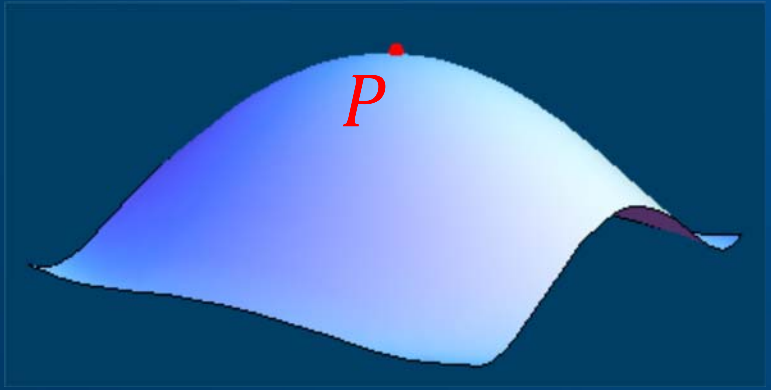
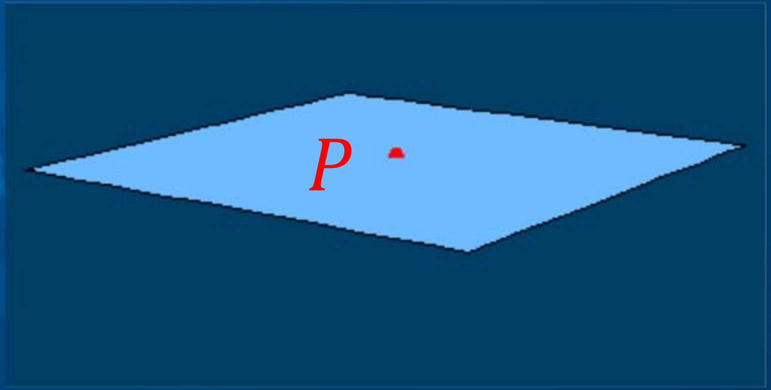
第71讲 多元函数的泰勒公式

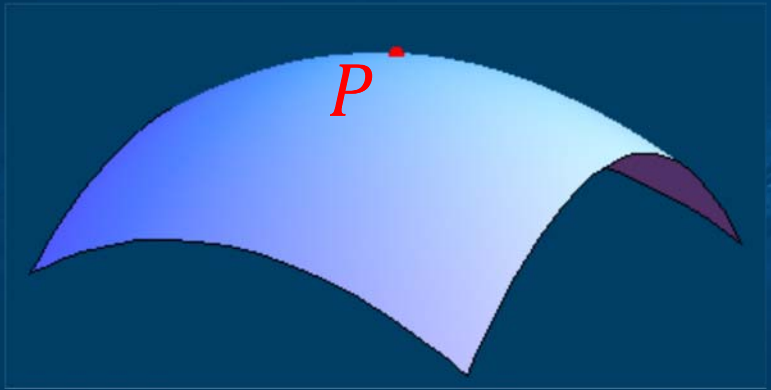
“以平代曲” → “以曲代曲”



