Practice on **Discrete Fourier Transform (DFT)**

CSE 220: Signals and Linear Systems

Restrictions

- You may use `numpy` (arrays, `exp`, `pi`, basic linear algebra) and `matplotlib`.
- You may **NOT** use `np.fft.*`, `scipy.fft*`, or built-in convolution/correlation routines.
- Implement DFT/IDFT by direct summation or the equivalent matrix form.

Definitions

Let $x[n]$ be a length-N sequence, $n = 0, \dots, N-1$.

DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \dots, N-1. \tag{1}$$

IDFT:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}, \quad n = 0, 1, \dots, N-1. \tag{2}$$

Signals to Use

Use $N = 64$ unless stated otherwise. For $n = 0, \dots, N-1$:

1. Rectangular pulse: $x_{\text{rect}}[n] = 1$ for $0 \le n < N/8$, else 0.
2. Cosine: $x_{\cos}[n] = \cos\big(\frac{2\pi m}{N} n\big)$ with $m = 5$.

Task 1: Implement DFT and IDFT + Spectra

1.1 Implementation

Complete:

- `dft(x)` $\to X$
- `idft(X)` $\to x$

1.2 Reconstruction Check (for both signals)

For each signal $x[n] \in \{x_{\text{rect}}[n], x_{\cos}[n]\}$:

1. Compute $X[k] = \text{DFT}\{x[n]\}$ and $\hat{x}[n] = \text{IDFT}\{X[k]\}$.
2. Report:

$$\max_n |x[n] - \hat{x}[n]|, \quad \frac{\|x - \hat{x}\|_2}{\|x\|_2 + \epsilon}.$$

3. Plot (discrete stem/marker style):

- $x[n]$ and $\Re\{\hat{x}[n]\}$ on the same figure (with legend).
- Magnitude spectrum $|X[k]|$.
- Phase spectrum $\angle X[k]$.

Task 2: Verify Circular Convolution Theorem

Circular convolution (length N):

$$y[n] = (x \circledast h)[n] = \sum_{m=0}^{N-1} x[m]\, h[(n-m) \bmod N]. \tag{3}$$

2.1 Implement Circular Convolution

Implement `circular_convolution(x,h)` for equal-length sequences.

2.2 Theorem Verification (Small Example)

Use $N = 4$:

$$x = [1, 2, 3, 4], \quad h = [4, 3, 2, 1].$$

1. Compute $y_{\text{time}} = x \circledast h$ using your time-domain function.
2. Compute $Y[k] = X[k]H[k]$ and $y_{\text{freq}} = \text{IDFT}\{Y[k]\}$.
3. Report error between y_{time} and $\Re\{y_{\text{freq}}\}$ and plot both on the same figure.

Task 3: Cross-Correlation via DFT

Let $x[n] = x_{\cos}[n]$ and let $y[n] = x[(n - n_s) \bmod N]$ with $n_s = 12$.
Compute circular cross-correlation using:

$$r_{xy}[n] = \text{IDFT}\{X[k] \cdot Y^*[k]\}. \tag{4}$$

1. Implement `cross_correlation_via_dft(x,y)`.
2. Plot $r_{xy}[n]$ (real part).
3. Report $n^\star = \arg\max_n \Re\{r_{xy}[n]\}$.
4. **Short answer (2–4 lines):** Explain how $n^\star$ relates to n_s in the circular setting.