GITAM
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DESIGN AND ANALYSIS OF ALGORITHMS

STRASSEN'S MATRIX MULTIPLICATION

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GITAM(DEEMED TO BE UNIVERSITY)

DIVIDE AND CONQUER – GENERAL IDEA

- **Divide a problem into subprograms of the same kind**
- **Solve subprograms using the same approach**
- **Combine partial solution (if necessary)**

MATRIX MULTIPLICATION

- The Problem

Multiply two matrices A and B, each of size $[n \times n]$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{n\times n} \cdot \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}_{n\times n} = \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix}_{n\times n}$$

MATRIX MULTIPLICATION – GENERAL METHOD

Mat A Mat B

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

Mat C = Mat A * Mat B

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

where,
c11=a11∗b11+a12∗b21+a13∗b31+a14∗b41

c12=a11∗b12+a12∗b22+a13∗b32+a14∗b42

c21=a21∗b11+a22∗b21+a23∗b31+a24∗b41

c22=a21∗b12+a22∗b22+a23∗b32+a24∗b42

and so on.

MATRIX MULTIPLICATION – THE TRADITIONAL WAY

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

$$\therefore T(n) = O(n^3)$$

MATRIX MULTIPLICATION – DIVIDE AND CONQUER WAY

$$\left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]_{C} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]_{A} \cdot \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]_{B}$$

$$C_{11} = \underline{A_{11} \cdot B_{11}} + \underline{A_{12} \cdot B_{21}}$$

$$C_{12} = \underline{A_{11} \cdot B_{12}} + \underline{A_{12} \cdot B_{22}}$$

$$C_{21} = \underline{A_{21} \cdot B_{11}} + \underline{A_{22} \cdot B_{21}}$$

$$C_{22} = \underline{A_{21} \cdot B_{12}} + \underline{A_{22} \cdot B_{22}}$$

$$\therefore T(n) = 8 \cdot T(\frac{n}{2}) + an^2$$

$$= O(n^3)$$

Where an^2 is for addition
So, it is no improvement compared with the traditional way

transform the problem of multiplying A and B, each of size [n×n] into 8 subproblems, each of size $\left[\frac{n}{2} \times \frac{n}{2}\right]$

EXAMPLE: MATRIX MULTIPLICATION USING DANDC

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

use Divide-and-Conquer way to solve it as following:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$$

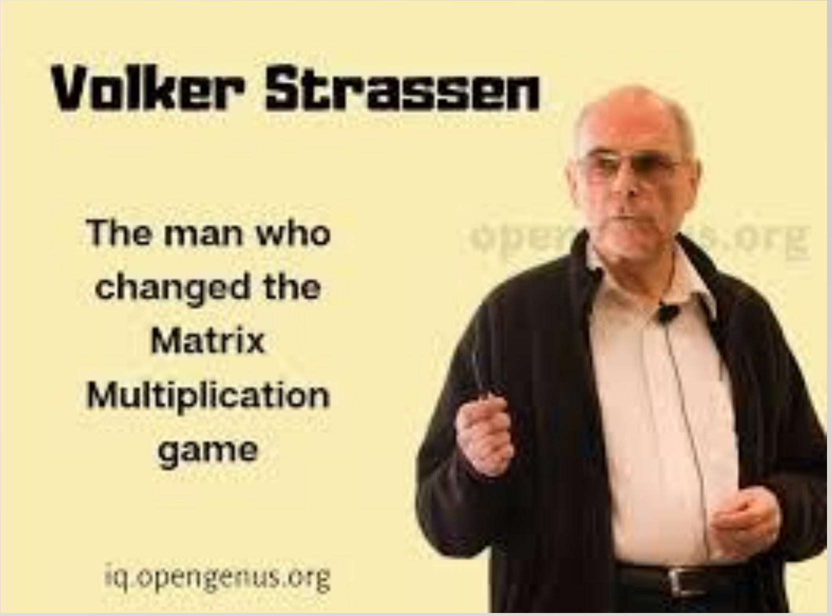
$$C_{21} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$C_{22} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

STRASSEN'S MATRIX MULTIPLICATION:

- Discover a way to compute the C_{ij}'s using 7 multiplications and 18 additions or subtractions

$$P = (A_{11} + A_{22})(B_{11} + B_{22})$$
$$Q = (A_{21} + A_{22})B_{11}$$
$$R = A_{11}(B_{12} - B_{22})$$
$$S = A_{22}(B_{21} - B_{11})$$
$$T = (A_{11} + A_{12})B_{22}$$
$$U = (A_{21} - A_{11})(B_{11} + B_{12})$$
$$V = (A_{12} - A_{22})(B_{21} + B_{22})$$
$$C_{11} = P + S - T + V$$
$$C_{12} = R + T$$
$$C_{21} = Q + S$$
$$C_{22} = P + R - Q + U$$



