Text Book
for
MATHEMATICS FOR DATA SCIENCE-1

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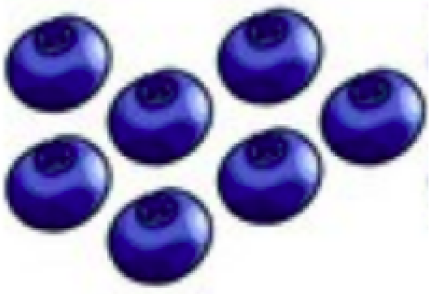
Contents

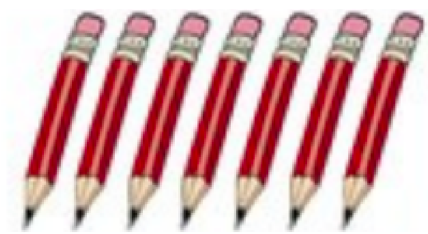
Chapter 1

1 Set theory

1.1 Natural numbers and integers

• Numbers keep a count of objects.





• '7' stones or '7' pencils. Here, 7 represents the count.

1.1.1 Natural numbers

• The set of natural numbers is denoted by $\mathbb{N}$.
• $\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$.
• The set of natural numbers includes 0.

1.1.2 Integers

• The set of integers is denoted by $\mathbb{Z}$.
• $\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$.

1.1.3 Arithmetic operations (+,-,×,÷,modulo)

• **Addition(+):** The addition operation gives the sum of two numbers. It involves combining two or more numbers into a single number.
Example $5 + 2 = 7$

• **Substraction(-):** The subtraction operation gives the difference between two numbers.
Example 9 - 4 = 5

• **Multiplication(×):** Multiplication is repeated addition.
Example $3 \times 4 = 12$
Here, $3 \times 4 = 3 + 3 + 3 + 3 = 12$.

• **Division(÷):** Division is repeated subtraction.

Example $18 \div 3 = 6$

• **Modulo:** Modulo operator or Remainder operator gives the remainder when one number is divided by another. It is denoted as "mod".

Example $10 \bmod 3 = 1$
Here, 1 is the remainder we get when we divide 10 by 3 so $10 \bmod 3 = 1$.

1.1.4 Factors

• a is a factor of b if $b \bmod a = 0$.
• b is a multiple of a.

Examples

(i) 2 is a factor of 6.

(ii) 5 is a factor of 10.

1.2 Rational numbers

• The numbers of the form $\frac{p}{q}$, where p, q are integers are called rational numbers.
• The set of rational numbers are denoted by $\mathbb{Q}$.
• The representation of rational numbers may not be unique! For example $\frac{1}{2} = \frac{2}{4} = \frac{10}{20}$.
• The reduced form of any rational number $\frac{p}{q}$ is when p, q have no factors in common.
• Rational numbers extend Natural numbers.

1.2.1 Greatest common divisor

The greatest common divisor (gcd) of two non-zero integers p and q is the greatest positive integer k such that k is a divisor of both p and q.

Examples

(i) gcd(9,12) $= 3$

(ii) gcd(15,45) $= 15$

(iii) gcd(0,q) $= q$, here 0 is multiple of every integer.

(iv) gcd(1,q) $= 1$, here 1 has no factors other than itself.

1.3 Real numbers

1.3.1 Irrational numbers

The numbers that cannot be written in the form of $\frac{p}{q}$, where p, q are integers are called irrational numbers. Simply, the numbers that are not rationals are called irrationals.

• $\sqrt{2}, \sqrt{3}, \pi$ are some examples of irrational numbers.

Real numbers: All the rational numbers including the irrational numbers are called real numbers.
• Real numbers extend rational numbers.

1.4 Sets

Definition 1.4-1 A *set* is a collection of well defined items.

Examples

Finite sets:

(i) The set of natural numbers less than 10 $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(ii) The set of all months in a year = {January, February, March, April, May, June, July, August, September, October, November, December}.

(iii) The set of all days of the week = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}.

Infinite sets:

(i) The set of all even natural numbers $= \{0, 2, 4, 6, 8, 10, \ldots\}$.

(ii) The set of integers ($\mathbb{Z}$).

• Items in a *set* are called *elements*.
• Order is not important in a set.
• Duplicates in a set does not matter.

Definition 1.4-2 The *Cardinality of a set* S is the number of elements in the set S.

Examples

(i) $S = \{1, 2, 5, 7, 9, 300\}$. The cardinality of the set S is 6.

(ii) $A =$ {Srikanth, Keerthana, Balloon, Cell phone, π}. The cardinality of the set A is 5.

1.4.1 Subsets

Definition 1.4.1-1 A set X is a *Subset* of another set Y if every element in X is also an element in Y. It is denoted by $X \subseteq Y$.

Examples

(i) $X = \{1, 2, 5, 7, 9, 300\}$ and $Y = \{0, 1, 2, 5, 7, 9, 40, 170, 300\}$. Here, $X \subseteq Y$.

(ii) $X = \{4, 16, 32, 64\}$ and $Y = \{64, 16, 32, 4\}$. Here, $X \subseteq Y$.

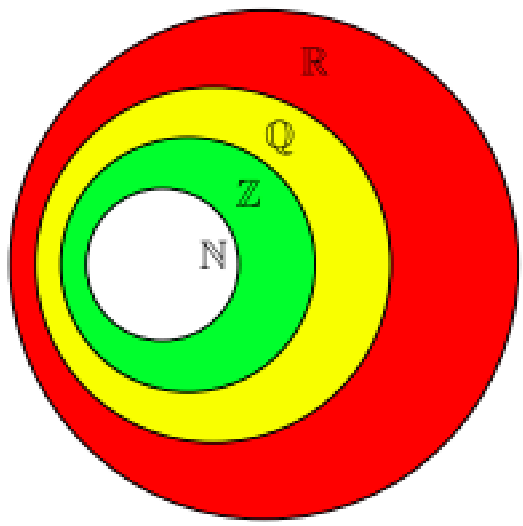
(iii) $\mathbb{N} \subseteq \mathbb{Z}, \mathbb{Z} \subseteq \mathbb{Q}, \mathbb{Q} \subseteq \mathbb{R}$.

Definition 1.4.1-2 A set X is a *proper subset* of another set Y if $X \subseteq Y$ but $X \neq Y$. It is denoted by $X \subset Y$.

(i) $X = \{1, 2, 5, 7, 9, 300\}$ and $Y = \{0, 1, 2, 5, 7, 9, 40, 170, 300\}$. Here, $X \subset Y$.

(ii) $X = \{4, 16, 32, 64\}$ and $Y = \{64, 16, 32, 4\}$. Here, $X = Y$ which implies that X is not a proper subset of Y.

