

[TITOLO]
[Sottotitolo]

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Chapter 1

Moduli Spaces

Set theoretic issues: whenever I write that something is an element of a class, what I mean is that that object satisfies the proposition that defines the class.

1.1 Introduction to moduli problems

Definition 1.1 (Presheaf).

A contravariant functor $F : \mathcal{C}^{op} \rightarrow \text{Set}$ is called a **presheaf** on \mathcal{C} .

Definition 1.2 (Moduli problem).

Let S be a scheme. A presheaf on Sch/S is called a **moduli problem**.

1.2 Fine and Coarse moduli spaces

1.3 Zariski sheaves and gluing of fine moduli spaces

Definition 1.3 (Equalizer).

Let \mathcal{C} be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \rightarrow C$. We say that the pair (A, h) is an **equalizer** of the diagram

$$B \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C$$

if $h : A \rightarrow B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \rightarrow A$ which makes the diagram commute

$$\begin{array}{ccccc} A & \xrightarrow{h} & B & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & C \\ \uparrow & \nearrow q & & & \\ Q & & & & \end{array}$$

Definition 1.4 (Zariski sheaf).

A moduli problem $F \in (\text{Sch}/S)^{op} \rightarrow \text{Set}$ is a **Zariski sheaf** if for any S -scheme X and any Zariski open cover $\{U_i \rightarrow X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Proposition 1.5 (Representable moduli functors are Zariski sheaves).

Let $F : (\text{Sch}/S)^{op} \rightarrow \text{Set}$ be a moduli problem, then if there exists a fine moduli space M for F it must be the case that F is a Zariski sheaf.

Proof.

By composing with the natural isomorphism we may assume $F = h_M$. Let X be an S -scheme and $\{U_i \rightarrow X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\text{Mor}(X, M) \longrightarrow \prod_k \text{Mor}(U_k, M) \rightrightarrows \prod_{i,j} \text{Mor}(U_i \cap U_j, M)$$

The arrows in this case correspond to restriction of morphisms, so the thesis is equivalent to the fact that restriction to a given set doesn't depend on the intermediate restrictions and that morphisms of schemes that coincide on double intersections glue to the union, both of which are true. \square

Definition 1.6 (Subfunctor).

Let $G : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. A **subfunctor** of G is a pair (F, i) consisting of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $i : F \rightarrow G$ such that $i_X : F(X) \rightarrow G(X)$ is a monomorphism for all $X \in \mathcal{C}$.

Remark 1.7.

If $\mathcal{D} = \text{Set}$ then we can express the same data equivalently as follows:

A functor $F : \mathcal{C} \rightarrow \text{Set}$ is a subfunctor of $G : \mathcal{C} \rightarrow \text{Set}$ if for all $X \in \mathcal{C}$ and for all $f \in \text{Mor}_{\mathcal{C}}(A, B)$

$$F(X) \subseteq G(X), \quad \text{and} \quad F(f) = G(f)|_{F(A)}.$$

In this case we write $F \subseteq G$.

Definition 1.8 (Fibered product of presheaves).

Let $F, G, H : \mathcal{C}^{op} \rightarrow \text{Set}$ be presheaves together with two natural transformations $\xi^1 : F \rightarrow H$ and $\xi^2 : G \rightarrow H$. We define their fibered product as follows:

If $X \in \mathcal{C}$ then

$$(F \times_H G)(X) = F(X) \times_{H(X)} G(X),$$

if $f : A \rightarrow B$ then¹

$$(F \times_H G)(f) : \begin{array}{ccc} F(B) \times_{H(B)} G(B) & \longrightarrow & F(A) \times_{H(A)} G(A) \\ (b_1, b_2) & \longmapsto & (F(f)(b_1), G(f)(b_2)) \end{array}.$$

Definition 1.9 (Open subfunctor).

Let $F : (\text{Sch}/S)^{op} \rightarrow \text{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S -scheme T and any natural transformation $h_T \rightarrow F$, the pullback $h_T \times_F G$ is representable by an open subscheme of T .

Remark 1.10.