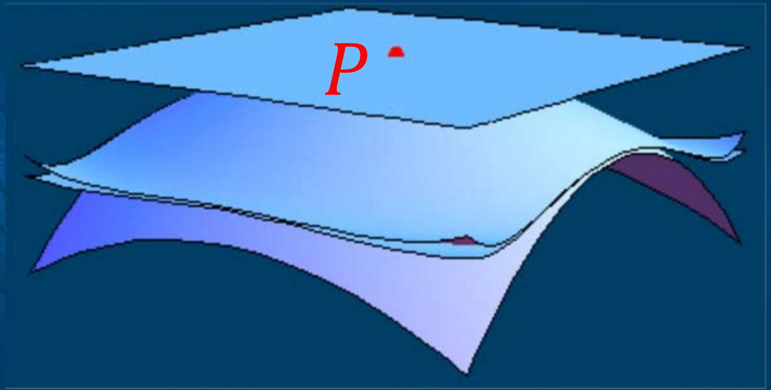
函数$z = f(x, y)$



一次多项式近似

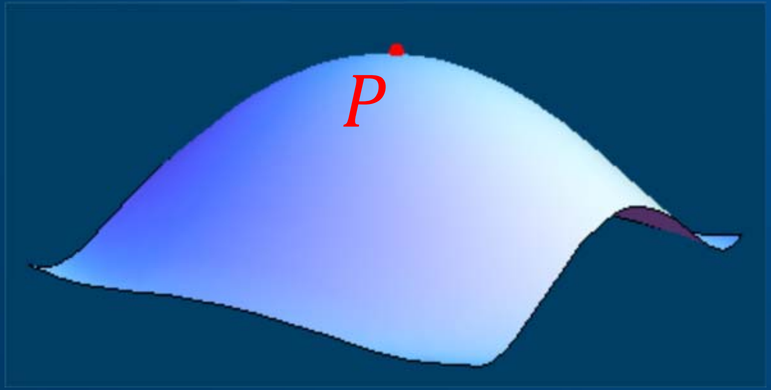


二次多项式近似

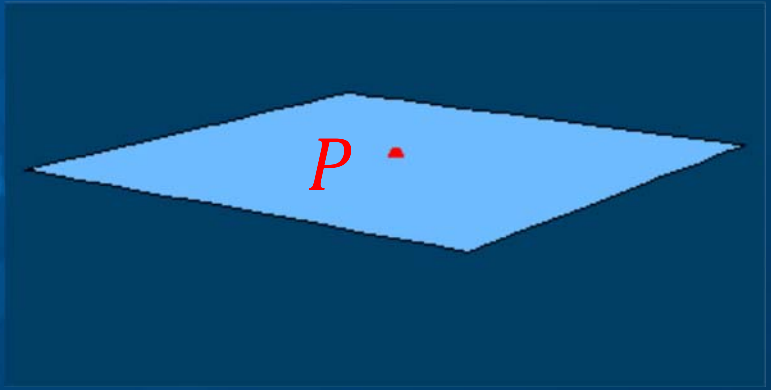


高次多项式近似

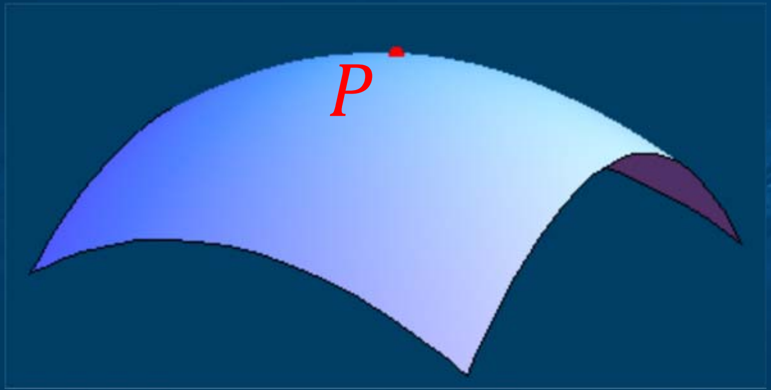
“以平代曲” “以曲代曲”



