THE SIMPLEX METHOD
STARTING SOLUTIONS

INDR 262-INTRODUCTION TO OPTIMIZATION METHODS

Metin Türkay
Department of Industrial Engineering
Koç University, Istanbul

ORIGIN IS NOT BASIC FEASIBLE

➢ **If the origin IS NOT basic feasible, how should we address the initialization?**

$$\begin{aligned} \text{maximize} \quad & Z = c^T x \\ \text{subject to} \quad & \\ & A_1 x \le b_1 \\ & A_2 x \ge b_2 \\ & A_3 x = b_3 \\ & x \ge 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \text{maximize} \quad & Z \\ & Z - c^T x = 0 \\ & A_1 x + I x_s = b_1 \\ & A_2 x - I x_e = b_2 \\ & A_3 x = b_3 \\ & x, x_s, x_e \ge 0 \end{aligned}$$

$\Downarrow$

If $b_1, b_2, b_3 \ge 0$, then we have an initialization problem. Since $x_e \ge 0$, selecting them would be violation of non-negativity constraints, and equality constraints do not have easily assigned basic variables.

ARTIFICIAL VARIABLES

➢ **Define artificial variables to address the initialization of complicating constraints ($\geq$ and $=$)**

$$
\begin{aligned}
&\text{maximize} \quad Z \\
&Z - c^T x && = 0 \\
&A_1 x + I x_s && = b_1 \\
&A_2 x \qquad - I x_e + I \bar{x}_2 && = b_2 \\
&A_3 x \qquad\qquad + I \bar{x}_3 && = b_3 \\
&x, x_s, x_e, \bar{x}_2, \bar{x}_3 \geq 0
\end{aligned}
$$

➢ When $x_s, x_e,$ and $\bar{x}_3$ are selected as initial basic variables, the initialization problem is solved.

➢ However, the articial variables are not natural to the problem and should be forced to become non-basic.

There are 2 different approaches to forcing the articial variables to become non-basic.

1. big-M formulation
2. 2-Phase method

BIG-M FORMULATION

- **The big-M formulation introduces very large penalty (*M*) for the artificial variables in the modified objective function.**

$$
\begin{array}{ll}
\text{maximize} & Z \\
& Z - c^T x \qquad\qquad + M\bar{x}_2 + M\bar{x}_3 = 0 \\
& A_1 x + I x_s \qquad\qquad\qquad = b_1 \\
& A_2 x \qquad - I x_e + I\bar{x}_2 \qquad = b_2 \\
& A_3 x \qquad\qquad\qquad + I\bar{x}_3 = b_3 \\
& x, x_s, x_e, \bar{x}_2, \bar{x}_3 \geq 0
\end{array}
$$

Then, the problem is solved with the simplex method.

2-PHASE METHOD

- **The problem is solved in 2 phases:**

  1. **Feasibility Problem where the objective function is replaced by minimization of the sum of artificial variables.**

$$
\begin{aligned}
&\text{minimize } Z \\
&Z \qquad\qquad\qquad\qquad - \bar{x}_2 - \bar{x}_3 = 0 \\
&A_1 x + I x_s \qquad\qquad\qquad\quad = b_1 \\
&A_2 x \qquad - I x_e + I \bar{x}_2 \qquad = b_2 \\
&A_3 x \qquad\qquad\qquad + I \bar{x}_3 = b_3 \\
&x, x_s, x_e, \bar{x}_2, \bar{x}_3 \geq 0
\end{aligned}
$$

When the optimal solution is found, then the feasibility is checked by considering the optimal z value (it should be 0 for a feasible solution to the original problem) and the artificial variables cannot be basic.

2-PHASE METHOD

2. **Optimality Problem where the original objective function is restored, the artifial variables are eliminated and the optimal solution of the Phase 1 problem is used (where $[A'\ S'\ E']$ and b' indicate the constraint matrix and right-hand-side at the optimal solution of the Phase 1 problem).**

$$\text{maximize } Z$$

$$\begin{aligned}
Z - c^T x \quad & =0 \\
A_1' x + S_1' x_s + E_1' x_e &= b_1' \\
A_2' x + S_2' x_s + E_2' x_e &= b_2' \\
A_3' x + S_3' x_s + E_3' x_e &= b_3' \\
x, x_s, x_e &\geq 0
\end{aligned}$$

The optimal solution is found starting from this basic feasible solution in Phase 1 to the original problem.