By the Yoneda lemma, giving a natural transformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows:

A subfunctor $G \subseteq F$ is open if for any S -scheme T and any family $\xi \in F(T)$ there

¹the map is well defined because $\xi_A^1(F(f)(b_1)) = H(f)(\xi_B^1(b_1)) = H(f)(\xi_B^2(b_2)) = \xi_A^2(G(f)(b_2))$.

exists an open subscheme $U \subseteq T$ such that the following diagram is natural in R and commutes

$$\begin{array}{ccc} \mathrm{Mor}(R, U) & \xrightarrow{G(\subseteq \circ \cdot)(\xi)} & G(R) \\ \subseteq \circ \downarrow & & \downarrow \subseteq \\ \mathrm{Mor}(R, T) & \xrightarrow{F(\cdot)(\xi)} & F(R) \end{array}$$

and a map $f \in \mathrm{Mor}(R, T)$ factors as $R \xrightarrow{g} U \subseteq T$ if and only if $F(f)(\xi) \in G(R)$ ².

Definition 1.11 (Open cover of a functor).

Let $F : (\mathrm{Sch}/S)^{op} \rightarrow \mathrm{Set}$ be a moduli problem. A collection of open subfunctors $\{F_i\}$ is an **open cover** of F if for any S -scheme T and any natural transformation $h_T \rightarrow F$, the open subschemes U_i that represent the pullbacks $h_T \times_F F_i$ form an open cover of T .

Remark 1.12.

Like above, we can rephrase the definition as follows:

A collections of open subfunctors $F_i \subseteq F$ form an open cover of F if for any S -scheme T and any family $\xi \in F(T)$, there exists an open cover $\{U_i\}$ of T such that $\xi|_{U_i} \in F_i(U_i)$ for all i .

The definitions above let us state the following criterion for representability:

Proposition 1.13 (Representability by open cover).

Let $F : (\mathrm{Sch}/S)^{op} \rightarrow \mathrm{Set}$ be a moduli problem which is a Zariski sheaf and let $\{F_i\}$ be an open cover of it by representable subfunctors, then F is representable.

Proof.

Let X_i be the fine moduli space for F_i and let $\xi_i \in F_i(X_i)$ be their universal families. Since F_j is an open subfunctor of F , there exists an open subscheme $U_{ij} \subseteq X_i$ such that the diagram is a cartesian square

$$\begin{array}{ccc} h_{U_{ij}} & \longrightarrow & h_{X_i} \\ \downarrow & \lrcorner & \downarrow \\ F_j & \longrightarrow & F \end{array}$$

Evaluating the diagram on U_{ij} yields

$$\begin{array}{ccc} \mathrm{Mor}(U_{ij}, U_{ij}) & \longrightarrow & \mathrm{Mor}(U_{ij}, X_i) \\ \downarrow & & \downarrow \\ F_j(U_{ij}) & \longrightarrow & F(U_{ij}) \end{array}$$

The top left set contains $id_{U_{ij}}$, which corresponds to the universal family in

$$F_j(U_{ij}) \times_{F(U_{ij})} \mathrm{Mor}(U_{ij}, X_i),$$

given by $(\xi|_{U_{ij}}, \iota_i)$, where $\iota_i : U_{ij} \rightarrow X_i$ is the inclusion and $\xi_j|_{U_{ij}} = F_j(f_i)(\xi_j)$ for

$$\begin{array}{ccccc} h_{U_{ij}}(U_{ij}) & \cong & F_j(U_{ij}) \times_{F(U_{ij})} h_{X_i}(U_{ij}) & \xrightarrow{pr_1} & F_j(U_{ij}) & \cong & \mathrm{Mor}(U_{ij}, X_j) \\ \Psi & & & & & & \Psi \\ id_{U_{ij}} & \longmapsto & & & & & f_i \end{array}$$

²the “only if” is trivially true by commutativity but for the “if” we are using the fact that $h_U \cong h_T \times_F G$.

The images of these two elements in $F(U_{ij})$ are $\xi_j|_{U_{ij}}$ (since $F_j \subseteq F$) and $\xi_i|_{U_{ij}} = \iota_i^* \xi_i$, so by commutativity

$$\xi_j|_{U_{ij}} = \xi_i|_{U_{ij}}.$$

We now evaluate the cartesian square that defines U_{ji} on U_{ij} :

$$\begin{array}{ccc} \text{Mor}(U_{ij}, U_{ji}) & \longrightarrow & \text{Mor}(U_{ij}, X_j) \\ \downarrow & & \downarrow \\ F_i(U_{ij}) & \longrightarrow & F(U_{ij}) \end{array}$$

