EXT 1: Calculus (Ext1), C2 Further Calculus Skills (Y12)

Integration By Substitution (Ext1)

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Exam Equivalent Time: 37.5 minutes *(based on HSC allocation of 1.5 minutes approx. per mark)*

Questions

1. Calculus, EXT1 C2 2013 HSC 5 MC

   Which integral is obtained when the substitution $u = 1 + 2x$ is applied to $\int x\sqrt{1+2x}\,dx$?

   (A) $\frac{1}{4}\int (u-1)\sqrt{u}\,du$

   (B) $\frac{1}{2}\int (u-1)\sqrt{u}\,du$

   (C) $\int (u-1)\sqrt{u}\,du$

   (D) $2\int (u-1)\sqrt{u}\,du$

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2. Calculus, EXT1 C2 2011 HSC 1d

   Using the substitution $u = \sqrt{x}$, evaluate $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. *(3 marks)*

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3. Calculus, EXT1 C2 2012 HSC 11d

   Use the substitution $u = 2 - x$ to evaluate $\int_1^2 x(2-x)^5\,dx$. *(3 marks)*

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4. Calculus, EXT1 C2 2015 HSC 11e

   Use the substitution $u = 2x - 1$ to evaluate $\int_1^2 \frac{x}{(2x-1)^2}\,dx$. *(3 marks)*

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5. Calculus, EXT1 C2 2016 HSC 11b

   Use the substitution $u = x - 4$ to find $\int x\sqrt{x-4}\,dx$. *(3 marks)*

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6. Calculus, EXT1 C2 2017 HSC 11e

   Evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$. *(3 marks)*

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7. Calculus, EXT1 C2 2019 HSC 13a

   Use the substitution $u = \cos^2 x$ to evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4 + \cos^2 x}\,dx$. *(3 marks)*

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8. Calculus, EXT1 C2 2013 HSC 11f

   Use the substitution $u = e^{3x}$ to evaluate $\int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x}+1}\,dx$. *(3 marks)*

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9. Calculus, EXT1 C2 2014 HSC 11d

   Evaluate $\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$ using the substitution $x = u^2 + 1$. *(3 marks)*

Worked Solutions

1. Calculus, EXT1 C2 2013 HSC 5 MC

$$\text{Let } u = 1 + 2x$$

$$\therefore x = \frac{1}{2}(u - 1)$$

$$\frac{du}{dx} = 2$$

$$\therefore dx = \frac{1}{2}\,du$$

$$\int x\sqrt{1+2x}\,dx$$

$$= \int \frac{1}{2}(u-1) \times u^{\frac{1}{2}} \times \frac{1}{2}\,du$$

$$= \frac{1}{4}\int (u-1)\sqrt{u}\,du$$

$$\Rightarrow A$$

Worked Solutions

2. Calculus, EXT1 C2 2011 HSC 1d

$$u = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$du = \frac{dx}{2\sqrt{x}}$$

$$\therefore 2du = \frac{dx}{\sqrt{x}}$$

$$\text{When } x = 4,\ u = 2$$

$$x = 1,\ \ x = 1$$

$$\therefore \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= \int_1^2 e^u \times 2\,du$$

$$= 2[e^u]_1^2$$

$$= 2\left[e^2 - e^1\right]$$

$$= 2e(e-1)$$

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3. Calculus, EXT1 C2 2012 HSC 11d

$$
\begin{aligned}
u &= 2 - x \\
\therefore x &= 2 - u \\
\frac{du}{dx} &= -1 \\
\therefore dx &= -du
\end{aligned}
$$

When $x = 2, \ u = 0$

When $x = 1, \ u = 1$

$$
\begin{aligned}
\therefore \int_1^2 x(2-x)^5 \, dx &= \int_1^0 -(2-u)u^5 \, du \\
&= \int_1^0 u^6 - 2u^5 \, du \\
&= \left[\frac{1}{7}u^7 - \frac{2}{6}u^6\right]_1^0 \\
&= \left[0 - \left(\frac{1}{7} - \frac{1}{3}\right)\right] \\
&= -\left(\frac{3}{21} - \frac{7}{21}\right) \\
&= \frac{4}{21}
\end{aligned}
$$

4. Calculus, EXT1 C2 2015 HSC 11e

$$
\begin{aligned}
u &= 2x - 1 \\
\Rightarrow 2x &= u + 1 \\
x &= \frac{1}{2}(u+1) \\
\frac{du}{dx} &= 2 \\
dx &= \frac{du}{2}
\end{aligned}
$$

