Fine moduli spaces
the case of Grassmannians

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Chapter 1

Moduli Spaces

Set theoretic issues: whenever I write that something is an element of a class, what I mean is that that object satisfies the proposition that defines the class.

Most definitions given in this chapter follow [1].

1.1 Introduction to moduli problems

Definition 1.1 (Presheaf).
A contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is called a **presheaf** on $\mathcal{C}$.

Definition 1.2 (Moduli problem).
Let S be a scheme. A presheaf on Sch/S is called a **moduli problem**.

Theorem 1.3 (Yoneda Lemma).
[TO DO]

Lemma 1.4.
The Yoneda embedding preserves limits.

Proof.
Suppose X is the limit of the diagram $\{f_{ij} : X_j \to X_i\}$. If we apply the Yoneda embedding to the diagram we obtain

$$\left\{\circ f_{ij} : h_{X_j} \to h_{X_i}\right\}$$

Let F be any presheaf on $\mathcal{C}$ and suppose that we have morphisms $F \to h_{X_i}$ which make the diagram commute, then for all $T \in \mathcal{C}$ we have compatible and natural $F(T) \to \mathrm{Hom}\,(T, X_i)$. If $f \in T(T)$ then these arrows define several $f_i \in \mathrm{Hom}\,(T, X_i)$ which compose respecting the diagram. By the universal property of limits this defines uniquely a morphism $f_\ell \in \mathrm{Hom}\,(T, X)$ and we see that the assignment $f \mapsto f_\ell$ is the unique map from $F(T)$ to $\mathrm{Hom}\,(T, X)$ which makes the diagram in Set commute. Since all that we have done is natural in T we have effectively constructed a morphism $F \to h_X$ as we desired. $\square$

1.2 Fine and Coarse moduli spaces

1.3 Zariski sheaves and gluing of fine moduli spaces

Definition 1.5 (Equalizer).
Let $\mathcal{C}$ be a category, $A, B, C \in \mathcal{C}$ and $f, g : B \to C$. We say that the pair (A, h) is an

equalizer of the diagram

$$B \overset{f}{\underset{g}{\rightrightarrows}} C$$

if $h : A \to B$ is such that $f \circ h = g \circ h$ and if (Q, q) is another such pair then there exists a unique morphism $Q \to A$ which makes the diagram commute

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ Q \arrow[u, dashed] \arrow[ur, "q"'] & & \end{tikzcd}$$

Definition 1.6 (Zariski sheaf)**.**
A moduli problem $F \in (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a **Zariski sheaf** if for any S-scheme X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer

$$F(X) \longrightarrow \prod_k F(U_k) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

where the arrows are induced by the inclusions.

Proposition 1.7 (Representable moduli functors are Zariski sheaves)**.**
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem, then if there exists a fine moduli space M for F it must be the case that F is a Zariski sheaf.

Proof.
By composing with the natural isomorphism we may assume $F = h_M$. Let X be an S-scheme and $\{U_i \to X\}$ a Zariski open cover for it. We want to show that the following diagram is an equalizer

$$\mathrm{Mor}(X, M) \longrightarrow \prod_k \mathrm{Mor}(U_k, M) \rightrightarrows \prod_{i,j} \mathrm{Mor}(U_i \cap U_j, M)$$

The arrows in this case correspond to restriction of morphisms, so the thesis is equivalent to the fact that restriction to a given set doesn't depend on the intermediate restrictions and that morphisms of schemes that coincide on double intersections glue to the union, both of which are true. $\square$

Definition 1.8 (Subfunctor)**.**
Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. A **subfunctor** of G is a pair (F, i) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $i : F \to G$ such that $i_X : F(X) \to G(X)$ is a monomorphism for all $X \in \mathcal{C}$.

Remark 1.9.
If $\mathcal{D} = \mathrm{Set}$ then we can express the same data equivalently as follows:
A functor $F : \mathcal{C} \to \mathrm{Set}$ is a subfunctor of $G : \mathcal{C} \to \mathrm{Set}$ if for all $X \in \mathcal{C}$ and for all $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$

$$F(X) \subseteq G(X), \qquad \text{and} \qquad F(f) = G(f)|_{F(A)}.$$

In this case we write $F \subseteq G$.

Definition 1.10 (Fibered product of presheaves).
Let $F, G, H : \mathcal{C}^{op} \to \mathrm{Set}$ be presheaves together with two natural transformations $\xi^1 : F \to H$ and $\xi^2 : G \to H$. We define their fibered product as follows:
If $X \in \mathcal{C}$ then

$$(F \times_H G)(X) = F(X) \times_{H(X)} G(X),$$

if $f : A \to B$ then[1]

$$(F \times_H G)(f) : \begin{array}{ccc} F(B) \times_{H(B)} G(B) & \longrightarrow & F(A) \times_{H(A)} G(A) \\ (b_1, b_2) & \longmapsto & (F(f)(b_1), G(f)(b_2)) \end{array}.$$

Definition 1.11 (Open subfunctor).
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. We say that a subfunctor $G \subseteq F$ is **open** if for any S-scheme T and any natural trasformation $h_T \to F$, the pullback $h_T \times_F G$ is representable by an open subscheme of T.

Remark 1.12.
By the Yoneda lemma, giving a natural transformation like in the above definition is equivalent to choosing a family $\xi \in F(T)$. We can thus rephrase the definition as follows:
A subfunctor $G \subseteq F$ is open if for any S-scheme T and any family $\xi \in F(T)$ there exists an open subscheme $U \subseteq T$ such that the following diagram is natural in R and commutes

$$\begin{tikzcd} \mathrm{Mor}(R, U) \arrow[r, "G(\subseteq\circ\ \cdot\ )(\xi)"] \arrow[d, hook, "\subseteq\circ"'] & G(R) \arrow[d, hook, "\subseteq"] \\ \mathrm{Mor}(R, T) \arrow[r, "F(\cdot)(\xi)"'] & F(R) \end{tikzcd}$$

and a map $f \in \mathrm{Mor}(R, T)$ factors as $R \xrightarrow{g} U \subseteq T$ <u>if and only if</u> $F(f)(\xi) \in G(R)$[2].

Definition 1.13 (Open cover of a functor).
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem. A collection of open subfunctors $\{F_i\}$ is an **open cover** of F if for any S-scheme T and any natural transformation $h_T \to F$, the open subschemes U_i that represent the pullbacks $h_T \times_F F_i$ form an open cover of T.

Remark 1.14.
Like above, we can rephrase the definition as follows:
A collections of open subfunctors $F_i \subseteq F$ form an open cover of F if for any S-scheme T and any family $\xi \in F(T)$, there exists an open cover $\{U_i\}$ of T such that $\xi|_{U_i} \in F_i(U_i)$ for all i.

The definitions above let us state the following criterion for representability:

Theorem 1.15 (Representability by open cover).
Let $F : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ be a moduli problem which is a Zariski sheaf and let $\{F_i\}$ be an open cover of it by representable subfunctors, then F is representable.

Proof.
For this proof we will mainly follow the version presented in [4] (Theorem 8.9 at page 212).

[1] the map is well defined because $\xi^1_A(F(f)(b_1)) = H(f)(\xi^1_B(b_1)) = H(f)(\xi^2_B(b_2)) = \xi^2_A(G(f)(b_2))$.
[2] the "only if" is trivially true by commutativity but for the "if" we are using the fact that $h_U \cong h_T \times_F G$.