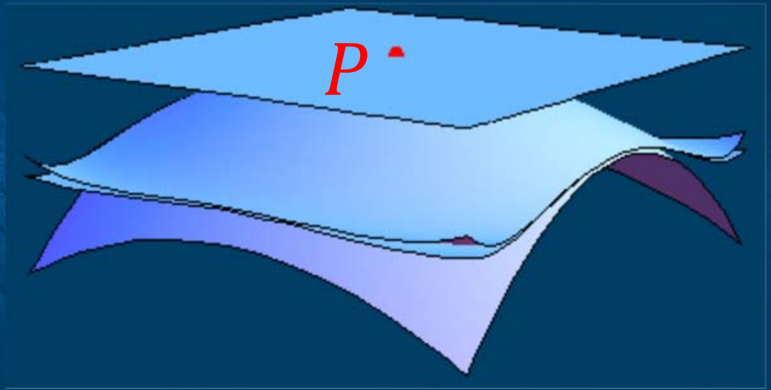
函数$z=f(x,y)$



一次多项式近似



二次多项式近似



高次多项式近似

海赛矩阵

多元函数的泰勒公式

近似计算

- 一元函数微分概念

$$f(x+\Delta x)-f(x)=\boxed{f'(x)}\Delta x+o(\Delta x)$$

一元函数的导数

- 二元函数微分概念

$$f(x+\Delta x,y+\Delta y)-f(x,y)=f_x'(x,y)\Delta x+f_y'(x,y)\Delta y+o(\rho)$$

$$=\boxed{\left(f_x'(x,y),\ f_y'(x,y)\right)}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}+o(\rho)$$

二元函数一阶导数

$$f'(x,y)=\left(f_x'(x,y),\ f_y'(x,y)\right)=\nabla f(x,y)$$ 梯度

二元函数一阶导数 $f'(x,y)=(f_x'(x,y),\ f_y'(x,y))$

$$(f_x'(x+\Delta x,y+\Delta y),f_y'(x+\Delta x,y+\Delta y))-(f_x'(x,y),f_y'(x,y))$$

$$=(f_x'(x+\Delta x,y+\Delta y)-f_x'(x,y),f_y'(x+\Delta x,y+\Delta y))-f_y'(x,y))$$

$$=(f_{xx}''(x,y)\Delta x+f_{xy}''(x,y)\Delta y+o(\rho),\ f_{yx}''(x,y)\Delta x+f_{yy}''(x,y)\Delta y+o(\rho))$$

$$=\left(f_{xx}''(x,y)\Delta x+f_{xy}''(x,y)\Delta y,\ f_{yx}''(x,y)\Delta x+f_{yy}''(x,y)\Delta y\right)+o(\rho)$$

$$=\boxed{\begin{pmatrix} f_{xx}''(x,y) & f_{xy}''(x,y) \\ f_{yx}''(x,y) & f_{yy}''(x,y) \end{pmatrix}}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}+o(\rho)$$

$f''(x,y)$ 二元函数的二阶导数（海赛矩阵）

设 n 元函数$f(\mathbf{x})$在点$\mathbf{x}(\mathbf{x}=(x_1,x_2,\cdots,x_n))$处对于自变量各分量的二阶偏导数 $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$ $(i,j=1,2,\cdots,n)$连续，则称矩阵

$$\mathbf{H}=\begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix} \triangleq \nabla^2 f(\mathbf{x})$$

为函数$f(\mathbf{x})$在点$\mathbf{x}$处的二阶导数或海赛矩阵.

例1 计算函数$f(x,y)=x^4+xy+(1+y)^2$的梯度与海赛矩阵，并求$\nabla f(0,0)$，$\nabla^2 f(0,0)$以及$\nabla f(0,-1),\nabla^2 f(0,-1)$.

例2 计算函数

$$f(x,y)=a+b_1x+b_2y+c_{11}x^2+2c_{12}xy+c_{22}y^2$$

在(0, 0) 处的梯度与海赛矩阵.

$$f(x,y)=f(0,0)+\nabla f(0,0)\begin{pmatrix}x\\y\end{pmatrix}+\frac{1}{2}(x,y)\nabla^2 f(0,0)\begin{pmatrix}x\\y\end{pmatrix}$$

定理1（1）设函数$z=f(x,y)$在$(0,0)$的某邻域U内存在一阶连续偏导数，则对于任意的$(x,y)\in U$，均存在$\theta\in(0,1)$使得

$$f(x,y)=f(0,0)+\nabla f(\theta x,\theta y)\binom{x}{y};$$

—— 0阶带拉格朗日余项的麦克劳林公式

（2）设函数$z=f(x,y)$在$(0,0)$的某邻域内存在二阶连续偏导数，则对于任意的$(x,y)\in U$，均存在$\theta\in(0,1)$使得

$$f(x,y)=f(0,0)+\nabla f(0,0)\binom{x}{y}+\frac{1}{2}(x,y)\nabla^2 f(\theta x,\theta y)\binom{x}{y}.$$

