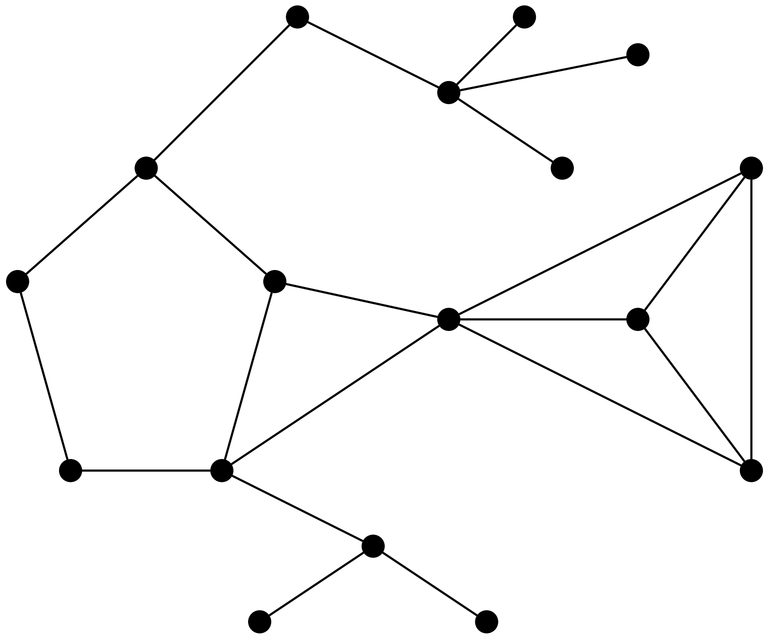
DAOUD SINIORA

Graph Theory

Lecture Notes



January 29, 2025

Contents

This set of notes is based on the textbook *"Combinatorics and Graph Theory"* by John Harris, Jeffry Hirst, and Michael Mossinghoff.

Chapter 1

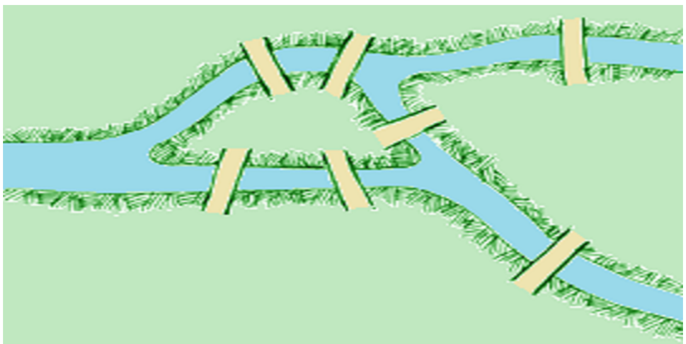
Graphs

Think of cities as points and the roads between them as lines. Hooray! You have a graph!

1.1 Bridges of Königsberg

The city once known as Königsberg was the capital of Prussia, a leading German state. During the Second World War (1939 – 1945) the city of Königsberg was heavily destroyed, and when the war was over the city became part of the Soviet Union and it was named Kaliningrad. In today's map, it is located in the Russian exclave on the Baltic Sea between Poland and Lithuania.

Through Königsberg passes the Pregolya river. In the 1700s there were seven bridges constructed across the river. The people of the city were wondering if it is ever possible to walk a tour which crosses each of these seven bridges exactly once.



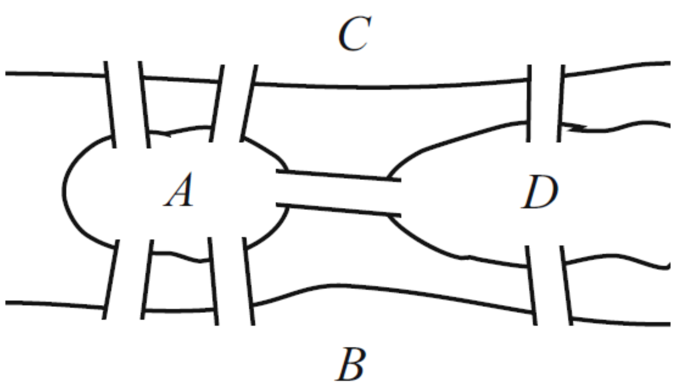
In 1736 the Swiss mathematician Leonhard Euler heard about the unsuccessful attempts by the people of Königsberg to have such a tour, and published an article showing that the problem of the *Seven Bridges of Königsberg* has no solution! Historically, the birth of Graph Theory dates back to this work of Euler.

The citizens of Königsberg were looking for a route which crosses every bridge of the city exactly once. However no one managed to find such a route.

> **Leonhard Euler** (1707 – 1783) was a Swiss mathematician and physicist. He was born in Basel, Switzerland, and spent most of his academic life in Saint Petersburg, Russia and in Berlin, Germany. Euler was one of the giants of 18^{th} century mathematics, and the most prolific in the field. He developed the concept of a function and introduced the notation $f(x)$. Euler's identity is one of the most beautiful mathematical formulas, it states that $e^{i\pi} + 1 = 0$.

In 1736, Leonhard Euler studied the Königsberg problem. He wrote the following expressing his ideas on attacking the problem.

"As far as the problem of the seven bridges of Königsberg is concerned, it can be solved by making an exhaustive list of all possible routes, and then finding whether or not any route satisfies the conditions of the problem. Because of the number of possibilities, this method of solution would be to difficult and laborious, and in other problems with more bridges it would be impossible... Hence I rejected it, and looked for another method concerned only with the problem of whether or not the specified route could be found; I considered that such a method would be much simpler."

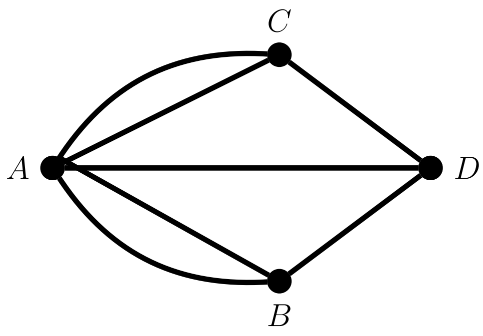


Euler used sequences of letters to describe routes in the city where no bridges are repeated. For instance, the sequence

$$ABACAD$$

represents the route which starts at A, crosses a bridge to B, then crosses the second bridge back to A, then crosses a bridge to C, then crosses the second bridge back to A, and finally crosses a bridge to D. This sequence has 6 letters and it uses 5 different bridges: AB, BA, AC, CA, AD, each exactly once.

We may use a multigraph to represent the land areas A, B, C, D together with the seven the bridges as follows.



With this way of thinking, the Königsberg Bridge Problem boils down to finding a sequence of 8 letters which uses each of the seven bridges exactly once. Euler described his approach as follows.

"The problem is therefore reduced to finding a sequence of eight letters, formed from the four letters A, B, C, D, in which the various pairs of letters occur the required number of times. Before I turn to the problem of finding such a sequence, it would be useful to find out whether or not it is even possible to arrange the letters in this way, for if it were possible to show that there is no such arrangement, then any work directed towards finding it would be wasted. I have therefore tried to find a rule which will be useful in this case, and in others, for determining whether or not such an arrangement can exist."

Euler argued that if a route which crosses each of the seven bridges exactly once does exist, then it must be represented by a sequence of 8 letters. Moreover, for this sequence to use all the three bridges connected to land area D, the letter D has to appear in this sequence exactly two times. Because if D does not appear then none of these three bridges is used, and if it appears once then at most two of these three bridges are used, and if D appears three times or more then some of these bridges are repeated. Using the same analysis, the letter B has to appear twice in the eight-letter sequence, the letter C has to appear twice as well. Finally, as land area A is connected to five bridges, the letter A must appear exactly three times in the sequence. It follows that the eight-letter sequence contains three As, two Bs, two Cs, and two Ds. In other words, the eight-letter sequence contains 9 letters, a contradiction! Euler concluded that

"It follows that such a journey cannot be undertaken across the seven bridges of Königsberg."

1.2 Graph Definition

To define what is a graph, we need sets. A *set* is a collection of objects. Such objects are called the elements or members of the set. When an object x is an element of a set S we write $x \in S$ and say "x belongs to S".

The set S of all objects x such that x has property P is expressed by writing

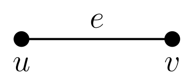
$$S = \{x \mid x \text{ has property } P\}.$$

We are now ready to present the definition of a graph.

Definition 1.2.1. (Graph)

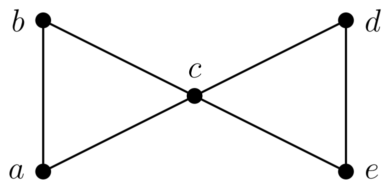
A *graph* G is a pair of sets V and E, where the elements of V are called *vertices* and the elements of E are called *edges*, and each edge is a set of exactly two distinct vertices.

We represent vertices by points, and an edge $e = \{u, v\}$ where u, v are distinct vertices is represented by the following diagram.



Given a graph G, the *order* of G is the cardinality of its vertex set, and the *size* of G is the cardinality of its edge set.

Example 1.1. The *bowtie graph* (or the *butterfly graph)* has vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \big\{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{c, e\}, \{d, e\}\big\}$. The bowtie graph has order 5 as it has five vertices, and size 6 as it has six edges.



Remark. Recall that a binary *relation* on a set V is a subset of $V \times V$. If we think of an edge $\{u, v\}$ as the two ordered pairs (u, v) and (v, u), then we may view a graph as a relation E on the set of vertices V as follows.

Definition 1.2.2. (Graph Relation)

A *graph* is a symmetric irreflexive relation E on a set V.

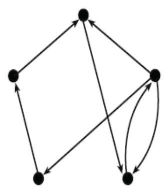
Notice that a graph is illustrated by a diagram of points and line segments where each line segment connects exactly two of the points. We may obtain new structures by modifying such an illustration in the following ways. See the figures below.

1. Directed graphs (or digraphs) are obtained when we require edges to be ordered pairs of vertices instead of sets of two vertices. So edges are replaced by arrows.

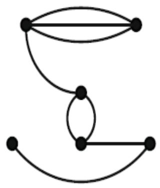
2. Multigraphs are obtained if we allow multiple edges between the same two vertices.

3. Pseudographs are obtained if we allow loops, where a loop is an edge from a vertex to itslef.

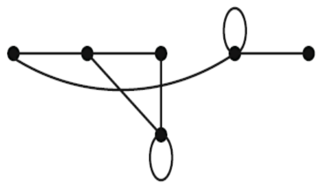
4. Hypergraphs are obtained if we allow edges to be sets of any number of vertices and not just two vertices.



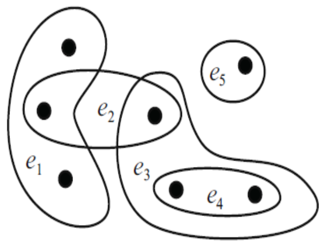
A digraph



A multigraph



A pseudograph



A hypergraph

In this course we focus on *finite simple graphs*, that is, the vertex set is finite, and moreover, the edges are undirected, and the graphs have no loops; precisely as in Definition 1.2.1 above.

One consequence of this definition is an upper bound on the number of edges. Suppose that $G = (V, E)$ is a graph where $|V| = n$; so G has n vertices. Since every edge in E is a set of exactly 2 vertices, the number of edges is at most the total number of subsets of V of cardinality 2, this is the number of 2-combinations from V. Thus,

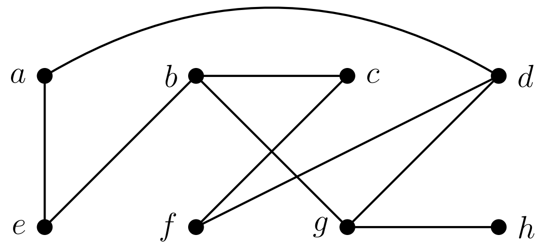
$$|E| \leq {}^{n}C_2 = \frac{n!}{(n-2)! \times 2!} = \frac{n(n-1)}{2}.$$

1.3 Handshaking Lemma

We start by introducing some important notation and terminology in graph theory. Let $G = (V, E)$ be a graph and $u, v \in V$ be two distinct vertices of G.