Let X_i be the fine moduli space for F_i and let $\xi_i \in F_i(X_i)$ be their universal families. Note that for all S-schemes T we have

$$(F_i \times_F F_j)(T) = F_i(T) \times_{F(T)} F_j(T) = F_i(T) \cap F_j(T) \subseteq F(T),$$

it follows that $F_i \times_F F_j = F_j \times_F F_i \doteqdot F_{i,j}$. We can define analogously $F_{i,j,k}$.

Since F_j is an open subfunctor of F, there exists an open subscheme $U_{ij} \subseteq X_i$ which represents $h_{X_i} \times_F F_j \cong F_i \times_F F_j = F_{i,j}$. We can define $U_{ji} \subseteq X_j$ similarly and since they are both moduli spaces for $F_{i,j}$ they are isomorphic. Let $\varphi_{ji} : U_{ij} \to U_{ji}$ be the isomorphism given by $\varphi_{ji} = \alpha_{U_{ij}}(id_{U_{ij}})$ for α natural isomorphism which makes the following diagram commute

$$\begin{array}{ccccc} h_{U_{ij}} & \cong & h_{X_i} \times_F F_j & \cong & F_{i,j} \\ \alpha \big\downarrow & & & & \| \\ h_{U_{ji}} & \cong & F_i \times_F h_{X_j} & \cong & F_{i,j} \end{array}$$

Note that if T is an S-scheme and $f \in h_{U_{ij}}(T)$ then

$$h_{\varphi_{ji}}(f) = \alpha_{U_{ij}}(id_{U_{ij}}) \circ f = \alpha_T(id_{U_{ij}} \circ f) = \alpha_T(f),$$

so α is the image of φ_{ji} under the Yoneda embedding.

We now want to show that the X_i can be glued along the U_{ij} using the isomorphisms φ_{ji}. First we need to show that $\varphi_{ji}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and then we have to verify the cocycle condition $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$.
The first condition follows immediately from the fact that F_k is an open subfunctor and our construction of the φ_{ji}.
Since the Yoneda embedding preserves limits (1.4) it preserves fibered products, so we see that the following diagram commutes

$$\begin{array}{ccccccc} h_{U_{ij} \cap U_{ik}} & \xrightarrow{\text{Yon.}} & h_{U_{ij}} \times_{h_{X_i}} h_{U_{ik}} & \cong & F_{i,j} \times_{F_i} F_{i,k} & \cong & F_{i,j,k} \\ {\scriptstyle \varphi_{ji}} \big\downarrow & & & & & & \| \\ h_{U_{ji} \cap U_{jk}} & \xrightarrow{\text{Yon.}} & h_{U_{ji}} \times_{h_{X_j}} h_{U_{jk}} & \cong & F_{i,j} \times_{F_j} F_{j,k} & \cong & F_{i,j,k} \end{array}$$

thus to prove that $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ it is enough to see that $id_{F_{i,j,k}} \circ id_{F_{i,j,k}} = id_{F_{i,j,k}}$.

We can thus define X to be the scheme obtained by gluing the X_i along the U_{ij}.
Observe that $\xi_i = \varphi_{ji}^* \xi_j$, so if we look at these families as elements of $F(X)$ we see that $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$. Since F is a Zariski sheaf, the ξ_i can be glued to a family $\xi \in F(X)$.

We now only need to verify that (X, ξ) is a fine moduli space for F:
Let T be an S-scheme and and let us consider a family $\zeta \in F(T)$. Since $\{F_i\}$ is an open cover of F, there exists an open cover $\{V_i\}$ of T such that $\zeta|_{V_i} \in F_i(V_i) \cong \mathrm{Mor}(V_i, X_i)$. Since F is a sheaf and $\zeta_i|_{V_i \cap V_j} = \zeta_j|_{V_i \cap V_j}$, the morphisms $V_i \to X_i$ corresponding to the ζ_i glue to a morphism $f : T \to X$ such that $f^* \xi = \zeta$ (by construction). $\square$

Chapter 2

Classical Grassmannians

2.1 First definitions and conventions

Definition 2.1 (Grassmannian)**.**
Let $k \leq n$ be a pair of positive integers. We define the (n,k)**-Grassmannian**, denoted[1] $\mathrm{Gr}(k,n,\mathbb{K})$, as the set of $(n-k)$-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^n$.

Remark 2.2.
We may equivalently define $\mathrm{Gr}(k,n)$ to be the following set:

$$\left\{\ker\varphi \mid \varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n,\mathbb{K}^k),\ \mathrm{rnk}\,\varphi = k\right\}.$$

Proof.
If $H \in \mathrm{Gr}(k,n)$, let $z_1,\cdots,z_n$ be a basis of $\mathbb{K}^n$ such that $H = \mathrm{Span}\,(z_1,\cdots,z_{n-k})$ and let $e_1,\cdots,e_k$ be any basis of $\mathbb{K}^k$, then we can view H as the kernel of the (rank k) linear map defined by

$$\varphi(z_i) = \begin{cases} 0 & \text{if } i \leq n-k \\ e_{i-n+k} & \text{otherwise} \end{cases}$$

Viceversa, if φ is a rank k linear map then by the rank-nullity theorem its kernel is an $n-k$ dimentional subspace of $\mathbb{K}^n$. $\square$

Lemma 2.3.
Let $\varphi,\psi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n,\mathbb{K}^k)$ be linear maps of full rank. The following conditions are equivalent:

1. $\ker\varphi = \ker\psi$,
2. *there exists $\theta \in \mathrm{GL}(\mathbb{K}^k)$ such that $\varphi = \theta\circ\psi$.*

Proof.
Let us prove both implications:

$\boxed{2. \implies 1.}$ $\ker(\theta\circ\psi) = \psi^{-1}(\ker\theta) = \psi^{-1}(\{0\}) = \ker\psi$.

$\boxed{1. \implies 2.}$ Let $z_1,\cdots,z_n$ be a basis of $\mathbb{K}^n$ such that $\ker\varphi = \ker\psi = \mathrm{Span}\,(z_1,\cdots,z_{n-k})$. By construction $\varphi(z_{n-k+1}),\cdots,\varphi(z_n)$ and $\psi(z_{n-k+1}),\cdots,\psi(z_n)$ are bases of $\mathbb{K}^k$. Let θ be the linear automorphism of $\mathbb{K}^k$ determined by $\theta(\psi(z_i)) = \varphi(z_i)$ for all $n-k < i \leq n$. By construction θ is nonsingular and φ agrees with $\theta\circ\psi$ on a basis of $\mathbb{K}^n$.

[1] we shall often omit the field when clear from context

□

Corollary 2.4.
We may redefine Grassmannians in terms of linear maps as follows:

$$\mathrm{Gr}(k,n) = {\left\{\varphi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \varphi \text{ surjective.}\right\}}\big/{\sim}$$

where $\varphi \sim \psi$ *if and only if* $\exists \theta \in \mathrm{GL}(\mathbb{K}^k)$ *such that* $\varphi = \theta \circ \psi$.

We end the prelude of this chapter by introducing some notation and conventions

Definition 2.5 (Multiindicies)**.**
We define a (k,n)-**multiindex** as an element of $\{1, \cdots, n\}^k$. Our notation for a multiindex I will usually be $I = (i_1, \cdots, i_k)$.
We denote the set of **ordered** (k,n)-**multiindicies** with

$$\omega(k,n) = \left\{(i_1, \cdots, i_k) \in \{1, \cdots, n\}^k \mid i_1 < \cdots < i_k\right\}.$$

If A is a $k \times n$ matrix and I is a (k,n)-multiindex, we denote the I-minor by A_I, i.e.

$$A_I = \begin{pmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{pmatrix}.$$

If B is an $\alpha \times \beta$ matrix, $i \in \{1, \cdots, \alpha\}$ and $j \in \{1, \cdots, \beta\}$ we use $B_{\times i, \times j}$ to denote the $(\alpha - 1) \times (\beta - 1)$ matrix obtained from B by deleting the i-th row and the j-th column.

