Fine moduli spaces
the case of Grassmannians

Francesco Sorce

Università di Pisa
Dipartimento di Matematica

Contents

1 Moduli Spaces **2**

1.1 Introduction to moduli problems 2

1.2 Fine and Coarse moduli spaces 2

1.3 Zariski sheaves and gluing of fine moduli spaces 2

2 Grassmannians **6**

2.1 Set-theoretic definition 6

2.1.1 The Plücker embedding 7

2.2 Definition as a projective scheme 9

2.3 Moduli functor 11

2.3.1 Affine cover 12

2.3.2 Representability of the Grassmannian functor 12

Chapter 1

Moduli Spaces

Set theoretic issues: whenever I write that something is an element of a class, what I mean is that that object satisfies the proposition that defines the class.

1.1 Introduction to moduli problems

Definition 1.1 (Presheaf).
A contravariant functor $F : \mathcal{C}^{op} \to \text{Set}$ is called a **presheaf** on $\mathcal{C}$.

Definition 1.2 (Moduli problem).
Let S be a scheme. A presheaf on Sch/S is called a **moduli problem**.

Theorem 1.3 (Yoneda Lemma).
[TO DO]

Lemma 1.4.
The Yoneda embedding preserves limits.

Proof.
Suppose X is the limit of the diagram $\{f_{ij} : X_j \to X_i\}$. If we apply the Yoneda embedding to the diagram we obtain

$$\left\{\circ f_{ij} : h_{X_j} \to h_{X_i}\right\}$$

Let F be any presheaf on $\mathcal{C}$ and suppose that we have morphisms $F \to h_{X_i}$ which make the diagram commute, then for all $T \in \mathcal{C}$ we have compatible and natural $F(T) \to \text{Hom}(T, X_i)$. If $f \in T(T)$ then these arrows define several $f_i \in \text{Hom}(T, X_i)$ which compose respecting the diagram. By the universal property of limits this defines uniquely a morphism $f_\ell \in \text{Hom}(T, X)$ and we see that the assignment $f \mapsto f_\ell$ is the unique map from $F(T)$ to $\text{Hom}(T, X)$ which makes the diagram in Set commute. Since all that we have done is natural in T we have effectively constructed a morphism $F \to h_X$ as we desired. $\square$

1.2 Fine and Coarse moduli spaces

1.3 Zariski sheaves and gluing of fine moduli spaces

Definition 1.5 (Equalizer).
Let $\mathcal{C}$ be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \to C$. We say that the pair (A, h) is an

equalizer of the diagram

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

if $h : A \to B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \to A$ which makes the diagram commute

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ Q \arrow[u, dashed] \arrow[ur, "q"'] & & \end{tikzcd}$$

Definition 1.6 (Zariski sheaf)**.**
A moduli problem $F \in (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a **Zariski sheaf** if for any S-scheme X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Proposition 1.7 (Representable moduli functors are Zariski sheaves)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem, then if there exists a fine moduli space M for F it must be the case that F is a Zariski sheaf.

Proof.
By composing with the natural isomorphism we may assume $F = h_M$. Let X be an S-scheme and $\{U_i \to X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\mathrm{Mor}(X, M) \longrightarrow \prod_k \mathrm{Mor}(U_k, M) \rightrightarrows \prod_{i,j} \mathrm{Mor}(U_i \cap U_j, M)$$

The arrows in this case correspond to restriction of morphisms, so the thesis is equivalent to the fact that restriction to a given set doesn't depend on the intermediate restrictions and that morphisms of schemes that coincide on double intersections glue to the union, both of which are true. $\square$

Definition 1.8 (Subfunctor)**.**
Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. A **subfunctor** of G is a pair (F, i) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $i : F \to G$ such that $i_X : F(X) \to G(X)$ is a monomorphism for all $X \in \mathcal{C}$.

Remark 1.9.
If $\mathcal{D} = \mathrm{Set}$ then we can express the same data equivalently as follows:
A functor $F : \mathcal{C} \to \mathrm{Set}$ is a subfunctor of $G : \mathcal{C} \to \mathrm{Set}$ if for all $X \in \mathcal{C}$ and for all $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$

$$F(X) \subseteq G(X), \qquad \text{and} \qquad F(f) = G(f)|_{F(A)}.$$

In this case we write $F \subseteq G$.