Given what we have already said, $(\xi_i|_{U_{ij}}, f_i) \in F_i(U_{ij}) \times_{F(U_{ij})} h_{X_j}(U_{ij})$, so it defines a morphism $\varphi_{ij} : U_{ij} \rightarrow U_{ji}$. We observe that φ_{ij} and φ_{ji} are inverses of each other. Indeed if we consider $\varphi_{ji} \circ \varphi_{ij}$ as an element of the top left set in the diagram

$$\begin{array}{ccc} \text{Mor}(U_{ij}, U_{ij}) & \longrightarrow & \text{Mor}(U_{ij}, X_i) \\ \downarrow & & \downarrow \\ F_j(U_{ij}) & \longrightarrow & F(U_{ij}) \end{array}$$

we notice that the two projections are given by

$$\iota_i \circ \varphi_{ji} \circ \varphi_{ij} = f_j \circ \varphi_{ij} = f_i$$

and

$$F_j(f_i \circ \varphi_{ji} \circ \varphi_{ij})(\xi_j) = F_j(f_j \circ \varphi_{ij})(\xi_j) = F_j(f_i)(\xi_j) = \xi_j|_{U_{ij}},$$

which are the same projections as $id_{U_{ij}}$, so $\varphi_{ji} \circ \varphi_{ij} = id_{U_{ij}}$. A symmetric argument yields $\varphi_{ij} \circ \varphi_{ji} = id_{U_{ji}}$. Note that with our notation $\varphi_{ii} = id_{U_{ii}} = id_{X_i}$. We also remark that

$$F(\varphi_{ij})(\xi_j) = F_j(\iota_j \circ \varphi_{ij})(\xi_j) = F_j(f_i)(\xi_j) = \xi_j|_{U_{ij}} = \xi_i|_{U_{ij}}.$$

We now want to show that the X_i can be glued along the U_{ij} using the isomorphisms φ_{ij} .

Intersection We want to check that

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}.$$

Given the symmetry of the indices and having already proven that $\varphi_{ji} = \varphi_{ij}^{-1}$, we just need to show inclusion. By the equivalent definition of open subfunctor, we know that the morphism $\varphi_{ij} : U_{ij} \cap U_{ik} \rightarrow U_{ji}$ factors through $U_{ji} \cap U_{jk}$ if and only if $\varphi_{ij}^* \xi_j|_{U_{ji} \cap U_{jk}} \in F(U_{ij} \cap U_{ik})$, which is true because

$$\varphi_{ij}^* \xi_j|_{U_{ji} \cap U_{jk}} = \xi_i|_{U_{ij} \cap U_{ik}}.$$

Cocycle cond. We now want to verify that

$$\varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}} = \varphi_{ik}|_{U_{ik} \cap U_{ij}},$$

but this simply follows from the fact that both maps pullback $\xi_k|_{U_{ki} \cap U_{kj}}$ to $\xi_i|_{U_{ij} \cap U_{ik}}$.

We can thus define X to be the scheme obtained by gluing the X_i along the U_{ij} . Moreover, since F is a Zariski sheaf, the ξ_i must glue to a family $\xi \in F(X)$ ³.

We now only need to verify that (X, ξ) is a fine moduli space for F :

Let T be an S -scheme and let us consider a family $\zeta \in F(T)$. Since $\{F_i\}$ is an open cover of F , there exists an open cover $\{V_i\}$ of T such that $\zeta|_{V_i} \in F_i(V_i) \cong \text{Mor}(V_i, X_i)$. Since F is a sheaf $\zeta_i|_{V_i \cap V_j} = \zeta_j|_{V_i \cap V_j}$, so the morphisms $V_i \rightarrow X_i$ corresponding to the ζ_i glue to a morphism $f : T \rightarrow X$ such that $f^*\xi = \zeta$ (by construction). \square

³because $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$.

Chapter 2

Grassmannians

2.1 Set-theoretic definition

Definition 2.1 (Grassmannian).

Let $k \leq n$ be a pair of positive integers. We define the (n, k) -**Grassmannian**, denoted¹ $\text{Gr}(k, n, \mathbb{K})$, as the set of $(n - k)$ -dimensional \mathbb{K} -vector subspaces of \mathbb{K}^n .

Remark 2.2 (Definition via quotients).

We may equivalently define $\text{Gr}(k, n)$ to be the following set:

$$\{\ker \varphi \mid \varphi \in \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k), \text{rk } \varphi = k\}.$$

Lemma 2.3.

