第81讲 重积分的一般变换

问题：设平面闭区域D由

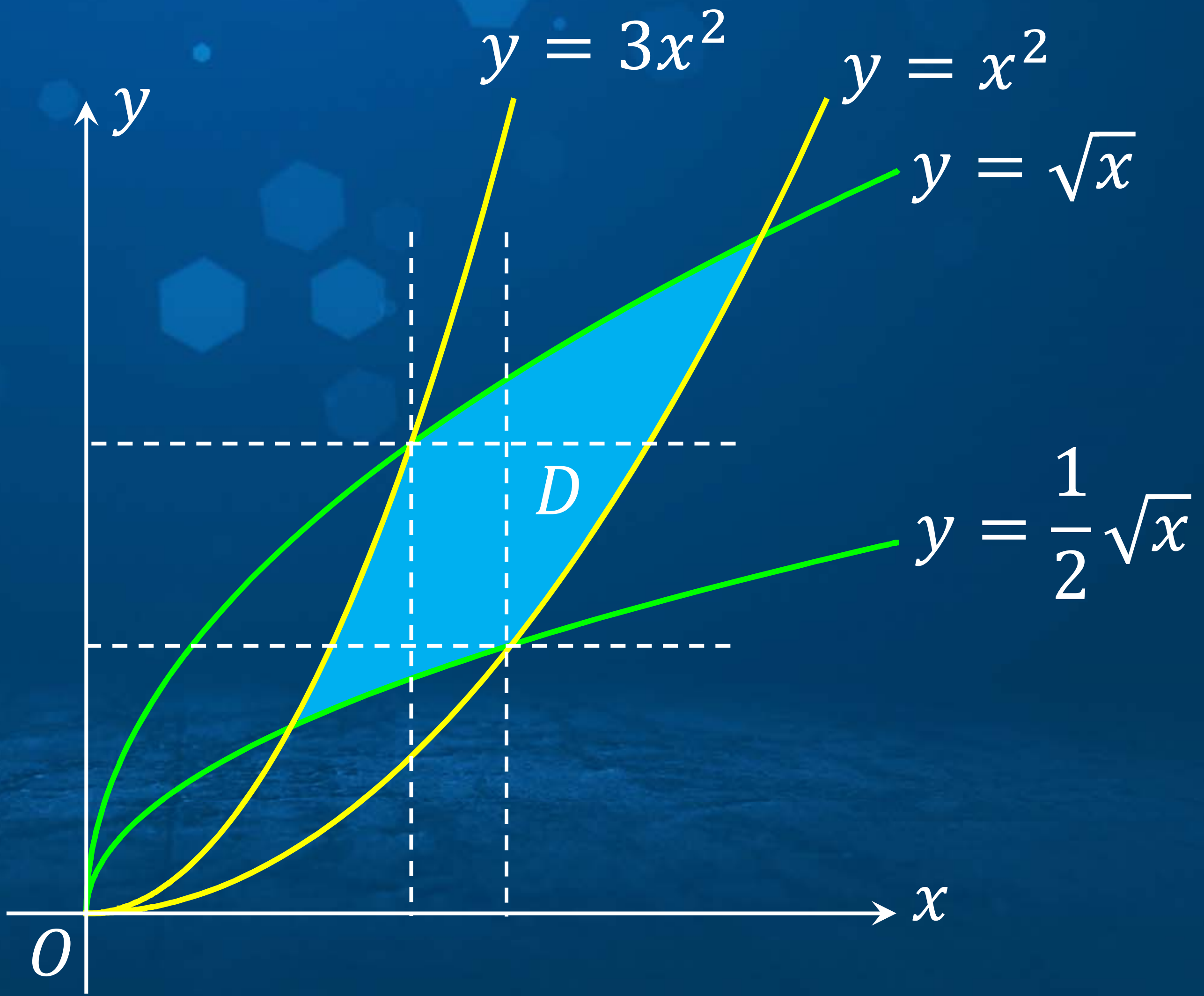
$$y = x^2 \text{ 和 } y = 3x^2$$

以及

$$y = \sqrt{x} \text{ 和 } y = \frac{1}{2}\sqrt{x}$$

所围成，求D的面积S.

