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On angles and distances between subspaces

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ABSTRACT

The angles and distances between two given subspaces of $\mathbb{C}_{n,1}$ are investigated on the basis of a joint decomposition of the corresponding orthogonal projectors. Several new results are established, with the particular attention paid to the notions of inclinedness and minimal angle. To demonstrate the usefulness of the approach utilized, some results known to be valid in Hilbert space are reestablished in $\mathbb{C}_{n,1}$, either in generalized form or with considerably shorter proofs than in the original sources.

1. Introduction

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbols $\mathbf{N}^*$, $\mathcal{R}(\mathbf{N})$, $\mathcal{N}(\mathbf{N})$, and $\mathrm{rk}(\mathbf{N})$ will stand for the conjugate transpose, column space, null space, and rank of $\mathbf{N} \in \mathbb{C}_{m,n}$, respectively. Moreover, $\mathbf{I}_n$ will be the identity matrix of order n, and for a given $\mathbf{N} \in \mathbb{C}_{n,n}$ we define $\overline{\mathbf{N}} = \mathbf{I}_n - \mathbf{N}$. Furthermore, $\zeta(\mathbf{N})$, $\rho(\mathbf{N})$, and $\xi(\mathbf{N})$ will mean the numbers of eigenvalues of $\mathbf{N} \in \mathbb{C}_{n,n}$ equal to, consecutively, zero, one, and belonging to the set $(0, 1)$, whereas, for Hermitian $\mathbf{N} \in \mathbb{C}_{n,n}$, the symbol

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$\lambda_{\max}(\mathbf{N})$ will be used for its largest eigenvalue. Customarily, the spectral norm of $\mathbf{N} \in \mathbb{C}_{m,n}$ will be denoted by $\|\mathbf{N}\|$, i.e., $\|\mathbf{N}\| = \sqrt{\lambda_{\max}(\mathbf{N}^*\mathbf{N})}$.

A crucial role in the considerations of the present paper is played by orthogonal projectors in $\mathbb{C}_{n,1}$ (Hermitian idempotent matrices of order n), whose set will be denoted by $\mathbb{C}_n^{\mathsf{OP}}$, i.e.,

$$\mathbb{C}_n^{\mathsf{OP}} = \{\mathbf{N} \in \mathbb{C}_{n,n}\colon\ \mathbf{N}^2 = \mathbf{N} = \mathbf{N}^*\}.$$

An essential property of any orthogonal projector is that $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ if and only if it is expressible as $\mathbf{N}\mathbf{N}^\dagger$ for some $\mathbf{N} \in \mathbb{C}_{n,m}$, where $\mathbf{N}^\dagger \in \mathbb{C}_{m,n}$ is the Moore–Penrose inverse of $\mathbf{N}$, i.e., the unique solution to the equations

$$\mathbf{N}\mathbf{N}^\dagger\mathbf{N} = \mathbf{N}, \quad \mathbf{N}^\dagger\mathbf{N}\mathbf{N}^\dagger = \mathbf{N}^\dagger, \quad (\mathbf{N}\mathbf{N}^\dagger)^* = \mathbf{N}\mathbf{N}^\dagger, \quad (\mathbf{N}^\dagger\mathbf{N})^* = \mathbf{N}^\dagger\mathbf{N}.$$

Then $\mathbf{N}\mathbf{N}^\dagger$ is the orthogonal projector onto $\mathcal{R}(\mathbf{N})$ and, consequently, $\mathbf{I}_n - \mathbf{N}\mathbf{N}^\dagger$ is the orthogonal projector onto the orthogonal complement of $\mathcal{R}(\mathbf{N})$, denoted by $\mathcal{R}(\mathbf{N})^\perp$, where $\mathbb{C}_{n,1} = \mathcal{R}(\mathbf{N}) \overset{\perp}{\oplus} \mathcal{R}(\mathbf{N})^\perp$, with the symbol $\overset{\perp}{\oplus}$ being used to indicate that the two subspaces involved in the direct sum are orthogonal. Similarly, $\mathbf{N}^\dagger\mathbf{N}$ and $\mathbf{I}_m - \mathbf{N}^\dagger\mathbf{N}$ are the orthogonal projectors onto $\mathcal{R}(\mathbf{N}^*)$ and $\mathcal{R}(\mathbf{N}^*)^\perp$, respectively, where $\mathbb{C}_{m,1} = \mathcal{R}(\mathbf{N}^*) \overset{\perp}{\oplus} \mathcal{R}(\mathbf{N}^*)^\perp$. What is important from the point of view of the present paper, is the fact that there is one-to-one correspondence between an orthogonal projector and a subspace onto which it projects. This fact ensures that many relationships between subspaces can be expressed within purely algebraical settings, in terms of the orthogonal projectors associated with the subspaces.

Let $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ be of rank r. It is known that there exists unitary $\mathbf{U} \in \mathbb{C}_{n,n}$ such that

$$\mathbf{P} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{1.1}$$

Clearly, any other orthogonal projector of order n, say $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$, can be represented as

$$\mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^*, \tag{1.2}$$

with $\mathbf{A} \in \mathbb{C}_{r,r}$ and $\mathbf{D} \in \mathbb{C}_{n-r,n-r}$ being Hermitian. Two particular versions of representation (1.2) are obtained when $r = 0$, in which case matrices $\mathbf{A}$ and $\mathbf{B}$ are absent, and when $r = n$, in which case matrices $\mathbf{D}$ and $\mathbf{B}$ are absent.

In what follows, the symbol $\mathbf{P}_\chi$ will denote the orthogonal projector, which projects (orthogonally) onto subspace χ. Furthermore, with regard to the orthogonal projectors onto the column spaces of submatrices of $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ given in (1.2), we will use the convention according to which $\mathbf{P}_{\mathbf{N}}$ stands for $\mathbf{P}_{\mathbf{N}} = \mathbf{N}\mathbf{N}^\dagger$ and $\widetilde{\mathbf{P}}_{\mathbf{N}}$ for $\widetilde{\mathbf{P}}_{\mathbf{N}} = \mathbf{I}_k - \mathbf{N}\mathbf{N}^\dagger$, where $\mathbf{I}_k$ is the identity matrix of an appropriate order and $\mathbf{N} \in \{\mathbf{A}, \overline{\mathbf{A}}, \mathbf{D}, \overline{\mathbf{D}}\}$.

The literature on angles and distances between subspaces is quite extensive. From among important publications dealing with the related topics, one should definitely mention Afriat [2], Ben-Israel and Greville [7], Deutsch [9], Galántai [14], Kato [21], and Meyer [25]. The present paper visits the angles and distances once again. By utilizing partitioned representations of a pair of orthogonal projectors, several new results in n-dimensional complex vector space are established. To demonstrate the usefulness of the approach utilized, some results known to be valid in Hilbert space are reestablished in $\mathbb{C}_{n,1}$, either in generalized form or with considerably shorter proofs than in the original sources.

The next section provides a collection of useful relationships involving matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{D}$ occurring in (1.2). Section 3 is devoted to the notion of inclinedness, whereas in Section 4 we investigate angles between subspaces. The final section of the paper contains considerations dealing with distances between subspaces.

2. Preliminary results

The following four lemmas concern relationships between submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ involved in matrix $\mathbf{Q}$ given in (1.2) and will be helpful in further considerations.

Lemma 1. *Let* $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *be partitioned as in* (1.2). *Then*:

(i) $\mathbf{A} = \mathbf{A}^2 + \mathbf{B}\mathbf{B}^*$ *or, equivalently*, $\mathbf{A}\overline{\mathbf{A}} = \mathbf{B}\mathbf{B}^*$,
(ii) $\mathbf{B} = \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{D}$ *or, equivalently*, $\mathbf{B}^* = \mathbf{B}^*\mathbf{A} + \mathbf{D}\mathbf{B}^*$,
(iii) $\mathbf{D} = \mathbf{D}^2 + \mathbf{B}^*\mathbf{B}$ *or, equivalently*, $\mathbf{D}\overline{\mathbf{D}} = \mathbf{B}^*\mathbf{B}$.

Proof. The three relationships are straightforward consequences of the condition $\mathbf{Q}^2 = \mathbf{Q}$. $\square$

It is noteworthy that conditions (i) and (iii) of Lemma 1 combined with the facts that $\mathbf{A}$ and $\mathbf{D}$ are Hermitian, respectively, ensure that $\mathbf{A}$ and $\mathbf{D}$ are both nonnegative definite.

Lemma 2. *Let* $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *be partitioned as in* (1.2). *Then*:

(i) $\overline{\mathbf{A}} = \overline{\mathbf{A}}^2 + \mathbf{B}\mathbf{B}^*$, (ii) $\mathbf{B}\mathbf{D} = \overline{\mathbf{A}}\mathbf{B}$,
(iii) $\mathbf{A}\mathbf{B} = \mathbf{B}\overline{\mathbf{D}}$, (iv) $\overline{\mathbf{D}} = \overline{\mathbf{D}}^2 + \mathbf{B}^*\mathbf{B}$,
(v) $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{A})$, (vi) $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\overline{\mathbf{A}})$,
(vii) $\mathcal{R}(\mathbf{B}^*) \subseteq \mathcal{R}(\mathbf{D})$, (viii) $\mathcal{R}(\mathbf{B}^*) \subseteq \mathcal{R}(\overline{\mathbf{D}})$,
(ix) $\mathbf{A}^\dagger\mathbf{B} = \mathbf{B}\overline{\mathbf{D}}^\dagger$, (x) $\overline{\mathbf{A}}^\dagger\mathbf{B} = \mathbf{B}\mathbf{D}^\dagger$.

Proof. Conditions (i)–(iv) follow directly from Lemma 1, while condition (v) is established on account of condition (i) of Lemma 1 by noting that

$$\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}\mathbf{A}^* + \mathbf{B}\mathbf{B}^*) = \mathcal{R}(\mathbf{A}\mathbf{A}^*) + \mathcal{R}(\mathbf{B}\mathbf{B}^*) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B}),$$

where the second equality is a consequence of the fact that $\mathbf{A}\mathbf{A}^*$ and $\mathbf{B}\mathbf{B}^*$ are both nonnegative definite. The next three conditions are obtained similarly.

