Fifth Order Improved Runge-Kutta Method for Solving Ordinary Differential Equations

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Abstract: In this paper, the fifth order Improved Runge-Kutta method (IRK5) that uses just five function evaluations per step is developed. The method proposed here are derived with only five stages which results in lower number of function evaluations. Therefore, IRK5 has a lower computational cost than the classical fifth order Runge-Kutta method (RK5). Here, the order conditions of the method are obtained up to order six and the coefficients of the fifth order method are determined by minimizing the error norm of the sixth order method. Numerical examples are given to illustrate the computational efficiency and accuracy of IRK5 in compared with RK5.

Key–Words: Improved Runge-Kutta methods; Two-step methods; Order conditions; Ordinary differential equations

1 Introduction

Consider the numerical solution of the initial value problem for the system of ordinary differential equation

$$y'(x) = f(x, y(x)), \qquad x \in [x_0\ X], \tag{1}$$
$$y(x_0) = y_0.$$

One of the most common method for solving numerically (1) is Runge-Kutta (RK) method. Most efforts to increase the order of RK method, have been accomplished by increasing the number of Taylor's series terms used and thus the number of function evaluations. The RK method of order p has a local error over the step size *h* of $O(h^{p+1})$.
Many authors have attempted to increase the efficiency of RK methods with a lower number of function evaluations required. As a result, Goeken et.al [1] proposed a class of Runge-Kutta method with higher derivatives approximations for the third and fourth-order method. Xinyuan [2] presented a class of Runge-Kutta formulae of order three and four with reduced evaluations of function. Phohomsiri and Udwadia [3] constructed the Accelerated Runge-Kutta integration schemes for the third-order method using two functions evaluation per step. Udwadia and Farahani [4] developed the Accelerated Runge-Kutta methods for higher orders. However most of the presented methods are obtained for the autonomous system while the Improved Runge-kutta methods (IRK) can be used for autonomous as well as non-autonomous systems. Rabiei et al ([5]) constructed the IRK methods and obtained the order conditions of the methods up to order five. The obtained methods were of order three and four, also the convergence and stability region of the methods were also discussed.

The IRK methods arise from the classical RK methods, can also be considered as a special class of two-step methods. That is, the approximate solution y_{n+1} is calculated using the values of y_n and y_{n-1}. Our method introduces the new terms of k_{-i}, which are calculated from k_i, $(i > 2)$ in the previous step. The scheme proposed herein has a lower number of function evaluations than the RK methods while maintaining the same order of local accuracy.
In section 2, we give a general idea of IRK method and order conditions of the method up to order six are given in section 3. In section 4, we derived the fifth order methods with five stages. Finally the numerical results and discussions are presented in section 5.

2 General form of IRK method

The general form of the proposed IRK method in this paper with *s*-stage for solving equation (1) has the form

$$\begin{aligned} y_{n+1} &= (1-\alpha)\, y_n - \alpha\, y_{n-1} \\ &\quad + h\left(b_1 k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i})\right), \end{aligned}$$

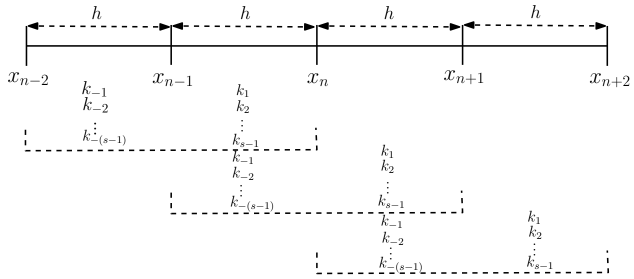

Figure 1: General construction of IRK method

for $1 \leq n \leq N-1$, where

$$\begin{aligned} k_1 &= f(x_n, y_n), \qquad\qquad k_{-1} = f(x_{n-1}, y_{n-1}), \\ k_i &= f(x_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij} k_j), \\ k_{-i} &= f(x_{n-1} + c_i h, y_{n-1} + h\sum_{j=1}^{i-1} a_{ij} k_{-j}). \end{aligned} \tag{2}$$