Definition 1.10 (Fibered product of presheaves)**.**
Let $F, G, H : \mathcal{C}^{op} \to \text{Set}$ be presheaves together with two natural transformations $\xi^1 : F \to H$ and $\xi^2 : G \to H$. We define their fibered product as follows:
If $X \in \mathcal{C}$ then

$$(F \times_H G)(X) = F(X) \times_{H(X)} G(X),$$

if $f : A \to B$ then[1]

$$(F \times_H G)(f) : \begin{array}{ccc} F(B) \times_{H(B)} G(B) & \longrightarrow & F(A) \times_{H(A)} G(A) \\ (b_1, b_2) & \longmapsto & (F(f)(b_1), G(f)(b_2)) \end{array}.$$

Definition 1.11 (Open subfunctor)**.**
Let $F : (\text{Sch}/S)^{op} \to \text{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S-scheme T and any natural trasformation $h_T \to F$, the pullback $h_T \times_F G$ is representable by an open subscheme of T.

Remark 1.12.
By the Yoneda lemma, giving a natural transformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows:
A subfunctor $G \subseteq F$ is open if for any S-scheme T and any family $\xi \in F(T)$ there exists an open subscheme $U \subseteq T$ such that the following diagram is natural in R and commutes

$$\begin{tikzcd} \text{Mor}(R, U) \arrow[r, "G(\subseteq \circ\ \cdot\ )(\xi)"] \arrow[d, hook, "\subseteq\circ"'] & G(R) \arrow[d, hook, "\subseteq"] \\ \text{Mor}(R, T) \arrow[r, "F(\cdot)(\xi)"'] & F(R) \end{tikzcd}$$

and a map $f \in \text{Mor}(R, T)$ factors as $R \xrightarrow{g} U \subseteq T$ <u>if and only if</u> $F(f)(\xi) \in G(R)$[2].

Definition 1.13 (Open cover of a functor)**.**
Let $F : (\text{Sch}/S)^{op} \to \text{Set}$ be a moduli problem. A collection of open subfunctors $\{F_i\}$ is an **open cover** of F if for any S-scheme T and any natural transformation $h_T \to F$, the open subschemes U_i that represent the pullbacks $h_T \times_F F_i$ form an open cover of T.

Remark 1.14.
Like above, we can rephrase the definition as follows:
A collections of open subfunctors $F_i \subseteq F$ form an open cover of F if for any S-scheme T and any family $\xi \in F(T)$, there exists an open cover $\{U_i\}$ of T such that $\xi|_{U_i} \in F_i(U_i)$ for all i.

The definitions above let us state the following criterion for representability:

Proposition 1.15 (Representability by open cover)**.**
Let $F : (\text{Sch}/S)^{op} \to \text{Set}$ *be a moduli problem which is a Zariski sheaf and let* $\{F_i\}$ *be an open cover of it by representable subfunctors, then* F *is representable.*

Proof.
Let X_i be the fine moduli space for F_i and let $\xi_i \in F_i(X_i)$ be their universal families. Note that for all S-schemes T we have

$$(F_i \times_F F_j)(T) = F_i(T) \times_{F(T)} F_j(T) = F_i(T) \cap F_j(T) \subseteq F(T),$$

[1] the map is well defined because $\xi^1_A(F(f)(b_1)) = H(f)(\xi^1_B(b_1)) = H(f)(\xi^2_B(b_2)) = \xi^2_A(G(f)(b_2))$.
[2] the "only if" is trivially true by commutativity but for the "if" we are using the fact that $h_U \cong h_T \times_F G$.

it follows that $F_i \times_F F_j = F_j \times_F F_i \doteqdot F_{i,j}$. We can define analogously $F_{i,j,k}$.