$$S = \iint_D \mathrm{d}x\mathrm{d}y$$



用Mathematica计算：

In[10]:= $\text{Integrate}\left[1, \left\{x, \frac{1}{6^{2/3}}, \frac{1}{3^{2/3}}\right\}, \left\{y, \frac{1}{2}\sqrt{x}, 3x^2\right\}\right]+$

$\text{Integrate}\left[1, \left\{x, \frac{1}{3^{2/3}}, \frac{1}{2^{2/3}}\right\}, \left\{y, \frac{1}{2}\sqrt{x}, \sqrt{x}\right\}\right]+$

$\text{Integrate}\left[1, \left\{x, \frac{1}{2^{2/3}}, 1\right\}, \left\{y, x^2, \sqrt{x}\right\}\right]$

Out[10]= $\boxed{\frac{1}{6}}$

- 重积分的一般坐标变换公式
- 广义极坐标与广义球坐标
- 一般变换的例子

定积分换元法

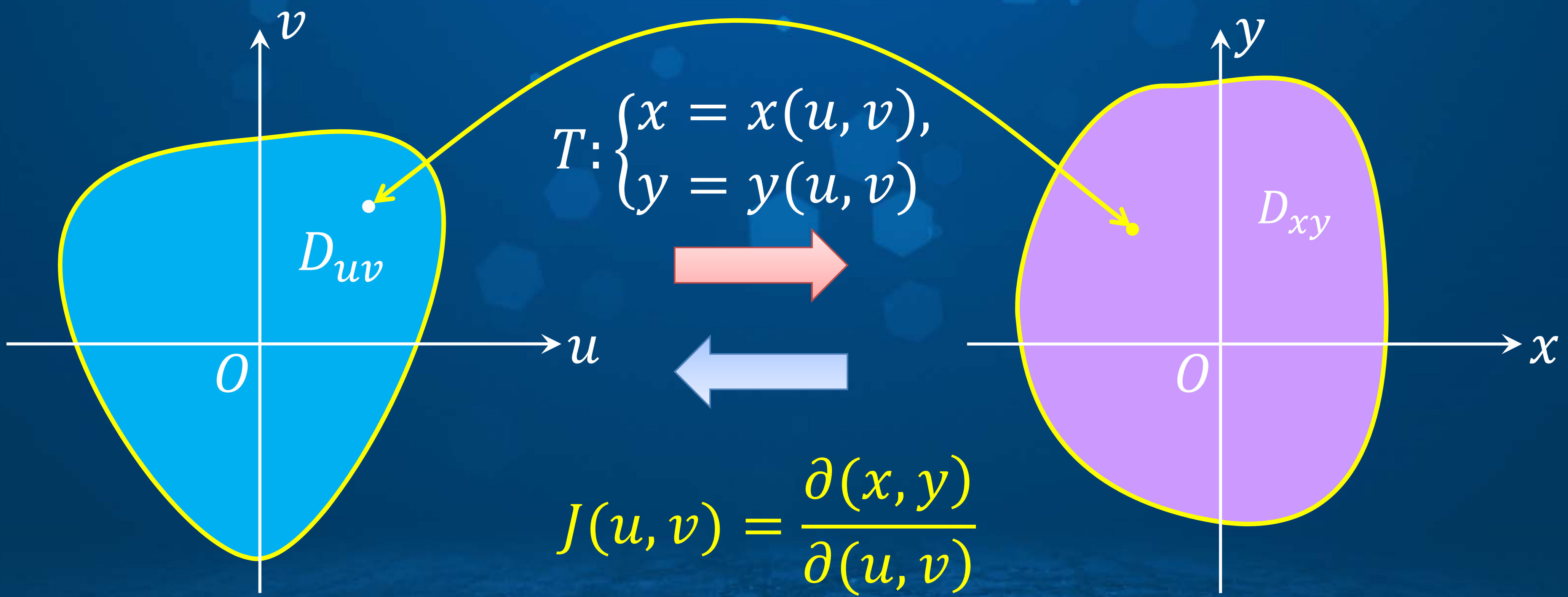
设$\varphi(t)$为$D_t=[\alpha,\beta]$上的单调连续可微函数，$f(x)$在$\varphi(\alpha),\varphi(\beta)$构成的闭区间$D_x$上连续，其中