- The vertex set V of G is denoted by V_G or $V(G)$.
- Similarly, the edge set E of G is denoted by E_G or $E(G)$.
- We denote the set $\{u, v\}$ by uv.
- When the set uv is an edge (that is, $uv \in E$), we say that u and v are *adjacent* or *neighbours*.
- When e is an edge and $e = uv$, we say that u and v are the *end vertices* of e.
- If $uv \notin E$, then we say that u and v are *nonadjacent*.
- If v is an end vertex of an edge e, then we say that e is *incident* with v.
- The *neighborhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v.
$$N(v) = \{w \in V \mid vw \in E\}.$$
- The closed neighborhood of v is $N[v] = \{v\} \cup N(v)$.
- The neighborhood of a set S of vertices is
$$N(S) = \bigcup_{v \in S} N(v).$$
- The closed neighborhood of a set S of vertices is $N[S] = S \cup N(S)$.
- The *degree* of v is the number of edges incident with v,
$$\deg(v) = |\{e \in E \mid v \in e\}| = |N(v)|.$$
- The maximum degree of a graph G is denoted by $\Delta(G)$.
$$\Delta(G) = \max\{\deg(v) \mid v \in V_G\}.$$
- The minimum degree of a graph G is denoted by $\delta(G)$.
$$\delta(G) = \min\{\deg(v) \mid v \in V_G\}.$$
- The *degree sequence* of a graph of order n is the sequence of length n listing all the vertex degrees in decreasing order.

Example 1.2. Consider the following graph G.



The order of G is 8, and its size is 9. Vertices a and d are adjacent, while b and f are nonadjacent. The degree sequence of G is $3, 3, 3, 2, 2, 2, 2, 1$.

$$\begin{aligned} \Delta(G) &= 3, & N(a) &= \{e, d\}, & N(b) &= \{e, g, c\}, \\ \delta(G) &= 1, & N[a] &= \{a, e, d\}, & N(h) &= \{g\}, \\ N(\{a, b\}) &= \{e, d, g, c\}, & N[\{a, b\}] &= \{a, b, e, d, g, c\}, & N(\{b, c\}) &= N[\{b, c\}]. \end{aligned}$$

The next lemma is our first result on graphs.

Lemma 1.3.1. (Handshaking Lemma)

Let $G = (V, E)$ be a finite graph. Then,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Consequently, G has an even number of vertices with odd degree.

Proof. When we sum the degrees of all the vertices, every single edge is counted exactly twice; it is counted once for each of its end vertices. Therefore, the sum of the degrees of all the vertices is twice the number of edges. To be more rigorous, notice that the degree of v is equal to the number of all pairs (v, b) where b is adjacent to v.

$$\begin{aligned} S = \sum_{v \in V} \deg(v) &= \sum_{v \in V} \big|\{(v, b) \mid vb \in E\}\big| \\ &= \Big|\bigcup_{v \in V} \{(v, b) \mid vb \in E\}\Big| \\ &= \Big|\bigcup_{e \in E} \{(v, b) \mid vb = e\}\Big| \\ &= \sum_{e \in E} \big|\{(v, b) \mid vb = e\}\big| \\ &= \sum_{e \in E} 2 = 2|E|. \end{aligned}$$

For the second part of the lemma, let S_0 be the sum of all even degrees and S_1 be the sum of all odd degrees. Therefore, $S = S_0 + S_1$. We know that S_0 is even and from above we know that that S is also even. It follows that S_1 is even as well. Since the sum of an odd number of odd integers is odd, and as S_1 is even, there must be even number of odd degrees. ■

Lemma 1.3.2.

In any graph with more than one vertex, there are two vertices with the same degree.

Proof. If there are two or more vertices of degree 0, we are done. If there is exactly one vertex of degree 0 then just ignore it (look at the remaining part of the graph). So we may assume that we have a graph G of n vertices, none of them has degree 0. Thus, for every vertex $v \in V_G$ we have that

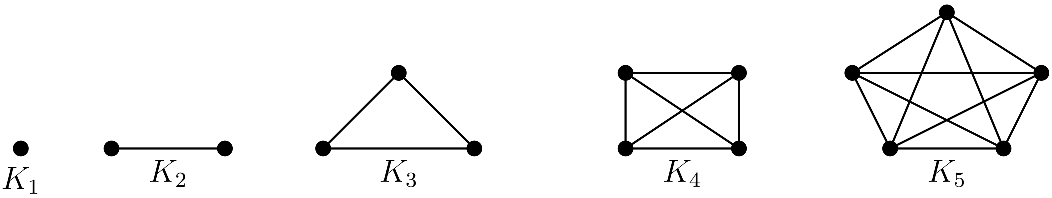
$$1 \leq \deg(v) \leq n - 1.$$

Now, think of the n vertices as pigeons, and the $n-1$ possible degrees as pigeonholes. Since we have more pigeons than pigeonholes, at least two vertices must have the same degree by the pigeonhole principle. ■

1.4 Special Graphs

♠ Complete Graphs

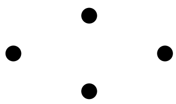
A *complete graph* K_n of order n is the graph with n vertices where every two distinct vertices are adjacent.



Clearly, the degree of any vertex in K_n is $n-1$.

♠ Empty Graphs

The *empty graph* E_n of order n is the graph with n vertices and no edges (the edge set is the empty set).

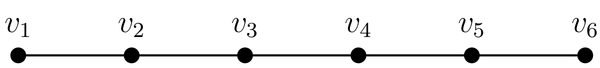


E_4

♠ Paths

A *path* P_n is a graph on n vertices $v_1, v_2, \ldots, v_n$ whose edge set is

$$E(P_n) = \{v_i v_{i+1} \mid 1 \le i \le n-1\}.$$

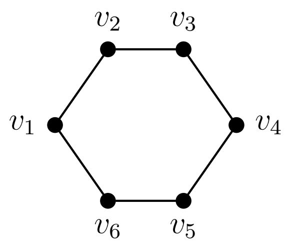


The path P_6

♠ Cycles

A *cycle* C_n is a graph on n vertices $v_1, v_2, \ldots, v_n$ whose edge set is

$$E(C_n) = \{v_i v_{i+1} \mid 1 \le i \le n-1\} \cup \{v_n v_1\}.$$



The cycle C_6

♠ Complements

Given a graph G, its complement $\overline{G}$ is the graph whose vertex set is the same as that of G, and whose edge set is the set of all sets uv such that uv is *not* an edge of G. So $V_{\overline{G}} = V_G$ and $E_{\overline{G}} = \{uv \mid u, v \in V_G, u \neq v, \text{ and } uv \notin E_G\}$.

Below is a graph G and its complement.

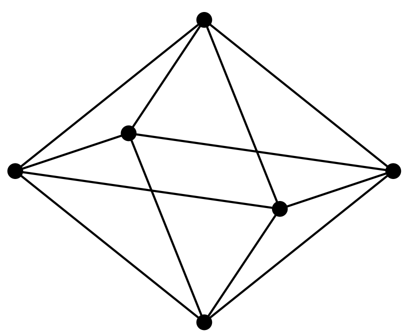
♠ Regular Graphs

A graph G is *regular* if each vertex has the same degree. If each vertex has degree r then we say the graph is *regular of degree* r, or we say r*-regular*.

Note that a graph G is regular if and only if $\delta(G) = \Delta(G)$. The complete graph K_n is regular of degree $n-1$, and the empty graph is regular of degree 0. By the Handshaking Lemma, we have the following fact.

Fact. An r-regular graph of order n has exactly $\frac{1}{2}nr$ edges.

Example 1.3. The following is a regular graph of degree 4. It has 6 vertices and 12 edges.



1.5 Subgraphs

There are two notions of a graph being part of another graph.

Definition 1.5.1. (Subgraph)

A graph H is a *subgraph* of a graph G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

When H is a subgraph of G we write $H \subseteq G$ and say G contains H. When the vertices are unlabeled, we say that $H \subseteq G$ if the vertices could be labeled in such a way that $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

Definition 1.5.2. (Induced Subgraph)

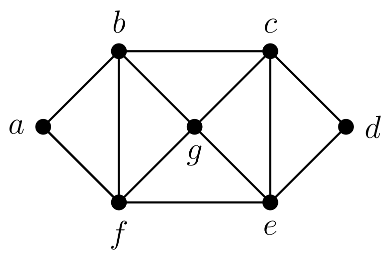
A graph H is an *induced subgraph* of a graph G if $V_H \subseteq V_G$ and

$$E_H = \{uv \in E_G \mid u, v \in V_H\}.$$

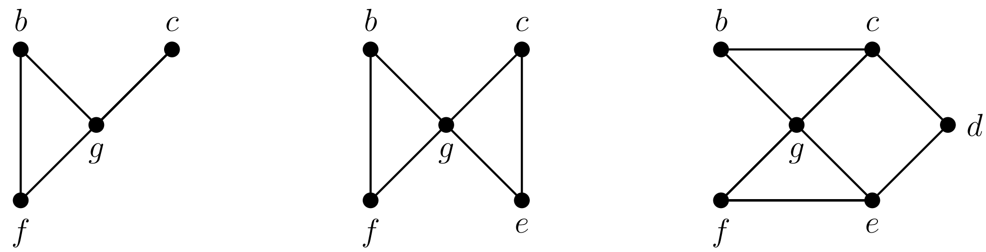
In other words, H is an induced subgraph of G if the vertices of H are also vertices of G, and the edges of H are all those edges of G whose both of their end vertices are in H. Clearly, every induced subgraph is a subgraph, but not every subgraph is induced.

Given any graph G and a subset $S \subseteq V_G$, there is only one induced subgraph of G with vertex set S. Namely the graph with vertex set S which has all edges of G whose end vertices are in S. We call this induced subgraph *the subgraph of* G *induced on* S and denote it by $\langle S \rangle$. Note that $\langle S \rangle$ is obtained from G by deleting all vertices not in S.

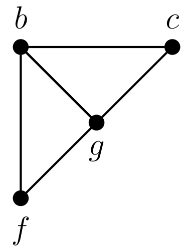
Example 1.4. Consider the following graph G.

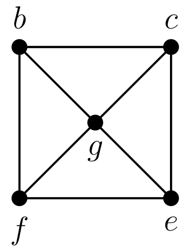


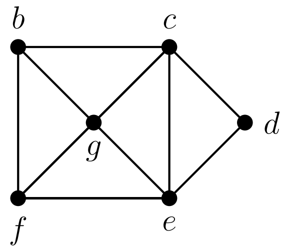
The following graphs are subgraphs of G but *not* induced subgraphs.



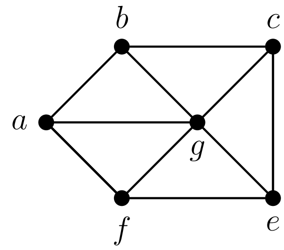
The following graphs are induced subgraphs of G.







The graph below is *not* a subgraph of G since its edge set is not a subset of $E(G)$.



1.6 Walks in Graphs

Definition 1.6.1. (Walk)

A *walk* in a graph is a sequence of vertices $(v_1, v_2, v_3, \ldots, v_k)$ such that any two consecutive vertices v_i and v_{i+1} are adjacent.

