeta : G \to H$. We define their fibered product as the following functor

$$F \times_H G : \begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathrm{Set} \\ X & \longmapsto & F(X) \times_{H(X)} G(X) \\ f : A \to B & \longmapsto & (b_1, b_2) \mapsto (F(f)(b_1), G(f)(b_2)) \end{array}$$

where the fibered product $F(X) \times_{H(X)} G(X)$ in defined through the maps η_X and ζ_X. The map $(F \times_H G)(f)$ is well defined because if $(b_1, b_2) \in F(B) \times_{H(B)} G(B)$ then

$$\eta_A(F(f)(b_1)) = H(f)(\eta_B(b_1)) \overset{\eta_B(b_1)=\zeta_B(b_2)}{=} H(f)(\zeta_B(b_2)) = \zeta_A(G(f)(b_2)).$$

Definition 1.29 (Open subfunctor)**.** Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S-scheme T and any natural transformation $h_T \to F$, there exists an open subscheme $U \subseteq T$ such that

$$\begin{tikzcd} U \arrow[r, "h_\bullet"] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & h_U \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & G \arrow[d] \\ T \arrow[r, "h_\bullet"] & h_T \arrow[r] & F \end{tikzcd}$$

i.e. U represents $h_T \times_F G$ and the map from h_U to h_T is given by the inclusion.

Because of the Yoneda lemma, giving a natural transformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows

Definition 1.30 (Open subfunctor v.2)**.** A subfunctor $G \subseteq F$ is open if for any S-scheme T and any family $\xi \in F(T)$ there exists an open subscheme $\iota : U \hookrightarrow T$ such that the following diagram is natural in R for all $R \in \mathrm{Sch}/S$, commutes and is cartesian[2]

$$\begin{tikzcd} \mathrm{Hom}\,(R, U) \arrow[r, "G \circ h_\iota(\cdot)(\xi)"] \arrow[d, hook, "h_\iota"'] \arrow[dr, phantom, "\ulcorner"] & G(R) \arrow[d, hook, "\subseteq"] \\ \mathrm{Hom}\,(R, T) \arrow[r, "F(\cdot)(\xi)"'] & F(R) \end{tikzcd}$$

Definition 1.31 (Open cover of a functor)**.** Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. A collection of open subfunctors $\{F_i \to F\}$ is an **open cover** of F if for any S-scheme T and any natural transformation $h_T \to F$, the open subschemes U_i of T determined by the F_i form an open cover of T.

Definition 1.32 (Restriction of a family)**.** If U is a subscheme of T and $\iota : U \to T$ is the inclusion morphism, then if $\xi \in F(T)$ we define its **restriction** to U to be

$$\xi|_U = F(\iota)(\xi).$$

Remark 1.33. If $\{F_i \to F\}$ is an open cover of the functor F then for any S-scheme T and any family $\xi \in F(T)$ there exists an open cover $\{U_i \to T\}$ of T such that $\xi|_{U_i} \in F_i(U_i)$ for all i.

[2]for any map $f : R \to U$ there exists a $g : R \to U$ such that $f = \iota \circ g$ if and only if $F(f)(\xi) \in G(R)$.

1.3.3 Representability criterion

Finally, we come to the main results of this chapter

Proposition 1.34. *Let F and G be Zariski sheaves, $\{F_i \to F\}$ and $\{G_i \to G\}$ be open covers with the same indicization and $f_i : F_i \to G_i$ be natural transformations such that*[3] $f_i|_{F_i \cap F_j} = f_j|_{F_i \cap F_j}$*. Then there exists a natural transformation $f : F \to G$ which restricts to f_i on F_i.*

Proof. Let T be a scheme and $\zeta : h_T \to F$ a natural transformation. Let $\{\iota_i : U_i \to T\}$ be the open cover induced by $\{F_i \to F\}$ through ζ by the definition of open subfunctor cover.

$$\begin{tikzcd} h_{U_i} \arrow[r, "\eta_i"] \arrow[d, "h_{\iota_i}"'] \arrow[dr, phantom, "\ulcorner", very near start] & F_i \arrow[r, "f_i"] & G_i \\ h_T \arrow[r, "\zeta"'] & F \arrow[u, phantom, "\cap" description, sloped] & G \arrow[u, phantom, "\cap" description, sloped] \end{tikzcd}$$

where η_i is the map $\zeta \circ h_{\iota_i}$ with its codomain restricted. This map is well defined because the square is cartesian. Let $g_i = f_i \circ \eta_i$ and note that

$$g_i|_{h_{U_i \cap U_j}} = f_i|_{F_i \cap F_j} \circ \eta_i|_{h_{U_i \cap U_j}} = f_j|_{F_i \cap F_j} \circ \eta_j|_{h_{U_i \cap U_j}} = g_j|_{h_{U_i \cap U_j}}.$$

Because G is a Zariski sheaf, there exists $\zeta' : h_T \to G$ such that $\zeta' \circ h_{\iota_i} = g_i$. We have thus constructed a map $\mathrm{Hom}\,(h_T, F) \to \mathrm{Hom}\,(h_T, G)$ which is functorial in T by naturality of the maps involved. Applying the Yoneda lemma (1.5) gives a map $F(T) \to G(T)$ which is functorial in T, i.e. $f : F \to G$. By construction it is also clear that $f|_{F_i} = f_i$. ☐

Corollary 1.35. *With the same setup as above, if each f_i is an isomorphism then f too is an isomorphism.*

Proof. Let f be the morphism $F \to G$ obtained as above and let $g : G \to F$ be the morphism obtained the same way but by gluing the $f_i^{-1} : G_i \to F_i$. It is easy to see that f and g are inverses with a local argument. ☐

Theorem 1.36 (Representability by open cover)**.** *Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a Zariski sheaf and let $\{F_i \to F\}$ be an open cover of it by representable subfunctors, then F is representable.*

Sketch. We fix schemes X_i and families $\xi_i \in F_i(X_i)$ such that (X_i, ξ_i) is a fine moduli space for F_i. For all S-schemes T we have

$$(F_i \times_F F_j)(T) = F_i(T) \times_{F(T)} F_j(T) = F_i(T) \cap F_j(T) \subseteq F(T),$$

thus $F_i \times_F F_j = F_j \times_F F_i \doteqdot F_{i,j}$.

Using the openness of F_j we find $U_{ij} \subseteq X_i$ which represents $h_{X_i} \times_F F_j \cong F_{i,j}$. By uniqueness of moduli spaces we see that there exists an isomorphism $\varphi_{ji} : U_{ij} \to U_{ji}$, which we can choose to correspond to the identity $F_{i,j} = F_{j,i}$.

Our choice for the maps φ_{ji} makes the cocycle condition $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ hold trivially. We can thus glue the X_i to a scheme X. Since $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$ by construction of φ_{ji}, we find a family $\xi \in F(X)$ by the sheaf property of F. It follows easily that (X, ξ) represents F. ☐

[3] for a natural trasformation $\zeta : F \to H$ and a subfunctor $G \subseteq F$, we define $\zeta|_G$ as the natural transformation $G \to H$ given by $(\zeta|_G)_T = \zeta_T|_{G(T)}$. Naturality follows from the naturality of f and the definition of subfunctor.

Chapter 2

Grassmannians as projective varieties

In this chapter we introduce Grassmannians from the point of view of classical algebraic geometry. We are interested in Grassmannians in the context of classification problems because their definition leads us to suspect that they are a moduli space for certain families of vector spaces. In the next chapter we will indeed find that they are fine moduli spaces for a functor that formalizes *families of k-vector subspaces of* $\mathbb{K}^n$.

We first define Grassmannians set-theoretically, then we will find a bijection between this set and a Zariski-closed subset of some projective space. This bijection will allow us to endow the Grassmannians with the structure of projective algebraic varieties.

2.1 First definitions and conventions

Notation 2.1. In this chapter we use V and W to denote a fixed n-dimensional and a fixed k-dimensional $\mathbb{K}$-vector space resprectively. Unless otherwise stated, we understand $\mathcal{B} = \{v_1, \cdots, v_n\}$ to be a basis of V and $\mathcal{D} = \{w_1, \cdots, w_k\}$ to be a basis of W. We use u_i and q_i to indicate general elements of V and W respectively.

When a basis $\mathcal{F}$ for a vector space U of dimension ℓ is fixed, we denote the isomorphism which sends $\mathcal{F}$ to the canonical basis of $\mathbb{K}^\ell$ by $[\cdot]_\mathcal{F} : U \to \mathbb{K}^\ell$. We denote the canonical basis of $\mathbb{K}^\ell$ by $\mathcal{C}\mathrm{an}_\ell = \{e_1, \cdots, e_\ell\}$.

