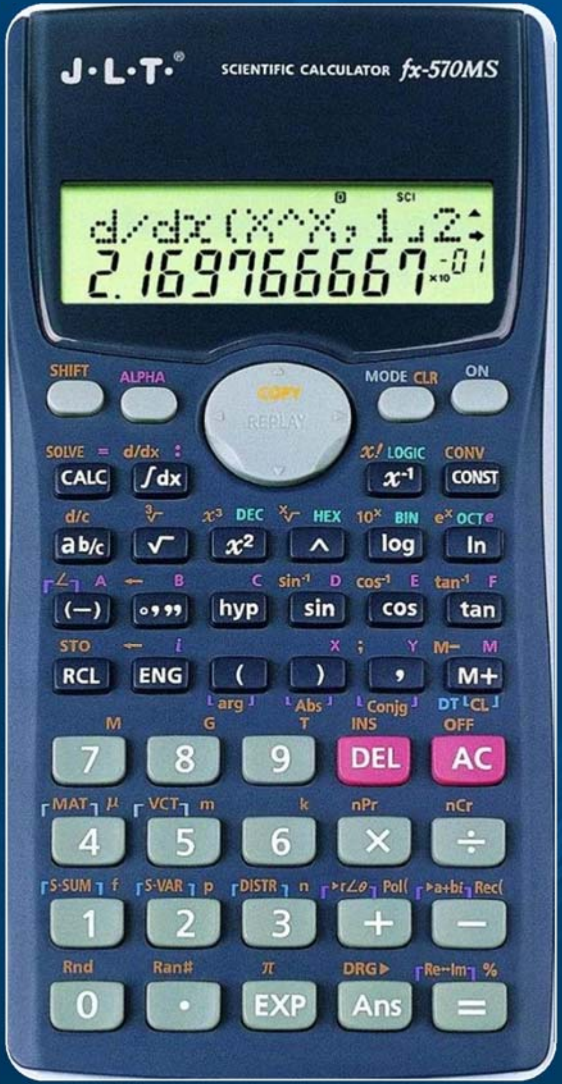
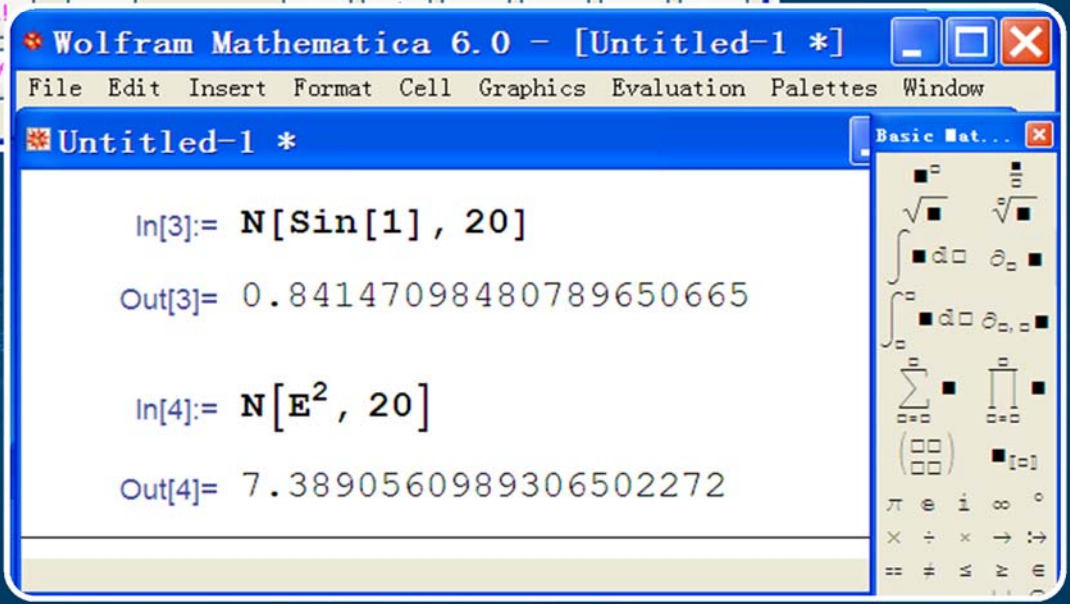
第31讲 函数的多项式逼近





