OG-USA: Documentation for the Large-scale Dynamic General Equilibrium Overlapping Generations Model for U.S. Policy Analysis

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Preface

The development of this model benefited from support from the Open Source Policy Center at the American Enterprise Institute as well as from the Open Source Macroeconomics Laboratory at the Becker Friedman Institute at the University of Chicago and the BYU Macroeconomics Laboratory at Brigham Young University. All Python code for the computational model is available at https://github.com/open-source-economics/OG-USA. This model benefited tremendously from work by Kerk L. Phillips and early student involvement by Evan Magnusson and Isaac Swift. Thanks to T.J. Alumbaugh, Peter Steinberg, Salim Furth, and Hank Doupe for significant code contributions. Contributions to the model code can also be viewed in detail on the OG-USA contributions page.

Contents

Part I

Introduction

Chapter 1

Introduction

The overlapping generations model is a workhorse of dynamic fiscal analysis. `OG-USA` is dynamic in that households in the model make consumption, savings, and labor supply decisions based on their expectations over their entire lifetime, not just the current period. Because `OG-USA` is a general equilibrium model, behavioral changes by households and firms can cause macroeconomic variables and prices to adjust. This characteristic has recently become a required component of fiscal policy analysis in the United States.[1]

But the main characteristic that differentiates the overlapping generations model from other dynamic general equilibrium models is its realistic modeling of the finite lifetimes of individuals and the cross-sectional age heterogeneity that exists in the economy. One can make a strong case that age heterogeneity and income heterogeneity are two of the main sources of diversity that explain much of the behavior in which we are interested for policy analysis.

`OG-USA` can be summarized as having the following characteristics.

- Households
  - overlapping generations of finitely lived households
  - households are forward looking and see to maximize their expected lifetime utility, which is a function of consumption, labor supply, and bequests
  - households choose consumption, savings, and labor supply every period.
  - The only uncertainty households face is with respect to their mortality risk
  - realistic demographics: mortality rates, fertility rates, immigration rates, population growth, and population distribution dynamics
  - heterogeneous lifetime income groups within each age cohort, calibrated from U.S. tax data
  - incorporation of detailed household tax data from `Tax-Calculator` microsimulation model
  - calibrated intentional and unintentional bequests by households to surviving generations

[1] For a summary of the House rule adopted in 2015 that requires dynamic scoring of significant tax legislation see this Politico article.

- Firms
  - representative perfectly competitive firm maximizes static profits with general CES production function by choosing capital and labor demand
  - exogenous productivity growth is labor augmenting technological change
  - firms face a corporate income tax as well as various depreciation deductions and tax treatments
- Government
  - government collects tax revenue from households and firms
  - government distributes transfers to households
  - government spends resources on public goods
  - government can run deficits and surpluses
  - a stabilization rule (budget closure rule) must be implemented at some point in the time path if government debt is growing at a rate permanently different from GDP.
- Aggregate, market clearing, and international
  - Aggregate model is deterministic (no aggregate shocks)
  - Three markets must clear: capital, labor, and goods markets
  -

We will update this document as more detail is added to the model. We are currently working on adding stochastic income, aggregate shocks, multiple industries, and a large open economy multi-country version of the model. There is much to do and, as any self-respecting open source project should, we welcome outside contributions.

Chapter 2

Exogenous Inputs and Endogenous Output

In this chapter, list the exogenous inputs to the model, options, and where the values come from (weak calibration vs. strong calibration). Point to the respective chapters for some of the inputs. Mention the code `parameters.py`.

Also go through the output of the model, endogenous variables, and potential tables and pictures to produce.

2.1 Exogenous Parameters

List all the exogenous parameters that are outputs of the model here.

Table 2.1: List of exogenous parameters and baseline calibration values

Symbol	Description	Value
S	Maximum periods in economically active household life	80
E	Number of periods of youth economically outside the model	$\text{round}\left(\frac{S}{4}\right) = 20$
R	Retirement age (period)	$E + \text{round}\left(\frac{9}{16}S\right) = 65$
T_1	Number of periods to steady state for initial time path guesses	160
T_2	Maximum number of periods to steady state for nonsteady-state equilibrium	160
ν	Dampening parameter for TPI	0.4
$\{\{\omega_{s,0}\}_{s=1}^{E+S}\}_{t=0}^{T_2+S-1}$	Initial population distribution by age	(see Ch. 3)
$\{f_s\}_{s=1}^{E+S}$	Fertility rates by age	(see Sec. 3.1)
$\{i_s\}_{s=1}^{E+S}$	Immigration rates by age	(see Sec. 3.2)
$\{\rho_s\}_{s=0}^{E+S}$	Mortality rates by age	(see Sec. 3.3)

2.2 Endogenous Variables

List all the endogenous variables that are outputs of the model here.

Part II

Households Theory

Chapter 3

Demographics

We start the `OG-USA` section on modeling the household with a description of the demographics of the model. Nishiyama (2015) and DeBacker et al. (2017) have recently shown that demographic dynamics are likely the biggest influence on macroeconomic time series, exhibiting more influence than fiscal variables or household preference parameters.

In this chapter, we characterize the equations and parameters that govern the transition dynamics of the population distribution by age. In `OG-USA`, we take the approach of taking mortality rates and fertility rates from outside estimates. But we estimate our immigration rates as residuals using the mortality rates, fertility rates, and at least two consecutive periods of population distribution data. This approach makes sense if one modeling a country in which in one is not confident in the immigration rate data. If the country has good immigration data, then the immigration residual approach we describe below can be skipped.

We define $\omega_{s,t}$ as the number of households of age s alive at time t. A measure $\omega_{1,t}$ of households is born in each period t and live for up to $E+S$ periods, with $S \geq 4$.[1] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$. We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{split}
\omega_{1,t+1} &= (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t} + i_1\omega_{1,t} \quad \forall t \\
\omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} + i_{s+1}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1\leq s \leq E+S-1
\end{split} \tag{3.1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific net immigration rate, ρ_s is an age-specific mortality hazard rate, and ρ_0 is an infant mortality rate.[2] The total population in the economy N_t at any period is simply the sum of households in the economy,

[1] Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), it is convenient for S to be at least 4.

[2] The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{3.2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3.3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{3.4}$$

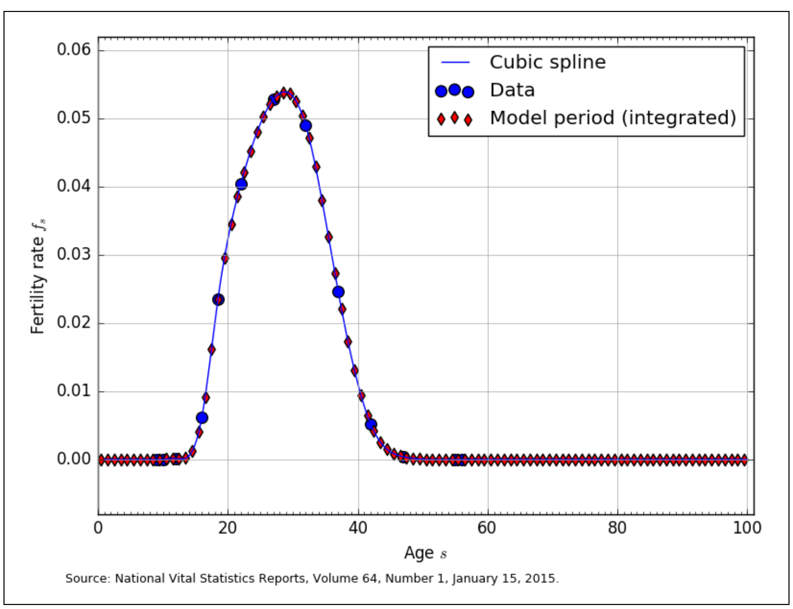
$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{3.5}$$

We discuss the approach to estimating fertility rates f_s, mortality rates ρ_s, and immigration rates i_s in Sections 3.1, 3.2, and 3.3.

3.1 Fertility rates

In `OG-USA`, we assume that the fertility rates for each age cohort f_s are constant across time. However, this assumption is conceptually straightforward to relax. Our data for U.S. fertility rates by age come from Martin et al. (2015, Table 3, p. 18) National Vital Statistics Report, which is final fertility rate data for 2013. Figure 3.1 shows the fertility-rate data and the estimated average fertility rates for $E+S=100$.

Figure 3.1: Fertility rates by age (f_s) for $E+S=$ 100



The large blue circles are the 2013 U.S. fertility rate data from Martin et al. (2015). These are 9 fertility rates $[0.3, 12.3, 47.1, 80.7, 105.5, 98.0, 49.3, 10.4, 0.8]$ that correspond to the midpoint ages of the following age (in years) bins $[10-14, 15-17, 18-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49]$. In order to get our cubic spline interpolating function to fit better at the endpoints we added to fertility rates of zero to ages 9 and 10, and we added two fertility rates of zero to ages 55 and 56. The blue line in Figure 3.1 shows the cubic spline interpolated function of the data.

The red diamonds in Figure 3.1 are the average fertility rate in age bins spanning households born at the beginning of period 1 (time $= 0$) and dying at the end of their 100th year. Let the total number of model years that a household lives be $E + S \leq 100$. Then the span from the beginning of period 1 (the beginning of year 0) to the end of period 100 (the end of year 99) is divided up into $E + S$ bins of equal length. We calculate the average fertility rate in each of the $E + S$ model-period bins as the average population-weighted fertility rate in that span. The red diamonds in Figure 3.1 are the average fertility rates displayed at the midpoint in each of the $E + S$ model-period bins.

3.2 Mortality rates

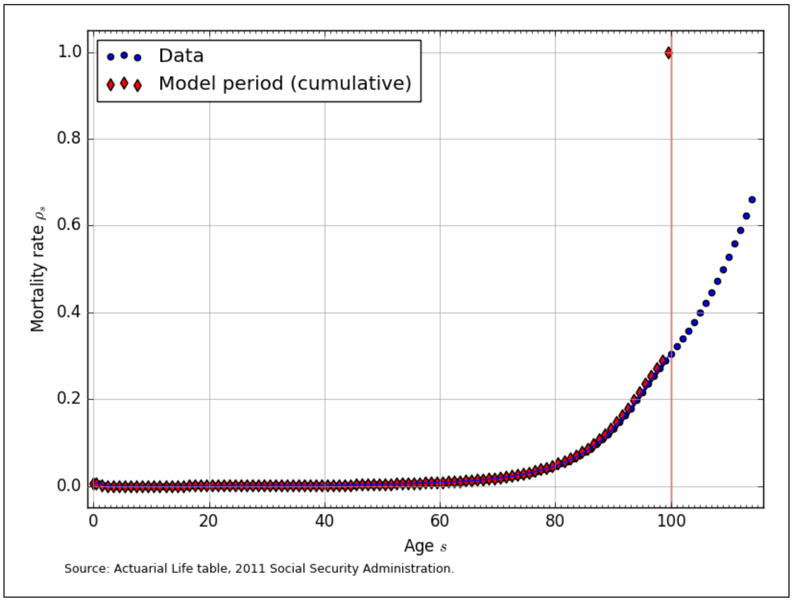
The mortality rates in our model ρ_s are a one-period hazard rate and represent the probability of dying within one year, given that an household is alive at the beginning of period s. We assume that the mortality rates for each age cohort ρ_s are constant across time. The infant mortality rate of $\rho_0 = 0.00587$ comes from the 2015 U.S. CIA World Factbook. Our data for U.S. mortality rates by age come from the Actuarial Life Tables of the U.S. Social Security Administration (see Bell and Miller, 2015), from which the most recent mortality rate data is for 2011. Figure 3.2 shows the mortality rate data and the corresponding model-period mortality rates for $E + S = 100$.

The mortality rates in Figure 3.2 are a population-weighted average of the male and female mortality rates reported in Bell and Miller (2015). Figure 3.2 also shows that the data provide mortality rates for ages up to 111-years-old. We truncate the maximum age in years in our model to 100-years old. In addition, we constrain the mortality rate to be 1.0 or 100 percent at the maximum age of 100.

3.3 Immigration rates

Because of the difficulty in getting accurate immigration rate data by age, we estimate the immigration rates by age in our model i_s as the average residual that reconciles the current-period population distribution with next period's population distribution given fertility rates f_s and mortality rates ρ_s. Solving equations (3.1) for the immigration rate i_s gives the following characterization of the immigration rates in given population levels in any two consecutive periods $\omega_{s,t}$ and $\omega_{s,t+1}$ and the fertility rates f_s and mortality rates ρ_s.

Figure 3.2: Mortality rates by age (ρ_s) for $E+S = 100$



$$
\begin{aligned}
i_1 &= \frac{\omega_{1,t+1} - (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t}}{\omega_{1,t}} \quad \forall t \\
i_{s+1} &= \frac{\omega_{s+1,t+1} - (1-\rho_s)\omega_{s,t}}{\omega_{s+1,t}} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned}
\tag{3.6}
$$

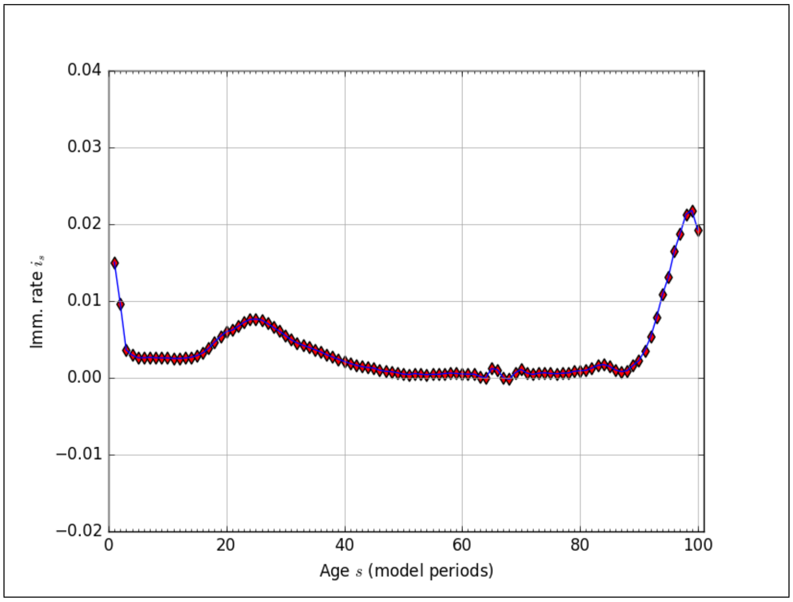
We calculate our immigration rates for three different consecutive-year-periods of population distribution data (2010 through 2013). Our four years of population distribution by age data come from Census Bureau (2015). The immigration rates i_s that we use in our model are the the residuals described in (3.6) averaged across the three periods. Figure 3.3 shows the estimated immigration rates for $E+S = 100$ and given the fertility rates from Section 3.1 and the mortality rates from Section 3.2.

At the end of Section 3.4, we describe a small adjustment that we make to the immigration rates after a certain number of periods in order to make computation of the transition path equilibrium of the model compute more robustly.

3.4 Population steady-state and transition path

This model requires information about mortality rates ρ_s in order to solve for the household's problem each period. It also requires the steady-state stationary population distribution $\bar{\omega}_s$ and population growth rate $\bar{g}_n$ as well as the full transition path of the stationary population distribution $\hat{\omega}_{s,t}$ and population grow rate $\tilde{g}_{n,t}$ from the current state to the steady-state. To solve for the steady-state and the transition path of the stationary population distribution, we write the stationary population dynamic equations (3.7) and their matrix representation

Figure 3.3: Immigration rates by age (i_s), residual, $E+S=100$



(3.8).

$$
\begin{aligned}
\hat{\omega}_{1,t+1} &= \frac{(1-\rho_0)\sum_{s=1}^{E+S} f_s\hat{\omega}_{s,t} + i_1\hat{\omega}_{1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \\
\hat{\omega}_{s+1,t+1} &= \frac{(1-\rho_s)\hat{\omega}_{s,t} + i_{s+1}\hat{\omega}_{s+1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \quad \text{and} \quad 1\leq s\leq E+S-1
\end{aligned} \tag{3.7}
$$

$$
\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...
$$
$$
\begin{bmatrix} (1-\rho_0)f_1+i_1 & (1-\rho_0)f_2 & (1-\rho_0)f_3 & \dots & (1-\rho_0)f_{E+S-1} & (1-\rho_0)f_{E+S} \\ 1-\rho_1 & i_2 & 0 & \dots & 0 & 0 \\ 0 & 1-\rho_2 & i_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & i_{E+S-1} & 0 \\ 0 & 0 & 0 & \dots & 1-\rho_{E+S-1} & i_{E+S} \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{3.8}
$$

We can write system (3.8) more simply in the following way.

$$
\hat{\boldsymbol{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}}\boldsymbol{\Omega}\hat{\boldsymbol{\omega}}_t \quad \forall t \tag{3.9}
$$

The stationary steady-state population distribution $\bar{\boldsymbol{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (3.9).

$$(1+\bar{g}_n)\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega}\bar{\boldsymbol{\omega}} \tag{3.10}$$

Proposition 3.1. If the age $s=1$ immigration rate is $i_1 > -(1-\rho_0)f_1$ and the other immigration rates are strictly positive $i_s > 0$ for all $s \geq 2$ such that all elements of $\boldsymbol{\Omega}$ are nonnegative, then there exists a unique positive real eigenvector $\bar{\boldsymbol{\omega}}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible. This can be easily shown. The matrix is of the form

$$\boldsymbol{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}; \quad \boldsymbol{\Omega}^{S+E-1} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

$$\boldsymbol{\Omega}^{S+E} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

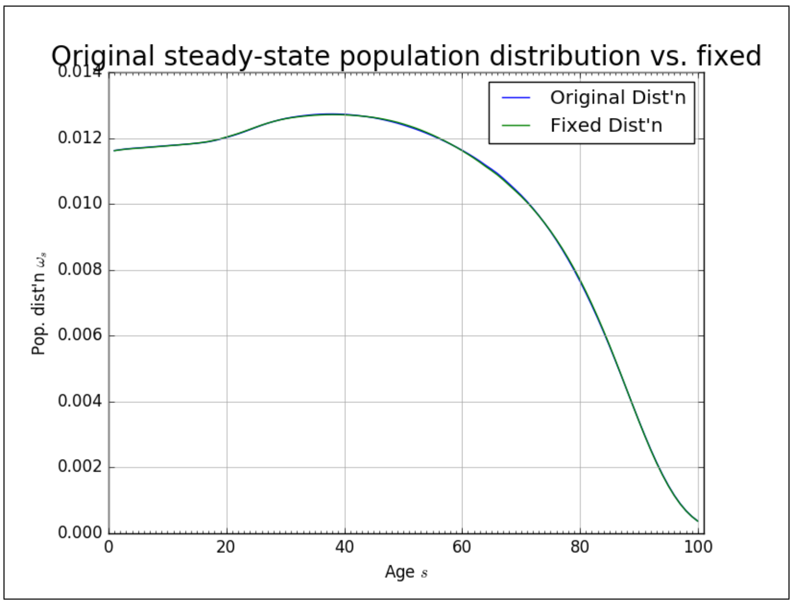
Existence of an $m \in \mathbb{N}$ such that $\left(\boldsymbol{\Omega}^{\mathbf{m}}\right)_{ij} \neq 0 \;\; (>0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\mathbf{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. $\square$

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$.

