《高等数学》全程教学视频课

第67讲 多元复合函数微分法

一元复合函数的求导法则

设$y=f(u),\ u=\varphi(v),\ v=\psi(x)\ \Leftrightarrow\ y=f(\varphi(\psi(x)))$

则 $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}=f'(u)\varphi'(v)\psi'(x)$$

因变量 $y \xrightarrow{\frac{\mathrm{d}y}{\mathrm{d}u}} u \xrightarrow{\frac{\mathrm{d}u}{\mathrm{d}v}} v \xrightarrow{\frac{\mathrm{d}v}{\mathrm{d}x}} x$ 自变量

“依次求导，沿线相乘”

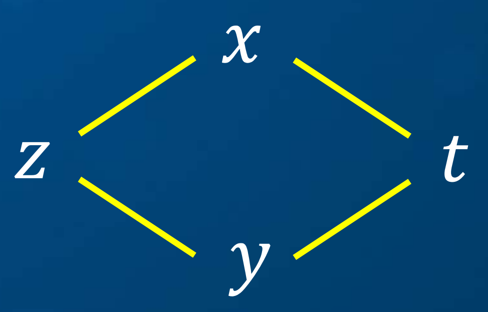
一元复合函数的求导的链式法则

多元复合函数的几种情形：

- 一个自变量的情形：

$z=f(x,y),x=\varphi(t),\ y=\psi(t)$

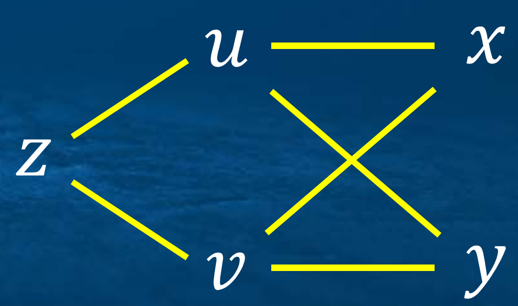
$z=f[\varphi(t),\ \psi(t)]$



- 多个自变量的情形：

$z=f(u,v),u=u(x,y),v=v(x,y)$

$z=f[u(x,y),v(x,y)]$



建立多元复合函数的求导法则！

多元复合函数的求导法则

多元函数一阶微分形式不变性

定理1（一个自变量的情形）设函数 $z=f(x,y)$ 在点 (x,y) 处可微，$x=\varphi(t),\ y=\psi(t)$ 在 t 处可导，则复合函数$z=f[\varphi(t),\ \psi(t)]$ 在点 t 处可导，且有

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$$

或

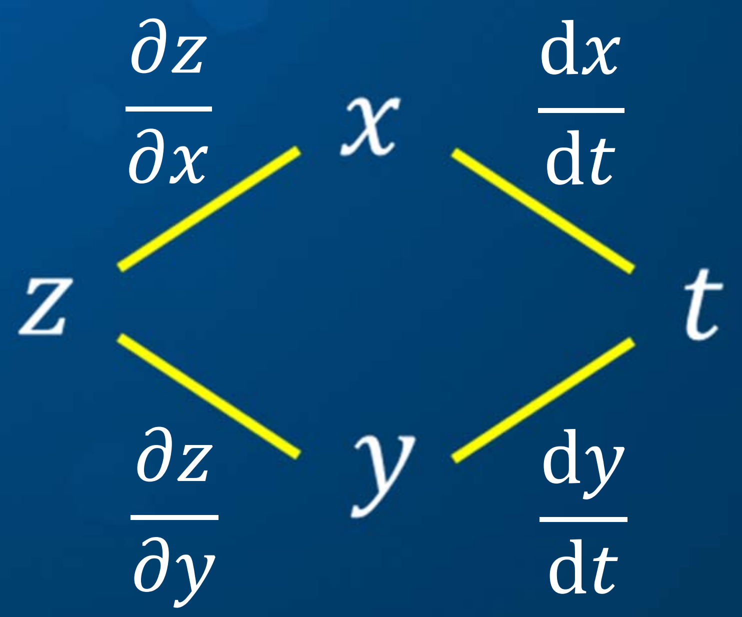
$$\frac{\mathrm{d}z}{\mathrm{d}t}=f_x'(x,y)\cdot\varphi'(t)+f_y'(x,y)\cdot\psi'(t)$$