Further, from condition (ii) of Lemma 1 it follows that $\mathbf{A}^\dagger\mathbf{B} = \mathbf{A}^\dagger(\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{D})$. Hence, on account of the inclusion $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{A}^*)$, being a modified version of condition (v) of the lemma, which is equivalent to $\mathbf{A}^\dagger\mathbf{A}\mathbf{B} = \mathbf{B}$, we get $\mathbf{A}^\dagger\mathbf{B} = \mathbf{B} + \mathbf{A}^\dagger\mathbf{B}\mathbf{D}$. In consequence, $\mathbf{B} = \mathbf{A}^\dagger\mathbf{B}\overline{\mathbf{D}}$. Postmultiplying this equation by $\overline{\mathbf{D}}^\dagger$ and utilizing condition (viii) of the lemma, which can equivalently be expressed as $\mathbf{B}\overline{\mathbf{D}}\,\overline{\mathbf{D}}^\dagger = \mathbf{B}$, we arrive at condition (ix). The last condition is established analogously. $\square$

Lemma 3. *Let* $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *be partitioned as in* (1.2). *Then*:

(i) $\mathbf{A} - \mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \widetilde{\mathbf{P}}_{\overline{\mathbf{A}}}$, (ii) $\overline{\mathbf{A}} + \mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \mathbf{P}_{\overline{\mathbf{A}}}$,
(iii) $\mathbf{D} - \mathbf{B}^*\mathbf{A}^\dagger\mathbf{B} = \widetilde{\mathbf{P}}_{\overline{\mathbf{D}}}$, (iv) $\overline{\mathbf{D}} + \mathbf{B}^*\mathbf{A}^\dagger\mathbf{B} = \mathbf{P}_{\overline{\mathbf{D}}}$,
(v) $\mathbf{D} + \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B} = \mathbf{P}_{\mathbf{D}}$, (vi) $\overline{\mathbf{D}} - \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B} = \widetilde{\mathbf{P}}_{\mathbf{D}}$,
(vii) $\mathbf{A} + \mathbf{B}\overline{\mathbf{D}}^\dagger\mathbf{B}^* = \mathbf{P}_{\mathbf{A}}$, (viii) $\overline{\mathbf{A}} - \mathbf{B}\overline{\mathbf{D}}^\dagger\mathbf{B}^* = \widetilde{\mathbf{P}}_{\mathbf{A}}$.

Proof. The proof will be limited to condition (i) only. On account of conditions (i) and (x) of Lemmas 1 and 2, respectively, it follows that $\mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \overline{\mathbf{A}}^\dagger\mathbf{A}\overline{\mathbf{A}}$. Hence, $\mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \overline{\mathbf{A}}^\dagger(\mathbf{I}_r - \overline{\mathbf{A}})\overline{\mathbf{A}}$, and taking into account that $\overline{\mathbf{A}}\,\overline{\mathbf{A}}^\dagger = \overline{\mathbf{A}}^\dagger\overline{\mathbf{A}}$ (being a consequence of $\overline{\mathbf{A}} = \overline{\mathbf{A}}^*$), we in turn get $\mathbf{A} - \mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \mathbf{I}_r - \overline{\mathbf{A}}^\dagger\overline{\mathbf{A}}$, establishing condition (i) of the lemma. $\square$

The next lemma refers to the notion of a contraction. Recall that $\mathbf{N} \in \mathbb{C}_{m,n}$ is called a contraction if the Euclidean norm of $\mathbf{N}\mathbf{x}$ is not greater than the Euclidean norm of $\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}_{n,1}$; see Exercise 43 in [7, Chapter 6].

Lemma 4. *Let* $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *be partitioned as in* (1.2). *Then*:

(i) $\mathrm{rk}(\overline{\mathbf{A}}) = r - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$, (ii) $\mathrm{rk}(\overline{\mathbf{D}}) = n - r + \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{D})$,
(iii) $\rho(\mathbf{A}) = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$, (iv) $\rho(\mathbf{D}) = \mathrm{rk}(\mathbf{D}) - \mathrm{rk}(\mathbf{B})$,
(v) $\mathbf{A}$ *is a contraction*, (vi) $\mathbf{D}$ *is a contraction*.

Proof. We establish the conditions in the left column only, for the ones in the right column are derived in a similar way. From (2.12) in [33] it follows that $\text{rk}(\mathbf{A}\overline{\mathbf{A}}) = \text{rk}(\mathbf{A}) + \text{rk}(\overline{\mathbf{A}}) - r$. Hence, on account of point (i) of Lemma 1, we get

$$\text{rk}(\overline{\mathbf{A}}) = r - \text{rk}(\mathbf{A}) + \text{rk}(\mathbf{BB}^*) = r - \text{rk}(\mathbf{A}) + \text{rk}(\mathbf{B}),$$

what is condition (i).

It is clear that $\rho(\mathbf{A}) = \zeta(\overline{\mathbf{A}})$, where, recall, $\rho(.)$ and $\zeta(.)$ are the numbers of eigenvalues of a matrix argument equal to one and zero, respectively. Since $\zeta(\overline{\mathbf{A}}) = r - \text{rk}(\overline{\mathbf{A}})$, utilizing condition (i) of the lemma leads to its point (iii).

We show that $\mathbf{A}$ is a contraction if we demonstrate that $\mathbf{I}_r - \mathbf{AA}^*$ is nonnegative definite. Since $\mathbf{A}^* = \mathbf{A}$, from condition (i) of Lemma 1 it follows that

$$\mathbf{I}_r - \mathbf{AA}^* = \mathbf{I}_r - (\mathbf{A} - \mathbf{BB}^*) = \overline{\mathbf{A}} + \mathbf{BB}^*, \tag{2.1}$$

with the right-hand side of (2.1) being the sum of two nonnegative definite matrices which is nonnegative definite itself. □

In what follows we will refer to the well-known fact that the eigenvalues of any Hermitian nonnegative definite contraction are in the set [0, 1]; see Groß [16, p. 142].

An important tool in constructing orthogonal projectors onto given column spaces is provided by the next lemma recalling two results known in the literature.

Lemma 5. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then*:

(i) $\mathbf{P} + \overline{\mathbf{P}}(\overline{\mathbf{P}}\mathbf{Q})^\dagger$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})$,
(ii) $\mathbf{P} - \mathbf{P}(\mathbf{P}\overline{\mathbf{Q}})^\dagger$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$.

Proof. Conditions (i) and (ii) constitute equivalences (3.1) $\Leftrightarrow$ (3.6) and (4.1) $\Leftrightarrow$ (4.8) in [27], respectively. □

Using Lemma 5 we obtain the following representations of the orthogonal projectors onto sums and intersections of certain subspaces, including their dimensions.

Lemma 6. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in* (1.2). *Then*:

(i) $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\mathbf{D} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] = r + \text{rk}(\mathbf{D})$,
(ii) $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})] = n + \text{rk}(\mathbf{B}) - \text{rk}(\mathbf{D})$,
(iii) $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] = n - r + \text{rk}(\mathbf{A})$,
(iv) $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})] = n - \text{rk}(\mathbf{A}) + \text{rk}(\mathbf{B})$.

Proof. We establish point (i) only, for the remaining ones are obtained analogously. On account of conditions (iii) and (vii) of Lemmas 1 and 2, respectively, direct verifications show that the Moore–Penrose inverse of

$$\overline{\mathbf{P}}\mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^* \tag{2.2}$$

is given by

$$(\overline{\mathbf{P}}\mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{BD}^\dagger \\ \mathbf{0} & \mathbf{P}_\mathbf{D} \end{pmatrix}\mathbf{U}^*. \tag{2.3}$$

Hence, from statement (i) of Lemma 5 it follows that the orthogonal projector onto $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})$ has the form claimed in point (i). The validity of the remaining part of this point is clearly seen. □

Lemma 7. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in* (1.2). *Then*:

(i) $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \widetilde{\mathbf{P}}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$,

(ii) $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \widetilde{\mathbf{P}}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})] = r - \mathrm{rk}(\mathbf{A})$,

(iii) $\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{P}}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\mathbf{D}) - \mathrm{rk}(\mathbf{B})$,

(iv) $\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{P}}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})] = n - r - \mathrm{rk}(\mathbf{D})$.

Proof. We again establish point (i) only. Direct verifications, with the use of conditions (iii) of Lemma 1, (vi), (x) of Lemma 2, and (ii) of Lemma 3, show that the Moore–Penrose inverse of

$$\mathbf{P}\overline{\mathbf{Q}} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \tag{2.4}$$

is given by

$$(\mathbf{P}\overline{\mathbf{Q}})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ -\mathbf{D}^\dagger\mathbf{B}^* & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{2.5}$$

Hence, from statement (ii) of Lemma 5 it follows that the orthogonal projector onto $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$ is of the form given in point (i) of the lemma. Furthermore, since $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{rk}[\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}]$, it is seen that

$$\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\widetilde{\mathbf{P}}_{\overline{\mathbf{A}}}) = \mathrm{rk}(\mathbf{I}_r - \overline{\mathbf{A}}\,\overline{\mathbf{A}}^\dagger) = r - \mathrm{rk}(\overline{\mathbf{A}}),$$

and the equality on the right-hand side of point (i) follows on account of condition (i) of Lemma 4. $\square$

The theorem below provides several characterizations involving $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ expressed in terms of ranks of submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$.

Theorem 1. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in* (1.2). *Then*:

(i) $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\} \Leftrightarrow \mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$,
(ii) $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1} \Leftrightarrow \mathrm{rk}(\mathbf{D}) = n - r$,
(iii) $\mathcal{R}(\mathbf{P}) \perp \mathcal{R}(\mathbf{Q}) \Leftrightarrow \mathrm{rk}(\mathbf{A}) = 0$,
(iv) $\mathcal{R}(\mathbf{P}) \oplus \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1} \Leftrightarrow \mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$ *and* $\mathrm{rk}(\mathbf{D}) = n - r$,
(v) $\mathcal{R}(\mathbf{P}) \overset{\perp}{\oplus} \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1} \Leftrightarrow \mathrm{rk}(\mathbf{A}) = 0$ *and* $\mathrm{rk}(\mathbf{D}) = n - r$.

Proof. Equivalences (i) and (ii) follow directly from points (i) of Lemmas 7 and 6, respectively. To establish the next condition, we utilize the fact that $\mathcal{R}(\mathbf{P}) \perp \mathcal{R}(\mathbf{Q}) \Leftrightarrow \mathbf{PQ} = \mathbf{0}$. As easy to see, $\mathbf{PQ} = \mathbf{0}$ if and only if $\mathbf{A} = \mathbf{0}$, i.e., $\mathrm{rk}(\mathbf{A}) = 0$. The proof is concluded with observations that condition (iv) is obtained by combining conditions (i) and (ii), whereas condition (v) follows by combining conditions (ii) and (iii). $\square$

It is known that $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ satisfy $\mathbf{PQ} = \mathbf{0} \Leftrightarrow \mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$; see Halmos [18, Theorem in §30]. Therefore, we conclude from the proof of Theorem 1 that the two conditions constituting equivalence (iii) are necessary and sufficient for $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$.

3. Inclinedness

The following definition introduces notions useful to describe relationships between subspaces of $\mathbb{C}_{n,1}$; see Definition 2 in [7, Chapter 6] or [2, p. 801].