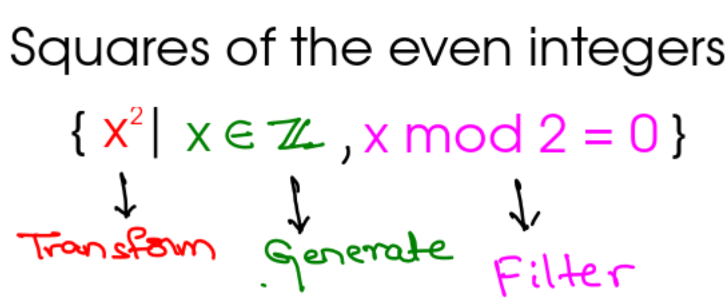
(iii) $\mathbb{N} \subset \mathbb{Z}, \mathbb{Z} \subset \mathbb{Q}, \mathbb{Q} \subset \mathbb{R}$.



1.4.2 Set comprehension

Definition 1.4.2-1 *Set comprehension* is a construction of a subset from the existing sets like $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ etc., by applying some filters on every element in the existing set. It is build with three main components. They are *generator, filter, transformer*

Example



1.5 Relations

1.5.1 Cartesian product

Definition 1.5.1-1 *Cartesian Product* of two non empty sets X and Y is defined as the set of all possible ordered pairs (x, y) such that $x \in X$ and $y \in Y$. The cartesian product is denoted as "$X \times Y$".

Example

$A = \{a, b\}$ and $B = \{1, 2, 3\}$.
The cartesian product $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

1.5.2 Binary relation

Definition 1.5.2-1 A *relation* between two sets (X and Y) is a collection of ordered pairs containing one element from each set. In other words, a *relation* R is a subset of cartesian product of X and Y ($R \subseteq X \times Y$).

Example

Suppose $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Now,
$R_1 = \{(a, 1), (b, 2), (b, 3)\}$.
$R_2 = \{(a, 2), (b, 1)\}$.
$R_3 = \{(b, 3)\}$.
$R_4 = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.
Here, R_1, R_2, R_3 and R_4 are relations from set A to set B.

NOTE Order of an element is important in a relation i.e., $(a, 1) \neq (1, a)$.

1.5.3 Properties of relation

(1) Reflexive relation:- Let R be a binary relation on a set S. Then, R is said to be reflexive if and only if for all $x \in S, (x, x) \in R$.

Example

$S = \{1, 2, 3, 4\}$. Now,
$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
$R_2 = \{(1, 1), (2, 3), (2, 2), (4, 4), (3, 4)\}$.
$R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 4), (4, 1)\}$.
Here, R_1 and R_3 are reflexive relations but R_2 is not a reflexive relation because (3,3) $\notin R_2$.
Also, R_1 is called as identity relation.

(2) Symmetric relation:- Let R be a binary relation on a set S. Then R is said to be symmetric if and only if for every $(x, y) \in R \implies (y, x) \in R$,where $x, y \in S$.

Example

$S = \{1, 2, 3, 4\}$. Now,

$R_1 = \{(1,2),(2,1),(3,4),(4,3),(2,2)\}$.
$R_2 = \{(1,2),(2,3),(2,1),(3,2),(4,3),(3,4)\}$.
$R_3 = \{(2,1),(1,2),(3,4),(2,4),(4,2)\}$.
Here, R_1 and R_2 are symmetric relations but R_3 is not a symmetric relation because (3,4) $\in R_3$ but $(4,3) \notin R_3$.

(3) Transitive relation:- Let R be a binary relation on a set S. Then R is said to be transitive if and only if for every pair of elements, (x,y) and $(y,z) \in R \implies (x,z) \in R$, where $x,y,z \in S$.

Example

$S = \{1,2,3,4\}$. Now,
$R_1 = \{(1,2),(2,3),(1,3)\}$.
$R_2 = \{(1,4)\}$.
$R_3 = \{(1,3),(1,4)\}$.
$R_4 = \{(2,4),(1,2),(1,4),(4,1),(1,1),(2,1)\}$.
$R_5 = \{(1,2),(2,1),(1,1),(2,3)\}$ Here, R_1, R_2, R_3 and R_4 are transitive relations but R_5 is not a transitive relation because (1,2) , (2,3) $\in R_5$ but (1,3) $\notin R_5$.

(4) Equivalence relation:- If R is Reflexive, Symmetric, and Transitive, then R is an equivalence relation.

Example

$S = \{1,2,3,4\}$. Now,
$R_1 = \{(1,1),(2,2),(3,3),(4,4),(2,3),(3,2),(4,1),(1,4)\}$.
Here, R_1 is an equivalence relation.

Problem:

Match the following relations with its properties

Relation defined on a set S = { 1,2,3}	Property
$R_1 = \{(1,3),(3,2),(1,1),(2,3),(3,1)\}$	(1) Reflexive relation
$R_2 = \{(1,1),(2,3),(3,2),(2,2),(3,3)\}$	(2) Symmetric relation
$R3 = \{(3,1),(1,1),(2,1),(2,3),(3,3),(2,2)\}$	(3) Transitive relation
$R4 = \{(3,2),(3,1),(2,1)\}$	(4) Equivalence relation

1.6 Functions

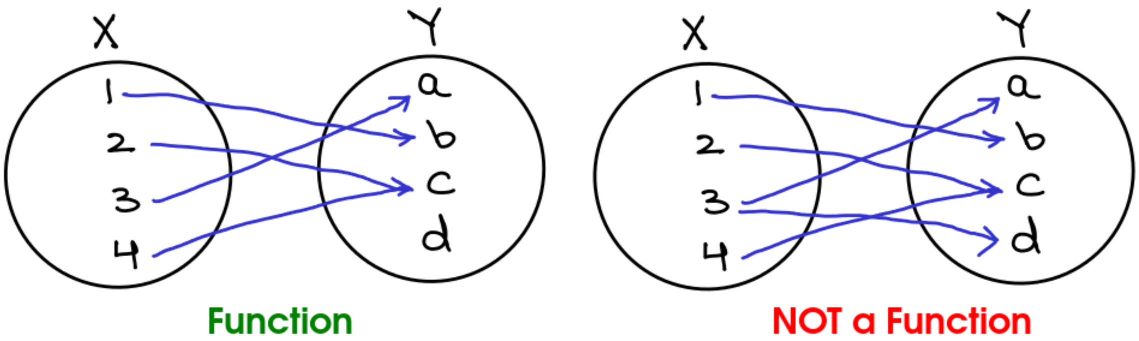
Definition 1.6-1 A *function* is a relation in which each element in set X is mapped/paired with exactly one element in set Y.

