

Riemann Zeta function

Riemann Zeta func: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
for $\operatorname{Re}(s) > 1$.

The function converges absolutely
and uniformly on compact
subsets of $\{\operatorname{Re}(n) > 3\}$, as
 $|n^s| = n^{\operatorname{Re}(s)}$.

$\zeta(s)$ is holomorphic function

on $\operatorname{Re}(s) > 1$.

Connection with primes

Euler product formula: for $\text{Re}(s) > 1$,

The infinite product $\prod_{p \in P} \left(1 - \frac{1}{p^s}\right)$

converges and

$$\frac{1}{S(s)} = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right),$$

where P is the set of positive

primes $\{2, 3, 5, 7, \dots\}$

Proof: Since $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $\text{Re}s > 1$, the prod

$\prod_{p \in P} \left(1 - \frac{1}{p^s}\right)$ also converges.

Fix s with $\Re s > 1$ if $\epsilon > 0$.

Let's choose N so large that

$$\sum_{n=N+1}^{\infty} \left| \frac{1}{n^s} \right| < \epsilon$$

Now $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$

$$\left(1 - \frac{1}{2^s} \right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots$$

$$\left(1 - \frac{1}{3^s} \right) \left(1 - \frac{1}{2^s} \right) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \dots$$

This is a well-known

number-theoretic procedure known

as sieve of Eratosthenes.

$$\text{Show: } \left(1 - \frac{1}{(p_N)^s}\right) \cdot \left(1 - \frac{1}{(p_{N-1})^s}\right) \cdots$$

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{(p_{N+1})^s} + \cdots$$

By the choice of N , if $n > N$ then:

$$\left| \left(\prod_{j=1}^n \left(1 - \frac{1}{(p_j)^s}\right) \right) \zeta(s) - 1 \right| < \epsilon.$$

as desired. Completing the

~~proof that~~ $\frac{1}{\zeta(s)} = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right)$

Relation to the gamma function

for $\operatorname{Re}(\beta) > 1$

$$S_{\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} dt \frac{t^{\beta-1} e^{-t}}{1 - e^{-t}}$$

proof: The integral converges for the same reason that the gamma function converges: For $j = 1, 2, \dots$

$$j^{-\beta} \int_0^{\infty} t^{\beta-1} e^{-t} dt \stackrel{(jt = t)}{=} \int_0^{\infty} t^{\beta-1} e^{-jt} dt$$

or

$$j^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} e^{-jt} dt$$

Summing over j gives

$$S_{\beta} = \sum_{j=1}^{\infty} j^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} dt t^{\beta-1} \sum_{j=1}^{\infty} e^{-jt}$$

thus: $S_{\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} dt \frac{t^{\beta-1} e^{-t}}{1 - e^{-t}}$

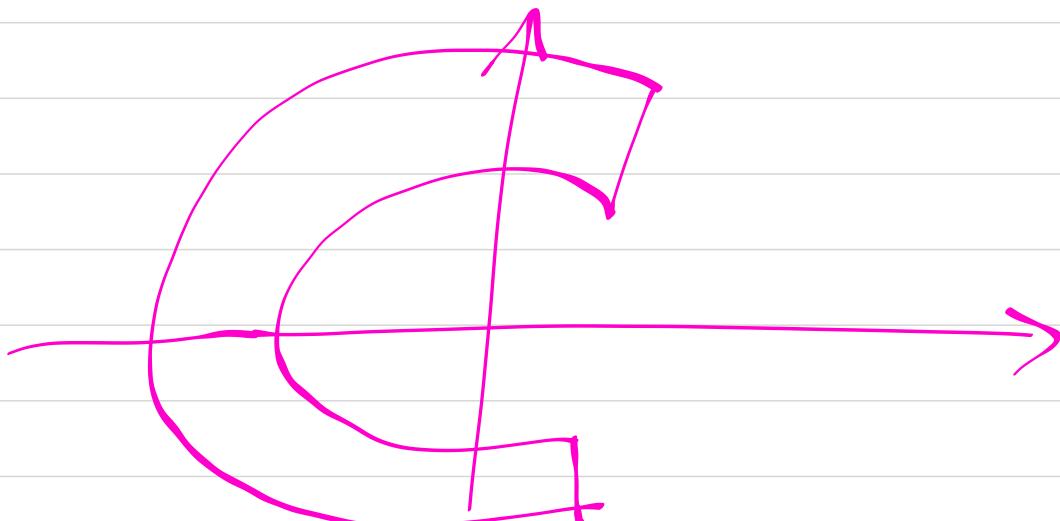
for $\beta \in \mathbb{C}$, define the analytic function $u(w) = \frac{(f-w)^{\beta-1}}{1-e^{-w}} e^{-w}$