多元复合函数求导数的链式法则

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}}$$

因变量z到自变量t的路径有：

$$\left.\begin{array}{ll} z\to x\to t & \dfrac{\partial z}{\partial x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t} \\[2ex] z\to y\to t & \dfrac{\partial z}{\partial y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}t} \end{array}\right\}\text{相加得 } \frac{\mathrm{d}z}{\mathrm{d}t}$$



树形图方法

“沿线相乘，分线相加”

定理2（两个自变量的情形） 设函数$u=u(x,y),v=v(x,y)$在点(x,y)处关于x和y的偏导数都存在，函数$z=f(u,v)$在点(x,y)对应的点(u,v)处可微, 则复合函数 $z=f[u(x,y),v(x,y)]$ 在点(x,y)处关于x和y的两个偏导数都存在，且有

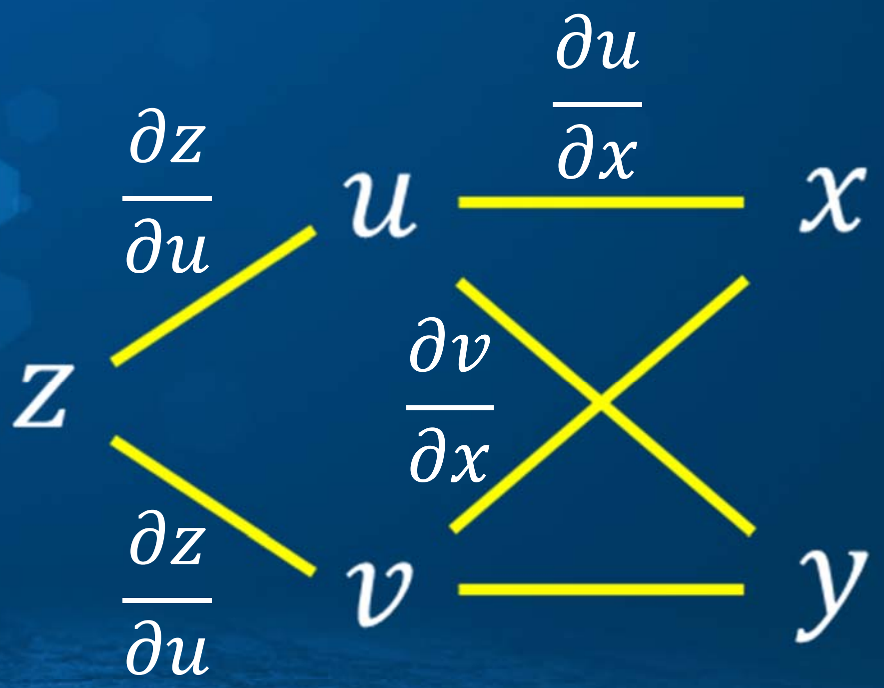
$$\boxed{\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\\ \frac{\partial z}{\partial y}&=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.\end{aligned}}$$

多元复合函数求偏导数的链式法则

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

“沿线相乘，分线相加”



