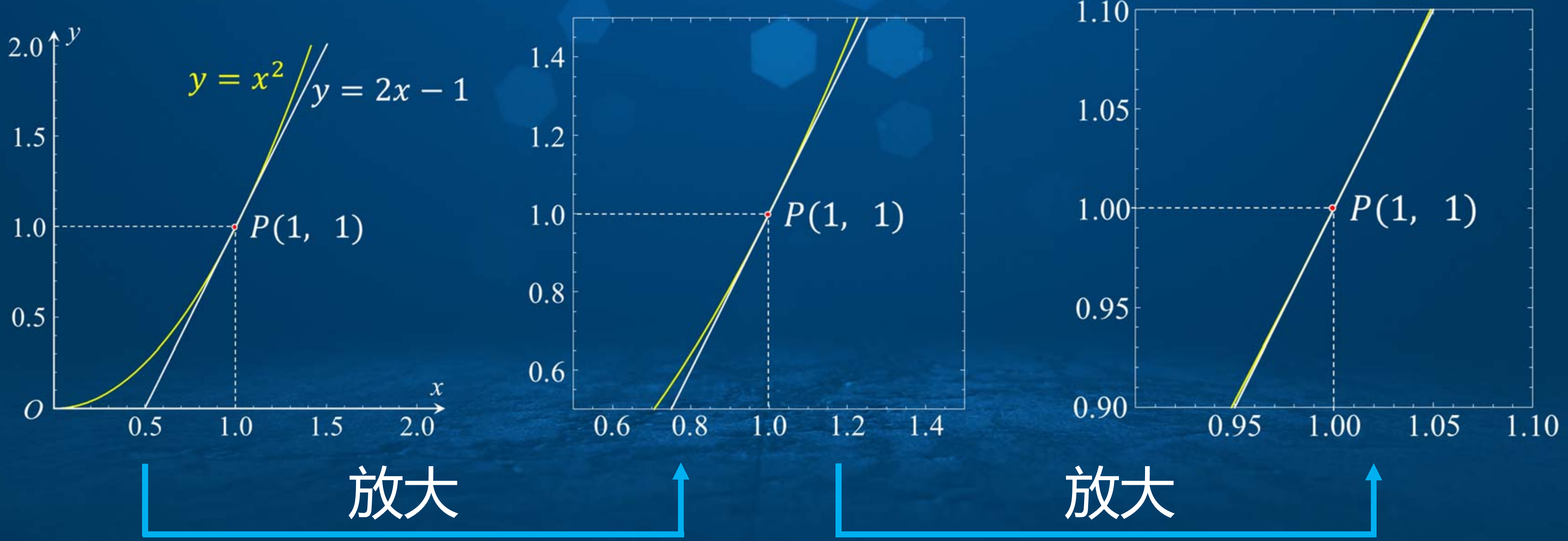
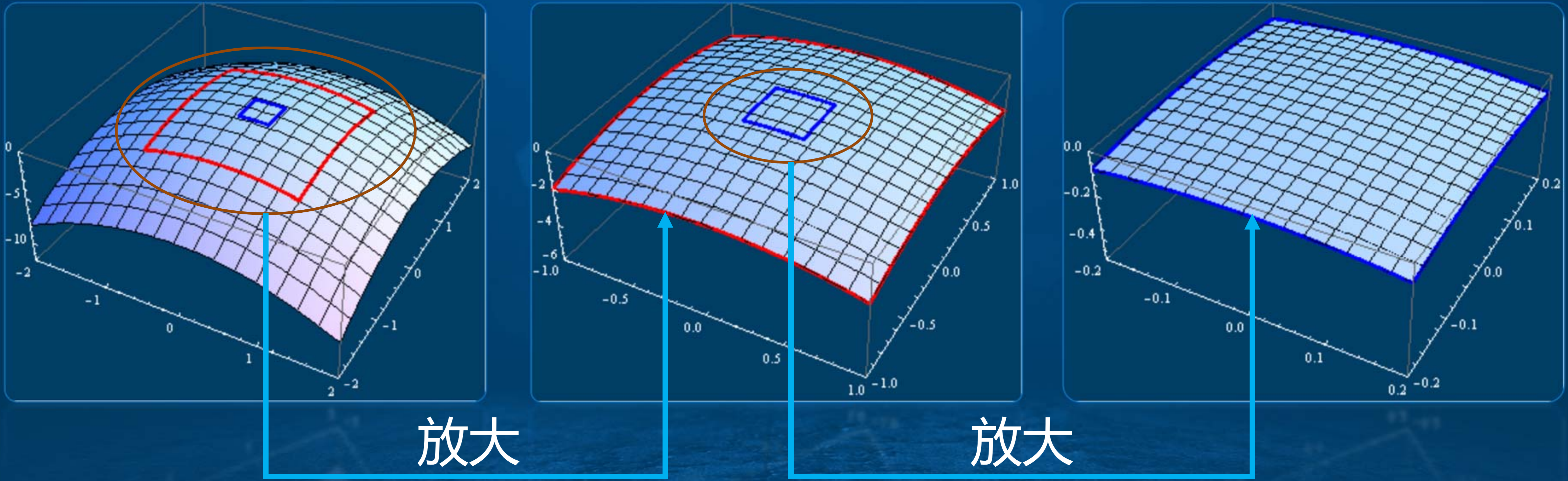
第65讲 全微分的概念

