Mock 1 Solution

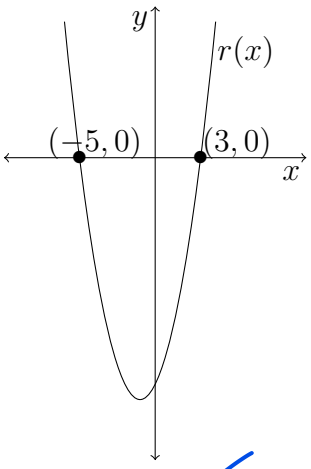
Mathematics for Data Science - 1

Max Marks: 50

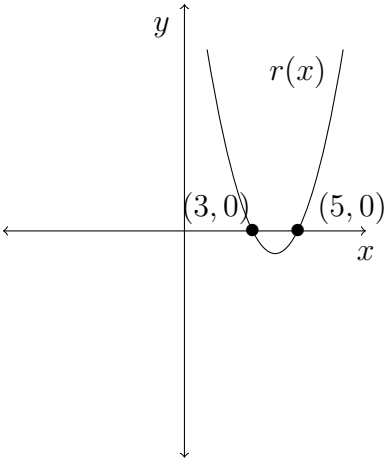
1 Instructions:

- There are some questions which have functions with discrete valued domains (such as day, month, year etc). For simplicity, we treat them as continuous functions.
- For NAT type question, enter only one right answer even if you get multiple answers for that particular question.
- **Notations:**
  - $\mathbb{R}$= Set of real numbers
  - $\mathbb{Q}$= Set of rational numbers
  - $\mathbb{Z}$= Set of integers
  - $\mathbb{N}$= Set of natural numbers
- The set of natural numbers includes 0.

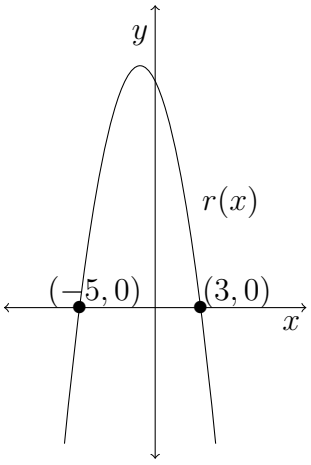
1. Let $r(x)$ be a polynomial function which is obtained as the quotient after dividing the polynomial $p(x) = -(x+5)(x-3)(x^2-16)$ by the polynomial $q(x) = -(x-4)(x+4)$. Choose the correct option(s) which represent(s) the polynomial $r(x)$ most appropriately. (MSQ)(Ans: Option 1) [Marks:3]



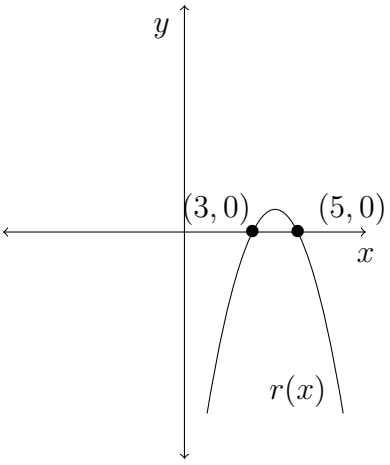
Option 1



Option 2



Option 3



Option 4

Solution

Given that $p(x) = -(x+5)(x-3)(x^2-16)$

$q(x) = -(x-4)(x+4)$

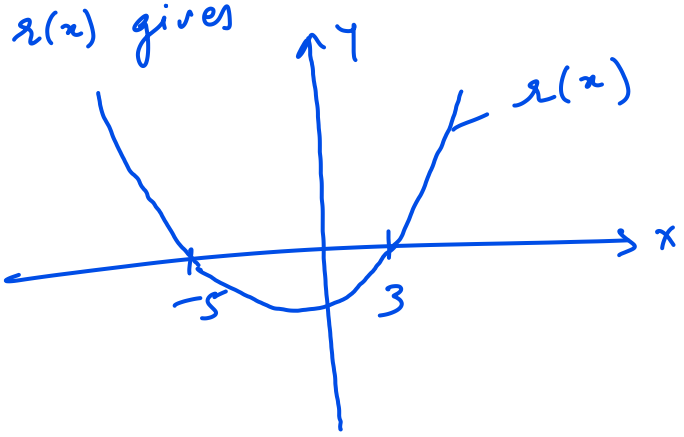
$r(x)$ is the quotient when $p(x)$ is divided by $q(x)$