因变量z到自变量x的路径有：

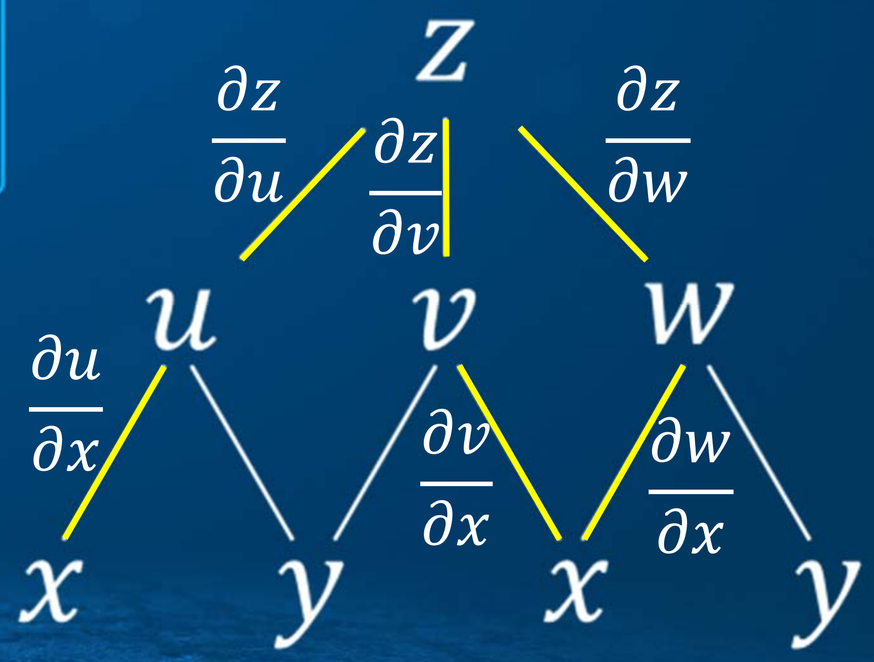
$$\left.\begin{array}{ll} z \to u \to x & \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} \\ z \to v \to x & \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \end{array}\right\} \text{相加得 } \frac{\partial z}{\partial x}$$

其他情形

$$z=f(u,v,w)$$
$$u=u(x,y),\ v=v(x,y),\ w=w(x,y)$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y}$$



例1 用树形图的方法写出下列复合函数的求导结果，设下面所涉及的函数都可微．

(1) $z = f(x,y)$，$y = \varphi(x)$;

(2) $z = f(u)$，$u = \varphi(x,y)$;

(3) $z = f(x,u)$，$u = \varphi(x,y)$.

例2 设$z = e^{u}\cos v, u = 2x - y, v = xy$，求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.

例3 设 $z = f\left(x, x+y, \dfrac{x}{y}\right)$，其中$f$为可微函数，求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.

例4 设函数$z = xy + xf\left(\frac{y}{x}\right)$, 其中$f(u)$为可微函数，证明

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z.$$

例5 设函数$z = f(xy, x^2 - y^2)$，其中f具有二阶连续偏导数，

求 $\frac{\partial^2 z}{\partial x \partial y}$.

一元函数$y=f(x)$中，无论 x 是自变量还是中间变量，都有

$$\mathrm{d}y=f'(x)\mathrm{d}x$$

一阶微分形式不变性

问：多元函数也具有一阶微分形式不变性吗?

二元函数$z=f(u,v)$ 的全微分为：

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

u和v是函数f的自变量

设二元函数$z=f(u,v),u=u(x,y),v=v(x,y)$, 则

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\underline{\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)}\mathrm{d}x+\underline{\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right)}\mathrm{d}y$$

$$=\frac{\partial f}{\partial u}\underline{\left(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\right)}+\frac{\partial f}{\partial v}\underline{\left(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y\right)}=\frac{\partial f}{\partial u}\mathrm{d}u+\frac{\partial f}{\partial v}\mathrm{d}v$$

$$\mathrm{d}z=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y=\frac{\partial f}{\partial u}\mathrm{d}u+\frac{\partial f}{\partial v}\mathrm{d}v$$ u和v是函数f的中间变量

全微分形式不变性

例6 设函数$z=f(3x+2y,x^2+y^2)$，其中f为可微函数，先用微分运算法则求z的全微分，再写出$\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial y}$的表达式．

【例6解】 根据一阶全微分形式的不变性有

$$\begin{aligned}\mathrm{d}z&=f'_1\cdot\mathrm{d}(3x+2y)+f'_2\cdot\mathrm{d}(x^2+y^2)\\&=f'_1\cdot(3\mathrm{d}x+2\mathrm{d}y)+f'_2\cdot(2x\mathrm{d}x+2y\mathrm{d}y)\\&=\underline{(3f'_1+2xf'_2)}\mathrm{d}x+\underline{(2f'_1+2yf'_2)}\mathrm{d}y=\boxed{\frac{\partial z}{\partial x}}\mathrm{d}x+\boxed{\frac{\partial z}{\partial y}}\mathrm{d}y\end{aligned}$$

比较得 $\frac{\partial z}{\partial x}=3f'_1+2xf'_2,\frac{\partial z}{\partial y}=2f'_1+2yf'_2$.