EXAMPLE

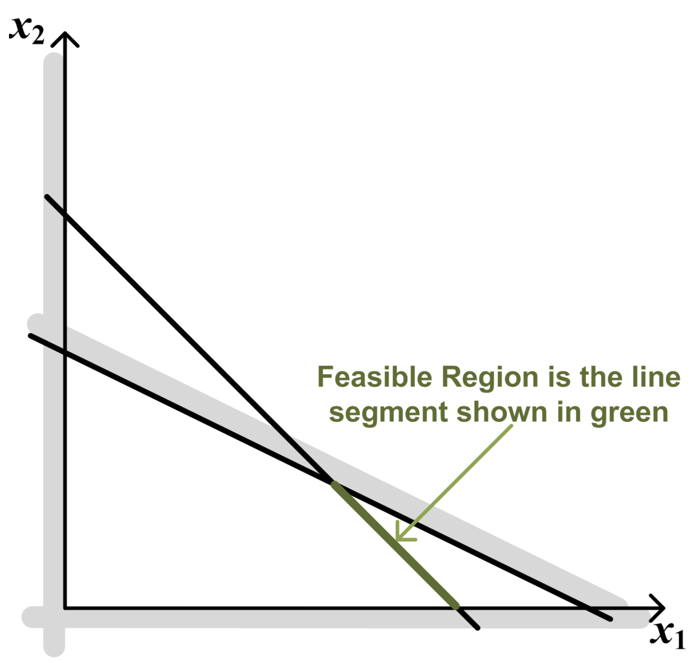
maximize $Z = 2x_1 + 3x_2$
subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ x_1 + x_2 &= 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$\Longrightarrow$

maximize $Z = 2x_1 + 3x_2$
subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 + x_2 &= 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$



BIG-M

maximize $Z = 2x_1 + 3x_2$
subject to

$$x_1 + 2x_2 + x_3 = 4$$
$$x_1 + x_2 = 3$$
$$x_1, x_2, x_3 \geq 0$$

$\Longrightarrow$

maximize $Z = 2x_1 + 3x_2 \quad -M\bar{x}_4$
subject to

$$x_1 + 2x_2 + x_3 = 4$$
$$x_1 + x_2 + \bar{x}_4 = 3$$
$$x_1, x_2, x_3 \geq 0$$

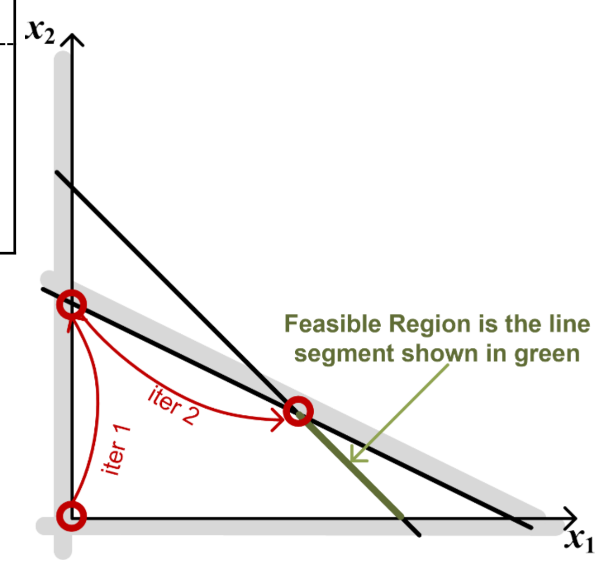
x_B	Z	x_1	x_2	x_3	$\bar{x}_4$	RHS	Ratio
Z	1	-2	-3	0	$\boxed{M}$	0	---
x_3	0	1	2	1	0	4	
$\bar{x}_4$	0	1	1	0	1	3	
Z	1	$-M-2$	$-M-3$	0	0	$-3M$	---
x_3	0	1	2	1	0	4	4/2=2
$\bar{x}_4$	0	1	1	0	1	3	3/1=3

- M in row 0 for a basic variable is inconsistent with the definition of basic variable!
- M can be enforced to 0 with a Type 2 ero.