$\Rightarrow p(x) = -(x+5)(x-3)(x+4)(x-4)$ (on factorizing (x^2-16))

$$\frac{p(x)}{q(x)} = \frac{-(x+5)(x-3)(x+4)(x-4)}{-(x-4)(x+4)}$$

$\Rightarrow r(x) = (x+5)(x-3)$

Graphing of $r(x)$ gives



Thus option 1 is correct

2. Let $R = \{(a, c), (d, b), (b, d), (b, c), (c, a)\}$ be a relation on the set $A = \{a, b, c, d\}$. The relation R is
(MSQ)(Answer: Option(d)) [Marks: 3]

- ◯ a function
- ◯ reflexive
- ◯ transitive
- ◯ not symmetric

Solution :-

option 1 :- R is not a function because there are two outputs for element 'b' i.e; (b,d) & (b,c)

option 2 : R is not reflexive because it does not contains elements such as (a,a), (b,b), (c,c), (d,d)

option 3: R is not transitive relation as for the pairs such as (a,c), (c,a) there is no (a,a) $\in R$

✓ option 4: It is not symmetric relation because for the element (b,c) there is no (c,b)

3. For the purposes of a research, a survey of 1000 students is conducted in a IITM BSc Degree Program. The results show that 52% liked CT, 45% liked Statistics and 60% liked Mathematics. In addition, 25% liked both CT and Statistics, 28% liked both Mathematics and Statistics and 30% liked both CT and Mathematics. 6% liked none of these subjects. Based on this information answer the following questions

Solution:- Step 4 :-

(a) How many students like all the three subjects? → 20% of 1000 = $\frac{20}{100} \times 1000 = 200$
(NAT)(Answer: 200) [Marks: 3]

(b) Find the number of students who like only one of the three subjects.
(NAT)(Answer: 510) → (17% + 12% + 22%) of 1000 [Marks: 3]
$= \frac{51}{100} \times 1000 = 510$

(c) Find the number of students who like at least two of the given subjects.
(NAT)(Answer: 430) → (10% + 5% + 8% + 20%) of 1000 = 430 [Marks: 3]

Step 1:

C n(CT) = % of students who like CT = 52%
F n(S) = " " " Statistics = 45%
R n(M) = " " " Mathematics = 60%

Given that, $n(CT \cap S) = 25\%$
$n(S \cap M) = 28\%$
$n(CT \cap M) = 30\%$

Since 6% like non of the given subjects so, $n(CT \cup S \cup M) = 94\%$

Step 2 :

Formula:

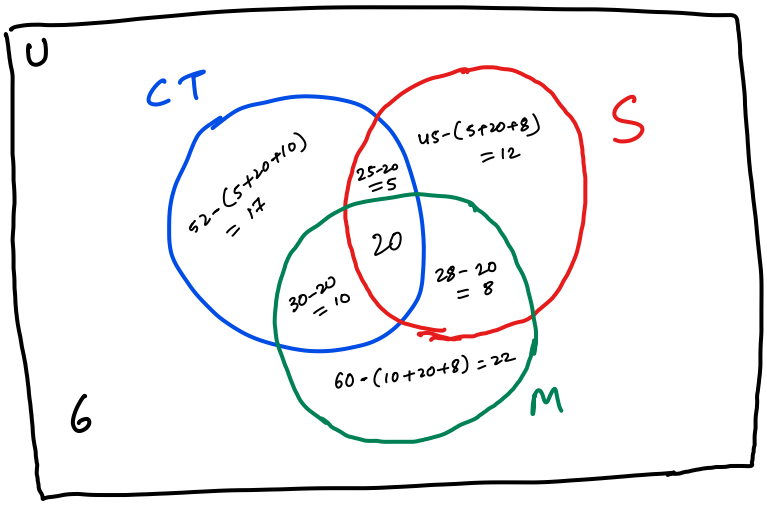
$$n(CT \cup S \cup M) = n(CT) + n(S) + n(M) - n(CT \cap S) - n(S \cap M) - n(CT \cap M) + n(CT \cap M \cap S)$$

$$94\% = 52\% + 45\% + 60\% - 25\% - 28\% - 30\% + n(CT \cap M \cap S)$$

$$\Rightarrow n(CT \cap M \cap S) = 20\%$$

Step 3 :

Venn Diagram related to the above situation is shown below



Step 4:- (see above in subquestions)

4. Consider a triangle ABC (See Figure 1), the line segments AB, BC, and CA represent the sides of the triangle ABC. It is given that angle $\angle ABC = \angle BCA = \alpha$, where α is an acute angle.