- In a function, there should not be two pairs with the same first element.
- In a function, a particular input is given to get a particular output. So, A function

$f : X \longrightarrow Y$ denotes that f is a function from set X to set Y, where X is the domain and Y is the co-domain.

- **Domain of** f= Set of all input values (set X).
  Co-domain of f= Set of all possible output values (set Y).
  Range of f= $\{y \mid y \in Y, y = f(x) \text{ for some } x \in X\}$.

Example



1.6.1 Types of functions

(1) Injective function: If each element in set X is mapped to a distinct element in set Y, then the function is called an injective function or one-to-one function.

(2) Surjective function: If the range is equal to the co-domain, then the function is called a surjective function or onto function.

(3) Bijective function: If the function is both injective and surjective, then it is called a bijective function.

1.6.2 Finding domain and range of a function

(1) If the co-domain is $\mathbb{R}$, then find the domain of the function $f(x) = \sqrt{x}$.
Solution: $\sqrt{x}$ is well-defined only if $x \geq 0$. So, the domain of f is $[0, \infty)$.

(2) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $f(x) = x^2$. Find the range of f and check whether the function f is surjective or not.
Solution: We know that set of all possible output values are positive real numbers including zero. Therefore, the range of the given function is $[0, \infty)$.

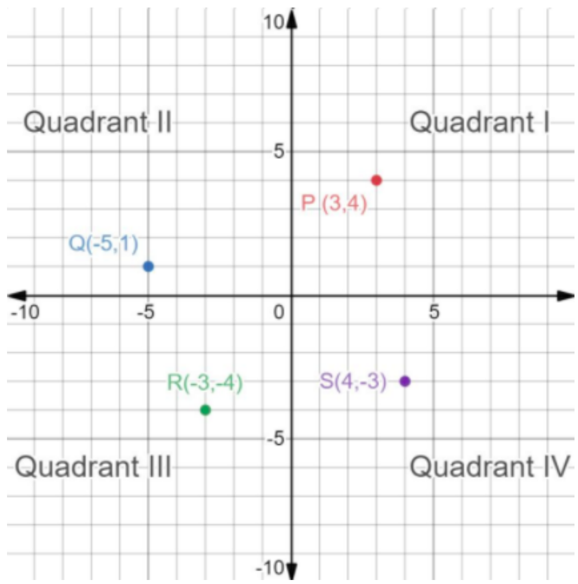
Now, the co-domain is $\mathbb{R}$ and range is $[0, \infty)$. Here co-domain is not equal to the range so the function is not surjective.

Chapter 2

2 Straight lines

2.1 Rectangular coordinate system

A **Cartesian coordinate system** is a coordinate system which specifies each point in a plane by a set of numerical coordinates. These coordinates are the signed distances to the point from two fixed perpendicular oriented lines, measured in the same unit of length. Each reference line is called an **axis** (plural axes) of the system, and the point where two axes meet is called **origin**, whose coordinate is (0, 0).



The coordinate axes split the coordinate plane into four quadrants and two axes.

Quadrant	Abscissa (X-axis)	Ordinate (Y-axis)
I	+ve	+ve
II	-ve	+ve
III	-ve	-ve
IV	+ve	-ve

2.2 Distance between any two points

The **distance** between any two points (x_1,y_1) and (x_2,y_2) in the Cartesian plane(XY plane) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Solved Examples**

(1) Find the distance between the two points (2,4) and (-4,12).

Solution:
We know that, the distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here, $x_1 = 2, y_1 = 4, x_2 = -4, y_2 = 12$.
By using the formula, the distance between the two points (2,4) and (-4,12) will be

$$\sqrt{((-4) - 2)^2 + (12 - 4)^2} = \sqrt{(-6)^2 + (8)^2} = \sqrt{36 + 64} = \sqrt{100} = \mathbf{10}$$

(2) If the distance between the two points (-3,y) and (1,4) is 5 units, then find the possible values of y.

Solution:
We know that, the distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here, $x_1 = -3, y_1 = y, x_2 = 1, y_2 = 4$.
By using the formula, the distance between the two points (-3,y) and (1,4) will be

$$\sqrt{(1 - (-3))^2 + (4 - y)^2} = \sqrt{(4)^2 + (4 - y)^2}$$

But, it is given that the distance is 5 units.
Therefore, $\sqrt{(4)^2 + (4 - y)^2} = 5 \implies (4 - y)^2 = 9 \implies 4 - y = \pm 3 \implies y = \mathbf{1, 7}$

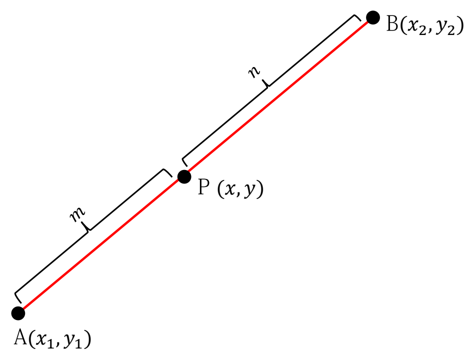
Hence, the possible values of y are 1 and 7.

2.3 Section formula:

If a point P (x, y) **internally** cuts the line segment AB, which connects two points A (x_1,y_1) and B (x_2,y_2) in a ratio $m : n$, then the value of x and y will be

$$x = \frac{mx_2 + nx_1}{m + n}$$

$$y = \frac{my_2 + ny_1}{m + n}$$

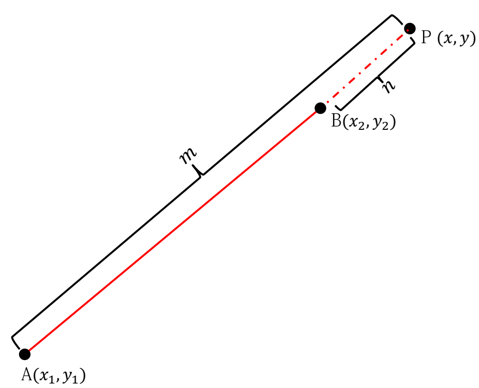


Internal section in the ratio $m : n$

If a point P (x, y) **externally** cuts the line segment AB, which connects two points A (x_1,y_1) and B (x_2,y_2) in a ratio $m : n$, then the value of x and y will be

$$x = \frac{mx_2 - nx_1}{m - n}$$

$$y = \frac{my_2 - ny_1}{m - n}$$



External section in the ratio $m : n$

- **Solved Examples**

(1) Find the point that divides the line segment $P(2, 5)$ and $Q(8, 8)$ internally in the ratio 1:2.