Remark 2.6.
The set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

forms a basis for $\bigwedge^k \mathbb{K}^n$. For brevity, for all multiindicies $I = (i_1, \cdots, i_k)$ we shall define

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Notation 2.7.
Whenever a base of $\mathbb{K}^\ell$ is fixed, we will identify $\bigwedge^\ell \mathbb{K}^\ell$ with $\mathbb{K}$ by sending the wedge of the ordered basis to $1 \in \mathbb{K}$.

2.2 The Plücker embedding

To make the study of Grassmannians easier, we want to identify $\mathrm{Gr}(k,n)$ with a projective variety.
Intuitively we seek to transform objects defined by several vectos into objects given by a single vector and then take the projective space construction. This conversion can be made by taking wedge products appropriately.

Definition 2.8 (Plücker map)**.**
Let $k \leq n$ be a pair of positive integers. We define the **Plücker map** as[2]:

$$\wedge^k : \begin{array}{ccc} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \\ \varphi & \longmapsto & \wedge^k \varphi \end{array},$$

[2] the map $\wedge^k \varphi$ is well defined because if we view it as a map $\wedge^k \varphi : (\mathbb{K}^n)^k \to \bigwedge^k \mathbb{K}^k$ then it is multilinear and alternating.

where $(\wedge^k \varphi)(v_1 \wedge \cdots \wedge v_k) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_k)$.
Note that if $e_1, \cdots, e_k$ is a basis of $\mathbb{K}^k$ then[3]

$$\wedge^k(\varphi)(v_1 \wedge \cdots \wedge v_k) = \det\big(\varphi(v_1)|\cdots|\varphi(v_k)\big)\, e_1 \wedge \cdots \wedge e_k,$$

which we will often identify with $\det\big(\varphi(v_1)|\cdots|\varphi(v_k)\big)$ when the choice of basis is clear.

Remark 2.9.
The codomain of the Plücker map is isomorphic to $\bigwedge^k \mathbb{K}^n$, indeed

$$\operatorname{Hom}_{\mathbb{K}}\left(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k\right) \cong \left(\bigwedge^k \mathbb{K}^n\right)^{\vee} \cong \bigwedge^k \mathbb{K}^n.$$

The isomorphism depends on the choice of basis for $\mathbb{K}^n$ and $\mathbb{K}^k$. If $e_1, \cdots, e_n$ is a basis of $\mathbb{K}^n$ and $e_1, \cdots, e_k$ is a basis of $\mathbb{K}^k$ then the isomorphism takes on the following form

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ \varphi & \longmapsto & \displaystyle\sum_{1 \le i_1 < \cdots < i_k \le n} \det(\varphi(e_{i_1}) \mid \cdots \mid \varphi(e_{i_k}))e_{i_1} \wedge \cdots \wedge e_{i_k} \end{array},$$

where the determinant is defined assuming that the $\varphi(e_j)$ are viewed as their coordinates in the base $e_1, \cdots, e_k$.

Remark 2.10.
The image of the Plücker map is a cone.

Proof.
For any $\lambda \in \mathbb{K}^*$ and any map $\varphi \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ we see that

$$\lambda \wedge^k(\varphi) = \wedge^k(\alpha \circ \varphi),$$

for any automorphism α of $\mathbb{K}^k$ with determinant[4] λ. □

Remark 2.11.
$\operatorname{rnk}\varphi < k$ if and only if $\wedge^k(\varphi) = 0$.

Proof.
$\wedge^k(\varphi)$ is the zero map if an only if the set $\{\varphi(v_1), \cdots, \varphi(v_k)\}$ is linearly dependent for any choice of $v_1, \cdots, v_k$, i.e. φ is not of full rank. □

Lemma 2.12.
Let $\varphi : \mathbb{K}^n \to \mathbb{K}^k$ be a full rank linear map, then

$$\ker\varphi = \left\{z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n,\ \wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0\right\}.$$

Proof.
If $\varphi(z) = 0$ then for any $w_2, \cdots, w_k \in \mathbb{K}^k$ we see that

$$\wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \wedge \varphi(w_2) \wedge \cdots \wedge \varphi(w_k) = 0.$$

[3]the columns of the determinant are the coordinates of the vectors given in the base $e_1, \cdots, e_k$.

[4]For example we may fix a base $e_1, \cdots, e_k$ of $\mathbb{K}^k$ and define $\alpha(e_i) = \begin{cases} \lambda e_1 & \text{if } i = 1 \\ e_i & \text{otherwise} \end{cases}$

Suppose now that $\varphi(z) \neq 0$ and let $v_2, \cdots, v_k$ be such that $\{\varphi(z), v_2, \cdots, v_k\}$ form a basis for $\mathbb{K}^k$. Since φ is surjective there exist $w_2, \cdots, w_k$ such that $\varphi(w_i) = v_i$ for all $2 \leq i \leq k$. By construction

$$\wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = \varphi(z) \wedge v_2 \wedge \cdots \wedge v_k \neq 0.$$

$\square$

Proposition 2.13.
Let $\sim$ be the equivalence relation defined in corollary (2.4), then for any two full rank linear maps $\varphi, \psi : \mathbb{K}^n \to \mathbb{K}^k$

$$\varphi \sim \psi \iff \exists \lambda \in \mathbb{K}^* \ s.t. \ \wedge^k(\varphi) = \lambda \wedge^k(\psi).$$

Proof.
Let us prove both implications:

$\boxed{\implies}$ If $\varphi = \theta \circ \psi$ for $\theta \in GL(\mathbb{K}^k)$ then it follows easily from known properties of the determinant that

$$\wedge^k(\varphi) = \wedge^k(\theta \circ \psi) = (\det \theta) \wedge^k(\psi).$$

$\boxed{\impliedby}$ From lemma (2.3) we see that it is enough to prove that $\ker \varphi = \ker \psi$. We conclude by applying lemma (2.12) as follows:

$$\begin{aligned} \ker \varphi &= \left\{ z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n, \ \wedge^k(\varphi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \right\} \\ &= \left\{ z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n, \ \lambda \wedge^k(\psi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \right\} = \\ &= \left\{ z \in \mathbb{K}^n \mid \forall w_2, \cdots, w_k \in \mathbb{K}^n, \ \wedge^k(\psi)(z \wedge w_2 \wedge \cdots \wedge w_k) = 0 \right\} = \ker \psi. \end{aligned}$$

$\square$

Remark 2.14.
Because of proposition (2.13) and remark (2.11) there exists a unique h such that the diagram commutes

$$\begin{tikzcd} \left\{ \varphi \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k) \mid \operatorname{rnk} \varphi = k \right\} \arrow[r, "{[\wedge^k]}"] \arrow[d, "\pi"] & \mathbb{P}(\operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)) \\ \operatorname{Gr}(k, n) \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Moreover, such an h must be injective by proposition (2.13).

