Statistical mechanics of black holes

Francesco Tarantelli
francesco.tarantelli@phd.unipi.it

University of Pisa

Introduction

- The application of quantum theory to the black hole solutions of general relativity leads to the remarkable conclusion that each black hole has a non-zero temperature, and an associated entropy.
- Schrödinger-Heisenberg quantum theory is compatible with general relativity and black holes, and chaotic many-particle quantum entanglement is the key to resolving the difficulties in the semiclassical description.
- The SYK model provides a remarkable description of the low temperature properties of certain black holes.

Presentation plan

Foundations by Boltzmann

$$S = \kappa_B \ln \mathrm{W} \ . \tag{1}$$

The thermodynamic entropy S is extensive $\longrightarrow$ proportional to the volume.

For quantum systems, we replace W with $D(E)$, i.e. the density of the energy eigenstates of the many-body quantum system per unit energy interval:

$$D(E) \sim \exp(S(E)/\kappa_B) \ . \tag{2}$$

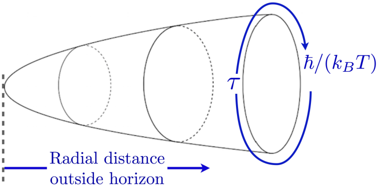
Partition function

Quantum Black Holes

- spacetime metric $g_{\mu\nu}$
- electromagnetic gauge field a_μ

$$\mathcal{Z} = \int \mathcal{D}g_{\mu\nu}\, \mathcal{D}a_\mu \, \exp\left(-\frac{1}{\hbar} \int d^d x \int_0^{\hbar/(\kappa_B T)} d\tau \sqrt{g} \mathcal{L}_d \left[g_{\mu\nu},\, a_\mu \right] \right) \; ; \quad (3)$$

with $\mathcal{L}_d$ the Einstein-Maxwell Lagrangian.



The constrain on imaginary time follows from:

$$U(t) = \exp\left(-i\mathcal{H}t/\hbar \right) \qquad \Longleftrightarrow \qquad \mathcal{Z} = \mathrm{Tr} \exp\left(-\mathcal{H}/(\kappa_B T) \right) \; ;$$

Semiclassical limit

Quantum Black Holes

Pathological path integral $\Longrightarrow$ semiclassical limit

Contribution of the saddle point imposing that spacetime is smooth at the horizon in imaginary time.

From this computation, the temperature T and the entropy S, for a neutral black hole of mass M in $d = 3$, are:

$$\frac{S}{\kappa_B} = \frac{Ac^3}{4G\hbar}\,, \qquad \frac{\kappa_B T}{\hbar} = \frac{c^3}{8\pi GM}\,, \tag{4}$$

with the *area* of the black hole horizon $A = 4\pi R^2$ and the horizon radius $R = 2GM/c^2$.

Questions:

1. are the results compatible with Boltzmann interpretation of entropy?
2. $\exists$ quantum Hamiltonian whose $D(E)$ yields this $S(E)$ and $\mathcal{Z}$?

$$\mathcal{Z} = \int_0^\infty dE\, D(E) \exp\left(-\frac{E}{\kappa_B T}\right)\,. \tag{5}$$

Holography

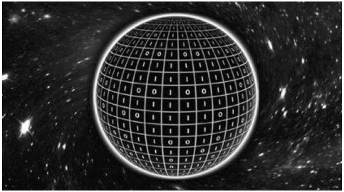
Quantum Black Holes

Boltzmann quantum entropy $\propto$ volume, while the black hole entropy $\propto$ $area$.

To understand this feature, we use the **holography**:

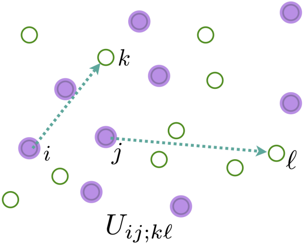
N qubits realize a many-body system which we can image residing on the black hole surface;

N required for quantum simulation of black hole $\propto$ its surface area;



2^N quantum states + $D(E) \sim \exp(S/\kappa_B) \implies S \propto area$.

The SYK model



$Q \cdot N$ fermions ψ_i where:

- $i = 1, \ldots, N$ indicates the position;
- $Q \approx 1/2$ is the density;
- $U_{ij;kl}$ are the random 2-body interaction terms.