Since F_j is an open subfunctor of F, there exists an open subscheme $U_{ij} \subseteq X_i$ which represents $h_{X_i} \times_F F_j \cong F_i \times_F F_j = F_{i,j}$. We can define $U_{ji} \subseteq X_j$ similarly and since they are both moduli spaces for $F_{i,j}$ they are isomorphic. Let $\varphi_{ji} : U_{ij} \to U_{ji}$ be the isomorphism given by $\varphi_{ji} = \alpha_{U_{ij}}(id_{U_{ij}})$ for α natural isomorphism which makes the following diagram commute

$$\begin{array}{ccccc} h_{U_{ij}} & \cong & h_{X_i} \times_F F_j & \cong & F_{i,j} \\ {\scriptstyle\alpha}\downarrow & & & & \| \\ h_{U_{ji}} & \cong & F_i \times_F h_{X_j} & \cong & F_{i,j} \end{array}$$

Note that if T is an S-scheme and $f \in h_{U_{ij}}(T)$ then

$$h_{\varphi_{ji}}(f) = \alpha_{U_{ij}}(id_{U_{ij}}) \circ f = \alpha_T(id_{U_{ij}} \circ f) = \alpha_T(f),$$

so α is the image of φ_{ji} under the Yoneda embedding.

We now want to show that the X_i can be glued along the U_{ij} using the isomorphisms φ_{ji}. First we need to show that $\varphi_{ji}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and then we have to verify the cocycle condition $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$.
The first condition follows immediately from the fact that F_k is an open subfunctor and our construction of the φ_{ji}.
Since the Yoneda embedding preserves limits (1.4) it preserves fibered products, so we see that the following diagram commutes

$$\begin{array}{ccccccc} h_{U_{ij} \cap U_{ik}} & \xrightarrow{\text{Yon.}} & h_{U_{ij}} \times_{h_{X_i}} h_{U_{ik}} & \cong & F_{i,j} \times_{F_i} F_{i,k} & \cong & F_{i,j,k} \\ {\scriptstyle\varphi_{ji}}\downarrow & & & & & & \| \\ h_{U_{ji} \cap U_{jk}} & \xrightarrow{\text{Yon.}} & h_{U_{ji}} \times_{h_{X_j}} h_{U_{jk}} & \cong & F_{i,j} \times_{F_j} F_{j,k} & \cong & F_{i,j,k} \end{array}$$

thus to prove that $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ it is enough to see that $id_{F_{i,j,k}} \circ id_{F_{i,j,k}} = id_{F_{i,j,k}}$.

We can thus define X to be the scheme obtained by gluing the X_i along the U_{ij}.
Observe that $\xi_i = \varphi_{ji}^* \xi_j$, so if we look at these families as elements of $F(X)$ we see that $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$. Since F is a Zariski sheaf, the ξ_i can be glued to a family $\xi \in F(X)$.

We now only need to verify that (X, ξ) is a fine moduli space for F:
Let T be an S-scheme and and let us consider a family $\zeta \in F(T)$. Since $\{F_i\}$ is an open cover of F, there exists an open cover $\{V_i\}$ of T such that $\zeta|_{V_i} \in F_i(V_i) \cong \mathrm{Mor}(V_i, X_i)$. Since F is a sheaf and $\zeta_i|_{V_i \cap V_j} = \zeta_j|_{V_i \cap V_j}$, the morphisms $V_i \to X_i$ corresponding to the ζ_i glue to a morphism $f : T \to X$ such that $f^*\xi = \zeta$ (by construction). $\square$

Chapter 2

Grassmannians

2.1 Set-theoretic definition

Definition 2.1 (Grassmannian)**.**
Let $k \leq n$ be a pair of positive integers. We define the (n,k)**-Grassmannian**, denoted[1] $\mathrm{Gr}(k,n,\mathbb{K})$, as the set of $(n-k)$-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^n$.

Remark 2.2 (Definition via quotients)*.*
We may equivalently define $\mathrm{Gr}(k,n)$ to be the following set:

$$\left\{\ker\varphi \mid \varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k),\ \operatorname{rnk}\varphi = k\right\}.$$

Lemma 2.3.
Let $\varphi, \psi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ be linear maps of full rank. The following conditions are equivalent:

1. $\ker\varphi = \ker\psi$*,*
2. *there exists $\theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta \circ \psi$.*

Proof.
We shall prove the two implications:

$\boxed{2. \implies 1.}$ $\ker(\theta\circ\psi) = \psi^{-1}(\ker\theta) = \psi^{-1}(\{0\}) = \ker\psi$.