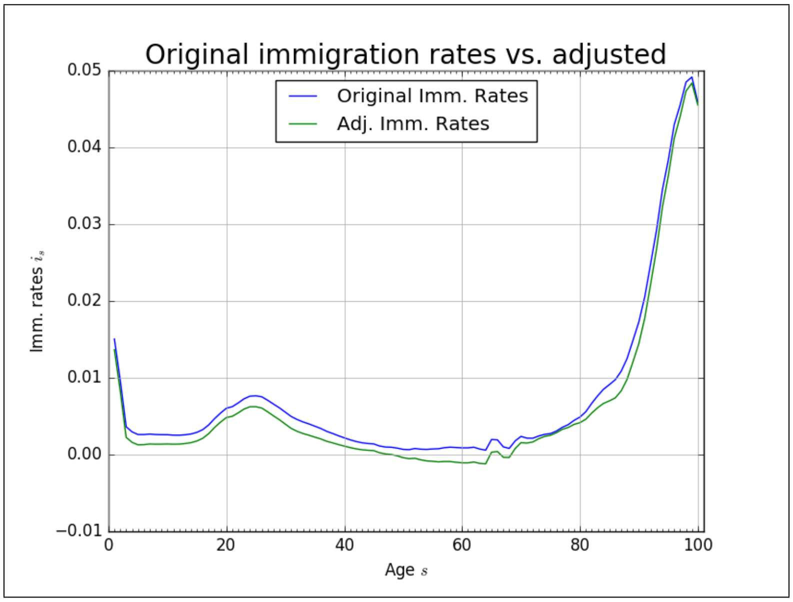
Figure 3.4 shows the steady-state population distribution $\bar{\boldsymbol{\omega}}$ and the population distribution after 120 periods $\hat{\boldsymbol{\omega}}_{120}$. Although the two distributions look very close to each other, they are not exactly the same.

Figure 3.4: Theoretical steady-state population distribution vs. population distribution at period $t = 120$



Further, we find that the maximum absolute difference between the population levels $\hat{\omega}_{s,t}$ and $\hat{\omega}_{s,t+1}$ was 1.3852×10^{-5} after 160 periods. That is to say, that after 160 periods, given the estimated mortality, fertility, and immigration rates, the population has not achieved its steady state. For convergence in our solution method over a reasonable time horizon, we want the population to reach a stationary distribution after T periods. To do this, we artificially impose that the population distribution in period $t = 120$ is the steady-state. As can be seen from Figure 3.4, this assumption is not very restrictive. Figure 3.5 shows the change in immigration rates that would make the period $t = 120$ population distribution equal be the steady-state. The maximum absolute difference between any two corresponding immigration rates in Figure 3.5 is 0.0028.

Figure 3.5: Original immigration rates vs. adjusted immigration rates to make fixed steady-state population distribution



The most recent year of population data come from Census Bureau (2015) population estimates for both sexes for 2013. We those data and use the population transition matrix (3.9) to age it to the current model year of 2015. We then use (3.9) to generate the transition path of the population distribution over the time period of the model. Figure 3.6 shows the progression from the 2013 population data to the fixed steady-state at period $t = 120$. The time path of the growth rate of the economically active population $\tilde{g}_{n,t}$ is shown in Figure 3.7.

Figure 3.6: Stationary population distribution at periods along transition path

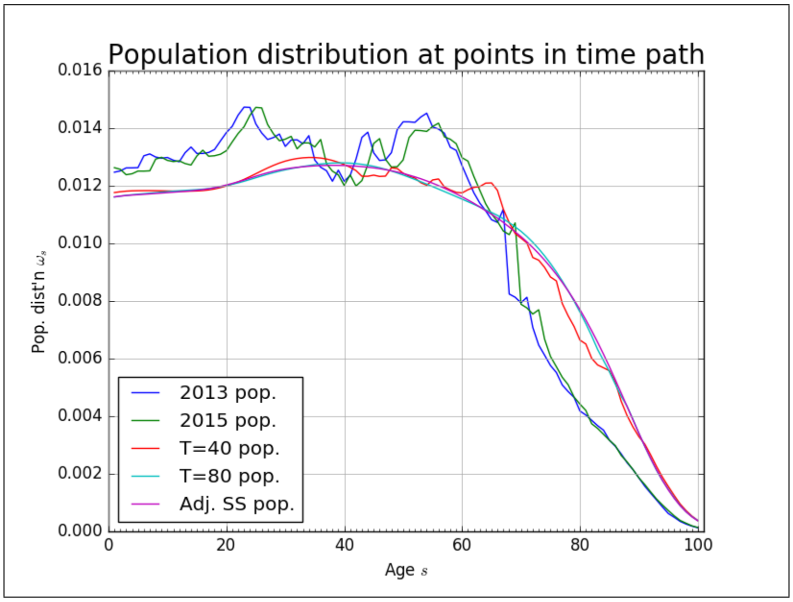
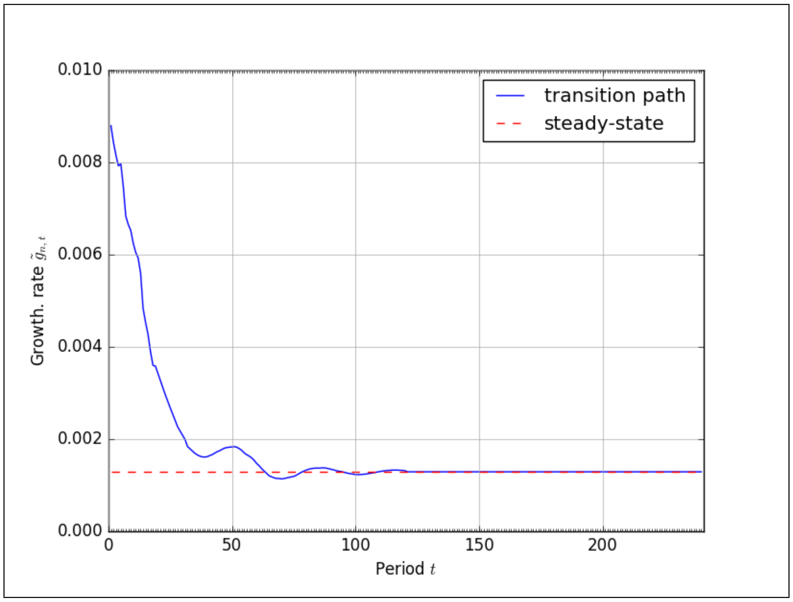


Figure 3.7: Time path of the population growth rate $\tilde{g}_{n,t}$



Chapter 4

Lifetime Earnings Profiles

Among households in `OG-USA`, we model both age heterogeneity and within-age ability heterogeneity. We use this ability or productivity heterogeneity to generate the income heterogeneity that we see in the data.

Differences among workers' productivity in terms of ability is one of the key dimensions of heterogeneity to model in a micro-founded macroeconomy. In this chapter, we characterize this heterogeneity as deterministic lifetime productivity paths to which new cohorts of agents in the model are randomly assigned. In `OG-USA`, households' labor income comes from the equilibrium wage and the agent's endogenous quantity of labor supply. In this section, we augment the labor income expression with an individual productivity $e_{j,s}$, where j is the index of the ability type or path of the individual and s is the age of the individual with that ability path.

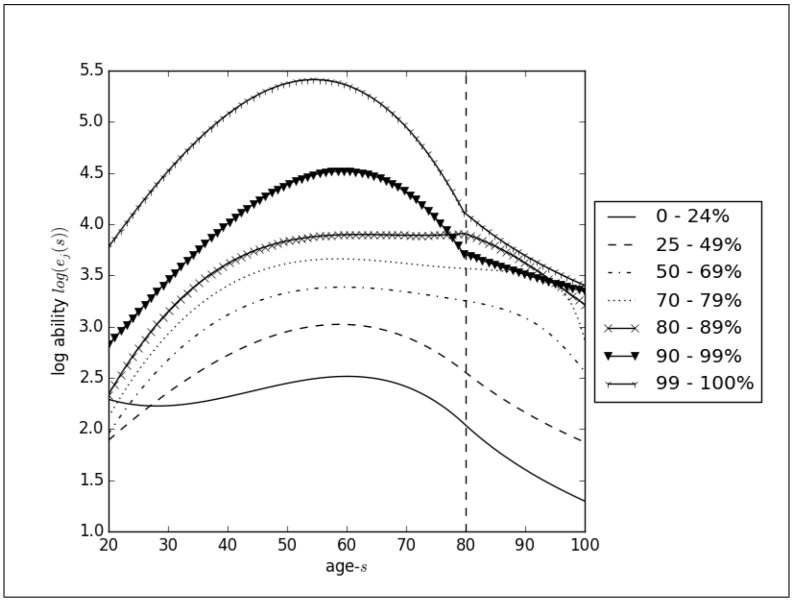
$$\text{labor income:} \quad x_{j,s,t} \equiv w_t e_{j,s} n_{j,s,t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{8.2}$$

In this specification, w_t is an equilibrium wage representing a portion of labor income that is common to all workers. Individual quantity of labor supply is $n_{j,s,t}$, and $e_{j,s}$ represents a labor productivity factor that augments or diminishes the productivity of a worker's labor supply relative to average productivity.

We calibrate deterministic ability paths such that each lifetime income group has a different life-cycle profile of earnings. The distribution on income and wealth are often focal components of macroeconomic models. As such, we use a calibration of deterministic lifetime ability paths from DeBacker et al. (2017b) that can represent U.S. earners in the top 1% of the distribution of lifetime income. Piketty and Saez (2003) show that income and wealth attributable to these households has shown the greatest growth in recent decades. The data come from the U.S. Internal Revenue Services's (IRS) Statistics of Income program (SOI) Continuous Work History Sample (CWHS). DeBacker et al. (2017b) match the SOI data with Social Security Administration (SSA) data on age and Current Population Survey (CPS) data on hours in order to generate a non-top-coded measure of hourly wage.

Figure 4.1 shows a calibration for $J = 7$ deterministic lifetime ability paths $e_{j,s}$ corresponding to labor income percentiles $\boldsymbol{\lambda} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Because there are few individuals above age 80 in the data, DeBacker et al. (2017b) extrapolate these estimates for model ages 80-100 using an arctan function.

Figure 4.1: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$



We calibrate the model such that each lifetime income group has a different life-cycle profile of earnings. Since the distribution on income and wealth are key aspects of our model, we calibrate these processes so that we can represent earners in the top 1 percent of the distribution of lifetime income. It is income and wealth attributable to these households that has shown the greatest growth in recent decades (see, for example, Piketty and Saez (2003)). In order to have observations on the earnings of those at very top of the distribution that are not subject to top-coding we use data from the Internal Revenue Services's (IRS) Statistics of Income program (SOI).

4.1 Continuous Work History Sample

The SOI data we draw from are the Continuous Work History Sample (CWHS). From this CWHS, we use a panel that is a 1-in-5000 random sample of tax filers from 1991 to 2009. For each filer-year observation we are able to observe detailed information reported on Form 1040 and the associated forms and schedules. We are also able to merge these tax data with Social Security Administration (SSA) records to get information on the age and gender of the primary and secondary filers. Our model variable of effective labor units maps into wage rates, because the market wage rate in the model, w_t, is constant across households. Earnings per hour thus depend upon effective labor units and equal $e_{j,s,t} \times w_t$ for household in lifetime income group j, with age s, in year t. Income tax data, however, do not contain information on hourly earnings or hours works. Rather, we only observe total earned income (wage and salaries plus self-employment income) over the tax year. In order to find hourly earnings for tax filers, we use an imputation procedure. This is described in detail in DeBacker and Ramnath (2017). The methodology applies an imputation for hours worked for a filing unit based on a model of hours worked for a filing unit estimated from the Current Population

Survey (CPS) for the years 1992-2010.[1] We then use the imputed hours to calculate hourly earnings rates for tax filing units in the CWHS.

We exclude from our sample filer-year observations with earned income (wages and salaries plus business income) of less than \$1,250. We further exclude those with positive annual wages, but with hourly wages below \$5.00 (in 2005\$). We also drop one observation where the hourly wage rate exceeds \$25,000.[2] Economic life in the model runs from age 21 to 100. Our data have few observations on filers with ages exceeding 80 years old. Our sample is therefore restricted to those from ages 21 to 80. After these restrictions, our final sample size is 333,381 filer-year observations.

4.2 Lifetime Income

In our model, labor supply and savings, and thus lifetime income, are endogenous. We therefore define lifetime income as the present value of lifetime labor endowments and not the value of lifetime labor earnings. Note that our data are at the tax filing unit. We take this unit to be equivalent to a household. Because of differences in household structure (i.e., singles versus couples), our definition of lifetime labor income will be in per adult terms. In particular, for filing units with a primary and secondary filer, our imputed wage represents the average hourly earnings between the two. When calculating lifetime income we assign single and couple households the same labor endowment. This has the effect of making our lifetime income metric a per adult metric, there is therefore not an over-representation of couple households in the higher lifetime income groups simply because their time endowment is higher than for singles. We use the following approach to measure the lifetime income.

First, since our panel data do not allow us to observe the complete life cycle of earnings for each household (because of sample attrition, death or the finite sample period of the data), we use an imputation to estimate wages in the years of the household's economic life for which they do not appear in the CWHS. To do this, we estimate the following equation, separately by household type (where household types are single male, single female, couple with male head, or couple with female head):

$$ln(w_{i,t}) = \alpha_i + \beta_1 age_{i,t} + \beta_2 age_{i,t}^2 + \beta_3 * age_{i,t}^3 + \varepsilon_{i,t} \tag{4.1}$$

The parameter estimates, including the household fixed effects, from Equation 4.1 are shown in Table 4.1. These estimates are then used to impute values for log wages in years of each households' economic life for which we do not have data. This creates a balanced panel of log wages of households with heads aged 21 to 80. The actual and imputed wage values are then used to calculate the net present value of lifetime labor endowments per adult for each household. Specifically, we define lifetime income for household i as:

$$LI_i = \sum_{t=21}^{80} \left(\frac{1}{1+r}\right)^{t-21} (w_{i,t} * 4000) \tag{4.2}$$

[1]The CPS survey asks retrospective questions about income in the last year and average hours worked per week (and weeks worked) in the last year). Therefore, these CPS surveys line up with tax years 1991-2009.

[2]This threshold is equivalent to \$50 million of wage income in one year at full time (40 hours per week) of work.

Table 4.1: Initial Log Wage Regressions

Dependent variables	Single males	Single females	Married, male head	Married, female head
Age	0.177***	0.143***	0.134***	0.065**
	(0.006)	(0.005)	(0.004)	(0.027)
Age^2	-0.003***	-0.002***	-0.002***	-0.000
	(0.000)	(0.000)	(0.000)	(0.001)
Age^3	0.000***	0.000***	0.000***	0.000
	(0.000)	(0.000)	(0.000)	(0.000)
Constant	-0.839***	-0.648***	-0.042	1.004***
	(0.072)	(0.070)	(0.058)	(0.376)
Adj R^2	-0.007	0.011	-0.032	-0.324
Observations	88,833	96,670	141,564	6,314

Source: CWHS data, 1991 – 2009.
** Significant at the 5 percent level ($p < 0.05$).
*** Significant at the 1 percent level ($p < 0.01$).

Note that households are all have the same time endowment in each year (4000 hours). Thus the amount of the time endowment scales lifetime income up or down, but does not change the lifetime income of one household relative to another. This is not the case with the interest rate, r, which we fix at 4%. Changes in the interest rate differentially impact the lifetime income calculation for different individuals because they may face different earnings profiles. For example, a higher interest rate would reduced the discounted present value of lifetime income for those individuals whose wage profiles peaked later in their economic life by a larger amount than it would reduce the discounted present value of lifetime income for individuals whose wage profiles peaked earlier.

4.3 Profiles by Lifetime Income

With observations of lifetime income for each household, we next sort households and find the percentile of the lifetime income distribution that each household falls in. With these percentiles, we create our lifetime income groupings.

$$\lambda_j = [0.25, 0.25, 0.2, 0.1, 0.1, 0.09, 0.01] \tag{4.3}$$

That is, lifetime income group one includes those in below the 25th percentile, group two includes those from the 25th to the median, group three includes those from the median to the 70th percentile, group four includes those from the 70th to the 80th percentile, group 5 includes those from the 80th to 90th percentile, group 6 includes those from the 90th to 99th percentile, and group 7 consists of the top one percent in the lifetime income distribution. Table 4.2 presents descriptive statistics for each of these groups.