BIG-M

x_B	Z	x_1	x_2	x_3	$\bar{x}_4$	RHS	Ratio
Z	1	$-1/2(M+1)$	0	$1/2(M+3)$	0	$-M+6$	---
x_2	0	1/2	1	1/2	0	2	2/(1/2)=4
$\bar{x}_4$	0	1/2	0	$-1/2$	1	1	1/(1/2)=2
Z	1	0	0	1	$M+1$	7	---
x_2	0	0	1	1	-1	1	
x_1	0	1	0	-1	2	2	

OPTIMAL SOLUTION: $Z^* = 7, x_1^* = 2, x_2^* = 1$



2-PHASE

maximize $Z = 2x_1 + 3x_2$
subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 \quad &= 4 \\ x_1 + \ x_2 \quad + \bar{x}_4 &= 3 \\ x_1, x_2, x_3, \bar{x}_4 &\geq 0 \end{aligned}$$

Phase 1: $\Longrightarrow$

maximize $-Z$ ($\equiv$minimize Z)
subject to

$$\begin{aligned} -Z \quad\quad\quad + \bar{x}_4 &= 0 \\ x_1 + 2x_2 + x_3 \quad &= 4 \\ x_1 + \ x_2 \quad + \bar{x}_4 &= 3 \\ x_1, x_2, x_3, \bar{x}_4 &\geq 0 \end{aligned}$$

x_B	Z	x_1	x_2	x_3	$\bar{x}_4$	RHS	Ratio
Z	-1	0	0	0	$\boxed{1}$	0	---
x_3	0	1	2	1	0	4	
$\bar{x}_4$	0	1	1	0	1	3	
Z	-1	-1	-1	0	0	-3	---
x_3	0	1	2	1	0	4	4/1=4
$\bar{x}_4$	0	1	1	0	1	3	3/1=3

- 1 in row 0 for a basic variable is inconsistent with the definition of basic variable!
- 1 can be enforced to 0 with a Type 2 ero.

2-PHASE

x_B	Z	x_1	x_2	x_3	$\bar{x}_4$	RHS	Ratio
Z	-1	0	0	0	1	$\boxed{0}$	---
x_3	0	0	1	1	-1	1	
x_1	0	1	1	0	1	3	

- Objective function value for Phase 1 problem is 0.
- No artificial variables in the basis.

- **OPTIMAL SOLUTION for the Phase 1 problem.**
- **This solution is basic feasible to the original problem.**
- **We can construct the Phase 2 problem from the optimal solution to the Phase 1 problem.**

