

Active Inference in continuous time notes

Federico Maggiore

September 25, 2020

Summary notation

- $\mathbf{x} = \{x_i\}_{i=1}^D$ environmental variables of the D -dimensional space constituting latent or hidden states;
- $\mathbf{s} = \{s_i\}_{i=1}^S$ body sensors input;
- $P(\mathbf{s}, \mathbf{x})$ *G-density*;
- $P(\mathbf{x}|\mathbf{s})$ *Posterior*;
- $P(\mathbf{s}|\mathbf{x})$ *Likelihood*;
- $P(\mathbf{x})$ *Prior*;
- $P(\mathbf{s}) = \int P(\mathbf{s}|\mathbf{x})P(\mathbf{x})d\mathbf{x}$ *marginal likelihood*
- $Q(\mathbf{x})$ *R-density*
- $F \equiv \int Q(\mathbf{x}) \ln \frac{Q(\mathbf{x})}{P(\mathbf{x}, \mathbf{s})} d\mathbf{x}$ *Variational Free Energy*

References: Manuel and Christopher 2019 Christopher L. et al. 2017

Free Energy Principle (FEP)

The goal of an agent is to determine the probability of the hidden states given some sensory inputs:

$$P(\mathbf{x}|\mathbf{s}) = \frac{P(\mathbf{s}, \mathbf{x})}{P(\mathbf{s})} = \frac{P(\mathbf{s}|\mathbf{x})P(\mathbf{x})}{P(\mathbf{s})} \quad (1)$$

with

- $P(\mathbf{s}, \mathbf{x})$ *G-density*, beliefs about the states assumed to be encoded by the agent;
- $P(\mathbf{x}|\mathbf{s})$ *Posterior*, i.e. probability of hidden causes x given observed sensory data;
- $P(\mathbf{s}|\mathbf{x})$ *Likelihood*, i.e. organism's assumptions about sensory input \mathbf{s} given the hidden causes \mathbf{x} ;

- $P(\mathbf{x})$ *Prior*, i.e. agent's beliefs about hidden causes **before** that \mathbf{s} are received;
- $P(\mathbf{s}) = \int P(\mathbf{s}|\mathbf{x})P(\mathbf{x})d\mathbf{x}$ *marginal likelihood*, i.e. normalization factor.

For the agent it's not necessary to compute the complete posterior distribution, it has only to find the hidden state -or at least a good approximation- that maximize the posterior, i.e. $\arg \max_{\mathbf{x}} P(\mathbf{x}|\mathbf{s})$. The problem with the exact Bayesian scheme, is that $P(\mathbf{s})$ is often impossible to calculate, and moreover $P(\mathbf{x}|\mathbf{s})$ may not take a standard shape and could not have a summary statistics.

A biologically plausible technique consist in using an auxiliary distribution $Q(\mathbf{x})$ called *recognition density* (*R-density*) that has to be optimized to become a good approximation of the posterior.

In order to do this the Kullback-Leibler divergence is minimized:

$$\begin{aligned}
D_{KL}(Q(\mathbf{x}) || P(\mathbf{x}|\mathbf{s})) &= \int Q(\mathbf{x}) \ln \frac{Q(\mathbf{x})}{P(\mathbf{x}|\mathbf{s})} d\mathbf{x} \\
&= \int Q(\mathbf{x}) \ln \frac{Q(\mathbf{x})P(\mathbf{s})}{P(\mathbf{x}, \mathbf{s})} d\mathbf{x} \\
&= \int Q(\mathbf{x}) \ln \frac{Q(\mathbf{x})}{P(\mathbf{x}, \mathbf{s})} d\mathbf{x} + \ln P(\mathbf{s}) \int Q(\mathbf{x}) d\mathbf{x} \\
&= F + \ln P(\mathbf{s})
\end{aligned} \tag{2}$$

where

- $F \equiv \int Q(\mathbf{x}) \ln \frac{Q(\mathbf{x})}{P(\mathbf{x}, \mathbf{s})} d\mathbf{x} = - \int Q(\mathbf{x}) \ln P(\mathbf{x}, \mathbf{s}) d\mathbf{x} + \int Q(\mathbf{x}) \ln Q(\mathbf{x}) d\mathbf{x}$ is the *Variational Free Energy*, a quantity that depends on the R-density and the knowledge about the environment i.e. the G-density $P(\mathbf{s}, \mathbf{x}) = P(\mathbf{s}|\mathbf{x})P(\mathbf{x})$ that we are assuming the agent has.
- $\ln P(\mathbf{s})$ is a term independent of the recognition density $Q(\mathbf{x})$ (\Rightarrow minimizing F with respect to $Q(\mathbf{x})$ will minimize the D_{KL})

Laplace approximation

Often optimizing F for arbitrary $Q(\mathbf{x})$ is particularly complex. Moreover, it is assumed that neural activity parametrise sufficient statistic. For these reasons, a common approximation is to assume that the R-density take a Gaussian form.