Definition 2.2 (Grassmannian)**.** Let $k \leq n$ be a pair of positive integers. We define the (n, k)**-Grassmannian** to be the following set

$$\mathrm{Gr}(k, V) = {}^{\displaystyle\{\varphi \in \mathrm{Hom}_\mathbb{K}(V, W) \mid \varphi \text{ surjective}\}}\!\Big/\!\sim$$

where $\varphi \sim \psi$ if and only if $\ker \varphi = \ker \psi$. To simplify notation we will usually write $\mathrm{Gr}(k, n)$.

Remark 2.3. We may equivalently define $\mathrm{Gr}(k, n)$ to be the following set:

$$\{\ker \varphi \mid \varphi \in \mathrm{Hom}_\mathbb{K}(V, W),\ \mathrm{rnk}\, \varphi = k\} = \{H \subseteq V \mid \dim H = n - k\}\,.$$

It is common in the literature to give this set the notation $\mathrm{Gr}(n - k, n)$ instead, but fixing a basis for V yields a bijection between $\mathrm{Gr}(k, n)$ and $\mathrm{Gr}(n - k, n)$, namely $H \mapsto H^\perp$.

Lemma 2.4. *Let $\varphi, \psi \in \operatorname{Hom}_{\mathbb{K}}(V, W)$ be linear maps of full rank. The following conditions are equivalent:*

1. $\ker \varphi = \ker \psi$,
2. *there exists $\theta \in \mathrm{GL}(W)$ such that $\varphi = \theta \circ \psi$.*

Proof. The implication 2. $\Longrightarrow$ 1. is a straight forward computation, the other can be derived by completing a basis of the kernels to a basis $\mathcal{B}$ of V and defining θ to be the change of basis between the images of $\mathcal{B}$ under φ and ψ. $\square$

We conclude this introductory section with some notation and conventions.

Definition 2.5 (Multiindicies)**.** We define a (k, n)**-multiindex** as an element of $\{1, \cdots, n\}^k$. Our notation for a multiindex I will usually be $I = (i_1, \cdots, i_k)$. We denote the set of **ordered** (k, n)**-multiindicies** by

$$\omega(k, n) = \left\{ (i_1, \cdots, i_k) \in \{1, \cdots, n\}^k \mid i_1 < \cdots < i_k \right\}.$$

If $I \in \omega(k, n)$, we write[1]

- $\widehat{I}$ for the element of $\omega(n-k, n)$ such that $I \cup \widehat{I} = \{1, \cdots, n\}$ and
- σ_I for the permutation that sends $\widehat{I} * I$ to $(1, \cdots, n)$.

Remark 2.6. If $I = (i_1, \cdots, i_k)$ is a (k, n)-multiindex and $u_1, \cdots, u_n$ are n vectors of V, we define

$$u_I = u_{i_1} \wedge \cdots \wedge u_{i_k}.$$

Note also that if $\mathcal{B}$ is a basis of V then

$$\{v_I \mid I \in \omega(k, n)\}$$

yields a basis of $\bigwedge^k V$, which we call the **basis induced by** $\mathcal{B}$ and denote by $\wedge^k \mathcal{B}$.

Notation 2.7. For $\mathcal{F}$ and $\mathcal{G}$ bases of U and Z respectively, we define

$$\eta_{\mathcal{F}} = [\cdot]_{\wedge^{\dim U} \mathcal{F}} : \bigwedge^{\dim U} U \to \mathbb{K}, \quad \eta_{\mathcal{G}}^{\mathcal{F}} = \eta_{\mathcal{G}}^{-1} \circ \eta_{\mathcal{F}} : \bigwedge^{\dim Z} Z \to \bigwedge^{\dim U} U.$$

2.2 The Plücker embedding

In this section we define an injection from the Grassmannian to a projective space. Our approach differs slightly from the usual one[2] because we consider equivalence classes of maps rather than equivalence classes of bases of subspaces.

Definition 2.8 (Plücker map)**.** Let $k \leq n$ be a pair of positive integers. We define the **Plücker map** as[3]

$$\wedge^k : \begin{array}{ccc} \operatorname{Hom}_{\mathbb{K}}(V, W) & \longrightarrow & \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k V, \bigwedge^k W) \\ \varphi & \longmapsto & \wedge^k \varphi \end{array},$$

where $(\wedge^k \varphi)(u_1 \wedge \cdots \wedge u_k) = \varphi(u_1) \wedge \cdots \wedge \varphi(u_k)$.

[1] $\cup$ and $*$ denote the union of the underlying sets and concatenation respectively.

[2] briefly illustrated in [2], pages 79 and 80

[3] the map $\wedge^k \varphi$ is well defined because if we view it as a map $\wedge^k \varphi : V^{\times k} \to \bigwedge^k W$ then it is multilinear and alternating.

Remark 2.9. The codomain of the Plücker map is isomorphic to $\bigwedge^k V$, indeed

$$\operatorname{Hom}_{\mathbb{K}}\left(\bigwedge^k V, \bigwedge^k W\right) \cong \left(\bigwedge^k V\right)^{\vee} \cong \bigwedge^k V.$$

For fixed bases of V and W we can write one such isomorphism concretely as

$$\zeta_{\mathcal{B},\mathcal{D}} : \begin{array}{ccc} \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k V, \bigwedge^k W) & \longrightarrow & \bigwedge^k V \\ \psi & \longmapsto & \displaystyle\sum_{I \in \omega(k,n)} \eta_{\mathcal{D}}(\psi(v_I)) v_I \end{array} .$$

When the bases are clear from context we simply write ζ.

Notation 2.10. We define $\phi_{\mathcal{B},\mathcal{D}} \doteqdot \zeta_{\mathcal{B},\mathcal{D}} \circ \wedge^k : \operatorname{Hom}_{\mathbb{K}}(V, W) \to \bigwedge^k V$

Proposition 2.11. *The image of the Plücker map is a cone.*

Proof. We have $\lambda \wedge^k \varphi = \wedge^k(\alpha \circ \varphi)$ for any $\alpha \in \mathrm{GL}(W)$ with determinant λ. □

Lemma 2.12. *If* $\varphi \in \operatorname{Hom}_{\mathbb{K}}(V, W)$ *then* $\operatorname{rnk} \varphi < k$ *if and only if* $\wedge^k(\varphi) = 0$.

Proof. $\wedge^k(\varphi)$ is the zero map if an only if the set $\{\varphi(u_1), \cdots, \varphi(u_k)\}$ is linearly dependent for any choice of $u_1, \cdots, u_k$, i.e. φ is not of full rank. □

Lemma 2.13. *Let* $\varphi : V \to W$ *be a full rank linear map, then*

$$\ker \varphi = \left\{ z \in V \mid \forall u_2, \cdots, u_k \in V,\ \wedge^k(\varphi)(z \wedge u_2 \wedge \cdots \wedge u_k) = 0 \right\}.$$

Proof. The inclusion $\subseteq$ is trivial. If $\varphi(z) \neq 0$ we can find $k-1$ vectors of the desired form by completing $\varphi(z)$ to a basis $\varphi(z), q_2, \cdots, q_k$ of W and then taking u_i to be any element of $\varphi^{-1}(q_i)$. This preimage is not empty by surjectivity of φ. □

Proposition 2.14 (Injectivity of the Plücker map up to scalars)**.** *Let* $\sim$ *be the equivalence relation on* $\operatorname{Hom}_{\mathbb{K}}(V, W)$ *which defines* $\mathrm{Gr}(k, n)$*, then for any two full rank linear maps* $\varphi, \psi : V \to W$

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{K}^* \text{ s.t. } \wedge^k(\varphi) = \lambda \wedge^k(\psi).$$

Proof. We prove both implications:

$\boxed{\Longrightarrow}$ By lemma (2.4), if $\varphi \sim \psi$ then there exists $\theta \in \mathrm{GL}(W)$ such that $\varphi = \theta \circ \psi$, thus

$$\wedge^k(\varphi) = \wedge^k(\theta \circ \psi) = (\det \theta) \wedge^k(\psi).$$

$\boxed{\Longleftarrow}$ It is enough to apply lemma (2.13) as follows:

$$\begin{aligned} \ker \varphi &= \left\{ z \in V \mid \forall u_2, \cdots, u_k \in V,\ \wedge^k(\varphi)(z \wedge u_2 \wedge \cdots \wedge u_k) = 0 \right\} = \\ &= \left\{ z \in V \mid \forall u_2, \cdots, u_k \in V,\ \lambda \wedge^k(\psi)(z \wedge u_2 \wedge \cdots \wedge u_k) = 0 \right\} = \\ &= \left\{ z \in V \mid \forall u_2, \cdots, u_k \in V,\ \wedge^k(\psi)(z \wedge u_2 \wedge \cdots \wedge u_k) = 0 \right\} = \ker \psi. \end{aligned}$$

□

Definition 2.15 (Plücker embedding)**.** Let us fix bases $\mathcal{B}$ and $\mathcal{D}$ of V and W. We define the **Plücker embedding** as follows

$$\mathrm{Pl}_{\mathcal{B}} : \begin{array}{ccc} \mathrm{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k V) \\ [\varphi] & \longmapsto & \left[\displaystyle\sum_{I \in \omega(k,n)} \eta_{\mathcal{D}}(\wedge^k \varphi(v_I)) v_I \right] \end{array}$$

The entries of the homogeneous $\binom{n}{k}$-tuple $\left[\left\{ \eta_{\mathcal{D}}(\wedge^k(\varphi(v_I))) \right\}_{I \in \omega(k,n)} \right]$ are called the **Plücker coordinates** of $[\varphi]$.