第31讲 函数的多项式逼近——问题的引入

“以直代曲”在近似计算中的应用

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x \qquad \sqrt{1.05}=1.02469507\cdots$$

$$f(x)=\sqrt{x}=\sqrt{x_0+\Delta x}\approx\sqrt{x_0}+\frac{1}{2\sqrt{x_0}}\cdot\Delta x$$

$$\sqrt{1.05}=\sqrt{1+0.05}\approx 1+\frac{1}{2}\cdot 0.05=1.025$$

二项式定理：

$$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots$$

$$\sqrt{x_0+\Delta x}=\sqrt{x_0}\left(1+\frac{\Delta x}{x_0}\right)^{\frac{1}{2}}\approx\sqrt{x_0}\left(1+\frac{1}{2}\cdot\frac{\Delta x}{x_0}\right)=\sqrt{x_0}+\frac{1}{2\sqrt{x_0}}\cdot\Delta x$$

"以曲代曲"在近似计算中的应用

$$\sqrt{x_0+\Delta x}=\sqrt{x_0}\left(1+\frac{\Delta x}{x_0}\right)^{\frac{1}{2}}$$

$$\sqrt{1.05}=1.02469507\cdots$$

$$\approx\sqrt{x_0}[1+\frac{1}{2}\cdot\frac{\Delta x}{x_0}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{\Delta x}{x_0}\right)^2]$$

$$\sqrt{1.05}\approx 1+\frac{1}{2}\cdot 0.05+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}0.05^2=1.0246875$$

$$\approx 1+\frac{1}{2}\cdot 0.05+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}0.05^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}0.05^3$$