Let $\varphi, \psi \in \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ be linear maps of full rank. The following conditions are equivalent:

1. $\ker \varphi = \ker \psi$,
2. there exists $\theta \in \text{GL}(\mathbb{K}^k)$ such that $\varphi = \theta \circ \psi$.

Proof.

We shall prove the two implications:

$$\boxed{2. \implies 1.} \quad \ker(\theta \circ \psi) = \psi^{-1}(\ker \theta) = \psi^{-1}(\{0\}) = \ker \psi.$$

$$\boxed{1. \implies 2.} \quad \text{Let } z_1, \dots, z_{n-k} \text{ be a basis of } \ker \varphi = \ker \psi \text{ and let } z_1, \dots, z_{n-k}, v_1, \dots, v_k \text{ be a completion of it to a basis of } \mathbb{K}^n. \text{ By construction } \varphi(v_1), \dots, \varphi(v_k) \text{ and } \psi(v_1), \dots, \psi(v_k) \text{ are bases of } \mathbb{K}^k. \text{ Let } \theta \text{ be the linear automorphism of } \mathbb{K}^k \text{ determined by } \theta(\psi(v_i)) = \varphi(v_i) \text{ for all } i. \text{ By construction } \theta \text{ is nonsingular and } \varphi \text{ agrees with } \theta \circ \psi \text{ on a basis of } \mathbb{K}^n.$$

□

Corollary 2.4.

We may identify Grassmannians in terms of linear maps as follows:

$$\text{Gr}(k, n) = \{\varphi \in \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \varphi \text{ surjective.}\} / \sim$$

where $\varphi \sim \psi$ if and only if $\exists \theta \in \text{GL}(\mathbb{K}^k)$ such that $\varphi = \theta \circ \psi$.

¹we shall often omit the field when clear from context

2.1.1 The Plücker embedding

To make the study of Grassmannians easier, we want to identify $\text{Gr}(k, n)$ with a subset of some projective space.

Notation 2.5.

We shall use the following notation for brevity:

$$\left\{ \alpha : (\mathbb{K}^n)^k \rightarrow \bigwedge^k \mathbb{K}^k \mid \alpha \text{ multilinear, alternating} \right\} = \bigwedge^k \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$$

Definition 2.6 (Plücker map).

Let $k \leq n$ be a pair of positive integers. We define the **Plücker map** as:

$$\phi : \begin{array}{ccc} \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \bigwedge^k \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \\ \varphi & \longmapsto & \bigwedge^k \varphi \end{array},$$

where $(\bigwedge^k \varphi)(v_1, \dots, v_k) = \varphi(v_1) \wedge \dots \wedge \varphi(v_k) = \det(\varphi(v_1) \mid \dots \mid \varphi(v_k)) e_1 \wedge \dots \wedge e_k$.

Remark 2.7.

The codomain of the Plücker map is isomorphic to $\bigwedge^k \mathbb{K}^n$, indeed

$$\bigwedge^k \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \xrightarrow{\text{Univ. Prop.}} \text{Hom}_{\mathbb{K}}\left(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k\right) \cong \left(\bigwedge^k \mathbb{K}^n\right)^{\vee} \cong \bigwedge^k \mathbb{K}^n.$$

Under this isomorphism the map takes on the following form

$$\begin{array}{ccc} \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ \varphi & \longmapsto & \sum_{1 \leq i_1 < \dots < i_k \leq n} \det(\varphi(e_{i_1}) \mid \dots \mid \varphi(e_{i_k})) e_{i_1} \wedge \dots \wedge e_{i_k}. \end{array}$$

Remark 2.8.

The image of the Plücker map is a cone.

Proof.

For any $\lambda \in \mathbb{K}^*$ and any map $\varphi \in \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ we see that

$$\lambda \phi(\varphi) = \phi(\alpha \circ \varphi),$$

for any automorphism α of \mathbb{K}^k with determinant² λ . □

Remark 2.9.

$\text{rk } \varphi < k$ if and only if $\phi(\varphi) = 0$.

Proof.

$\phi(\varphi)$ is the zero map if and only if $\varphi(v_1), \dots, \varphi(v_k)$ are always linearly dependent, i.e. if and only if φ is not of full rank. □

Lemma 2.10.