$$D_x=[\varphi(\alpha),\varphi(\beta)]\text{，如果}\varphi(\alpha)<\varphi(\beta)\text{；}$$

$$D_x=[\varphi(\beta),\varphi(\alpha)]\text{，如果}\varphi(\alpha)>\varphi(\beta)\text{；}$$

则有如下定积分换元公式

$$\int_{D_x} f(x)\,\boxed{\mathrm{d}x} \overset{x=\varphi(t)}{=} \int_{D_t} f[\varphi(t)]\,\boxed{|\varphi'(t)|\mathrm{d}t}$$



$$T:\begin{cases}x = x(u,v),\\ y = y(u,v)\end{cases}$$

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}$$

$$\iint_{D_{xy}} f(x,y)\,\boxed{\mathrm{d}x\mathrm{d}y} = \iint_{D_{uv}} f\boxed{[x(u,v),y(u,v)]}\,\boxed{|J(u,v)|\mathrm{d}u\mathrm{d}v}$$

$$\int_{D_x} f(x)\,\boxed{\mathrm{d}x} \overset{x=\varphi(t)}{=} \int_{D_t} f\boxed{[\varphi(t)]}\,|\varphi'(t)|\mathrm{d}t$$

定理 设$f(x,y)$在xOy面上的闭区域D_{xy}上连续，一对一的变换

$$T: x = x(u,v), y = y(u,v)$$

将uOv平面上的闭区域D_{uv}变成D_{xy}，且满足

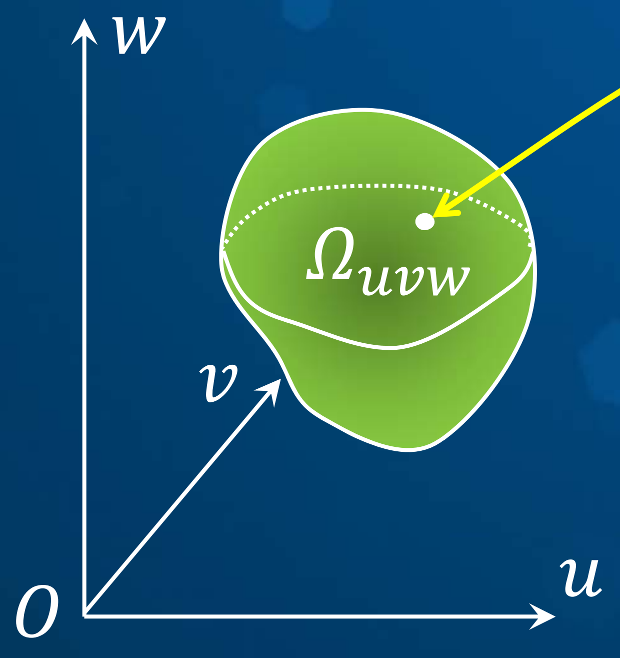
(1) $x(u,v), y(u,v)$在D_{uv}上具有一阶连续偏导数；

(2) 在D_{uv}上雅可比行列式 $J(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$，

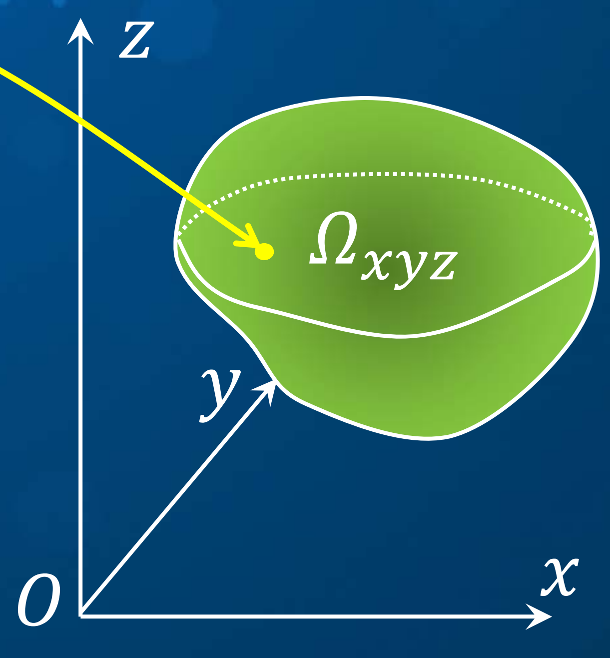
则有

二重积分换元法