$$=1.02469531\cdots$$

函数的多项式逼近

几个初等函数的麦克劳林多项式

逼近效果的图形演示

第31讲 函数的多项式逼近——主要内容

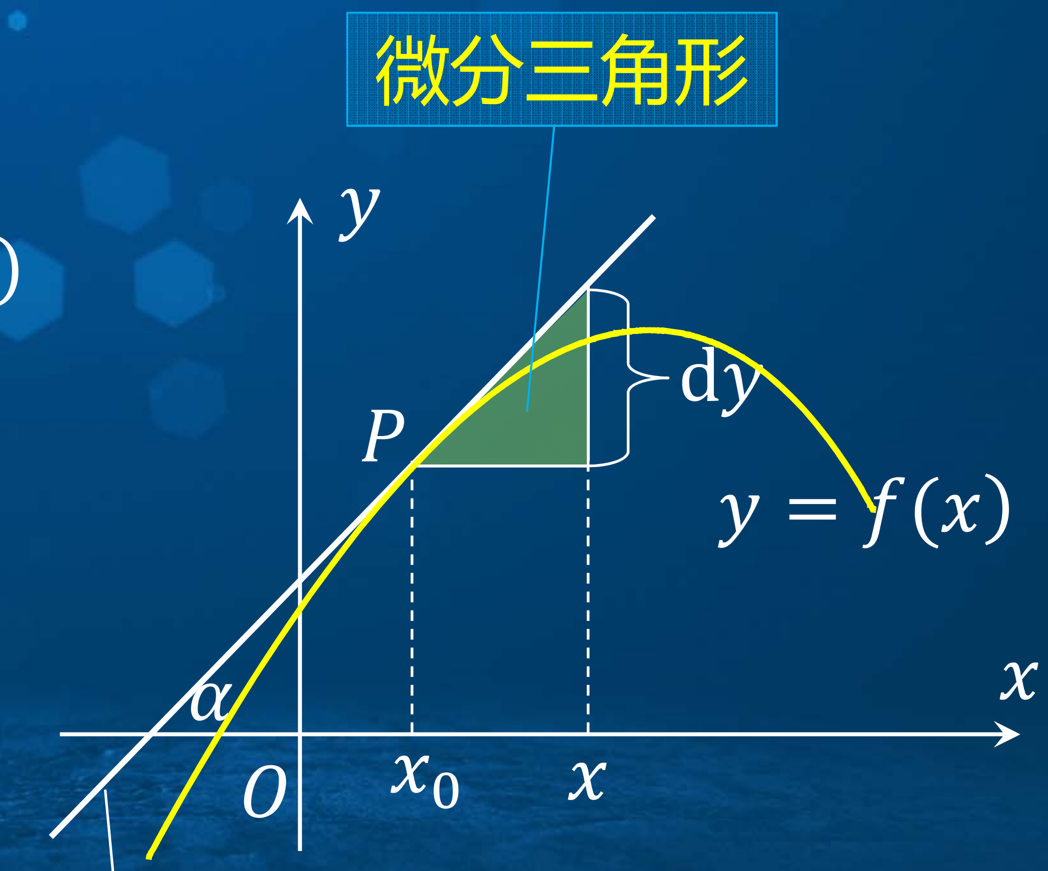
可微函数可以由线性函数逼近(局部线性化)，即

$$f(x) \approx f(x_0) + f(x_0)(x - x_0)$$

几何含义：

用曲线在点$P(x_0, f(x_0))$ 处的切线来近似代替曲线 $y = f(x)$ ，即所谓的“以直代曲”．

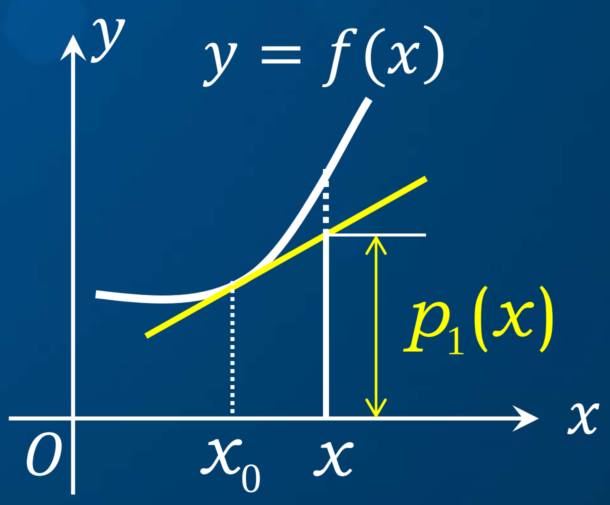
- 精度不高
- 精度不能控制



$$y = f(x_0) + f'(x_0)(x - x_0)$$

$f(x)$与$p_1(x)$的共同特征：

- $p_1(x_0) = f(x_0)$
- $p'_1(x_0) = f'(x_0)$



考虑在点x_0的某邻域内，用一个n次多项式 $p_n(x)$来逼近函数$f(x)$，要求：

$$p_n(x_0) = f(x_0), p_n'(x_0) = f'(x_0), \cdots, p_n^{(n)}(x_0) = f^{(n)}(x_0),$$

试求满足条件的$p_n(x)$.

