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Weighted Sliding Mode Control

Control of Multiple Surfaces via a Single Actuator

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***Abstract*— This paper presents a design method for controlling multiple sliding surfaces of nonlinear systems with matched uncertainty using a single actuator. Sliding mode control, in general, requires one actuator for every control objective. Compared to the vast literature on sliding mode control, there has been little work on general theories that can be applied to underactuated nonlinear control. A sufficient condition to guarantee simultaneous robust stabilization of multiple surfaces is derived. A corollary is presented for the case when the sliding surfaces are affine in the control. An example of an overhead crane is presented to validate the design method.**

I. INTRODUCTION

Traditional sliding mode control (SMC) [1], [2], [3] pairs a single control surface to a single actuator. This enables the designer to control, with little or no error, the same number of surfaces as available actuators. In general, when the the surfaces are not coupled and the uncertainty is matched, e.g. the sliding surfaces have the same relative degrees with respect to the control and disturbance channels, control design is straight forward. In spite of many developments in sliding mode design, little work has been done to control multiple surfaces with a single actuator. In most cases, the available theories have been developed for specific classes of systems [4], [5], [6] or for analyzing system attributes, such as the domain of attraction for an underactuated SMC controlled system [7]. Work has been done by combining multiple surfaces into a single weighted surface for control design [8], [9], although the global implications of the design technique have been refuted by multiple sources [10], [11]. Some development on controlling multiple sliding surfaces with mismatched uncertainties resulted in hierarchical sliding mode control [12].

The main challenge in underactuated SMC designs is to address the potential problems that result from interactions between the separate surfaces through control. In order to address these challenges, this paper proposes designing a two part control law that consists of a weighted summation of the single surface controls and a regulation term that guarantees the existence of a composite Lyapunov function.

This paper is organized as follows, Section II presents the system definition that forms the basis for the results that follows. Section III presents the main theoretical results of this paper in a theorem for the sufficient condition for global stability of a set of sliding surfaces controlled by a single actuator. A corollary is given that provides an interesting interpretation of the main theorem. An example is given in Section IV to demonstrate the capabilities of the control. Section V presents directions for future development as well as a concluding remark.

II. PROBLEM FORMULATION

A. System Definition

Consider a dynamic system (H) of the form

$$H : \begin{cases} \dot{x} = f(x, u, w) \\ y = h(x) \end{cases} \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}^p$

Define p sliding surfaces [1], [2], [3], $S_k(x, u)$, with the first order dynamics:

$$\dot{S}_k = \Omega_k(x, u) + \bar{w}_k(t) \tag{2}$$

where $k \in \{1, \dots, p\}$, $\Omega_k : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is an implicit function of $S_k : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, and $\bar{w}_k : \mathbb{R}_+ \mapsto \mathbb{R}$ is the exogenous input.

B. System Properties

The following assumptions are made with respect to the system (H) above:

- The equations have matched uncertainty. This means that all of the uncertainy and inputs (u) occur on the same order time derivatives of the outputs (y). This is a standard definition of nonlinear matched uncertainty [13].
- The disturbances are bounded above as follows: There exists $\hat{w}_k < \infty$ such that $|\bar{w}_k(it)| \leq \hat{w}_k \; \forall \; t \in \mathbb{R}_+$

III. MAIN RESULTS

In this section, the main results of the paper are presented. The main theorem for this paper can be interpreted as requiring that the interactions between the total control and the control on each surface must meet the standard negative definiteness requirements originally proposed by Lyapunov in his thesis [14]. The corollary extends this theorem to specific cases for control design, most notably when Ω_k from Equation (2) is an affine function of u.

Theorem 1. *Suppose that auxiliary control laws, u_k, $k \in \{1, \dots, p\}$, have been designed such that $V_k(S_k) : \mathbb{R}^n \to \mathbb{R}_+$ is a Lyapunov function for each of the sliding surfaces with the dynamics of Equation* (2).