Solution:

Let $S(x,y)$ be the point that divides the line segment PQ in the ratio 1:2.
By using the section formula, the $X-$ coordinate of the point S will be

$$x = \frac{mx_2 + nx_1}{m+n} = \frac{1(8)+2(2)}{1+2} = \frac{12}{3} = 4$$

Similarly, the $Y-$ coordinate of the point S will be

$$y = \frac{my_2 + ny_1}{m+n} = \frac{1(8)+2(5)}{1+2} = \frac{18}{3} = 6$$

Hence, the point **(4,6)** divides the given line segment PQ internally in the ratio 1:2.

(2) Find the coordinates of the midpoint of points P(4, -2) and Q(-1, -1).

Solution:
We know that, the mid point divides two points in the ratio 1:1.

Therefore, use the section formula by taking $m = 1$ and $n = 1$. The $X-$ coordinate of the mid point of points (4,-2) and (0,2) will be

$$x = \frac{1(0)+1(4)}{1+1} = \frac{4}{2} = 2$$

Similarly, the $Y-$ coordinate of the mid point of points (4,-2) and (0,2) will be

$$y = \frac{1(2)+1(-2)}{1+1} = \frac{0}{2} = 0$$

Hence, the mid point of the points P(4, -2) and Q(-1, -1) is **(2,0)**

2.4 Area of a triangle:

The **area of a triangle** (Δ) formed by three points, (x_1,y_1) , (x_2,y_2) and (x_3,y_3) in XY plane is

$$\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

We use modulus because area of any region is always positive.

• **Solved Examples**

(1) What is the area of the triangle formed by the points, $P(0,10), Q(-20,-30)$ and $R(10,30)$.

Solution:

The area of the triangle formed by the points (0,10), (-20,-30) and (10,30) will be

$$\begin{aligned}\Delta &= \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= \frac{1}{2}|0((-30) - 30) + (-20)(30 - 10) + 10(10 - (-30))| \\ &= \frac{1}{2}|0 - 20(20) + 10(40)| \\ &= \frac{1}{2}| - 400 + 400| \\ &= 0\end{aligned}$$

Hence, the area of the triangle formed by the points $P(0, 10), Q(-20, -30)$ and $R(10, 30)$ is **0**.

Note: If the area of the triangle formed by three points P, Q, and R is zero, then the three points P, Q and R are colinear.

(2) What is the area of the triangle formed by the midpoints of line segments PQ, QR, and RP where the coordinates of P, Q, and R are (0,0), (4,0), and (4,3) respectively?

Solution:
By using the section formula,
the mid points of the line segments PQ, QR, and RP are (2,0), (4,1.5) and (2,1.5) respectively.
Now, the area of the triangle formed by the points (2,0), (4,1.5) and (2,1.5) will be

$$\begin{aligned}\Delta &= \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= \frac{1}{2}|2(1.5 - 1.5) + 4(1.5 - 0) + 2(0 - 1.5)| \\ &= \frac{1}{2}|2(0) + 4(1.5) + 2(-1.5)| \\ &= \frac{1}{2}|0 + 6 - 3| \\ &= 1.5\end{aligned}$$

Hence, the area formed by the midpoints of the line segments PQ, QR, and RP is **1.5 sq.units**

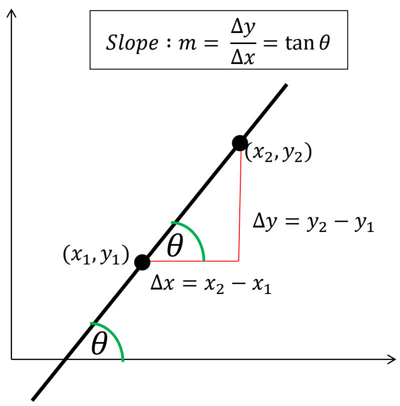
2.5 Straight lines

2.5.1 Slope of a straight line

Slope of a straight line (denoted by m) describes both direction and steepness of a line. Numerically, slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta,$$

where (x_1, y_1) and (x_2, y_2) are two points on the line and θ is the inclination of the line with respect to the positive X-axis.



- **Solved Example**

(1) What is the slope of the line passing through the origin and the point (-3,5)?

Solution:
The coordinates of the origin is (0,0). So, the slope of the line passing through the points (0,0) and (-3,5) is $\frac{5-0}{-3-0} = \frac{\mathbf{-5}}{\mathbf{3}}$

- **Characterization of parallel lines via slope:**

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β respectively. The two non-vertical lines l_1 and l_2 are parallel if and only if their slopes are equal, i.e. $m_1 = m_2$.

- **Characterization of perpendicular lines via slope:**

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β respectively. The two non-vertical lines l_1 and l_2 are perpendicular if and only if the product of their slopes is -1, i.e. $m_1 m_2 = -1$.

2.5.2 Equation of a straight line

In general the equation of the straight lines is often given in the *slope-intercept form*:

$$y = mx + c$$

where:
m is the slope or gradient of the line.
c is the y-intercept of the line.
Point-slope form of straight line
The equation of a straight line having slope m and passing through a point (x_1, y_1) is

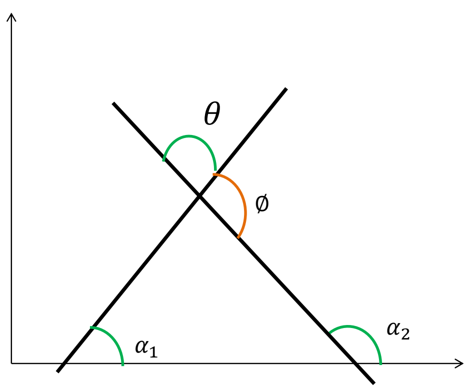
$$(y - y_1) = m(x - x_1)$$

Two-point form of straight line
The equation of a line connecting two points (x_1, y_1) and (x_2, y_2) is

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

2.5.3 Angle between two lines

Let l_1 and l_2 be two lines with slopes m_1 and m_2 and inclinations α_1 and α_2 respectively (let us assume, $\alpha_2 > \alpha_1$).



If l_1 and l_2 intersect each other forming adjacent angles ϕ and θ, then

$$tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

2.6 Different forms of equations of a straight line

Forms of equation of straight line	**Representation**
General form	$ax + by + c = 0$
Slope-point form	$(y - y_0) = m(x - x_0)$
Slope-intercept form (y-intercept)	$y = mx + c$
Slope-intercept form (x-intercept)	$y = m(x - d)$
Intercept form	$\frac{x}{a} + \frac{y}{b} = 1$
Two-point form	$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

2.7 Condition for parallel and perpendicular lines

Given two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $b_1, b_2 \neq 0$,

- The lines are parallel to each other, if $a_1 \times b_2 = a_2 \times b_1$.
- The lines are perpendicular to each other, if $a_1 \times a_2 = -b_1 \times b_2$.