Definition 1. Let $\mathcal{M}, \mathcal{N} \subseteq \mathbb{C}_{n,1}$. Then:

(i) subspaces $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal* whenever $\mathcal{M} \subseteq \mathcal{N}^\perp$ and $\mathcal{N} \subseteq \mathcal{M}^\perp$; otherwise $\mathcal{M}$ and $\mathcal{N}$ are *inclined*,
(ii) subspaces $\mathcal{M}$ and $\mathcal{N}$ are *orthogonally incident* whenever $\mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^\perp$ and $\mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^\perp$ are orthogonal,
(iii) subspace $\mathcal{M}$ is *completely inclined* to $\mathcal{N}$ whenever $\mathcal{M} \cap \mathcal{N}^\perp = \{\mathbf{0}\}$,
(iv) subspaces $\mathcal{M}$ and $\mathcal{N}$ are *totally inclined* whenever they are completely inclined to each other,
(v) the *dimension of inclination* between subspaces $\mathcal{M}$ and $\mathcal{N}$ is defined by $r(\mathcal{M}, \mathcal{N}) = \mathrm{rk}(P_{\mathcal{M}} P_{\mathcal{N}})$,
(vi) the *coefficient of inclination* between subspaces $\mathcal{M}$ and $\mathcal{N}$ is $R(\mathcal{M}, \mathcal{N}) = \mathrm{trace}(P_{\mathcal{M}} P_{\mathcal{N}})$.

By assuming that $\mathcal{M} = \mathcal{R}(\mathbf{P})$ and $\mathcal{N} = \mathcal{R}(\mathbf{Q})$, the lemma below expresses the notions given in Definition 1 in terms of submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ occurring in representation (1.2).

Lemma 8. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in* (1.2). *Then*:

(i) $\mathcal{R}(\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q})$ *are orthogonal if and only if* $\mathbf{A} = \mathbf{0}$; $\mathcal{R}(\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q})$ *are inclined if and only if* $\mathbf{A} \neq \mathbf{0}$,
(ii) $\mathcal{R}(\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q})$ *are orthogonally incident if and only if* $\mathbf{B} = \mathbf{0}$,
(iii) $\mathcal{R}(\mathbf{P})$ *is completely inclined to* $\mathcal{R}(\mathbf{Q})$ *if and only if* $\mathrm{rk}(\mathbf{A}) = r$,
(iv) $\mathcal{R}(\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q})$ *are totally inclined if and only if* $\mathrm{rk}(\mathbf{A}) = r$ *and* $\mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{D})$,
(v) *the dimension of inclination between* $\mathcal{R}(\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q})$ *is* $r[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\mathbf{A})$,
(vi) *the coefficient of inclination between* $\mathcal{R}(\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q})$ *is* $R[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \mathrm{trace}(\mathbf{A})$.

Proof. Point (i) of the lemma is just another version of point (iii) of Theorem 1, and, thus, requires no justification. From point (ii) of Definition 1 it follows that $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are orthogonally incident if and only if $\mathcal{R}(\mathbf{P}) \cap [\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]^\perp$ and $\mathcal{R}(\mathbf{Q}) \cap [\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]^\perp$ are orthogonal. Since $[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]^\perp = \mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})$, applying point (ii) of Lemma 5 to projectors $\mathbf{P}$ and $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})}$ given in (1.1) and point (iv) of Lemma 6, respectively, gives

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{U} \begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^*. \tag{3.1}$$

Similarly, by replacing $\mathbf{P}$ with $\mathbf{Q}$ given in (1.2), on account of point (vi) of Lemma 2, we obtain

$$\mathbf{P}_{\mathcal{R}(\mathbf{Q}) \cap [\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{U} \begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} - \overline{\mathbf{A}} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix} \mathbf{U}^*. \tag{3.2}$$

Hence, subspaces $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are orthogonally incident if and only if

$$\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} - \overline{\mathbf{A}} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix} = \mathbf{0}. \tag{3.3}$$

In view of points (i) and (vi) of Lemma 2, equality (3.3) is satisfied if and only if $\mathbf{B} = \mathbf{0}$, what establishes point (ii) of the lemma.

Next, according to point (iii) of Definition 1, subspace $\mathcal{R}(\mathbf{P})$ is completely inclined to $\mathcal{R}(\mathbf{Q})$ if and only if $\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q}) = \{\mathbf{0}\}$. In consequence, the statement in point (iii) of the lemma follows from point (ii) of Lemma 7. Analogously, on account of point (iii) of Lemma 7, it is seen that $\mathcal{R}(\mathbf{Q})$ is completely inclined to $\mathcal{R}(\mathbf{P})$ if and only if $\overline{\mathbf{D}}$ is nonsingular. Combining this fact with point (ii) of Lemma 4, we arrive at $\mathrm{rk}(\overline{\mathbf{D}}) = n - r \Leftrightarrow \mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{D})$. Hence, in the light of point (iv) of Definition 1, point (iv) of the lemma follows.

The remaining two points of Lemma 8 are established straightforwardly on account of points (v) and (vi) of Definition 1, respectively, with the first one obtained on account of condition (v) of Lemma 2. □

Lemma 8 is supplemented with a number of observations. Firstly, note that from (3.1) and (3.2) it follows that $\mathbf{P} + \mathbf{P}_{\mathcal{R}(\mathbf{Q}) \cap [\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{Q} + \mathbf{P}_{\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}$. The second remark concerns point (i) of the lemma, and reads $\mathbf{B} = \mathbf{0} \Leftrightarrow \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{PQ} = \mathbf{QP}$. (The fact that $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are orthogonally incident if $\mathbf{P}$ and $\mathbf{Q}$ commute was pointed out by Afriat [2, p. 801].) Finally, it can be shown that the nonsingularity of $\mathbf{A}$, being a condition occurring in point (ii) of Lemma 8, is satisfied if and only if $\mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{P})$, or, equivalently, $\mathcal{R}(\mathbf{PQ}) = \mathcal{R}(\mathbf{P})$.

In what follows we use the present approach to solve a more general version of Exercise 66 in [7, Chapter 6], with generalization obtained by relaxing the assumption that the two subspaces involved in it are inclined. Besides introducing the generalization, the theorem below is obtained from Exercise 66 in [7, Chapter 6] by taking $L = \mathcal{R}(\mathbf{P})$ and $M = \mathcal{R}(\mathbf{Q})$.

Theorem 2. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$. *Then*:

(i) $\mathcal{R}(\mathbf{P}) = \mathcal{R}(\mathbf{PQ}) \overset{\perp}{\oplus} [\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$,
(ii) $\mathcal{R}(\mathbf{Q}) = \mathcal{R}(\mathbf{QP}) \overset{\perp}{\oplus} [\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$,
(iii) $\mathcal{R}(\mathbf{PQ})$ *and* $\mathcal{R}(\mathbf{QP})$ *are totally inclined*,
(iv) $\dim[\mathcal{R}(\mathbf{PQ})] = \dim[\mathcal{R}(\mathbf{QP})] = r[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})]$,
(v) $\mathcal{R}(\mathbf{QP}) \perp [\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$,
(vi) $\mathcal{R}(\mathbf{PQ}) \perp [\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$,
(vii) $[\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})] \perp [\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$,

where $r[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})]$ *is the dimension of inclination between* $\mathcal{R}(\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q})$.

Proof. To establish relationship (i), first observe that, on account of conditions (i) of Lemma 1 and (v) of Lemma 2, the Moore–Penrose inverse of

$$\mathbf{PQ} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \tag{3.4}$$

is given by

$$(\mathbf{PQ})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{0} \\ \mathbf{B}^*\mathbf{A}^\dagger & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{3.5}$$

with

$$\mathbf{P}_{\mathcal{R}(\mathbf{PQ})} = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{3.6}$$

Applying Lemma 5 to the projectors given in point (ii) of Lemma 7 and (3.6) gives

$$\mathbf{P}_{\mathcal{R}(\mathbf{PQ})+[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \quad \text{and} \quad \mathbf{P}_{\mathcal{R}(\mathbf{PQ})\cap[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]} = \mathbf{0}.$$

Hence, noticing that $\mathbf{P}_{\mathcal{R}(\mathbf{PQ})}\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{0}$, leads to the asserted relationship.

Since condition (ii) of the theorem is a counterpart of (i) obtained by interchanging $\mathbf{P}$ and $\mathbf{Q}$, next we establish condition (iii). From point (e) of Definition 2 in [7, Chapter 6] it follows that $\mathcal{R}(\mathbf{PQ})$ and $\mathcal{R}(\mathbf{QP})$ are totally inclined if and only if

$$\mathcal{R}(\mathbf{PQ}) \cap \mathcal{N}(\mathbf{PQ}) = \{\mathbf{0}\} \quad \text{and} \quad \mathcal{R}(\mathbf{QP}) \cap \mathcal{N}(\mathbf{QP}) = \{\mathbf{0}\}. \tag{3.7}$$

The validity of relationships (3.7) is seen, for instance, from Solutions 29.5-1–29.5-5 [IMAGE – Bull. Int. Linear Algebra Soc. 30 (2003) pp. 24–25] to the problem posed by Groß and Trenkler [17].

Equalities given in point (iv) are established by combining statement (iv) of Lemma 8 with relationships $\mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{QP}) = \mathrm{rk}(\mathbf{A})$, obtained from (3.4) on account of point (v) of Lemma 2.

Condition (v) of the theorem is obtained from (vi) by interchanging $\mathbf{P}$ and $\mathbf{Q}$, and, thus, only the last two conditions are left to be shown. The first of them follows by noting that the projectors given in (3.6) and point (iii) of Lemma 7 satisfy $\mathbf{P}_{\mathcal{R}(\mathbf{PQ})}\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{0}$, whereas the second by observing that the projectors given in points (ii) and (iii) of Lemma 7 fulfil $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})}\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{0}$. □

4. Angles between subspaces

In what follows we quote two definitions provided in [9, p. 108] dealing with angles between subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space. They refer to the notions of an inner product of two elements of a Hilbert space, $\mathbf{x}$ and $\mathbf{y}$ say, denoted by $\langle\mathbf{x},\mathbf{y}\rangle$ and norm $\|\mathbf{x}\| = \sqrt{\langle\mathbf{x},\mathbf{x}\rangle}$.

