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GENERALIZED TENSOR COMPRESSIVE SENSING

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ABSTRACT

Compressive sensing (CS) has triggered enormous research activity since its first appearance. CS exploits the signal's sparseness or compressibility in a particular domain and integrates data compression and acquisition. While conventional CS theory relies on data representation in the form of vectors, many data types in various applications such as color imaging, video sequences, and multi-sensor networks, are intrinsically represented by higher-order tensors. Application of CS to higher-order data representation is typically performed by conversion of the data to very long vectors that must be measured using very large sampling matrices, thus imposing a huge computational and memory burden. In this paper, we propose Generalized Tensor Compressive Sensing (GTCS)–a unified framework for compressive sensing of higher-order tensors. GTCS offers an efficient means for representation of multidimensional data by providing simultaneous acquisition and compression from all tensor modes. In addition, we compare the performance of the proposed method with Kronecker compressive sensing (KCS). We demonstrate experimentally that GTCS outperforms KCS in terms of both accuracy and speed.

***Index Terms*—** Compressive sensing, convex optimization, multilinear algebra, higher-order tensor, generalized tensor compressive sensing

1. INTRODUCTION

Recent literature has witnessed an explosion of interest in sensing that exploits structured prior knowledge in the general form of sparsity, meaning that signals can be represented by only a few coefficients in some domain. Central to much of this recent work is the paradigm of compressive sensing (CS), also known under the terminology of compressed sensing, compressive sampling or compress sensing [1–3]. CS theory permits relatively few linear measurements of the signal while still allowing exact reconstruction via nonlinear recovery process. The key idea is that the sparsity helps in isolating the original vector. The first intuitive approach to a reconstruction algorithm consists in searching for the sparsest vector that is consistent with the linear measurements. However, this ℓ_0-minimization problem is NP-hard in general and thus computationally infeasible. There are essentially two approaches for tractable alternative algorithms. The first is convex relaxation, leading to ℓ_1-minimization [4], also known as basis pursuit [5], whereas the second constructs greedy algorithms. Besides, in image processing, the use of total-variation minimization which is closely connected to ℓ_1-minimization first appears in [6] and is widely applied later on. By now basic properties of the measurement matrix which ensure sparse recovery by ℓ_1-minimization are known: the null space property (NSP) [7] and the restricted isometry property (RIP) [8].

An intrinsic limitation in conventional CS theory is that it relies on data representation in the form of vector. In fact, many data types do not lend themselves to vector data representation. For example, images are intrinsically matrices. As a result, great efforts have been made to extend traditional CS to CS of data in matrix representation. A straightforward implementation of CS on 2D images recasts the 2D problem as traditional 1D CS problem by converting images to long vectors, such as in [9]. However, despite of considerably huge memory and computational burden imposed by long vector data and large sampling matrix, the sparse solutions produced by straightforward ℓ_1-minimization often incur visually unpleasant, high-frequency oscillations. This is due to the neglect of attributes known to be widely possessed by images, such as smoothness. In [10], instead of seeking sparsity in the transformed domain, they proposed a total variation-based minimization to promote smoothness of the reconstructed image. Later, as an alternative for alleviating the huge computational and memory burden associated with image vectorization, block-based CS (BCS) was proposed in [11]. In BCS, an image is divided into non-overlapping blocks and acquired using an appropriately-sized measurement matrix.

Another direction in the extension of CS to matrix CS

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generalizes CS concept and outlines a dictionary relating concepts from cardinality minimization to those of rank minimization [12–14]. The affine rank minimization problem consists of finding a matrix of minimum rank that satisfies a given set of linear equality constraints. It encompasses commonly seen low-rank matrix completion problem [14] and low-rank matrix approximation problem as special cases. [12] first introduced recovery of the minimum-rank matrix via nuclear norm minimization. [13] generalized the RIP in [8] to matrix case and established the theoretical condition under which the nuclear norm heuristic can be guaranteed to produce the minimum-rank solution.

Real-world signals of practical interest such as color imaging, video sequences and multi-sensor networks, are usually generated by the interaction of multiple factors or multimedia and thus can be intrinsically represented by higher-order tensors. Therefore, the higher-order extension of CS theory for multidimensional data has become an emerging topic. One direction attempts to find the best rank-R tensor approximation as a recovery of the original data tensor as in [15], they also proved the existence and uniqueness of the best rank-r tensor approximation in the case of 3rd order tensors. The other direction [16, 17] uses Kronecker product matrices in CS to act as sparsifying bases that jointly model the structure present in all of the signal dimensions as well as to represent the measurement protocols used in distributed settings. In this paper, we propose Generalized Tensor Compressive Sensing (GTCS)–a unified framework for compressive sensing of higher-order tensors. In addition, we propose two reconstruction procedures, a serial method (GTCS-S) and a parallelizable method (GTCS-P). Experimental results demonstrate the outstanding performance of GTCS in terms of both recovery accuracy and speed.