$$\boxed{\iint_{D_{xy}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{D_{uv}} f(x(u,v),y(u,v))\,|J(u,v)|\mathrm{d}u\mathrm{d}v}$$



$$T:\begin{cases} x = x(u,v,w), \\ y = y(u,v,w), \\ z = z(u,v,w) \end{cases}$$



$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$$\iiint_{\Omega_{xyz}} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \iiint_{\Omega_{uvw}} f[x(u,v,w), y(u,v,w), z(u,v,w)]|J|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

例如，二重积分直角坐标转化为极坐标时，

$$T: x=\rho\cos\theta\,,\ y=\rho\sin\theta \qquad D_{\rho\theta}\to D_{xy}$$

$$J(\rho,\theta)=\frac{\partial(x,y)}{\partial(\rho,\theta)}=\begin{vmatrix}\cos\theta & -\rho\sin\theta\\ \sin\theta & \rho\cos\theta\end{vmatrix}=\rho$$

$$\iint_{D_{xy}} f(x,y)\mathrm{d}x\mathrm{d}y=\iint_{D_{\rho\theta}} f(\rho\cos\theta,\rho\sin\theta)\,\rho\mathrm{d}\theta\mathrm{d}\rho$$

$$\boxed{\iint_{D_{xy}} f(x,y)\mathrm{d}x\mathrm{d}y=\iint_{D_{uv}} f(x(u,v),y(u,v))\,|J(u,v)|\mathrm{d}u\mathrm{d}v}$$

例如, 三重积分直角坐标转化为球坐标时,

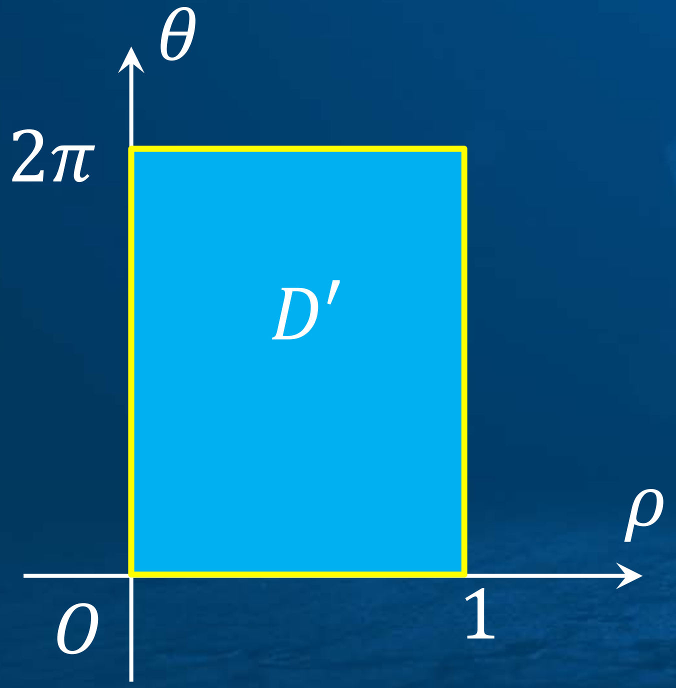
$$T:\begin{cases} x = r\sin\varphi\cos\theta, \\ y = r\sin\varphi\sin\theta, \\ z = r\cos\varphi. \end{cases}$$