Definition 2.15 (Plücker embedding)**.**
Let us fix a basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and a basis $e_1, \cdots, e_k$ of $\mathbb{K}^k$. We define the **Plücker embedding** as follows

$$\operatorname{Pl} : \begin{array}{ccc} \operatorname{Gr}(k, n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [\varphi]_\sim & \longmapsto & [(\det(\varphi(e_{i_1}) \mid \cdots \mid \varphi(e_{i_k})))_{1 \leq i_1 < \cdots < i_k \leq n}]_{\mathbb{K}^*} \end{array}.$$

Remark 2.16.
If we fix bases for $\mathbb{K}^n$ and $\mathbb{K}^k$ and ζ is the isomorphism $\operatorname{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k) \to \bigwedge^k \mathbb{K}^n$ discussed during remark (2.9), we see that the following diagram commutes

$$\begin{tikzcd} \mathbb{P}(\bigwedge^k \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)) \arrow[r, "\mathbb{P}(\zeta)"] & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ \operatorname{Gr}(k, n) \arrow[u, "h"] \arrow[ur, "\operatorname{Pl}"'] & \end{tikzcd}$$

This proves that the Plücker embedding is well defined and injective.

Remark 2.17.
If we define $\phi = \zeta \circ \wedge^k$ then we see that

$$\mathrm{Pl} \circ \pi = \mathbb{P}(\phi).$$

This form will be instrumental in the next chapter.

The entries of the homogeneous $\binom{n}{k}$-tuple associated to $[\varphi] \in \mathrm{Gr}(k, n)$ are called the **Plücker coordinates** of $[\varphi]$. The Plücker coordinates are unique up to multiplying each by the same nonzero scalar.

Remark 2.18.
The Plücker embedding depends on the choice of basis for $\mathbb{K}^n$ but not on the one for $\mathbb{K}^k$, since the effect of changing the basis of $\mathbb{K}^k$ is that of multiplying all Plücker coordinates by the same scalar (the determinant of the change of basis), which doesn't change the point they describe in $\mathbb{P}(\bigwedge^k \mathbb{K}^k)$.
The dependence on the base of $\mathbb{K}^n$ corresponds to the fact that $\mathrm{GL}(\mathbb{K}^n)$ acts transitively on $\mathrm{Gr}(k, n)$ viewed as the set of $(n-k)$-dimentional subspaces of $\mathbb{K}^n$.

2.3 The image of the Plücker embedding is closed

Thus far we have identified $\mathrm{Gr}(k, n)$ with a subset of some projective space. We now seek to show that it is a closed subset in the Zariski topology. Our approach mostly readapts parts of [6].

First we need some linear algebra results

Lemma 2.19.
Let $\omega \in \bigwedge^k \mathbb{K}^n$. For any given nonzero vector v there exists $\varepsilon \in \bigwedge^{k-1} \mathbb{K}^n$ such that $\omega = \varepsilon \wedge v$ if and only if $\omega \wedge v = 0$.

Proof.
If $\omega = \varepsilon \wedge v$ then $\omega \wedge v = \varepsilon \wedge v \wedge v = 0$.
Suppose now that $\omega \wedge v$. Let $v_1, \cdots, v_n$ be a basis of $\mathbb{K}^n$ such that $v_1 = v$. If we write

$$\omega = \sum_{I \in \omega(k,n)} p_I v_I$$

then we see that for any given multiindex I either $p_I = 0$ or $v_I \wedge v = 0$. Since $v_1, \cdots, v_n$ is a basis, $v_I \wedge v_1 = 0$ if and only if $1 \in I$, i.e. $v_I = v \wedge v_{(i_2, \cdots, i_k)}$, therefore

$$\omega = v \wedge \left(\sum_{2 \leq i_2 < \cdots i_k \leq n} p_{(1, i_2, \cdots, i_k)} e_{(i_2, \cdots, i_k)} \right)$$

□

Corollary 2.20 (Total decomposibility criterion)**.**
Let $\omega \in \bigwedge^k \mathbb{K}^n$. If $\dim \{v \in \mathbb{K}^n \mid \omega \wedge v = 0\} \geq k$ then $\omega = \lambda v_1 \wedge \cdots \wedge v_k$ for any set of linearly independent vectors $\{v_1, \cdots, v_k\}$ in $\{v \in \mathbb{K}^n \mid \omega \wedge v = 0\}$ and some scalar λ.

Proof.
The set $\{v \in \mathbb{K}^n \mid \omega \wedge v = 0\}$ is clearly a subspace of $\mathbb{K}^n$. Let $\{v_1, \cdots, v_k\}$ be linearly independent vectors of this space. By iterating the above lemma we see that

$$\omega = \lambda \wedge v_1 \wedge \cdots \wedge v_k$$

for some $\lambda \in \bigwedge^0 \mathbb{K}^n = \mathbb{K}$. □

Lemma 2.21.
A multilinear alternating form $\psi \in \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ is in the image of the Plücker map $\wedge^k$ if and only if there exists a basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and an element λ of $\mathbb{K}$ such that

$$\sum_{I \in \omega(n-k,n)} \psi(e_{\widehat{I}}) e_I = \lambda e_{(1,\cdots,n-k)}.$$

Proof.
Let us fix a basis $e_1, \cdots, e_k$ of $\mathbb{K}^k$. Using this basis we identify $\bigwedge^k \mathbb{K}^k$ with $\mathbb{K}$.
Suppose that $\psi = \wedge^k(\varphi)$ and let $e_1, \cdots, e_n$ be a basis of $\mathbb{K}^n$ such that $e_1, \cdots, e_{n-k}$ is a basis of $\ker \varphi$, then

$$\sum_{I \in \omega(n-k,n)} \wedge^k \varphi(e_{\widehat{I}}) e_I = \left(\varphi(e_{n-k+1}) \wedge \cdots \wedge \varphi(e_n)\right) e_{(1,\cdots,n-k)}.$$

Suppose now that we have the decomposibility above and let us define φ by

$$\varphi(e_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq n-k \\ \lambda e_1 & \text{if } i = n-k+1 \\ e_{i-n+k} & \text{if } i > n-k+1 \end{cases}$$

We now compute a second form for $\sum_{I \in \omega(n-k,n)} \psi(e_{\widehat{I}}) e_I$

$$\begin{aligned} \sum_{I \in \omega(n-k,n)} \psi(e_{\widehat{I}}) e_I =& \lambda e_{(1,\cdots,n-k)} = \\ =& \varphi(e_{n-k+1}) \wedge \cdots \wedge \varphi(e_n) e_{(1,\cdots,n-k)} = \\ =& \sum_{I \in \omega(n-k,n)} \wedge^k \varphi(e_{\widehat{I}}) e_I \end{aligned}$$

where the second equality follows from the construction of φ.
We have shown that for all $J \in \omega(k, n)$ we have

$$\psi(e_J) = \wedge^k \varphi(e_J),$$

so ψ and $\wedge^k(\varphi)$ agree on a basis of $\bigwedge^k \mathbb{K}^n$ and are thus the same map. □

Let us consider the following map: let $\psi : \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ be any alternating multilinear map, we define $\Phi(\psi)$ as

$$\Phi(\psi): \begin{array}{ccc} \mathbb{K}^n & \longrightarrow & \bigwedge^{n-k+1} \mathbb{K}^n \\ v & \longmapsto & \displaystyle\sum_{I \in \omega(n-k,n)} \psi(e_{\widehat{I}}) e_I \wedge v \end{array}$$

where $\widehat{I}$ is the ordered k-tuple of the indicies in $\{1, \cdots, n\}$ missing from I.