- 函数$f(x)$在点 x_0处的 n 阶泰勒多项式 条件？

$$p_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

函数 $f(x)$在点 x_0处的泰勒系数

特别地，若 $x_0=0$，则称

$$p_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k$$

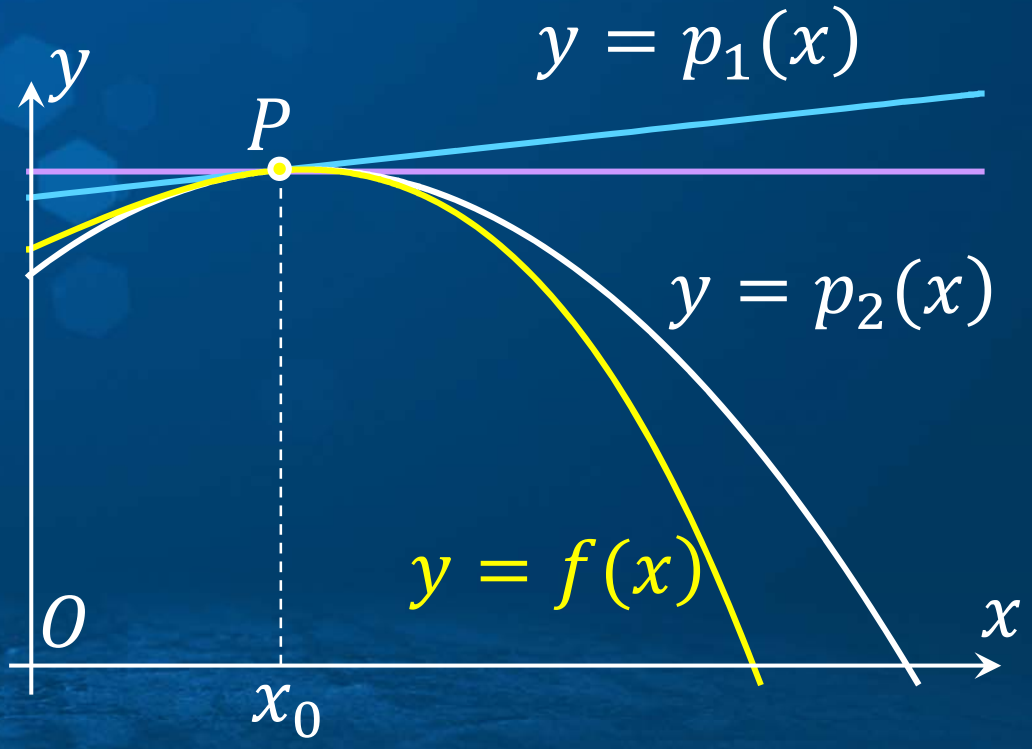
为函数 $f(x)$的 n 阶麦克劳林多项式.

函数$f(x)$在点 x_0处的 0,1,2 阶泰勒多项式

$$p_0(x) = f(x_0)$$

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$



$$\lim_{x\to 0}\frac{f(x) - p_2(x)}{(x - x_0)^2} = 0 \iff f(x) = p_2(x) + o((x - x_0)^2)$$

例1 求函数$f(x)=e^x$的 n 阶麦克劳林多项式.

$$e^x \sim p_n(x)=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$$

例2 求函数$f(x)=\sin x$ 的 n 阶麦克劳林多项式.

$$\sin x \sim p_{2m+1}(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^m\cdot\frac{x^{2m+1}}{(2m+1)!}.$$

例3 求函数$f(x)=\cos x$ 的 n 阶麦克劳林多项式.

$$\cos x \sim p_{2m}(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^m\cdot\frac{x^{2m}}{(2m)!}$$

例4 求函数$f(x)=\ln(1+x)$的 n 阶麦克劳林多项式.

