Zombie Apocalypse Flavia Trotolo

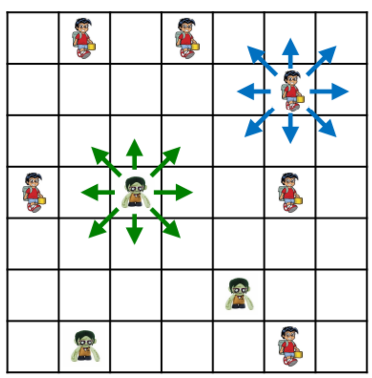
Zombie Apocalypse

Describe the Zombie Apocalypse Dynamics:

Zombie Ville is a square city made by square cells. Each cell can be occupied by either a zombie, a person or it can be empty. The state of the cell is determined by the variable X.
The row number is given by i the column number is given by j.
Hence, the state of the square i,j is named Zij
The state is a function of the time t
Hence at a point in time t, the state of a cell of position i, j can be written as:

- $X_{i,j}(t) = 0\ (if\ the\ cell\ is\ empty)$
- $X_{i,j}(t) = 1\ (if\ the\ cell\ is\ occupied\ by\ human)$
- $X_{i,j}(t) = -1\ (if\ the\ cell\ is\ occupied\ by\ zombie)$

If the cell at time t is occupied ($X_{i,j}(t) = 1$ or $X_{i,j}(t) = -1$) then the being can move in any adjacent cell (i.e. the being can move up, down, left, right, and diagonally(Nord-east, Nord-west, southwest, southwest))Hence, the neighborhood ($N^{i,j}$) of a cell i,j is composed by the cell i,j and the neighboring 8 cells as shown in picture below:



This report will mathematically formalize the rules for the game of the zombie apocalypse by defining laws of motions and describing mathematically the fight between species. It will also design experiments to test whether our mathematical description of the apocalypse is confirmed by simulation.

what the Zombie Apocalypse might represent in practice

This may represent the spreading of a certain virus for example coronavirus (similar to the zombies) on healthy individuals which are humans. Hence, zombie Ville can be used as a mathematical model to study real life situations. In evolution theory, similar models can be used to assess which animals will evolve with time and which would be defeated in battle by other animals.

A mathematical description of how the initial configuration is simulated

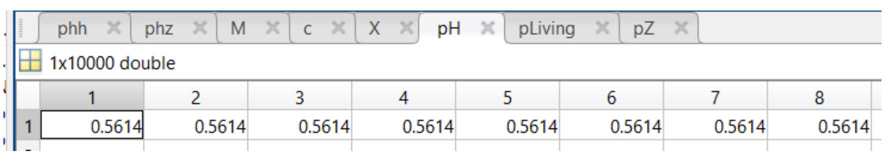
Determine the true probabilities via simulation

I set the parameters as follows to see the simulation values for the true probability of humans and zombies at t = 1

N = 100; [this indicates the number of rows]
M = 100; [this indicates number of columns]
T = 100; [this indicates number of time steps to simulate the battle]
phz = 0.6; [this indicates probability that a human kills a zombie in a fight]
phh = 0.0; [this indicates probability that a human targets and kills another human]
pLiving = 0.7; [this indicates probability of humans in human only matrix for initialization]
pZombies = 0.2; [this indicates probability of zombies in zombie only matrix for initialization]

In these parameters not that we assign to pLiving the probability of success of the human (success occurs when cell in human matrix is occupied by human, cell = 1). Similarly, pZombies is the probability of success of a zombie in the zombie matrix (success means the cell is occupied by a zombie (cell=1))

After running the script zombies.m we obtain that:
The true probability of humans at time = 1 is: pH = 0.561400000000000

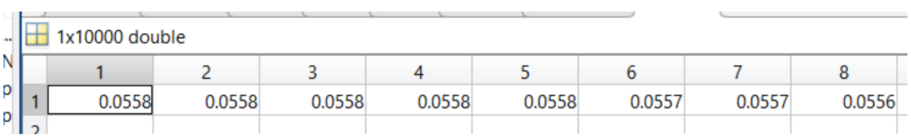


phh | phz | M | c | X | pH | pLiving | pZ

1x10000 double

	1	2	3	4	5	6	7	8
1	0.5614	0.5614	0.5614	0.5614	0.5614	0.5614	0.5614	0.5614

And, the true probability of Zombie at time = 1 is: pZ = 0.0558000000000000



1x10000 double

	1	2	3	4	5	6	7	8
1	0.0558	0.0558	0.0558	0.0558	0.0558	0.0557	0.0557	0.0556

Determine the true probabilities analytically

According to Bernouilli. the initial true probabilities of living and zombies in the matrix can be found as the subtraction of the two Bernouilli: $X = B(H) - B(Z)$
As per the table below:

B(H)	B(Z)	Values of X	Probability of X	True probabilities computation
0	0	$X = 0 - 0 = 0$	$(1 - p_{Living})(1 - p_{zombies})$	
0	1	$X = 0 - 1 = 1$	$(1 - p_{Living})(p_{zombies})$	$pZ = (1 - 0.7)(0.2) = 0.06$
1	0	$X = 1 - 0 = 1$	$(p_{living})(1 - p_{zombies})$	$pH = (0.7)(1 - 0.2) = 0.56$
1	1	$X = 1 - 1 = 1$	$(p_{living})(p_{zombies})$	

Notice that pH simulation = 0.561400000000000 is close to pH analytical= 0.56. Similarly, pZ simulation= 0.0558000000000000 is close to pZ analytical of 0.06

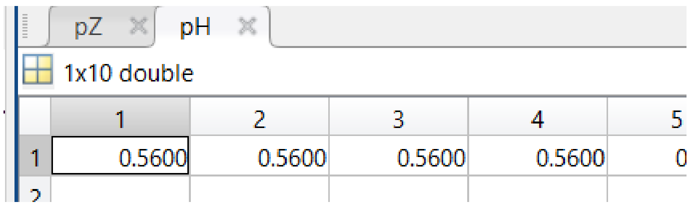
Bonus:

When I change the size of matrix as follows:

N = 10000; % number of rows
M = 10000; % number of columns

I obtained:

pH at t=1:

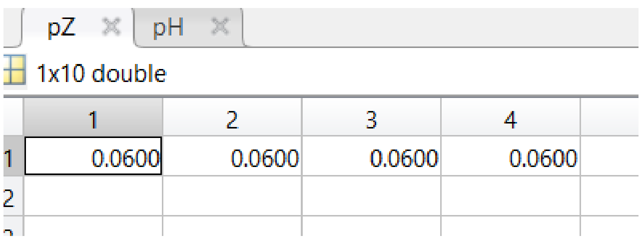


pZ | pH

1x10 double

	1	2	3	4	5
1	0.5600	0.5600	0.5600	0.5600	0
2					

pZ at t=1:



pZ | pH

1x10 double

	1	2	3	4
1	0.0600	0.0600	0.0600	0.0600
2				

The initial true probabilities of pH and Pz is 0.56 and 0.06 respectively. This corresponds to the predicted analytical value of pH and Pz of 0.56 and 0.06 respectively. It occurs because when the sample size grows by increasing the number of rows and columns, the average of populations becomes closer to the theoretical mean of pH and Pz.