Proposition 2.22.
An alternating multilinear map $\psi \in \mathrm{Hom}_{\mathbb{K}}(\bigwedge^k \mathbb{K}^n, \bigwedge^k \mathbb{K}^k)$ *is in the image of the Plücker map* $\wedge^k$ *if and only if* $\Phi(\psi)$ *has rank at most* k*.*

Proof.
Suppose that $\psi = \wedge^k(\varphi)$ and let $\{z_1, \cdots, z_{n-k}, z_{n-k+1}, \cdots, z_n\}$ be a basis of $\mathbb{K}^n$ such that the first $n-k$ vectors are a basis of $\ker\varphi$. From the proof of lemma (2.21) we see that if $v \in \ker\varphi$ then $v \in \ker\Phi(\psi)$, i.e. $\Phi(\psi)$ has a nullity of at least $n-k$ (or rank at most k).

Suppose now that $\{z_1 \cdots, z_{n-k}\}$ are linearly independent elements of $\ker\Phi(\psi)$. By the Total decomposibility criterion (2.20) we have that there exists $\lambda \in \mathbb{K}$ such that

$$\sum_{I\in\omega(n-k,n)} \psi(e_{\widehat{I}})e_I = \lambda z_1 \wedge \cdots \wedge z_{n-k}.$$

This concludes by lemma (2.21). □

Let us now consider the map

$$\Phi : \begin{array}{ccc} \bigwedge^k \mathbb{K}^n & \longrightarrow & \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}^n, \bigwedge^{n-k+1}\mathbb{K}^n\right) \\ \psi^* & \longmapsto & \Phi(\zeta^{-1}(\psi^*)) \end{array}$$

Note that Φ is linear, so we can represent Φ as a matrix with coefficients in $\bigwedge^k \mathbb{K}^n$: we fix a basis $\{e_I\}$ of $\bigwedge^k \mathbb{K}^n$ and write $\Phi(\sum a_I e_I) = \sum a_I \Phi(\zeta^{-1}e_I)$. Since each $\Phi(\zeta^{-1}e_I)$ is linear, it can be viewed as a matrix $\left(b^I_{i,j}\right)_{i,j}$ and so the matrix associated to Φ is $\left(\sum b^I_{i,j}e_I\right)_{i,j}$.

Proposition (2.22) tells us that the image of $\zeta \circ \wedge^k$ can by identified by imposing that the rank of the matrix representing Φ defined above is at most $n-k$, which is equivalent to the vanishing of its $(n-k+1)\times(n-k+1)$ minors, which is a closed condition.

It follows trivially that the projectivization[5] of this set (i.e. the image of Pl) is also closed in $\mathbb{P}(\bigwedge^k \mathbb{K}^n)$, so we found a bijection between $\mathrm{Gr}(k,n)$ and a projective variety, which we can use to endow $\mathrm{Gr}(k,n)$ with the structure of one.

[5] recall that $\mathrm{Imm}\wedge^k$ is a cone.

Chapter 3

Representability of the Grassmannian functor

In this chapter, unless otherwise specified, we have fixed a basis $e_1, \cdots, e_n$ of $\mathbb{K}^n$ and a basis $e_1, \cdots, e_k$ of $\mathbb{K}^k$. In case of ambiguity we will refer to these bases as *canonical*.

Having fixed a base, we will identify $\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^k)$ with the set of $k \times n$ matrices with coefficients in $\mathbb{K}$, which we will denote $\mathcal{M}(k,n)$. As a consequence of this we find yet another form for $\operatorname{Gr}(k,n)$:

$$\operatorname{Gr}(k,n) = {\{A \in \mathcal{M}(k,n) \mid \operatorname{rnk} A = k\}}\big/{\sim},$$

where $A \sim B \iff \exists P \in \mathrm{GL}(k)\ s.t.\ A = PB$.

We may rewrite the maps from the previous chapter as follows:

$$\phi^s : \begin{array}{ccc} \mathcal{M}(k,n) & \longrightarrow & \bigwedge^k \mathbb{K}^n \\ A & \longmapsto & \sum_{I \in \omega(k,n)} \det A_I e_I \end{array}$$

$$\mathrm{Pl}^s : \begin{array}{ccc} \operatorname{Gr}(k,n) & \longrightarrow & \mathbb{P}(\bigwedge^k \mathbb{K}^n) \\ [A]_\sim & \longmapsto & \left[\sum_{I \in \omega(k,n)} \det A_I e_I\right]_{\mathbb{K}^*} \end{array}$$

where we use the superscript s to distinguish these maps with the ones we will define for schemes.

3.1 Grassmannians as projective schemes

To connect Grassmannians to the world of representable functors we shall redefine them scheme-theoretically by emulating the construction from the prievious chapter using rings and ring homomorphisms.

Definition 3.1 (Braket ring).
We define the **braket ring**(see page 79 of [3]) as the ring of polynomial functions on $\bigwedge^k \mathbb{K}^n$, i.e.

$$\mathcal{B}_{k,n} \doteqdot \frac{\mathbb{K}[z_I \mid I \in \{1, \cdots, n\}^k]}{(\{z_I - \operatorname{sgn}(\sigma) z_{\sigma(I)}\}_{\sigma \in S_k})} \cong \mathbb{K}[z_I \mid I \in \omega(k,n)].$$

Definition 3.2 (Ring of generic matrices)**.**
Let $\mathbb{K}[X_{k,n}] \doteqdot \mathbb{K}[x_{1,1}, \cdots, x_{k,n}]$ denote the polynomial ring with $k \cdot n$ variables. We define the **generic matrix** by

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \cdots & x_{k,n} \end{pmatrix}$$

and by the same token we denote by X_I the generic $k \times k$ minor determined by the multiindex I and by $\det X_I$ the formal determinant of this minor.

Remark 3.3.
The ring $\mathbb{K}[X_{k,n}]$ is the coordinate ring of $\mathcal{M}(k,n)$.

Remark 3.4.
The familiar $\mathcal{M}(k,n)$ and $\bigwedge^k \mathbb{K}^n$ can be identified with the $\mathbb{K}$-points of the affine schemes $\operatorname{Spec} \mathbb{K}[X_{k,n}]$ and $\operatorname{Spec} \mathcal{B}_{k,n}$ respectively (Example 2.3.32 of [5]).

Definition 3.5 (Plücker ring homomorphism)**.**
We define the **Plücker ring homomorphism** or simply **Plücker homomorphism** as

$$\phi^{\#} : \begin{array}{ccc} \mathcal{B}_{k,n} & \longrightarrow & \mathbb{K}[X_{k,n}] \\ z_I & \longmapsto & \det X_I \end{array}$$

For brevity we will denote $\operatorname{Spec} \phi^{\#}$ by ϕ.
This definition is inspired by that of ϕ at page 79 of [3].

Proposition 3.6.
The kernel of the Plücker homomorphism is an homogeneous prime ideal which does not contain $(\{z_I\}_{I \in \omega(k,n)})$.