$$\Omega_{r\varphi\theta} \to \Omega_{xyz}$$

$$J(r,\varphi,\theta) = \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = \begin{vmatrix} x'_r & x'_\varphi & x'_\theta \\ y'_r & y'_\varphi & y'_\theta \\ z'_r & z'_\varphi & z'_\theta \end{vmatrix} = r^2\sin\varphi.$$

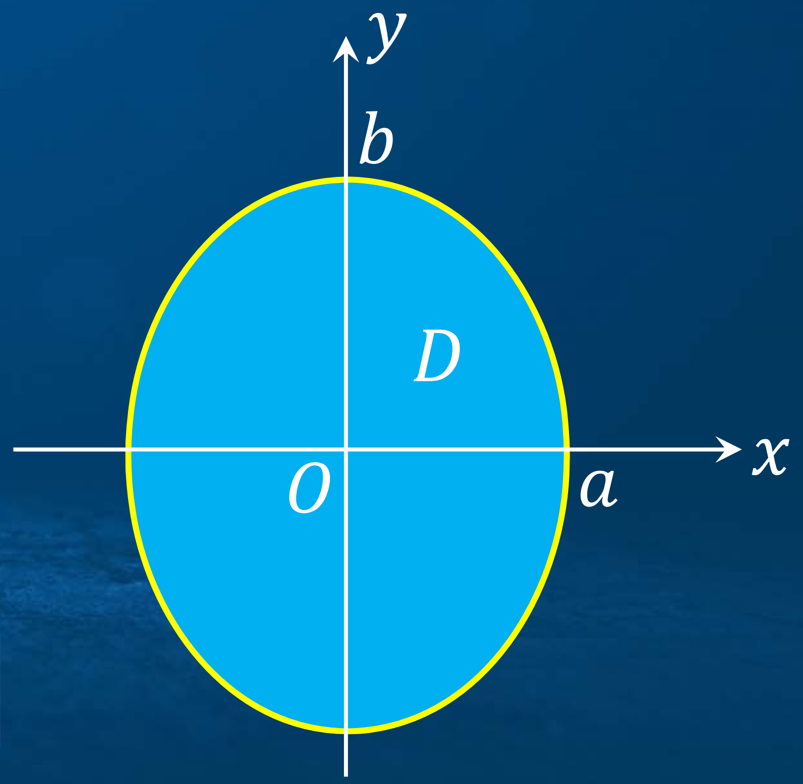
$$\iiint\limits_{\Omega_{xyz}} f(x,y,z)\mathrm{d}x\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{\Omega_{r\varphi\theta}} F(r,\varphi,\theta)\, r^2\sin\varphi\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$$

例1 计算 $I=\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,\mathrm{d}x\mathrm{d}y$，其中积分区域为 $D:\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1.$