2.8 Distance of a line from a given point

The distance(d) of a straight line $ax + by + c = 0$ from a point (x_1, y_1) is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

2.9 Distance between two parallel lines

The distance(D) between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is

$$D = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

2.10 Sum Squared Error (SSE)

SSE is the sum of the squares of the deviations of the predicted linear model from the actual data set. Numerically, if we are given a set of n points (x_i, y_i), $i = 1, 2, 3, ..., n$ and we have a line of fit $y = mx + c$, then the SSE will be calculated as

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

Solved examples

(1) A line fit $y = 2x + 2$ is given for the data as shown in the below table. Compute the sum squared error(SSE).

x	1	2	4	9
y	5	6	9	18

Solution

Here 4 points are given and $m = 2, c = 2$. So,

$$\begin{aligned}
\text{SSE} &= \sum_{i=1}^{4}(y_i - 2x_i - 2)^2 \\
&= (y_1 - 2x_1 - 2)^2 + (y_2 - 2x_2 - 2)^2 + (y_3 - 2x_3 - 2)^2 + (y_4 - 2x_4 - 2)^2 \\
&= (5 - 2(1) - 2)^2 + (6 - 2(2) - 2)^2 + (9 - 2(4) - 2)^2 + (18 - 2(9) - 2)^2 \\
&= (5 - 2 - 2)^2 + (6 - 4 - 2)^2 + (9 - 8 - 2)^2 + (18 - 18 - 2)^2 \\
&= (1)^2 + (0)^2 + (-1)^2 + (-2)^2 \\
&= 1 + 0 + 1 + 4 = \mathbf{6}
\end{aligned}$$

Chapter 3

3 Quadratic Function

A quadratic function is described as

$$f(x) = ax^2 + bx + c \text{ where } a \neq 0$$

- The curve representing any quadratic function is always a parabola. A simple example of parabola is shown in Figure 1.

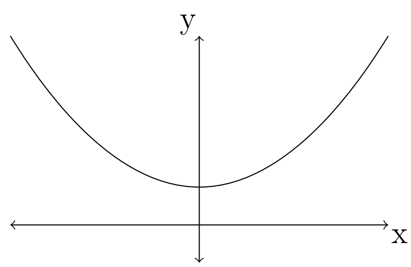


Figure 1 : A parabola

3.1 Important observations

3.1.1 Axis of symmetry

All parabolas have an axis of symmetry. That is, if the graph paper containing the graph of parabola is folded along the axis of symmetry the portion of parabola on either sides will exactly match each other.

- The equation of axis of symmetry of a parabola : $x = \dfrac{-b}{2a}$.

3.1.2 Vertex of a parabola

The point at which the axis of symmetry intersects the parabola is called the vertex.

- The x-coordinate of the vertex of a parabola is $\dfrac{-b}{2a}$.

- The y-coordinate of the vertex of a parabola is $f\left(\dfrac{-b}{2a}\right)$.

3.1.3 Types of parabola

A parabola will

- open towards positive y-axis and has minimum value, if $a > 0$. This is called **Upward parabola**.[Figure 2 : (I)]
- open towards negative y-axis and has maximum value if $a < 0$. This is called **Downward parabola**.[Figure 2 : (II)]

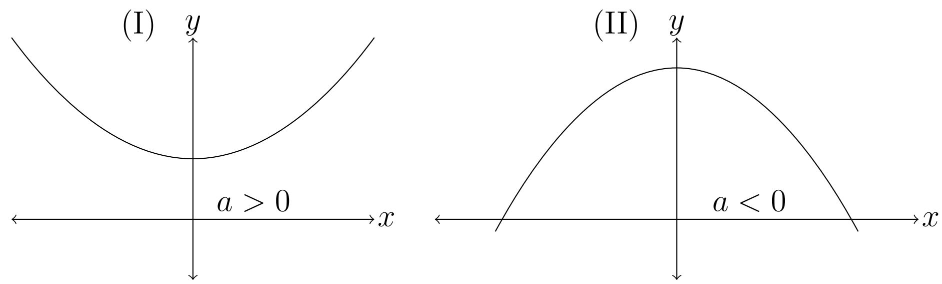


Figure 2 : Two parabolas for (I) $a > 0$ and (II) $a < 0$

3.2 Slope of a quadratic function

For the quadratic function described as $f(x) = ax^2 + bx + c$ where $a \neq 0$, the slope of f at any given point $(x, f(x))$ is $(2ax + b)$.

3.3 Quadratic equation

- If a quadratic function is set equal to a value, then the result is a quadratic equation.
- If $ax^2 + bx + c = 0$, with $a \neq 0$, and a, b, c are integers, then the quadratic equation is said to be in **standard form**.
- The solutions to a quadratic equation are called **roots** of the equation.
- One method for finding the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$ where $a \neq 0$ and a, b, c are integers, is to find **Zeros** of the quadratic function $f(x)$.

Note: Zeros of a quadratic function $f(x)$ are the x-intercepts of the curve represented by the function $f(x)$ and these are the solutions of the equation $f(x) = 0$.

3.3.1 Methods of solving a quadratic equation

(1) **Solve by factoring**

If the quadratic polynomial can be factored, the **Zero Product Property** may be used. This property states that when the product of two factors equals zero, then at least one of the factors is zero.

Steps to solve quadratic equations by factoring

- Write the equation in standard form (equal to 0).
- Factor the polynomial.
- Use the Zero Product Property to set each factor equal to zero.
- Solve each resulting linear equation

Solved examples

(i) Solve the quadratic equation $x^2 + 2x = 24$ by factoring.
Solution:
step-1: $x^2 + 2x - 24 = 0$
step-2:
$x^2 + 6x - 4x - 24 = 0 \implies x(x+6) - 4(x+6) = 0 \implies (x-4)(x+6) = 0$
step-3: $x - 4 = 0$ or $x + 6 = 0$
step-4: $x = 4$ or $x = -6$.

(ii) Solve the quadratic equation $4x^2 + 9x = 9$ by factoring.
Solution:
step-1: $4x^2 + 9x - 9 = 0$
step-2:
$4x^2 + 12x - 3x - 9 = 0 \implies 4x(x+3) - 3(x+3) = 0 \implies (4x-3)(x+3) = 0$
step-3: $4x - 3 = 0$ or $x + 3 = 0$
step-4: $x = \frac{3}{4}$ or $x = -3$.