Laws of Motion:

We now will create a score for the movements of the beings from a cell to another. The score to move from cell (i,j) to cell (k,l) is denoted as $\pi_{i,j->k,l}(X_{i,j}, X_{k,l})$. This depends on what is contained in cell i,j ($X_{i,j}(t)$) and what is contained in the cell k,l ($X_{k,l}(t)$)

- Consider that a zombie wants to go to a human (to eat it)
- Consider that a human wants not to go to a zombie (not to get eaten)

Hence, let's set a score for each move as follows:

- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 0\ if\ (k, l \notin N^{i,j})\ that\ is\ if\ the\ cell\ is\ not\ in\ the\ neighborhood$
- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 0\ if\ (X_{i,j}(t) = 0)\ if\ the\ cell\ is\ empty$
- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 1\ if\ (X_{i,j}(t) = -1\ and\ X_{k,l} = 0)\ \ if\ the\ zombie\ goes\ to\ empty$
- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 2\ \ if\ (X_{i,j}(t) = -1\ and\ X_{k,l} = 2)\ \ if\ the\ zombie\ goes\ to\ human$
- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 0\ \ if\ (X_{i,j}(t) = -1\ and\ X_{k,l} = -1)\ \ if\ the\ zombie\ goes\ to\ zombie$
- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 0\ \ if\ (X_{i,j}(t) = 1\ and\ X_{k,l} = -1)\ \ if\ the\ human\ goes\ to\ zombie$
- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 1\ \ if\ (X_{i,j}(t) = 1\ and\ X_{k,l} = 1)\ \ if\ the\ human\ goes\ to\ human$
- $\pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = 2\ \ if\ (X_{i,j}(t) = 1\ and\ X_{k,l} = 0)\ \ if\ the\ human\ goes\ to\ empty$

Let's denote with $T_{i,j}(t)$the target cell for a being in cell i,j at time t

Now, given the score information above, we can compute the probability associated with the target cell as:

$$p_{i,j}(k,i) = p(T_{i,j}(t) = (k,l) | X(t) = x) =$$

$$= \frac{\pi_{i,j->k,l}(X_{i,j}, X_{k,l})}{\Sigma_{(m,n)\in N^{i,j}} \pi_{i,j->m,n}(X_{i,j}, X_{m,nl})}$$

That is the score of going to target cell/ the score of going to other cells

Show that the probabilities you formulated respect the axioms of probability

Axiom 1: Probability to move from one cell to another cannot be negative

We explained that the probability to move to target cell is:

$$p_{i,j}(k,i) = p(T_{i,j}(t) = (k,l)|\, X(t) = x) = \\ = \frac{\pi_{i,j->k,l}(X_{i,j}, X_{k,l})}{\Sigma_{(m,n)\in N^{i,j}}\pi_{i,j->m,n}(X_{i,j}, X_{m,nl})}$$

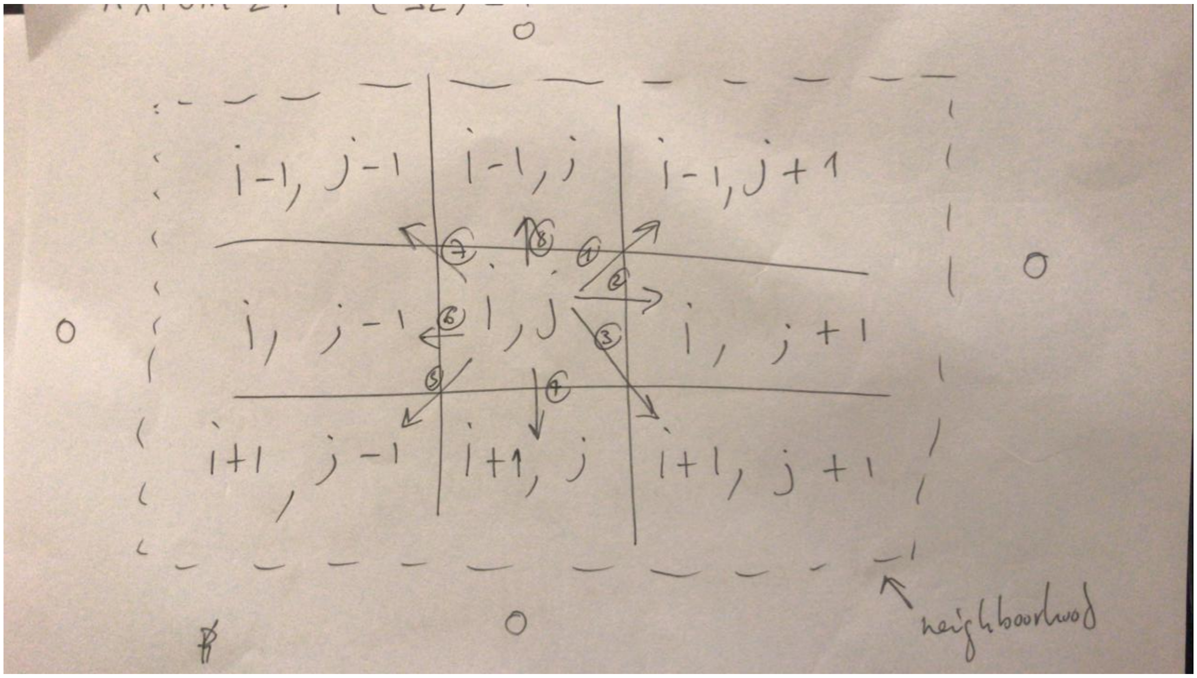
Notice that the score π for a species to moving to target cell is defined to be either 0 or a number greater than zero. So, both numerator and denominators are greater than zero