Definition 2. Let $\mathcal{M}$ and $\mathcal{N}$ be two subspaces of a Hilbert space. Then:

(i) the *angle* between $\mathcal{M}$ and $\mathcal{N}$ is the number in $\left[0,\frac{\pi}{2}\right]$ whose cosine is defined by

$$C(\mathcal{M},\mathcal{N}) = \sup\{|\langle\mathbf{x},\mathbf{y}\rangle| : \mathbf{x} \in \mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^{\perp},\ \|\mathbf{x}\| \leqslant 1, \mathbf{y} \in \mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^{\perp},\ \|\mathbf{y}\| \leqslant 1\},$$

(ii) the *minimal angle* between $\mathcal{M}$ and $\mathcal{N}$ is the number in $\left[0,\frac{\pi}{2}\right]$ whose cosine is defined by

$$C_0(\mathcal{M},\mathcal{N}) = \sup\{|\langle\mathbf{x},\mathbf{y}\rangle| : \mathbf{x} \in \mathcal{M},\ \|\mathbf{x}\| \leqslant 1,\ \mathbf{y} \in \mathcal{N},\ \|\mathbf{y}\| \leqslant 1\}. \tag{4.1}$$

The definition of the angle is attributed to Friedrichs [13], whereas the definition of the minimal angle is due to Dixmier [10]. It is clear that the notions of the angle and minimal angle coincide when subspaces $\mathcal{M}$ and $\mathcal{N}$ are disjoint. There are many applications of the notions of the angle and minimal angle. Three of them, arising in the considerations over: the rate of convergence of the method of cyclic projection, existence and uniqueness of abstract splines, and the product of operators with closed range were pointed out in [9]. Further interesting applications concern the strengthened Cauchy–Bunyakowski–Schwarz inequality [1] and canonical correlations of stochastic processes [22].

Note that a version of the definition of the minimal angle between two subspaces of $\mathbb{R}_{n,1}$ was provided by Ipsen and Meyer [20, Definition 2.1] and Meyer [25, p. 450]. It says that the minimal angle between nonzero $\mathcal{M},\mathcal{N} \subseteq \mathbb{R}_{n,1}$ is the number $\Theta_{\min}$ in $\left[0,\frac{\pi}{2}\right]$ for which

$$\cos\Theta_{\min} = \max_{\substack{\mathbf{u}\in\mathcal{M}\\ \|\mathbf{u}\|=1}} \max_{\substack{\mathbf{v}\in\mathcal{N}\\ \|\mathbf{v}\|=1}} \mathbf{v}'\mathbf{u}. \tag{4.2}$$

Actually, this definition is in accordance with point (ii) of Definition 2, for if $\mathcal{M},\mathcal{N} \subseteq \mathbb{R}_{n,1}$, then

$$\max_{\substack{\mathbf{u}\in\mathcal{M}\\ \|\mathbf{u}\|=1}} \max_{\substack{\mathbf{v}\in\mathcal{N}\\ \|\mathbf{v}\|=1}} \mathbf{v}'\mathbf{u} = \max_{\substack{\mathbf{u}\in\mathcal{M}\\ \|\mathbf{u}\|=1}} \max_{\substack{\mathbf{v}\in\mathcal{N}\\ \|\mathbf{v}\|=1}} |\mathbf{v}'\mathbf{u}|,$$

where "$\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 1$" can be replaced by "$\|\mathbf{u}\| \leqslant 1, \|\mathbf{v}\| \leqslant 1$". Thus, (4.1) and (4.2) coincide in the real case.

It is clear that subspaces $\mathcal{M}$ and $\mathcal{N}$ involved in Definition 2 can be thought of as subspaces of $\mathbb{C}_{n,1}$ characterized by $\mathcal{M} = \mathcal{R}(\mathbf{P})$ and $\mathcal{N} = \mathcal{R}(\mathbf{Q})$. Then, the present approach enables to establish without much effort a variety of properties of $C(\mathcal{M},\mathcal{N})$ and $C_0(\mathcal{M},\mathcal{N})$.

Several characteristics of $C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]$ and $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]$ were already listed in [9], including

$$C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})] = \|\mathbf{PQ} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\|, \tag{4.3}$$

$$C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})] = \|\mathbf{PQ}\| = \|\mathbf{PQP}\|^{1/2}, \tag{4.4}$$

provided in Lemma 10 therein; see also [14, p. 245 and Corollary 35]. Utilizing (1.1), (1.2), and the projector given in point (i) of Lemma 7, it can be shown that

$$\|\mathbf{PQ} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\| = \sqrt{\lambda_{\max}(\mathbf{BD}^{\dagger}\mathbf{B}^{*})} \quad \text{and} \quad \|\mathbf{PQ}\| = \sqrt{\lambda_{\max}(\mathbf{A})}, \tag{4.5}$$

with the first of these equalities established with the use of conditions (i) of Lemmas 2 and 3. In consequence,

$$C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})] = \sqrt{\lambda_{\max}(\mathbf{BD}^{\dagger}\mathbf{B}^{*})} \quad \text{and} \quad C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})] = \sqrt{\lambda_{\max}(\mathbf{A})}. \tag{4.6}$$

In the light of relationships (4.3) and (4.4), further expressions for $C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]$ and $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]$ can be derived. Observe, for instance, that combining condition (ii) of Lemma 3 with

(2.4) and (2.5) shows that $\mathbf{P}_{\mathcal{R}(\mathbf{P}\overline{\mathbf{Q}})}$ is of the same form as $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}$ given in (3.1). Moreover, by interchanging $\mathbf{P}$ and $\mathbf{Q}$, we also have $\mathbf{P}_{\mathcal{R}(\mathbf{Q}\overline{\mathbf{P}})} = \mathbf{P}_{\mathcal{R}(\mathbf{Q})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}$, with the projector on the right-hand side provided in (3.2). Whence, in view of condition (vi) of Lemma 2, we arrive at

$$\mathbf{P}\mathbf{P}_{\mathcal{R}(\mathbf{Q}\overline{\mathbf{P}})} = \mathbf{P}_{\mathcal{R}(\mathbf{P}\overline{\mathbf{Q}})}\mathbf{P}_{\mathcal{R}(\mathbf{Q}\overline{\mathbf{P}})} = \mathbf{PQ} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}.$$

In consequence, relationship (4.3) leads to

$$C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})] = \|\mathbf{P}\mathbf{P}_{\mathcal{R}(\mathbf{Q}\overline{\mathbf{P}})}\| = \|\mathbf{P}_{\mathcal{R}(\mathbf{P}\overline{\mathbf{Q}})}\mathbf{P}_{\mathcal{R}(\mathbf{Q}\overline{\mathbf{P}})}\|.$$

Similar derivations show that

$$\begin{aligned}\mathbf{PQ} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} &= \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}\mathbf{P}_{\mathcal{R}(\mathbf{Q})\cap[\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}\\ &= \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]}\mathbf{P}_{\mathcal{R}(\mathbf{Q})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}\end{aligned}$$

and

$$\mathbf{PQ} = \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]}\mathbf{P}_{\mathcal{R}(\mathbf{Q})\cap[\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]},$$

leading to additional representations of $C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]$ and $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]$.

It is worth pointing out in reference to equalities (4.5) that also further relationships of a similar type are valid. Two examples are given below.

Lemma 9. *Let* $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in* (1.2). *Then*:

(i) $\|\mathbf{P} - \mathbf{Q} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})}\|^2 = \lambda_{\max}(\mathbf{D})$,
(ii) $\|\mathbf{PQ} + \overline{\mathbf{P}}\,\overline{\mathbf{Q}}\|^2 = \max\{\lambda_{\max}(\mathbf{A}), \lambda_{\max}(\overline{\mathbf{D}})\}$.

Proof. On account of condition (viii) of Lemma 3, from (1.1), (1.2), and the projector given in point (ii) of Lemma 7, it follows that

$$\mathbf{P} - \mathbf{Q} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix}\mathbf{B}\overline{\mathbf{D}}^{\dagger}\mathbf{B}^* & -\mathbf{B}\\ -\mathbf{B}^* & -\mathbf{D}\end{pmatrix}\mathbf{U}^*.$$

Notice that this matrix is Hermitian and, in view of conditions (iii) of Lemma 1 and (iv), (viii) of Lemma 2, satisfies

$$(\mathbf{P} - \mathbf{Q} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})})^2 = \mathbf{U}\begin{pmatrix}\mathbf{B}\overline{\mathbf{D}}^{\dagger}\mathbf{B}^* & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{pmatrix}\mathbf{U}^*.$$

Let now $\mathbf{D}$ have a spectral decomposition of the form

$$\mathbf{D} = \mathbf{W}\mathrm{diag}(\underbrace{1,\ldots,1}_{s\text{ times}}, \delta_1,\ldots,\delta_t, \underbrace{0,\ldots,0}_{u\text{ times}})\mathbf{W}^*,$$

where $\mathbf{W} \in \mathbb{C}_{n-r}$ is unitary, $n - r = s + t + u$, and δ_j, $j = 1,\ldots,t$, satisfying $\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_t$, are eigenvalues of $\mathbf{D}$ belonging to the set $(0, 1)$, i.e., $s = \rho(\mathbf{D})$, $t = \xi(\mathbf{D})$, and $u = \zeta(\mathbf{D})$. Then,

$$\overline{\mathbf{D}} = \mathbf{W}\mathrm{diag}(\underbrace{0,\ldots,0}_{s\text{ times}}, 1-\delta_1,\ldots,1-\delta_t, \underbrace{1,\ldots,1}_{u\text{ times}})\mathbf{W}^*,$$

$$\overline{\mathbf{D}}^{\dagger} = \mathbf{W}\mathrm{diag}(\underbrace{0,\ldots,0}_{s\text{ times}}, \frac{1}{1-\delta_1},\ldots,\frac{1}{1-\delta_t}, \underbrace{1,\ldots,1}_{u\text{ times}})\mathbf{W}^*,$$

whence

$$\overline{\mathbf{D}}\,\overline{\mathbf{D}}^{\dagger}\mathbf{D} = \mathbf{W}\mathrm{diag}(\underbrace{0,\ldots,0}_{s\text{ times}}, \delta_1,\ldots,\delta_t, \underbrace{0,\ldots,0}_{u\text{ times}})\mathbf{W}^*.$$

On account of Theorem 2.8 in [36] and condition (iii) of Lemma 1, we have $\lambda_{\max}(\mathbf{B}\overline{\mathbf{D}}^{\dagger}\mathbf{B}^*) = \lambda_{\max}(\overline{\mathbf{D}}^{\dagger}\mathbf{B}^*\mathbf{B}) = \lambda_{\max}(\overline{\mathbf{D}}^{\dagger}\overline{\mathbf{D}}\mathbf{D})$. In consequence, $\lambda_{\max}(\mathbf{B}\overline{\mathbf{D}}^{\dagger}\mathbf{B}^*) = \delta_1$, and the validity of condition (i) of the lemma is clear.

Representations (1.1) and (1.2) yield

$$\mathbf{PQ}+\overline{\mathbf{P}}\,\overline{\mathbf{Q}}=\mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ -\mathbf{B}^* & \overline{\mathbf{D}}\end{pmatrix}\mathbf{U}^*,$$

from where, on account of conditions (i) of Lemma 1 and (iii), (iv) of Lemma 2, we get

$$(\mathbf{PQ}+\overline{\mathbf{P}}\,\overline{\mathbf{Q}})(\mathbf{PQ}+\overline{\mathbf{P}}\,\overline{\mathbf{Q}})^*=\mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \overline{\mathbf{D}}\end{pmatrix}\mathbf{U}^*.$$

Thus, condition (ii) is shown and the proof is complete. $\square$

From (4.4) we know that $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=0$ if and only if $\|\mathbf{PQ}\|=0$. In view of point (v) of Lemma 2, from (3.4) it is seen that the latter of this equalities is satisfied if and only if $\mathbf{A}=\mathbf{0}$. Equivalence (iii) of Theorem 1 further shows that $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=0 \Leftrightarrow \mathcal{R}(\mathbf{P})\perp\mathcal{R}(\mathbf{Q})$. Three conditions equivalent to $C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=0$ are given in the following theorem. One of them involves the difference $\mathbf{PQ}-\mathbf{P}_{\mathcal{R}(\mathbf{PQ})}$, which, according to our knowledge, so far did not occur in any considerations over angles between subspaces.