$\boxed{1. \implies 2.}$ Let $z_1, \cdots, z_{n-k}$ be a basis of $\ker\varphi = \ker\psi$ and let $z_1, \cdots, z_{n-k}, v_1, \cdots, v_k$ be a completion of it to a basis of $\mathbb{K}^n$. By construction $\varphi(v_1), \cdots, \varphi(v_k)$ and $\psi(v_1), \cdots, \psi(v_k)$ are bases of $\mathbb{K}^k$. Let θ be the linear automorphism of $\mathbb{K}^k$ determined by $\theta(\psi(v_i)) = \varphi(v_i)$ for all i. By construction θ is nonsingular and φ agrees with $\theta\circ\psi$ on a basis of $\mathbb{K}^n$.

□

Corollary 2.4.
We may identify Grassmannians in terms of linear maps as follows:

$$\mathrm{Gr}(k,n) = {\left\{\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \varphi \text{ surjective.}\right\}}\big/{\sim}$$

where $\varphi \sim \psi$ if and only if $\exists\theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta\circ\psi$.

[1] we shall often omit the field when clear from context

2.1.1 The Plücker embedding

To make the study of Grassmannians easier, we want to identify $\mathrm{Gr}(k, n)$ with a subset of some projective space.

Notation 2.5.
We shall use the following notation for brevity:

$$\left\{\alpha : (\mathbb{K}^n)^k \to \bigwedge^k \mathbb{K}^k \mid \alpha \text{ multilinear, alternating}\right\} = \bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$$

Definition 2.6 (Plücker map)**.**
Let $k \leq n$ be a pair of positive integers. We define the **Plücker map** as:

$$\phi : \begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \\ \varphi & \longmapsto & \wedge^k \varphi \end{array},$$

where $(\wedge^k \varphi)(v_1, \cdots, v_k) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_k) = \det\big(\varphi(v_1)|\cdots|\varphi(v_k)\big)\, e_1 \wedge \cdots \wedge e_k$.

Remark 2.7.
The codomain of the Plücker map is isomorphic to $\bigwedge^k \mathbb{K}^n$, indeed

$$\bigwedge^k \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}^n, \mathbb{K}^k\right) \overset{\text{Univ.Prop.}}{\cong} \mathrm{Hom}_{\mathbb{K}}\left(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k\right) \cong \left(\bigwedge^k \mathbb{K}^n\right)^{\vee} \cong \bigwedge^k \mathbb{K}^n.$$

Under this isomorphism the map takes on the following form

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ \varphi & \longmapsto & \displaystyle\sum_{1 \leq i_1 < \cdots < i_k \leq n} \det(\varphi(e_{i_1}) \mid \cdots \mid \varphi(e_{i_k})) e_{i_1} \wedge \cdots \wedge e_{i_k} \end{array}.$$

Remark 2.8.
The image of the Plücker map is a cone.

Proof.
For any $\lambda \in \mathbb{K}^*$ and any map $\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ we see that

$$\lambda \phi(\varphi) = \phi(\alpha \circ \varphi),$$

for any automorphism α of $\mathbb{K}^k$ with determinant[2] λ. □

Remark 2.9.
$\mathrm{rnk}\, \varphi < k$ if and only if $\phi(\varphi) = 0$.