Hence: $p_{i,j}(k,i) \geq 0$

That is the first axiom of probability. In addition, since the denominator is the sum of all scores for neighborhood then denominator is greater than denominator and $p_{i,j}(k,i) < 1$

Axiom 2: Probability of all the sample space is equal to 1:

Consider possible paths from cell to target cells in neighborhood with numbered paths below:



Now, let's write down the scores to move from central cell to target cell to each path

Path1: $\pi_{i,j->i-1,j+1}(X_{i,j}, X_{i-1,j+1})$

Path2: $\pi_{i,j->i,j+1}(X_{i,j}, X_{i,j+1})$

Path3: $\pi_{i,j->i+1,j+1}(X_{i,j}, X_{i+1,j+1})$

Path4: $\pi_{i,j->i+1,j}(X_{i,j}, X_{i+1,j})$

Path5: $\pi_{i,j->i+1,j-1}(X_{i,j}, X_{i+1,j-1})$

Path6: $\pi_{i,j->i,j-1}(X_{i,j}, X_{i,j-1})$

Path7: $\pi_{i,j->i-1,j-1}(X_{i,j}, X_{i-1,j-1})$

Path8: $\pi_{i,j->i-1,j}(X_{i,j}, X_{i-1,j})$

Now let's sum Paths from 1 to 8 to obtain the left hand side of the following equation:

$$\sum_{i+1,j-1}^{i-1,j+1} \pi_{i,j->k,l}(X_{i,j}, X_{k,l}) = \sum_{m,n \in N^{i,j}} \pi_{i,j->m,n}(X_{i,j}, X_{m,n})$$

Right hand side is the sum of all paths in neighborhood where m and n are the number of rows and columns in neighborhood. Now, revisiting probability to move to target cell:

$$p_{i,j}(k,i) = \frac{\pi_{i,j->k,l}(X_{i,j}, X_{k,l}\ )}{\Sigma_{(m,n)\in N^{i,j}}\pi_{i,j->m,n}(X_{i,j}, X_{m,nl}\ )}$$

Notice that the nominator is equal to the sum of all possible scores from current cell to target cell. This is equal to the scores of all paths within neighborhood (paths 1 to 8) + all the scores of all paths outside of neighborhood. The latter is equal to zero, since we said previously that the being in the cell can only move to target cell in neighborhood.

Hence,

$$\pi_{i,j\to k,l}\left(X_{i,j}, X_{k,l}\ \right) = \sum_{i+1,j-1}^{i-1,j+1} \pi_{i,j\to k,l}\left(X_{i,j}, X_{k,l}\ \right) + paths\ outside\ neighboorhood$$

$$= \sum_{i+1,j-1}^{i-1,j+1} \pi_{i,j\to k,l}(X_{i,j}, X_{k,l}\ )$$

Therefore, we have

$$p_{i,j}(k,i) = \frac{\pi_{i,j\to k,l}\left(X_{i,j}, X_{k,l}\ \right)}{\Sigma_{(m,n)\in N^{i,j}}\pi_{i,j\to m,n}\left(X_{i,j}, X_{m,nl}\ \right)}$$

$$= \frac{\sum_{i+1,j-1}^{i-1,j+1} \pi_{i,j\to k,l}\left(X_{i,j}, X_{k,l}\ \right)}{\sum_{i+1,j-1}^{i-1,j+1} \pi_{i,j\to k,l}\left(X_{i,j}, X_{k,l}\ \right)} = 1$$

Hence, the probability of all the sample space is equal to 1

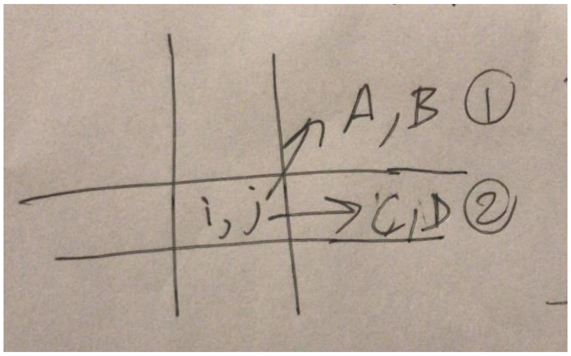
Axiom 3: **If two events A and B are mutually exclusive, then the probability of either A or B occurring is the probability of A occurring plus the probability of B occurring**

In mathematical terms:

$$\bigcup_{i=1}^{8} E_i = \sum_{i=1}^{8} P(E_i)$$

I am ignoring the outside of neighborhood since outside the events have probability zero

Consider the following figure below:



The being cannot move from present cell to more than one different target cells at same time, because the being cannot be in two positions at the following time step.

Hence we can write that

$$p\left(X_{i,j} \to X_{A,B}\right) \cap p\left(X_{i,j} \to X_{C,D}\right) = 0$$

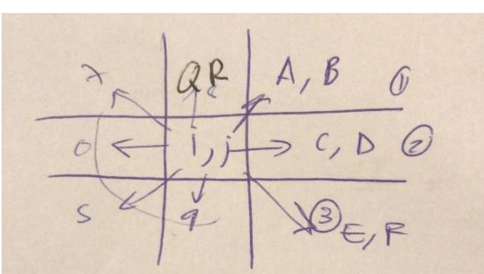
Now,

$$\begin{aligned} p\left(X_{i,j} \to X_{A,B}\right) \cup p\left(X_{i,j} \to X_{C,D}\right) \\ &= p\left(X_{i,j} \to X_{A,B}\right) + p\left(X_{i,j} \to X_{C,D}\right) - p\left(X_{i,j} \to X_{A,B}\right) \cap p\left(X_{i,j} \to X_{C,D}\right) \\ &= p\left(X_{i,j} \to X_{A,B}\right) + p\left(X_{i,j} \to X_{C,D}\right) \end{aligned}$$

Hence, for two cells we proved that:

$$\bigcup_{i=1}^{2} E_i = \sum_{i=1}^{2} P(E_i)$$

Now, consider the other neighboorhoods:



We have

$$\bigcup_{i=1}^{3} E_i = \sum_{i=1}^{2} p(E_i) \cup p\big(X_{i,j} \to X_{E,F}\big) = \sum_{i=1}^{2} p(E_i) + p\big(X_{i,j} \to X_{E,F}\big)$$

using the same reasoning as for the two cells case (i.e. species goes in one direction only)
This can be repeated until

$$\bigcup_{i=1}^{8} E_i = \sum_{i=1}^{7} p(E_i) \cup p\big(X_{i,j} \to X_{E,F}\big) = \sum_{i=1}^{2} p(E_i) + p\big(X_{i,j} \to X_{Q,R}\big)$$

Hence,

$$\bigcup_{i=1}^{8} E_i = \sum_{i=1}^{8} P(E_i)$$

More Laws of Motion for State Update

State Update

Furthermore, The Zombie apocalypse pdf provided further explains the state of initial cells $X_{i,j}$, $X_{k,l}$ and target cell $X_{i,j}(t+1)$ when the initials cells have same target: $X_{T_{i,j}} = X_{T_{k,l}}$

	$X_{i,j}$	$X_{k,l}$
	$X_{T_{i,j}}$	

If $X_{i,j} = X_{k,l}$ it means the cells are occupied by same species. The target cell state at time t+1 can be found with the following equation:

$$X_{T_{i,j}} = X_{T_{k,l}} = (1 - B_{same})X_{k,l} + B_{same}(X_{i,j})$$

Where $B_{same} = 0$ when $X_{k,l}$ wins, and $B_{same} = 1$ when $X_{i,j}$ wins. B_{same} is Bernoulli.

However, $X_{i,j} \neq X_{k,l}$ it means the cells are occupied by different species. The target cell state at time t+1 can be found with the following equation:

$$X_{T_{i,j}} = X_{T_{k,l}} = \left(1 - B_{fight}\right)X_{k,l} + B_{fight}(X_{i,j})$$

Where $B_{fight} = 0$ when $X_{k,l}$ wins, and $B_{fight} = 1$ when $X_{i,j}$ wins. B_{fight} is Bernoulli.

P_{same} is the probability parameter for the bernouilli B_{same}

P_{fight} is the probability parameter for the bernouilli B_{fight}

State Update state table, assuming target cell initially empty:

Cases	$X_{i,j}(t)$	$X_{k,l}(t)$	$X_{T_{i,j}}(t+1)$	$p(X_{T_{i,j}}(t+1))$	$X_{i,j}(t+1)$	$X_{k,l}(t+1)$	Description
empty	0	0	0	1	0	0	Empty cells
No conflict	1	0	1	1	0	0	Human goes to target
	-1	0	-1	1	0	0	Zombies goes to target
Conflict same species	1	1	1	p_{same}	0	1	One human wins and advance, other human remains in original position
				$1 - p_{same}$	1	0	
	-1	-1	-1	p_{same}	0	-1	One zombie wins and advance, other zombie remains in original position
				$1 - p_{same}$	-1	0	
Conflict different species	1	-1	1	p_{fight}	0	0	Human wins, defeated zombie dies
			-1	$1 - p_{fight}$	0	0	Zombie wins, defeated human dies

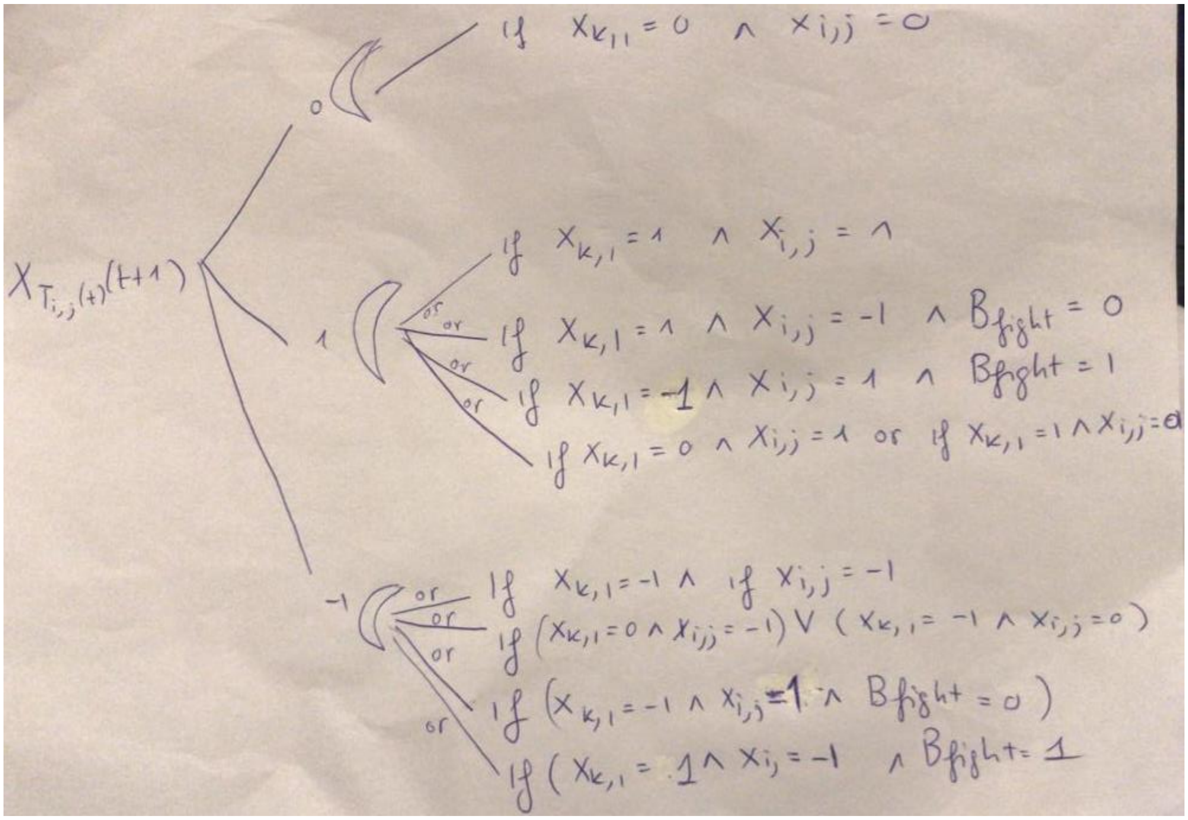
Note that

$$p_{same} + 1 - p_{same} = 1$$

And

$$p_{fight} + 1 - p_{fight} = 1$$

The Update state table can also be visualized as follows:



However, the description above changes the code and simulation. The following table shows what will happen between I and T where I is the initial cell and T is the Target Cell in the simulation in MATLAB which resolves resolutions in multiple time steps

Initial cell and target cell representation:

I(t)	T(t)	

State Table

I(t)	T(t)	I(t+1)	T(i+1)	Description
0	0	0	0	Initial cell is empty; hence target remains unvaried
0	-1	0	-1	
0	1	0	1	
1	-1	0	1	Human kills the zombie and conquers target cell
		-1	-1	The zombie wins and human gets converted to zombie
1	0	0	1	Human moves to target
1	1	1	1	Each human survives
		0	1	One human gets killed
-1	-1	-1	-1	Nothing happens, zombies do not interact
-1	1	0	1	Zombie dies
		-1	-1	Human converted to zombie
-1	0	0	-1	Zombie moves to target

A description of the experiments that will be carried out, what are the parameter settings, and why they were chosen

Experiment 1:

The first experiment tests many powerful humans who are friendly to each other and never kill each other fighting with smaller number of zombies. In addition, humans have a greater probability of zombies winning in a fight. Parameter settings are reassumed and explained in the table below

Experiment 1 parameters			
Parameter	value	meaning	Why I chose this
pLiving	0.7	Prob. Humans for initial human matrix	I have more humans initially
pZombies	0.2	Prob Zombies for initial zombies matrix	I have less humans initially
phz	0.6	Probability human kills zombie in fight	Humans more powerful than zombies
phh	0.05	Probability that human targets and kills other human	These humans are friendly to each other

Expected Results for experiment 1: Given that I have many humans, who are more powerful than zombies in fight, and who are friendly to each other, I expect humans to triumph over zombies.

For both experiment 1 and experiment 2, my matrix will have the size of number rows, N = 100; and number of columns, M = 100. This matrix is large enough for the result to better approximate analytical predicted values. In addition, I will run the code for 100000 steps. That is a large enough time period to see the behavior of humans, zombies in long run

Experiment 2:

The second experiment will be fairer in the initial probability of humans and zombies, to be more precise, I will start the apocalypse fight with as many humans as zombies in 50% probabilities. Then, I will design humans to be weak in fight such that zombies can more easily kill humans. Moreover, I will set the probability that humans kill humans to a very high value.

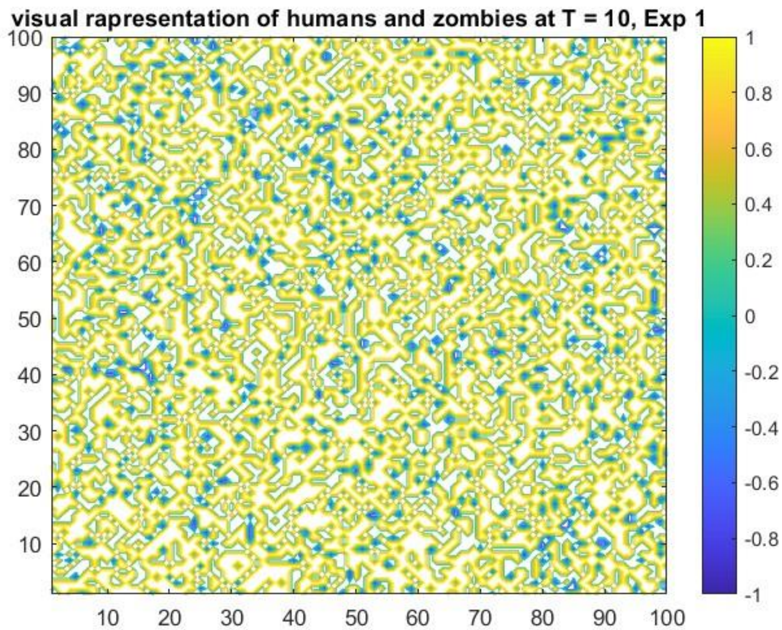
Experiment 2 parameters			
Parameter	value	meaning	Why I chose this
pLiving	0.5	Prob. Humans for initial human matrix	I have same number of humans and zombies
pZombies	0.5	Prob Zombies for initial zombie matrix	
phz	0.3	Probability human kills zombie in fight	Humans are less powerful than zombies in fight
phh	0.8	Probability that human targets and kills other human	These humans are killing each other with ease. Humans are enemies

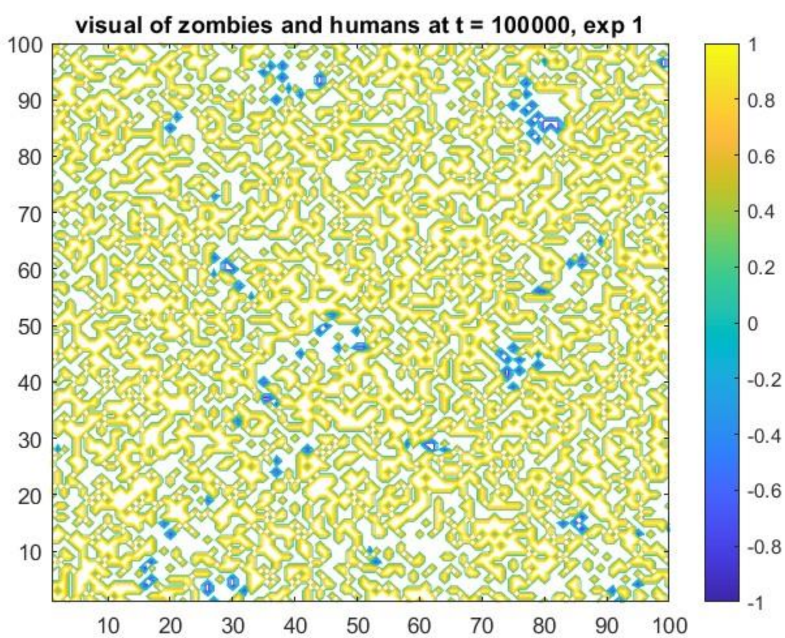
Expected results for experiment 2: Given that the initial number of humans and zombies is approximately equal, and humans are very weak in fight and kill each other, I expect zombies to triumph on humans with ease.

A description of the results in probabilistic terms.