2. BACKGROUND

Throughout the discussion, lower-case characters represent scalar values $(a, b, \ldots)$, bold-face characters represent vectors $(\mathbf{a}, \mathbf{b}, \ldots)$, capitals represent matrices $(A, B, \ldots)$ and calligraphic capitals represent tensors $(\mathcal{A}, \mathcal{B}, \ldots)$.

2.1. Multilinear algebra

A tensor is a multidimensional array. The order of a tensor is the number of modes.

Kronecker Product The Kronecker product of matrices $A \in \mathbb{R}^{I\times J}$ and $B \in \mathbb{R}^{K\times L}$ is denoted by $A \otimes B$. The result is a matrix of size$(I \cdot K) \times (J \cdot L)$ defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1J}B \\ a_{21}B & a_{22}B & \cdots & a_{2J}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{I1}B & a_{I2}B & \cdots & a_{IJ}B \end{pmatrix}.$$

Outer Product and Tensor Product In linear algebra, the outer product typically refers to the tensor product of two vectors. $\mathbf{u} \circ \mathbf{v} = \mathbf{u}\mathbf{v}^\top$. In this paper, we won't differentiate between outer product and tensor product. To distinguish from Kronecker product, we use $\circ$ to denote tensor product of two vectors. They can be related by $\mathbf{u} \circ \mathbf{v} = \mathbf{u} \otimes \mathbf{v}^\top$.

Mode-i Product The mode-i product of a tensor $\mathcal{X} = [x_{\alpha_1,\ldots,\alpha_d}] \in \mathbb{R}^{N_1\times\ldots\times N_d}$ and a matrix $U = [u_{j,\alpha_i}] \in \mathbb{R}^{J\times N_i}$ is denoted by $\mathcal{X} \times_i U$ and is of size $N_1 \times \ldots \times N_{i-1} \times J \times N_{i+1} \times \ldots \times N_d$. By element, we have $(\mathcal{X} \times_i U)_{\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\ldots,\alpha_d} = \sum_{\alpha_i=1}^{N_i} x_{\alpha_1,\ldots,\alpha_d} u_{j,\alpha_i}$.

Mode-i Fiber and Mode-i Unfolding The mode-i fiber of a tensor $\mathcal{X} = [x_{\alpha_1,\ldots,\alpha_d}] \in \mathbb{R}^{N_1\times\ldots\times N_d}$ is obtained by fixing every index but α_i. The mode-i unfolding $X_{(i)}$ of $\mathcal{X}$ arranges the mode-i fibers to be the columns of the resulting $N_i \times (N_1 \cdot \ldots \cdot N_{i-1} \cdot N_{i+1} \cdot \ldots \cdot N_d)$ matrix.

$\mathcal{Y} = \mathcal{X} \times_1 U_1 \times \ldots \times_d U_d$ is equivalent to $Y_{(i)} = U_i X_{(i)} (U_d \otimes \ldots \otimes U_{i+1} \otimes U_{i-1} \otimes \ldots \otimes U_1)^\top$.

Core Tucker Decomposition Let $\mathcal{X} \in \mathbb{R}^{N_1\times\ldots\times N_d}$ with mode-i unfolding $X_{(i)} \in \mathbb{R}^{N_i\times(N_1\cdot\ldots\cdot N_{i-1}\cdot N_{i+1}\cdot\ldots\cdot N_d)}$. Denote by $R_i(\mathcal{X}) \subset \mathbb{R}^{N_i}$ the column space of $X_{(i)}$ whose rank is r_i. Let $\mathbf{c}_{1,i}, \ldots, \mathbf{c}_{r_i,i}$ be a basis in $R_i(\mathcal{X})$. Then the subspace $\mathbf{V}(\mathcal{X}) := R_1(\mathcal{X}) \circ \ldots \circ R_d(\mathcal{X}) \subset \mathbb{R}^{N_1\times\ldots\times N_d}$ contains $\mathcal{X}$. Clearly a basis in $\mathbf{V}$ consists of the vectors $\mathbf{c}_{i_1,1} \circ \ldots \circ \mathbf{c}_{i_d,d}$ where $i_j \in [r_j] := \{1, \ldots, r_j\}$ and $j \in [d]$. Hence the core Tucker decomposition of $\mathcal{X}$ is

$$\mathcal{X} = \sum_{i_j\in[r_j],j\in[d]} \xi_{i_1,\ldots,i_d} \mathbf{c}_{i_1,1} \circ \ldots \circ \mathbf{c}_{i_d,d}. \qquad (1)$$

There are many ways to get a weaker decomposition as

$$\mathcal{X} = \sum_{i=1}^{K} \mathbf{a}_i^{(1)} \circ \ldots \circ \mathbf{a}_i^{(d)}, \quad \mathbf{a}_i^{(j)} \in R_j(\mathcal{X}), j \in [d]. \qquad (2)$$