(2) **Solve by completing the square**

Steps to solve quadratic equations by completing the square

- Transform the equation so that a perfect square is on one side and a constant is on the other side of the equation.
- Take square root on each side. REMEMBER that finding the square root of a constant yields positive and negative values.
- Solve each resulting equation. (If you are finding the square root of a negative number, then there is no real solution)

Solved examples

(i) Solve the quadratic equation $x^2 + 2x = 24$ by completing the square.
Solution:
step-1: $x^2 + 2x + 1 - 1 = 24$
step-2: $(x+1)^2 = 25 \implies x + 1 = \pm\sqrt{25}$
step-3: $x + 1 = +5$ or $x + 1 = -5$
step-4: $x = 4$ or $x = -6$.

(ii) Solve the quadratic equation $4x^2 + 9x = 9$ by completing the square.
Solution:
step-1: $4x^2 + 2.2.\frac{9}{4}x + \frac{81}{16} - \frac{81}{16} = 9$
step-2: $(2x + \frac{9}{4})^2 = \frac{225}{16} \implies 2x + \frac{9}{4} = \pm\sqrt{\frac{225}{16}}$
step-3: $2x + \frac{9}{4} = \frac{15}{4}$ or $2x + \frac{9}{4} = \frac{-15}{4}$
step-4: $x = \frac{3}{4}$ or $x = -3$.

(3) **Solve by quadratic formula**

The roots of a quadratic equation $ax^2 + bx + c = 0$ can be found directly by using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The above formula is called as quadratic formula and $b^2 - 4ac$ is called the discriminant.

Number of real roots depending on the value of the discriminant

Value of the discriminant	Number of real roots
$b^2 - 4ac > 0$	two real roots
$b^2 - 4ac = 0$	one real root
$b^2 - 4ac < 0$	No real roots

Solved examples

(i) Solve the quadratic equation $x^2 + 2x - 24 = 0$ by using the quadratic formula.
Solution:
Here, $a = 1, b = 2$, and $c = -24$. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-24)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{100}}{2} \\ &= \frac{-2 + 10}{2} \text{ or } \frac{-2 - 10}{2} \\ &= 4 \text{ or } -6 \end{aligned}$$

(ii) Solve the quadratic equation $4x^2 + 9x - 9 = 0$ by using the quadratic formula.
Solution:
Here, $a = 4, b = 9$, and $c = -9$. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-9 \pm \sqrt{9^2 - 4(4)(-9)}}{2(4)} \\ &= \frac{-9 \pm \sqrt{225}}{8} \\ &= \frac{-9 + 15}{8} \text{ or } \frac{-9 - 15}{8} \\ &= \frac{3}{4} \text{ or } -3 \end{aligned}$$

Chapter 4

4 Polynomial function

The word polynomial is derived from the word polynomen, which means poly (many), nomen (name). In this chapter we will be dealing with single variable polynomial.

4.1 Definition of polynomial function:

What is a Polynomial?
A Layman's Perspective: A polynomial is a mathematical expression that consists of numerous mathematical terms added together.
Definition: (A mathematician's Perspective) A polynomial is an algebraic expression in which the only arithmetic is addition, subtraction, multiplication and "natural" exponents of the variables.

- A polynomial function $f(x)$ of degree n is described as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots a_0 x^0, \text{ where } a_n \neq 0 \text{ and } n \in \mathbb{N}$$

  - The above expression can be treated as function from $\mathbb{R} \to \mathbb{R}$.
  - The domain of $f(x)$ is $\mathbb{R}$ and the range depends on the function $f(x)$.

Example: Identification of a polynomial:

4.2 Classification of polynomials:

4.2.1 Based on the number of variables:

Type of polynomial	**Example**
Polynomial in one variable	$5x^3 + x^2 + x$
Polynomial in two variable	$5x^3y + x^2 + xy$
Polynomial in more than two variable	$5x^3yz + x^2zy + xy$

4.2.2 Based on the degree of polynomial:

The degree of the polynomial

- The exponent on the variable in a term is called the degree of that variable in that term.
  Example: Consider a polynomial in two variables $x^3 + x^5y^4$, the degree of the variable x in the term x^5y^4 is 5.

- The degree of that term is the sum of the degrees of the variables in that term.
  Example: Consider a polynomial in two variables $x^3 + x^5y^4$, the degree of the term x^5y^4 is 9.

- The degree of the polynomial is the highest degree of any one of the terms with non-zero coefficients.
  Example: Consider $f(x) = x^3 + x^{10}$, the degree of the polynomial $f(x)$ is 10.

- The degree of zero polynomial is undefined.

- The various name of the polynomial based on the degree of the polynomial is shown below:

Degree	**Name**	**Example**
0	Constant polynomial	$5, 6, e, \pi$
1	Linear polynomial	$2x + 4, -3x$
2	Quadratic polynomial	$2x^2 + 4, -3x^2$
3	Cubic polynomial	$2x^3 + 4, -3x^3$
4	Quartic polynomial	$2x^4 + 4, -3x^4$

4.2.3 Based on the number of terms:

Name	**Explanation**	**Example**
Monomial	Polynomial with one term	$x^{10}, 6x^9, e, 5 + 4$
Binomial	Polynomial with two term	$x^{10} + 10, 6x^9 - x^2, x^2 + 3x + x$
Trinomial	Polynomial with three term	$x^{10} + x^2 + 10, 6x^9 - x^2 + 2, x^2 + x + e$

4.3 Operations on polynomial function:

4.3.1 Addition of polynomial:

When two polynomials are added, the like terms in the two polynomials are combined. We use the term "like terms" to refer to terms that have the same variable and exponent. For instance, two terms are similar only if they share the same variable.

$$\text{Let } p(x) = \sum_{k=0}^{n} a_k x^k \text{ and } q(x) = \sum_{j=0}^{m} b_j x^j. \text{ Then } p(x) + q(x) = \sum_{k=0}^{m \vee n} (a_k + b_k) x^k,$$

Here $m \vee n$ denotes whichever is maximum.
Solved example:
Q1. The sum of the two cubic polynomials $p(x) = \sum_{k=0}^{3} a_k x^k$ and $q(x) = \sum_{j=0}^{3} b_j x^j$, using the addition algorithm is
Solution:
We know that $p(x) + q(x) = \sum_{k=0}^{m \vee n} (a_k + b_k)\, x^k$, $m \vee n$ denotes whichever is maximum, so, $m \vee n = 3$. On expanding we get,
$\sum_{k=0}^{3} (a_k + b_k)\, x^k = (a_3 + b_3)\, x^3 + (a_2 + b_2)\, x^2 + (a_1 + b_1)\, x^1 + (a_0 + b_0)\, x^0$

Q2. The sum of the two cubic polynomials $p(x) = x^3 + 3x^2 + 5x - 10$ and $q(x) = 3x^3 + 5x^2 - 6x - 20$ is