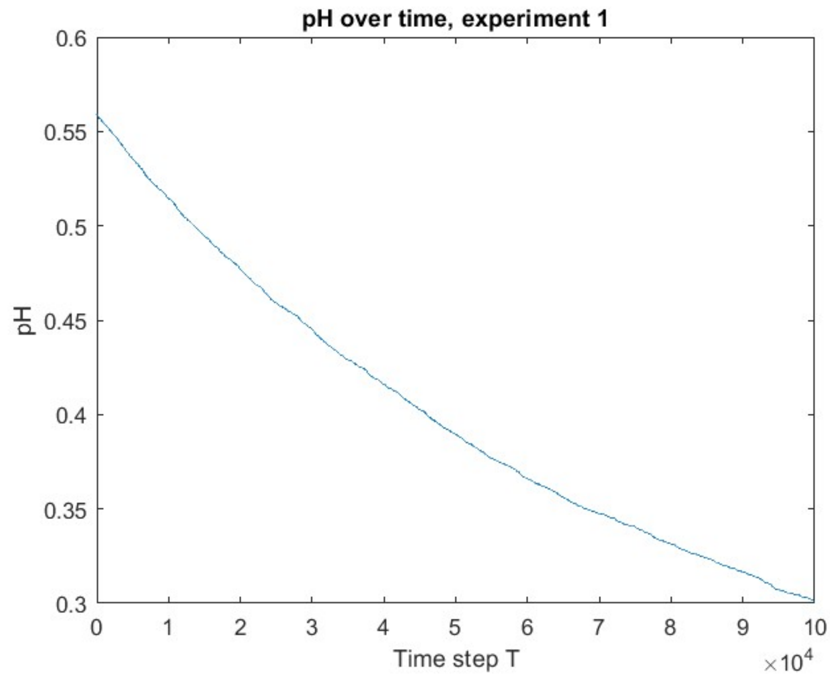
Experiment 1

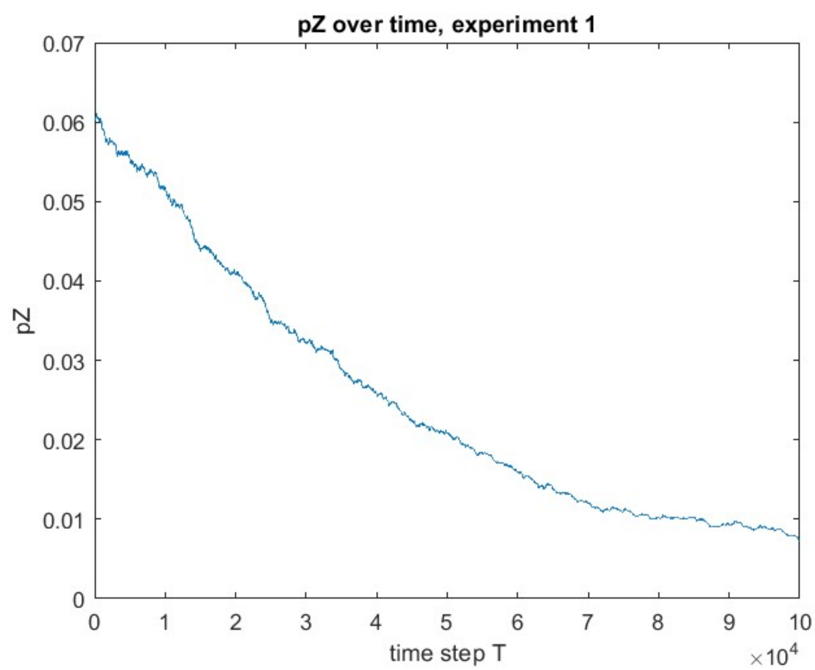
The following contours represent visually the abundance of zombies and humans at different time steps. They can give a quick visual idea of the situation





From the contour it is clear that humans (in yellow) are prevailing on zombies(in blue) At steps 100000 the number of humans greater by far than number of zombies, and zombies are almost all dead.
The curves below portray the variation of the true probabilities of humans and zombies over time

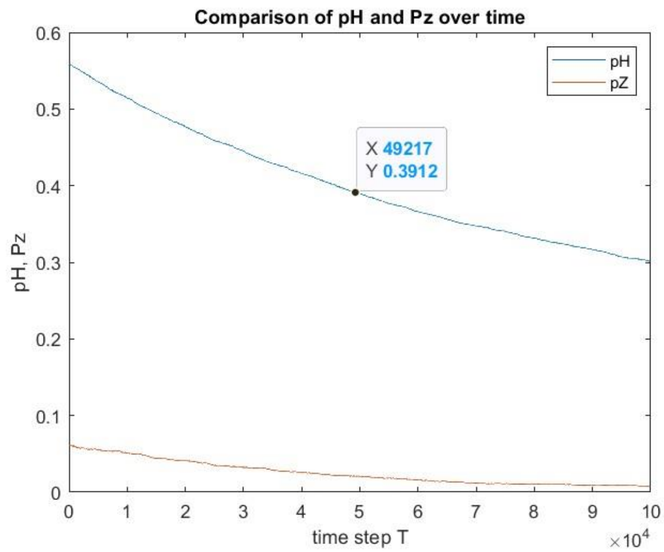




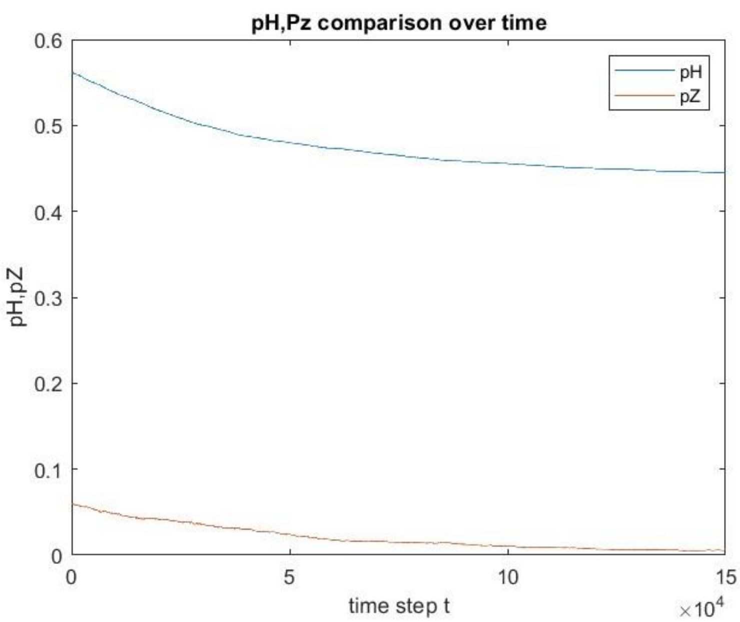
Empirical Probabilities at different time steps for experiment 1:

	T = 10	T= 100	T= 10000	T= 100000	T= 150000	T= 180000
pZ	0.058700000	0.0579000	0.0474000	0.01080000	0.00720000	0.0065000
pH	0.5579000	0.5576000	0.53720000	0.4692000	0.4538000	0.4470000

Now, we will compare the curves of pH and pZ and assess whether convergence is achieved:



Let’s increase the time step to better assess whether process converges. When trying to increase the time step:

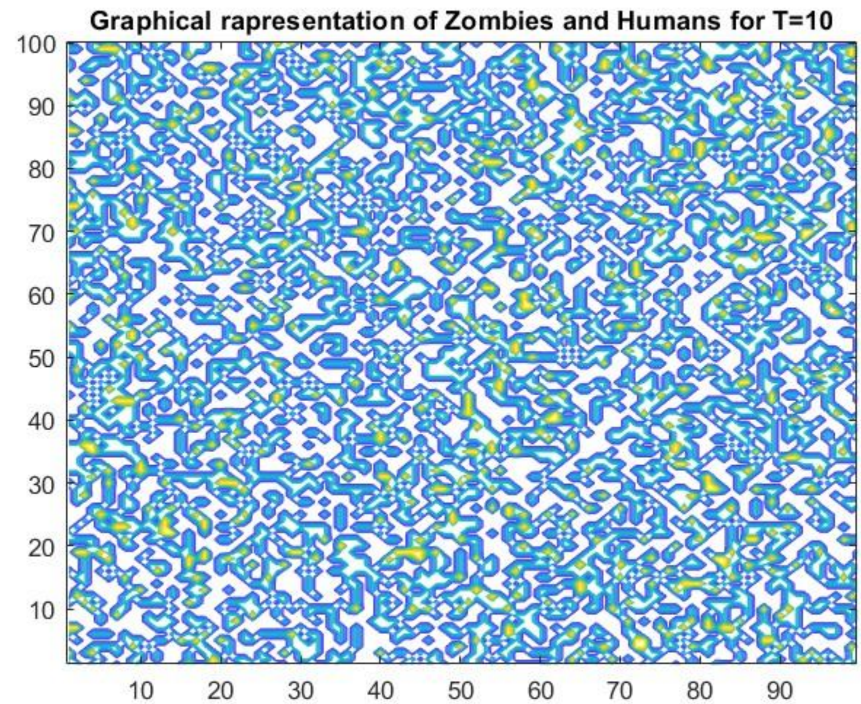


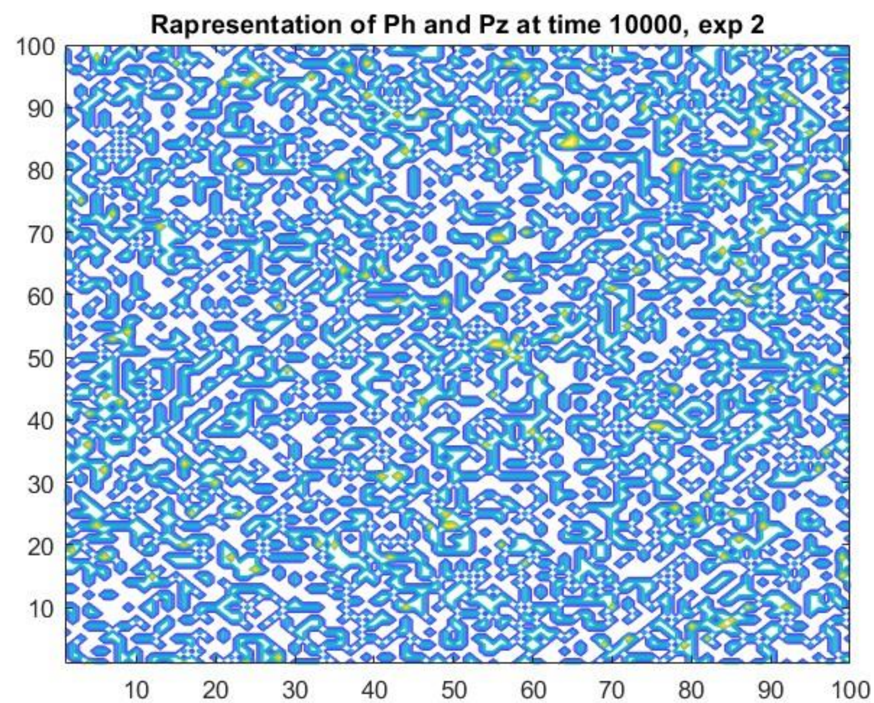
From the above graph and from the table of empirical probabilities for experiment 1, it seems that pH keeps falling over time, but at a slower rate. This is because as the number of zombies is reduced, the number of killed humans in fight with zombies is reduced. However, since humans still kill each other, the number of humans keep decreasing with time

Hence, the process does not converge as the number of humans keeps falling. The equilibrium is reached only for the number of zombies that will tend to human, which means eventually all zombies will be defeated due to their weakness in fight and smaller initial percentage to humans as per our set up.

Experiment 2

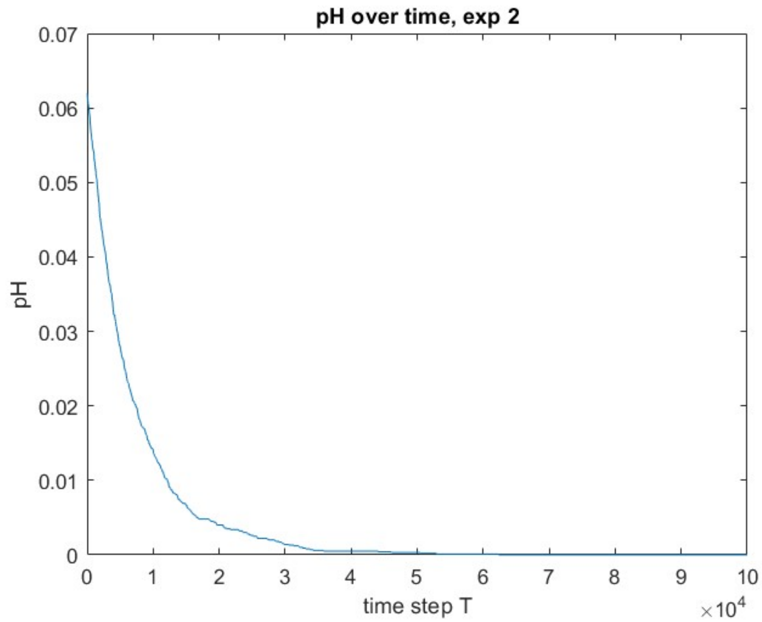
The following contours represent visually the abundance of zombies and humans at different time steps. They can give a quick visual idea of the situation