A simple constructive way is as follows. First decompose $X_{(1)}$ as $X_{(1)} = \sum_{j=1}^{r_1} \mathbf{c}_{j,1}\mathbf{g}_{j,1}^\top$ (e.g. by singular value decomposition (SVD)). Now each $\mathbf{g}_{j,1}$ can be viewed as a tensor of order $d-1 \in R_2(\mathcal{X}) \circ \ldots \circ R_d(\mathcal{X}) \subset \mathbb{R}^{N_2\times\ldots\times N_d}$. Unfold each $\mathbf{g}_{j,1}$ in mode 2 to obtain $\mathbf{g}_{j,1(2)}$ and decompose $\mathbf{g}_{j,1(2)} = \sum_{l=1}^{r_2} \mathbf{d}_{l,2,j}\mathbf{f}_{l,2,j}^\top, \quad \mathbf{d}_{l,2,j} \in R_2(\mathcal{X}), \mathbf{f}_{l,2,j} \in R_3(\mathcal{X}) \circ \ldots \circ R_d(\mathcal{X})$. Continuing in this way we get a decomposition of type (2). Note that if $\mathcal{X}$ is s-sparse then each vector in $R_i(\mathcal{X})$ is s-sparse and each rank r_i is at most s. So $K \leq s^{d-1}$.

CANDECOMP/PARAFAC Decomposition [18]For a tensor $\mathcal{X} \in \mathbb{R}^{N_1\times\ldots\times N_d}$, the CANDECOMP/PARAFAC (CP) decomposition is $\mathcal{X} \approx [\lambda; A^{(1)}, \ldots, A^{(d)}] \equiv \sum_{r=1}^{R} \lambda_r \mathbf{a}_r^{(1)} \circ \ldots \circ \mathbf{a}_r^{(d)}$, where $\lambda = [\lambda_1 \ldots \lambda_R]^\top \in \mathbb{R}^R$ and $A^{(i)} = [\mathbf{a}_1^{(i)} \cdots \mathbf{a}_R^{(i)}] \in \mathbb{R}^{N_i\times R}$ for $i = 1, \ldots, d$.

2.2. Compressive sensing

Compressive sensing is one of the ways to encode sparse information. A vector $\mathbf{x} \in \mathbb{R}^N$ is called s-sparse if it has at most s nonzero coordinates. The CS measurement protocol measures the signal $\mathbf{x}$ with the measurement matrix $A \in \mathbb{R}^{m \times N}$ where $m < N$ and transmits the encoded information $\mathbf{y} \in \mathbb{R}^m$ where $\mathbf{y} = A\mathbf{x}$. The receiver knows A and attempts to recover $\mathbf{x}$ from $\mathbf{y}$. Since $m < N$, there are usually infinitely many solutions for such under-constrained problem. However, if $\mathbf{x}$ is known to be sufficiently sparse, then exact recovery of $\mathbf{x}$ is possible, which establishes the fundamental tenet of CS theory. The recovery is done by finding a solution $\mathbf{z}^\star \in \mathbb{R}^N$ satisfying

$$\mathbf{z}^\star = \arg\min\{\|\mathbf{z}\|_1,\ A\mathbf{z} = \mathbf{y}\}. \tag{3}$$

Such $\mathbf{z}^\star$ coincides with $\mathbf{x}$. The following well known result states that each s-sparse solution can be recovered uniquely if A satisfies the null space property of order s, denoted as NSP_s. That is, if $A\mathbf{w} = \mathbf{0}, \mathbf{w} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, then for any subset $S \subset \{1, \ldots, N\}$ with cardinality $|S| = s$ it holds that $\|\mathbf{v}_S\|_1 < \|\mathbf{v}_{S^c}\|_1$, where $\mathbf{v}_S$ denotes the vector that coincides with $\mathbf{v}$ on the index set S and is set to zero on S^c.

A simple way to generate such A is to use A sampled randomly from Gaussian or Bernoulli distributions. Then there exists a universal constant c such that if

$$m \geq 2cs \ln \frac{N}{s} \tag{4}$$

then the recovery of $\mathbf{x}$ using (3) is successful with probability at least $1 - \exp(-\frac{m}{2c})$.