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If the actual control, u, is such that

$$\sum_k \alpha_k V_k'(S_k)\left[\Omega_k(x,u) - \Omega_k(x,u_k)\right] \le 0 \quad \forall\, S_k \tag{3}$$

for some $\alpha_k > 0$ and V_k' is the derivative of V_k with respect to S_k. Then, the composite function

$$V = \sum_k \alpha_k V_k \tag{4}$$

is a Lyapunov function for the combined sliding surfaces.

Proof:

$$V = \sum_k \alpha_k V_k(S_k) \tag{5}$$

$$\dot{V} = \sum_k \alpha_k V_k'(S_k)\dot{S}_k \tag{6}$$

$$= \sum_k \alpha_k V_k'(S_k)[\Omega_k(x,u) + \bar{w}_k] \tag{7}$$

$$= \sum_k \alpha_k V_k'(S_k)[\Omega_k(x,u_k) + \bar{w}_k] + \sum_k \alpha_k V_k'(S_k)[\Omega_k(x,u) - \Omega_k(x,u_k)] \tag{8}$$

After using the sufficient condition in Equation (3), the following inequality arises:

$$\dot{V} \le \sum_k \alpha_k V_k'(S_k)[\Omega_k(x,u_k) + \bar{w}_k] \tag{9}$$

$$= \sum_k \alpha_k \dot{V}_k \tag{10}$$

Because u_k has been designed such that V_k is a Lyapunov function for S_k; that is $\dot{V}_k < 0\ \forall\ S_k$. It follows then, from (10), that $\dot{V} < 0\ \forall\ S_k$.

Remark 1. *There are two stability properties that are important to note for Theorem 1*

- *V converging to zero implies asymptotic stability of each surface, S_k.*
- *V converging to a ball[1] implies that each surfaces, S_k, converges to a ball[2].*

Remark 2. *If the sufficient condition of Theorem 1 is not enforceable, in the sense that $0 \le \inf_u \sum_k \alpha_k V_k'(S_k)[\Omega_k(x,u) - \Omega_k(x,u_k)] \le M$ for some bounded $M \in \mathbb{R}$, the surfaces, at best, will converge to a ball, as mentioned in Remark 1. In light of this, one could set up an optimization problem to minimize the supremum function with a regularization term on u such as*

$$u = \operatorname*{argmin}_u \left\{ \sum_k \alpha_k V_k'(S_k)[\Omega_k(x,u) - \Omega_k(x,u_k)] + R(u) \right\} \tag{11}$$

in order to shape the resulting ball as desired.

Remark 3 (Singular Arc Situation)**.** *If Ω_k is affine in u, i.e*

$$\Omega_k(x,u) - \Omega_k(x,u_k) = \rho_k(x)(u - u_k), \tag{12}$$

where $\rho_k(x) \in \mathcal{L}_\infty$ is sign-definite, then the sufficient condition in (3) *reduces to*

$$\sum_k \rho_k(x)\alpha_k V_k'(S_k)[u - u_k] \le 0. \tag{13}$$

Situations may arise where the coefficient of u in (13) *vanishes over a period of time for nonzero values of S_k's. In this case, the condition $\sum_k \rho_k(x)\alpha_k V_k'(S_k) = 0$ is said to define a* Singular Arc Condition[3] *for that period. This is a degenerate situation in which no control exists that can simultaneously stabilize all surfaces in the sense of Lyapunov. Also, note that the condition is dependent on the Lyapunov function for each surface. As a result, the singular arc condition can be seen as some sort of "blind spot" for the Lyapunov function.*

Hence, implicit in the inequality in (13) *is the assumption that the Lyapunov functions for each of the sliding surfaces are chosen such that the system dynamics admits desirable boundedness properties along the singular arc.*

We remark also that the approach presented in this paper is not limited to the particular choice of the global Lyapunov function used in Theorem 1. For example, one could be interested in the non-smooth Lyapunov function[16] candidate

$$V = \max_k V_k(S_k). \tag{14}$$

While this Lyapunov function may be void of "blind spots", due to the non-smoothness, it introduces new challenges which may complicate the stability analysis or even create additional problem of high frequency switching between the surfaces.