To get a life-cycle profile of effective labor units for each group, we estimate the wage profile for each lifetime income group. We do this by estimating the following regression model separately for each lifetime income group using data on actual (not imputed) wages:

$$ln(w_{i,t}) = \alpha + \beta_1 age_{i,t} + \beta_2 age^2_{i,t} + \beta_3 * age^3_{i,t} + \varepsilon_{i,t} \tag{4.4}$$

Table 4.2: Descriptive Statistics by Lifetime Income Category

Lifetime Income Category:	1	2	3	4	5	6	7	All
Percentiles	0-25	25-50	50-70	70-80	80-90	90-99	99-100	0-100
Observations	65,698	101,484	74,253	33,528	31,919	24,370	2,129	333,381
Fraction Single								
Females	0.30	0.24	0.25	0.32	0.38	0.40	0.22	0.28
Males	0.18	0.22	0.30	0.35	0.38	0.37	0.20	0.26
Fraction Married								
Female Head	0.08	0.00	0.00	0.00	0.00	0.00	0.00	0.02
Male Head	0.45	0.53	0.45	0.32	0.23	0.23	0.57	0.39
Mean:								
Age, Primary	51.72	44.15	38.05	34.09	31.53	30.79	40.17	39.10
Hourly Wage	11.60	16.98	20.46	23.04	26.06	40.60	237.80	21.33
Annual Wages	25,178	44,237	54,836	57,739	61,288	92,191	529,522	51,604
Lifetime Income	666,559	1,290,522	1,913,029	2,535,533	3,249,287	5,051,753	18,080,868	2,021,298

* CWHS data, 1991-2009, all nominal values in 2005$.

The estimated parameters from equation (4.4) are given in Table 4.3. The life-cycle earnings profiles implied by these parameters are plotted in Figure 4.1. Note that there are few individuals above age 80 in the data. To extrapolate these estimates for model ages 80-100, we use an arctan function of the following form:

$$y = \left(\frac{-a}{\pi}\right) * arctan(bx + c) + \frac{a}{2} \tag{4.5}$$

where x is age, and a, b, and c are the parameters we search over for the best fit of the function to the following three criteria: 1) the value of the function should match the value of the data at age 80 2) the slope of the arctan should match the slope of the data at age 80 and 3) the value of the function should match the value of the data at age 100 times a constant. This constant is .5 for all lifetime income groups, except the 2nd highest ability is .7 (otherwise, the 2nd highest has a lower income than the 3rd highest ability group in the last few years).

Table 4.3: Log Wage Regressions, by Lifetime Income Group

Lifetime income groups (percentiles)	Constant	*Age*	*Age*^{2}	*Age*^{3}	Observations
0 to 25	3.41000000***	-0.09720122***	0.00247639***	-0.00001842***	65,698
	(0.08718100)	(0.00543339)	(0.00010901)	(0.00000071)	
25 to 50	0.69689692***	0.05995294***	-0.00004086	-0.00000521***	101,484
	(0.05020758)	(0.00345549)	(0.00007627)	(0.00000054)	
50 to 70	-0.78761958***	0.17654618***	-0.00240656***	0.00001039***	74,253
	(0.04519637)	(0.00338371)	(0.00008026)	(0.00000061)	
70 to 80	-1.11000000***	0.21168263***	-0.00306555***	0.00001438***	33,528
	(0.06838352)	(0.00530190)	(0.00012927)	(0.00000099)	
80 to 90	-0.93939272***	0.21638731***	-0.00321041***	0.00001579***	31,919
	(0.08333727)	(0.00664647)	(0.00016608)	(0.00000130)	
90 to 99	1.60000000***	0.04500235***	0.00094253***	-0.00001470***	24,370
	(0.11723131)	(0.00931334)	(0.00022879)	(0.00000176)	
99 to 100	1.89000000***	0.09229392**	0.00012902	-0.00001169*	2,129
	(0.50501510)	(0.03858202)	(0.00090072)	(0.00000657)	

Source: CWHS data, 1991 – 2009.
* Significant at the 10 percent level ($p < 0.10$).
** Significant at the 5 percent level ($p < 0.05$).
*** Significant at the 1 percent level ($p < 0.01$).

Chapter 5

Households

In this chapter, we describe what is arguably the most important economic agent in the `OG-USA` model: the household. We model households in `OG-USA` rather than individuals, because we want to abstract from the concepts of gender, marital status, and number of children. Furthermore, the household is the usual unit of account in tax data. Because `OG-USA` is primarily a fiscal policy model using U.S. data, it is advantageous to have the most granular unit of account be the household.

5.1 Budget Constraint

We described the derivation and dynamics of the population distribution in Chapter 3. A measure $\omega_{1,t}$ of households is born each period, become economically relevant at age $s = E+1$ if they survive to that age, and live for up to $E+S$ periods (S economically active periods), with the population of age-s individuals in period t being $\omega_{s,t}$. Let the age of a household be indexed by $s = \{1, 2, ...E+S\}$.

At birth, each household age $s = 1$ is randomly assigned one of J ability groups, indexed by j. Let λ_j represent the fraction of individuals in each ability group, such that $\sum_j \lambda_j = 1$. Note that this implies that the distribution across ability types in each age is given by $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ...\lambda_J]$. Once an household is born and assigned to an ability type, it remains that ability type for its entire lifetime. This is deterministic ability heterogeneity as described in Chapter 4. Let $e_{j,s} > 0$ be a matrix of ability-levels such that an individual of ability type j will have lifetime abilities of $[e_{j,1}, e_{j,2}, ...e_{j,E+S}]$. The budget constraint for the age-s household in lifetime income group j at time t is the following,

$$\begin{aligned} c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \zeta_{j,s}\frac{BQ_t}{\lambda_j \omega_{s,t}} + \eta_{j,s,t}\frac{TR_t}{\lambda_j \omega_{s,t}} - T_{s,t} \\ \forall j,t \quad \text{and} \quad s \geq E+1 \quad \text{where} \quad b_{j,E+1,t} = 0 \quad \forall j,t \end{aligned} \tag{5.1}$$

where $c_{j,s,t}$ is consumption, $b_{j,s+1,t+1}$ is savings for the next period, r_t is the interest rate (return on savings), $b_{j,s,t}$ is current period wealth (savings from last period), w_t is the wage, and $n_{j,s,t}$ is labor supply.

The next term on the right-hand-side of the budget constraint (5.1) represents the portion of total bequests BQ_t that go to the age-s, income-group-j household. Let $\zeta_{j,s}$ be the fraction of total bequests BQ_t that go to the age-s, income-group-j household, such that $\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\zeta_{j,s}=1$. We must divide that amount by the population of (j,s) households $\lambda_j\omega_{s,t}$. Chapter 6 details how to calibrate the $\zeta_{j,s}$ values from consumer finance data.

The last two terms on the right-hand-side of the budget constraint (5.1) have to do with government transfers and taxes, respectively. TR_t is total government transfers to households in period t and $\eta_{j,s,t}$ is the percent of those transfers that go to households of age s and lifetime income group j such that $\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\eta_{j,s,t}=1$. This term is divided by the population of type (j,s) households. We assume government transfers to be lump sum, so they do not create any direct distortions to household decisions.

The term $T_{s,t}$ is the total tax liability of the household. In contrast to government transfers $tr_{j,s,t}$, tax liability can be a function of labor income ($x_{j,s,t}\equiv w_t e_{j,s}n_{j,s,t}$) and capital income ($y_{j,s,t}\equiv r_t b_{j,s,t}$). The tax liability can, therefore, be a distortionary influence on household decisions. It becomes valuable to represent total tax liability as an effective tax rate τ^{etr} multiplied by total income,

$$T_{s,t}=\tau^{etr}_{s,t}(x_{j,s,t},y_{j,s,t})\left(x_{j,s,t}+y_{j,s,t}\right) \tag{8.3}$$

where the effective tax rate can be a function of both labor income and capital income $\tau^{etr}(x,y)$. Chapter 8 details exactly how we estimate these tax functions from microsimulation model data.

where many of the variables now have j subscripts. The variables with three subscripts (j,s,t) tell you to which ability type j and age s individual the variable belongs and in which period t.

5.2 Elliptical Disutility of Labor Supply

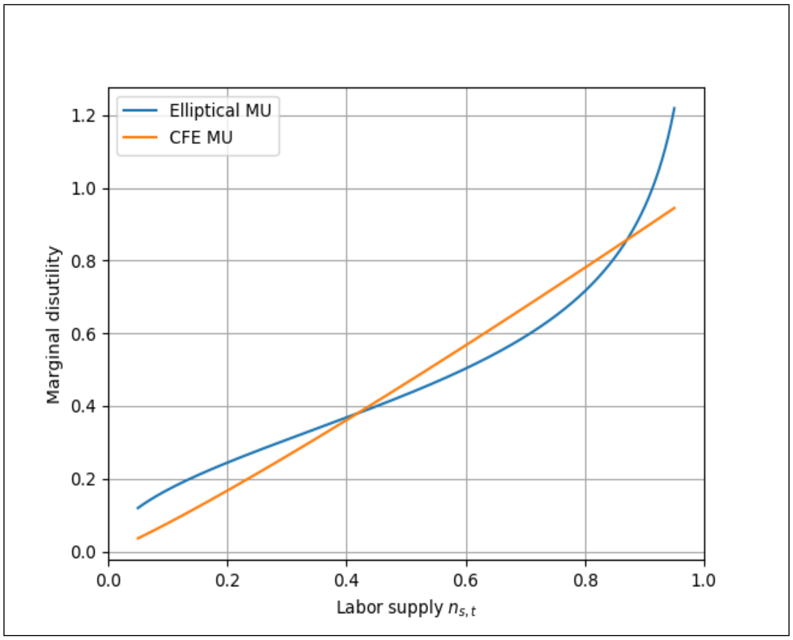
In `OG-USA`, the period utility function of each household is a function of consumption $c_{j,s,t}$, savings $b_{j,s+1,t+1}$, and labor supply $n_{j,s,t}$.[1] We detail this utility function, its justification, and functional form in Section 5.3. With endogenous labor supply $n_{j,s,t}$, we must specify how labor enters an agent's utility function and what are the constraints. Assume that each household is endowed with a measure of time $\tilde{l}$ each period that it can choose to spend as either labor $n_{j,s,t}\in[0,\tilde{l}]$ or leisure $l_{j,s,t}\in[0,\tilde{l}]$.

$$n_{j,s,t}+l_{j,s,t}=\tilde{l}\quad\forall s,t \tag{5.2}$$

The functional form for the utility of leisure or the disutility of labor supply has important implications for the computational tractability of the model. One difference of the household's labor supply decision $n_{j,s,t}$ from the consumption decision $c_{j,s,t}$ is that the consumption decision only has a lower bound $c_{j,s,t}\geq 0$ whereas the labor supply decision has both upper and lower bounds $n_{j,s,t}\in[0,\tilde{l}]$. Evans and Phillips (2017) show that many of the traditional functional forms for the disutility of labor—Cobb-Douglas, constant Frisch elasticty, constant relative risk aversion (CRRA)—do not have Inada conditions on both the upper and lower bounds of labor supply. To solve these in a heterogeneous agent model would require occasionally binding constraints, which is a notoriously difficult computational problem.

[1]Savings enters the period utility function to provide a "warm glow" bequest motive.

Figure 5.1: Comparison of CFE marginal disutility of leisure $\theta = 0.9$ to fitted elliptical utility



Evans and Phillips (2017) propose using an equation for an ellipse to match the disutility of labor supply to whatever traditional functional form one wants. Our preferred specification in `OG-USA` is to fit an elliptical disutility of labor supply function to approximate a linearly separable constant Frisch elasticity (CFE) functional form. Let $v(n)$ be a general disutility of labor function. A CFE disutility of labor function is the following,

$$v(n) \equiv \frac{n^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \quad \theta > 0 \tag{5.3}$$

where $\theta > 0$ represents the Frisch elasticity of labor supply. The elliptical disutility of labor supply functional form is the following,

$$v(n) = -b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}, \quad b, \upsilon > 0 \tag{5.4}$$

where $b > 0$ is a scale parameter and $\upsilon > 0$ is a curvature parameter. This functional form satisfies both $v'(n) > 0$ and $v''(n) > 0$ for all $n \in (0, 1)$. Further, it has Inada conditions at both the upper and lower bounds of labor supply $\lim_{n\to 0} v'(n) = 0$ and $\lim_{n\to\tilde{l}} v'(n) = -\infty$.

Because it is the marginal disutility of labor supply that matters for household decision making, we want to choose the parameters of the elliptical disutility of labor supply function (b, υ) so that the elliptical marginal utilities match the marginal utilities of the CFE disutility of labor supply. Figure 5.1 shows the fit of marginal utilities for a Frisch elasticity of $\theta = 0.9$ and a total time endowment of $\tilde{l} = 1.0$. The estimated elliptical utility parameters in this case are $b = 0.527$ and $\upsilon = 1.497$.[2]

[2] Peterman (2016) shows that in a macro-model that has only an intensive margin of labor supply and

5.3 Optimality Conditions

Households choose lifetime consumption $\{c_{j,s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{j,s,t+s-1}\}_{s=1}^S$, and savings $\{b_{j,s+1,t+s}\}_{s=1}^S$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints. The household period utility function is the following.

$$\begin{split} u(c_{j,s,t},n_{j,s,t},b_{j,s+1,t+1}) \equiv \frac{(c_{j,s,t})^{1-\sigma}-1}{1-\sigma} + e^{g_y t(1-\sigma)}\chi^n_s\left(b\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}}\right) + \\ \chi^b_j\rho_s\frac{(b_{j,s+1,t+1})^{1-\sigma}-1}{1-\sigma} \quad \forall j,t \quad \text{and} \quad E+1\leq s\leq E+S \end{split} \tag{5.5}$$

The period utility function (5.5) is linearly separable in $c_{j,s,t}$, $n_{j,s,t}$, and $b_{j,s+1,t+1}$. The first term is a constant relative risk aversion (CRRA) utility of consumption. The second term is the elliptical disutility of labor described in Section 5.2. The constant χ^n_s adjusts the disutility of labor supply relative to consumption and can vary by age s, which is helpful for calibrating the model to match labor market moments. See Chapter 15 for a discussion of the calibration.

It is necessary to multiply the disutility of labor in (5.5) by $e^{g_y(1-\sigma)}$ because labor supply $n_{j,s,t}$ is stationary, but both consumption $c_{j,s,t}$ and savings $b_{j,s+1,t+1}$ are growing at the rate of technological progress (see Chapter 12). The $e^{g_y(1-\sigma)}$ term keeps the relative utility values of consumption, labor supply, and savings in the same units.

The final term in the period utility function (5.5) is the "warm glow" bequest motive. It is a CRRA utility of savings, discounted by the mortality rate ρ_s.[3] Intuitively, it signifies the utility a household gets in the event that they don't live to the next period with probability ρ_s. It is a utility of savings beyond its usual benefit of allowing for more consumption in the next period. This utility of bequests also has constant χ^b_j which adjusts the utility of bequests relative to consumption and can vary by lifetime income group j. This is helpful for calibrating the model to match wealth distribution moments. See Chapter 15 for a discussion of the calibration. Note that any bequest before age $E+S$ is unintentional as it was bequeathed due an event of death that was uncertain. Intentional bequests are all bequests given in the final period of life in which death is certain $b_{j,E+S+1,t}$.

The household lifetime optimization problem is to choose consumption $c_{j,s,t}$, labor supply $n_{j,s,t}$, and savings $b_{j,s+1,t+1}$ in every period of life to maximize expected discounted lifetime utility, subject to budget constraints and upper-bound and lower-bound constraints.

$$\max_{\{(c_{j,s,t}),(n_{j,s,t}),(b_{j,s+1,t+1})\}_{s=E+1}^{E+S}} \sum_{s=1}^S \beta^{s-1}\left[\Pi_{u=E+1}^{E+s}(1-\rho_u)\right]u(c_{j,s,t+s-1},n_{j,s,t+s-1},b_{j,s+1,t+s}) \tag{5.6}$$

$$\text{s.t.} \quad c_{j,s,t}+b_{j,s+1,t+1} = (1+r_t)b_{j,s,t}+w_t e_{j,s}n_{j,s,t}+\zeta_{j,s}\frac{BQ_t}{\lambda_j\omega_{s,t}}+\eta_{j,s,t}\frac{TR_t}{\lambda_j\omega_{s,t}}-T_{s,t} \tag{5.1}$$

$$\text{and} \quad c_{j,s,t}\geq 0,\ n_{j,s,t}\in[0,\tilde{l}],\ \text{and}\ b_{j,1,t}=0 \quad \forall j,t,\ \text{and}\ E+1\leq s\leq E+S$$

no extensive margin and represents a broad composition of individuals supplying labor—such as `OG-USA`—a Frisch elasticity of around 0.9 is probably appropriate. He tests the implied macro elasticity when the assumed micro elasticities are small on the intensive margin but only macro aggregates—which include both extensive and intensive margin agents—are observed.

[3]See Section 3.2 of Chapter 3 for a detailed discussion of mortality rates in `OG-USA`.

The nonnegativity constraint on consumption does not bind in equilibrium because of the Inada condition $\lim_{c\to 0} u_1(c,n,b') = \infty$, which implies consumption is always strictly positive in equilibrium $c_{j,s,t} > 0$ for all j, s, and t. The warm glow bequest motive in (5.5) also has an Inada condition for savings at zero, so $b_{j,s,t} > 0$ for all j, s, and t. This is an implicit borrowing constraint.[4] And finally, as discussed in Section 5.2, the elliptical disutility of labor supply functional form in (5.5) imposes Inada conditions on both the upper and lower bounds of labor supply such that labor supply is strictly interior in equilibrium $n_{j,s,t} \in (0,\tilde{l})$ for all j, s, and t.