on the region $\mathcal{C} \setminus \{w : \operatorname{Re} w \geq 0, \text{ or } w=0\}$

The func. is well-defined if we take $-\pi < \arg(-w) < \pi$. Also define, for $0 < \epsilon \neq 2\pi k$, Hankel func.

$$H_\epsilon(\beta) = \int_{C_\epsilon} u(w) dw$$

where $C_\epsilon \supset C_\epsilon(s)$ is the Hankel contour shown below



The interpretation is the following:
 linear portions of the contour
 are understood to lie just
 above and just below the
 real axis (at distance δ), while
 the circular portion is understood
 to have radius ϵ .

Notice that, for $0 < \epsilon_1 < \epsilon_2 < 2\pi$

$H_{\epsilon_1}(\beta) = H_{\epsilon_2}(\beta)$ since the region
 bounded by $C_{\epsilon_1}, C_{\epsilon_2}$ contains no
 poles of u .

for $0 < \epsilon < 2\pi$ & $\operatorname{Re}(\beta) > 1$

we have

$$S(\beta) = - \frac{H_\epsilon(\beta)}{2i \sin(\pi\beta)} \frac{1}{\Gamma(\beta)}$$

As ω remains, $S(\beta)$ continues
analytically to $\sigma \setminus \gamma_1 \gamma_2$

Proof: Parametrizing C_ϵ :

$$H_\epsilon = \int_{-\infty}^{\tilde{\epsilon}} dt \quad \left(1 - e^{-(t+i\delta)} \right)^{-1}$$

$$+ \exp\left((\beta-1) \log(-(t+i\delta)) - (t+i\delta)\right)$$

$$+ \int_{\tilde{\epsilon}}^{\infty} (\text{same}) \, dt$$

integral

$$+ \int_{\tilde{\delta}}^{2\pi - \tilde{\delta}} \frac{i\epsilon e^{i\theta}}{1 - e^{-\epsilon e^{i\theta}}} (-\epsilon e^{i\theta})^{3-1}$$

$$+ e^{-\epsilon e^{i\theta}} = I + II + III$$

Here $\tilde{\delta}$ represents the radian measure
of the initial point of the

circular portion of the curve C_ϵ and $\tilde{\epsilon}$ represents the value of the parameter at which the linear portion of the curve C_ϵ meets the circular portion.

Now for small ϵ ,

$$\left| \frac{1 - e^{-\epsilon e^{i\theta}}}{1 - e^{-\epsilon}} \right| \geq \left| 1 - e^{-\epsilon} \right|$$

$\geq \epsilon/2$, thus

$$|\text{III}| \leq 2\pi \max_{\theta} \left| (\epsilon e^{i\theta})^{3-1} \right| \times \left| \frac{e^{-\epsilon e^{i\theta}}}{e^{-\epsilon}} \cdot \frac{-\epsilon}{\epsilon/2} \right|$$

$$\leq 4\pi \epsilon^{\operatorname{Re} \beta - 1} e^{-\theta \ln \epsilon} \epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$.