for $2 \leq i \leq s$, $c_2, \ldots, c_s \in [0\ 1]$ and f depends on both x and y while k_i and k_{-i} depend on the values of k_j and k_{-j} for $j = 1, ..., i-1$. Here *s* is the number of function evaluations performed at each step and increases with the order of local accuracy of the IRK method. In each step we only need to evaluate the values of $k_1, k_2, \ldots,$ while $k_{-1}, k_{-2}, \ldots$ are calculated from the previous step Figure (1) shows this idea.

The accelerated Runge-Kutta method (see [4]), is derived purposely for solving autonomous first order ODEs where the stage or function evaluation involved is of the form

$$k_i = f(y(x_n + a_{i-1} h k_{i-1}))$$

where k_i is a function of y only and the term involved k_{i-1}. There are two improvement used here, the first one is the function f is not autonomous, thus the method is not specific for $y' = f(y(x))$ but it can be used for solving both the autonomous equations as well as the more general differential equations $y' = f(x, y(x))$. The second improvement is that the internal stages k_i and k_{-i} contain more k values which are defined as $\sum_{j=1}^{i-1} a_j k_j$ for $i = 1, ..., s$, compared to the accelerated Runge-Kutta method in which their methods contain only one k value. This additional k values aimed to make the methods more accurate. Note that IRK method is not self-starting therefore a one-step method must provide the approximate solution y_1 at first step. The one-step method must be of appropriate order to ensure that the difference $y_1 - y(x_1)$ is order of *p* or higher. For example the Runge-Kutta method is one of the most popular one-step method that we can use to approximate the starting value for IRK method. In this paper, without loss the generality we derived the method with $\alpha = 0$, so the IRK method in formulae (2) can be represented by

$$y_{n+1} = y_n + h\left(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i})\right),$$

for $1 \leq n \leq N-1$, where

$$\begin{aligned} k_1 &= f(x_n, y_n), \qquad\qquad k_{-1} = f(x_{n-1}, y_{n-1}), \\ k_i &= f(x_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij} k_j), \\ k_{-i} &= f(x_{n-1} + c_i h, y_{n-1} + h\sum_{j=1}^{i-1} a_{ij} k_{-j}). \end{aligned} \tag{3}$$

for $2 \leq i \leq s$. It is convenient to represent (3) by Table 1. To determine the coefficients of method given by (3), the IRK method expression (3) is expanded using the Taylor's series expansion. After some algebraic simplifications this expansion is equated to the true solution $y(x_{n+1})$ at x_{n+1} that is given by Taylor's series expansion. This results will be in the system of nonlinear algebraic equations which is denoted as *order conditions*. We try to solve as many order conditions as possible because the highest power of *h* for which all of the order equations are satisfied is the order of the resulting IRK method. A great deal of algebraic and numeric calculations is required for the above process which were mainly performed using Maple.

Table 1: Table of coefficients for explicit IRK method ($\alpha = 0$)

$$\begin{array}{c|cccccc} 0 & & & & & & \\ c_2 & a_{21} & & & & & \\ c_3 & a_{31} & a_{32} & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ c_s & a_{s1} & a_{s2} & \ldots & a_{ss-1} & & \\ \hline b_{-1} & b_1 & b_2 & \ldots & b_{s-1} & b_s \end{array}$$

3 Order conditions

To find the order conditions for the method, first we used the Taylor's series expansion (see section 2). Rabiei et al [5] derived the order conditions for IRK method up to order five. Here, we developed the Taylor series expansion and obtained the order conditions up to order six as presented in Table 2.

Table 2: Order conditions of IRK method up to order six.