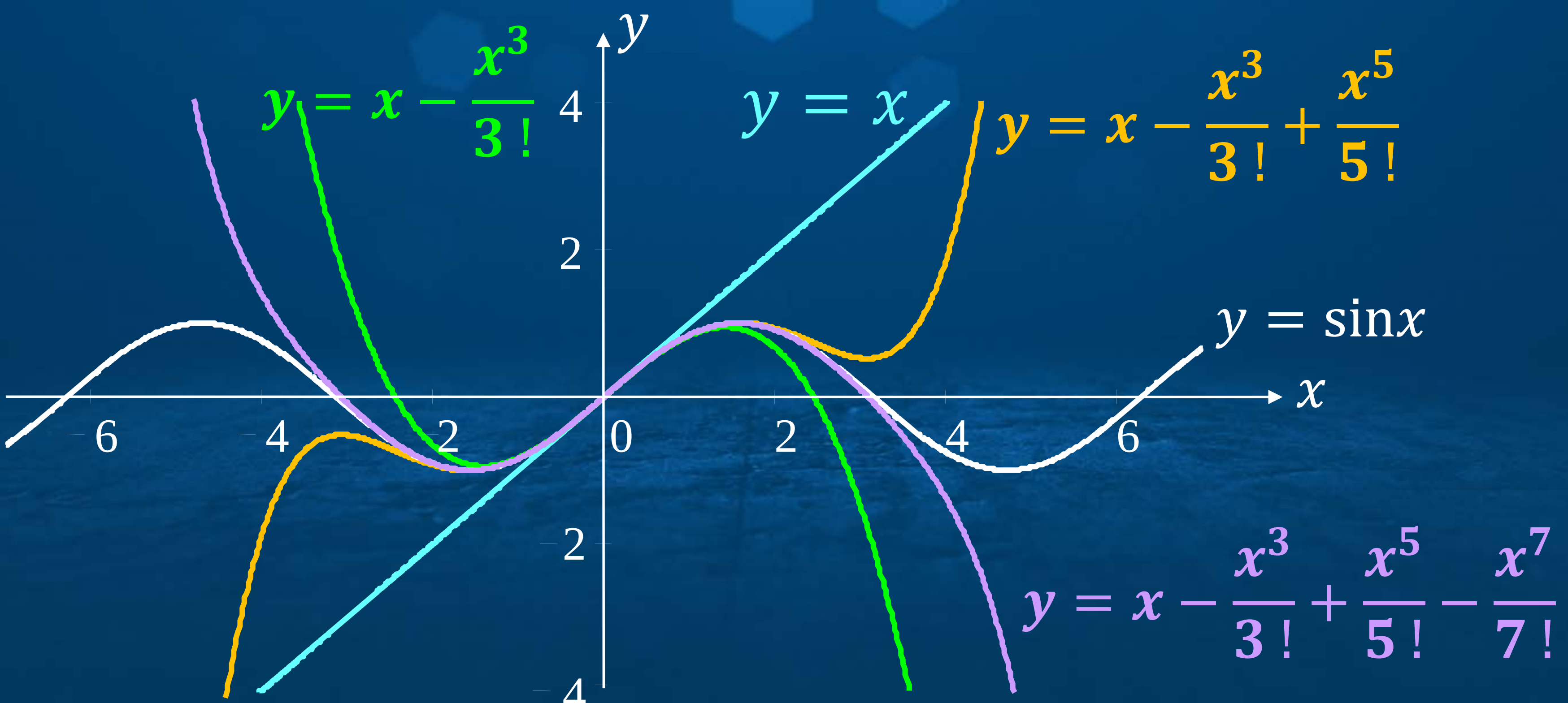
$$\ln(1+x)\sim p_n(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}$$

例5 求函数$f(x)=(1+x)^{\alpha}$的 n 阶麦克劳林多项式.

$$(1+x)^{\alpha}\sim p_n(x)=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