The household maximization problem can be further reduced by substituting in the household budget constraint, which binds with equality. This simplifies the household's problem to choosing labor supply $n_{j,s,t}$ and savings $b_{j,s+1,t+1}$ every period to maximize lifetime discounted expected utility. The $2S$ first order conditions for every type-j household that characterize the its S optimal labor supply decisions and S optimal savings decisions are the following.

$$w_t e_{j,s}\bigl(1-\tau^{mtrx}_{s,t}\bigr)(c_{j,s,t})^{-\sigma} = e^{g_y(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\ \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S \tag{5.7}$$

$$(c_{j,s,t})^{-\sigma} = \chi^b_j\rho_s(b_{j,s+1,t+1})^{-\sigma} + \beta\bigl(1-\rho_s\bigr)\Bigl(1+r_{t+1}\bigl[1-\tau^{mtry}_{s+1,t+1}\bigr]\Bigr)(c_{j,s+1,t+1})^{-\sigma} \\ \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S-1 \tag{5.8}$$

$$(c_{j,E+S,t})^{-\sigma} = \chi^b_j(b_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \quad \text{and} \quad s=E+S \tag{5.9}$$

The distortion of taxation on household decisions can be seen in Euler equations (5.7) and (5.8) in the terms that have a marginal tax rate $(1-\tau^{mtr})$. This comes from the expression for total tax liabilities as a function of the effective tax rate and total income as expressed in (8.3). Using the chain rule, we can break up the derivatives of total tax liability with respect to $n_{j,s,t}$ and $b_{j,s,t}$, respectively, into simpler functions of marginal tax rates. We discuss this in more detail in Chapter 8.

$$\frac{\partial T_{s,t}}{\partial n_{j,s,t}} = \frac{\partial T_{s,t}}{\partial w_t e_{j,s}n_{j,s,t}}\frac{\partial w_t e_{j,s}n_{j,s,t}}{\partial n_{j,s,t}} = \frac{\partial T_{s,t}}{\partial w_t e_{j,s}n_{j,s,t}}w_t e_{j,s} = \tau^{mtrx}_{s,t}w_t e_{j,s} \tag{8.6}$$

$$\frac{\partial T_{s,t}}{\partial b_{j,s,t}} = \frac{\partial T_{s,t}}{\partial r_t b_{j,s,t}}\frac{\partial r_t b_{j,s,t}}{\partial b_{j,s,t}} = \frac{\partial T_{s,t}}{\partial r_t b_{j,s,t}}r_t = \tau^{mtry}_{s,t}r_t \tag{8.7}$$

[4] It is important to note that savings also has an implicit upper bound $b_{j,s,t}\leq k$ above which consumption would be negative in current period. However, this upper bound on savings in taken care of by the Inada condition on consumption.

5.4 Expectations

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates, wages, and total bequests $\{r_u, w_u, BQ_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Chapters 13 and 14, the equilibrium interest rate r_t, wage w_t, and total bequests BQ_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \big\{b_{j,s,t}\big\}_{s=E+2}^{E+S} \quad \forall j,t \tag{5.10}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{5.11}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[5]

[5] In Chapter 14 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 14.1.

Chapter 6

Calibrated Bequests

This chapter describes how we calibrate the distribution of total bequests BQ_t to each living household of age s and lifetime income group j. The matrix that governs this distribution $\zeta_{j,s}$ is seen in the household budget constraint 5.1.

A large number of papers study the effects of different bequest motives and specifications on the distribution of wealth, though there is no consensus regarding the true bequest transmission process.[1]

[1]See De Nardi and Yang (2014), De Nardi (2004), Nishiyama (2002), Laitner (2001), Gokhale et al. (2000), Gale and Scholz (1994), Hurd (1989), Venti and Wise (1988), Kotlikoff and Summers (1981), and Wolff (2015).

Part III

Firms Theory

Chapter 7

Firms

The production side of the `OG-USA` model is populated by a unit measure of identical perfectly competitive firms that rent capital K_t and hire labor L_t to produce output Y_t. Firms also face a flat corporate income tax τ^{corp} as well as a tax on the amount of capital they depreciate τ^{δ}.

7.1 Production Function

Firms produce output Y_t using inputs of capital K_t and labor L_t according to a general constant elasticity (CES) of substitution production function,

$$Y_t = F(K_t, L_t) \equiv Z_t\biggl[(\gamma)^{\frac{1}{\varepsilon}}(K_t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(e^{g_y t}L_t)^{\frac{\varepsilon-1}{\varepsilon}}\biggr]^{\frac{\varepsilon}{\varepsilon-1}} \quad \forall t \tag{7.1}$$

where Z_t is an exogenous scale parameter (total factor productivity) that can be time dependent, γ represents the capital share of income, and ε is the constant elasticity of substitution between capital and labor. We have included constant productivity growth g_y as the rate of labor augmenting technological progress.

A nice feature of the CES production function is that the Cobb-Douglas production function is a nested case for $\varepsilon = 1$.

$$Y_t = Z_t(K_t)^{\gamma}(e^{g_y t}L_t)^{1-\gamma} \quad \text{for} \quad \varepsilon = 1 \quad \forall t \tag{7.2}$$

7.2 Optimality Conditions

The profit function of the representative firm is the following.

$$PR_t = (1-\tau^{corp})\Bigl[F(K_t, L_t) - w_t L_t\Bigr] - \bigl(r_t + \delta\bigr)K_t + \tau^{corp}\delta^{\tau}K_t \quad \forall t \tag{7.3}$$

Gross income for the firms is given by the production function $F(K, L)$ because we have normalized the price of the consumption good to 1. Labor costs to the firm are $w_t L_t$, and capital costs are $(r_t + \delta)K_t$. The per-period economic depreciation rate is given by δ.

Taxes enter the firm's profit function (7.3) in two places. The first is the corporate income tax rate τ^{corp}, which is a flat tax on corporate income. As is the case in the U.S., corporate income is defined as gross income minus labor costs. This will cause the corporate tax to only distort the firms' capital demand decision.

The next place where tax policy enters the profit function (7.3) is through a refund of a percent of depreciation costs δ^τ refunded at the corporate income tax rate τ^{corp}. When $\delta^\tau = 0$, no depreciation expense is deducted from the firm's tax liability. When $\delta^\tau = \delta$, all economic depreciation is deducted from corporate income.

Taking the derivative of the profit function (7.3) with respect to labor L_t and setting it equal to zero and taking the derivative of the profit function with respect to capital K_t and setting it equal to zero, respectively, characterizes the optimal labor and capital demands.

$$w_t = e^{g_y t}(Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[(1-\gamma)\frac{Y_t}{e^{g_y t}L_t}\right]^{\frac{1}{\varepsilon}} \quad \forall t \tag{7.4}$$

$$r_t = (1-\tau^{corp})(Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{Y_t}{K_t}\right]^{\frac{1}{\varepsilon}} - \delta + \tau^{corp}\delta^\tau \quad \forall t \tag{7.5}$$

We discuss how to calibrate the values of τ^{corp} and τ^δ from the `B-Tax` microsimulation model in Chapter 9.

Part IV

Government Theory

Chapter 8

Household Taxes and Tax-Calculator

The government is not an optimizing agent in `OG-USA`. The government levies taxes on households, provides transfers to households, levies taxes on firms, spends resources on public goods, and makes rule-based adjustments to stabilize the economy in the long-run. The government can run budget deficits or surpluses in a given year and must, therefore, be able to accumulate debt or savings.

The government sector influences households through two terms in the budget constraint (5.1)—government transfers TR_t and through the total tax liability function $T_{s,t}$, which can be decomposed into the effective tax rate times total income (8.3). In this chapter, we detail the household tax component of government activity $T_{s,t}$ in `OG-USA`, along with our method of incorporating detailed microsimulation data into a dynamic general equilibrium model.

Incorporating realistic tax and incentive detail into a general equilibrium model is notoriously difficult for two reasons. First, it is impossible in a dynamic general equilibrium model to capture all of the dimensions of heterogeneity on which the real-world tax rate depends. For example, a household's tax liability in reality depends on filing status, number of dependents, many types of income, and some characteristics correlated with age. A good heterogeneous agent DGE model tries to capture the most important dimensions of heterogeneity, and necessarily neglects the other dimensions.

The second difficulty in modeling realistic tax and incentive detail is the need for good microeconomic data on the individuals who make up the economy from which to simulate behavioral responses and corresponding tax liabilities and tax rates.

`OG-USA` follows the method of DeBacker et al. (2017) of generating detailed tax data on effective tax rates and marginal tax rates for a sample of tax filers along with their respective income and demographic characteristics and then using that data to estimate parametric tax functions that can be incorporated into `OG-USA`.

8.1 Effective and Marginal Tax Rates

Before going into more detail regarding how we handle these two difficulties in `OG-USA`, we need to define some functions and make some notation. For notational simplicity, we will use the variable x to summarize labor income, and we will use the variable y to summarize capital income.

$$x_{j,s,t} \equiv w_t e_{j,s} n_{j,s,t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{8.1}$$
$$y_{j,s,t} \equiv r_t b_{j,s,t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{8.2}$$

We can express total tax liability $T_{s,t}$ from the household budget constraint (5.1) as an effective tax rate multiplied by total income.

$$T_{s,t} = \tau^{etr}_{s,t}(x_{j,s,t}, y_{j,s,t})\left(x_{j,s,t} + y_{j,s,t}\right) \tag{8.3}$$

Rearranging (8.3) gives the definition of an effective tax rate (ETR) as total tax liability divided by unadjusted gross income, or rather, total tax liability as a percent of unadjusted gross income.

A marginal tax rate (MTR) is defined as the change in total tax liability from a small change income. In `OG-USA`, we differentiate between the marginal tax rate on labor income ($MTRx$) and the marginal tax rate on labor income ($MTRy$).

$$\tau^{mtrx} \equiv \frac{\partial T_{s,t}}{\partial w_t e_{j,s} n_{j,s,t}} = \frac{\partial T_{s,t}}{\partial x_{j,s,t}} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{8.4}$$
$$\tau^{mtry} \equiv \frac{\partial T_{s,t}}{\partial r_t b_{j,s,t}} = \frac{\partial T_{s,t}}{\partial y_{j,s,t}} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{8.5}$$

As we show in Section 5.3, the derivative of total tax liability with respect to labor supply $\frac{\partial T_{s,t}}{n_{j,s,t}}$ and the derivative of total tax liability next period with respect to savings $\frac{\partial T_{s+1,t+1}}{b_{j,s+1,t+1}}$ show up in the household Euler equations for labor supply (5.7) and savings (5.8), respectively. It is valuable to be able to express those marginal tax rates, for which we have no data, as marginal tax rates for which we do have data. The following two expressions show how the marginal tax rates of labor supply can be expressed as the marginal tax rate on labor income times the household-specific wage and how the marginal tax rate of savings can be expressed as the marginal tax rate of capital income times the interest rate.

$$\frac{\partial T_{s,t}}{\partial n_{j,s,t}} = \frac{\partial T_{s,t}}{\partial w_t e_{j,s} n_{j,s,t}}\frac{\partial w_t e_{j,s} n_{j,s,t}}{\partial n_{j,s,t}} = \frac{\partial T_{s,t}}{\partial w_t e_{j,s} n_{j,s,t}} w_t e_{j,s} = \tau^{mtrx}_{s,t} w_t e_{j,s} \tag{8.6}$$
$$\frac{\partial T_{s,t}}{\partial b_{j,s,t}} = \frac{\partial T_{s,t}}{\partial r_t b_{j,s,t}}\frac{\partial r_t b_{j,s,t}}{\partial b_{j,s,t}} = \frac{\partial T_{s,t}}{\partial r_t b_{j,s,t}} r_t = \tau^{mtry}_{s,t} r_t \tag{8.7}$$

8.2 Microeconomic Data

For `OG-USA`, we use an open source microsimulation model called `Tax-Calculator` that uses microeconomic data on U.S. households from the Internal Revenue Service (IRS) Statistics of Income (SOI) Public Use File (PUF).[1]

[1] `Tax-Calculator` is available through an open source repository https://github.com/open-source-economics/Tax-Calculator as well as through a web application https://www.ospc.org/taxbrain/. Documentation for `Tax-Calculator` is available at http://open-source-economics.github.io/Tax-Calculator/ and http://taxcalc.readthedocs.io/en/latest/public_api.html. For users that have not paid for access to the Public Use File (PUF), `Tax-Calculator` has an option to use a CPS matched dataset that is publicly available free of charge that has the same general properties as the PUF.

`Tax-Calculator` starts with the underlying population microeconomic data, in which each observation is a filer with a population weight that renders the sample representative. It then processes the relevant income and demographic characteristics in order to calculate the tax liability of each individual, according to all the rich tax law of the United States tax code. `Tax-Calculator` can then calculate effective tax rates for all of these individuals, thereby creating a sample of how ETR's are related to other variables in our `OG-USA` model, such as total income $x + y$, labor income x, and capital income y. `Tax-Calculator` can also generate marginal tax rates by adding a dollar to each filer's income of a particular type and calculate how the filer's tax liability changes. This is a finite difference calculation of a derivative.

Figure 8.1 shows a scatter plot of ETR's for 43-year-olds in 2017 and unadjusted gross income $x + y$. It is clear that ETR is positively related to income. It is also clear that a significant number of filers have a negative ETR. We will discuss in Section 8.3 the functional form `OG-USA` uses to best capture the main characteristics of these ETR data.

Figure 8.1: Plot of estimated ETR functions: $t =$ 2017 and $s = 43$ under current law

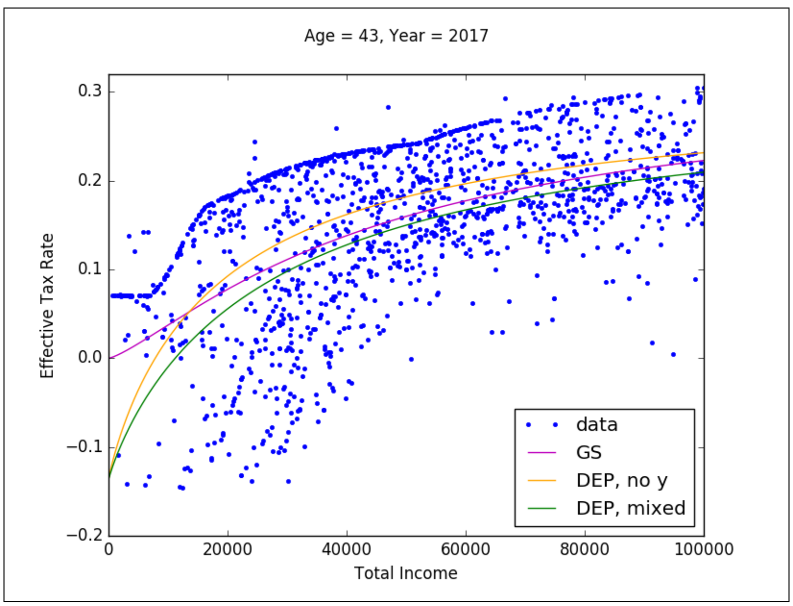
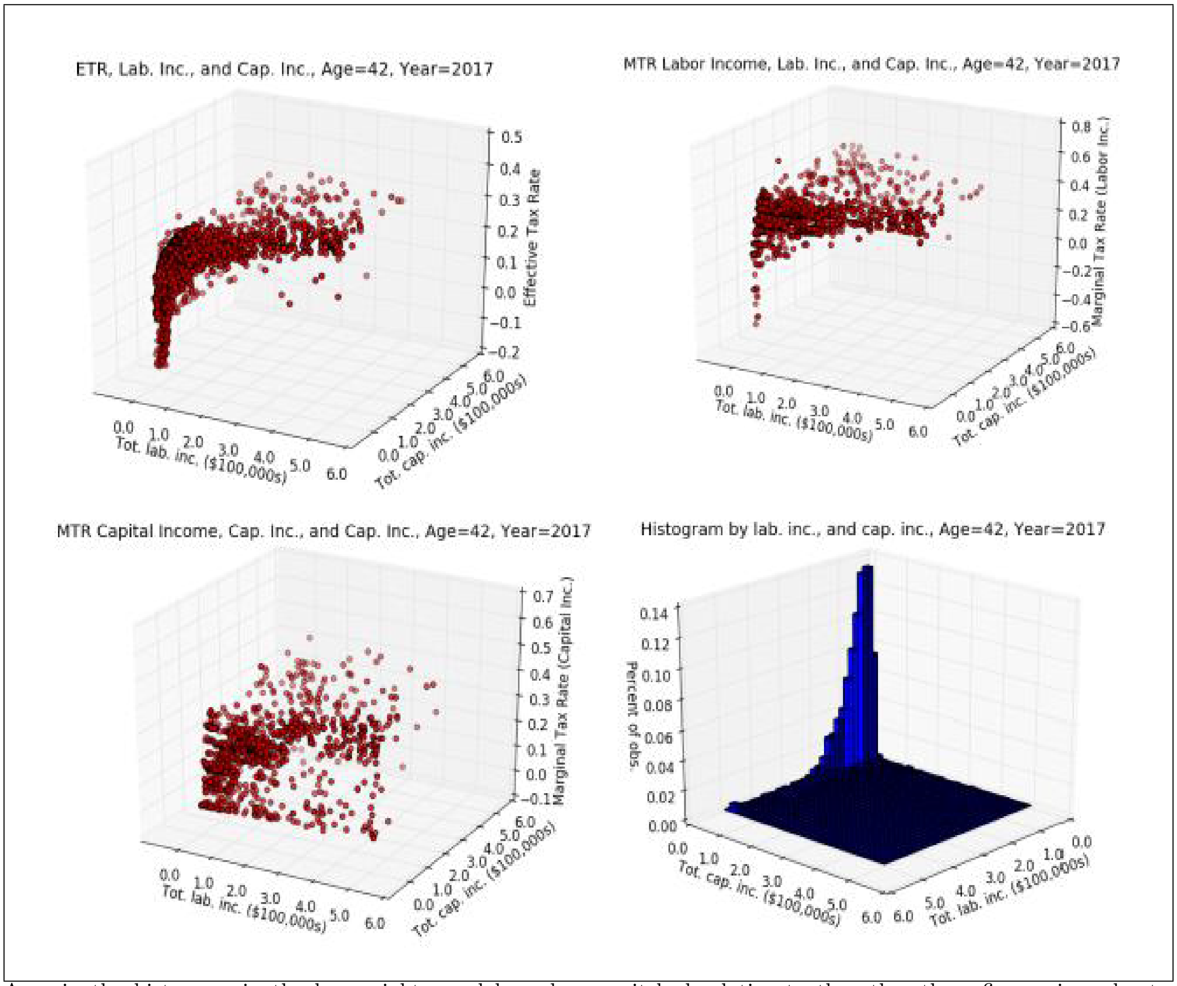


Figure 8.2 shows 3D scatter plots of ETR, $MTRx$, and $MTRy$ (and a histogram of the data) with the labor income and capital income, separately, of each age-42 filer in 2017, generated by `Tax-Calculator`. This figure presents the main visual evidence for the functional form we use to fit tax functions to these data in Section 8.3. Figure 8.2 presents strong evidence that the tax rate—ETR, $MTRx$, and $MTRy$—is most accurately modeled as a function of labor income and capital income, separately: $\tau(x, y)$.