Solution:

Using addition algorithm to find $p(x) + q(x)$

$\sum_{k=0}^{3}(a_k \ + \ b_k) \ x^k = (a_3 \ + \ b_3) \ x^3 \ + \ (a_2 \ + \ b_2) \ x^2 \ + \ (a_1 \ + \ b_1) \ x^1 \ + \ (a_0 \ + \ b_0) \ x^0$
$\implies (1+3) \ x^3 \ + \ (3+5) \ x^2 \ + \ (5+(-6)) \ x^1 \ + \ (-10+(-20)) \ x^0 \implies 4x^3+8x^2-x-30$

4.3.2 Subtraction of polynomial:

$$\text{Let } p(x) = \sum_{k=0}^{n} a_k x^k \text{ and } q(x) = \sum_{j=0}^{m} b_j x^j. \text{ Then } p(x) - q(x) = \sum_{k=0}^{m \,\vee\, n} (a_k - b_k) x^k,$$

Here $m \vee n$ denotes whichever is maximum.

Like in solved examples of addition of polynomial, a similar subtraction operation can be done here too.

Solved examples:

Q1. Subtract $q(x) = -3x^2 + 2x - 2$ from $p(x) = 1x^3 + 2x^2 + 8x$

$$\begin{array}{c} 1x^3 + 2x^2 + 8x + 0 \\ -(0x^3 - 3x^2 + 2x - 2) \\ \hline \end{array}$$

$$p(x) - q(x) = x^3 + (2+3)x^2 + (8-2)x + 2 = x^3 + 5x^2 + 6x + 2$$

4.3.3 Multiplication of polynomial:

$$\text{Let } p(x) = \sum_{k=0}^{n} a_k x^k \text{ and } q(x) = \sum_{j=0}^{m} b_j x^j. \text{ Then } p(x)q(x) = \sum_{k=0}^{m \,+\, n} \sum_{j=0}^{k} (a_j - b_{k-j}) x^k,$$

Solved example:

Q. Multiply the polynomials $p(x) = x^2 + x + 1$ and $q(x) = x^2 + 2x + 1$

Solution:

The product of $p(x)$ and $q(x)$ can be done using multiplication algorithm.

For different values of k, the corresponding a_k and b_k are shown the below table

k	a_k	b_k
0	1	1
1	1	2
2	1	1

And for different values of k, the corresponding coefficients and calculations are shown the below table

k	Coefficient	Calculations
0	a_0b_0	1
1	$a_1b_0 + a_0b_1$	$1 + 2 = 3$
2	$a_0b_2 + a_1b_1 + a_2b_0$	$1 + 2 + 1 = 4$
3	$a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$	$0 + 1 + 2 + 0 = 3$
4	$a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0$	$0 + 0 + 1 + 0 + 0 = 1$

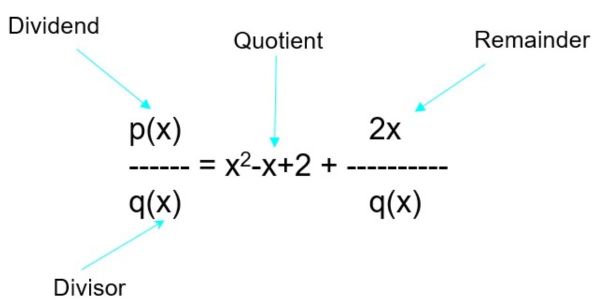
The resultant polynomial is: $p(x)q(x) = x^4 + 3x^3 + 4x^2 + 3x + 1$

4.3.4 Division of polynomial:

The division of a polynomial by a monomial, binomial, or another polynomial using various methods is known as polynomial division.
For example divide $p(x) = x^4 + 2x^2 + 3x + 2$ by $q(x) = x^2 + x + 1$.

$$
\begin{array}{r|rrrrr}
 & & & x^2 & -x & +2 \\
\hline
x^2+x+1 & x^4 & & +2x^2 & +3x & +2 \\
 & -x^4 & -x^3 & -x^2 & & \\
\hline
 & & -x^3 & +x^2 & +3x & \\
 & & x^3 & +x^2 & +x & \\
\hline
 & & & 2x^2 & +4x & +2 \\
 & & & -2x^2 & -2x & -2 \\
\hline
 & & & & 2x & \\
\end{array}
$$

Some terminologies:



One of the method to perform polynomial division is Division algorithm which is discussed below.

Division Algorithm:

Step 1. Arrange the terms in descending order of the degree and add the missing exponents with 0 as coefficient.
Step 2. Divide the first term of the dividend by the first term of the divisor and get the monomial.
Step 3. Multiply the monomial with divisor and subtract the result from the dividend.

Step 4.Check if the resultant polynomial has degree less than divisor. If true, write the remainder else go to Step 2.

Example:

Q1. If $p(x) = x^5 - 4x^4 + 3x^3 + 3x^2 - 4$ and $q(x) = (x-2)^2$, then $\frac{p(x)}{q(x)}$ is

- a $x^3 + x + 1$
- b $x^4 - x^2 - 1$
- c $\boldsymbol{x^3 - x - 1}$
- d $x^4 - x^2 + 1$

Solution:

The quotient when polynomial $p(x)$ is divided by another polynomial $q(x) = (x-2)^2 = x^2 - 4x + 4$ can be obtained by division algorithm.

$$
\begin{array}{r|l}
 & \quad x^3 \qquad\qquad - x - 1 \\
\hline
x^2 - 4x + 4 & x^5 - 4x^4 + 3x^3 + 3x^2 \qquad - 4 \\
 & -x^5 + 4x^4 - 4x^3 \\
\cline{2-2}
 & \qquad\qquad -x^3 + 3x^2 \\
 & \qquad\qquad x^3 - 4x^2 + 4x \\
\cline{2-2}
 & \qquad\qquad\qquad -x^2 + 4x - 4 \\
 & \qquad\qquad\qquad x^2 - 4x + 4 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Clearly, the answer is $\boldsymbol{x^3 - x - 1}$

Q2. What should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$?

- a $4x$
- b $\boldsymbol{4x - 3}$
- c $6x - 3$
- d $2x - 3$

Solution:

Using 4 step division algorithm, we find the remainder when $P(x)$ is divided by $2x^2 + x - 1$. If we subtract the obtained remainder from $P(x)$ then the resultant polynomial will be divisible by $2x^2 + x - 1$.