—— 1阶带拉格朗日余项的麦克劳林公式

一般情形的泰勒公式

设函数$z=f(x,y)$在(x_0,y_0)的某邻域U内存在一阶连续偏导数，则对于任意的$(x,y)\in U$，均有

$$f(x,y)=f(x_0,y_0)+\nabla f(\eta,\zeta)\begin{pmatrix}x\\y\end{pmatrix}$$

其中(η,ζ)为连接(x_0,y_0)与(x,y)线段上的某一点.

$$f(x,y)-f(x_0,y_0)=f_x'(\eta,\zeta)(x-x_0)+\ f_y'(\eta,\zeta)(y-y_0)$$

二元函数的拉格朗日中值公式

例3 设函数$z = f(x, y)$在区域D内存在偏导数，且对于D内任意的(x, y)，均有

$$f_x'(x, y) = f_y'(x, y) = 0.$$

证明：$f(x, y)$在区域D内为常数.

定理2 设$f(\mathbf{x})$是 n 元函数，$\mathbf{x}_0 \in \mathbb{R}^n$，如果$f(\mathbf{x})$在$\mathbf{x}_0$的某邻域内具有二阶连续偏导数，则对于点$\mathbf{x}_0$的某邻域内的点$\mathbf{x}$，存在常数$\theta\ (0<\theta<1)$，使得

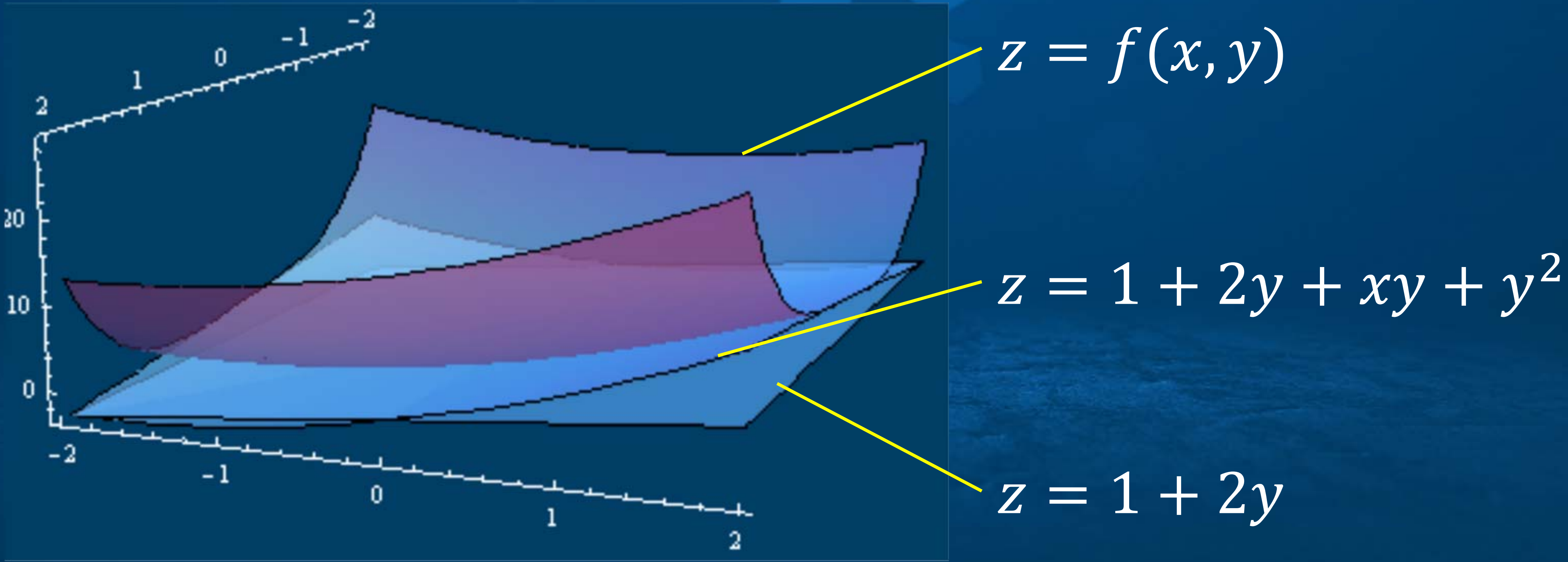
$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)\nabla^2 f(\mathbf{x}_0+\theta(\mathbf{x}-\mathbf{x}_0))(\mathbf{x}-\mathbf{x}_0)^T$$