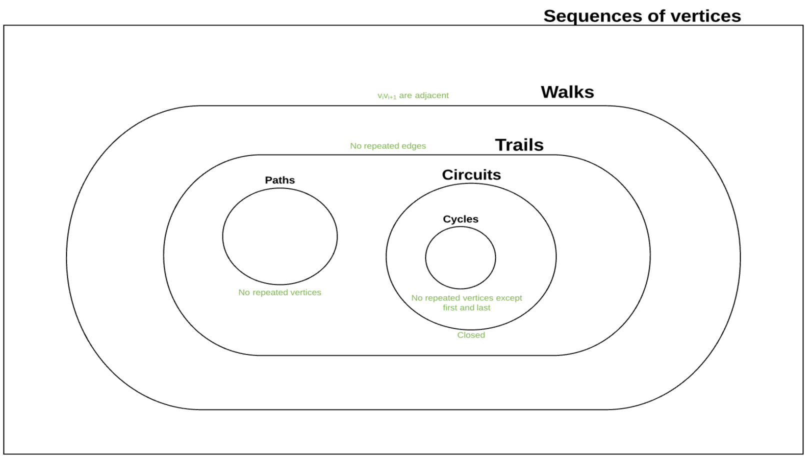
Such a walk is called a $v_1 v_k$-walk, and the vertices v_1 and v_k are called the end vertices of the walk. Note that in a walk the vertices are not necessarily distinct. A *closed walk* is walk which starts and ends at the same vertex. The *length* of a walk is the number of the edges in the walk, counting repetitions. So the length of the walk $(v_1, v_2, \ldots, v_k)$ is $k - 1$.

Definition 1.6.2. (Path; Cycle; Trail; and Circuit)

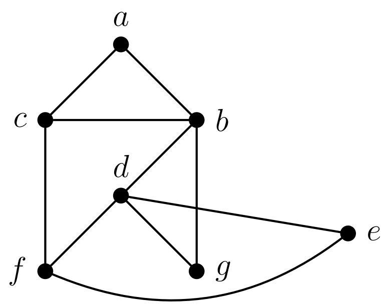
The following are special kinds of walks.

- A *path* is a walk whose vertices are distinct.
- A *cycle* is a closed walk $(v_1, v_2, \ldots, v_k, v_1)$ where $v_1, v_2, \ldots, v_k$ are all distinct and $k \geq 3$.
- A *trail* is a walk whose edges are distinct.
- A *circuit* is a closed trail.

Remark. Every path is a trail, but not every trail is a path. And every cycle is a circuit, and every circuit is a trail.



Example 1.5. Consider the following graph.



- The sequence (a, c, f, c, b, d) is an ad-walk of length 5.
- The sequence (a, b, d) is an ad-walk of length 2.
- The sequence (b, a, c, b, d) is a trail of length 4.
- The sequence (f, d, b, a, c, f) is a cycle of length 5.
- The sequence (d, g, b, a, c, f, e) is a path of length 6.
- The sequence (g, d, b, c, a, b, g) is a circuit of length 6.
- The sequence (a, b, c, a) is a cycle of length 3.
- The sequence (e, d, b, a, c, f, e) is a cycle of length 6.
- The sequence (f, c, a, b, e) is *not* a walk.

We say a walk W *contains* a walk V if the walk V can be obtained by deleting some vertices from W, and all the edges in V are also edges in W. For instance, in the graph above the walk (f, d, b, a, c, b, g) contains the walk (f, d, b, g).

Lemma 1.6.3.

If G is a graph with $\delta(G) \geq 2$, then G contains a cycle as a subgraph.

Proof. Let G be a graph with minimum degree 2. Let P be a path in G with maximum length. Say P is $v_1, v_2, \ldots, v_k$. Since $\delta(G) \geq 2$, we have that $k \geq 3$. Moreover, $\deg(v_1) \geq 2$, and so the first vertex v_1 has at least two neighbours. One of the neighbours is v_2. Let u be a neighbour of v_1 different from v_2. If u is distinct from all the vertices on P, then we can extend P to a longer path $u, v_1, v_2 \ldots, v_k$ but this contradicts that P is a longest path. Thus, $u = v_i$ for some $3 \leq i \leq k$, meaning that $v_1, v_2, \ldots, v_i, v_1$ is a cycle in G. ■

Corollary 1.6.4.

If a graph G has no cycles, then $\delta(G) \leq 1$.

Theorem 1.6.5.

Every uv-walk in a graph contains a uv-path.

Proof. We will prove the theorem by induction on the length of the walk. We will prove that for every $k \geq 1$, every uv-walk of length k contains some uv-path.

Base case. Any uv-walk of length one is itself a uv-path as vertices u, v must be adjacent and so distinct.

Induction step. Suppose that every uv-walk of length less than k contains a uv-path. Let W be a uv-walk of length k, say W is

$$u = w_0, w_1, w_2, \ldots, w_k = v.$$

If all the vertices in W are distinct, then W itself is a path and we are done. Otherwise, there are i, j with $i < j$ such that $w_i = w_j$. More precisely, W is

$$u = w_0, w_1, w_2, \ldots, w_i, w_{i+1} \ldots, w_j, w_{j+1} \ldots, w_k = v.$$

Since $w_i = w_j$ and $w_j w_{j+1}$ is an edge, then $w_i w_{j+1}$ is also an edge, and therefore the sequence V obtained by deleting $w_{i+1}, \ldots, w_j$ from W, that is, the sequence

$$u = w_0, w_1, w_2, \ldots, w_i, w_{j+1} \ldots, w_k = v$$

is a uv-walk of length strictly less than k. By the induction hypothesis, V contains a uv-path. This means that W too contains a uv-path, and the proof is complete. ■

1.7 Connectivity

Definition 1.7.1. (Connected Graph)

A graph is *connected* if every pair of vertices can be joined with a path.

Informally, G is connected if one can walk from any vertex in the graph to any other vertex. Another way to see it is that one can lift up the entire graph by grabbing just one vertex. A graph which is not connected is said to be *disconnected*. So a graph is disconnected if there are two vertices where there does not exist any paths between them. We call each maximal connected piece of a graph a *connected component*.

Definition 1.7.2. (Connected Component)

A *connected component* of a graph G is a connected induced subgraph Q such that there is no path in G from any vertex in Q to any vertex not in Q.

Note that a graph is connected if and only if it has exactly one connected component.

Remark. Define a relation on the vertex set by stating that a vertex u is related to a vertex v if there is a path from u to v. Then this gives an equivalence relation (check it!) whose equivalence classes are the connected components.

Let us introduce two important operations on graphs.

Definition 1.7.3. (Vertex Deletion)

Given a graph G, a vertex $v \in V_G$, and $S \subseteq V_G$.

(i) The graph $G - v$ is the graph obtained from G by removing the vertex v together with all the edges incident with v from G.

(ii) The graph $G - S$ is the graph obtained by removing every vertex in S and all its incident edges.

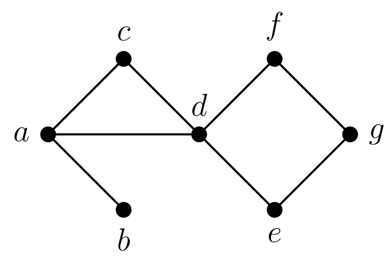
Definition 1.7.4. (Edge Deletion)

Given a graph G, an edge $e \in E_G$, and $T \subseteq E_G$.

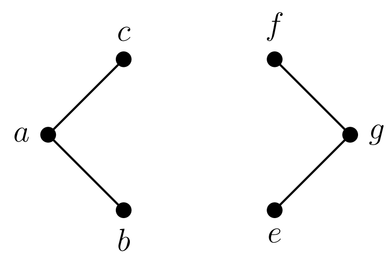
(i) The graph $G - e$ is the graph obtained from G by removing only the edge e from E_G. The end vertices of e stay in G.

(ii) The graph $G - T$ is the graph obtained by deleting all the edges in T from G.

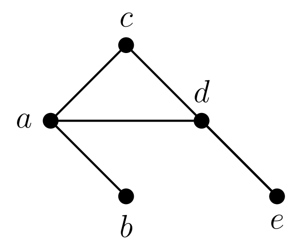
Example 1.6. Consider the graph G below.



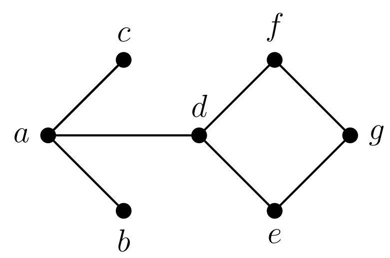
(i) The graph $G - d$ is given by the following diagram.



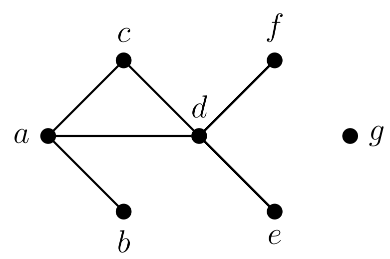
(ii) The graph $G - \{f, g\}$ is given by the following diagram.



(iii) The graph $G - cd$ is given by the following diagram.



(iv) The graph $G - \{eg, fg\}$ is given by the following diagram.

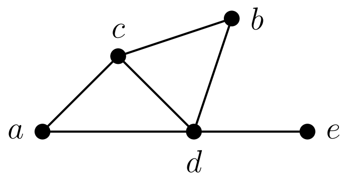


Definition 1.7.5. (Cut Vertex and Bridge)

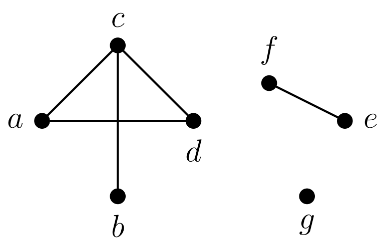
Let G be a graph.

- A vertex v is a *cut vertex* if $G - v$ has more connected components than G.
- An edge e is a *bridge* if $G - e$ has more connected components than G.

Example 1.7. The graph below is connected. It has one connected component. The vertex d is a cut vertex, while the vertex b is not. The edge de is the only bridge.



Example 1.8. The graph below is disconnected. It has 3 connected components, namely $\{a, b, c, d\}$, $\{f, e\}$, and $\{g\}$. The vertex c is a cut vertex, while the vertex f is not. The edges bc and fe are bridges.



Lemma 1.7.6.

A vertex v in a connected graph G is a cut vertex of G if and only if there are two vertices u and w distinct from v such that every uw-path passes through v.

A connected graph containing no cut vertices is called *nonseparable*. For example, all cycle graphs and complete graphs are nonseparable. The graphs K_2 and K_3 are the only nonseparable graphs of order 3 or less.

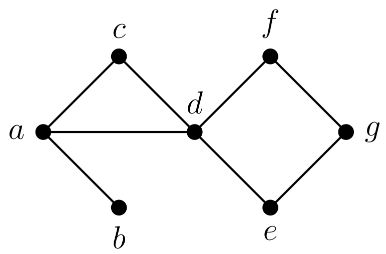
Lemma 1.7.7.

If G is nonseparable of order 3 or more, then $\delta(G) \geq 2$. Moreover, every vertex lies on a cycle.

Definition 1.7.8. (Cut Set)

A *cut set* is a proper subset $S \subseteq V_G$ such that $G - S$ is disconnected.

Example 1.9. The graph below is connected. The vertex d is a cut vertex as the number of connected components will increase from 1 to 2 after deleting d. The edge ab is the only bridge. The set $\{e, f\}$ is a cut set, as $G - \{e, f\}$ is disconnected.