Proof.
$\phi(\varphi)$ is the zero map if an only if $\varphi(v_1), \cdots, \varphi(v_k)$ are always linearly dependent, i.e. if and only if φ is not of full rank. □

Lemma 2.10.
Let $\varphi : \mathbb{K}^n \to \mathbb{K}^k$ be a full rank linear map, then

$$\ker \varphi = \{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \phi(\varphi)(z, w_2, \cdots, w_k) = 0\}.$$

[2]For example $\alpha(e_i) = \begin{cases} \lambda e_1 & \text{if } i = 1 \\ e_i & \text{otherwise} \end{cases}$

Proof.
If $\varphi(z) = 0$ then for any $w_2, \cdots, w_k \in \mathbb{K}^k$ we see that

$$\phi(\varphi)(z, w_2, \cdots, w_k) = 0 \wedge \varphi(w_2) \wedge \cdots \wedge \varphi(w_k) = 0.$$

Suppose now that $\varphi(z) \neq 0$ and let $v_2, \cdots, v_k$ be such that $\{\varphi(z), v_2, \cdots, v_k\}$ form a basis for $\mathbb{K}^k$. Since φ is surjective there exist $w_2, \cdots, w_k$ such that $\varphi(w_i) = v_i$ for all $2 \leq i \leq k$. By construction

$$\phi(\varphi)(z, w_2, \cdots, w_k) = \varphi(z) \wedge v_2 \wedge \cdots \wedge v_k \neq 0.$$

□

Proposition 2.11.
Let $\sim$ be the equivalence relation defined in corollary (2.4), then for any two full rank linear maps $\varphi, \psi : \mathbb{K}^n \to \mathbb{K}^k$

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{K}^* \ s.t. \ \phi(\varphi) = \lambda \phi(\psi).$$

Proof.
Let us prove both implications:

$\boxed{\Longrightarrow}$ If $\varphi = \theta \circ \psi$ for $\theta \in GL(\mathbb{K}^k)$ then it follows easily from known properties of the determinant that

$$\phi(\varphi) = \phi(\theta \circ \psi) = (\det \theta)\phi(\psi).$$

$\boxed{\Longleftarrow}$ From lemma (2.3) we see that it is enough to prove that $\ker \varphi = \ker \psi$. We conclude by applying lemma (2.10) as follows:

$$\begin{aligned}
\ker \varphi &= \{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n, \ \phi(\varphi)(z, w_2, \cdots, w_k) = 0\} \\
&= \{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n, \ \lambda\phi(\psi)(z, w_2, \cdots, w_k) = 0\} = \\
&= \{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n, \ \phi(\psi)(z, w_2, \cdots, w_k) = 0\} = \ker \psi.
\end{aligned}$$

□

Remark 2.12.
Because of proposition (2.11) and remark (2.9) there exists a unique h such that the diagram commutes

$$\begin{tikzcd}
\{\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \mathrm{rnk}\, \varphi = k\} \arrow[r, "{[\phi]}"] \arrow[d, "\pi"] & \mathbb{P}(\bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)) \\
\mathrm{Gr}(k, n) \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

Moreover, such an h must be injective by proposition (2.11).

Definition 2.13 (Plücker embedding)**.**
We define the **Plücker embedding** as follows

$$\mathrm{Pl} : \begin{array}{ccc} \mathrm{Gr}(k, n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [\varphi]_\sim & \longmapsto & [(\det(\varphi(e_{i_1}) \mid \cdots \mid \varphi(e_{i_k})))_{1 \leq i_1 < \cdots < i_k \leq n}]_{\mathbb{K}^*} \end{array}.$$

Remark 2.14.
If ζ is the isomorphism $\bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \to \bigwedge^k \mathbb{K}^n$ discussed during remark (2.7), we see that the following diagram commutes

$$\begin{tikzcd} \mathbb{P}(\bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)) \arrow[r, "\mathbb{P}(\zeta)"] & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ \mathrm{Gr}(k,n) \arrow[u, "h"] \arrow[ur, "\mathrm{Pl}"'] & \end{tikzcd}$$

This proves that the Plücker embedding is well defined and injective.