Solution:-

Step1: Let l_1, l_2 and l_3 represent the line segment AB, BC and CA respectively.

- Slope of l_1 is $\frac{1}{2}$
- Slope of l_2 is 2

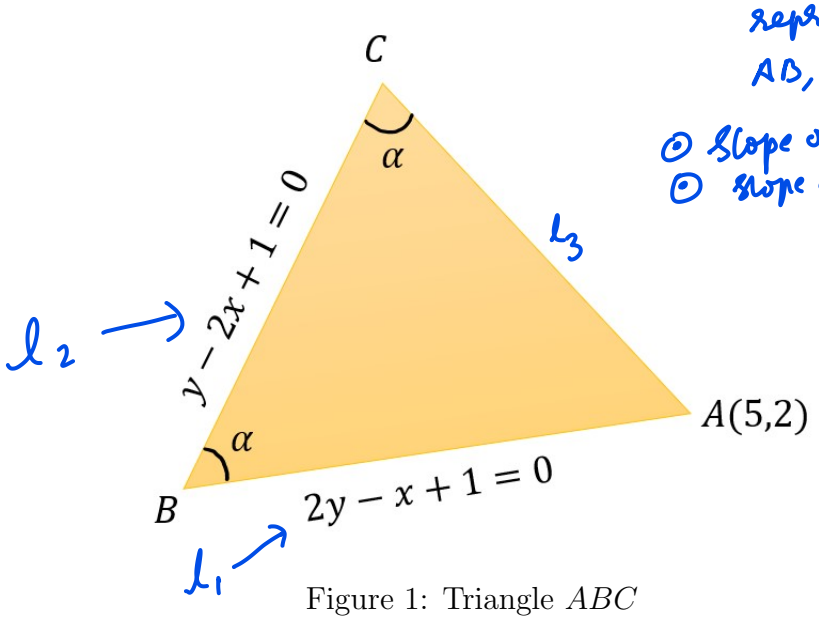


Figure 1: Triangle ABC

Based on this information answer the following questions

(a) Which of the following options are correct w.r.t triangle ABC (MSQ)(Answer: Option(a)(d)) [Marks: 3]

- ◯ Slope of the line segment AB is $\frac{1}{2}$
- ◯ Slope of the line segment BC is $-\frac{1}{2}$
- ◯ $\tan\alpha = -\frac{1}{2}$
- ◯ $\tan\alpha = \frac{3}{4}$

(b) Which of the following options represents the equation of the line AC (MSQ)(Answer: Option(a)(d)) [Marks: 3]

- ◯ $2y + 11x - 59 = 0$
- ◯ $6y = 2x + 2$
- ◯ $3y - x - 1 = 0$
- ◯ $y = -5.5x + 29.5$

Step 2: Angle b/w two lines l_1 and l_2 is α

So, $\tan\alpha = \dfrac{|m_1 - m_2|}{1 + m_1 m_2} = \dfrac{|1/2 - 2|}{1 + \frac{1}{2}\times 2} = \left|\dfrac{-3}{4}\right| = \boxed{\dfrac{3}{4}}$