Let $\varphi : \mathbb{K}^n \rightarrow \mathbb{K}^k$ be a full rank linear map, then

$$\ker \varphi = \{z \in \mathbb{K}^n \mid \forall w_2, \dots, w_k \in \mathbb{K}^n, \phi(\varphi)(z, w_2, \dots, w_k) = 0\}.$$

²For example $\alpha(e_i) = \begin{cases} \lambda e_1 & \text{if } i = 1 \\ e_i & \text{otherwise} \end{cases}$

Proof.

If $\varphi(z) = 0$ then for any $w_2, \dots, w_k \in \mathbb{K}^k$ we see that

$$\phi(\varphi)(z, w_2, \dots, w_k) = 0 \wedge \varphi(w_2) \wedge \dots \wedge \varphi(w_k) = 0.$$

Suppose now that $\varphi(z) \neq 0$ and let v_2, \dots, v_k be such that $\{\varphi(z), v_2, \dots, v_k\}$ form a basis for \mathbb{K}^k . Since φ is surjective there exist w_2, \dots, w_k such that $\varphi(w_i) = v_i$ for all $2 \leq i \leq k$. By construction

$$\phi(\varphi)(z, w_2, \dots, w_k) = \varphi(z) \wedge v_2 \wedge \dots \wedge v_k \neq 0.$$

□

Proposition 2.11.

Let \sim be the equivalence relation defined in corollary (2.4), then for any two full rank linear maps $\varphi, \psi : \mathbb{K}^n \rightarrow \mathbb{K}^k$

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{K}^* \text{ s.t. } \phi(\varphi) = \lambda \phi(\psi).$$

Proof.

Let us prove both implications:

\Rightarrow If $\varphi = \theta \circ \psi$ for $\theta \in GL(\mathbb{K}^k)$ then it follows easily from known properties of the determinant that

$$\phi(\varphi) = \phi(\theta \circ \psi) = (\det \theta) \phi(\psi).$$

\Leftarrow From lemma (2.3) we see that it is enough to prove that $\ker \varphi = \ker \psi$. We conclude by applying lemma (2.10) as follows:

$$\begin{aligned} \ker \varphi &= \{z \in \mathbb{K}^n \mid \forall w_2, \dots, w_k \in \mathbb{K}^k, \phi(\varphi)(z, w_2, \dots, w_k) = 0\} \\ &= \{z \in \mathbb{K}^n \mid \forall w_2, \dots, w_k \in \mathbb{K}^k, \lambda \phi(\psi)(z, w_2, \dots, w_k) = 0\} = \\ &= \{z \in \mathbb{K}^n \mid \forall w_2, \dots, w_k \in \mathbb{K}^k, \phi(\psi)(z, w_2, \dots, w_k) = 0\} = \ker \psi. \end{aligned}$$

□

Remark 2.12.

Because of proposition (2.11) and remark (2.9) there exists a unique h such that the diagram commutes

$$\begin{array}{ccc} \{\varphi \in \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \text{rk } \varphi = k\} & \xrightarrow{[\phi]} & \mathbb{P}(\bigwedge^k \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)) \\ \downarrow \pi & \nearrow h & \\ \text{Gr}(k, n) & & \end{array}$$

Moreover, such an h must be injective by proposition (2.11).

Definition 2.13 (Plücker embedding).

We define the **Plücker embedding** as follows

$$\text{Pl} : \begin{array}{ccc} \text{Gr}(k, n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [\varphi]_{\sim} & \longmapsto & [(\det(\varphi(e_{i_1}) \mid \dots \mid \varphi(e_{i_k})))_{1 \leq i_1 < \dots < i_k \leq n}]_{\mathbb{K}^*} \end{array}.$$

Remark 2.14.

If ζ is the isomorphism $\bigwedge^k \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \rightarrow \bigwedge^k \mathbb{K}^n$ discussed during remark (2.7), we see that the following diagram commutes

$$\begin{array}{ccc} \mathbb{P}(\bigwedge^k \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)) & \xrightarrow{\mathbb{P}(\zeta)} & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ \uparrow h & \nearrow \text{Pl} & \\ \text{Gr}(k, n) & & \end{array}$$

This proves that the Plücker embedding is well defined and injective.

We have thus identified $\text{Gr}(k, n)$ with a subset of some projective space. It is possible to show that this subset is in fact Zariski closed³

2.2 Definition as a projective scheme

In order to write \mathbb{K} -algebra morphisms that correspond to what we've done geometrically, we shall switch to the language of matrices.