Recently, the extension of CS theory for multidimensional signals has become an emerging topic. The objective of our paper is to consider the case where the s-sparse vector $\mathbf{x}$ is represented as an s-sparse tensor $\mathcal{X} = [x_{i_1,\ldots,i_d}] \in \mathbb{R}^{N_1 \times \ldots \times N_d}$. If we ignore the structure of $\mathcal{X}$ as a tensor, and view it as a vector of size $N = N_1 \cdot \ldots \cdot N_d$, clearly, we can transmit $\mathcal{X}$ as $\mathbf{x}$ by using $\mathbf{y} = A\mathbf{x}$. If we use a random A as described above, we need m to be at least of order

$$m \geq 2cs(-\ln s + \sum_{i=1}^{d} \ln N_i). \tag{5}$$

In [17], Kronecker compressive sensing (KCS) constructs A from Kronecker product $A := U_1 \otimes \ldots \otimes U_d$, where $U_i \in \mathbb{R}^{m_i \times N_i}$ for $i = 1, \ldots, d$ and each U_i has NSP_s property. Then $\mathbf{x}$ is recovered uniquely from $\mathbf{y} = A\mathbf{x}$ by ℓ_1-minimization. In this paper, we analyze the compression and reconstruction of tensor $\mathcal{X}$ from the tensor $\mathcal{Y} = \mathcal{X} \times_1 U_1 \times \ldots \times_d U_d \in \mathbb{R}^{m_1 \times \ldots \times m_d}$ using a sequence of ℓ_1-minimizations similar to the minimization in (3). The advantage of our method is that our recovery problems are in terms of each U_i, which are much smaller comparing with the recovery related to A as given by the minimization in (3). This means that the amount of computations of our method is much less than that given by (3). If we choose our matrices U_i at random then we have the condition

$$m_i \geq 2cs \ln \frac{N_i}{s}, \quad i = 1, \ldots, d. \tag{6}$$

3. TENSOR COMPRESSIVE SENSING

3.1. Tensor compressive sensing with serial recovery

We first discuss our method for matrices, i.e. $d = 2$ and then for tensors $d \geq 3$.

Theorem 3.1 *Let $X = [x_{ij}] \in \mathbb{R}^{N_1 \times N_2}$ be s-sparse. Let $U_i \in \mathbb{R}^{m_i \times N_i}$ and assume that U_i has NSP_s property for $i = 1, 2$. Define*

$$Y = [y_{pq}] = U_1 X U_2^\top \in \mathbb{R}^{m_1 \times m_2}. \tag{7}$$

Then X can be recovered uniquely using the following procedure. Let $\mathbf{y}_1, \ldots, \mathbf{y}_{m_2} \in \mathbb{R}^{m_1}$ be the columns of Y. Let $\mathbf{z}_i^\star \in \mathbb{R}^{N_1}$ be a solution of

$$\mathbf{z}_i^\star = \min\{\|\mathbf{z}_i\|_1,\ U_1\mathbf{z}_i = \mathbf{y}_i\}, \quad i = 1, \ldots, m_2. \tag{8}$$

Then each $\mathbf{z}_i^\star$ is unique and s-sparse. Let $Z \in \mathbb{R}^{N_1 \times m_2}$ be the matrix with columns $\mathbf{z}_1^\star, \ldots, \mathbf{z}_{m_2}^\star$. Let $\mathbf{w}_1^\top, \ldots, \mathbf{w}_{N_1}^\top$ be the rows of Z. Then the j^{th} row of X is the solution $\mathbf{u}_j^\star \in \mathbb{R}^{N_2}$ of

$$\mathbf{u}_j^\star = \min\{\|\mathbf{u}_j\|_1,\ U_2\mathbf{u}_j = \mathbf{w}_j\}, \quad j = 1, \ldots, N_1. \tag{9}$$

Proof Let $Z = XU_2^\top \in \mathbb{R}^{N_1 \times m_2}$. Assume that $\mathbf{z}_1^\star, \ldots, \mathbf{z}_{m_2}^\star$ are the columns of Z. Note that $\mathbf{z}_i^\star$ is a linear combination of the N_2 columns of X, given by the i^{th} row of U_2. Since X is s-sparse, $\mathbf{z}_i^\star$ has at most s nonzero entries. Note that $Y = U_1 Z$, it follows that $\mathbf{y}_i = U_1\mathbf{z}_i^\star$. Since U_1 has NSP_s, we deduce the equality (8). Observe next that $Z^\top = U_2 X^\top$. Hence the column $\mathbf{w}_j$ of $Z^\top$ is $\mathbf{w}_j = U_2\mathbf{u}_j^\star$. Since X is s-sparse, each $\mathbf{u}_j^\star$ is s-sparse. The assumption that U_2 has NSP_s property implies (9). This completes the proof. ■

If we choose U_1, U_2 to be random, then we need the assumption (6). We now make the following observation. Suppose we know that each column of XU_2^T is s_1-sparse and each row of X is s_2-sparse. Then from the proof of Theorem 3.1 it follows that we can recover X, on the assumption that U_1 has NSP_{s_1} and U_2 has NSP_{s_2}.

