UNIT-3

TRAVELLING SALESMAN PROBLEM

- Let G(V,E) be a directed graph with edge cost c_{ij} is defined such that $c_{ij} > 0$ for all i and j and $c_{ij} = \infty$,if $<i,j> \notin E$.
  Let $|V| = n$ and assume n>1.
- The traveling salesman problem is to find a tour of minimum cost.
- A tour of G is a directed cycle that include every vertex in V.
- The cost of the tour is the sum of cost of the edges on the tour.
- The tour is the shortest path that starts and ends at the same vertex (ie) 1.

APPLICATION :

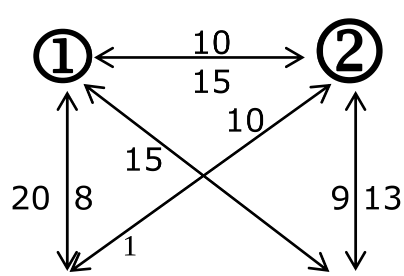
1. Suppose we have to route a postal van to pick up mail from the mail boxes located at 'n' different sites.
2. An n+1 vertex graph can be used to represent the situation.
3. One vertex represent the post office from which the postal van starts and return.
4. Edge <i,j> is assigned a cost equal to the distance from site 'i' to site 'j'.
5. the route taken by the postal van is a tour and we are finding a tour of minimum length.
6. every tour consists of an edge <1,k> for some $k \in V-\{\}$ and a path from vertex k to vertex 1.
7. the path from vertex k to vertex 1 goes through each vertex in V-{1,k} exactly once.
8. the function which is used to find the path is

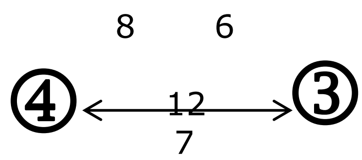
   $g(1,V-\{1\}) = \min\{ c_{ij} + g(j,s-\{j\})\}$

9. g(i,s) be the length of a shortest path starting at vertex i, going through all vertices in S,and terminating at vertex 1.
10. the function g(1,v-{1}) is the length of an optimal tour.

STEPS TO FIND THE PATH:

1. Find $g(i,\Phi) = c_{i1}$, 1<=i<n, hence we can use equation(2) to obtain g(i,s) for all s to size 1.
2. That we have to start with s=1,(ie) there will be only one vertex in set 's'.
3. Then s=2, and we have to proceed until |s| <n-1.
4. for example consider the graph.





Cost matrix

$$\begin{vmatrix} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 8 & 9 & 0 \end{vmatrix}$$

g(i,s) → set of nodes/vertex have to visited.

↓

starting position

$g(i,s) = \min\{c_{ij} + g(j, s-\{j\})$

STEP 1:

$g(1,\{2,3,4\})=\min\{c_{12}+g(2\{3,4\}),c_{13}+g(3,\{2,4\}),c_{14}+g(4,\{2,3\})\}$

$\min\{10+25,15+25,20+23\}$
$\min\{35,35,43\}$
$=35$

STEP 2:

$g(2,\{3,4\}) = \min\{c_{23}+g(3\{4\}),c_{24}+g(4,\{3\})\}$

$\min\{9+20,10+15\}$
$\min\{29,25\}$
$=25$

$g(3,\{2,4\}) =\min\{c_{32}+g(2\{4\}),c_{34}+g(4,\{2\})\}$

$\min\{13+18,12+13\}$
$\min\{31,25\}$
$=25$

$g(4,\{2,3\}) = \min\{c_{42}+g(2\{3\}),c_{43}+g(3,\{2\})\}$

$\min\{8+15,9+18\}$

min{23,27}
=23

STEP 3:

1. $g(3,\{4\}) = \min\{c_{34} + g\{4,\Phi\}\}$
   12+8 =20

2. $g(4,\{3\}) = \min\{c_{43} + g\{3,\Phi\}\}$
   9+6 =15

3. $g(2,\{4\}) = \min\{c_{24} + g\{4,\Phi\}\}$
   10+8 =18

4. $g(4,\{2\}) = \min\{c_{42} + g\{2,\Phi\}\}$
   8+5 =13

5. $g(2,\{3\}) = \min\{c_{23} + g\{3,\Phi\}\}$
   9+6=15

6. $g(3,\{2\}) = \min\{c_{32} + g\{2,\Phi\}\}$
   13+5=18

STEP 4:

$g\{4,\Phi\} = c_{41} = 8$

$g\{3,\Phi\} = c_{31} = 6$

$g\{2,\Phi\} = c_{21} = 5$

$|s| = 0.$

i =1 to n.

$g(1,\Phi) = c_{11} \Rightarrow 0$

$g(2,\Phi) = c_{21} \Rightarrow 5$

$g(3,\Phi) = c_{31} \Rightarrow 6$

$g(4,\Phi) = c_{41} \Rightarrow 8$

$|s| = 1$

$i = 2$ to 4

$g(2,\{3\}) = c_{23} + g(3,\Phi)$
$= 9+6 = 15$

$g(2,\{4\}) = c_{24} + g(4,\Phi)$
$= 10+8 = 18$

$g(3,\{2\}) = c_{32} + g(2,\Phi)$
$= 13+5 = 18$

$g(3,\{4\}) = c_{34} + g(4,\Phi)$
$= 12+8 = 20$

$g(4,\{2\}) = c_{42} + g(2,\Phi)$
$= 8+5 = 13$

$g(4,\{3\}) = c_{43} + g(3,\Phi)$
$= 9+6 = 15$

$|s| = 2$

$i \neq 1$, $1 \in s$ and $i \in s$.

$g(2,\{3,4\}) = \min\{c_{23}+g(3\{4\}),c_{24}+g(4,\{3\})\}$
$\min\{9+20,10+15\}$
$\min\{29,25\}$
$=25$

$g(3,\{2,4\}) = \min\{c_{32}+g(2\{4\}),c_{34}+g(4,\{2\})\}$
$\min\{13+18,12+13\}$
$\min\{31,25\}$
$=25$

$g(4,\{2,3\}) = \min\{c_{42}+g(2\{3\}),c_{43}+g(3,\{2\})\}$
$\min\{8+15,9+18\}$
$\min\{23,27\}$
$=23$

$|s| = 3$

$g(1,\{2,3,4\})=\min\{c_{12}+g(2\{3,4\}),c_{13}+g(3,\{2,4\}),c_{14}+g(4,\{2,3\})\}$
$\min\{10+25,15+25,20+23\}$
$\min\{35,35,43\}$
$=35$

optimal cost is 35

the shortest path is,

$g(1,\{2,3,4\}) = c_{12} + g(2,\{3,4\})$ => 1->2

$g(2,\{3,4\}) = c_{24} + g(4,\{3\})$ => 1->2->4

$g(4,\{3\}) = c_{43} + g(3\{\Phi\})$ => 1->2->4->3->1

so the optimal tour is **1 → 2 → 4 →3 → 1**