Phase 2: maximize Z

$$Z - 2x_1 - 3x_2$$

$$x_2 + x_3 = 1$$

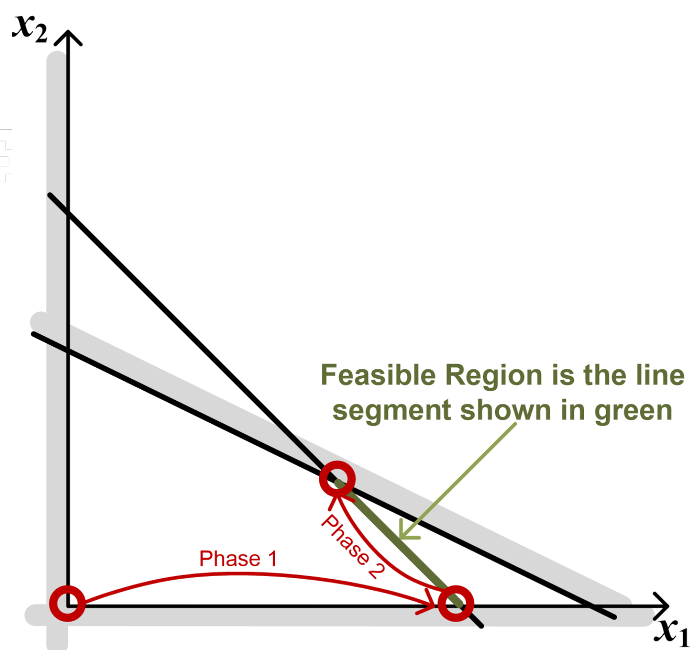
$$x_1 + x_2 = 3$$

$$x_1, x_2, x_3 \geq 0$$

2-PHASE

x_B	Z	x_1	x_2	x_3	RHS	Ratio
Z	1	$\boxed{-2}$	-3	0	0	---
x_3	0	0	1	1	1	
x_1	0	1	1	0	3	
Z	1	0	-1	0	6	---
x_3	0	0	1	1	1	$1/1=1$
x_1	0	1	1	0	3	$3/1=3$
Z	1	0	0	1	7	---
x_2	0	0	1	1	1	
x_1	0	1	0	-1	2	

- -2 in row 0 for a basic variable is inconsistent with the definition of basic variable!
- 1 can be enforced to 0 with a Type 2 ero.



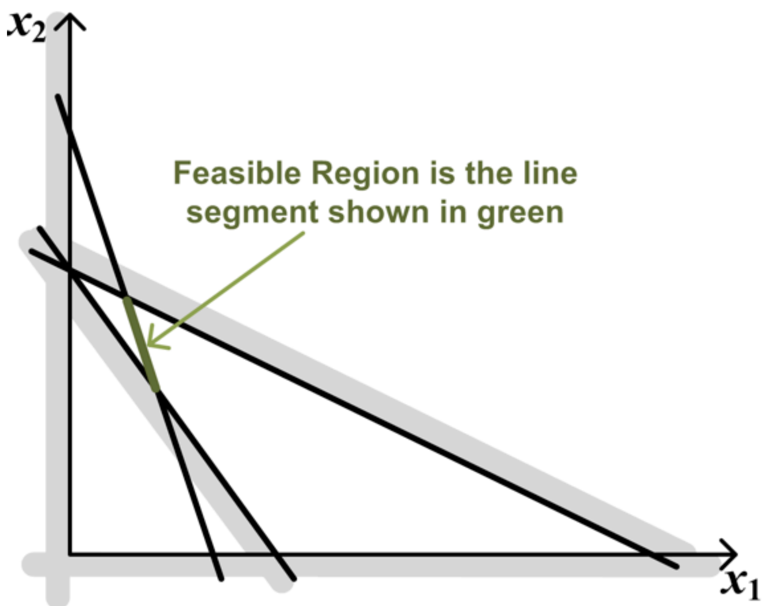
OPTIMAL SOLUTION: $Z^* = 7, x_1^* = 2, x_2^* = 1$

ANOTHER EXAMPLE

$$\text{minimize } Z = 4x_1 + x_2$$
subject to
$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$\Rightarrow$

$$\text{minimize } Z = 4x_1 + x_2$$
subject to
$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$



BIG-M

minimize $Z = 4x_1 + x_2$
subject to

$$
\begin{aligned}
3x_1 + x_2 &= 3\\
4x_1 + 3x_2 - x_3 &= 6\\
x_1 + 2x_2 + x_4 &= 4\\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

$\Longrightarrow$

maximize $-Z$ ($\equiv$minimize Z)

$$
\begin{aligned}
-Z + 4x_1 + x_2 + M\bar{x}_5 + M\bar{x}_6 &= 0\\
3x_1 + x_2 + \bar{x}_5 &= 3\\
4x_1 + 3x_2 - x_3 + \bar{x}_6 &= 6\\
x_1 + 2x_2 + x_4 &= 4\\
x_1, x_2, x_3, x_4, \bar{x}_5, \bar{x}_6 &\geq 0
\end{aligned}
$$