Remark. A vertex v in a connected graph G is a cut vertex if and only if the singleton $\{v\}$ is a cut set.

Recall that a graph is complete if every two vertices are adjacent. Clearly, a complete graph is connected, and moreover, it has no cut sets! On the other hand, a non-complete graph must have a cut set.

Definition 1.7.9. (Connectivity)

The *connectivity* $\kappa(G)$ of a non-complete graph G is the minimum cardinality of a cut set of G. When G is a complete graph of order n, we define $\kappa(G) = n - 1$.

Note that $\kappa(G)$ is the minimum number of vertices that have to be deleted from G to produce a disconnected graph. Thus, $\kappa(G) = 0$ if and only if G is disconnected. When G is a connected non-complete graph, then $1 \leq \kappa(G) \leq n - 2$.

We may view κ as a function with domain the set of all finite graphs and range the set of all natural numbers $\mathbb{N}$.

Remark. The larger the value of $\kappa(G)$, the more connected G is. Meaning that for larger κ, we need to work harder by deleting more vertices to make G disconnected.

Definition 1.7.10. (k-connected Graphs)

Let $k \geq 1$. We say a graph G is *k-connected* if $\kappa(G) \geq k$.

A graph being k-connected means that every cut set has cardinality at least k. Thus one needs to delete k vertices or more to make the graph disconnected. In other words, deleting strictly less than k vertices will *not* make the graph disconnected. Here are several facts that follow from the discussion above.

Lemma 1.7.11.

Let G be a graph.

(i) G *is connected if and only if* $\kappa(G) \geq 1$.

(ii) G *is 1-connected if and only if* G *is connected.*

(iii) G *is 2-connected if and only if* G *is connected and has no cut vertices.*

(iv) G *is 2-connected if and only if* G *is nonseparable.*

(v) If G *is* 2*-connected, then* G *contains at least one cycle.*

(vi) If G *is* k*-connected and* $j \leq k$*, then* G *is* j*-connected.*

(vii) $\kappa(G) \leq \delta(G)$.

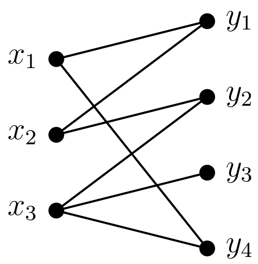
1.8 Bipartite Graphs

Definition 1.8.1. (Bipartite Graph)

A graph G is *bipartite* if there exist two disjoint subsets X and Y of vertices such that $V_G = X \cup Y$ and where every edge in G has one end vertex in X and the other in Y. The sets X and Y are called *partite sets*.

The cycle C_4 is bipartite. The cycle C_5 is not bipartite since it is impossible to partition its vertex set into two subsets where each edge of C_5 has one end vertex in the first set and the other end vertex in the second set. The empty graph E_n is bipartite. Given a bipartite graph with partite sets X and Y, draw the vertices of X on the left, say, and the vertices of Y on the right, then there will be no edge which lie entirely on the left side or entirely on the right side.

Example 1.10. Let G be a graph with vertex set $V = \{y_1, x_1, y_2, x_2, y_3, x_3, y_4\}$ and edge set $E = \{x_1y_1, x_1y_4, x_2y_1, x_2y_2, x_3y_2, x_3y_3, x_3y_4\}$. One can see that G is a bipartite graph with partite sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.



Lemma 1.8.2.

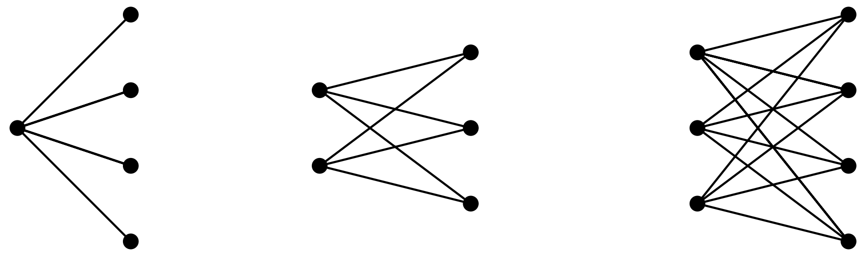
A graph G is bipartite if and only if the vertices of G can be coloured using two colours such that the end vertices of each edge are coloured differently.

Definition 1.8.3. (Complete Bipartite Graph)

The *complete bipartite graph* $K_{m,n}$ is the bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ and edge set

$$E = \{xy \mid x \in X \text{ and } y \in Y\}.$$

Example 1.11. The complete bipartite graphs $K_{1,4}$, $K_{2,3}$, and $K_{3,4}$, respectively.



How could we know if a given graph is bipartite or not. The following theorem gives a useful and interesting characterisation of bipartite graphs. Recall that when we say that G contains a cycle we mean G contains a cycle as a subgraph.

Theorem 1.8.4.

A graph G is bipartite if and only if G contains no odd cycles.

Proof. For the forward direction, suppose that G is bipartite with partite sets X and Y. For the sake of contradiction, suppose G contains some odd cycle C_k where k is odd. Say the vertices of the cycle are $\{v_1, v_2, v_3, \ldots, v_k\}$. Without loss of generality, let $v_1 \in X$. As v_1 and v_2 are adjacent, it follows that $v_2 \in Y$, and hence $v_3 \in X$, and hence $x_4 \in Y$, and so on. Therefore, $v_i \in X$ for odd i, and $v_i \in Y$ for even i. Since k is odd, we have that $v_k \in X$. But v_1 and v_k are adjacent and they both belong to X, this contradicts that G is bipartite.

For the reverse direction, suppose G contains no odd cycles. We will show that G is bipartite. Without loss of generality, we may assume that G is connected, for otherwise, the argument below is applied to each of its connected components separately.

Fix some vertex $v \in V$. Define the sets X, Y as follows.

$$X = \{x \in V_G \mid \text{the shortest length of } vx\text{-paths is even}\},$$

$$Y = \{y \in V_G \mid \text{the shortest length of } vy\text{-paths is odd}\}.$$

We will show that X and Y are partite sets of G, that is, no two vertices in X are adjacent, and similarly for Y. Note that $v \in X$ since the shortest path from v to itself has length 0 which is even. Let $x_1, x_2 \in X$ and suppose, for contradiction, that x_1 and x_2 are adjacent. Clearly, $v \neq x_1$, for if $v = x_1$ then the shortest vx_2-path has length 1 contradicting that $x_2 \in X$. Similarly, $v \neq x_2$.

Let P_1 be a shortest vx_1-path, say P_1 is

$$v = v_0, v_1, v_2, \ldots, v_{2k} = x_1.$$

Let P_2 be a shortest vx_2-path, say

$$v = w_0, w_1, w_2, \ldots, w_{2l} = x_2.$$

Suppose that $v_i = w_j$ for some vertex v_i in P_1 and some vertex w_j in P_2. If $i < j$, then $v = v_0, v_1, \ldots, v_i, w_{j+1}, \ldots, w_{2l} = x_2$ is a vx_2-path shorter than P_2, this cannot happen as P_2 is of shortest length. Similarly $j < i$ is impossible. Therefore, if $v_i = w_j$, then it must be that $i = j$. Now, let i be the largest such that $v_i = w_i$. Since x_1 is adjacent to x_2, that is, v_{2k} is adjacent to w_{2l} we have the following cycle C:

$$v_i, v_{i+1}, v_{i+2}, \ldots, v_{2k}, w_{2l}, v_{2l-1}, \ldots, w_{i+2}, w_{i+1}, w_i.$$

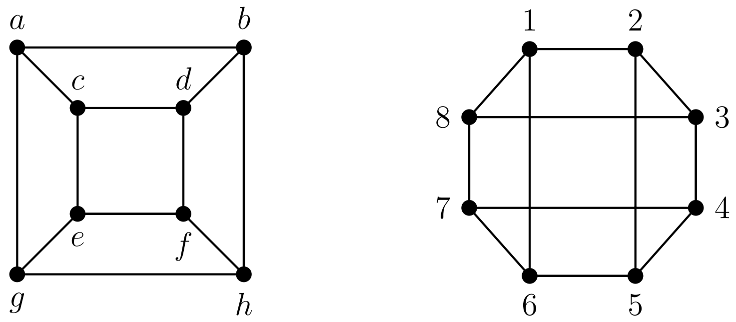
The length of the cycle C is equal to the length of the $v_i v_{2k}$-path plus the edge $v_{2k}w_{2l}$ plus the length of the $w_{2l}w_i$-path, that is the length of C is

$$(2k - i) + 1 + (2l - i) = 2k + 2l - 2i + 1 = 2(k + l - i) + 1.$$

This means that the cycle C has odd length! A contradiction, as G has no odd cycles. Therefore, x_1 and x_2 are nonadjacent. Similarly, no two vertices y_1, y_2 in Y can be adjacent to each other. This shows that X, Y are partite sets showing that G is a partite graph. ■

1.9 Graph Isomorphism

Sometimes we look at two graphs and we see them the same graph. For example, K_3 and C_3 are the same graph, they both look like a triangle. Or sometimes we just need to redraw the first graph and relabel its vertices, in order for it to look like the second graph. For example, consider the two graphs below. In the first graph move vertices c, e to the left of the edge ag and move vertices d, f to the right of the edge bh, and relabel the vertices to obtain the second graph.



Let us formalise the idea of two graphs being the same.

Definition 1.9.1. (Graph Isomorphism)

We say a graph G is *isomorphic* to a graph H if there exists a bijection $\phi : V_G \to V_H$ such that for any $u, v \in V_G$ we have that

$$\{u, v\} \in E_G \text{ if and only if } \{\phi(u), \phi(v)\} \in E_H.$$

The map ϕ above is called an *isomorphism* from graph G to graph H.

So G is isomorphic to H if one can find a bijection from the vertices of G to the vertices of H that preserves adjacencies, that is, the image of an edge in G is an edge in H, and the image of a non-edge in G is a non-edge in H.

When G is isomorphic to H we write $G \cong H$, or say "G is H" and write $G = H$. Note that if G is isomorphic to H, then H is also isomorphic to G.

In the example above, an isomorphism ϕ from the graph on the left to the one on the right is given by the bijection

$$\phi : \{a, b, c, d, e, f, g, h\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$$

where

$$\phi(a) = 1,\ \phi(b) = 2,\ \phi(c) = 8,\ \phi(d) = 3,\ \phi(e) = 7,\ \phi(f) = 4,\ \phi(g) = 6,\ \phi(h) = 5.$$

Definition 1.9.2. (Automorphism)

An isomorphism from a graph to itself is called an *automorphism*.

For a graph G, the set of all automorphisms of G forms a group under function composition denoted by $\mathrm{Aut}(G)$. The lemma below states some facts on isomorphic graphs.

Lemma 1.9.3.

Let the graphs G and H be isomorphic. Then the following are true.

(i) $|V_G| = |V_H|$.

(ii) $|E_G| = |E_H|$.

(iii) The degree sequence of G is the same as that of H.

(iv) G is connected if and only if H is connected.

(v) $\overline{G} \cong \overline{H}$.

It is possible for two graphs to have the same number of vertices and edges, and the same degree sequences, but *not* being isomorphic. The nonisomorphic graphs G, H below are such an example.

In general, it is a hard problem to determine whether two graphs are isomorphic or not. A common strategy is to start with comparing the number of vertices and edges, degree sequences, subgraphs, degrees of adjacent vertices, and complements.

Chapter 2

Distance in Graphs

The distance between us is the number of social connections from you to me.