Proof.
Since $\mathbb{K}[X_{k,n}]$ is an integral domain, $\ker \phi^{\#}$ is prime.
By definition of homogeneous ideal, we want to show that if $f = \sum f_d$ for d homogeneous and $f \in \ker \phi^{\#}$ then $f_d \in \ker \phi^{\#}$ for all d.
Looking at the definition of $\phi^{\#}$ we see that $\phi^{\#}(f_d)$ is a homogeneous polynomial of degree kd, in particular if $d \neq h$ then $\deg \phi^{\#}(f_d) \neq \deg \phi^{\#}(f_h)$. Since

$$0 = \phi^{\#}(f) = \sum \phi^{\#}(f_d)$$

this proves that $\phi^{\#}(f_d) = 0$ for all d.
Finally, observe that $\deg \phi^{\#}(z_I) = \deg(\det X_I) = k > 0$, so $z_I \notin \ker \phi^{\#}$. □

Proposition 3.7.
The induced map $\phi|_{\operatorname{Spec}(\mathbb{K}[X_{k,n}])(\mathbb{K})} : \operatorname{Spec}(\mathbb{K}[X_{k,n}])(\mathbb{K}) \to \operatorname{Spec}(\mathcal{B}_{k,n})(\mathbb{K})$ *is equal to* $\phi^s : \mathcal{M}(k,n) \to \bigwedge^k \mathbb{K}^n$ *under the identification mentioned above, i.e. for all matrices* $A \in \mathcal{M}(k,n)$ *with entries* $a_{i,j}$ *we have*

$$(\phi^{\#})^{-1}((x_{i,j} - a_{i,j})) = (z_I - \det A_I).$$

Proof.
First we observe that for any multiindex I

$$\det X_I - \det A_I \in (x_{i,j} - a_{i,j}),$$

thus $(z_I - \det A_I) \subseteq (\phi^{\#})^{-1}((x_{i,j} - a_{i,j}))$.
Since $(z_I - \det A_I)$ is a $\mathbb{K}$-point, it is in particular a maximal ideal of the Braket ring, thus we have the desired equality if $1 \notin (\phi^{\#})^{-1}((x_{i,j} - a_{i,j}))$, which is the case because otherwise $(x_{i,j} - a_{i,j})$ would not be proper. □

Proposition 3.8.
The $\mathbb{K}$-points of $V_+(\ker(\phi^{\#}))$ correspond to $\operatorname{Imm}\operatorname{Pl}^s$.

Proof.
First we note that

$$V_+(\ker \phi^{\#}) = \operatorname{Proj} \frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}} \subseteq \operatorname{Proj} \mathcal{B}_{k,n}.$$

Since ϕ becomes ϕ^s on $\mathbb{K}$-points we see that

$$Z(\ker \phi^{\#}) = \overline{\operatorname{Imm}\,(\operatorname{Spec} \phi^{\#})\,|_{\operatorname{Spec}\,(\mathbb{K}[X_{k,n}])(\mathbb{K})}} = \overline{\operatorname{Imm} \phi^s} \overset{\substack{\text{previous}\\ \text{chapter}}}{=} \operatorname{Imm} \phi^s.$$

It follows from Corollary 2.3.44 in [5] that the $\mathbb{K}$-points of $V_+(\ker \phi^{\#})$ correspond to

$$Z_+(\ker \phi^{\#}) = \mathbb{P}(Z(\ker \phi^{\#})) = \mathbb{P}(\operatorname{Imm} \phi^s) = \operatorname{Imm}\operatorname{Pl}^s.$$

□

This result allows us to redefine the Grassmannian as a projective scheme. We can obtain the Plücker embedding of the classical Grassmannian back by looking at $\mathbb{K}$-points.

From now on $\operatorname{Gr}(k,n)$ will denote $V_+(\ker \phi^{\#})$, while $\operatorname{Gr}(k,n)(\mathbb{K})$ will denote what we used to write as $\operatorname{Gr}(k,n)$.

3.1.1 Standard affine cover of the Grassmannian scheme

Recall that projective space admits a standard affine cover given by the locus of where one indeterminate does not vanish. In our case we see that

$$\operatorname{Proj} \mathcal{B}_{k,n} = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left((\mathcal{B}_{k,n})^0_{z_I} \right) = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left(\mathbb{K}\left[\frac{z_J}{z_I} \mid J \in \omega(k,n) \right] \right).$$

where the subscript denotes localization with multiplicative part $\{1, z_I, z_I^2, \cdots\}$ and the superscript 0 denotes the fact that we are only considering terms of degree 0 in this ring (this is the notation used in [5]).

This open affine cover of $\operatorname{Proj} \mathcal{B}_{k,n}$ induces an open cover on $\operatorname{Gr}(k,n)$ as follows:

$$\operatorname{Gr}(k,n) = V_+(\ker \phi^{\#}) = \bigcup_{I \in \omega(k,n)} \operatorname{Spec}\left(\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}} \right)^0_{z_I} \right).$$

Notation 3.9.
Let us fix $I \in \omega(k,n)$, then we denote the localization of $\phi^{\#}$ as

$$\phi_I^{\#} : \begin{array}{ccc} \mathbb{K}\left[\frac{z_J}{z_I} \mid J \in \omega(k,n) \right] & \longrightarrow & \mathbb{K}[X]^0_{\det X_I} \\ \frac{z_J}{z_I} & \longmapsto & \frac{\det X_J}{\det X_I} \end{array}$$

Remark 3.10.
The image of $\phi_I^{\#}$ is

$$\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right],$$

thus by the first isomorphism theorem we have

$$\frac{(\mathcal{B}_{k,n})^0_{z_I}}{\ker \phi_I^{\#}} \cong \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right].$$

Remark 3.11.
The following equality holds by the properties of localization

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}}\right)_{z_I} = \frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}},$$

thus

$$\left(\frac{\mathcal{B}_{k,n}}{\ker \phi^{\#}}\right)^0_{z_I} = \left(\frac{(\mathcal{B}_{k,n})_{z_I}}{(\ker \phi^{\#})_{z_I}}\right)^0 = \frac{(\mathcal{B}_{k,n})^0_{z_I}}{\ker \phi_I^{\#}} \cong \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right]$$

Putting what we have said together, we have shown that up to some canonical identifications

$$\mathrm{Gr}(k,n) = \bigcup_{I \in \omega(k,n)} \mathrm{Spec}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right]\right) \doteqdot \bigcup_{I \in \omega(k,n)} \mathrm{Gr}_I(k,n).$$

Proposition 3.12.
$\mathrm{Gr}_I(k,n)$ *is isomorphic to* $\mathbb{A}_{\mathbb{K}}^{k(n-k)}$ *as a scheme.*

Proof.
Since they are both affine schemes, it is enough to show that their coordinate rings are isomorphic.
To simplify the notation we will set $w_J = \frac{\det X_J}{\det X_I}$ and if S is a multiindex we will write S^i_j for the multiindex where the i-th entry is substituted by $j \in \{1, \cdots, n\}$.

Without loss of generality we may assume that $I = (1, \cdots, k)$. An analogous argument will work for any choice of multiindex.
First we will prove that

$$\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right] = \mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J = I^j_{\ell_j},\ j \in \{1, \cdots, k\},\ \ell_j \notin I\right],$$

then we will show that the RHS (which we will denote R for brevity) is isomorphic to $\mathbb{K}[Y_{k,n-k}] = \mathbb{K}[y_{1,1}, \cdots, y_{k,n-k}]$.