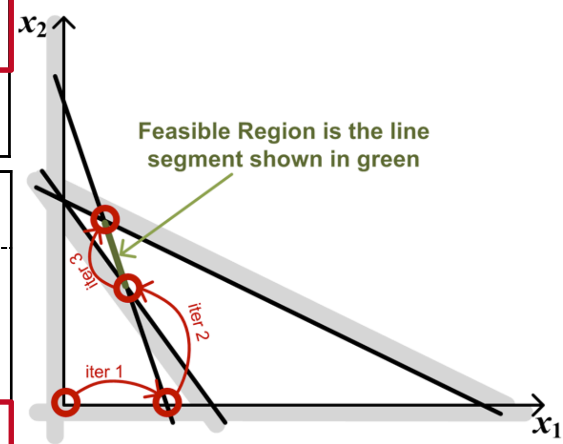
x_B	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	RHS	Ratio
Z	-1	4	1	0	0	$\boxed{M}$	$\boxed{M}$	0	---
$\bar{x}_5$	0	3	1	0	0	1	0	3	
$\bar{x}_6$	0	4	3	-1	0	0	1	6	
x_4	0	1	2	0	1	0	0	4	

- M in row 0 for a basic variable is inconsistent with the definition of basic variable!
- M can be enforced to 0 with a Type 2 ero.

x_B	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	RHS	Ratio
Z	-1	$-7M+4$	$-4M+1$	M	0	0	0	$-9M$	---
$\bar{x}_5$	0	3	1	0	0	1	0	3	3/3=1
$\bar{x}_6$	0	4	3	-1	0	0	1	6	6/4=3/2
x_4	0	1	2	0	1	0	0	4	4/1=4

BIG-M

x_B	Z	x_1	x_2	x_3	x_4	$\overline{x}_5$	$\overline{x}_6$	RHS	Ratio
Z	-1	0	$-1/3(5M+1)$	M	0	$1/3(7M-4)$	0	$-2M-4$	---
x_1	0	1	1/3	0	0	1/3	0	1	1/(1/3)=3
$\overline{x}_6$	0	0	5/3	-1	0	$-4/3$	1	2	2/(5/3)=6/5
x_4	0	0	5/3	0	1	$-1/3$	0	3	3/(5/3)=9/5
Z	-1	0	0	$-1/5$	0	M-8/5	M+1/5	-18/5	---
x_1	0	1	0	1/5	0	3/5	$-1/5$	3/5	3
x_2	0	0	1	$-3/5$	0	$-4/5$	3/5	6/5	---
x_4	0	0	0	1	1	4/3	1	1	1
Z	-1	0	0	0	1/5	M-4/3	M+2/5	-17/5	---
x_1	0	1	0	0	$-1/5$	1/3	$-2/5$	2/5	
x_2	0	0	1	0	3/5	0	6/5	9/5	
x_3	0	0	0	1	1	4/3	1	1	



OPTIMAL SOLUTION:

$z^* = 17/5$

$x_1^* = 2/5, x_2^* = 9/5$

2-PHASE

minimize $Z = 4x_1 + x_2$
subject to

$$
\begin{aligned}
3x_1 + x_2 &= 3\\
4x_1 + 3x_2 - x_3 &= 6\\
x_1 + 2x_2 + x_4 &= 4\\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Phase 1:

maximize $-Z$ ($\equiv$minimize Z)

$$
\begin{aligned}
-Z + \bar{x}_5 + \bar{x}_6 &= 0\\
3x_1 + x_2 + \bar{x}_5 &= 3\\
4x_1 + 3x_2 - x_3 + \bar{x}_6 &= 6\\
x_1 + 2x_2 + x_4 &= 4\\
x_1, x_2, x_3, x_4, \bar{x}_5, \bar{x}_6 &\geq 0
\end{aligned}
$$

x_B	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	RHS	Ratio
Z	−1	0	0	0	0	1	1	0	---
$\bar{x}_5$	0	3	1	0	0	1	0	3	
$\bar{x}_6$	0	4	3	−1	0	0	1	6	
x_4	0	1	2	0	1	0	0	4	

- 1 in row 0 for a basic variable is inconsistent with the definition of basic variable!
- 1 can be enforced to 0 with a Type 2 ero.

x_B	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	RHS	Ratio
Z	−1	−7	−4	1	0	0	0	−9	---
$\bar{x}_5$	0	3	1	0	0	1	0	3	3/3=1
$\bar{x}_6$	0	4	3	−1	0	0	1	6	6/4=3/2
x_4	0	1	2	0	1	0	0	4	4/1=4