Again, angle b/w lines l_2 and l_3 is α

So, $\tan\alpha = \dfrac{|m_2 - m_3|}{1 + m_2 m_3} = \dfrac{3}{4}$

Note: m_1, m_2 and m_3 are the slopes of the line l_1, l_2 & l_3 resp

$$\Rightarrow \frac{3}{4} = \frac{|2 - m_3|}{1 + 2m_3}$$

$$\Rightarrow \frac{2 - m_3}{1 + 2m_3} = \frac{3}{4}$$

$$8 - 4m_3 = 3 + 6m_3$$

$$\Rightarrow 10m_3 = 5$$

$$\boxed{m_3 = 1/2}$$

or

$$\frac{2 - m_3}{1 + 2m_3} = \frac{-3}{4}$$

$$8 - 4m_3 = -3 - 6m_3$$

$$2m_3 = -11$$

$$\boxed{m_3 = -11/2}$$

Step 3: As l_3 passes through $(5,2)$ and slope of l_3

Equation of the line l_3 is

$$y - 2 = \frac{1}{2}(x - 5)$$

$$\Rightarrow 2y - 4 = x - 5$$

$$\boxed{2y = x - 1}$$

Note this is line l_1

Equation of the line is

$$y - 2 = -\frac{11}{2}(x - 5)$$

$$2y - 4 = -11x + 55$$

$$\boxed{2y + 11x - 59 = 0}$$

or

$$\boxed{y = -5.5x + 29.5}$$

5. Rizwan wants to cross the river represented by $r(x) = 0.05(x-1)(x-5)(x-10) + k$, where k is an integer constant (Refer Figure 2). He chooses to cross the river $r(x)$ using the bridge. In order to cross the river he has to identify the coordinates of the point A. He realises that the coordinates of the point A can be identified in such a way that the curve $r(x)$ can be best fitted according to the data given in the table (refer Table 1). Consider the river and the bridge to be of negligible thickness. Based on these information answer the following questions

x	-10	-5	1	10	15
y	-150	-30	15	15	50

Table 1

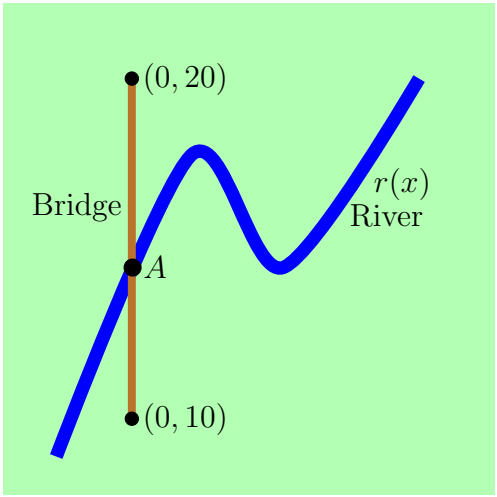


Figure 2: River and bridge

(a) Using the above information, find the value of k.
(NAT)(Answer: 15) [Marks: 4]

(b) Using the above information, what will be the $y-$coordinate of the point A?
(NAT)(Answer: 12.5) [Marks: 3]

Solution

It is given that $g(x)$ best fit the data given in the Table1.

Thus, SSE should be minimum.

x	y	$[y - f(x)]^2$
-10	-150	$\left[-150 - \{0.05(-10-1)(-10-5)(-10-10)+k\}\right]^2$ $= (15-k)^2$
-5	-30	$(15-k)^2$
1	15	$(15-k)^2$
10	15	$(15-k)^2$
15	50	$(15-k)^2$

$$SSE = 5(15-k)^2$$

This is quadratic equation in k, & it will be minimum at its vertex.

Thus $\boxed{k = 15}$

The equation of $g(x) = 0.05(x-1)(x-5)(x-10)+15$

From the figure it is clear that x-coordinate of 'A' is 0

So, y-coordinate will be

$g(0) = 0.05(-1)(-5)(-10) + 15 = 12.5$

6. Which of the following statements is (are) correct?
(MSQ)(Answer: Option(a)(b)(c)) [Marks: 4]

- (a) The product of the minimum value of the function $f(x) = 5|x| + 10$ and the maximum value of the function $g(x) = 10 - |x + 12|$ is 100.
- (b) There are infinitely many polynomial $p(x)$ of degree four such that $p(4) = 0$, $p(5) = 0$, $p(6) = 0$.
- (c) $y - 4 = (x + 5)^2$ is an equation of a parabola whose vertex is at $(-5, 4)$.
- (d) Elements in Cartesian product will only be pairs. → No, can be triplets etc. aswell.

Solution:-

option 1: Minimum value of modules function is zero.
Thus, minimum value of $f(x)$ is 10
maximum value of $g(x)$ is 10
So, minimum value of $f(x) \cdot g(x)$ is 100

option 2: Given that factors of $p(x)$ are $(x-4)(x-5)(x-6)$ and it is four degree polynomial. so $p(x)$ can be any polynomial depending on the other factor & the stretch factor.

option 3: The equation of quadratic equation in vertex form is given as $y - K = a(x-h)^2$ where (h, K) are the vertex of the quadratic equation.
Comparing, $y - 4 = (x+5)^2$ gives $h = -5$, $K = 4$
Thus vertex is $(-5, 4)$