On the real number line $\mathbb{R}$, there is a notion of distance between real numbers. For instance, the distance $d(x, y)$ between numbers x, y is defined to be $d(x, y) = |x-y|$. For example, $d(3, 7) = 4$ and $d(-7, 5) = 12$ and $d(6, 6) = 0$. In this setting, the distance is a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$. The distance function d is a called a *metric* and the pair $(\mathbb{R}, d)$ is called a *metric space*. We may also talk about the distance between two points in the Cartesian plane $\mathbb{R}^2$. In mathematics, a metric function satisfies the following properties for every x, y, z in the set.

(i) $d(x, y) \geq 0$;

(ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

In this chapter, we aim to define the concept of distance in graphs, and study its properties and applications. Once it is introduced, verify for yourself that the distance function in graphs is a metric, that is, it satisfies the properties above.

2.1 Eccentricity

A person is *"eccentric"* if they behave in an unusual or strange way. Geometrically, an object is eccentric if it is located elsewhere than at the center.

We will use paths to define distance in graphs. Recall that a path is a walk of distinct vertices, and the length of a path is the number of edges in the path.

Definition 2.1.1. (Graph Distance)

Let G be a connected graph, and let $u, v \in V_G$. The *distance* $d(u, v)$ from vertex u to vertex v is the length of a shortest uv-path in G.

$$d(u, v) = \min \left\{ \text{length}(P) \mid P \text{ is a } uv\text{-path} \right\}$$

A uv-path P of shortest length is called *geodesic*. So the length of any uv-geodesic is equal to $d(u, v)$. Given a disconnected graph, two vertices x, y in different connected components cannot be joined by a path. We say that the distance between them is infinity and write $d(x, y) = \infty$.

Exercise. Show that the graph distance d as defined above is a metric. Thus, for any graph G the pair (G, d) is a metric space.

Using the distance in a connected graph, we define the following concepts.

Definition 2.1.2. (Eccentricity)

Let G be a connected graph.

(i) The *eccentricity* of a vertex v, denoted $\text{ecc}(v)$, is the greatest distance from v to any vertex. That is,

$$\text{ecc}(v) = \max\{d(v, x) \mid x \in V_G\}.$$

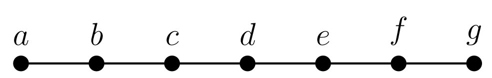
(ii) The *radius* of G, denoted $\text{rad}(G)$, is the smallest eccentricity in G.

(iii) The *diameter* of G, denoted $\text{diam}(G)$, is the largest eccentricity in G.

(iv) The *center* of G is the set $\{v \in V_G \mid \text{ecc}(v) = \text{rad}(G)\}$.

(v) The *periphery* of G is the set $\{v \in V_G \mid \text{ecc}(v) = \text{diam}(G)\}$.

Example 2.1. Consider the path P_7.

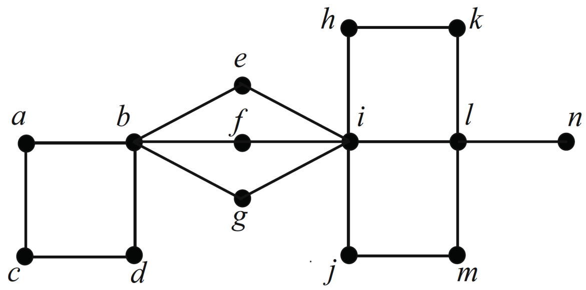


- $d(b, e) = 3, \ d(a, g) = 6, \ d(b, f) = 4, \ d(d, g) = 3.$
- $\text{ecc}(a) = 6, \ \text{ecc}(b) = 5, \ \text{ecc}(c) = 4, \ \text{ecc}(d) = 3, \ \text{ecc}(e) = 4, \ \text{ecc}(f) = 5, \ \text{ecc}(g) = 6.$
- $\text{rad}(P_7) = 3$ and $\text{diam}(P_7) = 6$.
- The center of P_7 is $\{d\}$.
- The periphery of P_7 is $\{a, g\}$.

Example 2.2. Find the radius and diameter of the complete graph K_5 and the cycle C_7.

- $\operatorname{rad}(K_5) = \operatorname{diam}(K_5) = 1$.
- $\operatorname{rad}(C_7) = \operatorname{diam}(C_7) = 3$.

Example 2.3. Consider the graph G below.



- $d(b,k) = 4,\ d(c,m) = 6,\ d(g,g) = 0,\ d(h,i) = 1$.
- $\operatorname{ecc}(a) = 5$ since the farthest vertices from a are vertices k, m, n and they are at a distance of 5 from a.
- $\operatorname{ecc}(e) = \operatorname{ecc}(f) = \operatorname{ecc}(g) = 3$ (smallest eccentricity).
- $\operatorname{ecc}(c) = \operatorname{ecc}(k) = \operatorname{ecc}(m) = \operatorname{ecc}(n) = 6$ (largest eccentricity).
- $\operatorname{rad}(G) = 3$ and $\operatorname{diam}(G) = 6$.
- The center of G is $\{e, f, g\}$.
- The periphery of G is $\{c, k, m, n\}$.

What is the relationship between the radius of a graph and its diameter?

Lemma 2.1.3.

Let G be a connected graph. Then

$$\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2\operatorname{rad}(G).$$

Proof. By definition, $\operatorname{rad}(G) \leq \operatorname{diam}(G)$. To establish the second inequality, choose two vertices u, v such that $d(u,v) = \operatorname{diam}(G)$. Next, choose some vertex c in the center of G. So $\operatorname{ecc}(c) = \operatorname{rad}(G)$. By the triangle inequality we have that

$$\operatorname{diam}(G) = d(u,v) \leq d(u,c) + d(c,v) \leq \operatorname{rad}(G) + \operatorname{rad}(G) = 2\operatorname{rad}(G).$$

■

Theorem 2.1.4. (Hedetneimi)

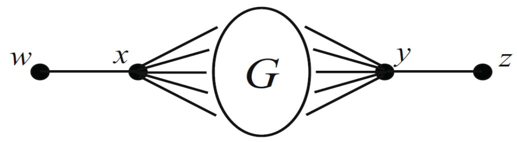
Any graph is isomorphic to the center of some graph.

Proof. Choose any graph G you like. We now construct a new graph H by adding to the vertices of G four new vertices w, x, y, z together with some new edges. Precisely,

$$V_H = V_G \cup \{w, x, y, z\} \text{ and,}$$

$$E_H = E_G \cup \{xv \mid v \in V_G\} \cup \{yv \mid v \in V_G\} \cup \{wx, yz\}.$$

The new graph H



Observe that if $u, v \in V_G$, then the distance between them computed in H is $d_H(u, v) \leq 2$. Let us compute the eccentricities of the vertices in H. We have that $\text{ecc}(w) = \text{ecc}(z) = 4$, and $\text{ecc}(x) = \text{ecc}(y) = 3$, and for any $v \in V_G$ we have $\text{ecc}(v) = 2$. This means that the radius of H is 2 and the center of H is G. ■

The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour the vertices of G such that no adjacent vertices have the same colour. For example, $\chi(K_4) = 4$ and $\chi(C_5) = 3$ and $\chi(C_6) = 2$. Observe that a graph G is bipartite if and only if $\chi(G) = 2$.

Stephan Hedetniemi formulated in 1966 in his doctoral thesis his famous conjecture, now called Hedetniemi's Conjecture. It states that the chromatic number of the tensor product $G \times H$ is equal to the minimum of the chromatic number of G and the chromatic number of H. In 2019, the Russian mathematician **Yaroslav Shitov** made a breakthrough by refuting the conjecture. He constructed a counterexample where $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$.

Theorem 2.1.5. (Bielak-Sysło)

A graph G is isomorphic to the periphery of some graph if and only if either every vertex in G has eccentricity 1 or no vertex has eccentricity 1.

Proof. We will show the contrapositive of the forward direction. So suppose that there is a vertex $u \in V_G$ with $\text{ecc}(u) = 1$ and a vertex $v \in V_G$ with $\text{ecc}(v) > 1$. This means that u is not in the periphery of G, and so G is not the periphery of itself. Since $\mathrm{ecc}(u) = 1$, the vertex u is adjacent to all other vertices in G.

For the sake of contradiction, suppose that G is the periphery of some graph H. Notice that H cannot be G, since G is not the periphery of itself. Moreover, the diameter of H cannot be 1, because if $\mathrm{diam}(H) = 1$, then every vertex in H has eccentricity 1, and so the periphery of H would be H, but we know that the periphery of H is G and $G \ncong H$. Therefore, $\mathrm{diam}(H) \geq 2$.

Now, consider the vertex $u \in V_G$ which has eccentricity 1 in G. In H, this vertex u belongs to the periphery of H. So there is a vertex $z \in V_H$ such that $d_H(u, z) = \mathrm{diam}(H)$. This means, that the vertex z has eccentricity $\mathrm{diam}(H)$ and so z also belongs to the periphery of H, that is, z belongs to G. But we know that u is adjacent to every other vertex in G, so u is adjacent to z, which means $d_H(u, z) = 1$. This contradicts that $d_H(u, z) = \mathrm{diam}(H) \geq 2$.

For the reverse direction, suppose first that every vertex in G has eccentricity 1. Then $\mathrm{diam}(G) = 1$ and so every vertex in G is in the periphery of G. So G is the periphery of itself. (Note that in this case G must be a complete graph.)

Second, assume that no vertex has eccentricity 1. Therefore, for every $u \in V_G$, there exists some vertex $v \in V_G$ such that u and v are nonadjacent. Now construct a new graph K by adding a new vertex z to G which is adjacent to every vertex in G. That is, $V_K = V_G \cup \{z\}$ and $E_K = E_G \cup \{zv \mid v \in V_G\}$. In K, the distance from z to any other vertex is 1, and so $\mathrm{ecc}(z) = 1$. Now, let $u, v \in V_G \subseteq V_K$. If u, v were adjacent, then $d_K(u, v) = 1$. Otherwise, if they were nonadjacent, then $d_K(u, v) = 2$, since a shortest path in K from u to v is (u, z, v) which has length 2. Thus, the eccentricity computed in K of any vertex in V_G is 2. Thus, the $\mathrm{rad}(K) = 1$ and $\mathrm{diam}(K) = 2$, and the center of K is $\{z\}$, and the periphery of K is precisely G. ■

2.2 Adjacency Matrix

Looking at the diagram of a graph was of great help to us. We used our vision to count the number of edges in paths and to measure the distances between vertices. We were able to do this merely because the graphs we encountered were small. Now, imagine large graphs with hundreds, thousands, or even millions of vertices. In this case, the visual strategy collapses dramatically, and we seek computers for assistance. Computers visualise graphs through their matrix representations.

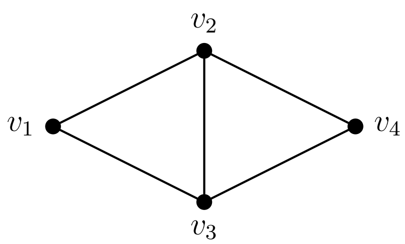
Recall that an $m \times n$ matrix A is a rectangular array of numbers where m is the number of rows and n is the number of columns. The entry of a matrix A located in the i^{th} row and j^{th} column is called the (i, j)-entry, and is denoted by $A(i, j)$. For a matrix A we also write $A = [a_{ij}]$ where a_{ij} is the (i, j)-entry of A.

Definition 2.2.1. (Adjacency Matrix)

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. The *adjacency matrix* of G is the $n \times n$ matrix $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.4. Find the adjacency matrix A of the following graph.



The adjacency matrix is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

In A the entry $A(3, 4) = 1$ since v_3, v_4 are adjacent, while $A(1, 4) = 0$ since v_1, v_4 are nonadjacent.