We have thus identified $\mathrm{Gr}(k, n)$ with a subset of some projective space. We now need to show that it is a closed subset in the Zariski topology.*********************** *******************************

Let us consider the following map: let $\psi : \bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ be any alternating multilinear map, we define $\Phi(\psi)$ as

$$\Phi(\psi) : \begin{array}{ccc} \mathbb{K}^n & \longrightarrow & \bigwedge^{k+1} \mathbb{K}^n \\ v & \longmapsto & \displaystyle\sum_{I \in \omega(k,n)} \psi(e_{i_1}, \cdots, e_{i_k}) e_I \wedge v \end{array} .$$

Proposition 2.15.
An alternating multilinear map $\psi \in \bigwedge^k \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ *is in the image of the Plücker map* ϕ *if and only if* $\Phi(\psi)$ *has rank at most* $n - k$*.*

Proof.
Suppose that $\psi = \phi(\varphi)$ and let $\{z_1, \cdots, z_k, z_{k+1}, \cdots, z_n\}$ be a basis of $\mathbb{K}^n$ such that the first k vectors are a basis of $\ker \varphi$. Then

$$\Phi(\psi)(v) = \sum_{I \in \omega(k,n)} \det(\varphi(z_{i_1} \mid \cdots \mid \varphi(z_{i_k}))) z_I \wedge v.$$

Notice that if $v \in \ker \varphi$ then for any choice of I either the determinant vanishes or $k < i_1$********************************* $\square$

2.2 Definition as a projective scheme

In order to write $\mathbb{K}$-algebra morphisms that correspond to what we've done geometrically, we shall switch to the language of matricies.

Definition 2.16 (Multiindicies)**.**
We define a (k, n)**-multiindex** as an element of $\{1, \cdots, n\}^k$. Our notation for a multiindex I will usually be $I = (i_1, \cdots, i_k)$.
If A is a $k \times n$ matrix and I is a (k, n)-multiindex, we denote the I-minor by A_I, i.e.

$$A_I = \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{pmatrix}.$$

We denote the set of **ordered** (k, n)**-multiindicies** with

$$\omega(k, n) = \left\{ (i_1, \cdots, i_k) \in \{1, \cdots, n\}^k \mid i_1 < \cdots < i_k \right\}.$$

Remark 2.17.
The set
$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$
forms a basis for $\bigwedge^k \mathbb{K}^n$. For brevity, for all multiindicies $I = (i_1, \cdots, i_k)$ we shall define
$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Under the isomorphism $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \cong \mathcal{M}(k, n)$ given by choosing a basis, we may redefine
$$\mathrm{Gr}(k, n) = \{A \in \mathcal{M}(k, n) \mid \mathrm{rnk}\, A = k\}/\sim,$$
where $A \sim B \iff \exists P \in \mathrm{GL}_k \ s.t.\ A = PB$,
$$\phi : \begin{array}{ccc} \mathcal{M}(k, n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \sum_{I \in \omega(k,n)} \det A_I e_I \end{array}$$
and
$$\mathrm{Pl} : \begin{array}{ccc} Gr(k, n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [A]_\sim & \longmapsto & [\sum_{I \in \omega(k,n)} \det A_I e_I]_{\mathbb{K}^*} \end{array}.$$

To connect Grassmannians to the world of representable functors we shall redefine them scheme-theoretically by mimicing the Plücker embedding using rings

Definition 2.18 (Braket ring)**.**
We define the **braket ring** as the ring of polynomial functions on $\bigwedge^k \mathbb{K}^n$, i.e.
$$\mathcal{B}_{k,n} \doteqdot \frac{\mathbb{K}[z_I \mid I \in \{1, \cdots, n\}^k]}{(\{z_I - \mathrm{sgn}\,(\sigma) z_{\sigma(I)}\}_{\sigma \in S_k})} \cong \mathbb{K}[z_I \mid I \in \omega(k, n)]$$

Notation 2.19.
Let $\mathbb{K}[x_{1,1}, \cdots, x_{k,n}]$ denote the polynomial ring with $k \cdot n$ variables. We will interpret this as the coordinate ring of $\mathcal{M}(k, n)$. Following this description we denote the **generic matrix** by
$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \cdots & x_{k,n} \end{pmatrix}$$
and by the same token we denote by X_I the generic $k \times k$ minor determined by the multiindex I and by $\det X_I$ the formal determinant of this minor.
We shall also denote $\mathbb{K}[x_{1,1}, \cdots, x_{k,n}]$ by the compacter notation $\mathcal{O}(\mathcal{M}(k, n))$.