Remark 2.16. If the Plücker embedding is well defined, it does not depend on the choice of basis for W. Indeed changing the basis of W simply multiplies all Plücker coordinates by the same nonzero scalar[4], so the resulting point in $\mathbb{P}(\bigwedge^k V)$ is left unchanged.

Proposition 2.17. *The Plücker embedding is well defined and injective.*

Proof. Because of proposition (2.14) and lemma (2.12), there exists a unique map p such that the diagram commutes

$$\begin{tikzcd} \{\varphi \in \mathrm{Hom}_{\mathbb{K}}(V,W) \mid \mathrm{rnk}\,\varphi = k\} \arrow[r, "\wedge^k"] \arrow[d, "\pi_\sim"] & \bigwedge^k \mathrm{Hom}_{\mathbb{K}}(V,W) \setminus \{0\} \arrow[d, "\mathbb{P}"] \\ \mathrm{Gr}(k,n) \arrow[r, dashed, "p"'] & \mathbb{P}(\bigwedge^k \mathrm{Hom}_{\mathbb{K}}(V,W)) \end{tikzcd}$$

It follows that $\mathrm{Pl}_{\mathcal{B}}$ is well defined because $\mathrm{Pl}_{\mathcal{B}} = \mathbb{P}(\zeta_{\mathcal{B},\mathcal{D}}) \circ p$.

By proposition (2.14) we have that p is injective, so $\mathrm{Pl}_{\mathcal{B}}$ must also be injective because $\zeta_{\mathcal{B},\mathcal{D}}$ is an isomorphism. □

Remark 2.18. $\mathrm{Pl}_{\mathcal{B}} \circ \pi_\sim = \mathbb{P}(\zeta_{\mathcal{B},\mathcal{D}} \circ \wedge^k) = \mathbb{P}(\phi_{\mathcal{B},\mathcal{D}})$.

2.2.1 Matrix form

Notation 2.19. If A is a $k \times n$ matrix and I is a (k,n)-multiindex, we denote the **I-minor of** A by A_I, i.e.

$$A_I = \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{pmatrix}.$$

If B is an $\alpha \times \beta$ matrix, $i \in \{1,\cdots,\alpha\}$ and $j \in \{1,\cdots,\beta\}$, we denote the $(\alpha-1)\times(\beta-1)$ matrix obtained from B by deleting the i-th row and the j-th column with $B_{\times i,\times j}$.

If we fix bases $\mathcal{B}$ for V and $\mathcal{D}$ for W we can identify V with $\mathbb{K}^n$, W with $\mathbb{K}^k$ and $\mathrm{Hom}_{\mathbb{K}}(V,W)$ with $\mathcal{M}(k,n)$. Under these identifications we have

$$\mathrm{Gr}(k,n) = \{A \in \mathcal{M}(k,n) \mid \mathrm{rnk}\,A = k\} / \sim,$$

where $A \sim B \iff \exists P \in \mathrm{GL}_k$ such that $A = PB$.

[4] the determinant of the change of basis

Because $\wedge^k\varphi(u_1 \wedge \cdots \wedge u_k) = \det\left([\varphi(u_1)]_{\mathcal{D}}|\cdots|[\varphi(u_k)]_{\mathcal{D}}\right) w_{(1,\cdots,k)}$ we have

$$\phi : \begin{array}{ccc} \mathcal{M}(k,n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \displaystyle\sum_{I\in\omega(k,n)} \det A_I e_I \end{array}$$

$$\text{Pl} : \begin{array}{ccccc} \text{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) & = & \mathbb{P}^{\binom{n}{k}-1} \\ [A]_\sim & \longmapsto & \left[\displaystyle\sum_{I\in\omega(k,n)} \det A_I e_I\right]_{\mathbb{K}^*} & = & \left[\{\det A_I\}_{I\in\omega(k,n)}\right]_{\mathbb{K}^*} \end{array}$$

2.3 The image of the Plücker embedding is closed

Thus far we have identified $\text{Gr}(k,n)$ with a subset of some projective space. We seek to show that this subset is closed in the Zariski topology.

2.3.1 Some linear algebra results

Definition 2.20 (Divisibility). We say that $\omega \in \bigwedge^k V$ is **divisible** by $v \in V$ if there exists $\varepsilon \in \bigwedge^{k-1} V$ such that $\omega = \varepsilon \wedge v$.

Lemma 2.21. *$\omega \in \bigwedge^k V$ is divisible by $v \in V \setminus \{0\}$ if and only if $\omega \wedge v = 0$.*

Proof. If $\omega = \varepsilon \wedge v$ then $\omega \wedge v = \varepsilon \wedge v \wedge v = 0$. If $\omega \wedge v = 0$ then by writing ω in a base containing v we can see that the simple multivectors with nonzero coefficients must contain v as a factor, so we can factor out v by multilinearity and get a decomposition of the form $\omega = \varepsilon \wedge v$. □

Corollary 2.22 (Total decomposability criterion). *Let $\omega \in \bigwedge^k V$ and define*

$$D_\omega = \{v \in V \mid \omega \wedge v = 0\}.$$

If $\dim D_\omega \geq k$ then $\omega = \lambda v_1 \wedge \cdots \wedge v_k$ for any set of linearly independent vectors $\{v_1, \cdots, v_k\}$ in D_ω and some scalar λ. Moreover $\lambda \neq 0$ if and only if $\dim D_\omega = k$.

Proof. For the first part of the result we may just iterate the above lemma. If $\lambda = 0$ then $D_\omega = V$, so its dimension is not k. If the dimension is greater than k then we may subtract two total decompositions differing only by one vector and use linear independence to check that the coefficients must have been zero. □

Proposition 2.23. *There is a canonical isomorphism between $\text{Hom}_{\mathbb{K}}(\bigwedge^k V, \bigwedge^n V)$ and $\bigwedge^{n-k} V$ given by*

$$\Xi : \begin{array}{ccc} \bigwedge^{n-k} V & \longrightarrow & \text{Hom}_{\mathbb{K}}(\bigwedge^k V, \bigwedge^n V) \\ \omega & \longmapsto & \omega \wedge \cdot \end{array}$$

For any basis $\mathcal{B}$ of V, the inverse of Ξ is given by

$$\Gamma_{\mathcal{B}} : \begin{array}{ccc} \text{Hom}_{\mathbb{K}}(\bigwedge^k V, \bigwedge^n V) & \longrightarrow & \bigwedge^{n-k} V \\ \psi & \longmapsto & \displaystyle\sum_{I\in\omega(n-k,n)} \operatorname{sgn}\sigma_I \eta_{\mathcal{B}}(\psi(v_{\widehat{I}}))v_I \end{array}$$

Proof. Ξ is clearly base independent and linear. Concluding from here is simply a matter of computing $\Gamma_{\mathcal{B}}(\Xi(\omega))$ by writing ω in terms of its coordinates in $\wedge^k\mathcal{B}$ and verifying that $\Xi(\Gamma_{\mathcal{B}}(\psi))$ and ψ agree on $\wedge^k\mathcal{B}$. □

Corollary 2.24. *Let $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k V, \bigwedge^k W)$. If $\mathcal{B} = \{v_1, \cdots, v_n\}$ and $\mathcal{B}' = \{v'_1, \cdots, v'_n\}$ are bases for V and $\mathcal{D} = \{w_1, \cdots, w_k\}$ and $\mathcal{D}' = \{w'_1, \cdots, w'_k\}$ are bases for W, there exists $\mu \in \mathbb{K}^*$ such that*

$$\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(v_{\widehat{I}}))v_I = \mu \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}'}(\psi(v'_{\widehat{I}}))v'_I.$$