“以直代曲”——以直线近似曲线



“以平代曲”——以平面近似曲面



问题：什么样的曲面可以“以平代曲”？ 如何确定这个平面？

二元函数的局部线性化

二元函数全微分的概念

具体函数可微性的判定

一元函数微分的概念：

如果$\Delta y = A(x)\Delta x + o(\Delta x)$，则$\mathrm{d}y = A(x)\Delta x$.

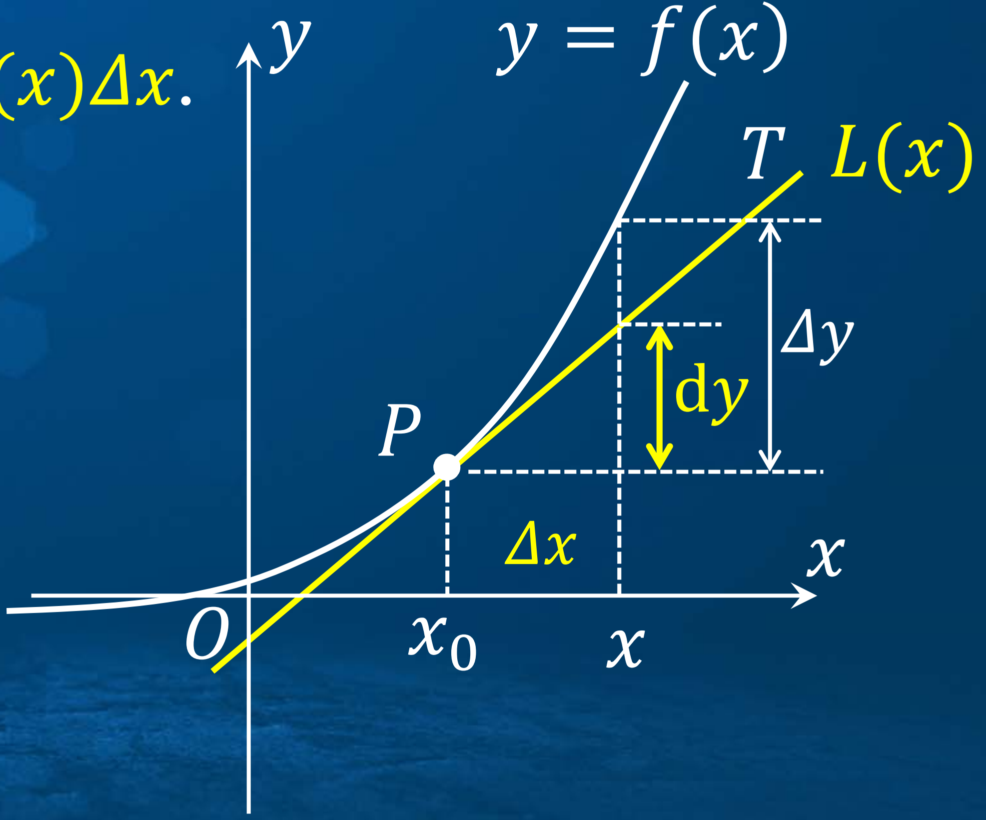
定义表明：

- $\mathrm{d}y$关于Δx是线性的
- 误差为Δx的高阶无穷小量

即$\mathrm{d}y$是Δy的线性逼近

一元函数可微与可导等价

局部线性化函数



$L(x_0+\Delta x) = f(x_0) + f'(x_0)\Delta x \approx f(x_0+\Delta x)$ “以直代曲”