Theorem 3. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$. *Then the following conditions are equivalent*:

(i) $C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=0$,
(ii) $\|\mathbf{PQ}-\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\|=0$,
(iii) $\|\mathbf{PQ}-\mathbf{P}_{\mathcal{R}(\mathbf{PQ})}\|=0$,
(iv) $\mathbf{PQ}$ *is an orthogonal projector*.

Proof. First observe that equivalence (i) $\Leftrightarrow$ (ii) is a trivial consequence of (4.3). Utilizing now the projectors given in point (i) of Lemma 7 and (3.4), it can be verified that equality $\mathbf{PQ}-\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}=\mathbf{0}$ is equivalent to the conjunction $\mathbf{B}=\mathbf{0}$ and $\mathbf{P}_{\overline{\mathbf{A}}}=\overline{\mathbf{A}}$. However, in view of condition (i) of Lemma 2, it is seen that $\mathbf{B}=\mathbf{0}$ implies $\overline{\mathbf{A}}^2=\overline{\mathbf{A}}$. Thus, $\overline{\mathbf{A}}$, in addition to being Hermitian, is also idempotent. This means that $\overline{\mathbf{A}}$ is an orthogonal projector, and necessarily satisfies $\mathbf{P}_{\overline{\mathbf{A}}}=\overline{\mathbf{A}}$. Concluding, $\|\mathbf{PQ}-\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\|=0$ is equivalent to $\mathbf{B}=\mathbf{0}$.

Similar derivations with the use of (3.4) and (3.6) show that also condition (iii) of the theorem is equivalent to $\mathbf{B}=\mathbf{0}$. The proof is thus complete, for equivalence $\mathbf{PQ}\in\mathbb{C}_n^{\mathrm{OP}}\Leftrightarrow\mathbf{B}=\mathbf{0}$ is established easily by direct calculations. $\square$

Note that another condition equivalent to $C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=0$ is $\mathbf{PQ}=\mathbf{QP}$; see [9, Lemma 10].

According to Theorem 15 in [9], if $\mathcal{R}(\mathbf{P})\oplus\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$, then $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=C_0[\mathcal{R}(\mathbf{P})^\perp,\mathcal{R}(\mathbf{Q})^\perp]$. Below we provide a stronger version of this result.

Theorem 4. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$. *Then* $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=C_0[\mathcal{R}(\mathbf{P})^\perp,\mathcal{R}(\mathbf{Q})^\perp]$ *if and only if* $\|\mathbf{PQ}\|=\|\overline{\mathbf{P}}\,\overline{\mathbf{Q}}\|$.

Proof. Since $\mathcal{R}(\mathbf{P})^\perp=\mathcal{R}(\overline{\mathbf{P}})$ and $\mathcal{R}(\mathbf{Q})^\perp=\mathcal{R}(\overline{\mathbf{Q}})$, it is clear that $C_0[\mathcal{R}(\mathbf{P})^\perp,\mathcal{R}(\mathbf{Q})^\perp]=C_0[\mathcal{R}(\overline{\mathbf{P}}),\mathcal{R}(\overline{\mathbf{Q}})]$. Hence, the assertion follows from the first equality in (4.4). $\square$

In a comment to Theorem 4, it is worth mentioning that from equivalence (iv) of Theorem 1 we have $\mathcal{R}(\mathbf{P})\oplus\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}\Leftrightarrow \mathrm{rk}(\mathbf{A})=\mathrm{rk}(\mathbf{B}),\mathrm{rk}(\mathbf{D})=n-r$, whereas direct calculations show that $\|\mathbf{PQ}\|=\|\overline{\mathbf{P}}\,\overline{\mathbf{Q}}\|\Leftrightarrow\lambda_{\max}(\mathbf{A})=\lambda_{\max}(\overline{\mathbf{D}})$.

Let us remark that also further results in [9] can be reestablished in $\mathbb{C}_{n,1}$ without much effort by utilizing the present approach. This, for example, concerns Theorem 16 in [9], according to which $C[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=C[\mathcal{R}(\mathbf{P})^\perp,\mathcal{R}(\mathbf{Q})^\perp]$ always holds.

The next result involves $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$, what is known to be an oblique projector (see Penrose [26, Lemma 2·3]), and shows that the results provided in [8, Lemma] and [28, Theorem 1] can be substantially generalized when formulated in the settings of $\mathbb{C}_{n,1}$. The theorem below will be afterwards used to establish yet another formula for $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]$.

Theorem 5. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$ *be such that* $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})=\{\mathbf{0}\}$. *Then*

$$\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|=\frac{1}{\sqrt{1-\|\mathbf{QP}\|^2}}.$$

Proof. From the right-hand side relationship in (4.5) we have $\|\mathbf{QP}\|^2=\lambda_{\max}(\mathbf{A})$. On the other hand, by utilizing conditions (i) and (x) of Lemma 2, formula (2.5) entails

$$(\overline{\mathbf{Q}}\mathbf{P})^\dagger[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]^*=\mathbf{U}\begin{pmatrix}\overline{\mathbf{A}}^\dagger & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

whence $\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2=\lambda_{\max}(\overline{\mathbf{A}}^\dagger)$. However, from point (i) of Lemma 7 it is known that $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})=\{\mathbf{0}\}$ ensures the nonsingularity of $\overline{\mathbf{A}}$. This means that $\overline{\mathbf{A}}^\dagger=\overline{\mathbf{A}}^{-1}$, and, furthermore, that $\lambda_{\max}(\mathbf{A})<1$. In consequence, the assertion will be established if we show that

$$\lambda_{\max}\left(\overline{\mathbf{A}}^{-1}\right)=\frac{1}{1-\lambda_{\max}(\mathbf{A})}. \tag{4.7}$$

For this purpose, consider a spectral decomposition of $\mathbf{A}$ in the form

$$\mathbf{A}=\mathbf{V}\mathrm{diag}(\underbrace{1,\ldots,1}_{k\text{ times}},\alpha_1,\ldots,\alpha_l,\underbrace{0,\ldots,0}_{m\text{ times}})\mathbf{V}^*, \tag{4.8}$$

where $\mathbf{V}\in\mathbb{C}_{r,r}$ is unitary, $r=k+l+m$, and $\alpha_j, j=1,\ldots,l$, satisfying $\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_l$, are eigenvalues of $\mathbf{A}$ belonging to the set $(0,1)$. Referring to the notation used in the present paper, we have $k=\rho(\mathbf{A})$, $l=\xi(\mathbf{A})$, and $m=\zeta(\mathbf{A})$. Needles to say, one or two from among the blocks indexed by k, l, and m in (4.8) may be absent, and, in particular, $\overline{\mathbf{A}}$ is nonsingular if and only if $k=0$. In such a case,

$$\overline{\mathbf{A}}^{-1}=\mathbf{V}\mathrm{diag}\left(\frac{1}{1-\alpha_1},\ldots,\frac{1}{1-\alpha_l},\underbrace{1,\ldots,1}_{m\text{ times}}\right)\mathbf{V}^*,$$

what means that (4.7) holds. □

As already mentioned, Theorem 5 is a generalized counterpart of Lemma in [8] and Theorem 1 in [28], which were originally formulated in an infinite dimensional Hilbert space. The generalization is obtained by replacing the assumptions that $\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]=\mathcal{R}(\mathbf{P})$ and $\mathcal{N}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]=\mathcal{R}(\mathbf{Q})$ with the weaker condition $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})=\{\mathbf{0}\}$. To show that this change is indeed essential, recall that the fact that $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ is an oblique projector was pointed out by Penrose [26, Lemma 2·3]; see also Greville [15]. This result was later supplemented by Baksalary and Trenkler [6, p. 9], who showed that in general $\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]=\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]$ and $\mathcal{N}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]=\mathcal{R}(\mathbf{Q})\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$. It is seen that $\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]=\mathcal{R}(\mathbf{P})$ if $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})=\{\mathbf{0}\}$ and $\mathcal{N}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]=\mathcal{R}(\mathbf{Q})$ if $\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$. Thus, the assumptions in Theorem 1 in [28] correspond to $\mathcal{R}(\mathbf{P})\oplus\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$.

A straightforward consequence of Theorem 5 is that disjointness of $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ ensures that

$$\|\mathbf{PQ}\|^2=\frac{\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2-1}{\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2},$$

being a generalized counterpart of Corollary 1 in [28]. Another form of this result is given in what follows; see also Ljance [24].

Corollary 1. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$ *be such that* $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})=\{\mathbf{0}\}$. *Then*

$$\{C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]\}^2=1-\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^{-2}.$$

Another result which can be relatively easily shown with the use of the present approach is that if $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are complementary, then $\|\mathbf{PQ}\|=\|\mathbf{P}-\overline{\mathbf{Q}}\|=\|\overline{\mathbf{P}}-\mathbf{Q}\|$. This implication can be also concluded from Corollary 2 in [28]. An alternative version of this result constitutes the corollary below.

Corollary 2. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *be such that* $\mathcal{R}(\mathbf{P}) \oplus \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$. *Then*

$$C_0[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \|\mathbf{P} - \overline{\mathbf{Q}}\|.$$

According to Remark 2.3 in Koliha [23], if $\mathcal{R}(\mathbf{P}) \oplus \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$, then the following inequalities hold

$$\|\mathbf{PQ}\| \leqslant \frac{\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|}{\sqrt{1 + \|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2}} < 1; \tag{4.9}$$

this result was originally given by Vidav [34, p. 158]. In a comment to relationships (4.9), Koliha [23] pointed out that "It would be interesting to find a purely algebraic proof" of them. The next theorem provides such a proof under a considerably weaker assumption that the column spaces of **P** and **Q** are merely disjoint. Surprisingly, the left-hand side inequality in (4.9) proves then to be sharp.