Proof. Note that

$$\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(v_{\widehat{I}}))v_I = \Xi^{-1}(\eta^{\mathcal{D}}_{\mathcal{B}} \circ \psi)$$

and similarly the other expression is $\Xi^{-1}(\eta^{\mathcal{D}'}_{\mathcal{B}'} \circ \psi)$. It is therefore enough to show that $\eta^{\mathcal{B}}_{\mathcal{D}} = \mu \eta^{\mathcal{B}'}_{\mathcal{D}'}$ for some $\mu \in \mathbb{K}^*$, which is true because $\dim_{\mathbb{K}} \operatorname{Hom}_{\mathbb{K}}(\bigwedge^n V, \bigwedge^k W) = 1$ and both $\eta^{\mathcal{B}}_{\mathcal{D}}$ and $\eta^{\mathcal{B}'}_{\mathcal{D}'}$ are not the zero map. □

2.3.2 Rank condition for the image

Lemma 2.25. *Fix bases $\mathcal{B}$ of V and $\mathcal{D}$ of W. A multilinear alternating form $\psi \in \operatorname{Hom}_{\mathbb{K}}(\bigwedge^k V, \bigwedge^k W)$ is in the image of the Plücker map $\wedge^k$ if and only if there exists $\lambda \in \mathbb{K}$ and linearly independent vectors $z_1, \cdots, z_{n-k}$ such that*

$$\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(v_{\widehat{I}}))v_I = \lambda z_{(1,\cdots,n-k)}.$$

Proof. We show both implications

$\boxed{\Longrightarrow}$ If $\psi = \wedge^k \varphi$, the equality follows by choosing $z_1, \cdots, z_{n-k}$ to be a basis of $\ker \varphi$. Completing this set to a basis of V and using corollary (2.24) gives the result after a simple calculation.

$\boxed{\Longleftarrow}$ Let $\mathcal{Z} = \{z_1, \cdots, z_n\}$ be a basis of V which extends the given $z_1, \cdots, z_{n-k}$. We can take φ to be

$$\varphi(z_i) = \begin{cases} 0 & \text{if } 1 \le i \le n-k \\ \left(\mu\lambda \operatorname{sgn} \sigma_{(1,\cdots,n-k)}\right) w_1 & \text{if } i = n-k+1 \\ w_{i-n+k} & \text{if } i > n-k+1 \end{cases}$$

where $\mu \in \mathbb{K}^*$ is such that $\eta^{\mathcal{B}}_{\mathcal{D}} = \mu \eta^{\mathcal{Z}}_{\mathcal{D}}$.

□

Definition 2.26. Let $\mathcal{B}$ be a basis of V. If $\omega = \sum_{J \in \omega(k,n)} p_J v_J$ we define

$$\Phi_{\mathcal{B}}(\omega) : \begin{array}{ccc} V & \longrightarrow & \bigwedge^{n-k+1} V \\ v & \longmapsto & \displaystyle\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I p_{\widehat{I}} v_I \wedge v \end{array}.$$

Remark 2.27. For any basis $\mathcal{D}$ of W we have $\Phi_{\mathcal{B}}(\omega)(v) = \Xi^{-1}(\eta^{\mathcal{D}}_{\mathcal{B}} \circ \zeta^{-1}_{\mathcal{B},\mathcal{D}}(\omega)) \wedge v$.

Proposition 2.28. *A k-multivector $\omega \in \bigwedge^k V$ is in the image of $\phi_{\mathcal{B},\mathcal{D}}$ if and only if $\Phi_{\mathcal{B}}(\omega)$ has rank at most k.*

Proof. $\omega \in \operatorname{Imm} \phi_{\mathcal{B},\mathcal{D}}$ if and only if $\zeta_{\mathcal{B},\mathcal{D}}^{-1}(\omega) \in \operatorname{Imm} \wedge^k$ by definition, so what we want to show is that $\psi \in \operatorname{Imm} \wedge^k$ if and only if the rank of the map

$$\Upsilon_{\mathcal{B},\mathcal{D}}(\psi) : v \mapsto \Xi^{-1}(\eta_{\mathcal{B}}^{\mathcal{D}} \circ \psi) \wedge v = \sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(v_{\hat{I}}))v_I \wedge v$$

is at most k.

For the $\implies$ arrow, let $\psi = \wedge^k \varphi$ and choose a basis $\mathcal{Z} = \{z_1, \cdots, z_n\}$ for V which extends a basis of $\ker \varphi$. Because of how we proved lemma (2.25), we see that if $v \in \ker \varphi$ then $\Upsilon_{\mathcal{B},\mathcal{D}}(\wedge^k \varphi)(v) = \lambda z_{(1,\cdots,n-k)} \wedge v$, which is zero by linear dependence. Thus the nullity of $\Upsilon_{\mathcal{B},\mathcal{D}}(\wedge^k \varphi)$ is at least $\dim \ker \varphi = n - k$.

Given $z_1, \cdots, z_{n-k}$ linearly independent vectors in $\ker \Upsilon_{\mathcal{B},\mathcal{D}}(\psi)$, by the total decomposability criterion (2.22) there exists $\lambda \in \mathbb{K}$ such that

$$\sum_{I \in \omega(n-k,n)} \operatorname{sgn} \sigma_I \eta_{\mathcal{D}}(\psi(v_{\hat{I}}))v_I = \lambda z_1 \wedge \cdots \wedge z_{n-k}.$$

This concludes by lemma (2.25). □

Theorem 2.29. *The image of $\phi_{\mathcal{B},\mathcal{D}}$ is a Zariski closed subset of $\bigwedge^k V$.*

Proof. We seek to translate the rank condition (2.28) into equations on the coordinates of $\bigwedge^k V$. Let $\mathbb{K}[z_I \mid I \in \omega(k,n)]$ be the coordinate ring of $\bigwedge^k \mathbb{K}^n$. If $B^I \in \mathcal{M}\left(\binom{n}{n-k+1}, n, \mathbb{K}\right)$ is the matrix which represents $\Phi_{\mathcal{B}}(v_I)$ in the bases induced by $\mathcal{B}$ and $\mathcal{D}$ then we define

$$M_{\mathcal{B},\mathcal{D}} = \sum_{I,\omega(k,n)} B^I z_I = \left(\sum_{I \in \omega(k,n)} (B^I)_{i,j} z_I \right)_{i,j}.$$

This matrix represents $\Phi_{\mathcal{B}}$ in the following way: if $\omega = \sum_{I \in \omega(k,n)} p_I v_I$,

$$\Phi_{\mathcal{B}}(\omega)(v) = \sum_{I \in \omega(k,n)} p_I \Phi_{\mathcal{B}}(v_I)(v) = \sum_{I \in \omega(k,n)} p_I B^I v = M_{\mathcal{B}}|_{z_I = p_I} v.$$

It follows that the coordinates of the k-multivectors in the image of $\phi_{\mathcal{B},\mathcal{D}c}$ are exactly those that satisfy the determinantal criterion for the rank being at most k, i.e.

$$\begin{aligned}\operatorname{Imm} \phi_{\mathcal{B},\mathcal{D}} = &\left\{ \sum_{I \in \omega(k,n)} p_I v_I \mid \operatorname{rnk}\ M_{\mathcal{B},\mathcal{D}}|_{z_I = p_I} < k+1 \right\} = \\ =&V(\{\det m \mid m \text{ is a } (k+1) \times (k+1) \text{ minor of } M\}).\end{aligned}$$

□

Corollary 2.30. $\mathrm{Pl}_{\mathcal{B}}$ *endows* $\mathrm{Gr}(k,n)$ *with the structure of a projective variety.*

Proof. Since $\operatorname{Imm} \phi_{\mathcal{B},\mathcal{D}}$ is a cone (2.11) and Zariski closed we see that $\mathbb{P}(\operatorname{Imm} \phi_{\mathcal{B},\mathcal{D}}) = \operatorname{Imm} \mathrm{Pl}_{\mathcal{B}}$ is Zariski closed. We conclude by recalling that $\operatorname{Imm} \mathrm{Pl}_{\mathcal{B}}$ is in bijection with $\mathrm{Gr}(k,n)$ by injectivity of $\mathrm{Pl}_{\mathcal{B}}$ (2.17). □

Remark 2.31. The determinants we used to show that the image of the Plücker embedding is closed do not generate the ideal of that variety. The most well known set of generators for that ideal are the **Plücker relations** (Theorem 2.4.3 in [2], page 80).