Definition 2.15 (Multiindices).

We define a (k, n) -**multiindex** as an element of $\{1, \dots, n\}^k$. Our notation for a multiindex I will usually be $I = (i_1, \dots, i_k)$.

If A is a $k \times n$ matrix and I is a (k, n) -multiindex, we denote the I -minor by A_I , i.e.

$$A_I = \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{pmatrix}.$$

We denote the set of **ordered** (k, n) -**multiindices** with

$$\omega(k, n) = \left\{ (i_1, \dots, i_k) \in \{1, \dots, n\}^k \mid i_1 < \dots < i_k \right\}.$$

Remark 2.16.

The set

$$\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$$

forms a basis for $\bigwedge^k \mathbb{K}^n$. For brevity, for all multiindices $I = (i_1, \dots, i_k)$ we shall define

$$e_I = e_{i_1} \wedge \dots \wedge e_{i_k}.$$

Under the isomorphism $\text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \cong \mathcal{M}(k, n)$ given by choosing a basis, we may redefine

$$\text{Gr}(k, n) = \{A \in \mathcal{M}(k, n) \mid \text{rk } A = k\} / \sim,$$

where $A \sim B \iff \exists P \in \text{GL}_k \text{ s.t. } A = PB$,

$$\phi : \begin{array}{ccc} \mathcal{M}(k, n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \sum_{I \in \omega(k, n)} \det A_I e_I \end{array}$$

and

$$\text{Pl} : \begin{array}{ccc} \text{Gr}(k, n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [A]_{\sim} & \longmapsto & [\sum_{I \in \omega(k, n)} \det A_I e_I]_{\mathbb{K}^*} \end{array}.$$

³CITAZIONE DA INSERIRE BENE, PER IL MOMENTO QUESTO È IL LINK AD UN PDF CHE TRATTA L'ARGOMENTO <https://www.math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Hudec.pdf>

Remark 2.17.

If we interpret $\mathcal{M}(k, n)$ as \mathbb{A}^{kn} and $\bigwedge^k \mathbb{K}^n$ as $\mathbb{A}^{\binom{n}{k}}$ then we see that ϕ is a polynomial map and thus a morphism of affine varieties.

To connect Grassmannians to the world of representable functors we shall redefine them scheme-theoretically by mimicing the Plücker embedding using rings

Definition 2.18 (Braket ring).

We define the **braket ring** as the ring of polynomial functions on $\bigwedge^k \mathbb{K}^n$, i.e.

$$\mathcal{B}_{k,n} \doteq \frac{\mathbb{K}[z_I \mid I \in \{1, \dots, n\}^k]}{(\{z_I - \operatorname{sgn}(\sigma) z_{\sigma(I)}\}_{\sigma \in S_k})} \cong \mathbb{K}[z_I \mid I \in \omega(k, n)]$$

Notation 2.19.

Let $\mathbb{K}[x_{1,1}, \dots, x_{k,n}]$ denote the polynomial ring with $k \cdot n$ variables. We will interpret this as the coordinate ring of $\mathcal{M}(k, n)$. Following this description we denote the **generic matrix** by

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \cdots & x_{k,n} \end{pmatrix}$$

and by the same token we denote by X_I the generic $k \times k$ minor determined by the multiindex I and by $\det X_I$ the formal determinant of this minor.

We shall also denote $\mathbb{K}[x_{1,1}, \dots, x_{k,n}]$ by the compacter notation $\mathcal{O}(\mathcal{M}(k, n))$.

Definition 2.20 (Plücker ring homomorphism).

We define the **Plücker ring homomorphism** as

$$\phi^\# : \begin{array}{ccc} \mathcal{B}_{k,n} & \longrightarrow & \mathcal{O}(\mathcal{M}(k, n)) \\ z_I & \longmapsto & \det X_I \end{array}$$

Proposition 2.21.

The induced map $\operatorname{Spec} \phi^\# : \mathbb{A}_{\mathbb{K}}^{kn}(\mathbb{K}) \rightarrow \mathbb{A}_{\mathbb{K}}^{\binom{n}{k}}(\mathbb{K})$ has the expected behaviour, i.e.

2.3 Moduli functor

2.3.1 Affine cover

2.3.2 Representability of the Grassmannian functor