Alternatively, by redefining the auxiliary control laws as $u_k(S_k - \lambda_k\mu)$, where μ is an integrator given by

$$\dot{\mu} = -\Gamma\mu + \sum_k \alpha_k\lambda_k V_k'(S_k - \lambda_k\mu), \quad \Gamma > 0 \tag{15}$$

and using the Lyapunov function candidate

$$V = \sum_k \alpha_k V_k(S_k - \lambda_k\mu) + \frac{\Gamma}{2}\mu^2, \tag{16}$$

the sufficient condition becomes

$$\sum_k \alpha_k\rho_k(x)V_k'(S_k - \lambda_k\mu)(u - u_k) \le 0 \tag{17}$$

from where it is seen that the existence, and certain properties of singular arcs can be modified by the choice of the

[1]The function $V(x(t)) > 0$ is said to converge to a ball of radius, r, if $\exists$ a $T > 0$ such that $V(x(T)) < r \Rightarrow V(x(t)) < r\ \forall\ t > T$.

[2]The signal, $s(t)$, is said to converge to a ball of radius, r', if $\exists$ a $T > 0$ such that $\|s(T)\| < r' \Rightarrow \|s(T)\| < r'\ \forall\ t > T$.

[3]Singular Arcs Conditions arise in optimal control problems that are difficult to solve because a direct application of Pontryagin's minimum principle fails to yield a complete solution. The most common situation is when the Hamiltonian depends linearly on the control. In such case, it is difficult to arrive at a logical conclusion of what the optimal control action should be at the points where the coefficient vanishes.

additional parameters $\Gamma, \lambda_k, k = 1, \ldots, p$.
Finally, we note the importance of carrying out further analysis to, say; determine the maximum duration of all singular arcs, characterize the boundedness of the surfaces along the singular arcs, and answer other questions of interest. These detailed analysis of the properties of the singular arc, and of the surfaces along the singular arcs are reserved for future work.

In the light of the above remark and for the purpose of this paper, we make the following assumptions for the case where the system is affine in the control.

Assumption 1. *The disturbance is matched. Hence, the sliding dynamics in* (2) *is of the form*

$$\dot{S}_k = \Omega_k(x) + \rho_k(x)_k(u + w_k), \tag{18}$$

where $\rho_k(x) \in \mathcal{L}_\infty$ is sign-definite.

Assumption 2. *The disturbances does not destabilize the system along singular arcs. That is, for any $\epsilon > 0$, there exists $r > 0$ such that $\sup |S_k| \leq r$, $k = 1, \ldots, p$ whenever $\left| \sum_k \alpha_k V_k'(S_k - \lambda_k \int_0^t \sum_j \alpha_j V_j'(S_j(\tau)) d\tau) \right| < \epsilon$ for some $\lambda_k, \alpha_k, k = 1, \ldots, p$.*

Corollary 1. *Suppose that each auxiliary control $u_k(S_k, x)$ in Theorem 1 has been designed such that there exist a continuosly differentiable, postive-definite, decrescent Lyapunov function V_k satisfying*

$$V_k'(S_k)\left(\Omega_k(x) + \rho_k(x) u_k(S_k, x)\right) < 0 \ \ \forall \ \ S_k \in \mathbb{R}. \tag{19}$$

If the actual control is designed as

$$\begin{aligned} \dot{\mu} &= -\Gamma\mu + \sum_k \alpha_k \lambda_k V_k'(S_k - \lambda_k \mu), \ \ \Gamma > 0 \\ u &= \sum_k u_k(S_k - \lambda_k \mu, x) + \beta_I + \beta_R, \end{aligned} \tag{20}$$

where

$$\beta_I = -\frac{\sum_k \left(\alpha_k \rho_k V_k'(S_k - \lambda_k \mu) \sum_{j \neq k} u_j \right)}{\mathit{sat}_\epsilon \left(\sum_k \left(\alpha_k \rho_k V_k'(S_k - \lambda_k \mu) \right) \right)} \tag{21}$$