Chapter 3

Representability of the Grassmannian functor

In this chapter we will work with $\mathbb{K}^n$ and $\mathbb{K}^k$ instead of abstract vector spaces. This means that we have canonical bases $\mathcal{C}\mathrm{an}_n = \{e_1, \cdots, e_n\}$ and $\mathcal{C}\mathrm{an}_k = \{e_1, \cdots, e_k\}$ and that we identify $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ with $\mathcal{M}(k, n)$.

To differentiate the scheme morphisms we define in this chapter from the morphisms of varieties defined previously we use a superscript s for the latter, i.e.

$$\phi^s : \begin{array}{ccc} \mathcal{M}(k,n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \displaystyle\sum_{I \in \omega(k,n)} \det A_I e_I \end{array} , \quad \mathrm{Pl}^s : \begin{array}{ccc} \mathrm{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [A]_\sim & \longmapsto & [\phi^s(A)]_{\mathbb{K}^*} \end{array}$$

Notation 3.1. Let I be an ideal of the ring A and J be a homogeneous ideal of the graded ring B. We adopt the following notation

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid I \subseteq \mathfrak{p}\}, \qquad V_+(J) = \{\mathfrak{p} \in \mathrm{Proj}\, B \mid I \subseteq \mathfrak{p}\}.$$

If the sets above are considered as closed subschemes we take the reduced structure.

3.1 Grassmannians as projective schemes

Definition 3.2 (Bracket ring)**.** We define the **bracket ring** (see page 79 of [2]) as the ring of polynomial functions on $\bigwedge^k \mathbb{K}^n$, i.e.

$$\mathcal{B}_{k,n} \doteqdot \frac{\mathbb{K}[z_I \mid I \in \{1, \cdots, n\}^k]}{(\{z_I - \mathrm{sgn}\,(\sigma) z_{\sigma(I)}\}_{\sigma \in S_k})} \cong \mathbb{K}[z_I \mid I \in \omega(k,n)].$$

We define $\mathcal{B}^+_{k,n}$ to be the ideal generated by the indeterminates z_I.

Definition 3.3 (Ring of generic matrices)**.** Let $\mathbb{K}[X_{k,n}] \doteqdot \mathbb{K}[x_{1,1}, \cdots, x_{k,n}]$ denote the polynomial ring with $k \cdot n$ variables. We define the **generic matrix** as

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \cdots & x_{k,n} \end{pmatrix}.$$

By the same token we use X_I to denote the generic $k \times k$ minor determined by the multiindex I and $\det X_I$ to write the formal determinant of this minor.

Remark 3.4. The ring $\mathbb{K}[X_{k,n}]$ is the coordinate ring of $\mathcal{M}(k,n)$.

Remark 3.5. The familiar $\mathcal{M}(k,n)$ and $\bigwedge^k \mathbb{K}^n$ can be identified with the $\mathbb{K}$-points of the affine schemes $\operatorname{Spec}\mathbb{K}[X_{k,n}]$ and $\operatorname{Spec}\mathcal{B}_{k,n}$ respectively (Example 2.3.32 of [4]). We will impose this identification for the rest of the chapter.

Definition 3.6 (Plücker ring homomorphism)**.** We define the **Plücker ring homomorphism** or simply **Plücker homomorphism** as

$$\phi^{\#}: \begin{array}{ccc} \mathcal{B}_{k,n} & \longrightarrow & \mathbb{K}[X_{k,n}] \\ z_I & \longmapsto & \det X_I \end{array}$$

For brevity we will denote $\operatorname{Spec}\phi^{\#}$ by ϕ.

Remark 3.7. It is clear by construction that

$$\phi|_{\mathcal{M}(k,n)}(A) = \sum_{I\in\omega(k,n)} \det A_I e_I = \phi^s(A).$$

Proposition 3.8. $\ker\phi^{\#}$ *is a homogeneous prime ideal and* $\mathcal{B}^+_{k,n} \not\subseteq \ker\phi^{\#}$*.*

Proof. $\ker\phi^{\#}$ is prime because $\mathbb{K}[X_{k,n}]$ is an integral domain and $z_I \notin \ker\phi^{\#}$ because $\deg\phi^{\#}(z_I) = \deg(\det X_I) = k > 0$. To show homogeneity let us note that if g is homogeneous of degree d then $\phi^{\#}(g)$ is homogeneous of degree kd. If follows that if f_d is the homogeneous component of f of degree d and $0 = \phi^{\#}(f) = \sum_{d\in\mathbb{N}}\phi^{\#}(f_d)$ then $\phi^{\#}(f_d) = 0$ for all $d \in \mathbb{N}$. □

Proposition 3.9. *Let* $t : \mathrm{Var}/\mathbb{K} \to \mathrm{Sch}/\mathbb{K}$ *be the fully faithfull functor defined as in Proposition 2.6 of [3]. Then* $V_+(\ker\phi^{\#}) \cong t(\operatorname{Imm}\mathrm{Pl}^s)$*.*

Proof. Because t is fully faithfull, we only need to show that $V_+(\ker(\phi^{\#}))(\mathbb{K}) \cong \operatorname{Imm}\mathrm{Pl}^s$. Passing to the corresponding cones, this is equivalent to

$$\operatorname{Imm}\phi^s \cong V(\ker\phi^{\#})(\mathbb{K}) = \overline{\operatorname{Imm}\phi|_{\mathcal{M}(k,n)}} = \overline{\operatorname{Imm}\phi^s},$$

which is true because $\operatorname{Imm}\phi^s \overset{(2.29)}{=} \overline{\operatorname{Imm}\phi^s}$. □

From now on $\mathrm{Gr}(k,n)$ will also have the scheme structure of $V_+(\ker\phi^{\#})$. What we used to write $\mathrm{Gr}(k,n)$ corresponds to $\mathrm{Gr}(k,n)(\mathbb{K})$.

3.1.1 Standard affine cover of the Grassmannian scheme

Recall that projective space admits a standard affine cover given by the loci where one indeterminate does not vanish. In our case we see that

$$\operatorname{Proj}\mathcal{B}_{k,n} = \bigcup_{I\in\omega(k,n)} \operatorname{Spec}\left((\mathcal{B}_{k,n})^0_{z_I}\right) = \bigcup_{I\in\omega(k,n)} \operatorname{Spec}\left(\mathbb{K}\left[\frac{z_J}{z_I} \mid J\in\omega(k,n)\right]\right),$$

where the subscript denotes localization with multiplicative part $\{1, z_I, z_I^2, \cdots\}$ and the superscript 0 denotes the fact that we are only considering terms of degree 0 in this ring (this is the notation used in [4]).

This open affine cover of $\operatorname{Proj}\mathcal{B}_{k,n}$ induces an open cover on $\mathrm{Gr}(k,n)$ as follows:

$$\mathrm{Gr}(k,n) = V_+(\ker\phi^{\#}) = \bigcup_{I\in\omega(k,n)} \operatorname{Spec}\left(\left(\frac{\mathcal{B}_{k,n}}{\ker\phi^{\#}}\right)^0_{z_I}\right).$$

Notation 3.10. Let us fix $I \in \omega(k,n)$, then we denote the restriction of $\phi^{\#}$ as

$$\phi_I^{\#} : \begin{array}{ccc} \mathbb{K}\left[\frac{z_J}{z_I} \mid J \in \omega(k,n)\right] & \longrightarrow & \mathbb{K}[X_{k,n}]^0_{\det X_I} \\ \frac{z_J}{z_I} & \longmapsto & \frac{\det X_J}{\det X_I} \end{array}$$

Remark 3.11. By the first isomorphism theorem we have

$$\frac{(\mathcal{B}_{k,n})^0_{z_I}}{\ker \phi_I^{\#}} \cong \operatorname{Imm} \phi_I^{\#} = \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right] \doteqdot \mathbb{K}\left[\frac{\det X_J}{\det X_I}\right].$$

Remark 3.12. Applying a property of localization we have

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}}\right)_{z_I} = \frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}},$$

thus

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}}\right)^0_{z_I} = \left(\frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}}\right)^0 = \frac{(\mathcal{B}_{k,n})^0_{z_I}}{\ker \phi_I^{\#}} \cong \mathbb{K}\left[\frac{\det X_J}{\det X_I}\right]$$

In summary we have shown that, up to some canonical identifications,

$$\operatorname{Gr}(k,n) = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right]\right) \doteqdot \bigcup_{I \in \omega(k,n)} \operatorname{Gr}_I(k,n).$$

Notation 3.13. Let I be a (k,n)-multiindex, $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, n\}$. We define I^i_j to be the multiindex which is the same as I but with the i-th entry replaced with j.

