[TITOLO]

[Sottotitolo]

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Contents

Chapter 1

Moduli Spaces

Set theoretic issues: whenever I write that something is an element of a class, what I mean is that that object satisfies the proposition that defines the class.

1.1 Introduction to moduli problems

Definition 1.1 (Presheaf).
A contravariant functor $F : \mathcal{C}^{op} \to \text{Set}$ is called a **presheaf** on $\mathcal{C}$.

Definition 1.2 (ModuHomli problem).
Let S be a scheme. A presheaf on Sch/S is called a **moduli problem**.

1.2 Fine and Coarse moduli spaces

1.3 Zariski sheaves and gluing of fine moduli spaces

Definition 1.3 (Equalizer).
Let $\mathcal{C}$ be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \to C$. We say that the pair (A, h) is an **equalizer** of the diagram

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

if $h : A \to B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \to A$ which makes the diagram commute

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ Q \arrow[u, dashed] \arrow[ur, "q"'] & & \end{tikzcd}$$

Definition 1.4 (Zariski sheaf).
A moduli problem $F \in (\text{Sch}/S)^{op} \to \text{Set}$ is a **Zariski sheaf** if for any S-scheme X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Proposition 1.5 (Representable moduli functors are Zariski sheaves)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ *be a moduli problem, then if there exists a fine moduli space* M *for* F *it must be the case that* F *is a Zariski sheaf.*

Proof.
By composing with the natural isomorphism we may assume $F = h_M$. Let X be an S-scheme and $\{U_i \to X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\mathrm{Mor}(X, M) \longrightarrow \prod_k \mathrm{Mor}(U_k, M) \rightrightarrows \prod_{i,j} \mathrm{Mor}(U_i \cap U_j, M)$$

The arrows in this case correspond to restriction of morphisms, so the thesis is equivalent to the fact that restriction to a given set doesn't depend on the intermediate restrictions and that morphisms of schemes that coincide on double intersections glue to the union, both of which are true. $\square$

Definition 1.6 (Subfunctor)**.**
Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. A **subfunctor** of G is a pair (F, i) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $i : F \to G$ such that $i_X : F(X) \to G(X)$ is a monomorphism for all $X \in \mathcal{C}$.

Remark 1.7.
If $\mathcal{D} = \mathrm{Set}$ then we can express the same data equivalently as follows:
A functor $F : \mathcal{C} \to \mathrm{Set}$ is a subfunctor of $G : \mathcal{C} \to \mathrm{Set}$ if for all $X \in \mathcal{C}$ and for all $f \in Mor_{\mathcal{C}}(A, B)$

$$F(X) \subseteq G(X), \qquad \text{and} \qquad F(f) = G(f)|_{F(A)}.$$

In this case we write $F \subseteq G$.

Definition 1.8 (Fibered product of presheaves)**.**
Let $F, G, H : \mathcal{C}^{op} \to \mathrm{Set}$ be presheaves together with two natural transformations $\xi^1 : F \to H$ and $\xi^2 : G \to H$. We define their fibered product as follows:
If $X \in \mathcal{C}$ then

$$(F \times_H G)(X) = F(X) \times_{H(X)} G(X),$$

if $f : A \to B$ then[1]

$$(F \times_H G)(f) : \begin{array}{ccc} F(B) \times_{H(B)} G(B) & \longrightarrow & F(A) \times_{H(A)} G(A) \\ (b_1, b_2) & \longmapsto & (F(f)(b_1), G(f)(b_2)) \end{array}.$$

Definition 1.9 (Open subfunctor)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S-scheme T and any natural trasformation $h_T \to F$, the pullback $h_T \times_F G$ is representable by an open subscheme of T.

Remark 1.10.
By Yonede's lemma, giving a natural trasnformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows:
A subfunctor $G \subseteq F$ is open if for any S-scheme T and any family $\xi \in F(T)$ there

[1] the map is well defined because $\xi^1_A(F(f)(b_1)) = H(f)(\xi^1_B(b_1)) = H(f)(\xi^2_B(b_2)) = \xi^2_A(G(f)(b_2))$.

exists an open subscheme $U \subseteq T$ such that the following diagram is natural in R and commutes

$$\begin{tikzcd}
\mathrm{Mor}(R,U) \arrow[r, "G(\subseteq\circ\ \cdot\ )(\xi)"] \arrow[d, hook, "\subseteq\circ"'] & G(R) \arrow[d, hook, "\subseteq"] \\
\mathrm{Mor}(R,T) \arrow[r, "F(\cdot)(\xi)"'] & F(R)
\end{tikzcd}$$

and a map $f \in \mathrm{Mor}(R,T)$ factors as $R \xrightarrow{g} U \subseteq T$ if and only if $F(f)(\xi) \in G(R)$[2].

[2]the "only if" is trivially true by commutativity but for the "if" we are using the fact that $h_U \cong h_T \times_F G$.

Chapter 2

Grassmanians

2.1 Classical view of Grassmannians

Definition 2.1 (Grassmannian)**.**
Let us fix two natural numbers $k \leq n$. We define the (n, k)**-Grasssmannian**, denoted $Gr(k, n, \mathbb{K})$ (or just $Gr(k, n)$ if the base field is clear), as the set of $(n-k)$-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^n$.

Remark 2.2 (Definition via quotients).
We may equivalently define $Gr(k, n)$ to be the following set:

$$\left\{\ker \varphi \mid \varphi \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k),\ \operatorname{rnk} \varphi = k\right\}.$$

To further lean into the definition we will adopt in the case of schemes, we may equivalently consider

$$\left\{\varphi \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \operatorname{rnk} \varphi = k\right\} / \sim,$$

where[1]

$$\varphi \sim \varphi' \iff \ker \varphi = \ker \varphi' \iff \exists \psi \in GL_k(\mathbb{K})\ s.t.\ \varphi = \psi \circ \varphi'.$$

2.2 Representability of the Grassmannian functor

2.3 Proprieties of the Grassmanian scheme

[1] if $\ker \varphi = \ker \varphi'$, $\mathcal{B}'$ is a basis for it and $\mathcal{B}$ is an extension of $\mathcal{B}'$ to a basis of $\mathbb{K}^n$, then we can define ψ as the automorphism of $\mathbb{K}^k$ that takes $\varphi'(\mathcal{B} \setminus \mathcal{B}')$ to $\varphi(\mathcal{B} \setminus \mathcal{B}')$.