Theorem 3.2 (GTCS-S) *Let $\mathcal{X} = [x_{i_1,\ldots,i_d}] \in \mathbb{R}^{N_1 \times \ldots \times N_d}$ be s-sparse. Let $U_i \in \mathbb{R}^{m_i \times N_i}$ and assume that U_i has NSP_s property for $i = 1, \ldots, d$. Define*

$$\mathcal{Y} = [y_{j_1,\ldots,j_d}] = \mathcal{X} \times_1 U_1 \times \ldots \times_d U_d \in \mathbb{R}^{m_1 \times \ldots \times m_d}. \tag{10}$$

Then $\mathcal{X}$ can be recovered uniquely using the following recursive procedure. Unfold $\mathcal{Y}$ in mode 1,

$$Y_{(1)} = U_1 X_{(1)} [\otimes_{k=d}^{2} U_k]^\top \in \mathbb{R}^{m_1 \times (m_2 \cdot \ldots \cdot m_d)}.$$

Let $\mathbf{y}_1, \ldots, \mathbf{y}_{m_2 \cdot \ldots \cdot m_d}$ be the columns of $Y_{(1)}$. Then $\mathbf{y}_i = U_1 \mathbf{z}_i$ where each $\mathbf{z}_i \in \mathbb{R}^{N_1}$ is s-sparse. Recover each $\mathbf{z}_i$ using (3). *Let $\mathcal{Z} = \mathcal{X} \times_2 U_2 \times \ldots \times_d U_d \in \mathbb{R}^{N_1 \times m_2 \times \ldots \times m_d}$ with its mode-1 fibers being $\mathbf{z}_1, \ldots, \mathbf{z}_{m_2 \cdot \ldots \cdot m_d}$. Unfold $\mathcal{Z}$ in mode 2,*

$$Z_{(2)} = U_2 X_{(2)} [\otimes_{k=d}^{3} U_k \otimes I]^\top \in \mathbb{R}^{m_2 \times (N1 \cdot m_3 \cdot \ldots \cdot m_d)}.$$

Let $\mathbf{w}_1, \ldots, \mathbf{w}_{N1 \cdot m_3 \cdot \ldots \cdot m_d}$ be the columns of $Z_{(2)}$. Then $\mathbf{w}_j = U_2 \mathbf{v}_j$ where each $\mathbf{v}_j \in \mathbb{R}^{N_2}$ is s-sparse. Recover each $\mathbf{v}_j$ using (3). *Continue the above procedure for mode $3, \ldots, d$ and $\mathcal{X}$ can be reconstructed in series.*

As for matrices, assume mode-i fibers of $\mathcal{X} \times_{i+1} U_{i+1} \times \ldots \times_d U_d$ is s_i-sparse for $i = 1, \ldots, d-1$ and mode-d fibers of $\mathcal{X}$ is s_d-sparse, then we can relax the condition such that U_i only has to satisfy NSP_{s_i} for $i = 1, \ldots, d$.

3.2. Tensor compressive sensing with parallelizable recovery

Theorem 3.3 (GTCS-P) *Let $\mathcal{X} = [x_{i_1,\ldots,i_d}] \in \mathbb{R}^{N_1 \times \ldots \times N_d}$ be s-sparse. Let $U_i \in \mathbb{R}^{m_i \times N_i}$ and assume that U_i has NSP_s property for $i = 1, \ldots, d$. Define $\mathcal{Y}$ as in* (10)*, then $\mathcal{X}$ can be recovered uniquely using the following procedure. Consider a decomposition of $\mathcal{Y}$ such that,*

$$\mathcal{Y} = \sum_{i=1}^{K} \mathbf{b}_i^{(1)} \circ \ldots \circ \mathbf{b}_i^{(d)}, \quad \mathbf{b}_i^{(j)} \in R_j(\mathcal{Y}) \subseteq U_j R_j(\mathcal{X}),$$
$$j \in [d]. \tag{11}$$

Let $\mathbf{w}_i^{(j)\star} \in R_j(\mathcal{X}) \subset \mathbb{R}^{N_j}$ be a solution of

$$\mathbf{w}_i^{(j)\star} = \min\{\|\mathbf{w}_i^{(j)}\|_1,\ U_j \mathbf{w}_i^{(j)} = \mathbf{b}_i^{(j)}\}, \quad i = 1, \ldots, K,$$
$$j = 1, \ldots, d. \tag{12}$$

Thus each $\mathbf{w}_i^{(j)\star}$ is unique and s-sparse. Then,

$$\mathcal{X} = \sum_{i=1}^{K} \mathbf{w}_i^{(1)} \circ \ldots \circ \mathbf{w}_i^{(d)}, \quad \mathbf{w}_i^{(j)} \in R_j(\mathcal{X}), j \in [d]. \tag{13}$$

Due to limited space, a detailed proof can be found in [19].

In fact, if all vectors $\in R_i(\mathcal{X})$ are s_i-sparse, then U_i only has to satisfy NSP_{s_i}. The above recovery procedure consists of two stages: the decomposition stage and the reconstruction stage where the latter for each tensor mode can be implemented in parallel.