由一元函数增量与微分的关系，对于二元函数$z=f(x,y)$有

$$\boxed{\begin{aligned}\Delta_x z &= f(x+\Delta x,y)-f(x,y)\\ \Delta_y z &= f(x,y+\Delta y)-f(x,y)\end{aligned}} = \boxed{\begin{aligned}f'_x(x,y)\Delta x\\ f'_y(x,y)\Delta y\end{aligned}} \begin{aligned}+o(\Delta x)\\ +o(\Delta y)\end{aligned}$$

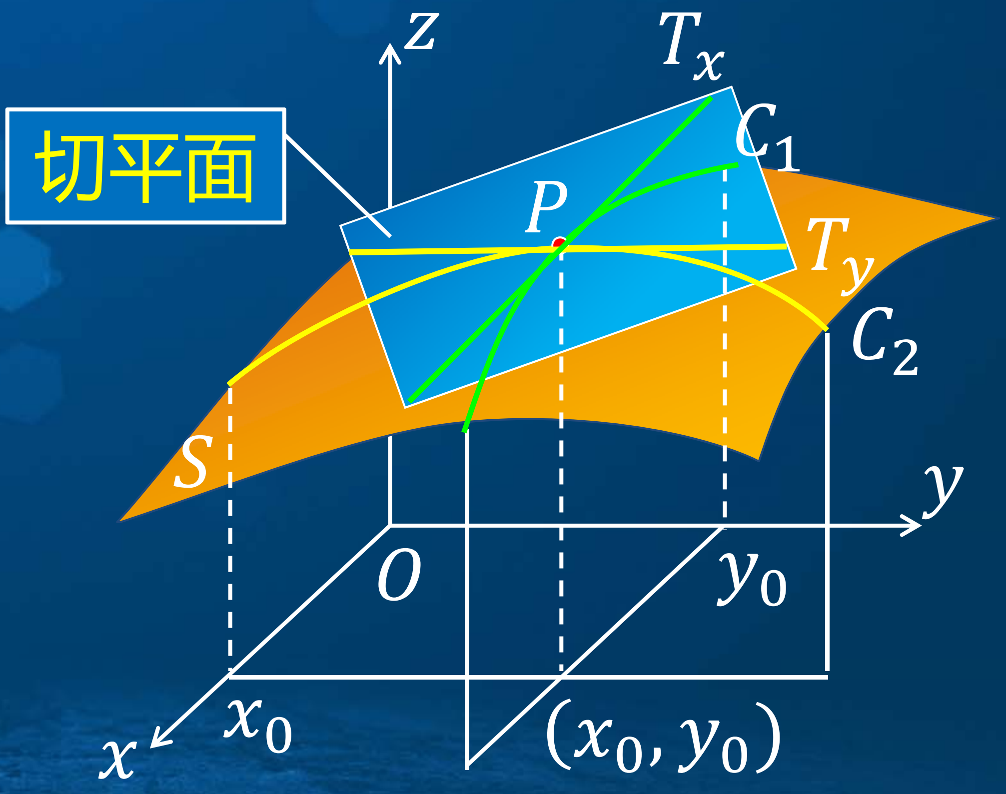
二元函数对x和y的偏增量

二元函数对x和y的偏微分

设$z=f(x,y)$在点$P(x,y)$的某邻域内有定义，$Q(x+\Delta x,y+\Delta y)$为邻域内的任意一点，定义全增量Δz为，即

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$$

设$z=f(x,y)$具有一阶连续偏导数，称由切线T_x和T_y所确定的平面为曲面 S 在点 P 处的切平面.



曲线C_1和C_2的方程分别为：

$$C_1:\begin{cases}x=x,\\ y=y_0,\\ z=f(x,y_0)\end{cases}\qquad C_2:\begin{cases}x=x_0,\\ y=y,\\ z=f(x_0,y)\end{cases}$$

曲线C_1和C_2的切向量分别为：

$$\mathbf{T}_x=(1,0,f_x'(x_0,y_0))\qquad \mathbf{T}_y=\left(0,1,f_y'(x_0,y_0)\right)$$

切平面的法向量可取

$$\mathbf{n} = \mathbf{T}_x \times \mathbf{T}_y = \left(-f_x'(x_0, y_0), -f_y'(x_0, y_0), 1\right)$$

因此曲面$z = f(x, y)$在点$P(x_0, y_0)$处的切平面方程为

$$f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

或 $$z = f(x_0, y_0) + f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0)$$

$f(x, y)$在(x_0, y_0)处的局部线性近似： $L(x, y)$局部线性化函数

$$f(x, y) \approx \underbrace{\boxed{f(x_0, y_0) + f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0)}}_{\parallel\ L(x,y)}$$

例1 求椭圆抛物面 $S: f(x,y)=2x^2+3y^2$在点$P(1,1,5)$处的切平面方程，写出其局部线性化函数$L(x,y)$，并分别计算$f(x,y)$和$L(x,y)$在(1.02,0.98)的函数值.