On the other hand, $I + II =$

$$\int_{\infty}^{\tilde{\epsilon}} dt \cdot (1 - e^{-t-i\delta})^{-1} \exp\left\{(\beta-1)$$

$$+ \left[\log \sqrt{t^2 + \delta^2} + i(\delta' - \pi) \right] - t - i\delta \left\{$$

$$+ \int_{\tilde{\epsilon}}^{\infty} dt \cdot (1 - e^{-t+i\delta})^{-1} \exp\left\{(\beta-1)$$

$$+ \left[\log \sqrt{t^2 + \delta^2} + i(\pi - \delta'') \right] - t + i\delta \left\{$$

where $\delta' = \delta'(t)$ and $\delta'' = \delta''(t)$

are chosen so that $(\delta' - \pi) \rightarrow (\pi - \delta'')$

as the args. of the initial and final points on the curve C_ϵ .

Note that the value of $H(\beta)$ is

independent of $\delta > 0$ as long as δ

is small - By uniform convergence

we may let $\delta \rightarrow 0^+$ so that

we may restrict ourselves to

$$\int_{\tilde{\epsilon}}^{\tilde{\epsilon}} dt (1 - e^{-t})^{-1} \exp\{(3-1) \log t - i\pi - t\}$$

$$+ \int_{\tilde{\epsilon}}^{\infty} dt (1 - e^{-t})^{-1} \exp\{(3-1) \log t + i\pi - t\}$$

$$= - \underbrace{(e^{i\pi} 3 - e^{-i\pi} 3)}_{-2i \sin(\pi 3)} \int_{\tilde{\epsilon}}^{\infty} dt \frac{t^{3-1} e^{-t}}{1 - e^{-t}}$$

$$- 2i \sin(\pi 3)$$

As $\tilde{\epsilon} \rightarrow 0^+$, the integral becomes

$\Gamma(3) \zeta(3)$. Thus, we have shown

that $H(\beta) \stackrel{\sim}{=} \lim_{\epsilon \rightarrow 0^+} H_\epsilon(2) = -2i \sin(\pi \beta) \Gamma(3) / \beta$

for the remaining, note that

for w a positive real axis and $w > 1$

$$|\kappa(w)| \leq Aw^{\Re z - 1} e^{-w}, \text{ w/A = A(3)}$$

Thus, κ will be integrable over

C_ϵ for any $z \in \mathbb{C}$ and H_z will define

an entire function of z by diff. under

integral sign. Thus, $S(z) = \underbrace{-H_z(z)}_{2i \sin(\pi z) P(z)}$

defines an analytic continuation

of $S(z)$ to $\mathbb{C} \setminus \mathbb{Z}$. However, $S(z)$

is holomorphic at $z = 2, 3, 4, \dots$ and

on $\{z : \operatorname{Re} z > 1\}$. Moreover, the simple

poles of $P(z)$ at $z = 0, -1, -2, \dots$ cancel the

simple zeros of $\sin \pi z$ at these values.

Therefore, the Riemann removable singularities

Theorem implies that the denominator continues holomorphically to a nonvanishing function on $\{z : \operatorname{Re} z \geq 3/2\}$, except for a zero at $z=1$. In conclusion,

$S(z)$ continues holomorphically to $\mathbb{C} \setminus \{1\}$.

Simple pole at $z=1$

Let us now demonstrate that $S(z)$ has a simple pole at $z=1$ with residue 1.

for $z=1$, $\sin \pi z = 0$, so that

$$I + II = 0 \quad \text{and}$$

$$II = \int_0^{2\pi} d\theta \frac{e^{-\epsilon e^{i\theta}}}{1 - e^{-\epsilon e^{i\theta}}} i \epsilon e^{i\theta}$$

$$= \int_0^{2\pi} d\theta \frac{i \epsilon e^{i\theta}}{\frac{e^{\epsilon e^{i\theta}} - 1}{e^{\epsilon e^{i\theta}}}}$$

$$= \int_0^{2\pi} d\theta \frac{i\varepsilon e^{i\theta}}{(1+\varepsilon e^{i\theta}+R)-1}, \quad |R| \leq C \cdot \varepsilon^2$$

$$= \int_0^{2\pi} d\theta \frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta} + R} \stackrel{\varepsilon \rightarrow 0^+}{=} 2\pi i$$

As a result:

$$\lim_{z \rightarrow 1} (\beta - 1) S(z) = \lim_{z \rightarrow 1} \frac{-H(z)}{P(z)} \frac{\beta - 1}{2i \sin(\pi z)}$$

$$= \frac{-2\pi i}{1 - 2\pi i} = 1.$$

Q.e.d.