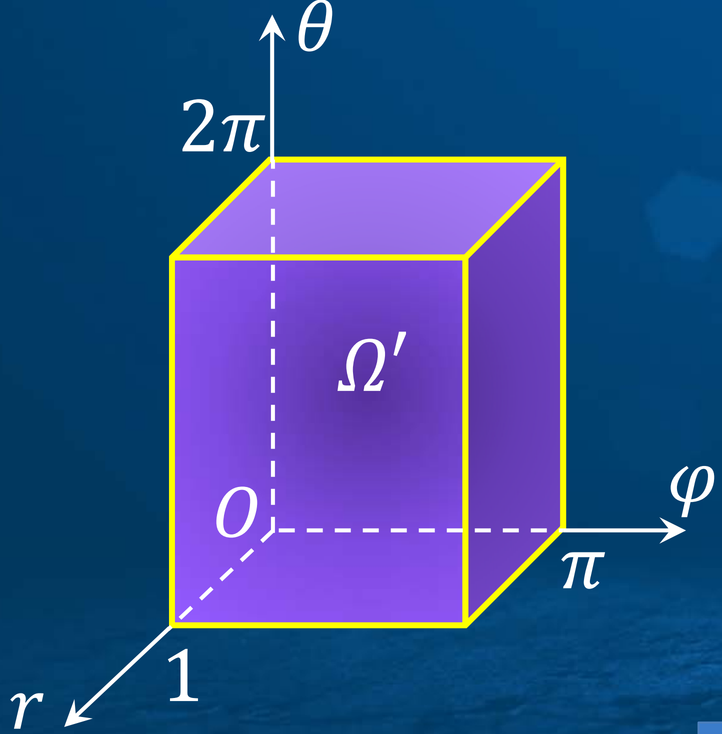


$$T:\begin{cases} x=a\rho\cos\theta, \\ y=b\rho\sin\theta \end{cases}$$

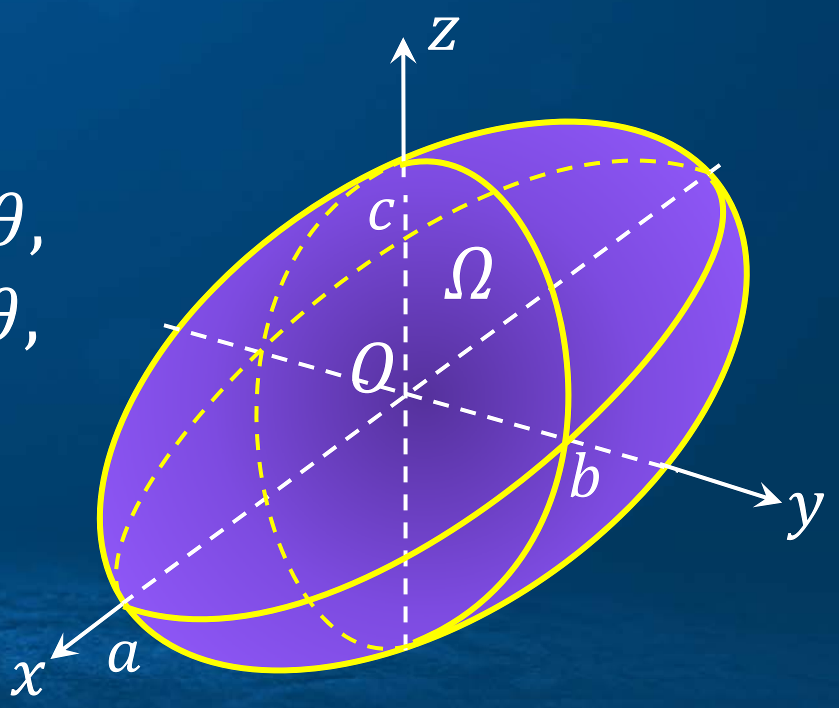
广义极坐标变换



例2 试计算椭球体 $\Omega: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} \le 1$ 的体积 V.