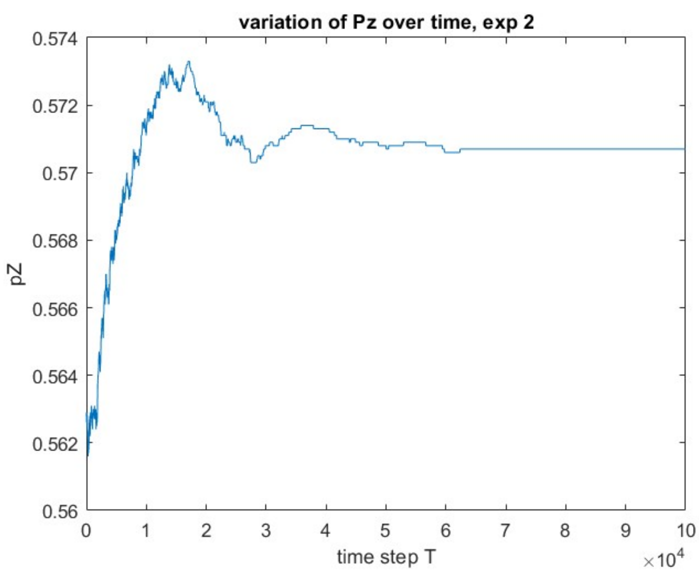


From the contour it is clear that zombies (in blue) are prevailing on humans (in yellow). At steps 10000 the number of zombies is clearly superior, and humans are almost all dead

The curves below portray the variation of the true probabilities of humans and zombies over time



From pH curve we see that humans number falls very fast, this makes sense because we programmed the humans to be weak in fight and kill each other

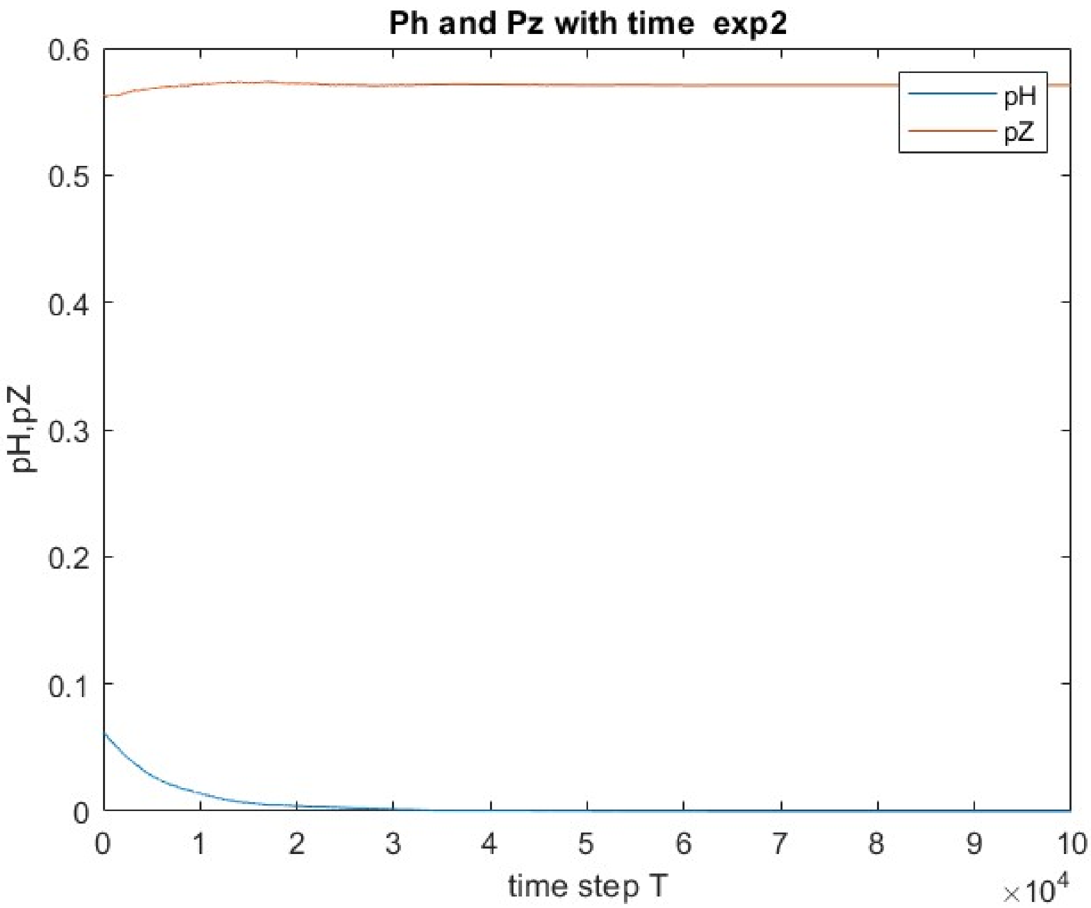


From PZ curve we see that the number of Zombies becomes constant after disturbance, the process is quick, probably because humans die quickly, and zombies stabilize

Empirical values of Ph and Pz, as follows:

	T = 10	T= 100	T=1000	T= 10000	T= 34718	T=100000
pZ	0.06190000	0.0612000	0.0542000	0.014000000	0.0006000	0
pH	0.562700000	0.56270000	0.56270000	0.57130000	0.5713000	0.5707000

Now, we will compare the curves of pH and pZ and assess whether convergence is achieved:



The process converges and there is an equilibrium state

From empirical probabilities and graph: pH is tending to zero, which is the equilibrium
Pz is tending to 0.57070000, which is the equilibrium for zombies. This means that stability of the system is eventually achieved. In this equilibrium humans have been destroyed and they are almost extinct to 0. At this point zombies can remain in their number as zombies do not kill each other.