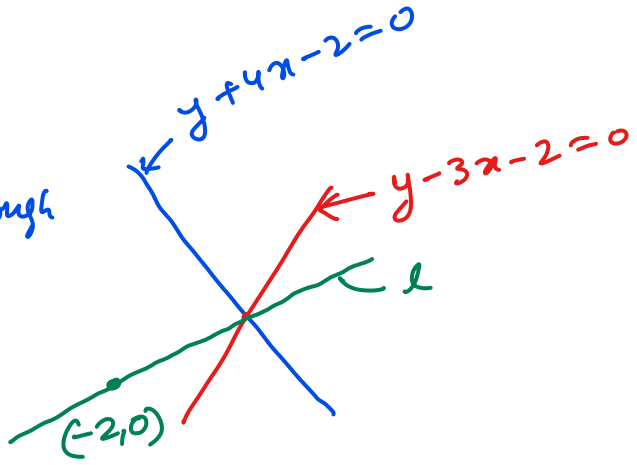
7. Choose the most appropriate option for the statement given below:

"The equation of the line joining the point $(-2, 0)$ to the point of intersection of the lines $y + 4x - 2 = 0$ and $y - 3x - 2 = 0$ is equidistant from the points $(0, 0)$ and $(2, 2)$."

(MSQ)(Answer: Option(a)) [Marks: 4]

- ◯ True
- ◯ False
- ◯ Data insufficient
- ◯ None of the above

Solution:-



Consider l passes through $(-2,0)$ and point of intersection of the given two lines.

(see rough sketch)

Step1: Finding point of intersection of

$y+4x-2=0$ & $y-3x-2=0$

$\downarrow$ Eq1 $\Rightarrow y = 3x+2 \rightarrow$ Eq 2

Substitute Eq 2 in Eq 1 we get

$3x+2+4x-2=0 \Rightarrow 7x=0 \Rightarrow \boxed{x=0}$

Substitute $x=0$ in Eq 1 we get $\boxed{y=2}$

Thus point of intersection of two lines is $(0,2)$

Step2: Eqn of line 'l' will be $y-x-2=0$ (using two point form)

Step3: Distance of point from a line is $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$

$d = \frac{|-2|}{\sqrt{2}} = \frac{2}{\sqrt{2}}$ (for point $(0,0)$) | $d = \frac{2}{\sqrt{2}}$ for $(2,2)$

Thus distance are same, so option(1) is correct

8. A total of ₹300 is raised by a group A, by collecting equal amounts from a certain number of people. Another group B, contains 5 more people and each person contributes ₹10 less to raise the same amount as that of group A. Based on this information answer the following questions

   (a) How many people actually contributed in group A?
   (NAT)(Answer: 10) [Marks: 3]

   (b) What is the contribution (₹) made by each person in group B?
   (NAT)(Answer: 20) [Marks: 1]

Solution:

Group A { Let the no. of people be x
Contribution of each person will be $\frac{300}{x}$

Group B { Contribution of each person is $\frac{300}{x} - 10$
No. of people is $x+5$

Group B too raises same amount i.e., 300.

So, $\left(\frac{300}{x} - 10\right)(x+5) = 300$

$\Rightarrow (300 - 10x)(x+5) = 300x$

$\Rightarrow 300x - 10x^2 + 1500 - 50x = 300x$

$\Rightarrow x^2 + 5x - 150 = 0$

$\Rightarrow x^2 + 15x - 10x - 150 = 0$

$\Rightarrow x(x+15) - 10(x+15) = 0 \Rightarrow (x-10)(x+15) = 0$

$x = 10$ or $x = -15$ (Not possible)

Thus, no. of people who actually contributed in group A is 10

Contribution made by each person in group B is $\frac{300}{10} - 10 = 20$

9. Suppose the function $f(x) = -x^2 + 4x - k$ and $g(x) = x^2 - kx + 4$ intersects at most at one point, where $k \in \mathbb{Z}$. Based on this information answer the following questions

(a) Which of the following could be the value of k.
(MSQ)(Answer: Option(a)) [Marks: 2]

- ◯ 4
- ◯ -5
- ◯ 10
- ◯ -10

(b) Find the possible number of values of k? (Also given full marks for $|k| = 2$)
(NAT)(Answer: 9) [Marks: 2]

Solution: Given that $f(x)$ & $g(x)$ intersects thus

$f(x) = g(x)$

$\Rightarrow -x^2 + 4x - k = x^2 - kx + 4$

$\Rightarrow 2x^2 - (k+4)x + k + 4 = 0$ — Eq 1

Given that they intersects at most at one point, therefore, two possibilities.