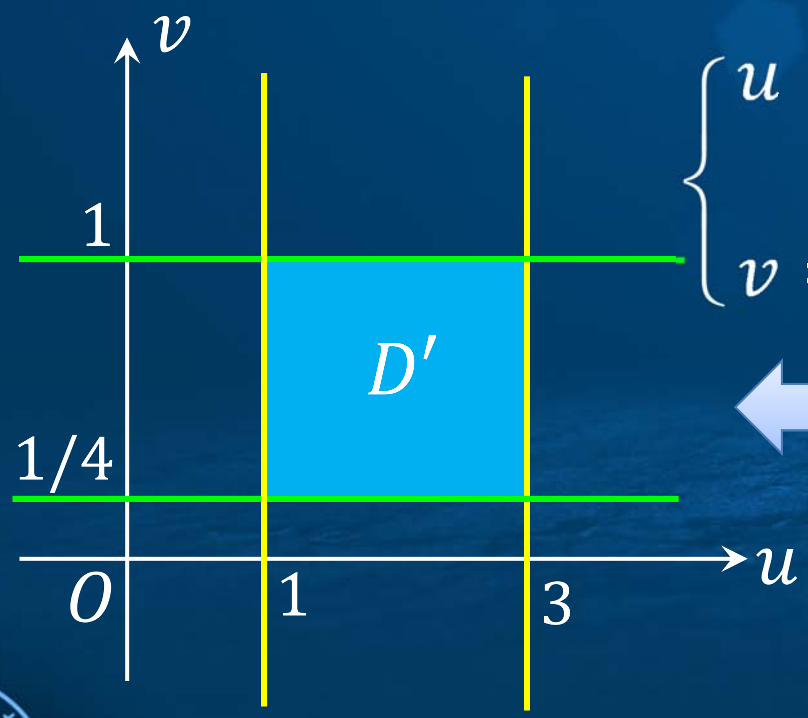


$$\begin{cases} x = ar\sin\varphi\cos\theta, \\ y = br\sin\varphi\sin\theta, \\ z = cr\cos\varphi. \end{cases}$$