where

$$\mathit{sat}_\epsilon(x) \triangleq \begin{cases} x & \text{if } x > \epsilon \\ \epsilon & \text{otherwise} \end{cases} \tag{22}$$

and

$$\beta_R = \max_k \{|\hat{w}_k\}\, \gamma \left(\sum_k \rho_k \alpha_k \rho_k V_k'(S_k - \lambda_k \mu) \right), \tag{23}$$

where $\gamma(.)$ is an odd function satisfying $z\gamma(z) \geq |z|, \forall \ z \in \mathbb{R}$, Then the sliding surfaces, S_k, $k \in \{1, \ldots, p\}$, are uniformly ultimately bounded (UUB) [1]..

Proof: Consider the composite Lyapunov function candidate

$$V = \sum_k \alpha_k V_k(S_k - \lambda_k \mu) + \frac{\Gamma}{2}\mu^2.$$

Taking first time derivative and rearranging terms yield

$$\begin{aligned} \dot{V} &= \sum_k \alpha_k V_k'(S_k - \lambda_k \mu)\left(\Omega_k(x) + \rho_k(x) u_k(S_k - \lambda_k \mu, x)\right) \\ &\quad + \sum_k \alpha_k V_k'(S_k - \lambda_k \mu) \rho_k(x) \left(u - u_k(S_k - \lambda_k \mu) + w_k\right) \\ &\quad + \dot{\mu}\left(\Gamma\mu - \sum_k \alpha_k \lambda_k V_k'(S_k - \lambda_k \mu) \right). \end{aligned} \tag{24}$$

Now, define the singular set

$$B_\epsilon(\mu) \triangleq \left\{ \mathbf{S} \in \mathbb{R}^p : \left| \sum_k \left(\alpha_k \rho_k V_k'(S_k - \lambda_k \mu) \sum_{j \neq k} u_j \right) \right| \leq \epsilon \right\}.$$

From Assumption 2, it is seen that S_k's are bounded whenever $[S_1, \ldots, S_p]^\top \in B_\epsilon(\mu)$. Next, we examine what happens outside of $B_\epsilon(\mu)$. After using the control laws in (20) with (23) and the relevant portion of (21), we have that

$$\begin{aligned} \dot{V} &\leq \sum_k \alpha_k V_k'(S_k - \lambda_k \mu)\left(\Omega_k(x) + \rho_k(x) u_k(S_k - \lambda_k \mu, x)\right) \\ &\quad - \left(\Gamma\mu - \sum_k \alpha_k \lambda_k V_k'(S_k - \lambda_k \mu) \right)^2 \end{aligned} \tag{25}$$

From which we conclude that the sliding surfaces are asymptotically stable outside the singular set $B_\epsilon(\mu)$. Thus, in the worst case, the surface are UUB.

Remark 4. *If $V_k = \frac{1}{2}S_k^2$ and $\dot{S}_k$ takes the form*

$$\dot{S}_k = \Phi_k(x) + g_k(x)(u + w_k(t)), \tag{26}$$

where $g_k(x) \neq 0$, $\forall x \in \mathbb{R}^n$. Then the auxiliary controls, u_k, of the form

$$u_k = \frac{1}{g_k(x)}\left[-K_k S_k - \Phi_k(x) \right], \ \ K_k > 0 \tag{27}$$

with u and β's as defined in Equations (20) *,*(21) *and* (23) *will satisfy the requirements of Corollary 1.*

IV. EXAMPLE

In this section, one example of the control design using Corollary 1 is presented. The example illustrates the validity of the control design method as well as the capabilities under different disturbances. The design is not rigorously developed for implementation, but presented as a tutorial and validation of the method in this paper. Because of this, the performance of the controller has not been compared to other known controllers. The merit of the method described is the freedom to independently define several desired behaviors in terms of sliding surfaces.

This example is adopted from [8] which was also studying underactuated sliding mode control of multiple surfaces. This example illustrates the stability of the design as well as other aspects of the design, including the ball that the steady state system converges to in the face of continuous disturbances and the effect that the ϵ term has on this ball.