Order of method	Order condition
first order	$b_1 - b_{-1} = 1$
second order	$b_{-1} + \sum_i b_i = \frac{1}{2}$
third order	$\sum_i b_i c_i = \frac{5}{12}$
fourth order	$\sum_i b_i c_i^2 = \frac{1}{3}$ $\sum_{i,j} b_i a_{ij} c_j = \frac{1}{6}$
fifth order	$\sum_{i,j} b_i c_i^3 = \frac{31}{120}$ $\sum_{i,j} b_i c_i a_{ij} c_j = \frac{31}{240}$ $\sum_{i,j} b_i a_{ij} c_j^2 = \frac{31}{360}$ $\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = \frac{31}{720}$
sixth order	$\sum_i b_i c_i^4 = \frac{1}{5}$ $\sum_{i,j} b_i c_i^2 a_{ij} c_j = \frac{1}{10}$ $\sum_{i,j,k} b_i a_{ij} c_j a_{ik} c_k = \frac{1}{20}$ $\sum_{i,j,k} b_i c_i a_{ij} c_j^2 = \frac{1}{15}$ $\sum_{i,j} b_i a_{ij} c_j^3 = \frac{1}{20}$ $\sum_{i,j,k} b_i c_i a_{ij} a_{jk} c_k = \frac{1}{30}$ $\sum_{i,j,k} b_i a_{ij} c_j a_{jk} c_k = \frac{1}{40}$ $\sum_{i,j,k} b_i a_{ij} a_{jk} c_k^2 = \frac{1}{60}$ $\sum_{i,j,k,m} b_i a_{ij} a_{jk} a_{km} c_m = \frac{1}{120}$

4 Fifth order method with five-stages

In s=3, the general form of IRK and table of coefficients are

$$
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
c_4 & a_{41} & a_{42} & a_{43} & & \\
c_5 & a_{51} & a_{52} & a_{53} & a_{54} & \\
\hline
b_{-1} & b_1 & b_2 & b_3 & b_4 & b_5
\end{array}
$$

$$
y_{n+1} = y_n + h\left(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{5} b_i (k_i - k_{-i})\right),
$$

$$
\begin{aligned}
k_1 &= f(x_n, y_n), \qquad k_{-1} = f(x_{n-1}, y_{n-1}),\\
k_i &= f(x_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij} k_j),\\
k_{-i} &= f(x_{n-1} + c_i h, y_{n-1} + h\sum_{j=1}^{i-1} a_{ij} k_{-j}).
\end{aligned} \tag{4}
$$

for $2 \le i \le 5$, $c_i \in [0\ 1]$ and $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 2, \ldots, 5$. To determine the coefficients of IRK5, they need to satisfy the order conditions of the method up to order five(see Table 2). Here, we choose $c_2 = \frac{1}{4}, c_3 = \frac{1}{4}, c_4 = \frac{1}{2}, c_5 = \frac{3}{4}$ and obtained the values of $b_{-1}, b_1, b_2, b_3, b_4$ and b_5 depend on other parameters by solving the first sixth equations of order conditions from Table 2.

$$
b_{-1} = \frac{1}{45},\ b_1 = \frac{46}{45},
$$

$$
\begin{aligned}
b_2 = -\frac{1}{90\ a_{32}}(6a_{32} - 58a_{52} + 60 + 9a_{42}\\
+9a_{43} - 58a_{53} - 116a_{54}),\\
b_3 = -\frac{1}{90\ a_{32}}(-58a_{52} + 60 + 9a_{42} + 9a_{43}\\
-58a_{53} - 116a_{54}),
\end{aligned}
$$

$$
b_4 = -\frac{1}{10},\ b_5 = \frac{29}{45}.
$$

Substitute the values of $b_{-1}, b_1, b_2, b_3, b_4$ and b_5 in the following reminder order conditions for the fifth order method

$$
\sum_{i=3}^{5}\sum_{j=2}^{i-1} b_i c_i a_{ij} c_j = \frac{31}{240},
$$

$$
\sum_{i=3}^{5}\sum_{j=2}^{i-1} b_i a_{ij} c_j^2 = \frac{31}{360},
$$

Table 3: Coefficients of IRK5 method

0					
0.25	0.25				
0.25	-0.0082	0.2586			
0.5	0.3860	0.5312	0.6444		
0.75	0.2060	-0.9002	0.8918	0.5517	
0.0222	1.0222	0.0403	-0.1070	-0.1	0.6444

$$\sum_{i=4}^{5}\sum_{j=3}^{i-1}\sum_{k=2}^{j-1} b_i a_{ij} a_{jk} c_k = \frac{31}{720}.$$