8.3 Fitting Tax Functions

In looking at the 2D scatter plot on effective tax rates as a function of total income in Figure 8.1 and the 3D scatter plots of ETR, $MTRx$, and $MTRy$ in Figure 8.2, it is clear that all

Figure 8.2: Scatter plot of ETR, MTRx, MTRy, and histogram as functions of labor income and capital income from microsimulation model: $t = 2017$ and $s = 42$ under current law



*Note: Axes in the histogram in the lower-right panel have been switched relative to the other three figures in order to see the distribution more clearly.

of these rates exhibit negative exponential or logistic shape. This empirical regularity allows us to make an important and nonrestrictive assumption. We can fit parametric tax rate functions to these data that are constrained to be monotonically increasing in labor income and capital income. This assumption of monotonicity is computationally important as it preserves a convex budget set for each household, which is important for being able to solve many household lifetime problems over a large number of periods.

`OG-USA` follows the approach of DeBacker et al. (2017) in using the following functional form to estimate tax functions for each age $s = E+1, E+2, ...E+S$ in each time period t. Equation 8.8 is written as a generic tax rate, but we use this same functional form for

ETR's, $MTRx$'s, and $MTRy$'s.

$$\begin{aligned}
\tau(x,y) =&\Bigl[\tau(x) + shift_x\Bigr]^\phi\Bigl[\tau(y) + shift_y\Bigr]^{1-\phi} + shift \\
&\text{where} \quad \tau(x) \equiv (max_x - min_x)\left(\frac{Ax^2 + Bx}{Ax^2 + Bx + 1}\right) + min_x \\
&\text{and} \quad \tau(y) \equiv (max_y - min_y)\left(\frac{Cy^2 + Dy}{Cy^2 + Dy + 1}\right) + min_y \\
&\text{where} \quad A, B, C, D, max_x, max_y, shift_x, shift_y > 0 \quad \text{and} \quad \phi \in [0,1] \\
&\text{and} \quad max_x > min_x \quad \text{and} \quad max_y > min_y
\end{aligned} \tag{8.8}$$

The parameters values will, in general, differ across the different functions (effective and marginal rate functions) and by age, s, and tax year, t. We drop the subscripts for age and year from the above exposition for clarity.

By assuming each tax function takes the same form, we are breaking the analytical link between the the effective tax rate function and the marginal rate functions. In particular, one could assume an effective tax rate function and then use the analytical derivative of that to find the marginal tax rate function. However, we've found it useful to separately estimate the marginal and average rate functions. One reason is that we want the tax functions to be able to capture policy changes that have differential effects on marginal and average rates. For example, a change in the standard deduction for tax payers would have a direct effect on their average tax rates. But it will have secondary effect on marginal rates as well, as some filers will find themselves in different tax brackets after the policy change. These are smaller and second order effects. When tax functions are are fit to the new policy, in this case a lower standard deduction, we want them to be able to represent this differential impact on the marginal and average tax rates. The second reason is related to the first. As the additional flexibility allows us to model specific aspects of tax policy more closely, it also allows us to better fit the parameterized tax functions to the data.

The key building blocks of the functional form Equation (8.8) are the $\tau(x)$ and $\tau(y)$ univariate functions. The ratio of polynomials in the $\tau(x)$ function $\frac{Ax^2+Bx}{Ax^2+Bx+1}$ with positive coefficients $A, B > 0$ and positive support for labor income $x > 0$ creates a negative-exponential-shaped function that is bounded between 0 and 1, and the curvature is governed by the ratio of quadratic polynomials. The multiplicative scalar term $(max_x - min_x)$ on the ratio of polynomials and the addition of min_x at the end of $\tau(x)$ expands the range of the univariate negative-exponential-shaped function to $\tau(x) \in [min_x, max_x]$. The $\tau(y)$ function is an analogous univariate negative-exponential-shaped function in capital income y, such that $\tau(y) \in [min_y, max_y]$.

The respective $shift_x$ and $shift_y$ parameters in Equation (8.8) are analogous to the additive constants in a Stone-Geary utility function. These constants ensure that the two sums $\tau(x) + shift_x$ and $\tau(y) + shift_y$ are both strictly positive. They allow for negative tax rates in the $\tau(\cdot)$ functions despite the requirement that the arguments inside the brackets be strictly positive. The general $shift$ parameter outside of the Cobb-Douglas brackets can then shift the tax rate function so that it can accommodate negative tax rates. The Cobb-Douglas share parameter $\phi \in [0,1]$ controls the shape of the function between the two univariate functions $\tau(x)$ and $\tau(y)$.

This functional form for tax rates delivers flexible parametric functions that can fit the tax rate data shown in Figure 8.2 as well as a wide variety of policy reforms. Further, these functional forms are monotonically increasing in both labor income x and capital income y. This characteristic of monotonicity in x and y is essential for guaranteeing convex budget sets and thus uniqueness of solutions to the household Euler equations. The assumption of monotonicity does not appear to be a strong one when viewing the the tax rate data shown in Figure 8.2. While it does limit the potential tax systems to which one could apply our methodology, tax policies that do not satisfy this assumption would result in non-convex budget sets and thus require non-standard DGE model solutions methods and would not guarantee a unique equilibrium. The 12 parameters of our tax rate functional form from (8.8) are summarized in Table 8.1.

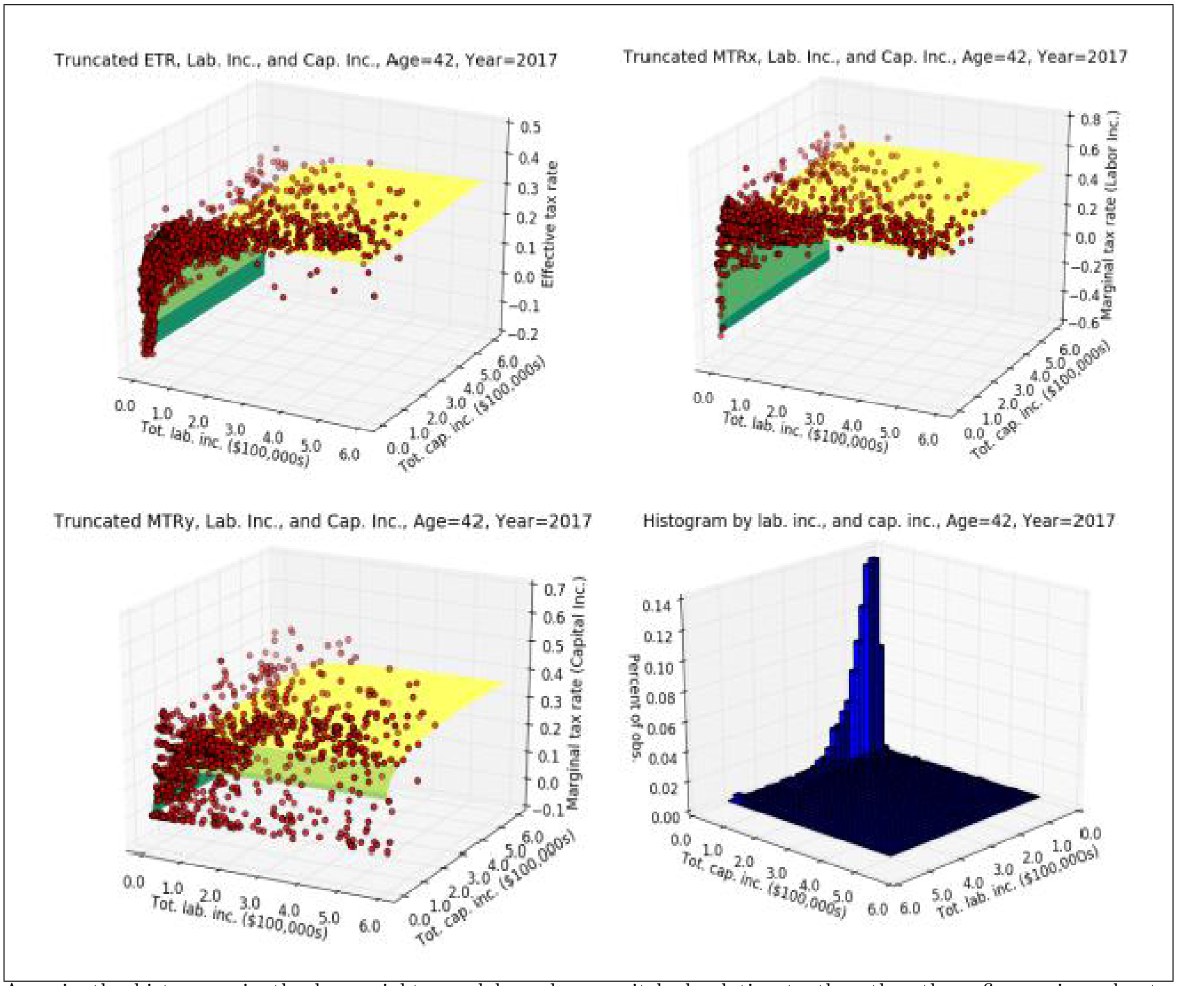
Table 8.1: Description of tax rate function $\tau(x,y)$ parameters

Symbol	Description
A	Coefficient on squared labor income term x^2 in $\tau(x)$
B	Coefficient on labor income term x in $\tau(x)$
C	Coefficient on squared capital income term y^2 in $\tau(y)$
D	Coefficient on capital income term y in $\tau(y)$
max_x	Maximum tax rate on labor income x given $y=0$
min_x	Minimum tax rate on labor income x given $y=0$
max_y	Maximum tax rate on capital income y given $x=0$
min_y	Minimum tax rate on capital income y given $x=0$
$shift_x$	shifter $>\lvert min_x\rvert$ ensures that $\tau(x)+shift_x>0$ despite potentially negative values for $\tau(x)$
$shift_y$	shifter $>\lvert min_y\rvert$ ensures that $\tau(y)+shift_y>0$ despite potentially negative values for $\tau(y)$
$shift$	shifter (can be negative) allows for support of $\tau(x,y)$ to include negative tax rates
ϕ	Cobb-Douglas share parameter between 0 and 1

Let $\boldsymbol{\theta}_{s,t} = (A,B,C,D,max_x,min_x,max_y,min_y,shift_x,shift_y,shift,\phi)$ be the full vector of 12 parameters of the tax function for a particular type of tax rate, age of filers, and year. We first directly specify min_x as the minimum tax rate and max_x as the maximum tax rate in the data for age-s and period-t individuals for capital income close to 0 ($\$0<y<\$3,000$), and min_y as the minimum tax rate and max_y as the maximum tax rate for labor income close to 0 ($\$0<x<\$3,000$). We then set $shift_x = \min(0,|min_x|)+\varepsilon$ and $shift_y = \min(0,|min_y|)+\varepsilon$ so that the respective arguments in the brackets of (8.8) are strictly positive. Then let $shift$ be be the minimum tax rate in the corresponding data minus ε. Let $\bar{\boldsymbol{\theta}}_{s,t} = \{min_x,max_x,min_y,max_y,shift_x,shift_y,shift\}$ be the set of parameters we take directly from the data in this way.

We then estimate five remaining parameters $\tilde{\boldsymbol{\theta}}_{s,t} = (A,B,C,D,shift,\phi)$ using the fol-

Figure 8.3: Estimated tax rate functions of ETR, MTRx, MTRy, and histogram as functions of labor income and capital income from microsimulation model: $t = 2017$ and $s = 42$ under current law



*Note: Axes in the histogram in the lower-right panel have been switched relative to the other three figures in order to see the distribution.

lowing nonlinear weighted least squares criterion,

$$\hat{\boldsymbol{\theta}}_{s,t} = \tilde{\boldsymbol{\theta}}_{s,t} : \quad \min_{\tilde{\boldsymbol{\theta}}_{s,t}} \sum_{i=1}^{N} \Big[\tau_i - \tau_{s,t}\big(x_i, y_i | \tilde{\boldsymbol{\theta}}_{s,t}, \bar{\boldsymbol{\theta}}_{s,t}\big)\Big]^2 w_i, \qquad \text{subject to} \quad A, B, C, D > 0 \quad \text{and} \quad \phi \in [0,1] \tag{8.9}$$

where τ_i is the tax rate for observation i from the microsimulation output, $\tau_{s,t}(x_i, y_i|\tilde{\boldsymbol{\theta}}_{s,t}, \bar{\boldsymbol{\theta}}_{s,t})$ is the predicted tax rate for filing-unit i with x_i labor income and y_i capital income given parameters $\boldsymbol{\theta}_{s,t}$, and w_i is the CPS sampling weight of this observation. The number N is the total number of observations from the microsimulation output for age s and year t. Figure 8.3 shows the typical fit of an estimated tax function $\tau_{s,t}\big(x, y|\hat{\boldsymbol{\theta}}_{s,t}\big)$ to the data. The data in Figure 8.3 are the same age $s = 42$ and year $t = 2017$ as the data Figure 8.2.

The underlying data can limit the number of tax functions that can be estimated. For

Table 8.2: Estimated baseline current law tax rate function $\tau_{s,t}(x,y)$ parameters for $s = 42$, $t = 2017$

Parameter	ETR	$MTRx$	$MTRy$
A	6.28E-12	3.43E-23	4.32E-11
B	4.36E-05	4.50E-04	5.52E-05
C	1.04E-23	9.81E-12	5.62E-12
D	7.77E-09	5.30E-08	3.09E-06
max_x	0.80	0.71	0.44
min_x	-0.14	-0.17	0.00E+00
max_y	0.80	0.80	0.13
min_y	-0.15	-0.42	0.00E+00
$shift\_x$	0.15	0.18	4.45E-03
$shift\_y$	0.16	0.43	1.34E-03
$shift$	-0.15	-0.42	0.00E+00
$share$	0.84	0.96	0.86
Obs. (N)	3,105	3,105	1,990
SSE	9122.68	15041.35	7756.54

example, we use the age of the primary filer from the PUF-CPS match to be equivalent to the age of the DGE model household. The DGE model we use allows for individuals up to age 100, however the data contain few primary filers with age above age 80. Because we cannot reliably estimate tax functions for $s > 80$, we apply the tax function estimates for 80 year-olds to those with model ages 81 to 100. In the case certain ages below age 80 have too few observations to enable precise estimation of the model parameters, we use a linear interpolation method to find the values for those ages $21 \leq s < 80$ that cannot be precisely estimated.[2]

In `OG-USA`, we estimate the 12-parameter functional form (8.8) using weighted nonlinear least squares to fit an effective tax rate function ($\tau^{etr}_{s,t}$), a marginal tax rate of labor income function ($\tau^{mtrx}_{s,t}$), and a marginal tax rate of capital income function ($\tau^{mtry}_{s,t}$) for each age $E + 1 \leq s \leq E + S$ and each of the first 10 years from the current period.[3] That means we have to perform 2,400 estimations of 12 parameters each. Figure 8.3 shows the predicted surfaces for $\tau^{etr}_{s=42,t=2017}$, $\tau^{mtrx}_{s=42,t=2017}$, and $\tau^{mtry}_{s=42,t=2017}$ along with the underlying scatter plot data from which those functions were estimated. Table 8.2 shows the estimated values of those functional forms.

The full set of estimated values are calculated in the `OG-USA/ogusa/txfunc.py` module in the `OG-USA` repository. And the estimated values are stored in the `TxFuncEst_baseline.pkl` file.

[2]We use two criterion to determine whether the function should be interpolated. First, we require a minimum number of observations of filers of that age and in that tax year. Second, we require that that sum of squared errors meet a predefined threshold.

[3]We assume that whatever parameters the tax functions have in year 10 persist forever.

8.4 Factor Transforming Income Units

The tax functions $\tau^{etr}_{s,t}$, $\tau^{mtrx}_{s,t}$, and $\tau^{mtry}_{s,t}$ are estimated based on current U.S. tax filer reported incomes in dollars. However, the consumption units of the `OG-USA` model are not in the same units as the real-world U.S. incomes data. For this reason, we have to transform the income by a $factor$ so that it is in the same units as the income data on which the tax functions were estimated.

The tax rate functions are each functions of capital income and labor income $\tau(x,y)$. In order to make the tax functions return accurate tax rates associated with the correct levels of income, we multiply the model income x^m and y^m by a $factor$ so that they are in the same units as the real-world U.S. income data $\tau(factor \times x^m, factor \times y^m)$. We define the $factor$ such that average steady-state household total income in the model times the $factor$ equals the U.S. data average total income.

$$factor\left[\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\lambda_j\bar{\omega}_s\left(\bar{w}e_{j,s}\bar{n}_{j,s}+\bar{r}\bar{b}_{j,s}\right)\right] = \text{Avg. household income in data} \tag{8.10}$$

We do not know the steady-state wage, interest rate, household labor supply, and savings *ex ante*. So the income $factor$ is an endogenous variable in the steady-state equilibrium computational solution. We hold the factor constant throughout the nonsteady-state equilibrium solution.