A. Modeling and Control Design

This example uses a low order model of a crane with a suspended load that is modeled as a cart with an attached pendulum and a force actuator that acts on the cart mass. A diagram of the system can be seen in Figure 1.

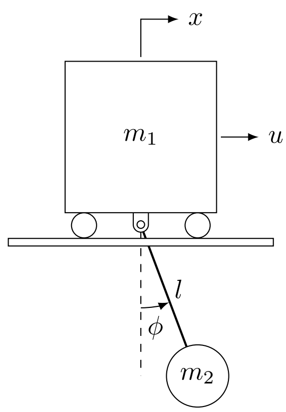

Fig. 1: Example 2 - Overhead Crane Diagram

The equations of motion can easily be obtained using Lagrange equations. The derivation is omitted for brevity. The equations are as follows

$$\dot{v} = \frac{m_2 l\omega^2 \sin\phi + m_2 g \sin\phi\cos\phi}{m_1 + m_2 \sin^2\phi} + \frac{1}{m_1 + m_2\sin^2\phi}u + w_v(t) \tag{28}$$

$$\dot{\phi} = \omega \tag{29}$$

$$\dot{\omega} = -\frac{m_2 l\omega^2 \sin\phi\cos\phi + (m_1+m_2)g\sin\phi}{l(m_1 + m_2\sin^2\phi)} - \frac{\cos\phi}{l(m_1+m_2\sin^2\phi)}u + w_\omega(t) \tag{30}$$

The states are cart speed (v), load angle (ϕ), and angular speed (ω). The input (u) represents a force actuator that causes horizontal movement in the crane. The terms $w_v(t)$ and $w_\omega(t)$ in Equations (28) and (30) represent disturbances in the linear and angular load momenta.

The outputs that are to be controlled are

$$y_1 = x \tag{31}$$

$$y_2 = \phi \tag{32}$$

Because y_1 (cart position) is not a state[4], a free integrator has to be added to the system of equations of the form

$$\dot{y}_1 = \dot{x} = v \tag{33}$$

[4]This system usually appears as a 4$^{\text{th}}$ order system in the literature but in reality is only 3$^{\text{rd}}$. This is readily apparent if one uses a bond graph [15] to model the system.

The sliding surfaces used for control design in this problem are of the form

$$S_k = \dot{e}_k + \sigma_k e_k \tag{34}$$

where y_k is the k$^{\text{th}}$ output, $y_{k,d}$ is the k$^{\text{th}}$ desired output, and $e_k = y_k - y_{k,d}$. The two surfaces that are controlled track a desired trolley position (x_d) and regulate the angle (θ) to 0. The desired values are: $x_d = 2\ m$, $v_d = 0$, $\theta_d = 0$, and $\omega_d = 0$. Substituting these values into Equation (34) yields the following two sliding surfaces.

$$S_1 = v + \sigma_1(x - x_d) \tag{35}$$

$$S_2 = \omega + \sigma_2\phi \tag{36}$$

Differentiating and using the form of Remark 4, results in the following auxiliary control laws

$$u_1 = \Delta\left[-K_1S_1 - \sigma_1 v - \frac{m_2 l\omega^2\sin\phi + m_2 g\sin\phi\cos\phi}{\Delta}\right] \tag{37}$$

$$u_2 = -\frac{l\Delta}{\cos\theta}\left[-K_2S_2 - \sigma_2\omega + \frac{m_2 l\omega^2\sin\phi\cos\phi + (m_1+m_2)g\sin\phi}{l\Delta}\right] \tag{38}$$

where $\Delta = m_1 + m_2\sin^2\phi$. Again, these two control laws are used to implement the total control of the form of Equations (20), (21), and (23), with $g_1(x) = \frac{1}{m_1+m_2\sin^2\phi}$, and $g_2(x) = -\frac{\cos\phi}{l(m_1+m_2\sin^2\phi)}$. For implementation of β, V_k were taken as $\frac{1}{2}S_k^2$.