ALGORITHM: MATRIX MULTIPLICATION STRASSEN'S METHOD

Algorithm Strassen *(n, A, B, C)*

// *n* is size, *A*,*B* the input matrices, *C* output matrix

begin

if *n* == 2

$$\begin{cases} C_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} \\ C_{12} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ C_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} \\ C_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{cases}$$

else

(cont.)

else

Partition A into 4 submatrices: $A_{11}, A_{12}, A_{21}, A_{22}$;

Partition B into 4 submatrices: $B_{11}, B_{12}, B_{21}, B_{22}$;

call Strassen $(\frac{n}{2}, A_{11} + A_{22}, B_{11} + B_{22}, P)$;

call Strassen $(\frac{n}{2}, A_{21} + A_{22}, B_{11}, Q)$;

call Strassen $(\frac{n}{2}, A_{11}, B_{12} - B_{22}, R)$;

call Strassen $(\frac{n}{2}, A_{22}, B_{21} - B_{11}, S)$;

call Strassen $(\frac{n}{2}, A_{11} + A_{12}, B_{22}, T)$;

call Strassen $(\frac{n}{2}, A_{21} - A_{11}, B_{11} + B_{12}, U);$

call Strassen $(\frac{n}{2}, A_{12} - A_{22}, B_{21} + B_{22}, V);$

$C_{11} = P + S - T + V;$

$C_{12} = R + T;$

$C_{21} = Q + S;$

$C_{22} = P + R - Q + U;$

end;

TIME COMPLEXITY

$$T(n) = \begin{cases} 7 \cdot T(\frac{n}{2}) + an^2 & n > 2 \\ b & n \le 2 \end{cases}$$

$$\begin{aligned} T(n) &= an^2[1 + 7/4 + (7/4)^2 + \cdots + (7/4)^{k-1}] + 7^k T(1) \\ &\le cn^2(7/4)^{\log_2 n} + 7^{\log_2 n}, \quad c \text{ a constant} \\ &= cn^{\log_2 4 + \log_2 7 - \log_2 4} + n^{\log_2 7} \\ &= O(n^{\log_2 7}) \approx O(n^{2.81}) \end{aligned}$$

TIME COMPLEXITY:

$$T(n) = \begin{cases} 7 \cdot T(\frac{n}{2}) + an^2 & n > 2 \\ b & n \leq 2 \end{cases}$$

$$T(n) = 7 \cdot T(\frac{n}{2}) + an^2$$

$$= 7^2 \cdot T(\frac{n}{2^2}) + (\frac{7}{4})an^2 + an^2$$

$$= 7^3 \cdot T(\frac{n}{2^3}) + (\frac{7}{4})^2 \cdot an^2 + (\frac{7}{4}) \cdot an^2 + an^2$$

$$\cdots$$

Assume $n = 2^k$ for some integer k

$$= 7^{k-1} \cdot T(\frac{n}{2^{k-1}}) + an^2 \cdot \left[(\frac{7}{4})^{k-2} + \cdots + 1 \right]$$

$$= 7^{k-1} \cdot b + an^2 \left[\frac{(\frac{7}{4})^{k-1} - 1}{\frac{7}{4} - 1} \right]$$