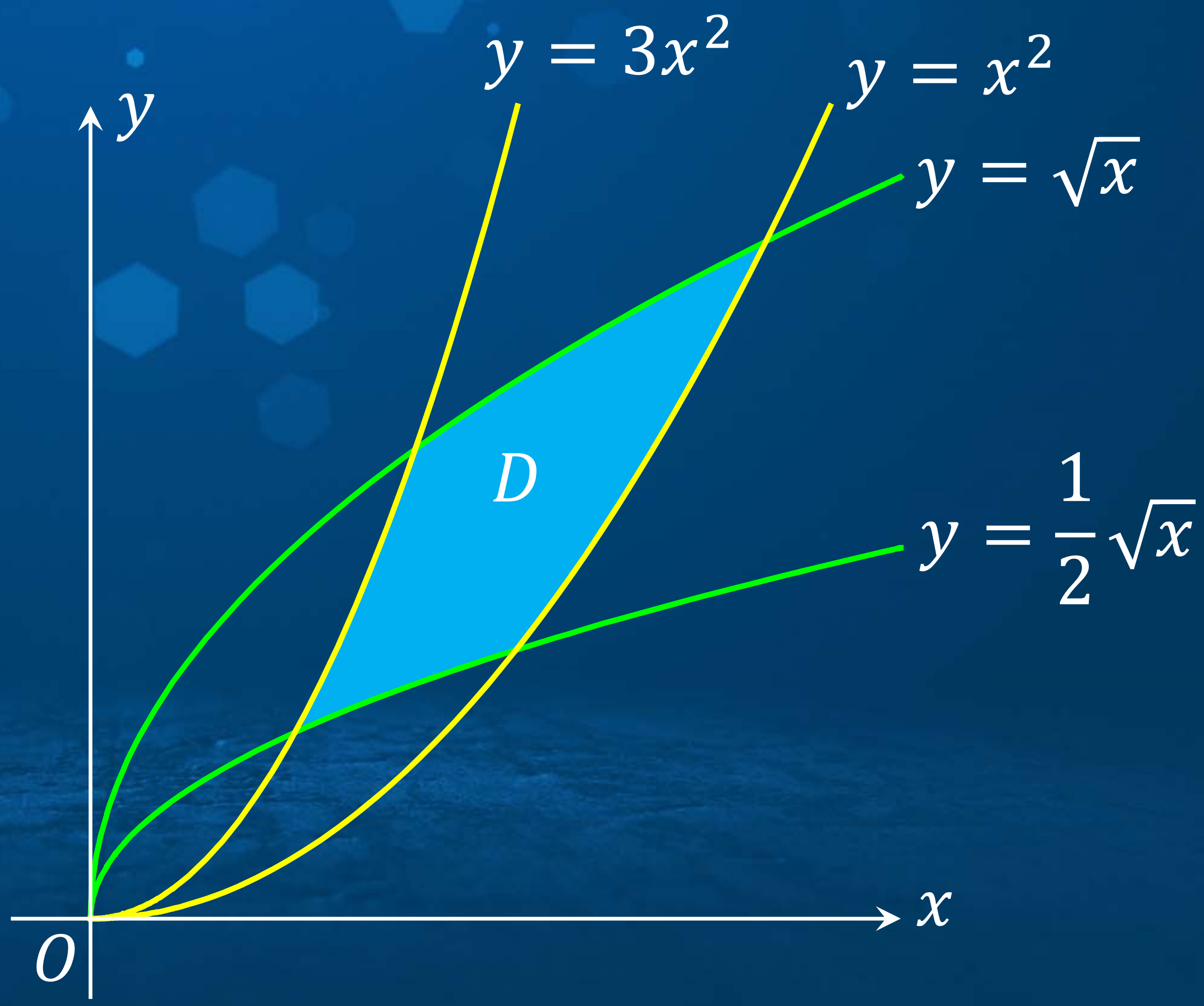


广义球坐标变换

例3 计算抛物线所围成闭区域 D 的面积 S .

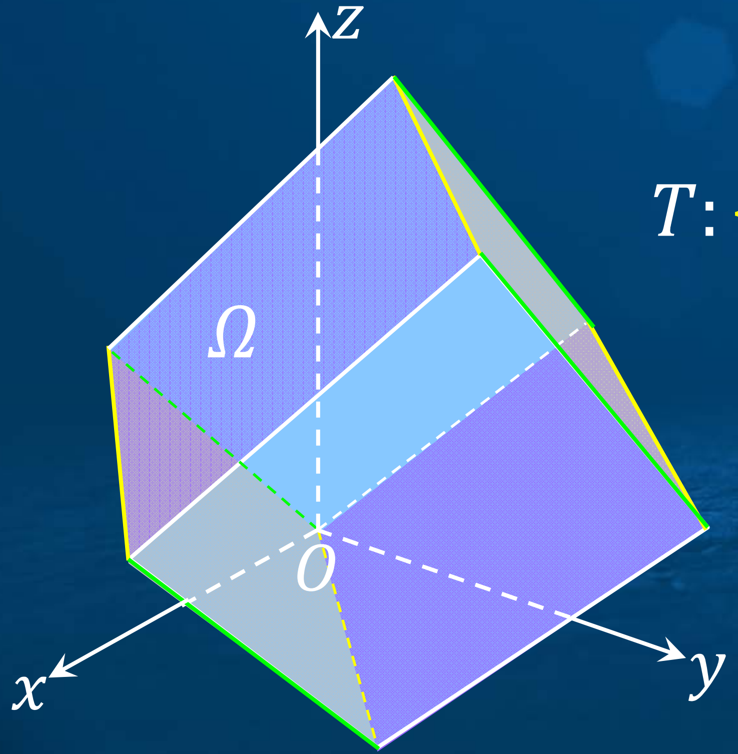


$$\begin{cases} u = \dfrac{y}{x^2} \\ v = \dfrac{y^2}{x} \end{cases}$$



例4 试计算 $I=\iiint_{\Omega}(x+y-z)(-x+y+z)(x-y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ ，

其中 $\Omega: 0\leq x+y-z\leq 1, 0\leq -x+y+z\leq 1, 0\leq x-y+z\leq 1.$



$$T:\begin{cases} u=x+y-z,\\ v=-x+y+z,\\ w=x-y+z \end{cases}$$