B. Results

All the system and control parameters required to produce the results in this section are given in Table I. The crane was simulated using the system and control parameters given in V with the initial conditions $x(0) = 0\,m$, $v(0) = 0\,m/s$, $\phi(0) = 0\,rad$, and $\omega(0) = 0\,rad/sec$.
Figures 2-4 show the sliding surface and system responses when $w_v(t) = 0.1g_1\cos(10t)$ and $w_\omega(t) = 0.1g_2\sin(10t)$. These figures show that the system remains stable according to the claims of Corollary 1. Figure 5 shows the auxiliary control actions as well as the aggregate control authority. Also, in Figure 6, the *interconnection* and the robustness β terms are shown together with the total β signal. Looking at both figures, it is seen that β_I controls the blending of the auxiliary controls in accordance with the weighted Lyapunov (or energy) function. Moreover, the profile of the robustness term β_R in the middle plot of Figure 6 elucidates the switching action responsible for the robustness property of the ensuing control system.

V. Conclusions and Future Works

In this paper, a novel method was proposed to achieve stable underactuated control of multiple objectives using the sliding mode technique. A theorem was presented for a sufficient condition for the stability of a general nonlinear system where the outputs are only a function of the states and the resulting objective sliding surfaces having matched

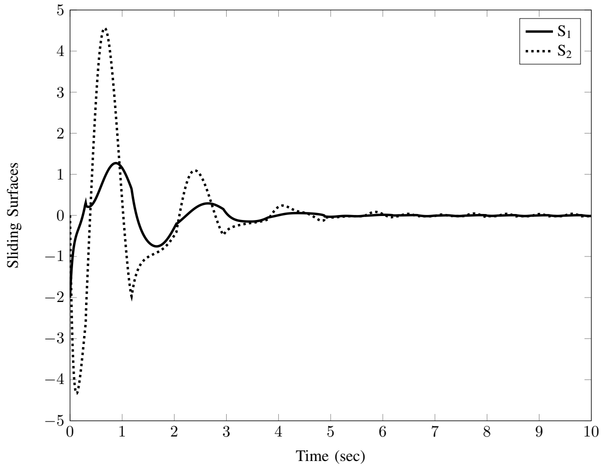

Fig. 2: Sliding Surfaces

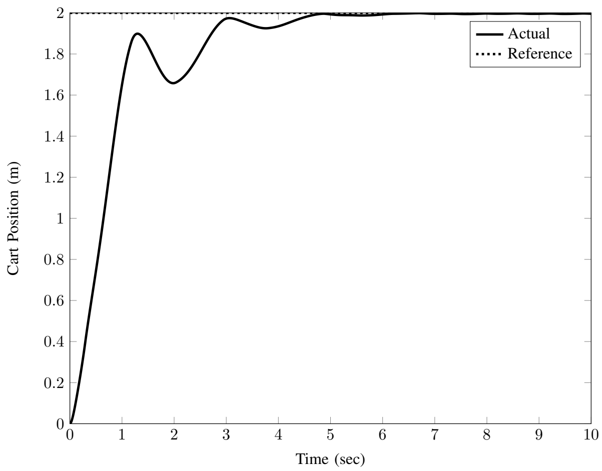

Fig. 3: Cart Position

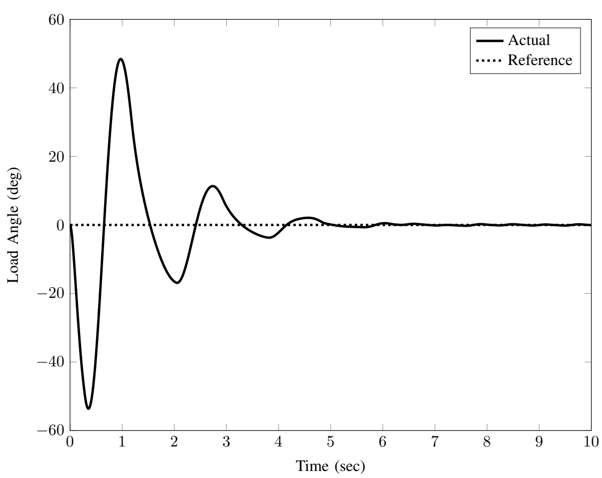

Fig. 4: Load Angle

uncertainty. A corollary was presented showing the resulting control requirements when the Ω_k terms are affine in u and the total control u is a weighted sum of the individual