Observe that if we are satisfied with recovering a rank-R approximation of $\mathcal{X}$, we only need to fit a rank-R approximation of $\mathcal{Y} = \sum_{r=1}^{R} \mathbf{b}_r^{(1)} \circ \ldots \circ \mathbf{b}_r^{(d)}$ (e.g. by CP decomposition) and subsequently recover each $\mathbf{w}_r^{(j)}$ for $j = 1, \ldots, d$ and $r = 1, \ldots, R$. Then a rank-R approximation of $\mathcal{X}$ would be $\hat{\mathcal{X}} = \sum_{r=1}^{R} \mathbf{w}_r^{(1)} \circ \ldots \circ \mathbf{w}_r^{(d)}$[1].

[1] It has recently come to our attention that a different proof of this observation has been proposed to derive a similar result, referred to as Multi-Way Compressed Sensing in [20].

4. EXPERIMENTAL RESULTS

We experimentally demonstrate the performance of GTCS methods on sparse image and video sequence. In [17], KCS has shown its outstanding performance for compression of multidimensional signals in comparison with several other methods such as independent measurements and partitioned measurements. Therefore, we choose KCS as a comparison to the proposed GTCS methods. Our experiments use the ℓ_1-minimization solvers from [21]. We set the same threshold to determine the termination of ℓ_1-minimization in all subsequent experiments. All simulations are executed on a desktop with 2.4 GHz Intel Core i5 CPU and 8GB RAM.

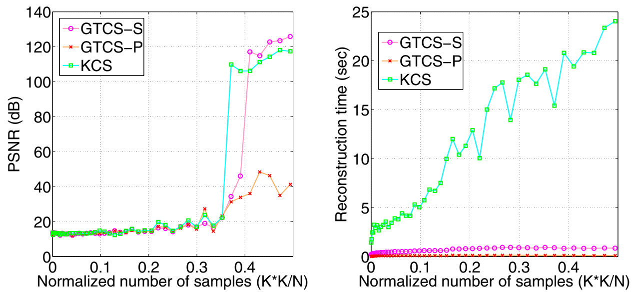
4.1. Sparse image representation

As shown in Figure 2(a), the original black and white image is of size 64×64 ($N = 4096$ pixels). Its columns are 14-sparse and rows are 18-sparse. The image itself is 178-sparse. We let the number of measurements evenly split among the two modes, that is, for each mode, the randomly constructed Gaussian matrix U is of size $K \times 64$. Therefore the KCS measurement matrix $U \otimes U$ is of size $K^2 \times 4096$. Thus the total number of samples is K^2. We define the normalized number of samples to be $\frac{K^2}{N}$. For GTCS, both the serial recovery method GTCS-S and the parallelizable recovery method GTCS-P are implemented. In the matrix case, we simply conduct SVD on the compressed image in the decomposition stage of GTCS-P. Although the reconstruction stage of GTCS-P is parallelizable, we here recover each vector in series. We comprehensively examine the performance of all the above methods by varying K from 1 to 45.

Figure 1(a) and 1(b) compare the peak signal to noise ratio (PSNR) and the recovery time respectively. Both KCS and GTCS methods achieve PSNR over 30dB when $K = 39$. As K increases, GTCS-S tends to outperform KCS in terms of both accuracy and efficiency. Although PSNR of GTCS-P is the lowest among the three methods, it is most time efficient. Moreover, with parallelization of GTCS-P, the recovery procedure can be further accelerated considerably. The reconstructed images when $K = 38$, that is, using 0.35 normalized number of samples, are shown in Figure 2(b)2(c)2(d). Though GTCS-P recovers much noisier image, it is good at recovering the non-zero structure of the original image.

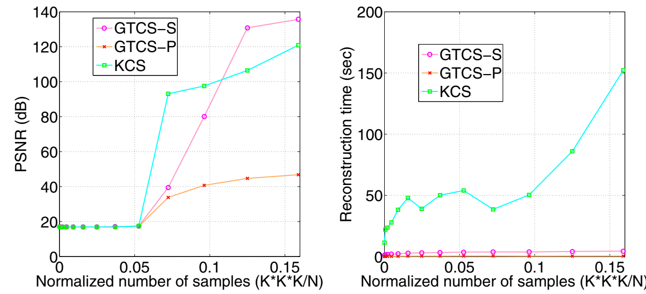
4.2. Sparse video representation

We next compare the performance of GTCS and KCS on video data. Each frame of the video sequence is preprocessed to have size 24×24 and we choose the first 24 frames. The video data together is represented by a $24 \times 24 \times 24$ tensor and has $N = 13824$ voxels in total. To obtain a sparse tensor, we manually keep only $6 \times 6 \times 6$ nonzero entries in the center of the video tensor data and the rest are set to zero. Therefore, the video tensor itself is 216-sparse and its mode-i fibers