8.5 Household Transfers

Total transfers to households by the government in a given period t is TR_t. The percent of those transfers given to all households of age s and lifetime income group j is $\eta_{j,s}$ such that $\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\eta_{j,s,t} = 1$. `OG-USA` currently has the transfer distribution function set to distribute transfers uniformly among the population.

$$\eta_{j,s,t} = \frac{\lambda_j\omega_{s,t}}{\tilde{N}_t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{8.11}$$

However, this distribution function $\eta_{j,s,t}$ could also be modified to more accurately reflect the way transfers are distributed in the United States.

Chapter 9

Corporate Taxes and B-Tax

TODO: Have Jason update this chapter to describe how `B-Tax` is used to calibrate the values of τ^{corp} and δ^{τ} from the firm's profit function in (7.3).

Chapter 10

Unbalanced Government Budget Constraint

In `OG-USA`, the government enters by levying taxes on households, providing transfers to households, levying taxes on firms, spending resources on public goods, and making rule-based adjustments to stabilize the economy in the long-run. It is this last activity that is the focus of this chapter.

10.1 Government Tax Revenue

We see from the household's budget constraint that taxes $T_{s,t}$ and transfers TR_t enter into the household's decision,

$$
\begin{split}
c_{j,s,t} + b_{j,s+1,t+1} = (1 + r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \zeta_{j,s}\frac{BQ_t}{\lambda_j\omega_{s,t}} + \eta_{j,s,t}\frac{TR_t}{\lambda_j\omega_{s,t}} - T_{s,t} \\
\forall j,t \quad \text{and} \quad s\geq E+1 \quad \text{where} \quad b_{j,E+1,t}=0 \quad \forall j,t
\end{split}
\tag{5.1}
$$

where we defined the tax liability function $T_{s,t}$ in (8.3) as an effective tax rate times total income and the transfer distribution function $\eta_{j,s,t}$ is uniform across all households as in (8.11). And government revenue from the corporate income tax rate τ^{corp} and the tax on depreciation expensing τ^{δ} enters the firms' profit function.

$$
PR_t = (1 - \tau^{corp})\bigl(Y_t - w_t L_t\bigr) - \bigl(r_t + \delta\bigr)K_t + \tau^{corp}\delta^\tau K_t \quad \forall t \tag{7.3}
$$

We define total government revenue from taxes as the following.

$$
Rev_t = \underbrace{\tau^{corp}\bigl[Y_t - w_t L_t\bigr] - \tau^{corp}\delta^\tau K_t}_{\text{corporate tax revenue}} + \underbrace{\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\lambda_j\omega_{s,t}\tau^{etr}_{s,t}\left(x_{j,s,t},y_{j,s,t}\right)\bigl(x_{j,s,t}+y_{j,s,t}\bigr)}_{\text{household tax revenue}} \quad \forall t \tag{10.1}
$$

10.2 Government Budget Constraint

Let the level of government debt in period t be given by D_t. The government budget constraint requires that government revenue Rev_t plus the budget deficit $(D_{t+1} - D_t)$ equal expenditures on interest of the debt, government spending on public goods G_t, and total transfer payments to households TR_t every period t.

$$D_{t+1} + Rev_t = (1 + r_t)D_t + G_t + TR_t \quad \forall t \tag{10.2}$$

We assume that total government transfers to households are a fixed fraction of GDP each period.

$$TR_t = \alpha_{tr}\, Y_t \quad \forall t \tag{10.3}$$

We also assume that government spending is a fixed fraction of GDP each period in the initial periods $G_t = \alpha_g\, Y_t$. We make this more specific in equation (10.4) in the next section.

10.3 Budget Closure Rule

If total government transfers to households TR_t and government spending on public goods G_t are both fixed fractions of GDP, one can imagine corporate and household tax structures that cause the debt level of the government to either tend toward infinity or to negative infinity, depending on whether too little revenue or too much revenue is raised, respectively.

A virtue of dynamic general equilibrium models is that the model must be stationary in order to solve it. That is, no variables can be indefinitely growing as time moves forward. The labor augmenting productivity growth g_y from Chapter 7 and the potential population growth $\tilde{g}_{n,t}$ from Chapter 3 render the model nonstationary. But we show how to stationarize the model against those two sources of growth in Chapter 12. However, even after stationarizing the effects of productivity and population growth, the model could be rendered nonstationary and, therefore, not solvable if government debt were becoming too positive or too negative too quickly.

We specify a closure rule that is automatically implemented after some period T_{G1} to stabilize government debt as a percent of GDP (debt-to-GDP ratio). Let α_D represent the long-run debt-to-GDP ratio at which we want the economy to eventually settle.

$$G_t = \begin{cases} \alpha_g\, Y_t & \text{if} \quad t < T_{G1} \\ \left[\rho_G \alpha_D Y_t + (1-\rho_G)D_t\right] - (1+r_t)D_t - TR_t + Rev_t & \text{if} \quad T_{G1} \leq t < T_{G2} \\ \alpha_D Y_t - (1+r_t)D_t - TR_t + Rev_t & \text{if} \quad t \geq T_{G2} \end{cases} \tag{10.4}$$

The first case in (10.4) says that government spending G_t will be a fixed fraction α_g for every period before T_{G1}. The second case specifies that, starting in period T_{G1} and continuing until before period T_{G2}, government spending be adjusted to set tomorrow's debt D_{t+1} to be a convex combination between $\alpha_D Y_t$ and the current debt level D_t, where α_D is a target debt-to-GDP ratio and $\rho_G \in (0, 1]$ is the percent of the way to jump toward the target $\alpha_D Y_t$ from the current debt level D_t. The last case specifies that, for every period after T_{G2}, government spending G_t is set such that the next-period debt be a fixed target percentage α_D of GDP.

This rule allows the government to run increasing deficits or surpluses in the short run (before period T_{G1}). But then the adjustment rule is implemented gradually beginning in period $t = T_{G1}$ to return the debt-to-GDP ratio back to its long-run target of α_D. Then the rule is implemented exactly in period T_{G2} by adjusting government spending G_t to set the debt D_{t+1} such that it is exactly α_D proportion of GDP Y_t.

10.4 Some Caveats and Alternatives

`OG-USA` adjusts government spending G_t as its closure rule instrument because of its simplicity and lack of distortionary effects. Since government spending does not enter into the household's utility function, its level does not affect the solution of the household problem. As alternatives, one could choose to adjust taxes or transfers to close the budget (or a combination of all of the government fiscal policy levers).

There is no guarantee that closure rule (10.4) is sufficient to stabilize the debt-to-GDP ratio in the long run. For large and growing deficits, the convex combination parameter ρ_G might be too gradual, or the budget closure initial period T_{G1} might be too far in the future, or the target debt-to-GDP ratio α_D might be too low. The existence of any of these problems might be manifest in the steady state computation stage. However, it is possible for the steady-state to exist, but for the time path to never reach it. These problems are often avoided by choosing conservative values for T_{G1}, ρ_G, and α_D that close the budget quickly.

And finally, since government spending is doing all of the lifting to hit the target debt to GDP ratio, it is also possible that government spending is forced to be less than zero to make this happen. This would be the case if tax revenues bring in less than is needed to financed transfers and interest payments on the national debt. None of the equations we've specified above preclude that result, but it does raise conceptual difficulties. Namely, what does it mean for government spending to be negative? Is the government selling off pubic assets? We caution those using this budget closure rule to consider carefully how the budget is closed in the long run given their parameterization. We also note that such difficulties present themselves across all budget closure rules when analyzing tax or spending proposals that induce structural budget deficits. In particular, you probably need a different closure instrument if government spending must be negative in the steady-state to hit your long-term debt-to-GDP target.

Part V

Market clearing and Stationarization

Chapter 11

Market Clearing

Three markets must clear in `OG-USA`—the labor market, the capital market, and the goods market. By Walras' Law, we only need to use two of those market clearing conditions because the third one is redundant. In the model, we choose to use the labor market clearing condition and the capital market clearing condition, and to ignore the goods market clearing condition. But we present all three market clearing conditions here. Further, the redundant goods market clearing condition—sometimes referred to as the resource constraint—makes for a nice check on the solution method to see if everything worked.

We also characterize here the law of motion for total bequests BQ_t. Although it is not technically a market clearing condition, one could think of the bequests law of motion as the bequests market clearing condition.

11.1 Market Clearing Conditions

Labor market clearing (11.1) requires that aggregate labor demand L_t measured in efficiency units equal the sum of household efficiency labor supplied $e_{j,s}n_{j,s,t}$.

$$L_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{11.1}$$

Capital market clearing (11.2) requires that aggregate capital demand from firms K_t and from the government D_t equal the sum of capital savings and investment by households $b_{j,s,t}$.

$$K_t + D_t = \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\omega_{s-1,t-1}\lambda_j b_{j,s,t} + i_s\omega_{s,t-1}\lambda_j b_{j,s,t}\Big) \quad \forall t \tag{11.2}$$

Note that the capital demand side of the capital market clearing equation (11.2) includes both capital demand by firms K_t and capital demand by government D_t. It is here that we can see the potential of government deficits to crowd out investment.

Aggregate consumption C_t is defined as the sum of all household consumptions, and aggregate investment is defined by the resource constraint $Y_t = C_t + I_t + G_t$ as shown in

(11.3).

$$\begin{aligned} Y_t = C_t + K_{t+1} - \left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J} i_s\omega_{s,t}\lambda_j b_{j,s,t+1}\right) - (1-\delta)K_t + G_t \quad \forall t \\ \text{where} \quad C_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j c_{j,s,t} \end{aligned} \tag{11.3}$$

Note that the extra terms with the immigration rate i_s in the capital market clearing equation (11.2) and the resource constraint (11.3) accounts for the assumption that age-s immigrants in period t bring with them (or take with them in the case of out-migration) the same amount of capital as their domestic counterparts of the same age. Note also that the term in parentheses with immigration rates i_s in the sum acts is equivalent to a net exports term in the standard equation $Y = C + I + G + NX$. That is, if immigration rates are positive, then immigrants are bringing capital into the country and the term in parentheses has a negative sign in front of it. Negative exports are imports.

11.2 Total Bequests Law of Motion

Total bequests BQ_t are the collection of savings of household from the previous period who died at the end of the period. These savings are augmented by the interest rate because they are returned after being invested in the production process.

$$BQ_t = (1+r_t)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\omega_{s-1,t-1}b_{j,s,t}\right) \quad \forall t \tag{11.4}$$

Because the form of the period utility function in (5.5) ensures that $b_{j,s,t} > 0$ for all j, s, and t, total bequests will always be positive $BQ_{j,t} > 0$ for all j and t.

Chapter 12

Stationarization

The previous chapters derive all the equations necessary to solve for the steady-state and nonsteady-state equilibria of this model. However, because labor productivity is growing at rate g_y as can be seen in the firms' production function (7.1) and the population is growing at rate $\tilde{g}_{n,t}$ as defined in (3.5), the model is not stationary. Different endogenous variables of the model are growing at different rates. We have already specified a budget closure rule on government spending G_t (10.4) that stationarizes the debt-to-GDP ratio.

Table 12.1 lists the definitions of stationary versions of these endogenous variables. Variables with a "ˆ" signify stationary variables. The first column of variables are growing at the productivity growth rate g_y. These variables are most closely associated with individual variables. The second column of variables are growing at the population growth rate $\tilde{g}_{n,t}$. These variables are most closely associated with population values. The third column of variables are growing at both the productivity growth rate g_y and the population growth rate $\tilde{g}_{n,t}$. These variables are most closely associated with aggregate variables. The last column shows that the interest rate r_t and household labor supply $n_{j,s,t}$ are already stationary.

Table 12.1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{c_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_{j,t} \equiv \frac{BQ_{j,t}}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{j,s,t} \equiv \frac{y_{j,s,t}}{e^{g_y t}}$		$\hat{C}_t \equiv \frac{C_t}{e^{g_y t}\tilde{N}_t}$	
$\hat{T}_{s,t} \equiv \frac{T_{j,s,t}}{e^{g_y t}}$		$\hat{TR}_t \equiv \frac{TR_t}{e^{g_y t}\tilde{N}_t}$	

[a] The interest rate r_t in (7.5) is already stationary because Y_t and K_t grow at the same rate. Household labor supply $n_{j,s,t} \in [0,\tilde{l}]$ is stationary.

The usual definition of equilibrium would be allocations and prices such that households optimize (5.7), (5.8), and (5.9), firms optimize (7.4) and (7.5), and markets clear (11.1) and

(11.2), and (11.4). In this chapter, we show how to stationarize each of these characterizing equations so that we can use our fixed point methods described in Sections 13.2 and 14.2 to solve for the equilibria in Definitions 13.1 and 14.1.

12.1 Stationarized Household Equations

The stationary version of the household budget constraint (5.1) is found by dividing both sides of the equation by $e^{g_y t}$. For the savings term $b_{j,s+1,t+1}$, we must multiply and divide by $e^{g_y(t+1)}$, which leaves an $e^{g_y} = \frac{e^{g_y(t+1)}}{e^{g_y t}}$ in front of the stationarized variable.

$$\hat{c}_{j,s,t} + e^{g_y}\hat{b}_{j,s+1,t+1} = (1+r_t)\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \zeta_{j,s}\frac{\hat{BQ}_t}{\lambda_j\hat{\omega}_{s,t}} + \eta_{j,s,t}\frac{\hat{TR}_t}{\lambda_j\hat{\omega}_{s,t}} - \hat{T}_{s,t} \quad \forall j,t \quad\text{and}\quad s\geq E+1 \quad\text{where}\quad b_{j,E+1,t}=0 \quad \forall j,t \tag{12.1}$$

Because total bequests BQ_t and total government transfers TR_t grow at both the labor productivity growth rate and the population growth rate, we have to multiply and divide each of those terms by the economically relevant population $\tilde{N}_t$. This stationarizes total bequests $\hat{BQ}_t$, total transfers $\hat{TR}_t$, and the respective population level in the denominator $\hat{\omega}_{s,t}$.

We stationarize the Euler equations for labor supply (5.7) by dividing both sides by $e^{g_y(1-\sigma)}$. On the left-hand-side, e^{g_y} stationarizes the wage $\hat{w}_t$ and $e^{-\sigma g_y}$ goes inside the parentheses and stationarizes consumption $\hat{c}_{j,s,t}$. On the right-and-side, the $e^{g_y(1-\sigma)}$ terms cancel out.

$$\hat{w}_t e_{j,s}\big(1-\tau^{mtrx}_{s,t}\big)(\hat{c}_{j,s,t})^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j,t, \quad\text{and}\quad E+1\leq s\leq E+S \tag{12.2}$$

We stationarize the Euler equations for savings (5.8) and (5.9) by dividing both sides of the respective equations by $e^{-\sigma g_y t}$. On the right-hand-side of the equation, we then need to multiply and divide both terms by $e^{-\sigma g_y(t+1)}$, which leaves a multiplicative coefficient $e^{-\sigma g_y}$.

$$(\hat{c}_{j,s,t})^{-\sigma} = e^{-\sigma g_y}\left[\chi^b_j\rho_s(\hat{b}_{j,s+1,t+1})^{-\sigma} + \beta\big(1-\rho_s\big)\Big(1+r_{t+1}\big[1-\tau^{mtry}_{s+1,t+1}\big]\Big)(\hat{c}_{j,s+1,t+1})^{-\sigma}\right] \quad \forall j,t, \quad\text{and}\quad E+1\leq s\leq E+S-1 \tag{12.3}$$

$$(\hat{c}_{j,E+S,t})^{-\sigma} = e^{-\sigma g_y}\chi^b_j(\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \quad\text{and}\quad s=E+S \tag{12.4}$$

12.2 Stationarized Firms Equations

The nonstationary production function (7.1) can be stationarized by dividing both sides by $e^{g_y t}\tilde{N}$. This stationarizes output $\hat{Y}_t$ on the left-hand-side. Because the general CES production function is homogeneous of degree 1, $F(xK,xL) = xF(K,L)$, which means the right-hand-side of the production function is stationarized by dividing by $e^{g_y t}\tilde{N}_t$.

$$\hat{Y}_t = F(\hat{K}_t,\hat{L}_t) \equiv Z_t\biggl[(\gamma)^{\frac{1}{\varepsilon}}(\hat{K}_t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\hat{L}_t)^{\frac{\varepsilon-1}{\varepsilon}}\biggr]^{\frac{\varepsilon}{\varepsilon-1}} \quad \forall t \tag{12.5}$$

Notice that the growth term multiplied by the labor input drops out in this stationarized version of the production function. We stationarize the nonstationary profit function (7.3) in the same way, by dividing both sides by $e^{g_y t}\tilde{N}_t$.

$$\hat{PR}_t = (1-\tau^{corp})\Bigl[F(\hat{K}_t,\hat{L}_t) - \hat{w}_t\hat{L}_t\Bigr] - \bigl(r_t+\delta\bigr)\hat{K}_t + \tau^{corp}\delta^\tau\hat{K}_t \quad \forall t \tag{12.6}$$

The firms' first order equation for labor demand (7.4) is stationarized by dividing both sides by $e^{g_y t}$. This stationarizes the wage $\hat{w}_t$ on the left-hand-side and cancels out the $e^{g_y t}$ term in front of the right-hand-side. To complete the stationarization, we multiply and divide the $\frac{Y_t}{e^{g_y t}L_t}$ term on the right-hand-side by $\tilde{N}_t$.

$$\hat{w}_t = (Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[(1-\gamma)\frac{\hat{Y}_t}{\hat{L}_t}\right]^{\frac{1}{\varepsilon}} \quad \forall t \tag{12.7}$$