In the adjacency matrix of a graph, $A(i, i) = 0$ for all $1 \leq i \leq n$. That is, all the entries on the main diagonal are zeros since no vertex is adjacent to itself. Observe that the number of ones in row i is the degree of vertex v_i and so the number of ones in an adjacency matrix is equal to twice the number of edges. Moreover, the adjacency matrix is symmetric (equal to its transpose).

Notice that a single graph may have several adjacency matrices. Once we change the ordering of the vertices we may obtain a different adjacency matrix of the same graph.

Let us compute higher powers of the matrix A of the previous example.

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 4 & 5 & 5 \\ 5 & 5 & 4 & 5 \\ 2 & 5 & 5 & 2 \end{bmatrix}.$$

In A the entry $A(1,4) = 0$ meaning that v_1 and v_4 are nonadjacent. However, in A^2 we have $A^2(1,4) = 2$. We obtain 2 by taking the dot product of the first row of A with the fourth column of A. Here are the details,

$$\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = (0 \cdot 0) + (1 \cdot 1) + (1 \cdot 1) + (0 \cdot 0) = 2.$$

The second position of the first row is 1, and this means that v_1, v_2 are adjacent. The second position of the fourth column is also 1, which means that v_2, v_4 are adjacent. Therefore we have a walk v_1, v_2, v_4 of length 2 (see the graph above). Similarly, there is a 1 in the third position of the first row and the fourth column, meaning that there is a walk v_1, v_3, v_4 of length 2. The entry $A^2(1,4)$ is equal to the number of walks of length 2 from v_1 to v_4.

Theorem 2.2.2.

Let G be a graph whose vertices are labeled $v_1, v_2, \ldots, v_n$, and let A be its corresponding adjacency matrix. Then the entry $A^k(i,j)$ is equal to the number of walks of length k from v_i to v_j.

Proof. We prove the theorem by induction on k. For $k = 1$, the matrix A^1 is the adjacency matrix of G, and so the $A(i,j) = 1$ if and only if v_i, v_j are adjacent if and only if there is exactly one walk of length 1 from v_i to v_j.

For the induction step, suppose the statement of the theorem holds for some positive integer k. We will show that the statement is true for $k+1$. That is, we will show that the (i,j)-entry of A^{k+1} is the number of walks of length $k+1$ from v_i to v_j.

Let $B = [b_{ij}] = A^k$, and $C = [c_{ij}] = A^{k+1}$. Notice that $C = A^{k+1} = A^k A = BA$.

By definition of matrix multiplication, the (i, j)-entry of A^{k+1}, denoted by c_{ij}, is equal to the dot product of the i^{th} row of B with the j^{th} column of A. That is,

$$c_{ij} = \sum_{h=1}^{n} b_{ih} a_{hj}.$$

The entry b_{ih} is the (i, h)-entry of A^k, and by the induction hypothesis it is the number of walks of length k from v_i to v_h.

Observe that, every walk from v_i to v_j of length $k + 1$ is a walk of length k from v_i to some neighbour v_h of v_j together with the edge $v_h v_j$. This means that the total number of $v_i v_j$-walks of length $k + 1$ is equal to the total number of all walks of length k from v_i to any neighbour v_h of v_j.

Therefore, the total number of walks of length $k + 1$ from v_i to v_j is equal to

$$\sum_{v_h \in N(v_j)} b_{ih}.$$

By the definition of the adjacency matrix A, the entry $a_{hj} = 1$ if v_h, v_j are adjacent, otherwise $a_{hj} = 0$. Therefore,

$$\sum_{v_h \in N(v_j)} b_{ih} = b_{i1}a_{1j} + b_{i2}a_{2j} + b_{i3}a_{3j} + \ldots + b_{in}a_{nj} = \sum_{h=1}^{n} b_{ih}a_{hj} = c_{ij}.$$

This proves the theorem. ■

Corollary 2.2.3.

Let G be a graph with vertices labeled $v_1, v_2, \ldots, v_n$, and let A be its adjacency matrix. The distance $d(v_i, v_j)$ is the least integer k such that $A^k(i, j) \neq 0$.

Proof. Let k be the least integer such that $A^k(i, j) \neq 0$. From Theorem 2.2.2 we know that $A^k(i, j)$ is equal to the number of walks of length k from v_i to v_j. Since this entry is nonzero, there is at least one $v_i v_j$-walk of length k, call this walk W. Moreover, since the $A^r(i, j) = 0$ for all $r < k$, there are no $v_i v_j$-walks of lengths less than k. This means that the shortest length of a $v_i v_j$-walk in G is k. So W is a $v_i v_j$-walk of shortest length. Clearly, a walk of shortest length must be a path, thus W is a path from v_i to v_j. Observe that we cannot have a path from v_i to v_j of length less than k simply because every path is a walk. Therefore, k is the length of a shortest path from v_i to v_j, in other words, $d(v_i, v_j) = k$. ■

Let G be a graph of order n, and let A be its adjacency matrix. Recall that I_n is the $n \times n$ identity matrix, and $A^0 = I_n$. For $k \geq 0$, the *stroll matrix* of G is given by

$$S_k = \sum_{r=0}^{k} A^r = I_n + A + A^2 + A^3 + \ldots + A^k.$$

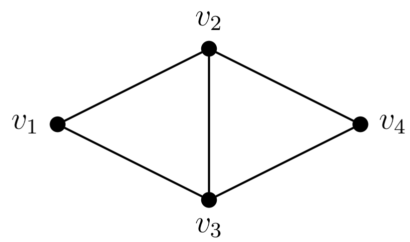
It follows that

$$S_k(i,j) = I_n(i,j) + A(i,j) + A^2(i,j) + A^3(i,j) + \ldots + A^k(i,j).$$

The entry $S_k(i,j)$ is the number of v_iv_j-walks of length at most k.

Clearly, all entries in S_k are nonnegative, and $S_k(i,j) \le S_{k+1}(i,j)$.

Example 2.5. Consider our previous graph and its adjacency matrix A.



Where $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Let us compute the stroll matrices S_0, S_1, S_2, S_3.

$$S_0 = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } S_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 4 & 3 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \text{ and } S_3 = \begin{bmatrix} 5 & 7 & 7 & 4 \\ 7 & 8 & 8 & 7 \\ 7 & 8 & 8 & 7 \\ 4 & 7 & 7 & 5 \end{bmatrix}.$$

$S_2(3,3) = 4$ means that there are 4 walks from v_3 to itself of length at most 2. And $S_3(2,4) = 7$ means that there are 7 walks from v_2 to v_4 of length at most 3.

The theorem below follows from the fact that $S_k(i,j)$ is the number of walks of length at most k from v_i to v_j.

Theorem 2.2.4.

Let G be a graph of order n with vertices labeled $v_1, v_2, \ldots, v_n$, and let A be its adjacency matrix and S_k its stroll matrix.

(i) $\mathrm{d}(v_i, v_j)$ *is the least integer k such that $S_k(i, j) \neq 0$.*

(ii) $\mathrm{ecc}(v_i)$ *is the least integer k such that the i^{th} row of S_k contains no zeros.*

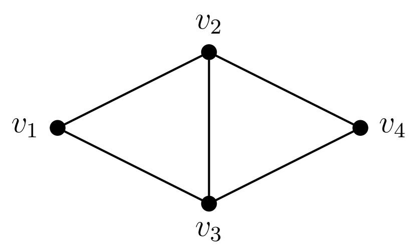
(iii) $\mathrm{rad}(G)$ *is the least integer k where S_k contains at least one row with no zeros.*

(iv) $\mathrm{diam}(G)$ *is the least integer k such that all entries of S_k are nonzeros.*

(v) G is disconnected if and only if S_{n-1} contains a zero.

Let G be a connected graph with vertices $v_1, v_2, \ldots, v_n$. The *distance matrix* of the graph G is the $n \times n$ matrix M where $M(i, j) = d(v_i, v_j)$. From the discussion above $M(i, j) = k$ where k is the least integer such that $S_k(i, j) \neq 0$.

Example 2.6. Consider the graph below again.



The distance matrix is

$$M = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

2.3 Erdős Number

Let us introduce the graph of mathematicians. The vertices of this graph are researchers (living or dead). The edge relation is defined by stating that two researchers are adjacent if they have coauthored a paper together. One of the vertices is called Paul Erdős.

Paul Erdős (1913 – 1996) was a Hungarian mathematician, considered one of the greatest and most prolific mathematicians of the twentieth century. He published over 1500 papers in mathematics covering topics in graph theory, Ramsey theory, set theory, combinatorics, analysis, geometry, and number theory. He collaborated with more than 500 mathematicians. He believed that God had a book that contained the most beautiful proof for every mathematical theorem. He called this book, *"The Book"*. The highest compliment he would say to the work of other mathematicians was, "That's straight from The Book." Defying the myth that mathematics is a game for young people, Erdős went on being active in the field until his death at the age of 83. The film *"N Is a Number: A Portrait of Paul Erdős"* is a 1993 biographical documentary about the life of Erdős.

The *Erdős number* of a researcher is the graph theoretic distance between the vertex of Erdős and the vertex of the researcher. Erdős himself is assigned Erdős number 0. Any mathematician who coauthored a paper with Erdős has Erdős number 1. There are more than 500 mathematicians with Erdős number 1. A researcher has Erdős number 2 if they did not coauthor a paper with Erdős but coauthored a paper with a researcher who coauthored with Erdős. There are more than 11000 researchers with Erdős number 2. Albert Einstein has Erdős number 2. The Iranian mathematician Maryam Mirzakhani who won the Fields Medal in 2014 has Erdős number 3.

Chapter 3

Trees

In nature, once the branches of a tree split they never rejoin.

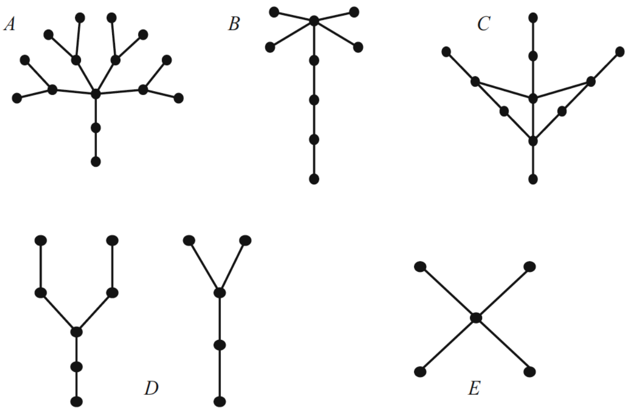
3.1 Tree Definition

A graph which contains no cycles is called *acyclic*. Observe that acyclic graphs are bipartite as they do not have odd cycles.

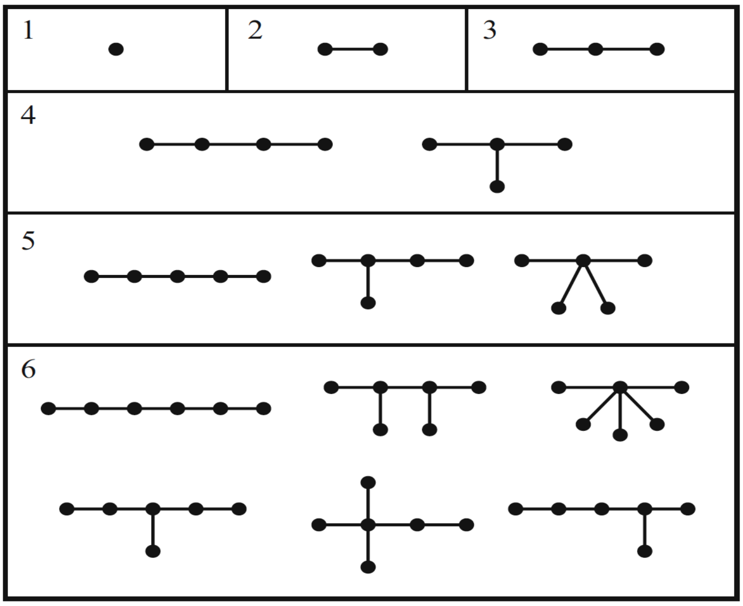
Definition 3.1.1. (Forest)

A *tree* is a connected acyclic graph. A *forest* is an acyclic graph. A *leaf* is a vertex of degree 1.