Now,

$$
\begin{array}{r|rrrrr}
\multicolumn{1}{r}{} & & & 3x^2 & +x & +1 \\ \cline{2-6}
2x^2+x-1 & 6x^4 & +5x^3 & & +4x & -4 \\
 & -6x^4 & -3x^3 & +3x^2 & & \\ \cline{2-4}
 & & 2x^3 & +3x^2 & +4x & \\
 & & -2x^3 & -x^2 & +x & \\ \cline{3-5}
 & & & 2x^2 & +5x & -4 \\
 & & & -2x^2 & -x & +1 \\ \cline{4-6}
 & & & & 4x & -3
\end{array}
$$

Therefore, when $P(x)$ is divided by $2x^2 + x - 1$, we get $4x - 3$ as the remainder.
Hence, $\boldsymbol{4x - 3}$ should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$.

Q3. By dividing a polynomial $p(x)$ with another polynomial $q(x)$ we get $h(x)$ as the quotient and $r(x)$ as the remainder.

(a) The maximum degree of $r(x)$ can be,

- ◯ $deg\ p(x)$
- ◯ $deg\ (p(x)) - 1$
- ◯ $deg\ q(x)$
- ◯ $\boldsymbol{deg\ (q(x)) - 1}$

(b) If $deg\ p(x) < deg\ q(x)$, then choose the set of correct answers:

- ◯ $\boldsymbol{h(x) = 0}$
- ◯ $deg\ h(x) = deg\ q(x)$
- ◯ $deg\ r(x) = deg\ q(x)$
- ◯ $\boldsymbol{deg\ r(x) = deg\ p(x)}$

Solution:

(a) The degree of the remainder $r(x)$ should be strictly less than the degree of the polynomial $q(x)$. So the maximum degree of $r(x)$ is $deg\ (q(x)) - 1$.

(b) If $deg\ p(x) < deg\ q(x)$, then quotient will be zero polynomial, hence $deg\ h(x) = 0$. The remainder will be $p(x)$ itself. So $deg\ r(x) = deg\ p(x)$.

Q4. If the polynomials $x^3 + ax^2 + 5x + 7$ and $x^3 + 2x^2 + 3x + 2a$ leave the same remainder when divided by $(x - 2)$, then the value of a is

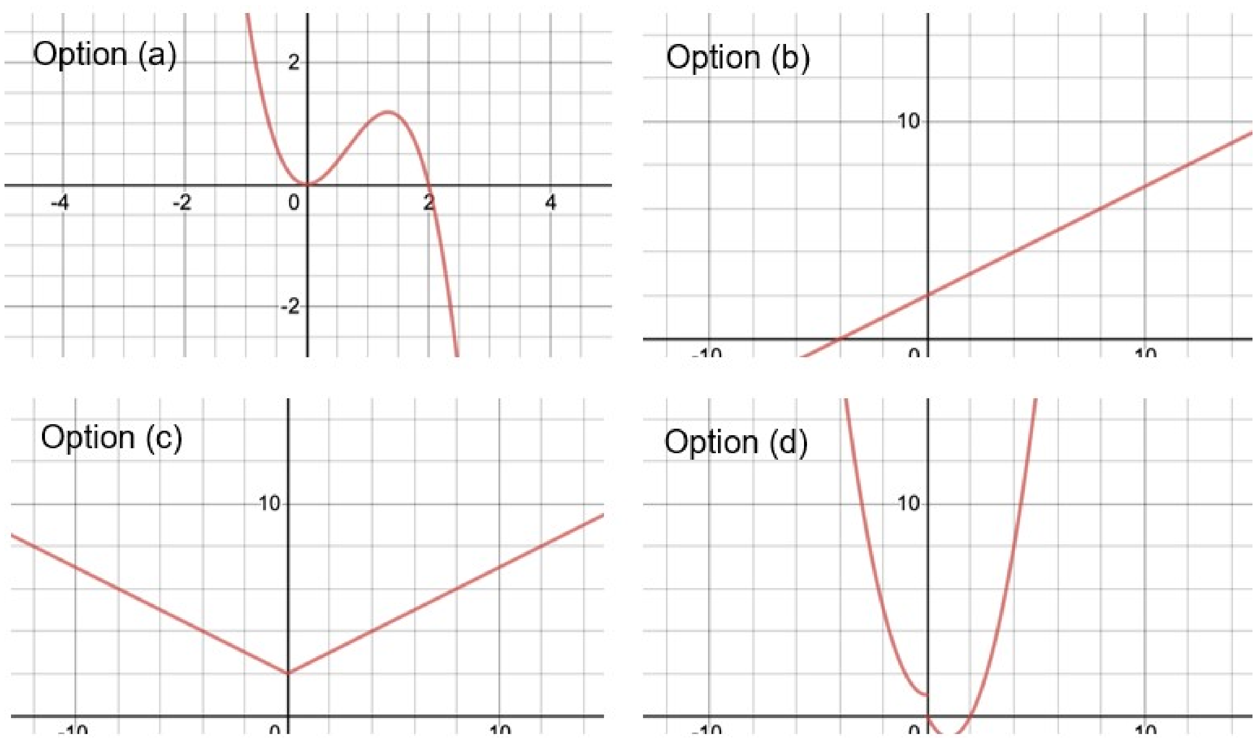
Solution: Given that both the polynomials leave same remainder when divided by $(x - 2)$.
By substituting $x = 2$ both the polynomial should have same value.
By substituting $x = 2$ in $x^3 + ax^2 + 5x + 7$, we get $8 + 4a + 10 + 7 = 4a + 25$.
By substituting $x = 2$ in $x^3 + 2x^2 + 3x + 2a$, we get $8 + 8 + 6 + 2a = 2a + 22$.

So we have,

$$4a + 25 = 2a + 22$$
$$2a = -3$$
$$a = -\frac{3}{2}$$

4.4 Characteristics of polynomial function:

Polynomial functions of second degree or higher have graphs that do not have sharp corners. That is, the graphs are smooth curves.
Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous.
Sample question: Which of the graphs given below, represent polynomial functions?



Clearly, the graph represented in option (a) and option (b) are smooth and continuous. Thus it represents polynomial a function. Also, note that graph in option (c) have sharp corner at $x = 0$ and and option (d) is not continuous thus it doesn't represents polynomial function.

4.5 Zeroes of polynomial function:

Recall: If f is a polynomial function, the values of x for which $f(x) = 0$ are called zeros of f.

- If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.
- Also, any value $x = a$ that is a zero of a polynomial function yields a factor of the polynomial, of the form $(x - a)$.
- Given the equation of a polynomial function, we can use this method to find $x-$intercepts because at the $x-$intercepts we find the input values whose output value is zero.
- For general polynomials, this can be a challenging prospect. However quadratic functions can be solved using the quadratic formula.
- The corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember. And formulas do not exist for general higher-degree polynomials.