Lemma 3.14. *If $I \in \omega(k,n)$ then the following equality holds in $\mathbb{K}[X_{k,n}]_{\det X_I}$*

$$X_I^{-1} X = \begin{pmatrix} w_{I^1_1} & \cdots & w_{I^1_n} \\ \vdots & \ddots & \vdots \\ w_{I^k_1} & \cdots & w_{I^k_n} \end{pmatrix}, \qquad \textit{where } w_J = \frac{\det X_J}{\det X_I}$$

Proof. Recall that if $\operatorname{Adj}(X_I)$ is the adjugate matrix of X_I then

$$(X_I)^{-1} = \frac{1}{\det X_I} \operatorname{Adj}(X_I) = \frac{1}{\det X_I}\left((-1)^{i+j} \det(X_I)_{\times j \times i}\right)_{\substack{1 \le i \le k \\ 1 \le j \le k}}.$$

We can verify the identity for each element:

$$\frac{(\operatorname{Adj}(X_I)X)_{i,j}}{\det X_I} = \frac{1}{\det X_I} \sum_{\ell=1}^{k} \left((-1)^{i+\ell} \det(X_I)_{\times \ell, \times i}\right) x_{\ell,j} = \frac{\det X_{I^i_j}}{\det X_I} = w_{I^i_j}.$$

□

Remark 3.15. $(X_I^{-1}X)_J = X_I^{-1} X_J$, in particular $(X_I^{-1}X)_I$ is the identity matrix.

Proposition 3.16. *$\operatorname{Gr}_I(k,n)$ is isomorphic to $\mathbb{A}^{k(n-k)}_{\mathbb{K}}$ as a scheme.*

Proof. Since both schemes are affine, it is enough to show that their coordinate rings are isomorphic. Without loss of generality we may assume that $I = (1, \cdots, k)$. For brevity we set $w_J = \frac{\det X_J}{\det X_I}$.

Let M be the formal matrix whose (i,j)-entry is $w_{I^i_j}$. Lemma (3.14) shows that $M = X_I^{-1}X$, so $\det M_J = \det X_I^{-1} \det X_J = w_J$. This shows that

$$\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right] = \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J = I^j_{\ell_j},\ j \in \{1, \cdots, k\},\ \ell_j \notin I\right].$$

Let R denote this ring. To conclude we want to show that it is isomorphic to $\mathbb{K}[Y_{k,n-k}] = \mathbb{K}[y_{1,1}, \cdots, y_{k,n-k}]$.

Let us consider the following ring homomorphism

$$\chi : \begin{array}{ccc} \mathbb{K}[Y_{k,n-k}] & \longrightarrow & R \\ y_{i,j} & \longmapsto & w_{I^i_{j+k}} \end{array}.$$

It is surjective by construction, so we just need to show that it is injective to find the desired isomorphism.

Suppose by contradiction that there exists a nonzero polynomial $p \in \mathbb{K}[Y_{k,n-k}]$ which maps to 0. If $\overline{\mathbb{K}}$ is an algebraic closure[1] of $\mathbb{K}$ we can consider the lift

$$\widetilde{\chi} : \begin{array}{ccc} \overline{\mathbb{K}}[Y_{k,n-k}] & \longrightarrow & \widetilde{R} = \overline{\mathbb{K}}[w_{I^i_j}] \\ y_{i,j} & \longmapsto & w_{I^i_{j+k}} \end{array}$$

Note that if $\chi(p) = 0$ then $\widetilde{\chi}(p) = 0$ because $R \subseteq \widetilde{R}$ and $\widetilde{\chi}|_{\mathbb{K}[Y_{k,n-k}]} = \chi$. Consider now any matrix of the form

$$A = \left(I_k \mid \widetilde{A}\right) = (a_{i,j})_{i,j}$$

where I_k is the $k \times k$ identity matrix and $\widetilde{A} \in \mathcal{M}(k, n-k, \overline{\mathbb{K}})$. From what we have said above it follows that $\det A_{I^i_j} = a_{i,j}$, so

$$p(\widetilde{A}) = p\left(\left(\det A_{I^i_j}\right)_{\substack{i\in\{1,\cdots,k\},\\ j\in\{k+1,\cdots,n\}}}\right) = \widetilde{\chi}(p)(A) = 0.$$

This shows that p has infinitely many roots in $\overline{K}$, so if we fix the value of $k(n-k)-1$ indeterminates the resulting polynomial is the 0 polynomial. If we reiterate this reasoning we eventually prove that $p = 0$ in $\overline{\mathbb{K}}[Y_{k,n-k}]$, but $0 \in \mathbb{K}[Y_{k,n-k}] \subseteq \overline{\mathbb{K}}[Y_{k,n-k}]$, so p is the zero polynomial in the original ring, contradicting our hypothesis. $\square$

Remark 3.17. Since $\mathrm{Gr}_I(k,n)$ and $\mathrm{Gr}_J(k,n)$ are affine and $\mathrm{Gr}(k,n)$ is projective and thus separated, $\mathrm{Gr}_I(k,n) \cap \mathrm{Gr}_J(k,n)$ is affine for any choice of multiindices.

3.2 Grassmannian moduli problem

Let us consider the following moduli problem

$$\mathbb{G}(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q\}\big/\sim \\ f : S \to T & \longmapsto & (\alpha : \mathcal{O}_T^n \to Q) \mapsto (f^*\alpha : \mathcal{O}_S^n \to f^*Q) \end{array}$$

[1] we can take any field extension $\mathbb{K} \subseteq \mathbb{F}$ where $\mathbb{F}$ is an infinite field.

where Q is a locally free sheaf of rank k on T and two surjections $\alpha : \mathcal{O}_T^n \twoheadrightarrow Q$, $\beta : \mathcal{O}_T^n \twoheadrightarrow V$ are equivalent if and only if there exist an isomorphism of sheaves $\theta : Q \to V$ such that the diagram commutes

$$\begin{tikzcd}
\mathcal{O}_T^n \arrow[r, two heads, "\alpha"] \arrow[dr, two heads, "\beta"'] & Q \arrow[d, "\theta"] \\
& V
\end{tikzcd}$$

We have functoriality because of the composition properties of pullbacks.

Remark 3.18. This functor formalizes the classification problem of $(n-k)$-dimensional subspaces of an n-dimensional space. Indeed

$$\mathbb{G}(k,n)(\operatorname{Spec}\mathbb{K}) = \left\{\alpha : \mathcal{O}_{\operatorname{Spec}\mathbb{K}}^n \twoheadrightarrow Q\right\}\big/\sim \;\cong \left\{\varphi : \mathbb{K}^n \twoheadrightarrow \mathbb{K}^k\right\}\big/\sim \;= \operatorname{Gr}(k,n)(\mathbb{K}).$$

For the middle isomorphism we used the fact that sheaves over a point are skyscrapers and that $\mathcal{O}_{\operatorname{Spec}\mathbb{K},\operatorname{Spec}\mathbb{K}} = \mathbb{K}$. The last equality is our first definition for the Grassmannian up to the choice of a basis.

In this this section we prove that the Grassmann scheme represents this functor.

3.2.1 Open subfunctor cover of the Grassmannian

Notation 3.19. For any multiindex $I \in \omega(k,n)$ and any scheme T we define the following morphism of sheaves

$$s_I^T : \begin{array}{ccc} \mathcal{O}_T^k & \longrightarrow & \mathcal{O}_T^n \\ e_j & \longmapsto & e_{i_j} \end{array}.$$

If there is no ambiguity we write s_I.

Definition 3.20 (Principal subfunctors of the Grassmannian)**.** Fixed a multiindex $I \in \omega(k,n)$ we define the following functor

$$\mathbb{G}_I(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \left\{\mathcal{O}_T^n \overset{\alpha}{\twoheadrightarrow} Q \mid \alpha \circ s_I \text{ surjective}\right\}\big/\sim \\ f & \longmapsto & \alpha \mapsto f^*\alpha \end{array}$$

where the equivalence relation is the same as the one defined for $\mathbb{G}(k,n)$.

Proposition 3.21. *The functor $\mathbb{G}_I(k,n)$ is well defined.*

Proof. First we observe that $\mathbb{G}_I(k,n)(T)$ is well defined because if $\psi = \theta \circ \alpha$ with θ isomorphism of sheaves then on each stalk we have

$$\psi_x \circ (s_I)_x = \theta_x \circ \varphi_x \circ (s_I)_x,$$

which is surjective if and only if $\varphi_x \circ (s_I)_x$ is surjective.