The path P_9, the complete graphs K_1 and K_2, and the complete bipartite graph $K_{1,4}$ are all trees. Clearly every tree is a forest. A forest is a graph whose connected components are all trees. Every subgraph of a forest is a forest, and every connected subgraph of a forest is a tree. In the diagrams below, graphs A, B, E are all trees. Graph D is a forest but not a tree as it is disconnected. Graph C is not a forest as it contains a cycle.



Any tree of order 6 or less is isomorphic to one of the following trees.



3.2 Properties of Trees

Trees in graph theory always have leaves!

Lemma 3.2.1.

Any tree with more than one vertex has at least two leaves.

Proof. Let T be a tree with two or more vertices. So T is a connected graph with no cycles. Let P be a path in T with maximum length k. Say P is given by

$$(v_0, v_1, v_2, \dots, v_{k-1}, v_k).$$

Since T contains at least two vertices, it must be that $k \geq 1$, and so v_0 and v_k are distinct. Consider the last vertex v_k. We know that $\deg(v_k) \geq 1$. We will show that v_k is a leaf. Suppose for contradiction that $\deg(v_k) \geq 2$, so v_k has at least two neighbours. One of the neighbours is v_{k-1}. Let u be another neighbour of v_k. If u is distinct from all the vertices in P, then we can extend P to a longer path

$$(v_0, v_1, \dots, v_{k-1}, v_k, u)$$

but this contradicts that P is a longest path. Otherwise, $u = v_i$ for some $0 \leq i \leq k-2$ meaning that $(v_i, v_{i+1}, \dots, v_k, v_i)$ is a cycle in T, a contradiction as T is acyclic. Therefore, $\deg(v_k) = 1$, meaning that v_k is a leaf. By a similar argument v_0 is another leaf. ∎

Theorem 3.2.2.

Let T be a tree of order n. Then T has exactly $n-1$ edges.

Proof. We will proceed by induction on the order n. When $n = 1$, then any tree of order 1 is K_1 which has 0 edges.

Fix some integer $n \geq 1$. Suppose that *any* tree of order n has exactly $n-1$ edges. Choose an arbitrary tree T of order $n+1$. We will show that T has exactly n edges.

By Lemma 3.2.1, the tree T must contain a leaf u. Clearly u is not a cut vertex (why?). So $T - u$ is connected. Moreover, $T - u$ is acyclic as well (again why?). Thus, $T - u$ is a tree of order n, and by the induction hypothesis, $T - u$ contains $n-1$ edges. The edges of T are all the edges in $T - u$ together with the single edge incident with the leaf u. It follows that T has $(n-1)+1$ edges, and so T has exactly n edges. ∎

Corollary 3.2.3.

A forest of order n and k connected components has $n-k$ edges.

Proof. Let F be a forest of order n, and let its connected components be $Q_1, Q_2, \ldots, Q_k$. Let n_i be the order of Q_i. Observe that each connected component Q_i must be a tree and so, by Theorem 3.2.2, has $n_i - 1$ edges. Thus the total number of edges in the forest is

$$\sum_{i=1}^{k} E(Q_i) = \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k.$$

■

Theorem 3.2.4.

A graph G of order n is a tree if and only if G is connected and has $n-1$ edges.

Proof. The forward direction is immediate. For the reverse direction, suppose that G is connected and has $n-1$ edges. It remains to show that G is acyclic. Suppose for the sake of contradiction that G has a cycle. Let e be some edge from this cycle. Clearly $G - e$ is still connected with $n-2$ edges.

If cycles still exist, we keep deleting edges from existing cycles, one edge at a time. In each time, the resulting graph will stay connected. At the end we obtain a connected graph H with no cycles and with strictly less than $n-1$ edges. But H is a tree and of order n (no vertices of G were deleted), and so by Theorem 3.2.2, the graph H has $n-1$ edges, a contradiction. Therefore, G has no cycles, so G is a tree. ■

Theorem 3.2.5.

A graph G of order n is a tree if and only if G is acyclic and contains $n-1$ edges.

Proof. The forward direction is immediate. For the reverse direction, let G be an acyclic graph with $n-1$ edges. So G is a forest. Let k be the number of connected components of G. By Corollary 3.2.3, G has $n-k$ edges, implying that $k=1$. This means that G has one connected component, implying that G is a connected graph. Thus, G is a tree. ■

Corollary 3.2.6.

Let G be a graph of order n. Then the following are equivalent.

(i) G is connected and acyclic.

(ii) G is connected and has $n-1$ edges.

(iii) G is acyclic and has $n-1$ edges.

Lemma 3.2.7.

Every edge in a tree is a bridge.

Proof. Let T be a tree. Choose any edge e of T where $e = uv$. Suppose for contradiction that e is not a bridge, and so $T - e$ is a connected graph. Then there is a path P starting with vertex u and ending with vertex v in $T - e$. It follows that the path P followed with the edge e form a cycle in T, but T has no cycles, which is a contradiction. Therefore, $T - e$ is disconnected and e is a bridge. ∎

Lemma 3.2.8.

Any two vertices in a tree are joined by a unique path.

Proof. Let T be a tree. By definition, T is connected and so every pair of vertices are joined by some path. We next show that such path is unique. Suppose for the sake of contradiction that there are two vertices a, b of T that are joined by two distinct paths $P_1 = (a, u_1, u_2, \ldots, u_k)$ and $P_2 = (a, v_1, v_2, \ldots, v_l)$ where $b = u_k = v_l$. Since P_1 and P_2 are different, let t be the least such that $u_t \neq v_t$ (this means that $u_{t-1} = v_{t-1}$). And let i be the least integer such that

$$u_i \in \{u_t, u_{t+1}, \ldots, u_k\} \cap \{v_t, v_{t+1}, \ldots, v_l\}.$$

Notice that such i is guaranteed to exist because this intersection is nonempty as $u_k = v_l$. Let j be such that $u_i = v_j$. Then the sequence

$$(u_{t-1}, u_t, u_{t+1}, \ldots, u_i, v_{j-1}, v_{j-2}, \ldots, v_{t+1}, v_t, v_{t-1})$$

is a cycle (check it), which is a contradiction as T is acyclic. Therefore, every two vertices in T are joined by exactly one path. ∎

Corollary 3.2.9.

Let T be a tree and u be a vertex of T. Then, the eccentricity of u is the length of a longest path starting at u.

Lemma 3.2.10.

Suppose that T is a tree with at least 2 vertices. Let u and v be vertices of T such that $d(u, v) = \operatorname{ecc}(u)$. Then, v is a leaf of T.

Proof. Let T be a tree with at least 2 vertices. Choose vertices u and v of T such that $d(u, v) = \operatorname{ecc}(u)$. So, $d(u, v)$ is the greatest distance from u to any vertex in T. Let P be the unique uv-path in T, and notice that $d(u, v)$ is equal to the length of

P. Assume v is not a leaf. Then, $\deg(v) \geq 2$, and so v has a neighbour w not on P, as otherwise there will be a cycle in T. We now extend the path P by the edge vw to obtain a uw-path $\hat{P}$ longer than P. As $\hat{P}$ is the only uw-path in T, it follows that the length of $\hat{P}$ is equal to the distance between u and w, and so $d(u, w) > d(u, v)$, contradicting that $d(u, v) = \operatorname{ecc}(u)$. Therefore, v must be a leaf. ■

Tree pruning. Given a tree T of order at least 3, denote by T^- the tree obtained from T by deleting all of its leaves. Obviously, T^- is a subgraph of T. For example, if T is the path P_6, then T^- is P_4.

Lemma 3.2.11.

Let T be a tree with more than two vertices. Then

$$\operatorname{Center}(T) = \operatorname{Center}(T^-).$$

Proof. First, we will show that there are no leaves in the center of T. Let l be a leaf in T, and let b be its unique neighbour. As T has more than two vertices, it must be that b is not a leaf (otherwise T will be K_2). One can see that $\operatorname{ecc}(l) = \operatorname{ecc}(b) + 1$. This means $\operatorname{ecc}(l) > \operatorname{ecc}(b)$ and so the center of T contains no leaves. Thus, no central vertex is deleted when forming T^- and so $\operatorname{Center}(T) \subseteq T^-$.

Now choose any vertex u in T which is not a leaf (so u belongs to T^-). Let v be a vertex in T such that $d(u, v) = \operatorname{ecc}(u)$. By the lemma above, v is a leaf. Let P be the unique uv-path in T, and notice that v is the only leaf on P. So all vertices of P will stay in T^- except v, as the leaf v is the only vertex deleted from P when creating T^-. As this holds for all paths starting at u of length $\operatorname{ecc}(u)$, it follows that

$$\operatorname{ecc}_{T^-}(u) = \operatorname{ecc}_T(u) - 1.$$

Since the above is true for all non-leaves of T (which are the vertices of T^-), the eccentricity of each vertex in T^- is one less than it was in T, and so those vertices of minimum eccentricity in T are the same vertices of minimum eccentricity in T^-. In other words, we have shown that $\operatorname{Center}(T) = \operatorname{Center}(T^-)$. ■

Theorem 3.2.12.

Let T be a tree, then the center of T is either K_1 or K_2.

Proof. We will proceed by induction on the order n of a tree T. If $n = 1$, then T is K_1 and the center of K_1 is itself. If $n = 2$, then T is K_2 and the center of K_2 is itself.

Let $n \geq 3$ and suppose that the theorem is true for all trees of order strictly less than n. Let T be a tree of order n. We know that T has at least two leaves and so T^- has order strictly less than n. By induction hypothesis the center of T^- is either K_1 or K_2, and by Lemma 3.2.11, we know that $\text{Center}(T) = \text{Center}(T^-)$. This completes the proof. ■

Theorem 3.2.13.

Let T be a tree of order n and G be a graph where $n - 1 \leq \delta(G)$. Then G contains T as a subgraph.

Proof. We induct on n. Let T be a tree of order 1 and G be any graph. Then T is K_1, and K_1 is a subgraph of any graph.

Fix some natural number $n \geq 2$. Suppose that the theorem is true for $n - 1$. Let T be a tree of order n and G be a graph with $n - 1 \leq \delta(G)$. By Theorem 3.2.1, we know that T contains leaves. Let u be a leaf of T, and let b be its unique neighbour. Consider the tree $T - u$ which contains b. The order of $T - u$ is $n - 1$ and

$$(n-1) - 1 = n - 2 < n - 1 \leq \delta(G).$$

By induction hypothesis, $T - u$ is a subgraph of G. We may think of $T - u$ sitting inside G, and so b is a vertex of G. We know that $\deg_G(b) \geq \delta(G) \geq n-1$. Moreover, the degree of b in $T - u$ is at most $n - 2$. Thus, there must be a vertex $g \in G$ such that g is not in $T - u$ and g is adjacent to b. Finally, the subgraph $T - u$ together with vertex g and edge gb form the tree T as a subgraph of G. ■

Let us collect what we know about trees in one place.