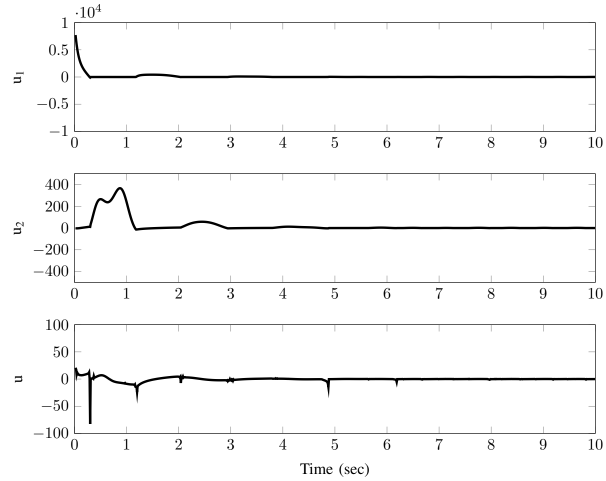

Fig. 5: Control Authority

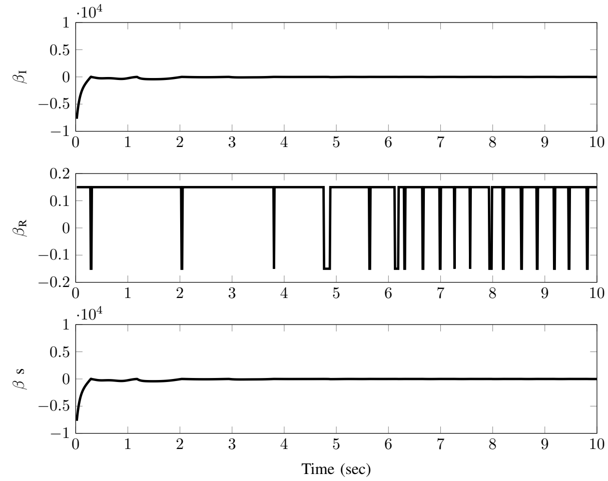

Fig. 6: Robustness and Interconnection terms

controls plus an interconnection and robustness terms given by Equation (21) and (23). An example was presented showing the resulting design for a nonlinear crane. This example illustrated the validity of the control design and the effect that the choice of ϵ has on the smoothness of the control as well as the resulting ball that the steady state lives in.

We conclude by giving some directions for future works:

1) Considering additional corollaries to Theorem 1 which would give rise to different control laws and comparing the performance of the controllers with additional examples.
2) Including actuator dynamics in the design.
3) Accounting for parameter adaptation would be beneficial. This will include developing a switching adaptation law with an update law outside of B_ϵ and a modified update law within B_ϵ.
4) Extending the theoretical development to a full underactuated MIMO system where there are multiple actuators that affect multiple surfaces at the same time.

5) Analysing the singular arc condition further to identify the least restrictive conditions on the sliding surfaces such that the control laws in Corollary 1 can be applied. Such analysis would also encompass the quest for the "optimal" action along a singular arc.

APPENDIX

TABLE I: System Parameters

Parameter	Symbol	Value
Cart Mass	m_1	$1\,kg$
Load Mass	m_2	$0.8\,kg$
Hoist Rope Length	l	$0.305\,m$
Gravity	g	$9.81\,m/s^2$

TABLE II: Control Parameters

Parameter	Symbol	Value
x Decay Rate	σ_1	1
ϕ Decay Rate	σ_2	5
Surface 1 Weight	α_1	1
Surface 2 Weight	α_2	10
Disturbance bound	$\hat{w}_1, \hat{w}_2$	0.5
Integrator constant	Γ	1
Integrator constant	λ_1	1
Integrator constant	λ_2	1
Singularity level	ϵ	0.01

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