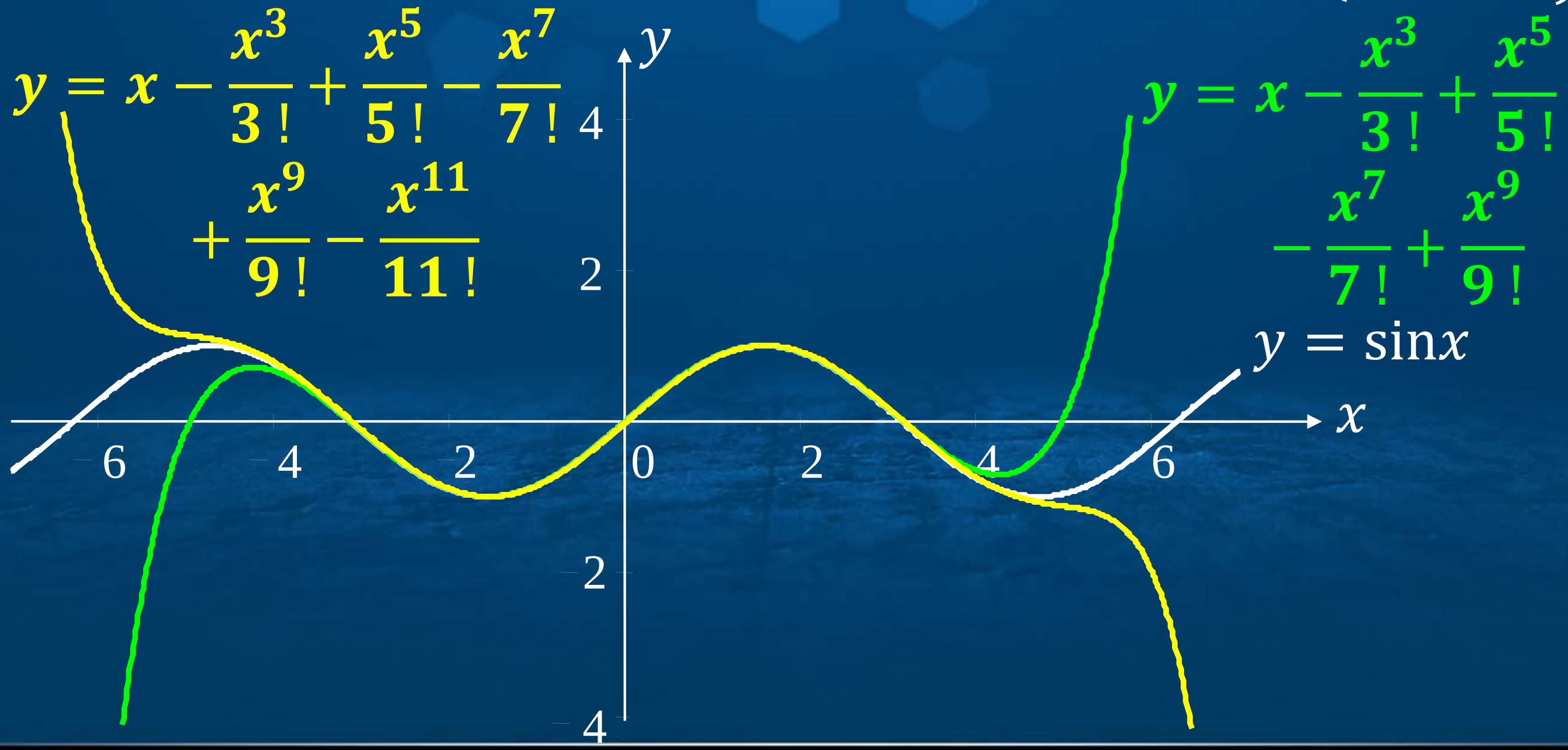
泰勒多项式逼近$\sin x$:

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}$$

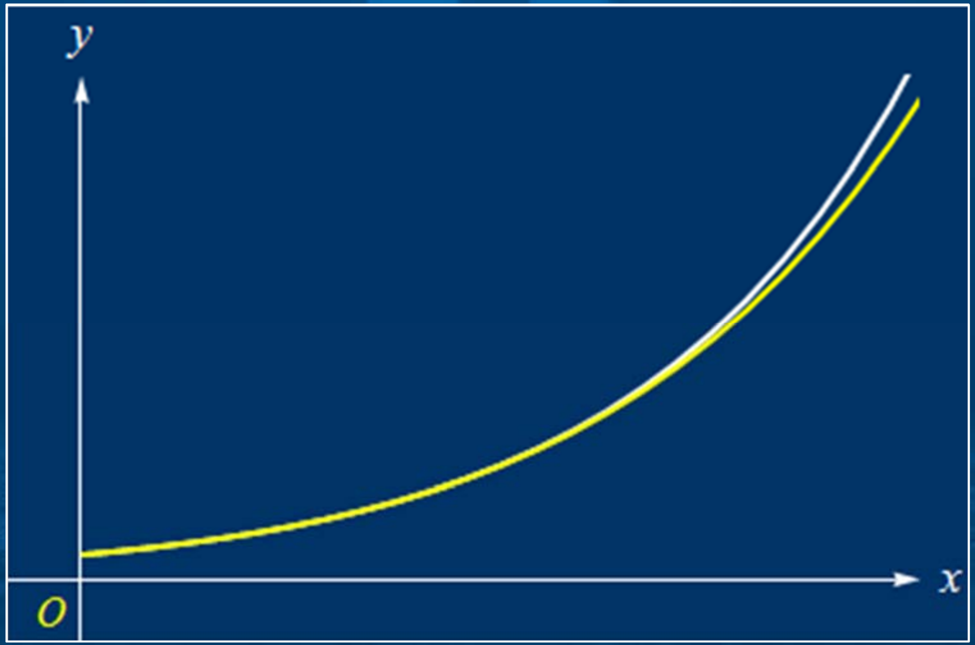


泰勒多项式逼近$\sin x$:

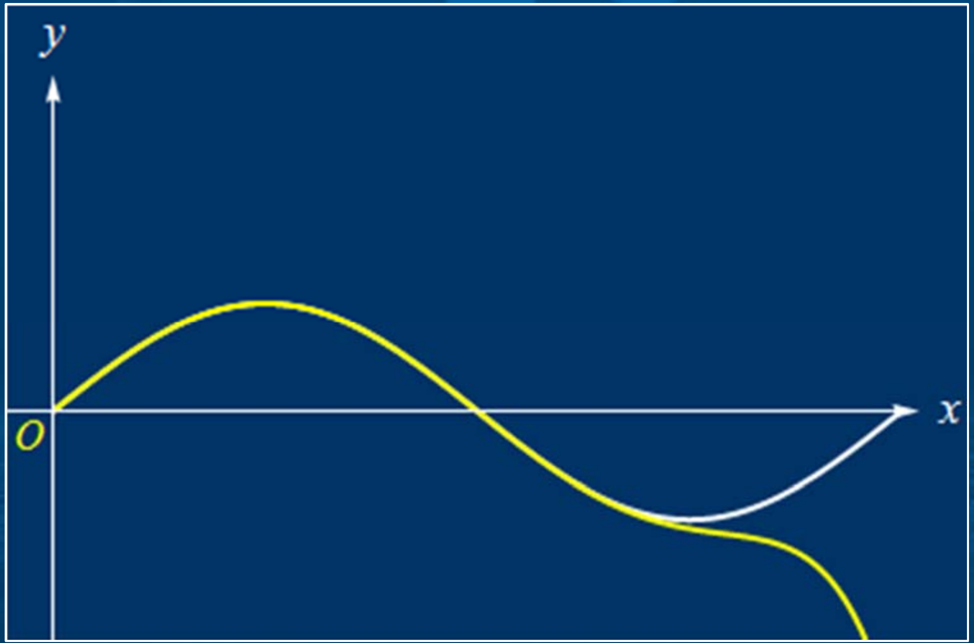
$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}$$