Consider now a morphism $f : S \to T$, then

$$f^*\alpha \circ s_I^S = f^*\alpha \circ f^* s_I^T = f^*(\alpha \circ s_I^T)$$

is surjective if and only if it is surjective on all stalks, i.e. if and only if for all $s \in S$ we have that the following map is surjective

$$f^*(\alpha \circ s_I^T)_s = (\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{\mathcal{O}_{S,s}},$$

which is true because the tensor product is right-exact. $\square$

Proposition 3.22. *The $\mathbb{G}_I(k,n)$ are open subfunctors of $\mathbb{G}(k,n)$.*

Proof. The inclusion $\mathbb{G}_I(k,n)(T) \subseteq \mathbb{G}(k,n)(T)$ is apparent, so we just need to show that if we fix a quotient $[\alpha : \mathcal{O}_T^n \twoheadrightarrow Q]$ in $\mathbb{G}(k,n)(T)$ then we can find an open subscheme of T which represents $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$.

Let us fix a representative α for the given quotient. The locus where $\alpha \circ s_I : \mathcal{O}_T^k \to Q$ is surjective is the complement of the support of its cokernel sheaf $\mathcal{K}$, i.e.

$$(\alpha \circ s_I)_x \text{ surjective} \iff x \notin \operatorname{Supp} \mathcal{K}.$$

Note that by the definition of $\sim$ and properties of isomorphisms of sheaves, the first condition does not depend of the choice of representative for $[\alpha]$, so $\operatorname{Supp}\mathcal{K}$ only depends on $[\alpha]$. Note that $\mathcal{K}$ is of finite type because the codomains are locally free of finite rank, so $\operatorname{Supp}\mathcal{K}$ is closed[2] and hence $U_I = T \setminus \operatorname{Supp}\mathcal{K}$ is open.

We now want to show that U_I represents the functor $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$, that is we want to show that if $f : S \to T$ is a morphism of $\mathbb{K}$-schemes then f factors through U_I if and only if $[f^*\alpha : \mathcal{O}_S^n \to f^*Q] \in \mathrm{Gr}_I(S)$.

Note that $f(s) \in U_I$ if and only if $(\alpha \circ s_I^T)_{f(s)}$ is surjective which, by Nakayama's lemma applied to the cokernels, is equivalent to the surjectivity of

$$(\alpha \circ s_I^T)|_{f(s)} : k(f(s))^n \to Q_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} k(f(s)).$$

Observe that, up to standard identifications,

$$\begin{aligned}
f^*(\alpha \circ s_I^T)|_s =& f^*(\alpha \circ s_I^T)_s \otimes_{\mathcal{O}_{S,s}} id_{k(s)} = \\
=& ((\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{\mathcal{O}_{S,s}}) \otimes_{\mathcal{O}_{S,s}} id_{k(s)} = \\
=& (\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{k(s)} = \\
=& ((\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} id_{k(f(s))}) \otimes_{k(f(s))} id_{k(s)} = \\
=& (\alpha \circ s_I^T)|_{f(s)} \otimes_{k(f(s))} id_{k(s)}.
\end{aligned}$$

Note that we used the fact that $\mathcal{O}_{T,f(s)} \to \mathcal{O}_{S,s} \to k(s) = \mathcal{O}_{T,f(s)} \to k(f(s)) \to k(s)$. Since field extensions do not change the rank of linear maps, this shows that

$$f^*(\alpha \circ s_I^T)|_s \text{ is surjective} \quad \iff \quad (\alpha \circ s_I^T)|_{f(s)} \text{ is surjective.}$$

By Nakayama's lemma we can again consider equivalently $f^*(\alpha \circ s_I^T)_s = (f^*\alpha)_s \circ (s_I^S)_s$.

We have thus shown that $f(s) \in U_I$ if and only if $(f^*\alpha)_s \circ (s_I^S)_s$ is surjective, i.e. f factors through U_I if and only if $(f^*\alpha) \circ s_I^S$ is surjective, i.e. $f^*\alpha \in \mathbb{G}_I(k,n)(S)$. ☐

Proposition 3.23. *The collection $\{\mathbb{G}_I(k,n)\}$ is a Zariski open cover of $\mathbb{G}(k,n)$.*

Proof. For any $\mathbb{K}$-scheme S and any quotient $[\alpha] \in \mathrm{Gr}(k,n)(S)$ (without loss of generality we choose a representative α) we need to show that for any $s \in S$ there exists a multiindex I such that $s \in U_I$ defined as in the previous proposition.

We are therefore looking for a multiindex I such that $(\alpha \circ s_I)_s$ is surjective. By Nakayama's lemma this is equivalent to showing that there exists an I such that

$$k(s)^k \xrightarrow{s_I} k(s)^n \xrightarrow{\alpha_s} Q_s \otimes_{\mathcal{O}_{S,s}} k(s)$$

is surjective, which is trivially true since $\operatorname{rnk} \alpha_s = k$. ☐

[2] For more detail see Section 01B4 in [7]

3.2.2 Representability of the Grassmannian functor

Lemma 3.24. *Let T be a scheme and $[\alpha : \mathcal{O}_T^n \twoheadrightarrow Q], [\beta : \mathcal{O}_T^n \twoheadrightarrow Q'] \in \mathbb{G}(k,n)$. If $[\alpha] = [\beta]$ then the isomorphism $\theta : Q \to Q'$ such that $\beta = \theta \circ \alpha$ is unique.*

Proof. First, observe that if $\alpha = \beta$ then by surjectivity and commutativity $\theta = id_Q$. Let $\theta, \eta : Q \to Q'$ be isomorphisms such that $\beta = \theta \circ \alpha$ and $\beta = \eta \circ \alpha$. Then $\theta^{-1} \circ \eta : Q \to Q$ is an isomorphism such that $\theta^{-1} \circ \eta \circ \alpha = \theta^{-1} \circ \beta = \alpha$, so $\theta^{-1} \circ \eta = id_Q$ and thus $\theta = \eta$. □

Proposition 3.25. *The Grassmannian functor $\mathbb{G}(k,n)$ is a Zariski sheaf.*

Proof. Consider a $\mathbb{K}$-scheme T and an open cover $\{U_i \to T\}$. Let $\alpha_i : \mathcal{O}_{U_i}^n \twoheadrightarrow Q_i$ be representatives of quotients such that

$$\alpha_i|_{U_i \cap U_j} \sim \alpha_j|_{U_i \cap U_j}.$$

Because of lemma (3.24), the isomorphism giving the equivalence above is unique. Let $\varphi_{ji} : Q_i|_{U_i \cap U_j} \to Q_j|_{U_i \cap U_j}$ be this isomorphism. Because of the uniqueness $\varphi_{ii} = id_{Q_i}$ and $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$, so we have the data to glue the Q_i to a locally free sheaf of rank k over T, which we denote by Q.

By construction $\alpha_i : \mathcal{O}_{U_i}^n \twoheadrightarrow Q|_{U_i}$ for all i. Let $V \subseteq T$ be an open subset. For any section $s \in \mathcal{O}_T^n(V)$ we can define $\alpha_V(s)$ by gluing the $(\alpha_i)_V(s|_{U_i})$, which we can do by construction[3] of Q. It is well known that a sheaf morphism is determined by its restrictions to open sets. □

Proposition 3.26. *The affine scheme $\mathrm{Gr}_I(k,n)$ represents the functor $\mathbb{G}_I(k,n)$.*

Proof. First we prove that for any $\mathbb{K}$-scheme T, $\mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathbb{G}_I(T)$, then we need to check naturality.

