ANLY601 Assignment1

Shaoyu Feng (sf865)
Collobrator: Mengtong Zhang
Collobrator: Yunjia Zeng

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1 Fundamentals and Review

1. **Exercise 1 (Likelihood Estimation)**
1. What is the maximum likelihood estimate for θ when $X_i \sim$ Geometric(θ)?

Ans: The likelihood function for geometric distribution is given by $L(\theta) = \theta^n(1-\theta)^{\sum_1^n (x_i) - n}$. Take the log of the likelihood function: $ln(L(\theta)) = nln(\theta) + (\sum_1^n x_i - n)ln(1-\theta)$. Take the derivative of the log likelihood function, and let it equal to 0, we have:

$$\frac{d(ln(L(\theta)))}{d\theta} = \frac{n}{\theta} - \frac{\sum_1^n x_i - n}{1-\theta} = 0$$

$$\theta = \frac{n}{\sum_1^n x_i}$$

Therefore the maximum likelihood esimation for θ for geometric distribution is $\frac{1}{X}$

2. What is the maximum likelihood estimate for a and b when $X_i \sim$ Unif (a, b)?

Ans: The likelihood function for uniform distribution with a, b is:

$$L(a, b) = \frac{1}{(b-a)^n}$$

Take the natural log for this likelihood function:

$$ln(L(a, b)) = -nln(b-a)$$

Take the derivative with respect to a and b respectively:

$$\frac{d}{da} ln(L(a, b)) = \frac{n}{b-a}$$

$$\frac{d}{db} ln(L(a, b)) = \frac{-n}{b-a}$$

We can see that the derivative with respect to a is monotonically increasing, So we take the largest a possible which is $a_{MLE} = min(x_i)$. Similarly, We can see that the derivative with respect to b is monotonically decreasing, So we take the smallest possible of b which is $b_{MLE} = max(x_i)$.

2. **Exercise 2 (Loss Function)**

1. Show that squared error loss (L2 loss) is equivalent to the negative log likelihood of a Y $\sim$ N(μ, σ^2) where σ is known.

Ans:
Log likelihood for Gaussian is given by:

$$LL = \sum_{n=1}^{N} log(N(x_n|\mu, \sigma^2) = \sum_{n=1}^{N} log(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}} \frac{(x_n - \mu)^2}{\sigma^2})$$

we then have

$$LL = \sum_{n=1}^{N} (-log(\sqrt{2\pi\sigma^2}) + (-0.5)\frac{(x_n - \mu)^2}{\sigma^2})$$

$$LL = -\frac{N}{2} log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{n-1} N(x_n - \mu)^2$$

From above notation, we can easily see that the negative log likelihood of a Gaussian is same with L2 Loss (given σ is a constant).

2. Show that the mean absolute error (L1 loss) is equivalent to the negative log likelihood of a Y $\sim$ LaPlace(θ)

Ans:
Log likelihood for Laplace isgiven by:

$$LL = \sum_{n=1} Nlog(L(x_n|\mu, \theta)) = \sum_{n=1}^{N} \frac{1}{2\theta} e^(- \frac{|x - \mu|}{\theta}))$$

$$LL = -Nlog(2\theta) - \frac{1}{b} \sum_{n=1}^{N} |x - \mu|$$

Similarly, we can see that L1 loss is equivalent to negative log likelihood of Laplace(θ)

3. **Exercise 3 (Decision Rules)**
Suppose that X has mean μ and variance $\sigma^2 < \infty$ show that:

1. Show that the mean is optimal decision rule for the mean squared error when the decision rule is unbiased

Ans:

For mean square error, loss is written as :

$$L(\theta, \delta(X)) = E[(\theta - \delta(X))^2] = VAR(\delta(X)) + E(\delta(X) - \theta)^2$$

If the decision rule is unbiased, the second terms on the right hand side is 0, we can simply choose mean value as a decision rule to minimize the loss.

2. Show the median is the optimal decision rule for the mean absolute error.
Ans:
To minimize $E(|X-a|)$, we see that

$$E|X-a| = \int_{-\infty}^{\infty} |x-a|f(x)dx = \int_{-\infty}^{a} -(x-a)f(x)dx + \int_{a}^{\infty}(x-a)f(x)dx$$

$$= \int_{-\infty}^{a} f(x)dx - \int_{a}^{\infty} f(x)dx$$

In order to minimize such loss, we can choose a as median as it will make the loss to be 0.