Theorem 6. *Let* $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero and let* $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *be such that* $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$. *Then*

$$\|\mathbf{PQ}\| < \frac{\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|}{\sqrt{1 + \|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2}} < 1. \tag{4.10}$$

Proof. On account of point (i) of Theorem 1, $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\} \Leftrightarrow \mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$, with the equality on the right-hand side of the equivalence ensuring that $\overline{\mathbf{A}}$ is nonsingular. In such a case, $\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2 = \lambda_{\max}(\overline{\mathbf{A}}^{-1})$, with $\lambda_{\max}(\overline{\mathbf{A}}^{-1})$ satisfying (4.7). Hence, in view of $\lambda_{\max}(\mathbf{A}) < 1$,

$$\frac{\|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2}{1 + \|(\overline{\mathbf{Q}}\mathbf{P})^\dagger\|^2} = \frac{\frac{1}{1-\lambda_{\max}(\mathbf{A})}}{1 + \frac{1}{1-\lambda_{\max}(\mathbf{A})}} = \frac{1}{2 - \lambda_{\max}(\mathbf{A})} < 1,$$

showing the validity of the right-hand side inequality in (4.10). The left-hand side inequality therein is established by observing that $\|\mathbf{PQ}\|^2 = \lambda_{\max}(\mathbf{A})$ satisfies

$$\lambda_{\max}(\mathbf{A}) < \frac{1}{2 - \lambda_{\max}(\mathbf{A})}.$$

The proof is complete. □

It is known that if either of orthogonal projectors **P** or **Q** is nonzero, then $\|\mathbf{P} + \mathbf{Q}\| = 1 + \|\mathbf{PQ}\|$; see e.g., [12, Theorem 7], [14, Lemma 47], or [35, Theorem]. A direct consequence of this result is that $C_0[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \|\mathbf{P} + \mathbf{Q}\| - 1$, what was pointed out in [14, Corollary 24]. The theorem below provides yet another relationship involving spectral norm of **PQ**.

Theorem 7. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$\|\mathbf{PQ}\| = \|\mathbf{P} + \mathbf{Q} - \mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})}\|. \tag{4.11}$$

Proof. As can be verified by direct calculations with the use of conditions (iii) of Lemma 1, (vi), (vii), (x) of Lemma 2, and (i), (ii) of Lemma 3,

$$(\mathbf{P} + \mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - \frac{1}{2}\widetilde{\mathbf{P}}_{\overline{\mathbf{A}}} & -\mathbf{B}\mathbf{D}^\dagger \\ -\mathbf{D}^\dagger\mathbf{B}^* & 2\mathbf{D}^\dagger - \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*,$$

with $\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})}$ being of the same form as $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})}$ given in point (i) of Lemma 6. In consequence,

$$\mathbf{P} + \mathbf{Q} - \mathbf{P}_{\mathcal{R}(\mathbf{Q}+\mathbf{P})} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} - \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*.$$

Notice that this matrix is Hermitian and, in view of Lemma 1, conditions (vii) and (v) of Lemmas 2 and 3, respectively, satisfies

$$(\mathbf{P}+\mathbf{Q}-\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})})^2 = \mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B}\end{pmatrix}\mathbf{U}^*.$$

On account of Theorem 2.8 in [36] and condition (i) of Lemma 1, we have $\lambda_{\max}(\mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B}) = \lambda_{\max}(\overline{\mathbf{A}}^\dagger\mathbf{A}\overline{\mathbf{A}})$. However, from (4.8) it follows that $\lambda_{\max}(\overline{\mathbf{A}}^\dagger\mathbf{A}\overline{\mathbf{A}}) = \alpha_1$. In consequence, $\|\mathbf{P}+\mathbf{Q}-\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})}\|^2 = \lambda_{\max}(\mathbf{A})$, and, in the light of the right-hand side relationship in (4.5), the assertion follows. □

By referring to the right-hand side formula in (4.6), Theorem 7 leads to what follows.

Corollary 3. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$. *Then* $C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})] = \|\mathbf{P}+\mathbf{Q}-\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})}\|$.

Since $\mathbf{P},\mathbf{Q}$ are nonnegative definite, they satisfy $\mathcal{R}(\mathbf{P}+\mathbf{Q}) = \mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})$. Thus, Theorem 7 remains true if (4.11) is replaced by $\|\mathbf{PQ}\| = \|\mathbf{P}+\mathbf{Q}-\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})}\|$.

The notions of the angle and minimal angle between subspaces can be used to characterize various properties of matrices. The theorem below provides the corresponding characteristics of nilpotent, EP, and bi-EP matrices. Recall that nonzero $\mathbf{N}\in\mathbb{C}_{n,n}$ is: nilpotent of index 2 whenever $\mathbf{N}^2=\mathbf{0}$, EP whenever $\mathbf{NN}^\dagger = \mathbf{N}^\dagger\mathbf{N}$, and bi-EP whenever $\mathbf{NN}^\dagger\mathbf{N}^\dagger\mathbf{N} = \mathbf{N}^\dagger\mathbf{NNN}^\dagger$.

Theorem 8. *Let* $\mathbf{N}\in\mathbb{C}_{n,n}$. *Then*:

(i) $\mathbf{N}$ *is nilpotent of index* 2 *if and only if* $C_0[\mathcal{R}(\mathbf{N}),\mathcal{R}(\mathbf{N}^*)] = 0$,
(ii) $\mathbf{N}$ *is EP if and only if* $C_0[\mathcal{R}(\mathbf{N}),\mathcal{N}(\mathbf{N})] = 0$,
(iii) $\mathbf{N}$ *is bi-EP if and only if* $C[\mathcal{R}(\mathbf{N}),\mathcal{N}(\mathbf{N})] = 0$, *or, equivalently*, $C[\mathcal{R}(\mathbf{N}),\mathcal{R}(\mathbf{N}^*)] = 0$.

Proof. On account of Corollary 6 in Hartwig and Spindelböck [19], any matrix $\mathbf{N}\in\mathbb{C}_{n,n}$ of rank r can be represented as

$$\mathbf{N} = \mathbf{U}\begin{pmatrix}\mathbf{\Sigma K} & \mathbf{\Sigma L}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \tag{4.12}$$

where $\mathbf{U}\in\mathbb{C}_{n,n}$ is unitary, $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1\mathbf{I}_{r_1},\ldots,\sigma_t\mathbf{I}_{r_t})$ is the diagonal matrix of singular values of $\mathbf{N}$, $\sigma_1>\sigma_2>\cdots>\sigma_t>0$, $r_1+r_2+\cdots+r_t=r$, and $\mathbf{K}\in\mathbb{C}_{r,r}$, $\mathbf{L}\in\mathbb{C}_{r,n-r}$ satisfy

$$\mathbf{KK}^*+\mathbf{LL}^* = \mathbf{I}_r. \tag{4.13}$$

With the use of this representation, several useful characterizations of matrix $\mathbf{N}$ can be established. For instance, direct calculations show that $\mathbf{N}$ satisfies $\mathbf{N}^2=\mathbf{0}$ if and only if $\mathbf{\Sigma K\Sigma K}=\mathbf{0}$ and $\mathbf{\Sigma K\Sigma L}=\mathbf{0}$, or, equivalently, $\mathbf{K\Sigma K}=\mathbf{0}$ and $\mathbf{K\Sigma L}=\mathbf{0}$. By combining modified versions of these conditions, obtained by postmultiplying them by $\mathbf{K}^*$ and $\mathbf{L}^*$, respectively, on account of (4.13) we obtain $\mathbf{K}=\mathbf{0}$. Since the fact that $\mathbf{K}=\mathbf{0}\Rightarrow\mathbf{N}^2=\mathbf{0}$ is clearly seen, we conclude that $\mathbf{N}$ is nilpotent of index 2 if and only if $\mathbf{K}=\mathbf{0}$. Further, it is known that $\mathbf{N}$ is EP if and only if $\mathbf{L}=\mathbf{0}$, whereas $\mathbf{N}$ is bi-EP if and only if $\mathbf{L}^*\mathbf{K}=\mathbf{0}$, or, equivalently, $\mathbf{K}$ is a partial isometry, i.e., $\mathbf{K}^*=\mathbf{K}^\dagger$; see [19, Corollary 6] and [3, Lemma 1].

To establish equivalence (i) first observe that the former equality in (4.4) ensures that $C_0[\mathcal{R}(\mathbf{N}),\mathcal{R}(\mathbf{N}^*)] = \|\mathbf{P}_{\mathcal{R}(\mathbf{N})}\mathbf{P}_{\mathcal{R}(\mathbf{N}^*)}\|$. From (4.12) and

$$\mathbf{N}^\dagger = \mathbf{U}\begin{pmatrix}\mathbf{K}^*\mathbf{\Sigma}^{-1} & \mathbf{0}\\ \mathbf{L}^*\mathbf{\Sigma}^{-1} & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

it follows that $\mathbf{P}_{\mathcal{R}(\mathbf{N})} = \mathbf{NN}^\dagger$ and $\mathbf{P}_{\mathcal{R}(\mathbf{N}^*)} = \mathbf{N}^\dagger\mathbf{N}$ are of the forms

$$\mathbf{P}_{\mathcal{R}(\mathbf{N})} = \mathbf{U}\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^* \quad\text{and}\quad \mathbf{P}_{\mathcal{R}(\mathbf{N}^*)} = \mathbf{U}\begin{pmatrix}\mathbf{K}^*\mathbf{K} & \mathbf{K}^*\mathbf{L}\\ \mathbf{L}^*\mathbf{K} & \mathbf{L}^*\mathbf{L}\end{pmatrix}\mathbf{U}^*.$$

Whence, we obtain

$$C_0[\mathcal{R}(\mathbf{N}), \mathcal{R}(\mathbf{N}^*)] = 0 \Leftrightarrow \mathbf{K}^*\mathbf{K} = \mathbf{0}, \mathbf{K}^*\mathbf{L} = \mathbf{0}. \tag{4.14}$$

An obvious conclusion that the conjunction on the right-hand side of equivalence (4.14) can be replaced with $\mathbf{K} = \mathbf{0}$, completes the proof of point (i).

The proof of equivalence (ii) is established in a similar way. On account of $\mathbf{P}_{\mathcal{N}(\mathbf{N})} = \mathbf{I}_n - \mathbf{P}_{\mathcal{R}(\mathbf{N}^*)}$, we arrive at

$$C_0[\mathcal{R}(\mathbf{N}), \mathcal{N}(\mathbf{N})] = 0 \Leftrightarrow \mathbf{K}^*\mathbf{K} = \mathbf{I}_r, \mathbf{K}^*\mathbf{L} = \mathbf{0}. \tag{4.15}$$

In the light of (4.13), it is seen that conjunction on the right-hand side of equivalence (4.15) can be replaced with $\mathbf{L} = \mathbf{0}$. As mentioned above, $\mathbf{L} = \mathbf{0}$ is equivalent to the requirement that $\mathbf{N}$ is EP.