Functional Relation

We keep motivating ourselves to acquire more explicit information on the continuation of $S(z)$ to $\mathbb{C} \setminus \{1\}$. In keeping on this direction, let us derive the

functional relation

$$\zeta(1-z) = 2 \zeta(z) \Gamma(z) \cos\left(\frac{\pi}{2} z\right) (2\pi)^{-z}$$

We start by considering $z \in \{0, -1, -2, \dots\}$

Suppose $\operatorname{Re} z > 0$. Let $0 < \varepsilon < 2\pi$ and let n be a positive integer. The idea is to relate $H(z)$ to $H_{2n+1}\pi(z)$ using the calculus of residues and then let $n \rightarrow \infty$. Thus,

$$\frac{1}{2\pi i} \left[H_{2n+1}\pi(z) - H_\varepsilon(z) \right]$$

is the sum of residues of u in the region $\operatorname{Re} z_n = \{z : \varepsilon < |z| < (2n+1)\pi\}$.

The poles of u are simple: $\pm 2\pi k i$.

The residue at $\pm 2\pi k i$ ($k > 0$) is

$$\lim_{\omega \rightarrow \pm 2\pi k i} (\omega \mp 2\pi k i) \cdot \frac{(-\omega)^{z-1} e^{-\omega}}{1 - e^{-\omega}}$$

$$\text{But } \frac{(\omega \mp 2\pi k i)^{-\omega}}{1 - e^{-\omega}} \rightarrow 1$$

so the residue of μ at $\pm 2\pi k i$ is

$$\lim_{\omega \rightarrow \pm 2\pi k i} (\omega)^{\beta-1} = (\mp 2\pi k i)^{\beta-1}$$

where we continue to use the principal branch of the logarithm. Hence, we have

$$\exp((\beta-1) \log(\mp 2\pi k i))$$

$$= \exp\left((\beta-1) \left(\log 2\pi + \frac{i\pi}{2}\right)\right)$$

$$= \exp\left(\mp i(\beta-1) \frac{\pi}{2} (2\pi k)^{\beta-1}\right)$$

In summary,

$$H_{(2m+1)}(\pi)(\beta) - H_E(\beta) = 2\pi i \sum \text{of residues of } \mu \text{ in } \mathbb{R}_{\text{even}}$$

$$= 4\pi i \cos \frac{\pi}{2}(\beta-1) \sum_{k=1}^n (2\pi k)^{\beta-1}$$

Since $|U(\omega)| \leq A_3 |\omega|^{Re z - 1} \frac{e^{-Re \omega}}{|1 - e^{-\omega}|}$

So, on $\operatorname{Re} z > 0$:

$$\begin{aligned} -H_E(z) &= 4\pi i \cos\left(\frac{\pi}{2}(z-1)\right) (2\pi)^{z-1} \sum_{k=1}^{\infty} k^{z-1} \\ &= 4\pi i (2\pi)^{z-1} \sin\left(\frac{\pi}{2}z\right) \zeta(1-z) \end{aligned}$$

Therefore, $S(z) = \frac{-H_E(z)}{2i \sin(\pi z)} \frac{1}{\Gamma(z)}$

$$= (2\pi)^z \frac{\sin\left(\frac{\pi}{2}z\right) \zeta(1-z)}{\sin(\pi z) \Gamma(z)}$$

$$= (2\pi)^z \frac{1}{2 \cos\left(\frac{\pi}{2}z\right)} \frac{\zeta(1-z)}{\Gamma(z)}$$

g.e.d.