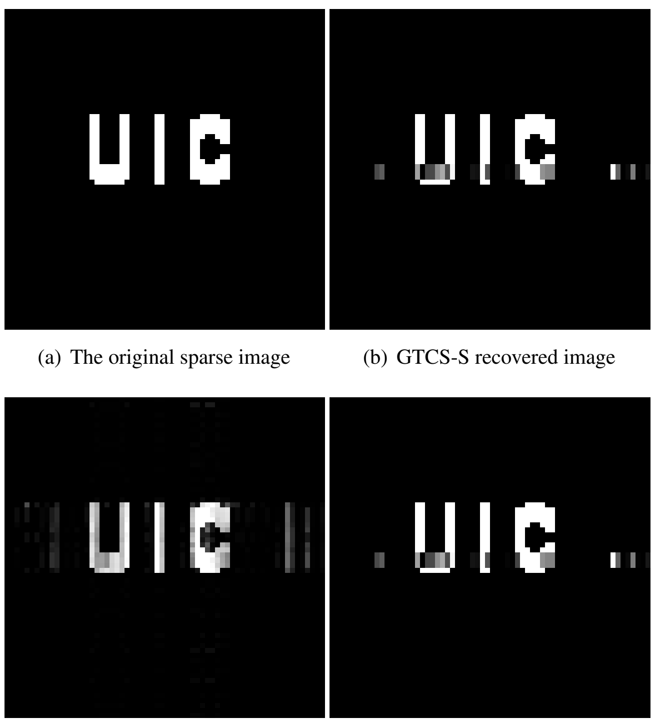
(a) PSNR comparison (b) Recovery time comparison

Fig. 1. PSNR and reconstruction time comparison on sparse image.



(a) PSNR comparison (b) Recovery time comparison

Fig. 3. PSNR and reconstruction time comparison on sparse video.



(a) The original sparse image (b) GTCS-S recovered image

(c) GTCS-P recovered image (d) KCS recovered image

Fig. 2. The original image and the recovered images by GTCS and KCS using 0.35 normalized number of samples.

are all 6-sparse for $i = 1, \ldots, 3$. The randomly constructed Gaussian measurement matrix for each mode is now of size $K \times 24$ and the total number of samples is K^3. Therefore, the normalized number of samples is $\frac{K^3}{N}$. In the decomposition stage of GTCS-P, we employ a decomposition described in Section 2.1 to obtain a weaker form of the core Tucker decomposition. We vary K from 1 to 13.

Figure 3(a) depicts PSNR of the first non-zero frame recovered by all three methods. All methods exhibit an abrupt increase in PSNR at $K = 10$ (using 0.07 normalized number of samples). Also, Figure 3(b) summarizes the recovery time. In comparison to the image case, the time advantage of GTCS becomes more important in the reconstruction of higher-order tensor data.

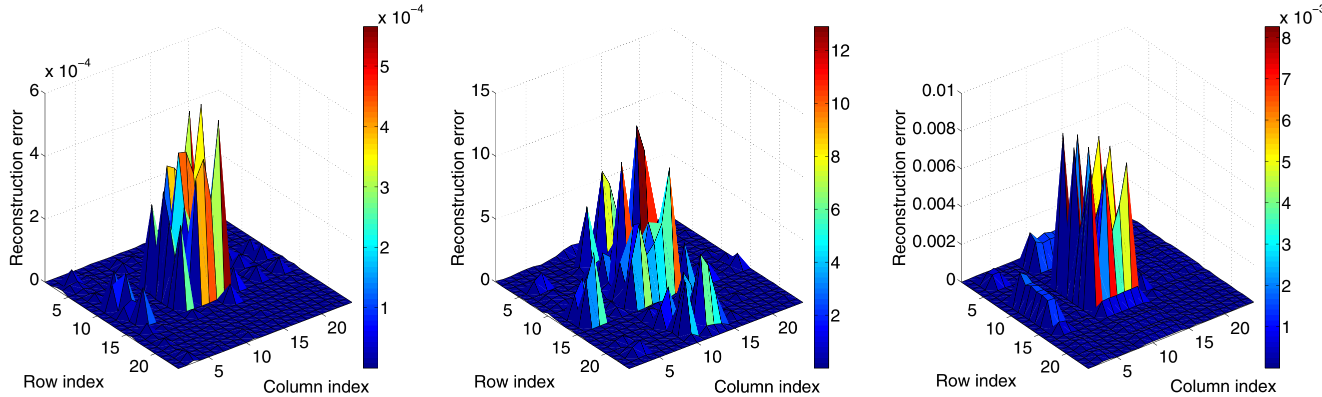
We specifically look into the recovered frames of all three methods when $K = 12$. Since all the recovered frames achieve a PSNR higher than 40 dB, it is hard to visually observe any difference compared to the original video frame. Therefore, we display the reconstruction error image defined as the absolute difference between the reconstructed image and the original image. Figures 4(a)4(b)4(c) visualize the reconstruction errors of all three methods. Compared to KCS, GTCS-S achieves much lower reconstruction error using much less time.