【例1解】 $f'_x(x,y)=4x,\ f'_y(x,y)=6y$

$$f'_x(1,1)=4,\ \ f'_y(1,1)=6$$

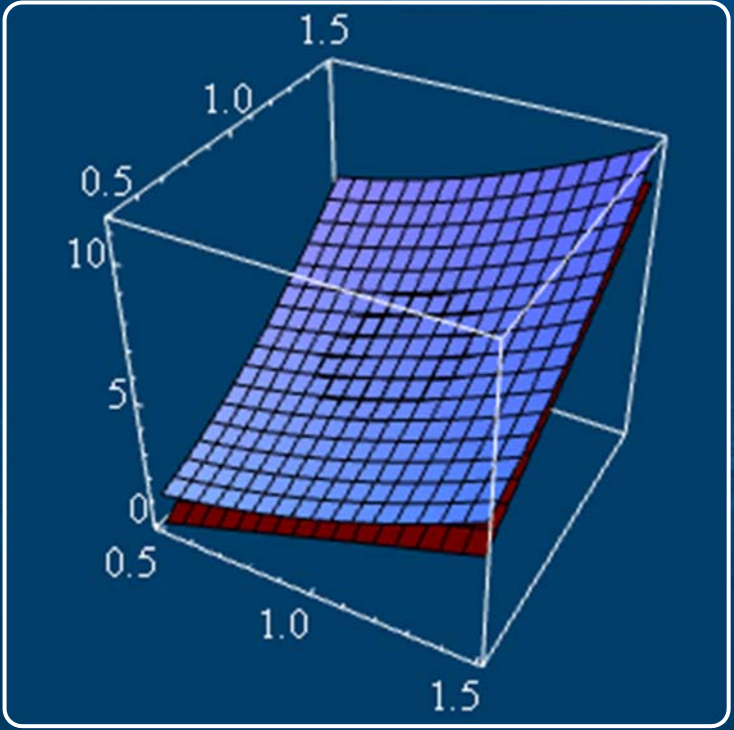
$f(x,y)$在点(1,1)处的局部线性化函数

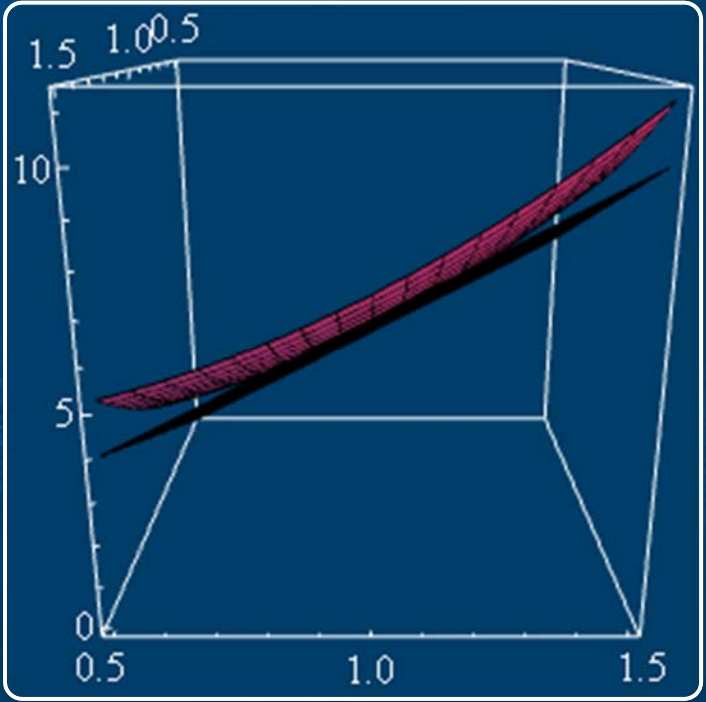
$$L(x,y)=f(1,1)+f'_x(1,1)(x-1)+f'_y(1,1)(y-1)$$

$$=5+4(x-1)+6(y-1)=4x+6y-5$$

$$f(1.02,0.98)=4.962 \qquad L(1.02,0.98)=4.96$$

例1 求椭圆抛物面 $S: f(x,y)=2x^2+3y^2$在点$P(1,1,5)$处的切平面方程，写出其局部线性化函数$L(x,y)$，并分别计算$f(x,y)$和$L(x,y)$在$(1.02,0.98)$的函数值.