4. **Exercise 4 (Convexity)**
Suppose Y $\sim$ Bernoulli(p) where p = 1/(1 + exp(-βx)) For a fixed x show that:
1. The cross entropy loss $L(y,p) = -(ylog(p)) + (1-y)log(1-p))$ is convex with respect to β.
Ans:
First order derivative of the loss function is given by:

$$\frac{dL}{d\beta} = \frac{dL}{dp} * \frac{dp}{d\beta} = -(\frac{y}{p} - \frac{1-y}{1-p}) * (\frac{\beta e^{-\beta x}}{(e^{-\beta x}+1)^2})$$

Second order derivative is given by:

$$\frac{d^2L}{d\beta^2} = \frac{x^2 e^{\beta x}}{(e^{\beta x}+1)^2}$$

For the second order derivative, we can see that for a fixed x, the results is always bigger or equal to 0, which satisfies second order theorem of convexity. Thus we say the cross entropy loss is convex with respect to β.

2. The mean squared error loss L(y, p) = (y p)^2 is not convex in β
Ans:
First order derivative of the loss function is given by:

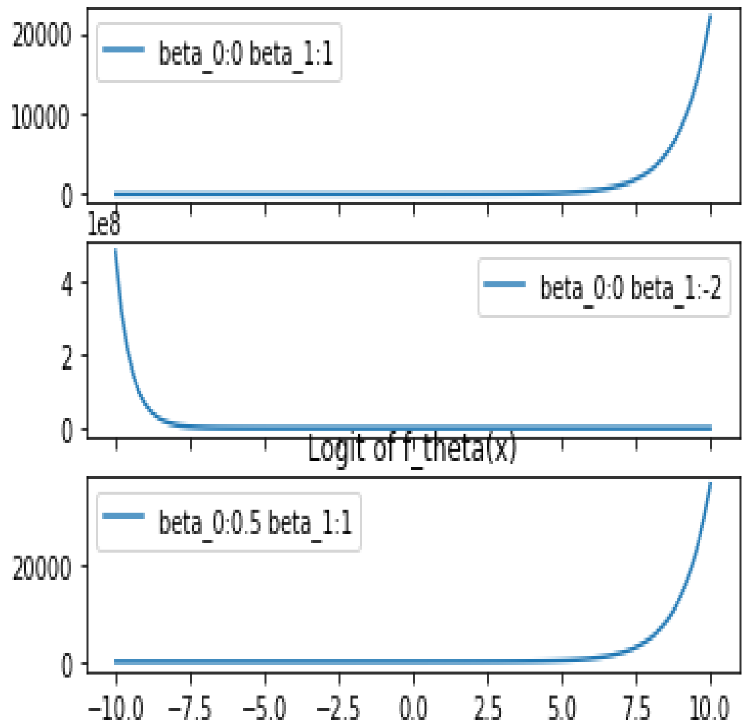
$$\frac{dL}{d\beta} = \frac{dL}{dp} * \frac{dp}{d\beta} = -2(y-p)p(1-p)x$$

Second order derivative is given by:

$$\frac{d^2L}{d\beta^2} = -2[y - 2yp - 2p + 3p^2]x^2 p(1-p)$$

This does not satisfy second order convexity. One counter example is that when y=0, the second order derivative is positive only when p is in range [0, 2/3], this disprove the convexity of Mean squared loss.

5. **Exercise 5 (decision Boundary)**
Ans:
For part2, the logit for $f_\theta(x)$ with different parameter is given by:



We have $logit(f_\theta(x)) = log(\frac{f_\theta(x)}{1-f_\theta(x)}) = log(\frac{1}{exp(-\beta x)}) = \beta x$. Logit function is a monotonous function, we can say that $\theta x = \theta_0 + \theta_1 x$ is a linear seperating hyperplane.

2 Parametric learning

1. **Exercise 6 (Sufficient Statistic))**
Suppose $X_{i=1}^N \sim N(\mu, \sigma^2), \sigma < \infty$ and is known. Show that the sample mean T(X) = $\bar{X}$ is a sufficient statistic for μ.
Ans:

Based on the Factorization Theorem, let $f(x|\theta)$ denote the joint pdf or pmf of a sample X. A statistic is a sufficient statistic for θ if and only if there exists a factorization of the function $f(x|\theta)$ into two functions, h(x) and $g(t|)$ for all sample points x and all parameter points : $f(x|\theta) = g(T(x)|\theta)h(x)$.