5. CONCLUSION

In this paper, we propose Generalized Tensor Compressive Sensing (GTCS)–a unified framework for compressive sensing of higher-order tensors. In addition, we propose two reconstruction procedures, a serial method (GTCS-S) and a parallelizable method (GTCS-P). We then compare the performance of the proposed method with Kronecker compressive sensing (KCS) on both image and video data. Experimental results show that GTCS outperforms KCS in terms of both accuracy and efficiency. The advantage of our method mainly comes from the fact that our recovery problems are in terms of each tensor mode, which are much smaller comparing with the recovery related to the vectorization of all tensor modes. Such advantage becomes more important as the order of the data increases. We state our theorems for sparse tensors. However, most real-world data are not really sparse, instead, they are only compressible in some domain. Future work will focus on demonstrating the effectiveness of GTCS on compressible higher-order data.

6. REFERENCES

[1] E.J. Candes, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from



(a) Reconstruction error of GTCS-S (b) Reconstruction error of GTCS-P (c) Reconstruction error of KCS

Fig. 4. Visualization of the reconstruction error in the recovered video frame 9 using 0.125 normalized number of samples.

highly incomplete frequency information," *Information Theory, IEEE Transactions on*, feb. 2006.

[2] E.J. Candes and T. Tao, "Near-optimal signal recovery from random projections: Universal encoding strategies?," *Information Theory, IEEE Transactions on*, vol. 52, pp. 5406–5425, dec. 2006.

[3] D.L. Donoho, "Compressed sensing," *Information Theory, IEEE Transactions on*, apr. 2006.

[4] D.L. Donoho and B.F. Logan, "Signal recovery and the large sieve," *SIAM Journal on Applied Mathematics*, vol. 52, pp. 577–591, 1992.

[5] S. Chen, D.L. Donoho, and M.A. Saunders, "Atomic decomposition by basis pursuit," *SIAM Journal on Scientific Computing*, vol. 20, pp. 33–61, 1996.

[6] L. Rudin, S. Osher, and E. Fatemi, "Nonlinear total variation based noise removal algorithms," *Physica D: Nonlinear Phenomena*, vol. 60, pp. 259–268, 1992.

[7] A. Cohen, W. Dahmen, and R. Devore, "Compressed sensing and best k-term approximation," *J. Amer. Math. Soc*, pp. 211–231, 2009.

[8] E. Candes, "The restricted isometry property and its implications for compressed sensing," *Comptes Rendus Mathematique*, vol. 346, pp. 589–592, 2008.

[9] M.B. Wakin, J.N. Laska, M.F. Duarte, D. Baron, S. Sarvotham, D. Takhar, K.F. Kelly, and R.G. Baraniuk, "An architecture for compressive imaging," in *IEEE ICIP*, oct. 2006, pp. 1273–1276.

[10] E. J. Candes, J. K. Romberg, and T. Tao, ," *Comm. Pure Appl. Math.*, aug.

[11] L. Gan, "Block compressed sensing of natural images," in *15th ICDSP 2007*, jul. 2007, pp. 403–406.

[12] M. Fazel, *Matrix rank minimization with applications*, Ph.D. thesis, 2002.

[13] B. Recht, M. Fazel, and P. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM Rev.*, vol. 52.

[14] E. Candes and B. Recht, "Exact matrix completion via convex optimization," *Commun. ACM*, jun. 2012.

[15] L.H. Lim and P. Comon, "Multiarray signal processing: Tensor decomposition meets compressed sensing," *CoRR*, vol. abs/1002.4935, 2010.

[16] M.F. Duarte and R.G. Baraniuk, "Kronecker product matrices for compressive sensing," in *IEEE ICASSP 2010*, mar. 2010, pp. 3650–3653.

[17] M.F. Duarte and R.G. Baraniuk, "Kronecker compressive sensing," *Image Processing, IEEE Transactions on*, vol. 21, pp. 494–504, feb. 2012.

[18] T. G. Kolda and B. W. Bader, "Tensor decompositions and applications," *SIAM REVIEW*, vol. 51, 2009.

[19] Qun Li, *Multilinear Algebra in High-Order Data Analysis: Retrieval, Classification and Representation*, Ph.D. thesis, 2013.

[20] N.D. Sidiropoulos and A. Kyrillidis, "Multi-way compressed sensing for sparse low-rank tensors," *Signal Processing Letters, IEEE*, vol. 19, 2012.

[21] E. J. Candes and J. K. Romberg, "The l_1 magic toolbox," *Available online: http://www.l1-magic.org*.