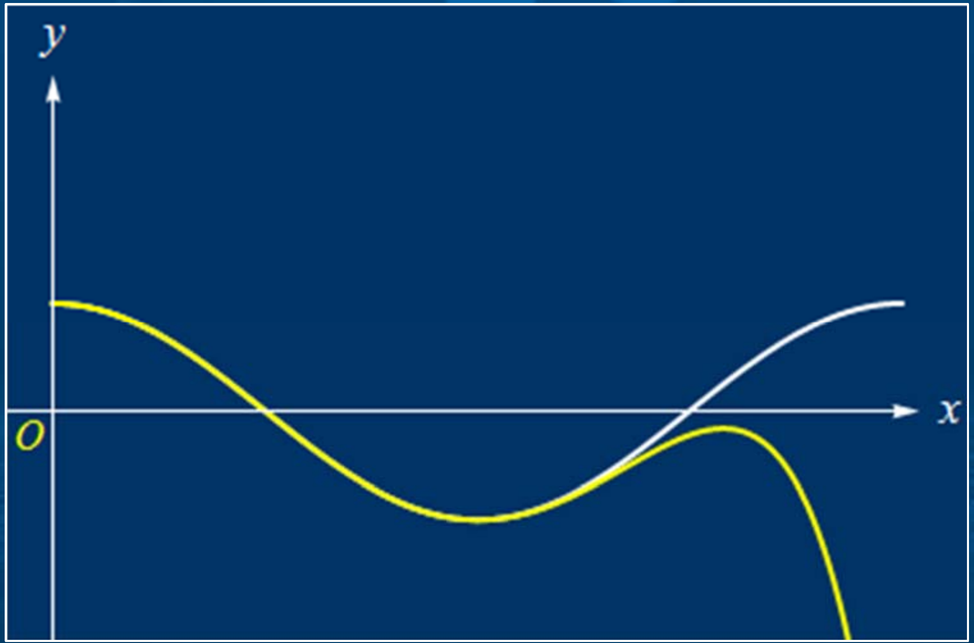
$$e^x \sim p_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$



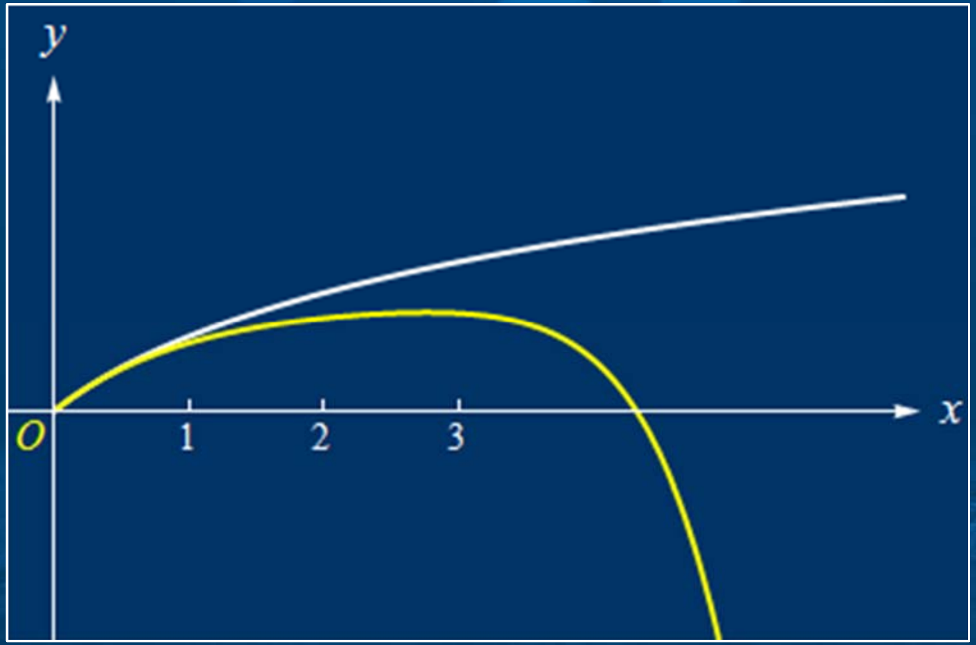
$$\sin x \sim p_{2m+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \cdot \frac{x^{2m+1}}{(2m+1)!}.$$



$$\cos x \sim p_{2m}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \cdot \frac{x^{2m}}{(2m)!}$$



$$\ln(1+x) \sim p_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}$$



$$(1+x)^{\alpha} \sim p_n(x)$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

取$\alpha = \frac{1}{2}$，有