(i) Both the curves do not intersect at all.

(ii) Both the curves intersect at exactly one point.

Thus, discriminant (D) of Eq 1 should be $D \leq 0$

$(k+4)^2 - 4(2)(k+4) \leq 0$

$(k+4)(k+4-8) \leq 0$

$(k+4)(k-4) \leq 0$

rough figure

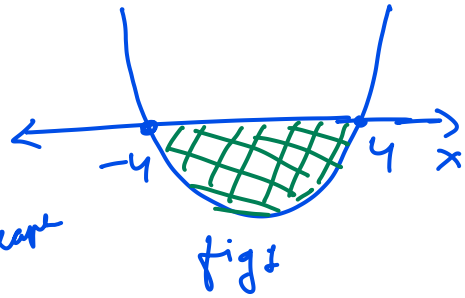


fig 1

Thus, $k \in [-4, 4]$ or

$k = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ ($\because k \in \mathbb{Z}$

$|k| = 9$

10. Consider two polynomial function $p(x) = 0.1\,(x-1)\,(x-5)\,(x-10)$ and $q(x) = 0.4\,(x-1)\,(x-10)$ defined in the interval $(1, 10)$. A line $l(x)$ passes through $x-$intercept of $p(x)$ and the intersection point of $p(x)$ and $q(x)$.

(a) Which of the following are correct.
(MSQ)(Answer: Option(a)(b)(c)(d)) [Marks: 4]

- ✓ The $x-$intercept of $p(x)$ in the given domain is 5.
- ✓ In the given domain $p(x)$ has 2 turning points.
- ✗ The $x-$coordinate of the vertex of $q(x)$ is 5.5.
- ✓ The $x-$coordinate of the intersection point of $p(x)$ and $q(x)$ is 9.

(b) What is the slope of the line $l(x)$?
(NAT)(Answer: -0.8) [Marks: 2]

Solution: A rough graph of $p(x)$ & $q(x)$ is shown below

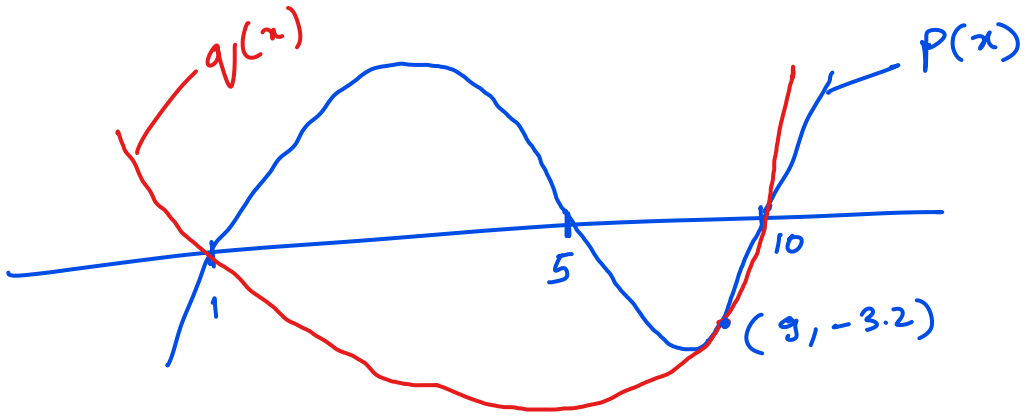


Fig 1.

Note the domain of both the f^n is $(1, 10)$

Intersection point of $p(x)$ & $q(x)$ can be found as follow

$p(x) = q(x)$

$\Rightarrow 0.1(x-1)(x-5)(x-10) = 0.4(x-1)(x-10)$

$\Rightarrow (x-1)(x-5)(x-10) - 4(x-1)(x-10) = 0$

$(x-1)(x-10)\{(x-5) - 4\} = 0$

$x = 1, 9, 10$ (1, 10 are not in domain)

Thus x-coordinate of point of intersection of $p(x)$ & $q(x)$ is 9.

y-coordinate of " is -3.2

Slope of the line will be $\frac{-3.2 - 0}{9 - 5} = -0.8$