When $x = 2, u = 3$

$x = 1, u = 1$

$$
\begin{aligned}
\therefore \int_1^2 \frac{x}{(2x-1)^2} \, dx & \\
&= \int_1^3 \frac{1}{2}(u+1) \cdot \frac{1}{u^2} \cdot \frac{du}{2} \\
&= \frac{1}{4}\int_1^3 \left(\frac{u+1}{u^2}\right) du \\
&= \frac{1}{4}\int_1^3 \frac{1}{u} + u^{-2} du \\
&= \frac{1}{4}\left[\ln u - u^{-1}\right]_1^3 \\
&= \frac{1}{4}\left[\left(\ln 3 - \frac{1}{3}\right) - (\ln 1 - 1)\right] \\
&= \frac{1}{4}\left(\ln 3 - \frac{1}{3} + 1\right) \\
&= \frac{1}{4}\left(\ln 3 + \frac{2}{3}\right)
\end{aligned}
$$

5. Calculus, EXT1 C2 2016 HSC 11b

$$
\begin{aligned}
u &= x - 4 \\
&\Rightarrow x = u + 4 \\
\frac{du}{dx} &= 1 \\
\Rightarrow dx &= du \\
\\
\therefore &\int x\sqrt{x-4}\, dx \\
&= \int (u+4) \cdot u^{\frac{1}{2}}\, du \\
&= \int u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\, du \\
&= \frac{2}{5}u^{\frac{5}{2}} + 4 \cdot \frac{2}{3}u^{\frac{3}{2}} + c \\
&= \frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + c
\end{aligned}
$$

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6. Calculus, EXT1 C2 2017 HSC 11e

$$
\begin{aligned}
x &= u^2 - 1 \\
u^2 &= x + 1 \\
u &= \sqrt{x+1} \\
du &= \frac{1}{2\sqrt{x+1}}\, dx \\
\\
\text{If} \quad x &= 3, \quad u = 2 \\
x &= 0, \quad u = 1 \\
\therefore \int_0^3 \frac{x}{\sqrt{x+1}}\, dx &= 2\int_1^2 u^2 - 1\, du \\
&= 2\left[\frac{u^3}{3} - u\right]_1^2 \\
&= 2\left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right] \\
&= 2\left(\frac{2}{3} + \frac{2}{3}\right) \\
&= \frac{8}{3}
\end{aligned}
$$

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7. Calculus, EXT1 C2 2019 HSC 13a

$$
\begin{aligned}
u &= \cos^2 x \\
\frac{du}{dx} &= -2\sin x \cos x \\
&= -\sin 2x \\
du &= -\sin 2x \, dx
\end{aligned}
$$

When $x = \frac{\pi}{4}$, $u = \frac{1}{2}$

When $x = 0$, $u = 1$

$$
\begin{aligned}
\therefore \int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4 + \cos^2 x} \, dx &= -\int_1^{\frac{1}{2}} \frac{du}{4 + u} \\
&= -\left[\ln(4 + u)\right]_1^{\frac{1}{2}} \\
&= -(\ln 4.5 - \ln 5) \\
&= -\ln \frac{9}{10} \\
&= \ln \frac{10}{9}
\end{aligned}
$$

8. Calculus, EXT1 C2 2013 HSC 11f

MARKER'S COMMENT: Many students did not calculate in radians and incorrectly got an answer of 8.2. *BE CAREFUL!* Note that converting your answer to 0.14 is also correct but not required.

$$
\begin{aligned}
&\text{Let } u = e^{3x} \\
&\frac{du}{dx} = 3e^{3x} \\
&\therefore dx = \frac{du}{3e^{3x}}
\end{aligned}
$$

When $x = \frac{1}{3}$, $u = e^{3 \times \frac{1}{3}} = e$

$x = 0$, $u = e^0 = 1$

$$
\begin{aligned}
\therefore \int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x} + 1} \, dx &= \int_1^e \frac{e^{3x}}{u^2 + 1} \times \frac{du}{3e^{3x}} \\
&= \frac{1}{3}\int_1^e \frac{1}{u^2 + 1} \, du \\
&= \frac{1}{3}\left[\tan^{-1} u\right]_1^e \\
&= \frac{1}{3}\left[\tan^{-1} e - \tan^{-1} 1\right] \\
&= \frac{1}{3}\left(\tan^{-1} e - \frac{\pi}{4}\right)
\end{aligned}
$$

9. Calculus, EXT1 C2 2014 HSC 11d

$$
\begin{aligned}
x &= u^2 + 1 \\
u^2 &= x - 1 \\
u &= \sqrt{x-1} \\
\frac{du}{dx} &= \frac{1}{2}(x-1)^{-\frac{1}{2}} \\
&= \frac{1}{2\sqrt{x-1}} \\
\Rightarrow 2du &= \frac{dx}{\sqrt{x-1}}
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{When} \quad x = 5, \;\; u = 2 \\
&\qquad\quad x = 2, \;\; u = 1 \\
&\therefore \int_2^5 \frac{x}{\sqrt{x-1}}\, dx \\
&= 2\int_1^2 u^2 + 1 \, du \\
&= 2\left[\frac{u^3}{3} + u\right]_1^2 \\
&= 2\left[\left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right)\right] \\
&= \frac{20}{3}
\end{aligned}
$$