Remark 2.20.
The familiar $\mathcal{M}(k, n)$ and $\bigwedge^k \mathbb{K}^n$ can be identified with the $\mathbb{K}$-points of the affine schemes $\mathrm{Spec}\, \mathcal{O}(\mathcal{M}(k, n))$ and $\mathrm{Spec}\, \mathcal{B}_{k,n}$ respectively[3].

Definition 2.21 (Plücker ring homomorphism)**.**
We define the **Plücker ring homomorphism** as
$$\phi^\# : \begin{array}{ccc} \mathcal{B}_{k,n} & \longrightarrow & \mathcal{O}(\mathcal{M}(k, n)) \\ z_I & \longmapsto & \det X_I \end{array}$$

[3] Example 2.3.32 from Qing Liu

Proposition 2.22.
The induced map $\operatorname{Spec}\phi^\# : \mathbb{A}^{kn}(\mathbb{K}) \to \mathbb{A}^{\binom{n}{k}}(\mathbb{K})$ *is equal to the Plücker map* $\phi : \mathcal{M}(k,n) \to \bigwedge^k \mathbb{K}^n$ *under the afformentioned identification, i.e. for any matrix* $A \in \mathcal{M}(k,n)$ *with entries* $a_{i,j}$ *we have*

$$(\phi^\#)^{-1}((x_{i,j} - a_{i,j})) = (z_I - \det A_I).$$

Proof.
First we observe that for any multiindex I

$$\det X_I - \det A_I \in (x_{i,j} - a_{i,j}),$$

thus $(z_I - \det A_I) \subseteq (\phi^\#)^{-1}((x_{i,j} - a_{i,j}))$.
Since $(z_I - \det A_I)$ is a $\mathbb{K}$-point, it is in particular a maximal ideal of the Braket ring, thus we have the desired equality if $1 \notin (\phi^\#)^{-1}((x_{i,j} - a_{i,j}))$, which is the case because otherwise $(x_{i,j} - a_{i,j})$ would not be proper. $\square$

Lemma 2.23.
The kernel of the Plücker homomorphism is homogeneous.

Proof.
By definition of homogeneous ideal, we want to show that if $f = \sum f_d$ for d homogeneous and $f \in \ker \phi^\#$ then $f_d \in \ker \phi^\#$ for all d.
Looking at the definition of $\phi^\#$ we see that $\phi^\#(f_d)$ is a homogeneous polynomial of degree kd, in particular if $d \neq h$ then $\deg \phi^\#(f_d) \neq \deg \phi^\#(f_h)$. Since

$$0 = \phi^\#(f) = \sum \phi^\#(f_d)$$

this proves that $\phi^\#(f_d) = 0$ for all d. $\square$

Since $\operatorname{Imm}\phi$ is closed and[4] $V(\ker \phi^\#) = \overline{\operatorname{Imm}\phi}$, we can identify $\operatorname{Gr}(k,n)$ with $V_+(\ker \phi^\#)$. This identification corresponds to the equality

$$\mathbb{P}(\operatorname{Imm}\phi) = V_+(\ker \phi^\#)(\mathbb{K}).$$

2.3 Moduli functor

Let us consider the following functor

$$\mathbb{G}(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & Set \\ T & \longmapsto & \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q\}/\sim \end{array}$$

where Q is a locally free sheaf of rank k on T and two surjections $\alpha : \mathcal{O}_T^n \twoheadrightarrow Q$, $\beta : \mathcal{O}_T^n \twoheadrightarrow V$ are equivalent if and only if there exist an isomorphism of sheaves $\theta : Q \to V$ such that the diagram commutes

$$\begin{tikzcd} \mathcal{O}_T^n \arrow[r, two heads, "\alpha"] \arrow[dr, "\beta"'] & Q \arrow[d, "\theta"] \\ & V \end{tikzcd}$$

In this this section we will prove that the Grassmannian scheme $V_+(\ker \phi^\#)$ defined above represents this functor.

[4]Known result of algebraic geometry

2.3.1 Affine cover

2.3.2 Representability of the Grassmannian functor