By choosing a_{32}, a_{43} and a_{53} as free parameters we have

$$a_{42} = \frac{9}{32}a_{43}a_{32} - \frac{29}{16}a_{53}a_{32} - a_{43} + \frac{31}{64},$$

$$a_{54} = \frac{151}{7424} + \frac{81}{3712}a_{43}a_{32} - \frac{9}{64}a_{53}a_{32} - a_{53},$$

$$a_{54} = \frac{16}{29}.$$

To minimize the error norm of the sixth order method we substitute all the parameters into as many as possible order conditions for the sixth order method. Here, we choose the second and third equations from the order conditions of the sixth order method as follows:

$$f_6 = \sum_{i=4}^{5}\sum_{j=3}^{i-1}\sum_{k=2}^{j-1} b_i c_i a_{ij} a_{jk} c_k - \frac{1}{30},$$

$$f_8 = \sum_{i=4}^{5}\sum_{j=3}^{i-1}\sum_{k=2}^{j-1} b_i a_{ij} a_{jk} c_k^2 - \frac{1}{60},$$

Define

$$\tau = \sqrt{f_6^2 + f_8^2}.$$

By minimizing the value of τ we obtained the values of free parameters using Maple software.

$$a_{32} = 0.2586,\ a_{43} = 0.6444,\ a_{53} = 0.8918.$$

Substituting the free parameters we obtained the values of other coefficients which are presented in table 3

Table 4: Coefficients of Butcher's RK5

0					
$\frac{1}{4}$	$\frac{1}{4}$				
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$			
$\frac{1}{2}$	0	$\frac{-1}{2}$	1		
$\frac{3}{4}$	$\frac{3}{16}$	0	0	$\frac{9}{16}$	
1	$\frac{-3}{7}$	$\frac{2}{7}$	$\frac{12}{7}$	$\frac{-12}{7}$	$\frac{8}{7}$
	$\frac{7}{90}$	0	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$

5 Numerical examples

In this section, we tested a standard set of initial value problems to show the efficiency and accuracy of the proposed methods. The exact solution $y(x)$ is used to estimate the global error as well as to approximate the starting value of y_1 at the first step $[x_0\ x_1]$. The following problems are solved for $x \in [0\ 10]$.

Problem 1 (an oscillatory problem, see [7])

$$y' = ycos(x), \qquad y(0) = 1,$$

Exact solution:

$$y(x) = e^{sin(x)}.$$

Problem 2 (see [8])

$$\begin{aligned} y_1' &= -2y_1 + y_2 + 2sin(x), \\ y_2' &= y_1 - 2y_2 + 2(cos(x) - sin(x)), \\ y_1(0) &= 2, \quad y_2(2) = 3, \end{aligned}$$

Exact solution:

$$\begin{aligned} y_1(x) &= 2e^{-x} + sin(x), \\ y_2(x) &= 2e^{-x} + cos(x). \end{aligned}$$

The number of function evaluations versus the log(maximum global error) for the tested problems are shown in Figures 2 and 3.