It can be seen from the firms' first order equation for capital demand (7.5) that the interest rate is already stationary. If we multiply and divide the $\frac{Y_t}{K_t}$ term on the right-hand-side by $e^{t_y t}\tilde{N}_t$, those two aggregate variables become stationary. In other words, Y_t and K_t grow at the same rate and $\frac{Y_t}{K_t} = \frac{\hat{Y}_t}{\hat{K}_t}$.

$$\begin{aligned} r_t &= (1-\tau^{corp})(Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{\hat{Y}_t}{\hat{K}_t}\right]^{\frac{1}{\varepsilon}} - \delta + \tau^{corp}\delta^\tau \quad \forall t \\ &= (1-\tau^{corp})(Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{Y_t}{K_t}\right]^{\frac{1}{\varepsilon}} - \delta + \tau^{corp}\delta^\tau \quad \forall t \end{aligned} \tag{7.5}$$

12.3 Stationarized Government Equations

Each of the tax rate functions $\tau^{etr}_{s,t}$, $\tau^{mtrx}_{s,t}$, and $\tau^{mtry}_{s,t}$ is stationary. The total tax liability function $T_{s,t}$ is growing at the rate of labor productivity growth g_y This can be see by looking at the decomposition of the total tax liability function into the effective tax rate times total income (8.3). The effective tax rate function is stationary, and household income is growing at rate g_y. So household total tax liability is stationarized by dividing both sides of the equation by $e^{g_y t}$.

$$\begin{aligned} \hat{T}_{s,t} &= \tau^{etr}_{s,t}(\hat{x}_{j,s,t},\hat{y}_{j,s,t})\,(\hat{x}_{j,s,t}+\hat{y}_{j,s,t}) \qquad \forall t \quad\text{and}\quad E+1\leq s\leq E+S \\ &= \tau^{etr}_{s,t}(\hat{w}_t e_{j,s}n_{j,s,t}, r_t\hat{b}_{j,s,t})\Bigl(\hat{w}_t e_{j,s}n_{j,s,t} + r_t\hat{b}_{j,s,t}\Bigr) \quad \forall t \quad\text{and}\quad E+1\leq s\leq E+S \end{aligned} \tag{12.8}$$

We can stationarize the simple expression for total transfers to households TR_t in (10.3) by dividing both sides by $e^{g_y t}\tilde{N}_t$.

$$\hat{TR}_t = \alpha_{tr}\,\hat{Y}_t \quad \forall t \tag{12.9}$$

We can stationarize the expression for total government revenue Rev_t in (10.1) by dividing both sides of the equation by $e^{g_y t}\tilde{N}_t$.

$$\hat{Rev}_t = \underbrace{\tau^{corp}\big[\hat{Y}_t - \hat{w}_t\hat{L}_t\big] - \tau^{corp}\delta^\tau\hat{K}_t}_{\text{corporate tax revenue}} + \underbrace{\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\lambda_j\hat{\omega}_{s,t}\tau^{etr}_{s,t}\left(\hat{x}_{j,s,t},\hat{y}_{j,s,t}\right)\left(\hat{x}_{j,s,t}+\hat{y}_{j,s,t}\right)}_{\text{household tax revenue}} \quad \forall t \tag{12.10}$$

Every term in the government budget constraint (10.2) is growing at both the productivity growth rate and the population growth rate, so we stationarize it by dividing both sides by $e^{g_y t}\tilde{N}_t$. We also have to multiply and divide the next period debt term D_{t+1} by $e^{g_y(t+1)}\tilde{N}_{t+1}$, leaving the term $e^{g_y}(1+\tilde{g}_{n,t+1})$.

$$e^{g_y}(1+\tilde{g}_{n,t})\hat{D}_{t+1} + \hat{Rev}_t = (1+r_t)\hat{D}_t + \hat{G}_t + \hat{TR}_t \quad \forall t \tag{12.11}$$

The budget closure rule is the last government equation to stationarize. In each of the cases, we simply divide both sides by $e^{g_y t}\tilde{N}_t$.

$$\hat{G}_t = \begin{cases} \alpha_g\,\hat{Y}_t & \text{if} \quad t < T_{G1} \\ \Big[\rho_G\alpha_D\hat{Y}_t + (1-\rho_G)\hat{D}_t\Big] - (1+r_t)\hat{D}_t - \hat{TR}_t + \hat{Rev}_t & \text{if} \quad T_{G1} \le t < T_{G2} \\ \alpha_D\hat{Y}_t - (1+r_t)\hat{D}_t - \hat{TR}_t + \hat{Rev}_t & \text{if} \quad t \ge T_{G2} \end{cases} \tag{12.12}$$

12.4 Stationarized Market Clearing Equations

The labor market clearing equation (11.1) is stationarized by dividing both sides by $\tilde{N}_t$.

$$\hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{12.13}$$

The capital market clearing equation (11.2) is stationarized by dividing both sides by $e^{g_y t}\tilde{N}_t$. Because the right-hand-side has population levels from the previous period $\omega_{s,t-1}$, we have to multiply and divide both terms inside the parentheses by $\tilde{N}_{t-1}$ which leaves us with the term in front of $\frac{1}{1+\tilde{g}_{n,t}}$.

$$\hat{K}_t + \hat{D}_t = \frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\hat{\omega}_{s-1,t-1}\lambda_j\hat{b}_{j,s,t} + i_s\hat{\omega}_{s,t-1}\lambda_j\hat{b}_{j,s,t}\Big) \quad \forall t \tag{12.14}$$

We stationarize the goods market clearing (11.3) condition by dividing both sides by $e^{g_y t}\tilde{N}_t$. On the right-hand-side, we must multiply and divide the K_{t+1} term by $e^{g_y(t+1)}\tilde{N}_{t+1}$ leaving the coefficient $e^{g_y}(1+\tilde{g}_{n,t+1})$. And the term that subtracts the sum of imports of next period's immigrant savings we must multiply and divide by $e^{g(t+1)}$, which leaves the term e^{g_y}.

$$\hat{Y}_t = \hat{C}_t + e^{g_y}(1+\tilde{g}_{n,t+1})\hat{K}_{t+1} - e^{g_y}\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J} i_s\hat{\omega}_{s,t}\lambda_j\hat{b}_{j,s,t+1}\right) - (1-\delta)\hat{K}_t + \hat{G}_t \quad \forall t$$
$$\text{where} \quad \hat{C}_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j\hat{c}_{j,s,t} \tag{12.15}$$

We stationarize the law of motion for total bequests BQ_t in (11.4) by dividing both sides by $e^{g_y t}\tilde{N}_t$. Because the population levels in the summation are from period $t-1$, we must multiply and divide the summed term by $\tilde{N}_{t-1}$ leaving the term in the denominator of $1+\tilde{g}_{n,t}$.

$$\hat{BQ}_t = \left(\frac{1+r_t}{1+\tilde{g}_{n,t}}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\hat{\omega}_{s-1,t-1}\hat{b}_{j,s,t}\right) \quad \forall t \tag{12.16}$$

Part VI

Equilibrium Definitions and Solution Methods

Chapter 13

Stationary Steady-state Equilibrium

In this chapter, we define the stationary steady-state equilibrium of the OG-USA model. Chapters 3 through 11 derive the equations that characterize the equilibrium of the model. However, we cannot solve for any equilibrium of the model in the presence of nonstationarity in the variables. Nonstationarity in OG-USA comes from productivity growth g_y in the production function (7.1), population growth $\tilde{g}_{n,t}$ as described in Chapter 3, and the potential for unbounded growth in government debt as described in Chapter 10.

We implemented an automatic government budget closure rule using government spending G_t as the instrument that stabilizes the debt-to-GDP ratio at a long-term rate in (10.4). And we showed in Chapter 12 how to stationarize all the other characterizing equations.

13.1 Stationary Steady-State Equilibrium Definition

With the stationarized model, we can now define the stationary steady-state equilibrium. This equilibrium will be long-run values of the endogenous variables that are constant over time. In a perfect foresight model, the steady-state equilibrium is the state of the economy at which the model settles after a finite amount of time, regardless of the initial condition of the model. Once the model arrives at the steady-state, it stays there indefinitely unless it receives some type of shock or stimulus.

These stationary values have all the growth components from productivity growth and population growth removed as defined in Table 12.1. Because the productivity growth rate g_y and population growth rate series $\tilde{g}_{n,t}$ are exogenous. We can transform the stationary equilibrium values of the variables back to their nonstationary values by reversing the identities in Table 12.1.

Definition 13.1 (**Stationary steady-state equilibrium**). A non-autarkic stationary steady-state equilibrium in the OG-USA model is defined as constant allocations of stationary household labor supply $n_{j,s,t} = \bar{n}_{j,s}$ and savings $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ for all j, t, and $E+1 \leq s \leq E+S$, and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all t such that the following conditions hold:

i. the population has reached its stationary steady-state distribution $\hat{\omega}_{s,t} = \bar{\omega}_s$ for all s and t as characterized in Section 3.4,

ii. households optimize according to (12.2), (12.3), and (12.3),

iii. firms optimize according to (12.7) and (7.5),

iv. Government activity behaves according to (12.11) and (12.12), and

v. markets clear according to (12.13), (12.14), and (12.16).

13.2 Stationary Steady-state Solution Method

This section describes the solution method for the stationary steady-state equilibrium described in Definition 13.1. The steady-state is characterized by $2JS$ equations and $2JS$ unknowns. However, because some of the other equations cannot be solved for analytically and substituted into the Euler equations, we must take a two-stage approach to the equilibrium solution. We first make a guess at steady-state interest rate $\bar{r}$, wage $\bar{w}$, total bequests $\overline{BQ}$, total household transfers $\overline{TR}$, and income multiplier $factor$. We call these five steady-state guesses the "outer loop" of the steady-state solution method. They are the macroeconomic variables necessary to solve the household's problem.

The "inner loop" of the steady-state solution method is to solve for the steady-state household decisions $\bar{b}_{j,s}$ and labor supply $\bar{n}_{j,s}$ for all j and $E+1 \leq s \leq E+S$ given the values of the outer-loop variables. Because the lifetime optimization problem of each household of type j is a highly nonlinear system of $2S$ equations and $2S$ unknowns, we break the inner loop problem into two stages, the first of which is a univariate convex optimization problem and the second of which is a serial series of univariate convex optimization problems.

The first stage of the inner loop is to guess an initial steady-state consumption $\bar{c}_{j,1}$ for each household of type j. The second stage of the inner loop is to solve for each period household optimization problem recursively given the initial consumption guess from the first stage. We update the first stage guess for $\bar{c}_{j,1}$ until the implied consumption in the last period $\bar{c}_{j,S}$ and savings in the last period $\bar{b}_{j,S+1}$ satisfy the last period savings Euler equation (12.3). We outline this algorithm in the following steps.

1. Use the techniques from Section 3.4 to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ and steady-state growth rate $\bar{g}_n$ of the exogenous population process.

2. Choose an initial guess for the values of the steady-state interest rate $\bar{r}^i$, wage $\bar{w}^i$, total bequests $\overline{BQ}^i$, total household transfers $\overline{TR}^i$, and income multiplier $factor^i$, where superscript i is the index of the iteration number of the guess.

   (a) Note that if the production function is Cobb-Douglas ($\varepsilon = 1$), then you only have to guess the steady-state values of the steady-state interest rate $\bar{r}^i$, total bequests $\overline{BQ}^i$, total household transfers $\overline{TR}^i$, and income multiplier $factor^i$. In this case, the steady-state wage $\bar{w}$ is determined by the interest rate using equations (7.5) and (12.7). In this Cobb-Douglas case ($\varepsilon = 1$), choosing both $\bar{r}$ and $\bar{w}$ in the outer loop can cause the solution method to not converge.

3. Given guesses for $\bar{r}^i$, $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $factor^i$, solve for the steady-state household labor supply $\bar{n}_{j,s}$ and savings $\bar{b}_{j,s}$ decisions for all j and $E+1 \leq s \leq E+S$ using two-stage approach.

   (a) Given $\bar{r}^i$, $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $factor^i$, guess an initial steady-state consumption $\bar{c}^m_{j,E+1}$ for each type-j household, where m is the index of the inner-loop iteration.

      i. Given $\bar{r}^i$ and $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, $factor^i$, and $\bar{c}^m_{j,E+1}$, and the fact that $\bar{b}_{j,E+1} = 0$, we can use the household labor supply Euler equation (12.2) to solve for $\bar{n}_{j,E+1}$ for all j. This problem is a univariate root finder in $\bar{n}_{j,E+1}$.

$$\bar{w}^i e_{j,s}\big(1-\tau^{mtrx}_s\big)(\bar{c}_{j,s})^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall j, \quad \text{and} \quad E+1 \leq s \leq E+S$$

      ii. Given $\bar{c}^m_{j,E+1}$, $\bar{b}_{j,E+1} = 0$, and $\bar{n}_{j,E+1}$, we can use the household budget constraint (12.1) to solve analytically for $\bar{b}_{j,E+2}$ for all j.

$$\bar{b}_{j,s+1} = e^{-g_y}\left[(1+\bar{r}^i)\bar{b}_{j,s} + \bar{w}^i e_{j,s}\bar{n}_{j,s} + \zeta_{j,s}\frac{\overline{BQ}^i}{\lambda_j\bar{\omega}_s} + \eta_{j,s}\frac{\overline{TR}^i}{\lambda_j\bar{\omega}_s} - \bar{T}_s - \bar{c}_{j,s}\right]$$
$$\forall j \quad \text{and} \quad E+1 \leq s \leq E+S$$

      iii. Given $\bar{c}^m_{j,E+1}$ and $\bar{b}_{j,E+2}$, use the household labor supply Euler equation (12.3) to solve analytically for $\bar{c}_{j,E+2}$ for all j.

$$\bar{c}_{j,s+1} = \left[\frac{e^{\sigma g_y}(\bar{c}_{j,s})^{-\sigma} - \chi^b_j\rho_s(\bar{b}_{j,s+1})^{-\sigma}}{\beta(1-\rho_s)\Big(1+\bar{r}^i\big[1-\tau^{mtry}_{s+1}\big]\Big)}\right]^{-\frac{1}{\sigma}}$$
$$\forall j, \quad \text{and} \quad E+1 \leq s \leq E+S-1$$

      iv. Repeat in serial steps (i) through (iii) until solved for the all households' steady-state lifetime decisions $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ for all j.

   (b) Given household lifetime decisions $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ for all j based on guesses for initial period consumption $\bar{c}^m_{j,E+1}$ for all j and outer loop guesses $\bar{r}^i$, $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $factor^i$, check the error in the last period savings Euler equation (12.4) based on $\bar{c}_{j,E+S}$ and $\bar{b}_{j,E+S+1}$.

$$error_j \equiv e^{-\sigma g_y}\chi^b_j(\bar{b}_{j,E+S+1,t+1})^{-\sigma} - (\bar{c}_{j,E+S})^{-\sigma} \quad \forall j$$

   (c) If the error is greater than some small positive tolerance $error_j > toler_c$ for some j, then update the guesses for initial consumption $\bar{c}^{m+1}_{j,1}$ and repeat steps (a) and (b).

   (d) If the error is less than some small positive tolerance $error_j \leq toler_c$ for all j, then $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ is the full set of partial equilibrium household steady-state solutions given guesses for $\bar{r}^i$, $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $factor^i$.

4. Given partial equilibrium household steady-state solutions $\{\bar{c}_{j,s}, \bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ based on macroeconomic variable guesses $\bar{r}^i$, $\bar{w}^i$, $\overline{BQ}^i$, $\overline{TR}^i$, and $factor^i$, check the errors in the five equations that characterize each of the macroeconomic variable guesses.