- Let us consider the formal matrix

$$M = \begin{pmatrix} 1 & & & w_{I^1_{k+1}} & \cdots & w_{I^1_n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & w_{I^k_{k+1}} & \cdots & w_{I^k_n} \end{pmatrix}$$

We can see that $M = (X_I)^{-1} X$, where $(X_I)^{-1}$ is the formal inverse of X_I, which exists because $\det X_I$ is invertible. More precisely

$$(X_I)^{-1} = \frac{1}{\det X_I} \mathrm{Adj}\,(X_I)$$

where $\mathrm{Adj}\,(X_I)$ is the adjugate matrix to X_I. The equality $M = (X_I)^{-1} X$ holds for the first k columns by definition of inverse, for the other columns we can see that they agree on every entry:

$$\begin{aligned}\frac{1}{\det X_I}(\mathrm{Adj}\,(X_I)X)_{i,j} =& \frac{1}{\det X_I}\sum_{\ell=1}^{k}\left((-1)^{i+\ell}\det\,(X_I)_{\times\ell,\times i}\right)x_{\ell,j} = \\ =& \frac{1}{\det X_I}\det X_{I^i_j} = \\ =& w_{I^i_j}\end{aligned}$$

We have thus proven that for any (k,n)-multiindex J we have

$$\det M_J = \det(X_I^{-1}X)_J = \frac{1}{\det X_I}\det X_J = w_J.$$

Since $\det M_J$ is a polynomial expression in the ring R by definition of M and the w_J are generators for $\mathbb{K}\left[\frac{\det X_J}{\det X_I} \mid J \in \omega(k,n)\right]$, we have shown the nontrivial inclusion and thus equality we seeked to prove.

- Let us consider the following ring homomorphism

$$\chi : \begin{array}{ccc} \mathbb{K}[Y_{k,n-k}] & \longrightarrow & R \\ y_{i,j} & \longmapsto & w_{I^i_j} \end{array}.$$

It is obviously surjective so we just need to show that it is injective to find the desired isomorphism.

Suppose by contraddiction that there exists a nonzero polynomial $p \in \mathbb{K}[Y_{k,n-k}]$ which maps to 0. If $\overline{\mathbb{K}}$ is an algebraic closure[1] of $\mathbb{K}$ we can consider the lift $\widetilde{\chi} : \overline{\mathbb{K}}[Y_{k,n-k}] \to \widetilde{R} = \overline{\mathbb{K}}[w_{I^i_j}]$. Observe that if p maps to 0 according to the original map, then it would also map to 0 according to this extension. Consider now any matrix of the form

$$A = \left(I_k \mid \widetilde{A}\right) = (a_{i,j})_{i,j}$$

where I_k is the $k \times k$ identity matrix and $\widetilde{A} \in \mathcal{M}(k, n-k, \overline{\mathbb{K}})$. From what we have said above it follows that $\det A_{I^i_j} = a_{i,j}$, so

$$p(\widetilde{A}) = p\left(\left(\det A_{I^i_j}\right)_{\substack{i\in\{1,\cdots,k\},\\ j\in\{k+1,\cdots,n\}}}\right) = \widetilde{\chi}(p)(A) = 0.$$

We have shown that p has infinitely many roots, so if we fix the value of $k(n-k)-1$ indeterminates the resulting polynomial is the 0 polynomial. If we reiterate this reasoning we eventually prove that $p = 0$ in $\overline{\mathbb{K}}[Y_{k,n-k}]$, but $0 \in \mathbb{K}[Y_{k,n-k}] \subseteq \overline{\mathbb{K}}[Y_{k,n-k}]$, so p is the zero polynomial in the original ring, contraddicting our hypothesis.

□

3.2 Grassmannian moduli functor

Let us consider the following functor

$$\mathbb{G}(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \{\alpha : \mathcal{O}_T^n \twoheadrightarrow Q\}/\sim \\ f : S \to T & \longmapsto & (\alpha : \mathcal{O}_T^n \to Q) \mapsto (f^*\alpha : \mathcal{O}_S^n \to f^*Q) \end{array}$$

[1] we can take any field extension $\mathbb{K} \subseteq \mathbb{F}$ where $\mathbb{F}$ is an infinite field.

where Q is a locally free sheaf of rank k on T and two surjections $\alpha : \mathcal{O}_T^n \twoheadrightarrow Q$, $\beta : \mathcal{O}_T^n \twoheadrightarrow V$ are equivalent if and only if there exist an isomorphism of sheaves $\theta : Q \to V$ such that the diagram commutes

$$\begin{tikzcd} \mathcal{O}_T^n \arrow[r, two heads, "\alpha"] \arrow[dr, two heads, "\beta"'] & Q \arrow[d, "\theta"] \\ & V \end{tikzcd}$$

We have functoriality because of the composition properties of pullbacks.

In this this section we will prove that the Grassmann scheme represents this functor.

3.2.1 Open subfunctor cover of the Grassmannian

Notation 3.13.
For any multiindex I and any scheme T we define the following morphism of sheaves

$$s_I^T : \begin{array}{ccc} \mathcal{O}_T^k & \longrightarrow & \mathcal{O}_T^n \\ e_j & \longmapsto & e_{i_j} \end{array}.$$

If there is no ambiguity on the scheme we will write s_I.

Definition 3.14 (Principal subfunctor of the Grassmannian)**.**
Fixed a multiindex $I \in \omega(k, n)$ we define the following association

$$\mathbb{G}_I(k,n) : \begin{array}{ccc} (\mathrm{Sch}/\mathbb{K})^{op} & \longrightarrow & \mathrm{Set} \\ T & \longmapsto & \left\{ \mathcal{O}_T^n \overset{\alpha}{\twoheadrightarrow} Q \mid \alpha \circ s_I \text{ surjective} \right\} \Big/ \sim \\ f & \longmapsto & \alpha \mapsto f^*\alpha \end{array}$$

where the equivalence relation is the same as the one defined for $\mathbb{G}(k, n)$.

Remark 3.15.
The association $\mathbb{G}_I(k, n)$ is functor.

Proof.
First we observe that $\mathbb{G}_I(k, n)(T)$ is well defined because if $\psi = \theta \circ \alpha$ with θ isomorphism of sheaves then on each stalk we have

$$\psi_x \circ (s_I)_x = \theta_x \circ \varphi_x \circ (s_I)_x,$$

which is surjective if and only if $\varphi_x \circ (s_I)_x$ is surjective.

Consider now a map $f : S \to T$, then

$$f^*\alpha \circ s_I^S = f^*\alpha \circ f^* s_I^T = f^*(\alpha \circ s_I^T)$$

is surjective if and only if it is surjective on all stalks, i.e. if and only if for all $s \in S$ we have that the following map is surjective

$$f^*(\alpha \circ s_I^T)_s = (\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} \mathcal{O}_{S,s}$$

which is true because the tensor product is right-exact. □

Lemma 3.16.
The support of a finite type quasicoherent sheaf $\mathcal{F}$ on a scheme X is a closed subset[2].

[2]This statement is presented as Exercise 4.D. in [7]

Proof.
Since the support is a local notion and we can take an open affine cover of any scheme, we may assume $X = \operatorname{Spec} A$.
From the theory of quasicoherent sheaves on affine schemes we know that there exists a finitely generated A-module M such that $\mathcal{F} = \widetilde{M}$. Let $m_1, \cdots, m_k$ be generators for M, then

$$\operatorname{Supp} M = \bigcup_{i=1}^{k} \operatorname{Supp} m_i.$$

Since we have written $\operatorname{Supp} M$ as a finite union of sets, it is enough to show that $\operatorname{Supp} m$ is closed for all $m \in M$, which is true because

$$\operatorname{Supp} m = V(\operatorname{Ann}_A m),$$

indeed

$$0 = m_{\mathfrak{p}} \iff \exists s \in A \setminus \mathfrak{p} \ s.t. \ sm = 0 \iff \exists t \in \operatorname{Ann}_A m \setminus \mathfrak{p},$$

thus $\mathfrak{p} \in \operatorname{Supp} m$ if and only if $\operatorname{Ann}_A m \subseteq \mathfrak{p}$, i.e. $\mathfrak{p} \in V(\operatorname{Ann}_A m)$. □

Proposition 3.17.
The $\mathbb{G}_I(k,n)$ are open subfunctors of $\mathbb{G}(k,n)$.