6 Discussion and conclusion

From Figures 2 and 3 we observe that for the both tested problems the new method has lower number of function evaluations compared to the existing method. As a conclusion, the fifth order improved Runge-Kutta

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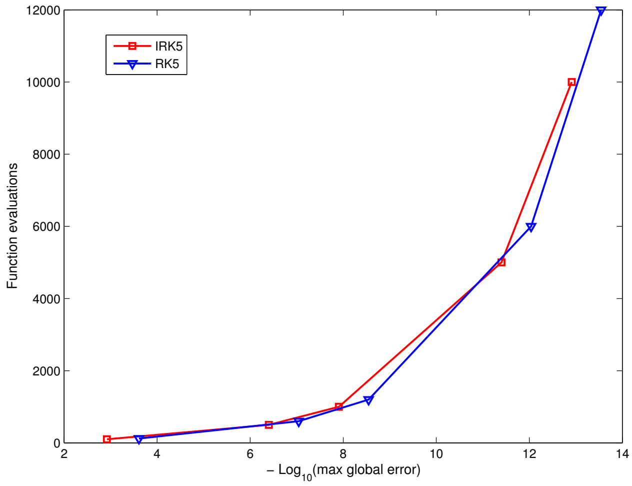

Figure 2: Number of function evaluations versus of maximum global error for problem 1

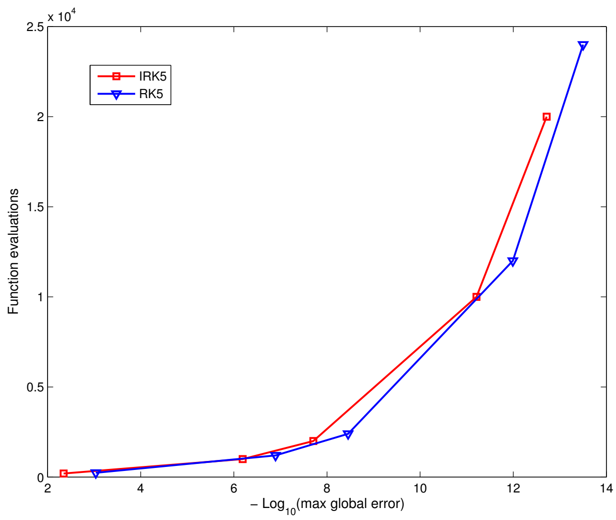

Figure 3: Number of function evaluations versus of maximum global error for problem 2

methods have been developed for numerical integration of first order ordinary differential equations with reduced number of function evaluations required per step. The order conditions of the new method are derived up to order six and by satisfying the appropriate order conditions we obtained the methods of order five with the lower stages. The IRK5 method is almost two-step in nature and is computationally more efficient compared to the existing classical RK5 method .

References:

[1] D. Goeken, O. Johnson, Runge-Kutta with higher order derivative approximations. Appl. Numer. Math , 34, 207-218 (2000)

[2] W. Xinyuan, A class of Runge-Kutta formulae of order three and four with reduced evaluations of function. Appl. Math. Comput, 146, 417-432 (2003)

[3] P. Phohomsiri, F. E. Udwadia, Acceleration of Runge-Kutta integration schemes. Discrit. Dynamic. Nature. Soci. 2, 307-314 (2004)

[4] F.E. Udwadia, A. Farahani, Accelerated Runge-Kutta methods. Discrit. Dynamic. Nature. Soci. 2008, doi:10.1155/2008/790619 (2008)

[5] F. Rabiei, F. Ismail, M. Suleiman, N. Arifin, Improved Runge-Kutta method for solving ordinary differential equation. Math. Computer. Apllication, submettied (2011)

[6] J. C. Butcher, On Runge-Kutta processes of highh order. J. Australian. Math. Soci. 4, 179-194 (1964)

[7] T.e. Hull, W. H. Enright, B. M. Fellen ,A. E. Sedgwick, Comparing numerical methods for ordinary differential equations, SIAM J. Numer. Anal. vol 9, no4, 603-637,(1980)

[8] R. D'Ambrosio, M. Ferro, Z. Jackiewicz, B. paternoster, Collocation based two-step runge-Kutta method for ordinary differential equations, ICCSA 2008, Part II, LNCS 5073, pp. 736-751, (2008)

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