In this case, we know the population follows normal with known variance, we have:

$$f(x_1, ..., x_n|\mu) = (2\pi)^{-n/2}\sigma^{-n}exp^{(\frac{-1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2)}$$

$$= (2\pi)^{-n/2}\sigma^{-n}exp^{(\frac{-1}{2\sigma^2}\sum_{i=1}^n x_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^n x_i - \frac{n\mu^2}{2\sigma^2})}$$

Since σ^2 is known, we let:

$$h(x) = 2\pi)^{-n/2}\sigma^{-n}exp^{(\frac{-1}{2\sigma^2}\sum_{i=1}^n x_i^2}$$

and

$$g(r(x_1, x_2, ..., x_n), \mu) = exp^{\frac{\mu}{\sigma^2}r(x_1,x_2,...,x_n) - \frac{n\mu^2}{2\sigma^2}}$$

where

$$r(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i$$

By the factorization theorem this shows that $\sum_{i=1}^{n} x_i$ is a sufficient statistics, It follows that the sample mean is also a sufficient statistic.

2. **Exercise 7 (Ancilliarity)**
   Let $X_{i_{i=1}}^{n}$ be independent and identically distributed observations from a location parameter family with cumulative distribution function $F(x - \theta)$. Show that range of the distribution of $R = max_i(X_i) - min_i(X_i)$ does not depend on the parameter θ.
   Ans:

   Given the fact that X_i is an independent and identically distributed observations from a location parameter family. we then have the fact that$X_1 = Z_1 + \theta, ..., X_n = Z_n + \theta$ and $min_i(X_i) = min_i(Z_i + \theta), maxi(Xi) = max_i(Z_i + \theta)$, where $Z_{i_{i=1}}^{n}$ are independent and identically distributed observations from F(x).
   In such a case, we say that $R = max_i(X_i) - min_i(X_i) = max(Z_i) - min(Z_i)$ is also location invariant, this means that $R = max_i(X_i) - min_i(X_i)$ is ancillary and thus does not depend on the parameter θ.

3. **Exercise 8 (Completeness)**
   Show that $N(\mu, \mu^2)$ has a sufficient statistic but is not complete.
   Ans:
   We have

$$f(x_1, ..., x_n|\mu) = (2\pi)^{-n/2}\mu^{-n}exp^{(\frac{-1}{2\mu^2}\sum_{i=1}^{n}(x_i-\mu)^2)}$$
$$= (2\pi)^{-n/2}\mu^{-n}exp^{(\frac{-n}{2|\mu|}(x_i-\mu)^2)}exp^{-\frac{s^2}{2\mu^2}}$$

   It is trivial to say that $(\bar{x}, s^2)$ is a sufficient statistic. Then we have $h(T) = \bar{x}^2 - \frac{n+1}{n}s^2$.

$$E(h(T)) = E((\bar{x})))^2 + Var(\bar{x}) - \frac{n+1}{n}E(s^2) = \mu^2 + \frac{\mu^2}{n} - \frac{n+1}{n}\mu^2 = 0$$

   But h(T) is not travially 0 for all θ

4. **Exercise 9 (Regular exponential family)**
   Show that the Poisson distribution is part of the regular exponential family.
   Ans:
   f_θ us said to be an exponential family if it has following form:

$$f(x|\theta) = h(x)e^{\psi(\theta)T(X)-A(\theta)}$$

   where $A(\theta)$ is the cumulant, T(X) is the sufficient statistics for the parameter. The canonical form can be rewritten as

$$f(X = x|\eta) = h(x)^{\eta T(X)-B(\eta)}$$

   For Poisson distribution, we have probablitue mass function given by:

$$p(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$
$$= \frac{1}{x!}e^{xlog\lambda-\lambda}$$

   In such a case, we show that poisson distribution is an exponential family distribution with $\eta = log\lambda$ and T(x)=x and $B(\eta) = \lambda$ and $h(x) = \frac{1}{x!}$