称上式为$f(\mathbf{x})$在点$\mathbf{x}_0$处的一阶带拉格朗日余项的泰勒公式．

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)\nabla^2 f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T + o(|\mathbf{x}-\mathbf{x}_0|^2)$$

皮亚诺余项

例4 写出函数$f(x,y)=x^4+xy+(1+y)^2$在点(0,0)处的带皮亚诺余项的一阶及二阶泰勒公式．



由$f(\mathbf{x})$在点$\mathbf{x}_0$处带皮亚诺余项的一阶、二阶泰勒公式，分别有$\mathbf{x}_0$某邻域内$\mathbf{x}$的函数值$f(\mathbf{x})$的近似计算公式：

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T$$

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)\nabla^2 f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T$$

二元函数的情形

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0, y_0) + f_x'(x_0, y_0)\Delta x + f_y'(x_0, y_0)\Delta y$$

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0, y_0) + f_x'(x_0, y_0)\Delta x + f_y'(x_0, y_0)\Delta y + \frac{1}{2}[f_{xx}''(x_0, y_0)\Delta x^2 + 2f_{xy}''(x_0, y_0)\Delta x\Delta y + f_{yy}''(x_0, y_0)\Delta y^2]$$

例5 分别使用一阶和二阶泰勒公式近似计算$1.1^{1.8}$的值(其保留11位有效数字的近似值为1.1871533798...)．

【例5解】 令$f(x,y)=x^y$, $x_0=1$, $y_0=2$, $\Delta x=0.1$, $\Delta y=-0.2$

$$f'_x(1,2)=yx^{y-1}|_{(1,2)}=2 \qquad f'_y(1,2)=x^y\ln x|_{(1,2)}=0$$

$$f''_{xx}(1,2)=y(y-1)x^{y-2}|_{(1,2)}=2$$

$$f''_{xy}(1,2)=[x^{y-1}+yx^{y-1}\ln x]|_{(1,2)}=1$$

$$f''_{yy}(1,2)=x^y\ln^2 x|_{(1,2)}=0$$

【例5解】 令$f(x,y)=x^y$, $x_0=1$, $y_0=2$, $\Delta x=0.1$, $\Delta y=-0.2$

$f_x'(1,2)=2$ $f_y'(1,2)=0$ $f_{xx}''(1,2)=2$ $f_{xy}''(1,2)=1$ $f_{yy}''(1,2)=0$

$$1.1^{1.8}=f(x_0+\Delta x,y_0+\Delta y)$$

一次近似

$$\approx f(x_0,y_0)+f_x'(x_0,y_0)\Delta x+f_y'(x_0,y_0)\Delta y$$

$$=1+2\cdot 0.1+0\cdot(-0.2)=1.2$$

$$1.1^{1.8}=f(x_0+\Delta x,y_0+\Delta y)$$

二次近似

$\boxed{1.1^{1.8}=1.1871533798\ldots}$

$$\approx 1.2++\frac{1}{2}[f_{xx}''(x_0,y_0)\Delta x^2+2f_{xy}''(x_0,y_0)\Delta x\Delta y+f_{yy}''(x_0,y_0)\Delta y^2]$$

$$=1.2+0.5[2\cdot 0.1^2+2\cdot 1\cdot 0.1\cdot(-0.2)+0\cdot(-0.2)^2]=1.19$$