The proof of point (iii) is based on relationships

$$C[\mathcal{R}(\mathbf{N}), \mathcal{N}(\mathbf{N})] = \|\mathbf{P}_{\mathcal{R}(\mathbf{N})}\mathbf{P}_{\mathcal{N}(\mathbf{N})} - \mathbf{P}_{\mathcal{R}(\mathbf{N})\cap\mathcal{N}(\mathbf{N})}\|, \tag{4.16}$$

$$C[\mathcal{R}(\mathbf{N}), \mathcal{R}(\mathbf{N}^*)] = \|\mathbf{P}_{\mathcal{R}(\mathbf{N})}\mathbf{P}_{\mathcal{R}(\mathbf{N}^*)} - \mathbf{P}_{\mathcal{R}(\mathbf{N})\cap\mathcal{R}(\mathbf{N}^*)}\|, \tag{4.17}$$

obtained from (4.3). With the use of the formula $\mathbf{X}^\dagger = (\mathbf{X}^*\mathbf{X})^\dagger\mathbf{X}^*$, which holds for every $\mathbf{X} \in \mathbb{C}_{m,n}$ (see e.g., [29, p. 67]), from point (ii) of Lemma 5, we obtain

$$\mathbf{P}_{\mathcal{R}(\mathbf{N})\cap\mathcal{N}(\mathbf{N})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - \mathbf{K}^\dagger\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{N})\cap\mathcal{R}(\mathbf{N}^*)} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - (\mathbf{I}_r - \mathbf{K}^*\mathbf{K})(\mathbf{I}_r - \mathbf{K}^*\mathbf{K})^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*.$$

Hence, formulae (4.16) and (4.17) entail

$$C[\mathcal{R}(\mathbf{N}), \mathcal{N}(\mathbf{N})] = 0 \Leftrightarrow \mathbf{K}^\dagger\mathbf{K} = \mathbf{K}^*\mathbf{K},\ \mathbf{L}^*\mathbf{K} = \mathbf{0}, \tag{4.18}$$

$$C[\mathcal{R}(\mathbf{N}), \mathcal{R}(\mathbf{N}^*)] = 0 \Leftrightarrow \mathbf{I}_r - \mathbf{K}^*\mathbf{K} = (\mathbf{I}_r - \mathbf{K}^*\mathbf{K})(\mathbf{I}_r - \mathbf{K}^*\mathbf{K})^\dagger,\ \mathbf{L}^*\mathbf{K} = \mathbf{0}. \tag{4.19}$$

Premultiplying the former condition on the right-hand side of equivalence (4.18) by $\mathbf{K}$ gives $\mathbf{K} = \mathbf{K}\mathbf{K}^*\mathbf{K}$, which is an alternative way of expressing the fact that $\mathbf{K}$ is a partial isometry. Since, on the one hand, trivially $\mathbf{K}^* = \mathbf{K}^\dagger \Rightarrow \mathbf{K}^\dagger\mathbf{K} = \mathbf{K}^*\mathbf{K}$, and, on the other hand, $\mathbf{K}^* = \mathbf{K}^\dagger \Leftrightarrow \mathbf{L}^*\mathbf{K} = \mathbf{0}$, we conclude that the two conditions on the right-hand side of equivalence (4.18) are equivalent and each of them holds if and only if $\mathbf{N}$ is bi-EP. This establishes the first equivalence in point (iii) of the theorem. To show the second equivalence, we again refer to the fact that $\mathbf{K}^* = \mathbf{K}^\dagger \Leftrightarrow \mathbf{L}^*\mathbf{K} = \mathbf{0}$. Replacing $\mathbf{K}^*$ with $\mathbf{K}^\dagger$ in the former condition on the right-hand side of equivalence (4.19) gives $\mathbf{I}_r - \mathbf{K}^\dagger\mathbf{K} = (\mathbf{I}_r - \mathbf{K}^\dagger\mathbf{K})(\mathbf{I}_r - \mathbf{K}^\dagger\mathbf{K})^\dagger$. Since $\mathbf{I}_r - \mathbf{K}^\dagger\mathbf{K} \in \mathbb{C}_r^{\mathsf{OP}}$, we have $\mathbf{I}_r - \mathbf{K}^\dagger\mathbf{K} = (\mathbf{I}_r - \mathbf{K}^\dagger\mathbf{K})^\dagger$, what in turn means that the former condition on the right-hand side of equivalence (4.19) is implied by the latter one. In consequence, we see that $C[\mathcal{R}(\mathbf{N}), \mathcal{R}(\mathbf{N}^*)] = 0$ if and only if $\mathbf{N}$ is bi-EP. The proof is complete. □

5. Distances between subspaces

In the present section we explore distances between subspaces. Point (i) of the definition below is obtained from Definition 4 in [7, Chapter 6], whereas its points (ii) and (iii) follow from [25, p. 435] and [25, pp. 452, 453], respectively.

Definition 3. Let $\mathcal{M}, \mathcal{N} \subseteq \mathbb{C}_{n,1}$. Then:

(i) the *distance* between $\mathcal{M}$ and $\mathcal{N}$ is defined by

$$\mathrm{dist}(\mathcal{M}, \mathcal{N}) = \|\mathbf{P}_\mathcal{M} - \mathbf{P}_\mathcal{N}\|, \tag{5.1}$$

(ii) the *orthogonal distance* between $\mathbf{m} \in \mathbb{C}_{n,1}$ and $\mathcal{N}$ is defined by $\mathrm{dist}(\mathbf{m}, \mathcal{N}) = \|\overline{\mathbf{P}}_{\mathcal{N}}\mathbf{m}\|$,
(iii) the *direct distance* between $\mathcal{M}$ and $\mathcal{N}$ is defined by

$$\delta(\mathcal{M}, \mathcal{N}) = \max_{\substack{\mathbf{m}\in\mathcal{M} \\ \|\mathbf{m}\|=1}} \mathrm{dist}(\mathbf{m}, \mathcal{N}) = \max_{\substack{\mathbf{m}\in\mathcal{M} \\ \|\mathbf{m}\|=1}} \|\overline{\mathbf{P}}_{\mathcal{N}}\mathbf{m}\|.$$

It is noteworthy that quantity $\mathrm{dist}(\mathcal{M}, \mathcal{N})$ defined in (5.1) is often called in literature the *gap* between subspaces $\mathcal{M}$ and $\mathcal{N}$. Another relevant remark is that the spectral norm involved in (5.1) is unitarily invariant, which is an essential property from the point of view of problems occurring in the perturbation analysis; see e.g., [30,31,32,37].

Similarly as in the preceding section, also now we attribute $\mathcal{M} = \mathcal{R}(\mathbf{P})$ and $\mathcal{N} = \mathcal{R}(\mathbf{Q})$. Then, with respect to point (i) of Definition 3 it is useful to note that, in view of Lemma 1, representations (1.1) and (1.2) entail

$$(\mathbf{P} - \mathbf{Q})^2 = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}\mathbf{U}^*.$$

Hence, $\|\mathbf{P} - \mathbf{Q}\|^2 = \max\{\lambda_{\max}(\overline{\mathbf{A}}), \lambda_{\max}(\mathbf{D})\}$, and taking into account that $\mathbf{A}$ and $\mathbf{D}$ are Hermitian nonnegative definite contractions, we conclude that

$$0 \leqslant \mathrm{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] \leqslant 1;$$

cf. [14, Corollary 27].

Alternative representations of the distance are attainable. For example, from (2.2) and (2.4) we get

$$\overline{\mathbf{P}}\mathbf{Q}(\overline{\mathbf{P}}\mathbf{Q})^* = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}\mathbf{U}^* \quad \text{and} \quad (\overline{\mathbf{Q}}\mathbf{P})^*\overline{\mathbf{Q}}\mathbf{P} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

respectively. In consequence,

$$\mathrm{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \max\{\|\overline{\mathbf{Q}}\mathbf{P}\|, \|\overline{\mathbf{P}}\mathbf{Q}\|\}; \tag{5.2}$$

see [25, p. 454]. Furthermore, from (3.1) and (3.2) it follows that

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} - \mathbf{P}_{\mathcal{R}(\mathbf{Q})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ -\mathbf{B}^* & -\mathbf{D} \end{pmatrix}\mathbf{U}^*,$$

what coincides with $\mathbf{P} - \mathbf{Q}$. Thus, we obtain

$$\mathrm{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \mathrm{dist}\{\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})], \mathcal{R}(\mathbf{Q}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]\}.$$

The subsequent considerations refer to point (iii) of Definition 3. It was shown in [25, pp. 453, 454] that

$$\delta[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \|\overline{\mathbf{Q}}\mathbf{P}\| = \|\mathbf{P}\overline{\mathbf{Q}}\|, \quad \delta[\mathcal{R}(\mathbf{Q}), \mathcal{R}(\mathbf{P})] = \|\overline{\mathbf{P}}\mathbf{Q}\| = \|\mathbf{Q}\overline{\mathbf{P}}\|, \tag{5.3}$$

where, in general, $\delta[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] \neq \delta[\mathcal{R}(\mathbf{Q}), \mathcal{R}(\mathbf{P})]$. Combining (5.2) and (5.3), we get

$$\mathrm{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \max\{\delta[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})], \delta[\mathcal{R}(\mathbf{Q}), \mathcal{R}(\mathbf{P})]\},$$

and, since (5.3) entails that $\delta[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \delta[\mathcal{N}(\mathbf{P}), \mathcal{N}(\mathbf{Q})]$, we arrive at

$$\mathrm{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \mathrm{dist}[\mathcal{N}(\mathbf{P}), \mathcal{N}(\mathbf{Q})];$$

for a collection of results on distance between subspaces see also [21, Chapter IV].

The next theorem provides an alternative representation of the direct distance between $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$.

Theorem 9. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$\delta[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] = \delta\{\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})], \mathcal{R}(\mathbf{Q}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]\}.$$

Proof. Let $\mathcal{M}_1 = \mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$ and $\mathcal{N}_1 = \mathcal{R}(\mathbf{Q}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$. From the left-hand side formulae in (5.3) it follows that

$$\delta(\mathcal{M}_1, \mathcal{N}_1) = \|\mathbf{P}_{\mathcal{M}_1} \overline{\mathbf{P}}_{\mathcal{N}_1}\|,$$

where $\mathbf{P}_{\mathcal{M}_1} = \mathbf{P}_{\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]}$ and $\mathbf{P}_{\mathcal{N}_1} = \mathbf{P}_{\mathcal{R}(\mathbf{Q}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]}$ are of the forms (3.1) and (3.2), respectively. Hence, in the light of point (vi) of Lemma 2,

$$\mathbf{P}_{\mathcal{M}_1} \overline{\mathbf{P}}_{\mathcal{N}_1} = \mathbf{U} \begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^*. \tag{5.4}$$

By noticing that $\mathbf{P}_{\mathcal{M}_1} \overline{\mathbf{P}}_{\mathcal{N}_1}$ given in (5.4) coincides with $\mathbf{P}\overline{\mathbf{Q}}$ given in (2.4), the assertion follows on account of the left-hand side formulae in (5.3). $\square$

The theorem below generalizes and extends Exercise 68 in [7, Chapter 6]. Similarly as in the case of Theorem 2 inspired by Exercise 66 in [7, Chapter 6], the generalization is included in relaxing the assumption that $\mathcal{M}$ and $\mathcal{N}$ are inclined. The extension consists in introducing conditions (iv)–(vii).