5. **Exercise 10 (Regular exponential family)**
Ans:
Recall that we have:
$$B(\eta) = log \int_x h(x)e^{\eta T(X)}dx$$
Differentiating with respect to η_i yields
$$\frac{\delta}{\delta\eta_i}B(\eta) = \frac{\int_x T_i(x)h(x)e^{\eta T(X)}dx}{\int_x h(x)e^{\eta T(X)}dx} = E_i[T_i(X)]$$
Let $Z(\eta) = \int_x T_i(x)h(x)e^{\eta T(X)}dx$, differentiating this expression again with respect to η_j, we then have:
$$\frac{\delta^2}{\delta\eta_i\eta_j}B(\eta) = \frac{\int_x T_i(x)T_j(x)h(x)e^{\eta T(X)}dx}{Z(\eta)} - \frac{(\frac{\delta}{\delta\eta_i})(\frac{\delta}{\delta\eta_j})}{Z(\eta)^2}$$
$$= E_\eta[T_i(X)T_j(X)] - E_\eta[T_i(X)]E_\eta[T_j(X)] = Cov_\eta[T_i(X), T_j(X)]$$

6. **Exercise 11 (Delta Method)**
Ans:
We have $X \sim Bernoulli(p)$, then $n\bar{x}$ follows binomial distribution. We then have $E(\bar{x}) = p$ and $var(\bar{x}) = np(1-p)/n^2 = \frac{p(1-p)}{n}$
Based on Delta Method, we say that
$$Var(\bar{x}(1-\bar{x})) = (1-2p)^2 Var(\bar{x}) = \frac{(1-2p)^2(1-p)p}{n^2}$$
Therefore we have approximate distribution for τ is $N(0, (1-2p)^2(1-p)p)$

3 Fundamentals and Review

1. **Exercise 12 (Joint Entropy)**
1.Compute the joint entropy H(X,Y) of X and Y.
Ans:
$$H(X,Y) = \sum_x \sum_y P(x,y) log_2(P(X,y))$$
$$= 2 * 1/4 * log_2(1/4) + 2 * 1/6 * log_2(1/6) + 2 * 1/12 * log_2(1/12) = -2.45$$
2. Find the Marginal distribution of X and the conditional entropy H(Y—X)
Ans:
Marginal Distribution for X is given by: P(X=0)=P(X=1)=P(X=2)=1/3.
For conditional Entropy $P(Y=0|X=0) = 3/4$
$P(Y=0|X=1) = 1/4$
$P(Y=0|X=2) = 1/2$
$P(Y=1|X=0) = 1/4$
$P(Y=1|X=1) = 3/4$
$P(Y=1|X=2) = 1/2$
and Therefore: $H(Y|X=0) = 3/4 * log_2(3/4) + 1/4 * log_2(1/4) = -0.81$
$H(Y|X=1) = 1/4 * log_2(1/4) + 3/4 * log_2(3/4) = -0.81$
$H(Y|X=2) = log_2(1/2) = -1$
$H(Y|X) = 1/3 * (-0.81 * 2 - 1) = -0.87$

3. Verify the entropy results above by using the chain rules that relates $H(X,Y) to H(X) and H(Y|X)$.
Ans:
$H(X) = 1/3 * 3 * log_2(1/3) = -1.58$ Therefore $H(X,Y) = H(X) + H(Y|X)$

2. **Exercise 13 (Differential Entropy)**
Find the differential entropy (this is the continuous version of entropy) of a multivariate normal distribution.
Ans:

$$DiferentialEntropy = -\int_{-\infty}^{+\infty} N(x|\mu,\sum)ln(N(x|\mu,\sum))dx = -E[ln(N(x|\mu,\sum))]$$

$$= -E[ln((2\pi)^{-\frac{D}{2}}|\sum|^{-0.5}e^{-0.5(x-\mu)^T\sum^{-1}(x-\mu)})]$$

$$= \frac{D}{2}ln(2\pi) + \frac{1}{2}ln|\sum| + \frac{1}{2}E[(x-\mu)^T\sum^{-1}(x-\mu)]$$

we have

$$E[(x-\mu)^T\sum^{-1}(x-\mu)] = E[tr((x-\mu)^T\sum^{-1}(x-\mu))] = tr(E[\sum^{-1}(x-\mu)(x-\mu)^T]) = tr(\sum^{-}1\sum) = tr(I) = D$$

$$DiferentialEntropy = \frac{D}{2}ln(2\pi) + \frac{1}{2}ln|\sum| + D$$

where D is the number of dimensions.