2-PHASE

x_B	z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	RHS	Ratio
Z	−1	0	−5/3	1	0	7/3	0	−2	---
x_1	0	1	1/3	0	0	1/3	0	1	1/(1/3)=3
$\bar{x}_6$	0	0	5/3	−1	0	−4/3	1	2	2/(5/3)=6/5
x_4	0	0	5/3	0	1	−1/3	0	3	3/(5/3)=9/5
Z	−1	0	0	0	0	1	1	0	---
x_1	0	1	0	1/5	0	3/5	−1/5	3/5	
x_2	0	0	1	−3/5	0	−4/5	3/5	6/5	
x_4	0	0	0	1	1	1	−1	1	

- Objective function value for Phase 1 problem is 0.
- No artificial variables in the basis.

- **OPTIMAL SOLUTION for the Phase 1 problem.**
- **This solution is basic feasible to the original problem.**
- **We can construct the Phase 2 problem from the optimal solution to the Phase 1 problem.**

2-PHASE

Phase 2: minimize $-Z$ ($\equiv$minimize Z)

$$-Z + 4x_1 + x_2 = 0$$
$$x_1 + 1/5x_3 = 3/5$$
$$x_2 - 3/5x_3 = 6/5$$
$$x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

x_B	Z	x_1	x_2	x_3	x_4	RHS	Ratio
Z	-1	$\boxed{4}$	$\boxed{1}$	0	0	0	---
x_1	0	1	0	1/5	0	3/5	
x_2	0	0	1	$-3/5$	0	6/5	
x_4	0	0	0	1	1	1	

x_B	Z	x_1	x_2	x_3	x_4	RHS	Ratio
Z	-1	0	0	$-1/5$	0	$-18/5$	---
x_1	0	1	0	1/5	0	3/5	3
x_2	0	0	1	$-3/5$	0	6/5	---
x_4	0	0	0	1	1	1	1

- Positive values in row 0 for basic variables is inconsistent with the definition of a basic variable!
- They can be enforced to 0 with a Type 2 ero.

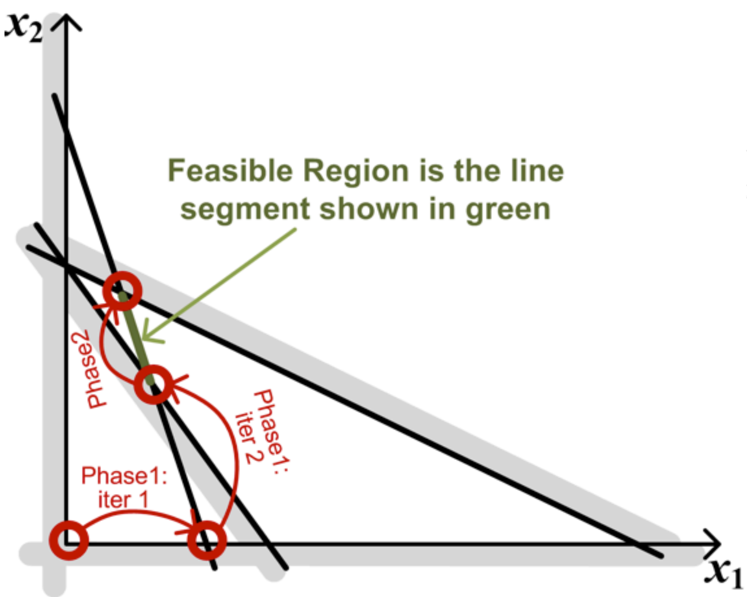
2-PHASE

x_B	Z	x_1	x_2	x_3	x_4	RHS	Ratio
Z	-1	0	0	0	1/5	$-17/5$	---
x_1	0	1	0	0	$-1/5$	2/5	
x_2	0	0	1	0	3/5	9/5	
x_3	0	0	0	1	1	1	

OPTIMAL SOLUTION:

$z^* = 17/5$

$x_1^* = 2/5$, $x_2^* = 9/5$



BIG-M EXAMPLE

(1)

maximize $z = -x_1 + x_2$
subject to
$x_1 + x_2 \geq 1$
$3x_1 + 2x_2 = 6$

maximize $z = -x_1 + x_2 \quad - M\bar{x}_4 - M\bar{x}_5$
subject to
$x_1 + x_2 - x_3 + \bar{x}_4 = 1$
$3x_1 + 2x_2 + \bar{x}_5 = 6$
$x_1, x_2, x_3, \bar{x}_4, \bar{x}_5 \geq 0$

$z = -x_1 + x_2 - M\bar{x}_4 - M\bar{x}_5$
$z + x_1 - x_2 + M\bar{x}_4 + M\bar{x}_5 = 0$

x_B	z	x_1	x_2	x_3	$\bar{x}_4$	$\bar{x}_5$	RHS	
z	1	1	-1	0	M	M	0	
$\bar{x}_4$	0	1	1	-1	1	0	1	
$\bar{x}_5$	0	3	2	0	0	1	6	
z	1	-4M+1	-3M-1	M	0	0	-7M	
$\bar{x}_4$	0	1	1	-1	1	0	1	1
$\bar{x}_5$	0	3	2	0	0	1	6	2
z	1	0	M-2	-3M+1	4M-1	0	-3M-1	
x_1	0	1	1	-1	1	0	1	—
$\bar{x}_5$	0	0	-1	3	-3	1	3	1
z	1	0	-5/3	0	M	M-1/3	-2	
x_1	0	1	2/3	0	0	1/3	2	3
x_3	0	0	-1/3	1	-1	1/3	1	—
z	1	5/2	0	0	M	M+1/2	3	
x_2	0	3/2	1	0	0	1/2	3	
x_3	0	1/2	0	1	-1	1/2	2	

OPTIMAL

$z^* = 3$
$x_1^* = 0, \; x_2^* = 3$

2-PHASE EXAMPLE

maximize $z = -x_1 + x_2$

subject to

$$x_1 + x_2 \geq 1$$
$$3x_1 + 2x_2 = 6$$
$$x_1, x_2 \geq 0$$

<u>Phase 1:</u>

minimize $z = + \bar{x}_4 + \bar{x}_5$ ②

subject to

$$x_1 + x_2 - x_3 + \bar{x}_4 = 1$$
$$3x_1 + 2x_2 + \bar{x}_5 = 6$$
$$x_1, x_2, x_3, \bar{x}_4, \bar{x}_5 \geq 0$$

x_B	z	x_1	x_2	x_3	$\bar{x}_4$	$\bar{x}_5$	RHS	
z	1	0	0	0	1	1	0	
$\bar{x}_4$	0	1	1	-1	1	0	1	
$\bar{x}_5$	0	3	2	0	0	1	6	
z	1	-4	-3	1	0	0	-7	
$\bar{x}_4$	0	1	1	-1	1	0	1	1
$\bar{x}_5$	0	3	2	0	0	1	6	2
z	1	0	1	-3	4	0	-3	
x_1	0	1	1	-1	1	0	1	—
$\bar{x}_5$	0	0	-1	3	-3	1	3	1
z	1	0	0	0	1	1	0	
x_1	0	1	2/3	0	0	1/3	2	
x_3	0	0	-1/3	1	-1	1/3	1	

FEASIBLE SOLUTION

maximize $z = -x_1 + x_2$

$$x_1 + \tfrac{2}{3} x_2 = 2$$
$$-\tfrac{1}{3} x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Phase 2 problem

2-PHASE EXAMPLE

Phase 2:

maximize $z = -x_1 + x_2$

subject to

$$x_1 + \tfrac{2}{3} x_2 = 2$$

$$-\tfrac{1}{3} x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

(3)

x_B	z	x_1	x_2	x_3	RHS
z	1	1	-1	0	0
x_1	0	1	2/3	0	2
x_3	0	0	-1/3	1	1
z	1	0	-5/3	0	-2
x_1	0	1	2/3	0	2
x_3	0	0	-1/3	1	1
z	1	5/2	0	0	3
x_2	0	3/2	1	0	3
x_3	0	1/2	0	1	2

$\frac{2}{2/3} = 3$

$-$

OPTIMAL $z^* = 3, \; x_1^* = 0, \; x_2^* = 3$