   (a) If we substitute the two market clearing conditions (12.13) and (12.14), the firm's production function (12.5), and the assumption from Chapter 10 that the long-run debt-to-GDP ratio is α_D into the firm's first order condition for capital demand (7.5), we get an expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the interest rate $\bar{r}^{i'}$.

$$\bar{r}^{i'} = (1-\tau^{corp})(\bar{Z})^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{\bar{Y}}{\bar{K}}\right]^{\frac{1}{\varepsilon}} - \delta + \tau^{corp}\delta^\tau$$
$$\text{where} \quad \bar{Y} = \bar{Z}\left[(\gamma)^{\frac{1}{\varepsilon}}(\bar{K})^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\bar{L})^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$
$$\text{and} \quad \bar{L} = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\bar{\omega}_s\lambda_j e_{j,s}\bar{n}_{j,s}$$
$$\text{and} \quad \bar{K} + \alpha_D\bar{Y} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\bar{\omega}_{s-1}\lambda_j\bar{b}_{j,s} + i_s\bar{\omega}_s\lambda_j\bar{b}_{j,s}\Big)$$

   The error for this variable is the percent difference between the initial guess for the interest rate $\bar{r}^i$ and the steady-state interest rate implied by household optimization based on the initial guess $\bar{r}^{i'}$.

$$error_r = \frac{\bar{r}^{i'} - \bar{r}^i}{\bar{r}^i}$$

   (b) If we substitute the two market clearing conditions (12.13) and (12.14), the firm's production function (12.5), and the assumption from Chapter 10 that the long-run debt-to-GDP ratio is α_D into the firm's first order condition for labor demand (12.7), we get an expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the wage $\bar{w}^{i'}$.

$$\bar{w}^{i'} = (\bar{Z})^{\frac{\varepsilon-1}{\varepsilon}}\left[(1-\gamma)\frac{\bar{Y}}{\bar{L}}\right]^{\frac{1}{\varepsilon}}$$
$$\text{where} \quad \bar{Y} = \bar{Z}\left[(\gamma)^{\frac{1}{\varepsilon}}(\bar{K})^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\bar{L})^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$
$$\text{and} \quad \bar{L} = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\bar{\omega}_s\lambda_j e_{j,s}\bar{n}_{j,s}$$
$$\text{and} \quad \bar{K} + \alpha_D\bar{Y} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\bar{\omega}_{s-1}\lambda_j\bar{b}_{j,s} + i_s\bar{\omega}_s\lambda_j\bar{b}_{j,s}\Big)$$

The error for this variable is the percent difference between the initial guess for the wage $\bar{w}^i$ and the steady-state wage implied by household optimization based on the initial guess $\bar{w}^{i'}$.

$$error_w = \frac{\bar{w}^{i'} - \bar{w}^i}{\bar{w}^i}$$

(c) The stationarized law of motion for total bequests (12.16) provides the expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the wage $\overline{BQ}^{i'}$. Note that we need all the household decisions here because $\bar{r}^{i'}$ enters the equation on the right-hand-side.

$$\overline{BQ}^{i'} = \left(\frac{1+\bar{r}^{i'}}{1+\bar{g}_n}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\bar{\omega}_{s-1}\bar{b}_{j,s}\right)$$

The error for this variable is the percent difference between the initial guess for total bequests $\overline{BQ}^i$ and the steady-state total bequests implied by household optimization based on the initial guess $\overline{BQ}^{i'}$.

$$error_{bq} = \frac{\overline{BQ}^{i'} - \overline{BQ}^i}{\overline{BQ}^i}$$

(d) In equation (12.9), we defined total household transfers as a fixed percentage of GDP ($\hat{TR}_t = \alpha_{tr}\hat{Y}_t$). Similar to our approach with $\bar{r}^{i'}$ and $\bar{w}^{i'}$, if we substitute the two market clearing conditions (12.13) and (12.14), the firm's production function (12.5), and the assumption from Chapter 10 that the long-run debt-to-GDP ratio is α_D into the stationary expression of total household transfers (12.9), we get an expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for total household transfers $\overline{TR}^{i'}$.

$$\begin{aligned}
\overline{TR}^{i'} &= \alpha_{tr}\,\bar{Y} \\
&\quad \text{where}\quad \bar{Y} = \bar{Z}\left[(\gamma)^{\frac{1}{\varepsilon}}(\bar{K})^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\bar{L})^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&\quad \text{and}\quad \bar{L} = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\bar{\omega}_s\lambda_j e_{j,s}\bar{n}_{j,s} \\
&\quad \text{and}\quad \bar{K} + \alpha_D\bar{Y} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\bar{\omega}_{s-1}\lambda_j\bar{b}_{j,s} + i_s\bar{\omega}_s\lambda_j\bar{b}_{j,s}\Big)
\end{aligned}$$

The error for this variable is the percent difference between the initial guess for total household transfers $\overline{TR}^i$ and the steady-state total household transfers implied by household optimization based on the initial guess $\overline{TR}^{i'}$.

$$error_{tr} = \frac{\overline{TR}^{i'} - \overline{TR}^i}{\overline{TR}^i}$$

(e) The $factor$ that transforms the model units to U.S. dollar units for the tax functions (8.10) is already defined in terms of steady-state variables. The following is an expression in which household decisions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the steady-state $factor^{i'}$. Note, as with the equation for $\overline{BQ}^{i'}$, that we include the updated values of $\bar{w}^{i'}$ and $\bar{r}^{i'}$ on the right-hand-side of the equation.

$$factor^{i'} = \frac{\text{Avg. household income in data}}{\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\lambda_j\bar{\omega}_s\left(\bar{w}^{i'}e_{j,s}\bar{n}_{j,s} + \bar{r}^{i'}\bar{b}_{j,s}\right)}$$

The error for this variable is the percent difference between the initial guess for the $factor^i$ and the steady-state factor implied by household optimization based on the initial guess $factor^{i'}$.

$$error_f = \frac{factor^{i'} - factor^i}{factor^i}$$

5. If the maximum absolute error among the five outer loop error terms is greater than some small positive tolerance $toler_{out}$,

$$\max\left|\left(error_r, error_w, error_{bq}, error_{tr}, error_f\right)\right| > toler_{out}$$

then update the guesses for the outer loop variables as a convex combination governed by $\xi_{ss} \in (0,1]$ of the respective initial guesses and the new implied values and repeat steps (3) through (5).

$$\begin{aligned}\left[\bar{r}^{i+1}, \bar{w}^{i+1}, \overline{BQ}^{i+1}, \overline{TR}^{i+1}, factor^{i+1}\right] = \xi_{ss}\left[\bar{r}^{i'}, \bar{w}^{i'}, \overline{BQ}^{i'}, \overline{TR}^{i'}, factor^{i'}\right] + \\ (1-\xi_{ss})\left[\bar{r}^{i}, \bar{w}^{i}, \overline{BQ}^{i}, \overline{TR}^{i}, factor^{i}\right]\end{aligned}$$

6. If the maximum absolute error among the five outer loop error terms is less-than-or-equal-to some small positive tolerance $toler_{ss,out}$,

$$\max\left|\left(error_r, error_w, error_{bq}, error_{tr}, error_f\right)\right| \leq toler_{ss,out}$$

then the steady-state has been found.

(a) Make sure that steady-state government spending is nonnegative $\bar{G} \geq 0$. If steady-state government spending is negative, that means the government is getting resources to supply the debt from outside the economy each period to stabilize the debt-to-GDP ratio. $\bar{G} < 0$ is a good indicator of instability.

(b) Make sure that the resource constraint (goods market clearing) (12.15) is satisfied. It is redundant, but this is a good check as to whether everything worked correctly.

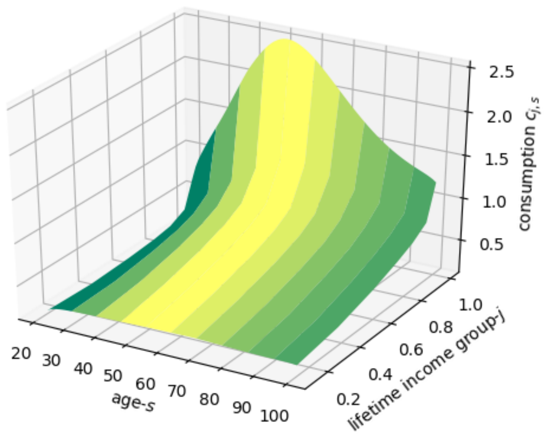
(c) Make sure that the government budget constraint (12.11) binds.

(d) Make sure that all the $2JS$ household Euler equations are solved to a satisfactory tolerance.

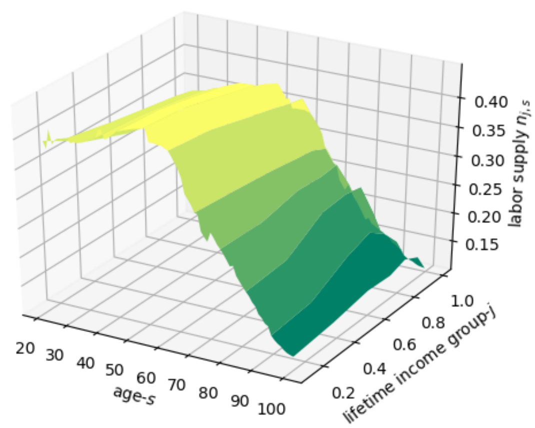
13.3 Baseline Steady-state Results

In this section, we use the baseline calibration described in Chapter 15, which includes the baseline tax law from `Tax-Calculator`, to show some steady-state results from `OG-USA`. Figure 13.1 shows the household steady-state variables by age s and lifetime income group j.

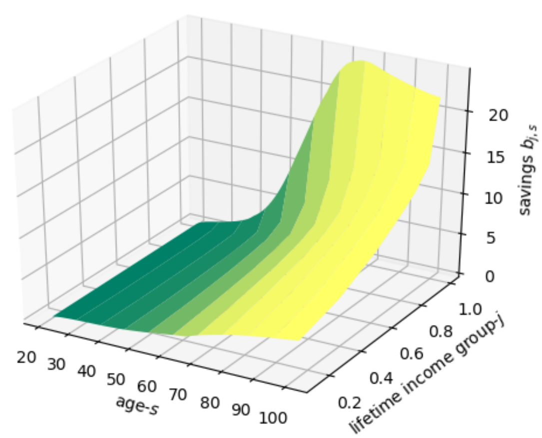
Figure 13.1: Steady-state distributions of household consumption $\bar{c}_{j,s}$, labor supply $\bar{n}_{j,s}$, and savings $\bar{b}_{j,s+1}$



(a) Consumption $\bar{c}_{j,s}$



(b) Labor supply $\bar{n}_{j,s}$



(c) Savings $\bar{b}_{j,s+1}$

Table 13.1 lists the steady-state prices and aggregate variable values along with some of the maximum error values from the characterizing equations.

Table 13.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Variable	Value
$\bar{r}$	0.058	$\bar{w}$	1.148
$\bar{Y}$	0.630	$\bar{C}$	0.462
$\bar{I}$	0.144	$\bar{K}$	1.810
$\bar{L}$	0.357	$\bar{B}$	2.440
$\overline{BQ}$	0.106	$factor$	141,580
$\overline{Rev}$	0.096	$\overline{TR}$	0.057
$\bar{G}$	0.023	$\bar{D}$	0.630
Max. abs. labor supply Euler error	4.57e-13	Max. abs. savings Euler error	8.52e-13
Resource constraint error	-4.39e-15	Serial computation time	1 hr. 25.9 sec.*

* The steady-state computation time does not include any of the exogenous parameter computation processes, the longest of which is the estimation of the baseline tax functions which computation takes 1 hour and 15 minutes.

Chapter 14

Stationary Non Steady-state Equilibrium

In this chapter, we define the stationary nonsteady-state equilibrium of the `OG-USA` model. Chapters 3 through 11 derive the equations that characterize the equilibrium of the model. We also need the steady-state solution from Chapter 13 to solve for the nonsteady-state equilibrium transition path. As with the steady-state equilibrium, we must use the stationarized version of the characterizing equations from Chapter 12.

14.1 Stationary Nonsteady-State Equilibrium Definition

We define a stationary nonsteady-state equilibrium as the following.

Definition 14.1 (Stationary Nonsteady-state functional equilibrium). A non autarkic nonsteady-state functional equilibrium in the `OG-USA` model is defined as stationary allocation functions of the state $\bigl\{n_{j,s,t} = \phi_s\bigl(\hat{\mathbf{\Gamma}}_t\bigr)\bigr\}_{s=E+1}^{E+S}$ and $\bigl\{\hat{b}_{j,s+1,t+1} = \psi_s\bigl(\hat{\mathbf{\Gamma}}_t\bigr)\bigr\}_{s=E+1}^{E+S}$ for all j and t and stationary price functions $\hat{w}(\hat{\mathbf{\Gamma}}_t)$ and $r(\hat{\mathbf{\Gamma}}_t)$ for all t such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (5.11), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u} = \hat{\mathbf{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\mathbf{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (12.2), (12.3), and (12.3),

iii. firms optimize according to (12.7) and (7.5),

iv. Government activity behaves according to (12.11) and (12.12), and

v. markets clear according to (12.13), (12.14), and (12.16).

14.2 Stationary Nonsteady-state Solution Method

This section describes the solution method for the stationary nonsteady-state equilibrium described in Definition 14.1. We use the time path iteration (TPI) method. This method was originally outlined in a series of papers between 1981 and 1985[1] and in the seminal book Auerbach and Kotlikoff (1987, ch. 4) for the perfect foresight case and in Nishiyama and Smetters (2007, Appendix II) and Evans and Phillips (2014, Sec. 3.1) for the stochastic case. The intuition for the TPI solution method is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey et al. (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 13.1 in a finite number of periods $T < \infty$ regardless of the initial state $\hat{\boldsymbol{\Gamma}}_1$. The first step in solving for the nonsteady-state equilibrium transition path is to solve for the steady-state using the method described in Section 13.2. The next step is a transition path "outer loop" step, analogous to the outer loop described in the steady-state solution method. Guess transition paths for aggregate variables $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i, \hat{\boldsymbol{TR}}^i\}$, where $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, $\hat{\boldsymbol{BQ}}^i = \left\{\hat{BQ}_1^i, \hat{BQ}_2^i, ...\hat{BQ}_T^i\right\}$, and $\hat{\boldsymbol{TR}}^i = \left\{\hat{TR}_1^i, \hat{TR}_2^i, ...\hat{TR}_T^i\right\}$. The only requirement on these transition paths is that the initial total bequests $\hat{BQ}_1^i$ conform to the initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$, and that the economy has reached the steady-state by period $t = T$ $\{r_T^i, \hat{w}_T^i, \hat{BQ}_T^i, \hat{TR}_T^i\} = \{\bar{r}, \bar{w}, \overline{BQ}, \overline{TR}\}$.

The "inner loop" of the nonsteady-state transition path solution method is to solve for the full set of lifetime savings decisions $\bar{b}_{j,s+1,t+1}$ and labor supply decisions $\bar{n}_{j,s,t}$ for every household that will be alive between periods $t = 1$ and $t = T$. Because we know the initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$ in the transition path and we know the long-run steady-state $\bar{\boldsymbol{\Gamma}}$, we do not have to use the two-stage inner-loop method for solving the households' problems that we used in Section 13.2. Because we know the neighborhood where the solutions live, we can simply solve for the $2JS$ equations and unknowns for each household's lifetime decisions using a multivariate root finder. This is much faster than the two-stage method describe in Section 13.2. We outline this algorithm in the following steps.

1. Compute the steady-state solution $\{\bar{n}_{j,s}, \bar{b}_{j,s}\}_{s=E+1}^{E+S}$ corresponding to Definition 13.1.
2. Given initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$ and steady-state solutions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$, guess transition paths of outer loop macroeconomic variables $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i, \hat{\boldsymbol{TR}}^i\}$ such that $\hat{BQ}_1^i$ is consistent with $\hat{\boldsymbol{\Gamma}}_1$ and $\{r_t^i, \hat{w}_t^i, \hat{BQ}_t^i, \hat{TR}_t^i\} = \{\bar{r}, \bar{w}, \overline{BQ}, \overline{TR}\}$ for all $t \geq T_1$.
    (a) We choose two long-run time periods, T_1 and T_2. The first time period $t = T_1$ is the period in which the time paths of all the macroeconomic guesses hit their steady-state and stay at their steady-state thereafter. The second time period $t = T_2 > T_1$ is the period after which all the endogenous inner loop household

[1] See Auerbach et al. (1981, 1983), Auerbach and Kotlikoff (1983c,b,a), and Auerbach and Kotlikoff (1985).

variables hit their steady-state and stay at their steady-state thereafter. These two periods should be different because it requires time periods for the endogenous variables to hit the steady-state after the macroeconomic time path guesses have hit their steady-state.

3. Given initial condition $\hat{\boldsymbol{\Gamma}}_1$, outer-loop guesses for the aggregate time paths $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i, \hat{\boldsymbol{TR}}^i\}$, solve for the inner loop lifetime decisions of every household that will be alive across the time path $\{n_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j and $1 \leq t \leq T_2$.

   (a) Given time path guesses $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i, \hat{\boldsymbol{TR}}^i\}$, solve for each household's lifetime decisions $\{\hat{n}_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j, $E+1 \leq s \leq E+S$, and $1 \leq t \leq T_2+S-1$.

      i. In the transition path equilibrium solution method, the household problem can be solved with a multivariate root finder solving the $2S$ equations and unknowns at once for all j and $1 \leq t \leq T_2 + S - 1$, as opposed to the two-stage method for the steady-state solution described in Section 13.2. Use $2S$ household Euler equations (12.2), (12.3), and (12.4) to solve for each household's $2S$ lifetime decisions.

      ii. If one solves for each household's problem serially from the oldest households alive in period $t = 1$ to the youngest and then for every household born in period $t = 1, 2, ...T_2 - 1$, one can use the equilibrium guesses of the previous generation as initial guesses for the solver. This speeds up computation further and makes the initial guess for the highly nonlinear system of equations start closer to the solution value.

4. Given partial equilibrium household nonsteady-state solutions $\{n_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j and $1 \leq t \leq T_2$ based on macroeconomic variable time path guesses $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i, \hat{\boldsymbol{TR}}^i\}$, check the errors across the four time paths in the four equations that characterize each of the macroeconomic variable guesses.

   (a) If we substitute the two market clearing conditions (12.13) and (12.14), the firm's production function (12.5), the government budget constraint (12.11), and the assumption from Chapter 10 that the long-run debt-to-GDP ratio is α_D into the firm's first order condition for capital demand (7.5), we get an expression in which household decisions $\{\bar{n}_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ imply a values for the interest rate $\boldsymbol{r}^{i'}$.

$$r_t^{i'} = (1 - \tau^{corp})(Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{\hat{Y}_t}{\hat{K}_t}\right]^{\frac{1}{\varepsilon}} - \delta + \tau^{corp}\delta^{\tau}$$

$$\text{where} \quad \hat{Y}_t = Z_t\left[(\gamma)^{\frac{1}{\varepsilon}}(\hat{K}_t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\hat{L}_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\text{and} \quad \hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\bar{\omega}_s\lambda_j e_{j,s}\bar{n}_{j,s}$$

$$\text{and} \quad \bar{K} + D_t = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\bar{\omega}_{s-1}\lambda_j\bar{b}_{j,s} + i_s\bar{\omega}_s\lambda_j\bar{b}_{j,s}\Big)$$

The error for this variable is the percent difference between the initial guess for the interest rate $\bar{r}^i$ and the steady-state interest rate implied by household optimization based on the initial guess $\bar{r}^{i'}$.

$$error_r = \frac{\bar{r}^{i'} - \bar{r}^i}{\bar{r}^i}$$

14.3 Baseline Nonsteady-state Results

asdf

Part VII

Calibration and International Options

Chapter 15

Calibration

Chapter 16

Small Open Economy Option

TODO: Include chapter describing small open economy option. Fixed interest rate r^*, foreign capital K_t^F in capital market clearing equation.

Appendices

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