Qubit system : $\quad |0\rangle \quad \text{and} \quad \psi_i^\dagger\,|0\rangle\,.$

Properties

The SYK model

- At large N, $\quad D(E) \sim \exp\big(N s_0\big)\sinh\left(\sqrt{2N\gamma E}\right)$.

  with $s_0 = 0.4648\ldots$ for $Q = 1/2$ and $\gamma \sim 1/U$

  $$U/N^{3/2} = \ll U_{ij;kl} \gg ;$$

- $$\frac{S(T)}{\kappa_B} = N(s_0 + \gamma\kappa_B T) - \frac{3}{2}\ln\left(\frac{N^{1/3}U}{\kappa_B T}\right) + \ldots \qquad (6)$$

  hence, the energy level spacing exponentially small in N near ground state: $\lim_{T\to 0}\lim_{N\to\infty} S(T)/(\kappa_B T) = s_0$.

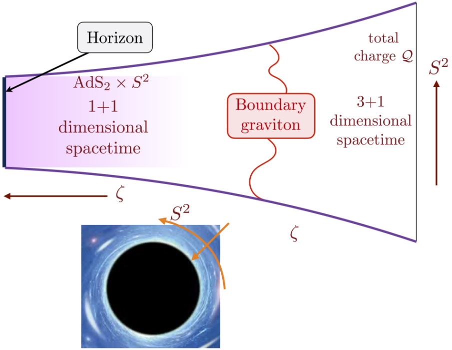
- The E dependence of $D(E)$ is associated with a time representation mode $f(\tau)$; considering the low energy and the phase mode $\phi(\tau)$ quantum fluctuations:

  $$\mathcal{Z}_{SYK} = e^{Ns_0}\int \mathcal{D}f\mathcal{D}\phi \exp\left(-\frac{1}{\hbar}\int_0^{\hbar/(\kappa_B T)} d\tau \mathcal{L}_{SYK}[f,\phi]\right) . \qquad (7)$$

From the SYK Model to Black Holes

Can we evaluate the path integral over spacetime metrics of black holes from the path integral over time reparameterizations of the SYK model?

Yes, for black hole with fixed charge Q.



The saddle-point solution of the Einstein-Maxwell action for a charged black hole.

$\lim\ T \to 0$ yields:

unimportant angular momentum modes $+$ theory of quantum gravity in 1+1 spacetime dimensions

Jackiw-Teitelboim (JT) gravity:

$$\mathcal{Z}_{JT} = \int \mathcal{D}g_{\mu\nu}\,\mathcal{D}a_\mu \exp\left(-\frac{1}{\hbar}\int d\zeta \int_0^{\hbar/(\kappa_B T)} d\tau\,\sqrt{g}\mathcal{L}_1\left[g_{\mu\nu}, a_\mu\right]\right). \tag{8}$$

Holographic realization of $\mathcal{Z}_{JT}$ is exactly mapped in the $0+1$ dimensional $\mathcal{Z}_{SYK}$:

- in the boundary region, fluctuations of $g_{\mu\nu} \implies f(\tau)$;
- the boundary value of a_μ becomes the phase field $\phi(\tau)$.

Black Hole Density of States $D(E)$

From the SYK Model to Black Holes

From this mapping,
parameters in the black hole $D(E)$ are deduced by comparing the SYK entropy with the low T limit of entropy of charged black hole:

$$\frac{S(T)}{\kappa_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2} \frac{\kappa_B T}{\hbar}\right) - \\ - \frac{3}{2} \ln\left(\frac{c^2}{A_0^{1/6} (\hbar G)^{1/3} (\kappa_B T/\hbar)}\right) + \dots \; ; \tag{9}$$

A_0 is the *area* of the black hole horizon at $T = 0$.

Then, from the SYK $D(E)$, we obtain the black hole $D(E)$:

$$D(E) \sim \exp\left(\frac{A_0 c^3}{4\hbar G}\right) \sinh\left(\left[\sqrt{\pi} A_0^{3/2} \frac{c^3}{\hbar G} \frac{E}{\hbar c}\right]^{1/2}\right) . \tag{10}$$

Conclusions

- The SYK density of states $D(E)$, with precise and discrete energy levels, is described at low energies by a gravitational theory expressed in terms of the semiclassical fluctuations of a spacetime metric.

- The connection between the SYK model and black holes does not imply that the ultimate high energy and short distance physics is described by the SYK model.

- Nevertheless, the SYK model provides a much simpler quantum simulation of the low energy physics in certain cases, including the complex quantum entanglement and the maximal many-body quantum chaos.