For now the D=1 case in detail, leaving the formulation of the multivariate case for later. Let's assume that the R-density $Q(x)$ has a peak at point μ . The Taylor-expansion of the logarithm around this peak is

$$\ln Q(x) \simeq \ln Q(\mu) - \frac{1}{2} \frac{(x - \mu)^2}{\Sigma} \tag{3}$$

with

$$\frac{1}{\Sigma} = - \frac{\partial^2}{\partial x^2} \ln P(x) \Big|_{x=\mu} \tag{4}$$

Now it is possible to approximate the probability distribution $Q(x)$ with the distribution

$$\mathcal{N}(x; \mu, \Sigma) = \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{(x-\mu)^2}{2\Sigma}} \tag{5}$$

i.e. a Gaussian distribution that has been normalized using the factor $Q(\mu)\sqrt{2\pi\Sigma}$.
Now the free energy can be written as follow

$$\begin{aligned}
F &= \int \mathcal{N}(x; \mu, \Sigma) \left(-\frac{1}{2} \ln 2\pi\Sigma - \frac{(x - \mu)^2}{2\Sigma} \right) dx - \int \mathcal{N}(x; \mu, \Sigma) \ln P(x, s) dx \\
&= -\frac{1}{2} \ln 2\pi\Sigma - \frac{1}{2\Sigma} \int \mathcal{N}(x; \mu, \Sigma) (x - \mu)^2 dx - \int \mathcal{N}(x; \mu, \Sigma) \ln P(x, s) dx \\
&= -\frac{1}{2} \ln 2\pi\Sigma - \frac{1}{2} - \int \mathcal{N}(x; \mu, \Sigma) \ln P(x, s) dx
\end{aligned} \tag{6}$$

Laplace encoded free energy

In this case indeed this approximation is used in the following manner (in the following only the univariate case is presented in detail since it captures all the relevant assumptions).

First of all the R-densities are assumed Gaussian distributions

$$Q(x) \equiv \mathcal{N}(x; \mu, \Sigma) = \mathcal{N}(x; \mu, \Sigma) = \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{(x-\mu)^2}{2\Sigma}}$$

Generalizing for a density $Q(\mathbf{x})$ over a K-dimensional space \mathbf{x} with peak at $\boldsymbol{\mu}$

$$\ln Q(\mathbf{x}) \simeq \ln Q(\boldsymbol{\mu}) - \frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \quad (7)$$

with

$$[\boldsymbol{\Sigma}^{-1}]_{i,j} = -\frac{\partial^2}{\partial x_i \partial x_j} \ln P(\mathbf{x}) \Big|_{\mathbf{x}=\boldsymbol{\mu}} \quad (8)$$

is possible to approximate $Q(\mathbf{x})$ with the multivariate Gaussian distribution

$$\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^K \det \boldsymbol{\Sigma}}} e^{\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})} \quad (9)$$

This approximation is particularly useful to approximate integrals.

Laplace encoded free energy

Let's first rewrite F as follow:

$$F = \langle \ln P(\mathbf{x}, \mathbf{s}) \rangle_Q - \langle \ln Q(\mathbf{x}) \rangle_Q$$

with L called *Laplace encoded energy* and the second term is referred as entropy. Often optimizing F for arbitrary $Q(x)$ is particularly complex, so a common approximation

References

- Christopher L., Buckley et al. (Dec. 2017). “The free energy principle for action and perception: A mathematical review”. In: *Journal of Mathematical Psychology* 81, pp. 55–79. DOI: 10.1016/j.jmp.2017.09.004. URL: <https://doi.org/10.1016/j.jmp.2017.09.004>.
- Manuel, Baltieri and Buckley Christopher (Mar. 2019). “PID Control as a Process of Active Inference with Linear Generative Models”. In: *Entropy* 21.3, p. 257. DOI: 10.3390/e21030257. URL: <https://doi.org/10.3390/e21030257>.