By definition $\mathrm{Gr}_I(k,n) = \mathrm{Spec}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right]\right)$, so

$$\mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right).$$

For a map $\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k$, we define $M(U)$ as the matrix which represents $\alpha_U : \mathcal{O}_T^n(U) \to \mathcal{O}_T^k(U)$ in the canonical bases. We define the following maps

$$\begin{array}{ccc} \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) & \longleftrightarrow & \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} \\ \varphi & \longmapsto & \eta(\varphi) \\ \rho(\alpha) : \frac{\det X_J}{\det X_I} \mapsto \frac{\det(M(T))_J}{\det(M(T))_I} & \longleftarrow & \alpha \end{array}$$

where $\eta(\varphi)$ is defined on an open subset V of T by

$$\eta(\varphi)_V(e_j) = \sum_{i=1}^{k} (\mathrm{res}_V^T \circ \varphi)\left(\frac{\det X_{I_j^i}}{\det X_I}\right) e_r \overset{(3.14)}{=} (\mathrm{res}_V^T \circ \varphi)\left(X_I^{-1} X\right) e_j.$$

[3] More precicely, the φ_{ji} are the gluing functions on Q and

$$\alpha_j(s|_{U_j})|_{U_i \cap U_j} = \alpha_j(s|_{U_i \cap U_j}) = \varphi_{ji} \circ \alpha_i(s|_{U_i \cap U_j}) = \varphi_{ji}(\alpha_j(s|_{U_j})|_{U_i \cap U_j}).$$

The maps are well defined because $\alpha \circ s_I = id_{\mathcal{O}_T^k} \iff M(T)_I = I_k$ and

$$\frac{\det X_{I_{i_s}^r}}{\det X_I} = \delta_{r,s} \implies \eta(\varphi) \circ s_I = id_{\mathcal{O}_T^k}.$$

We can see that η and ρ are inverses via the following computations:

$$\mathrm{res}_V^T \circ \rho(\alpha)(X_I^{-1}X) = \mathrm{res}_V^T(M(T)_I{}^{-1}M(T)) = \mathrm{res}_V^T(I_k^{-1}M(T)) = M(V),$$

$$\rho(\eta(\varphi))\left(\frac{\det X_J}{\det X_I}\right) = \frac{\det((\mathrm{res}_T^T \circ \varphi)\left(X_I^{-1}X\right)_J)}{1} = \varphi(\det((X_I^{-1}X)_J)) = \varphi\left(\frac{\det X_J}{\det X_I}\right).$$

Observe now that

$$\begin{array}{ccc} \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} & \longleftrightarrow & \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q \mid \alpha \circ s_I \text{ isomorphism}\}/\sim \\ \alpha & \longmapsto & [\alpha] \\ (\beta \circ s_I)^{-1} \circ \beta & \longleftarrow & [\beta] \end{array}$$

is a bijection. The second map is well defined because if $\beta = \theta \circ \beta'$ then

$$(\beta \circ s_I)^{-1} \circ \beta = (\beta' \circ s_I)^{-1} \circ \theta^{-1} \circ \theta \circ \beta' = (\beta' \circ s_I)^{-1} \circ \beta'$$

and they are inverses because $\beta \sim (\beta \circ s_I)^{-1} \circ \beta$ by definition of $\sim$ and if $\alpha \circ s_I = id_{\mathcal{O}_T^k}$ then $(\alpha \circ s_I)^{-1} \circ \alpha = \alpha$. We conclude by noticing that

$$\{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q \mid \alpha \circ s_I \text{ isomorphism}\}/\sim\, = \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q \mid \alpha \circ s_I \text{ surjective}\}/\sim$$

because on all stalks $\alpha \circ s_I$ is an endomorphism of finitely generated modules.

To prove naturality we consider a morphism $f : S \to T$ of $\mathbb{K}$-schemes. Recall that

$$\begin{array}{ccc} \mathbb{G}_I(k,n)(T) & \longrightarrow & \mathbb{G}_I(k,n)(S) \\ [\alpha] & \longmapsto & [f^*\alpha] \end{array}.$$

Under the bijection above, imposing naturality gives

$$\begin{array}{ccc} \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} & \longrightarrow & \left\{\beta : \mathcal{O}_S^n \to \mathcal{O}_S^k \mid \beta \circ s_I = id_{\mathcal{O}_S^k}\right\} \\ \alpha & \longmapsto & f^*\alpha \end{array}$$

since $f^*\alpha \circ s_I^S = f^*(\alpha \circ s_I^T) = f^*(id_{\mathcal{O}_T^k}) = id_{\mathcal{O}_S^k}$. If we impose naturality again we get

$$\begin{array}{ccc} \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) & \longrightarrow & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_S(S)\right) \\ \varphi & \longmapsto & \rho(f^*\eta(\varphi)) \end{array}$$

We claim that $\rho(f^*(\eta(\varphi))) = f^\#(T) \circ \varphi$. Since η is the inverse of ρ, it is enough to prove that $f^*(\eta(\varphi)) = \eta(f^\#(T) \circ \varphi)$. Equality holds because for all $s \in S$ both of the maps induced on stalks are represented by the matrix

$$f_s^\#\left(\left(\varphi(X_I^{-1}X)\right)_{f(s)}\right).$$

We conclude by recalling that the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \arrow[r, "\mathrm{Spec}"] \arrow[d, "h_{\mathrm{Gr}_I(k,n)}(f)"'] & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) \arrow[d, "\mathrm{Hom}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], f^\#(T)\right)"] \\ \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(S, \mathrm{Gr}_I(k,n)) \arrow[r, "\mathrm{Spec}"] & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_S(S)\right) \end{tikzcd}$$

□

Theorem 3.27. *The Grassmann scheme* $\mathrm{Gr}(k,n)$ *is a fine moduli space for the Grassmann functor* $\mathbb{G}(k,n)$.

Proof. We know that $\{\mathbb{G}_I(k,n) \to \mathbb{G}(k,n)\}$ is an open cover (3.23), that $\mathbb{G}(k,n)$ is a Zariski sheaf (3.25) and that $h_{\mathrm{Gr}_I(k,n)} \cong \mathbb{G}_I(k,n)$ (3.26). If we can show that these isomorphisms restrict well to double intersections we have the desired result by proposition (1.34). Let T be a scheme and let us consider a morphism

$$f \in \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}\left(T, \mathrm{Gr}_I(k,n) \cap \mathrm{Gr}_J(k,n)\right) = \mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}\left(T, \mathrm{Gr}(k,n) \cap D_+(z_I z_J)\right).$$

Applying a standard result for morphisms towards an affine scheme[4] we get

$$f^\#(T) \in \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\left(\mathbb{K}\left[\det X_L\right]_{\det X_I \det X_J}\right)^0, \mathcal{O}_T(T)\right).$$

By the universal property of localization, we may identify this set with

$$\left\{\beta \in \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_L}{\det X_I}\right], \mathcal{O}_T(T)\right) \mid \beta\left(\frac{\det X_J}{\det X_I}\right) \in \mathcal{O}_T(T)^*\right\}.$$

Applying the functor η defined during the proof of proposition (3.26), which we will denote η^I to emphasize which determinant we consider at the denominator, we obtain[5]

$$\eta^I(f^\#(T)) \in \left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k},\ \alpha \circ s_J \text{ isomorphism}\right\}.$$

Observe that we can identify this set with

$$\{\alpha : \mathcal{O}_T^n \to Q \mid \alpha \circ s_I \text{ and } \alpha \circ s_J \text{ surjective}\}_{/\sim} = (\mathbb{G}_I(k,n) \times_{\mathbb{G}(k,n)} \mathbb{G}_J(k,n))(T),$$

so to conclude the proof we just need to verify that $\eta^I(f^\#(T)) \sim \eta^J(f^\#(T))$ in $\mathbb{G}(k,n)$. By lemma (3.14), the matrix $X_J^{-1} X_I$ can be described only using elements in $\mathbb{K}\left[\det X_L\right]^0_{\det X_I \det X_J}$. We can thus define θ by setting $\theta_V(e_j) = f^\#(V)(X_J^{-1} X_I) e_j$. It is clear by construction that $\theta \circ \eta^I(f^\#(T)) = \eta^J(f^\#(T))$. Defining δ from $X_I^{-1} X_J$ analogously yields an inverse of θ, realizing the sought out equivalence. □

---

[4] see remark (3.17).

[5] the condition on the image of $\frac{\det X_J}{\det X_I}$ corresponds to $\det(\alpha \circ s_J)$ being invertible, and thus to $\alpha \circ s_J$ being an isomorphism.

Bibliography

[1] Jarod Alper. Stacks and moduli. https://sites.math.washington.edu/~jarod/moduli.pdf, 2024.

[2] A. Björner. *Oriented Matroids*. EBSCO ebook academic collection. Cambridge University Press, 1999.

[3] Robin Hartshorne. *Schemes*, pages 60–200. Springer New York, New York, NY, 1977.

[4] Qing Liu. *Algebraic geometry and arithmetic curves*. Oxford Graduate Texts in Mathematics. Oxford University Press, London, England, June 2006.

[5] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*. Ergebnisse der Mathematik und Ihrer Grenzgebiete, 3 Folge/A Series of Modern Surveys in Mathematics Series. Springer Berlin Heidelberg, 1994.

[6] BERNHARD RIEMANN. Theorie der abelsehen funktionen. *J. reine angew. Math*, 54:88–144, 1857.

[7] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2024.

[8] Oswald Teichmüller. Extremale quasikonforme abbildungen und quadratische differentialen. *Abh. Preuss. Akad. Wiss.*, 22:3–197, 1939.