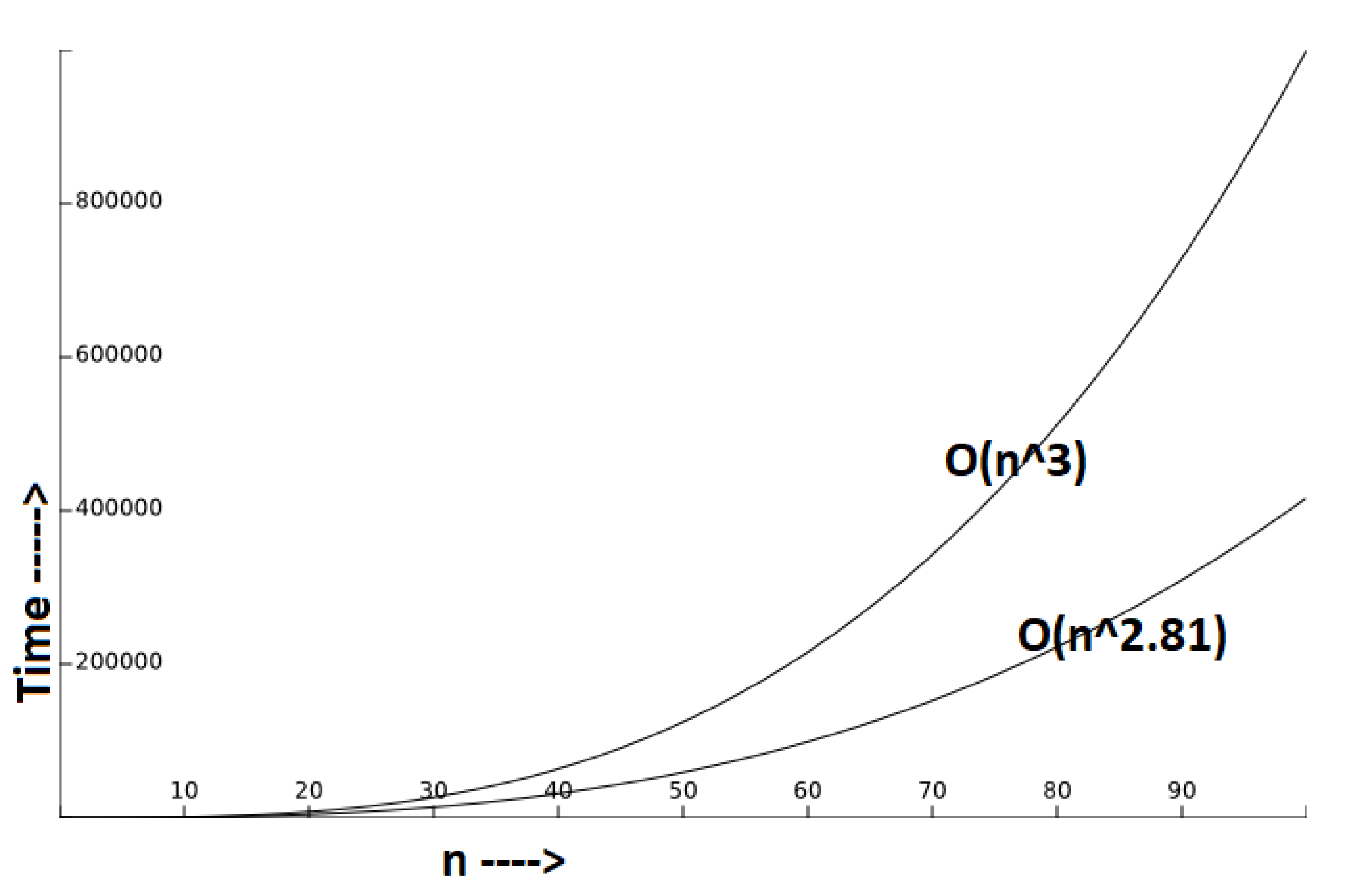
$$\leq b \cdot 7^k + c \cdot n^2 \cdot (\frac{7}{4})^k$$

$$= b \cdot 7^{Lgn} + cn^2 \cdot (\frac{7}{4})^{Lgn} = b \cdot 7^{Lgn} + cn^2 (n)^{Lg\frac{7}{4}}$$

$$= b \cdot n^{Lg7} + cn^{Lg7} = (b + c) \cdot n^{Lg7}$$

$$= O(n^{Lg7}) = O(n^{2.81})$$

COMPARISON : STRAIGHT METHOD VS. STRASSEN'S METHOD



IMPROVEMENTS / OTHER METHODS

1978	**V. Y. Pan.** Strassen's algorithm is not optimal.	**$O(n^{2.796})$**
1979	**D. Bini et al.** $O(n^{2.7799})$ complexity for nxn approximate matrix multiplication.	**$O(n^{2.7799})$**
1981	**A. Schönhage.** Partial and total matrix multiplication.	**$O(n^{2.522})$**

First to break the 2.5 barrier:

1981	**Copper Smith and Winograd** On the asymptotic complexity of matrix multiplication.	**$O(n^{2.496})$**
1986	**Volker Strassen**	**$O(n^{2.479})$**
1989	**CopperSmith and Winograd** Matrix multiplication via arithmetic progressions.	**$O(n^{2.376})$**

2011	**Virginia Vassilevska Williams**: Breaking the Coppersmith-Winograd barrier	**$O(n^{2.3727})$**

- Q&A