Proof.
We will follow the approach showcased in [2].
The inclusion $\mathbb{G}_I(k,n)(T) \subseteq \mathbb{G}(k,n)(T)$ is apparent so we just need to show that if we fix a quotient $[\alpha : \mathcal{O}_T^n \twoheadrightarrow Q]$ in $\mathbb{G}(k,n)(T)$ then we can find an open subscheme of T which represents $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$.

Let us fix a representative α for the given quotient and define $\alpha \circ s_I : \mathcal{O}_T^k \to Q$. The locus where this map is surjective is the complement of the support of its cokernel sheaf $\mathcal{K}$, i.e. $\alpha \circ s_I$ is surjective on $\mathcal{O}_{T,x}^k$ if and only if $x \notin \operatorname{Supp} \mathcal{K}$. By lemma (3.16) the set U_I where $\alpha \circ s_I$ is surjective is an open subset of T ($\mathcal{K}$ is of finite type because locally it is the sheaf associated to the quotient of finite rank free modules).
We now want to show that U_I represents the functor $h_T \times_{\mathbb{G}(k,n)} \mathbb{G}_I(k,n)$, that is we want to show that if $f : S \to T$ is a morphism of $\mathbb{K}$-schemes then f factors through U_I if and only if $f^*\alpha : \mathcal{O}_S^n \to f^*Q \in \operatorname{Gr}_I(S)$, which is the case if and only if

$$(\alpha \circ s_I^T)_{f(s)} \otimes_{\mathcal{O}_{T,f(s)}} \mathcal{O}_{S,s}$$

is surjective for all $s \in S$.
Let us fix $x \in S$ and $y \in T$ so that $f(x) = y$. By definition $y \in U_I$ if and only if $(\alpha \circ s_I)_y$ is surjective on $\mathcal{O}_{T,y}^k$, which by Nakayama's lemma is the same as asking that the following map is surjective

$$(\alpha \circ s_I)\big|_y : k(y)^k \to Q_y \otimes_{\mathcal{O}_{T,y}} k(y).$$

If we now pull back via f we obtain a map

$$(f^*\alpha \circ f^*s_I)\big|_x : k(x)^k \to f^*Q_x \otimes_{\mathcal{O}_{S,x}} k(x)$$

[I DON'T GET WHY THIS MAP SHOULD BE SURJECTIVE]

□

Proposition 3.18.
The collection $\{\mathbb{G}_I(k,n)\}$ is a Zariski open subfunctor cover of $\mathbb{G}(k,n)$.

Proof.
For any $\mathbb{K}$-scheme S and any quotient $[\alpha] \in \mathrm{Gr}(k,n)(S)$ (without loss of generality we choose a representative α) we need to show that for any $s \in S$ there exists a multiindex I such that $s \in U_I$ defined as in the previous proposition.
We are therefore looking for a multiindex I such that $(\alpha \circ s_I)_s$ is surjective. By Nakayama's lemma this is equivalent to showing that there exists and I such that

$$k(s)^k \xrightarrow{s_I} k(s)^n \xrightarrow{\alpha_s} Q_s \otimes_{\mathcal{O}_{S,s}} k(s),$$

which is trivially true since $\operatorname{rnk} \alpha_s = k$. $\square$

3.2.2 Representability of the Grassmannian functor

Proposition 3.19.
The grassmannian functor $\mathbb{G}(k,n)$ is a Zariski sheaf.

Proof.
Consider a $\mathbb{K}$-scheme T and an open cover $\{U_i\}$. Consider now quotiens $\alpha_i : \mathcal{O}^n_{U_i} \twoheadrightarrow Q_i$ such that

$$\alpha_i|_{U_i \cap U_j} \sim \alpha_j|_{U_i \cap U_j}.$$

By definition of $\sim$ there exist isomorphisms of sheaves $\varphi_{ji} : Q_i|_{U_i\cap U_j} \to Q_j|_{U_i \cap U_j}$. If we define $\varphi_{ii} = id_{Q_i}$ and fix the isomorphisms in such a way that $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ we have the data to glue the Q_i to a locally free sheaf of rank k over T, which we denote by Q. Now, up to isomorphism let us consider $\alpha_i : \mathcal{O}^n_{U_i} \twoheadrightarrow Q|_{U_i}$ for all i. If we fix any open set $V \subseteq T$ we see that if $s \in \mathcal{O}^n_T(V)$ is a section, we can define $\alpha_V(s)$ by gluing the $\alpha_i(s|_{U_i})$, which we can do by construction of Q and the choice of representative for the α_i. By construction $\alpha_{U_i} = \alpha_i$ and it is in fact the only such morphism, so we have verified the gluing property of sheaves for $\mathbb{G}(k,n)$. $\square$

Proposition 3.20.
The affine scheme $\mathrm{Gr}_I(k,n)$ represents the functor $\mathbb{G}_I(k,n)$.

Proof.
First we prove that for any $\mathbb{K}$-scheme $\mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathbb{G}_I(T)$, then we need to check that given a map $f : S \to T$ the pullbacks behave well.

- From our work in the first section of this chapter we can see that

$$\mathrm{Hom}_{\,\mathrm{Sch}/\mathbb{K}}(T, \mathrm{Gr}_I(k,n)) \cong \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right).$$

Let us now consider the following maps:

$$\begin{array}{llcl} \eta: & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) & \longrightarrow & \left\{\alpha : \mathcal{O}^n_T \to \mathcal{O}^k_T \mid \alpha \circ s_I = id_{\mathcal{O}^k_T}\right\} \\ & \varphi & \longmapsto & \eta(\varphi) \\ \rho: & \left\{\alpha : \mathcal{O}^n_T \to \mathcal{O}^k_T \mid \alpha \circ s_I = id_{\mathcal{O}^k_T}\right\} & \longrightarrow & \mathrm{Hom}_{\,\mathbb{K}\text{-alg}}\left(\mathbb{K}\left[\frac{\det X_J}{\det X_I}\right], \mathcal{O}_T(T)\right) \\ & \alpha & \longmapsto & \frac{\det X_J}{\det X_I} \mapsto \frac{\det \alpha_J}{\det \alpha_I} \end{array}$$

where $\det \alpha_L$ is the determinant of the L minor of the matrix associated to α_T in the canonical basis and for all $V \subseteq T$ open subsets

$$\eta(\varphi)_V(e_j) = \sum_{i=1}^{k} (\mathrm{res}^T_V \circ \varphi)\left(\frac{\det X_{I^i_j}}{\det X_I}\right) e_i.$$

By construction η and ρ are inverses.

We can reach $\mathbb{G}_I(T)$ as follows:

$$\left\{\alpha : \mathcal{O}_T^n \to \mathcal{O}_T^k \mid \alpha \circ s_I = id_{\mathcal{O}_T^k}\right\} = {\left\{\alpha : \mathcal{O}_T^n \twoheadrightarrow \mathcal{O}_T^k \mid \alpha \circ s_I \text{ isomorphism}\right\}}\big/{\sim}.$$

Since on all stalks $\alpha \circ s_I$ is an endomorphism of a finitely generated module, $\alpha \circ s_I$ is an isomorphism if and only if it is surjective.

- Let $f : S \to T$ be a morphism of $\mathbb{K}$-schemes. ******************************

$\square$

Theorem 3.21.
The Grassmann scheme $\mathrm{Gr}(k, n)$ *is a fine moduli space for the Grassmann functor* $\mathbb{G}(k, n)$.

Proof.
TODO $\square$

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