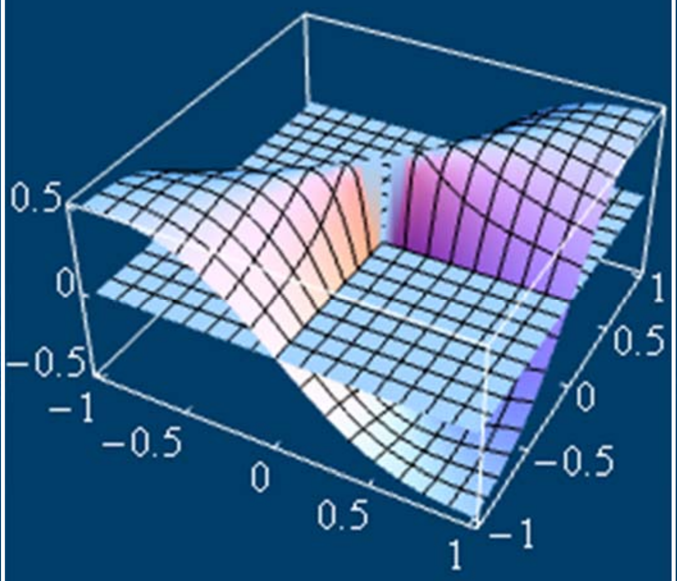
例2 写出函数$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$，在点(0,0)处的局部线性函数$L(x,y)$，讨论使用$L(x,y)$是否可以用它来近似计算(0,0)附近的$f(x,y)$的函数值.

$f(x,y)$在点(0,0)处的局部线性化函数

$$L(x,y)=0$$

$$f(0.1,0.1)=0.5$$

$$f(0.01,0.01)=0.5$$



局部线性近似函数：

$$f(x,y) \approx f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

$$\underbrace{f(x,y) - f(x_0,y_0)}_{\Delta z} \approx \boxed{f'_x(x_0,y_0)\underbrace{(x-x_0)}_{\Delta x} + f'_y(x_0,y_0)\underbrace{(y-y_0)}_{\Delta y}}$$

关于Δx，Δy的线性函数

一元函数 $y = f(x)$ 在点x_0处可微时，$\Delta y = f'(x_0)\Delta x + o(\Delta x)$

猜测：$\Delta z = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y +$ 高阶无穷小

定义1 设函数$z = f(x,y)$在点(x_0,y_0)的某邻域内有定义，且偏导数$f_x'(x_0,y_0)$和$f_y'(x_0,y_0)$均存在，若函数在点(x_0,y_0)的全增量

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

能够表示为

$$\Delta z = f_x'(x_0,y_0)\Delta x + f_y'(x_0,y_0)\Delta y + o(\rho),$$

其中$\rho = \sqrt{(\Delta x)^2+(\Delta y)^2}$，则称$z = f(x,y)$在点$(x_0,y_0)$处可微 (或可微分)，而$f_x'(x_0,y_0)\Delta x + f_y'(x_0,y_0)\Delta y$称为$f(x,y)$在点$(x_0,y_0)$处的全微分，记作

$$\mathrm{d}\, z|_{(x_0,y_0)} = f_x'(x_0,y_0)\Delta x + f_y'(x_0,y_0)\Delta y.$$

$z=f(x,y)$在点(x_0,y_0)可微，即$f(x,y)$在点(x_0,y_0)存在偏导数，且

$$\boxed{\Delta z=f_x'(x_0,y_0)\Delta x+f_y'(x_0,y_0)\Delta y+o(\rho)}$$

$$\Updownarrow$$

$$\boxed{\lim_{\rho\to 0}\frac{\Delta z-[f_x'(x_0,y_0)\Delta x+f_y'(x_0,y_0)\Delta y]}{\rho}=0}$$

其中$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$，$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$

例3 证明函数$f(x,y)=2x^2+3y^2$在$(1,1)$处可微，并求函数在该点处的全微分.

例4 试利用定义验证函数$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0)\end{cases}$

在点$(0,0)$处的不可微.