Theorem 10. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following statements are equivalent*:

(i) $\operatorname{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] < 1$,
(ii) $\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q}) = \{\mathbf{0}\}$ *and* $\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$,
(iii) $\operatorname{rk}(\mathbf{P}) = \operatorname{rk}(\mathbf{PQ})$ *and* $\operatorname{rk}(\mathbf{Q}) = \operatorname{rk}(\mathbf{PQ})$,
(iv) $\mathcal{R}(\mathbf{P}) = \mathcal{R}(\mathbf{PQ})$ *and* $\mathcal{R}(\mathbf{Q}) = \mathcal{R}(\mathbf{QP})$,
(v) $\operatorname{rk}(\mathbf{P} + \mathbf{Q}) = \operatorname{rk}(\mathbf{PQ} + \mathbf{QP})$,
(vi) $\operatorname{rk}(\mathbf{P} - \mathbf{Q}) = \operatorname{rk}(\mathbf{PQ} - \mathbf{QP})$,
(vii) $\mathbf{P} + \mathbf{Q}$ *has no eigenvalues equal to* 1.

Proof. On account of (5.1) it is seen that $\operatorname{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] < 1$ means that $(\mathbf{P} - \mathbf{Q})^2$ has no eigenvalues equal to one. In such a situation, from point (iii) of Theorem 4 in [5] we get $r - \operatorname{rk}(\mathbf{A}) - \operatorname{rk}(\mathbf{B}) + \operatorname{rk}(\mathbf{D}) = 0$. Since $\operatorname{rk}(\mathbf{A}) \leqslant r$ and $\operatorname{rk}(\mathbf{B}) \leqslant \operatorname{rk}(\mathbf{D})$, it follows that

$$\operatorname{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] < 1 \;\Leftrightarrow\; \operatorname{rk}(\mathbf{A}) = r,\ \operatorname{rk}(\mathbf{B}) = \operatorname{rk}(\mathbf{D}). \tag{5.5}$$

On the other hand, points (ii) and (iii) of Lemma 7 ensure that, respectively, $\mathbf{P}_{\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})} = \mathbf{0}$ if and only if $\operatorname{rk}(\mathbf{A}) = r$ and $\mathbf{P}_{\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})} = \mathbf{0}$ if and only if $\operatorname{rk}(\mathbf{B}) = \operatorname{rk}(\mathbf{D})$. Thus, equivalence (i) $\Leftrightarrow$ (ii) is established.

To show that (i) $\Leftrightarrow$ (iii) holds, note that combining $\operatorname{rk}(\mathbf{P}) = r$ and $\operatorname{rk}(\mathbf{Q}) = \operatorname{rk}(\mathbf{A}) - \operatorname{rk}(\mathbf{B}) + \operatorname{rk}(\mathbf{D})$, given in [5, Theorem 1], with $\operatorname{rk}(\mathbf{PQ}) = \operatorname{rk}(\mathbf{A}) = \operatorname{rk}(\mathbf{QP})$, being a direct consequence of (3.6), leads to $\operatorname{rk}(\mathbf{P}) = \operatorname{rk}(\mathbf{PQ}) \Leftrightarrow \operatorname{rk}(\mathbf{A}) = r$ and $\operatorname{rk}(\mathbf{Q}) = \operatorname{rk}(\mathbf{PQ}) \Leftrightarrow \operatorname{rk}(\mathbf{B}) = \operatorname{rk}(\mathbf{D})$, respectively. (Observe that conjunction in statement (iii) of the theorem can actually be replaced with $\operatorname{rk}(\mathbf{P}) = \operatorname{rk}(\mathbf{Q})$ and $\operatorname{rk}(\mathbf{Q}) = \operatorname{rk}(\mathbf{PQ})$.)

To establish the next equivalence, we need orthogonal projectors onto $\mathcal{R}(\mathbf{PQ})$ and $\mathcal{R}(\mathbf{QP})$. The former of them is given in (3.6), and it is easily seen that $\mathcal{R}(\mathbf{P}) = \mathcal{R}(\mathbf{PQ}) \Leftrightarrow \operatorname{rk}(\mathbf{A}) = r$. Substituting conjugate transposes of matrices given in (3.4) and (3.5) to $\mathbf{P}_{\mathcal{R}(\mathbf{QP})} = \mathbf{QP}(\mathbf{QP})^{\dagger}$, and using condition (v) of Lemma 2, leads to

$$\mathbf{P}_{\mathcal{R}(\mathbf{QP})} = \mathbf{U} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B} \end{pmatrix} \mathbf{U}^*.$$

Hence, $\mathcal{R}(\mathbf{Q}) = \mathcal{R}(\mathbf{QP}) \Leftrightarrow \mathbf{D} = \mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B}$. On account of point (iii) of Lemma 3 we see that $\mathbf{D} = \mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B} \Leftrightarrow \operatorname{rk}(\overline{\mathbf{D}}) = n - r$, what, in view of condition (ii) of Lemma 4, is equivalent to $\operatorname{rk}(\mathbf{B}) = \operatorname{rk}(\mathbf{D})$.

The proof is concluded with observations that parts (i) $\Leftrightarrow$ (v) and (i) $\Leftrightarrow$ (vi) follow directly from Lemma 6 in [5], whereas equivalence (i) $\Leftrightarrow$ (vii) is a straightforward consequence of point (iii) of Theorem 6 in [5]. $\square$

Recall that statement (ii) of Theorem 10 expresses the fact that subspaces $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are totally inclined. Another observation is that three further conditions equivalent to $\operatorname{dist}[\mathcal{R}(\mathbf{P}), \mathcal{R}(\mathbf{Q})] < 1$ were given in [4, Theorem 12] and include

$$\mathcal{R}(\mathbf{P}-\mathbf{Q})=\mathcal{R}(\mathbf{PQ}-\mathbf{QP}),\quad \mathcal{R}(\mathbf{P}+\mathbf{Q})=\mathcal{R}(\mathbf{PQ}+\mathbf{QP}),\quad \mathcal{R}(\overline{\mathbf{P}}-\mathbf{Q})=\mathbb{C}_{n,1}.$$

It is also worth pointing out that equalities on the right-hand side of equivalence (5.5) can be expressed in alternative forms. Two lemmas below list some examples.

Lemma 10. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$ and let $\mathbf{Q}$ be partitioned as in (1.2). Then the following conditions are equivalent*:

(i) $\mathrm{rk}(\mathbf{A})=r$, (ii) $\lambda_{\max}(\overline{\mathbf{A}})<1$,
(iii) $\mathrm{rk}(\overline{\mathbf{A}})=\mathrm{rk}(\mathbf{B})$, (iv) $\overline{\mathbf{A}}=\mathbf{B}\overline{\mathbf{D}}^{\dagger}\mathbf{B}^*$,
(v) $\overline{\mathbf{Q}}=\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})}$.

Proof. Equivalence (i) $\Leftrightarrow$ (ii) is seen directly from decomposition (4.8), whereas relationships (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv) follow from points (i) of Lemma 4 and (viii) of Lemma 3, respectively. To show that the nonsingularity of $\mathbf{A}$ is equivalent also to condition (v), we need the orthogonal projector onto $\mathcal{R}(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})$. It can be shown with the use of formulae given in [6, p. 28] that $\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})}$ is of the form

$$\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})}=\mathbf{U}\begin{pmatrix}\mathbf{B}\overline{\mathbf{D}}^{\dagger}\mathbf{B}^* & -\mathbf{B}\\ -\mathbf{B}^* & \overline{\mathbf{D}}\end{pmatrix}\mathbf{U}^*.$$

Hence, the assertion is established by referring to point (iv) of the lemma. □

Lemma 11. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$ and let $\mathbf{Q}$ be partitioned as in (1.2). Then the following conditions are equivalent*:

(i) $\mathrm{rk}(\overline{\mathbf{D}})=n-r$, (ii) $\lambda_{\max}(\mathbf{D})<1$,
(iii) $\mathrm{rk}(\mathbf{B})=\mathrm{rk}(\mathbf{D})$, (iv) $\mathbf{D}=\mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B}$,
(v) $\mathbf{Q}=\mathbf{P}_{\mathcal{R}(\mathbf{QP})}$.

Proof. The proof is based on analogous arguments to the ones utilized in the proof of Lemma 10. □

Further conditions necessary for $\mathrm{dist}[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]<1$ were given by Kato [21, Chapter I], who observed, for instance, that if $\|\overline{\mathbf{Q}}\mathbf{P}\|=\delta<1$, then either $\mathbf{Q}$ maps $\mathcal{R}(\mathbf{P})$ onto $\mathcal{R}(\mathbf{Q})$ one-to-one and bicontinuously and $\|\mathbf{P}-\mathbf{Q}\|=\delta$ or $\mathbf{Q}$ maps $\mathcal{R}(\mathbf{P})$ onto a proper subspace of $\mathcal{R}(\mathbf{Q})$ one-to-one and bicontinuously and $\|\mathbf{P}-\mathbf{Q}\|=1$. This result remains valid also for infinite dimensional Hilbert space.

It follows from [25, p. 454] that if $\dim(\mathcal{M})=\dim(\mathcal{N})$ holds along with either $\mathcal{M}\cap\mathcal{N}^{\perp}\neq\{\mathbf{0}\}$ or $\mathcal{M}^{\perp}\cap\mathcal{N}\neq\{\mathbf{0}\}$, then $\mathrm{dist}(\mathcal{M},\mathcal{N})=1$. From Theorem 10 it is straightforwardly seen that this result can be generalized by relaxing the assumption that the two subspaces involved have the same dimension. This fact, also noted in [11, Corollary 7], constitutes the corollary below.

Corollary 4. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. Then $\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})\neq\{\mathbf{0}\}\Rightarrow\mathrm{dist}[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=1$ and $\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})\neq\{\mathbf{0}\}\Rightarrow\mathrm{dist}[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=1$.*

Further characterizations of the distance between $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are possible. For instance, from points (ii) and (iii) of Theorem 5 in [5] it follows that for $\mathrm{dist}[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=1$ it is sufficient that either -1 or 1 are not among eigenvalues of $\mathbf{P}-\mathbf{Q}$.

The last theorem of the paper concerns relationship between minimal angle and direct distance.

Theorem 11. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$ be such that $\mathcal{R}(\mathbf{P})\oplus\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$. Then*

$$C_0[\mathcal{R}(\mathbf{P}),\mathcal{R}(\mathbf{Q})]=\mathrm{dist}[\mathcal{R}(\mathbf{P}),\mathcal{N}(\mathbf{Q})].$$

Proof. The assertion follows by combining point (i) of Definition 3 with Corollary 2. □

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