- Any tree of order n contains exactly $n - 1$ edges.
- Any tree of order greater than 1 contains at least two leaves.
- A graph G of order n is a tree if and only if G is connected and contains $n - 1$ edges.
- A graph G of order n is a tree if and only if G is acyclic and contains $n - 1$ edges.
- A graph G is a tree if and only if any pair of vertices are joined by exactly one path.
- The center of any tree is either K_1 or K_2.
- Any graph G contains every tree of order at most $\delta(G) + 1$ as a subgraph.

3.3 Spanning Trees

Given a connected graph G, start deleting edges from its cycles, one edge at a time, until no cycles exist. At the end, when no cycles are left, what will we get? We will obtain a subgraph G which is a tree that contains all the vertices of G.

Definition 3.3.1. (Spanning Tree)

A *spanning tree* of a graph G is a tree T such that T is a subgraph of G that contains every vertex of G.

Notice that if G is a graph of order n, then any spanning tree of G has n vertices and $n-1$ edges.

Lemma 3.3.2.

Every connected graph contains at least one spanning tree.

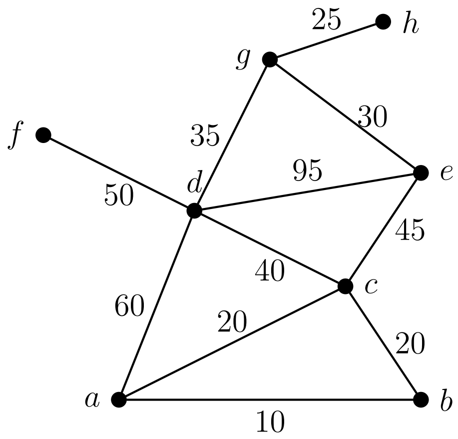
Proof. Let G be a connected graph. Choose a subgraph T of G which is a tree of maximum order. If T contains all vertices of G, we are done. Otherwise, and as G is connected, there exists a vertex v in G that is not in T which is adjacent to a vertex u in T. Now consider the subgraph $\hat{T}$ where $V_{\hat{T}} = V_T \cup \{v\}$ and $E_{\hat{T}} = E_T \cup \{uv\}$. So $\hat{T}$ is the graph formed from T by adding the new vertex v and the new edge uv. Clearly, $\hat{T}$ is connected since T is connected. Moreover, $\hat{T}$ has no cycles. To see this observe that T had no cycles, and the degree of v in $\hat{T}$ is 1, so v will not be part of any cycles in $\hat{T}$. Thus, $\hat{T}$ is a tree which is a subgraph of G of order one more than the order of T, contradicting that T is a tree of maximum order. Therefore, T must be a spanning tree of G. ■

Exercise. Prove the above lemma by induction on the order of the graph.

Definition 3.3.3. (Weight Function)

Let G be a graph. A *weight function* is a function W that assigns to every edge of G some real number, that is, $W : E_G \to \mathbb{R}$. A graph together with a weight function is called a *weighted graph*.

Consider the following weighted graph. For example, the weight of edge de is 95, that is, $W(de) = 95$, and the weight of edge ab is 10, that is, $W(ab) = 10$.



In real life, one might think of the weighted graph above as a transportation map. The vertices represent cities, and edges represent roads. The weight of an edge represents the difficulty of passing through the road due to distance, traffic, steepness, and curvature of the road.

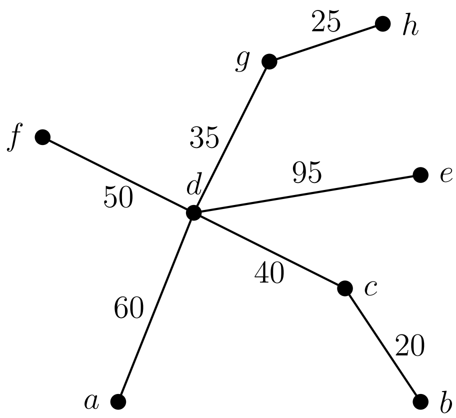
Suppose we want to construct a railway system to connect all of the eight cities where we will lay tracks along some of these roads. We want every city to be accessible from every other city via the railway system (not necessarily a direct connection).

Which roads shall we choose for constructing the railway? Keep in mind we need to minimize cost and time of travel. The longer and steeper the road is, the more expensive the track is.

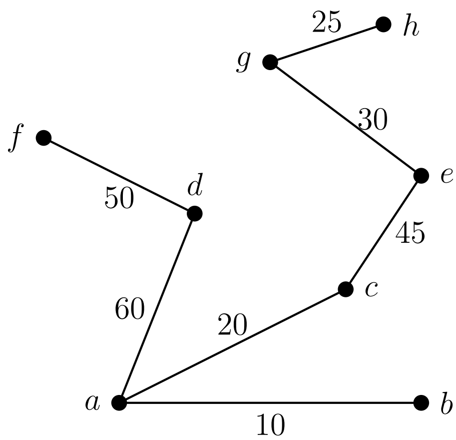
We may lay tracks along every road, but this is expensive and unnecessary. In fact, to minimize the cost, the roads we shall choose to lay the track along should not form a cycle and should connect all cities. In graph-theoretic terms, what we are looking for is a spanning tree of the graph above.

Here are two suggestions for a spanning tree.

Total weight = 325



Total weight = 240



Definition 3.3.4. (Minimum-Weight Spanning Tree)

Let G be a connected weighted graph. A *minimum-weight spanning tree* is a spanning tree T such that the sum of weights of the edges of T is less than or equal to the sum for any other spanning tree of G.

A tree T is a minimum-weight spanning tree of G if for any spanning tree T' we have that

$$\sum_{e \in E_T} W(e) \leq \sum_{e \in E_{T'}} W(e).$$

The economic solution for our railway system boils down to finding a minimum-weight spanning tree. Here is how we find one.

3.3.1 Kruskal's Algorithm

Here is an algorithm to find a minimum-weight spanning tree.

Algorithm 3.3.5. (Kruskal's Algorithm)

Given a connected weighted graph G.

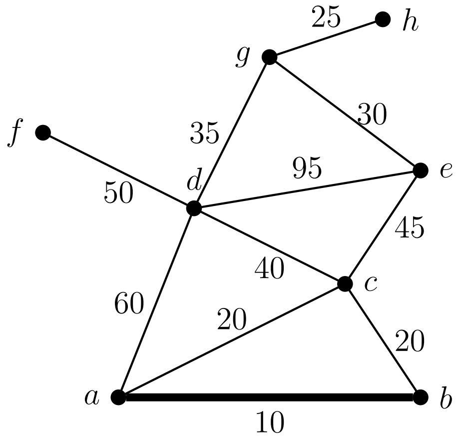
(i) Find an edge of minimum weight and mark it.

(ii) Among all unmarked edges which do not form a cycle with the marked edges, choose an edge of minimum weight and mark it.

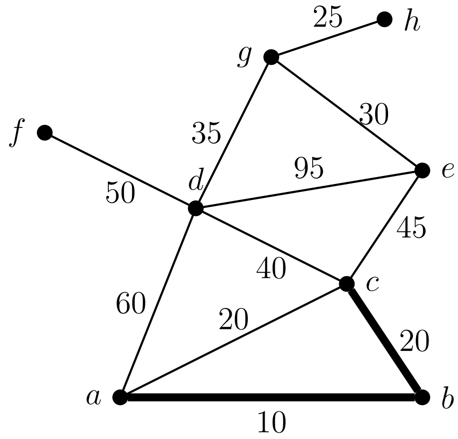
(iii) If the marked edges form a spanning tree, then stop.
Otherwise repeat step (ii).

Let us apply Kruskal's algorithm to the transportation graph above.

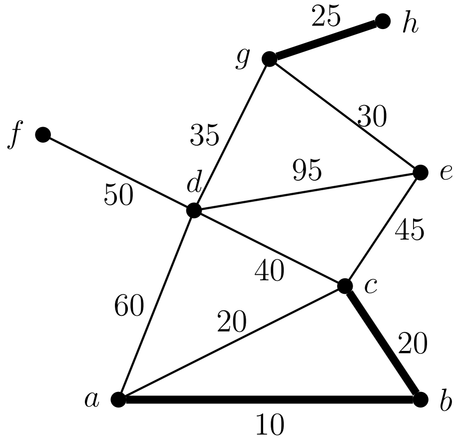
Step 1



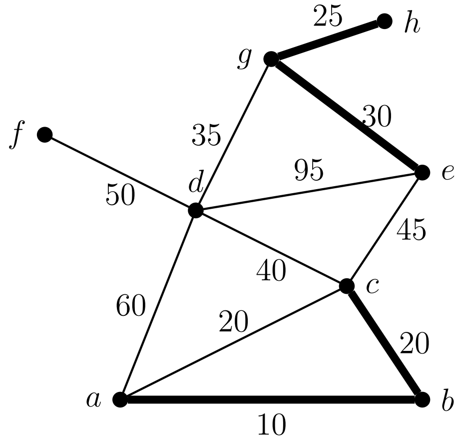
Step 2



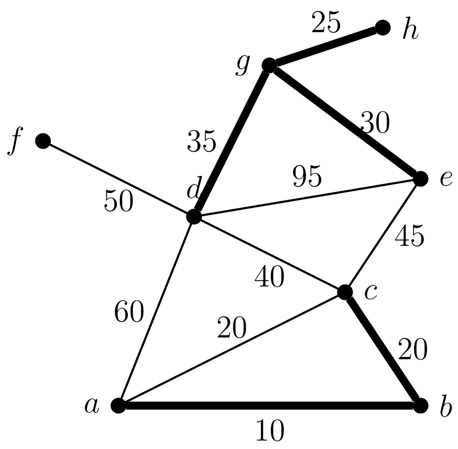
Step 3



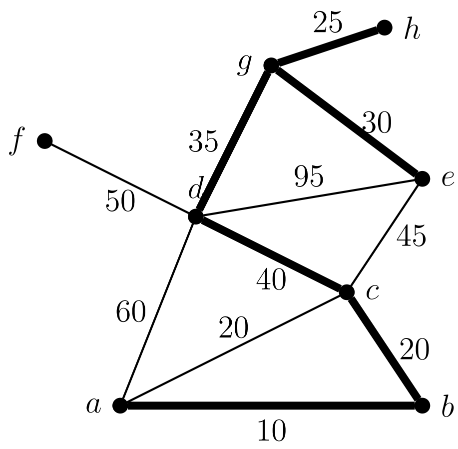
Step 4



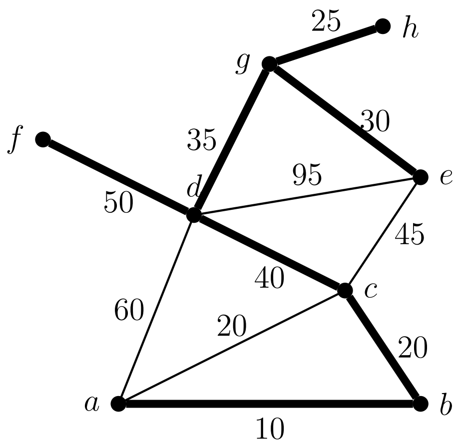
Step 5



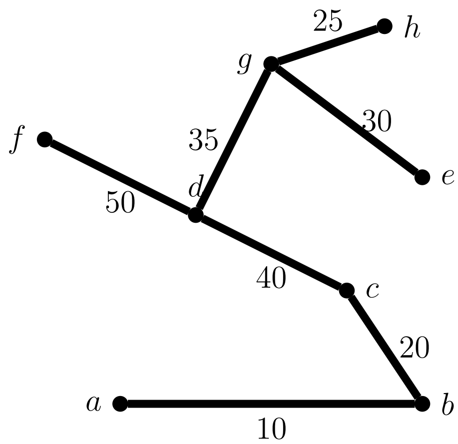
Step 6



STEP 7 and then STOP



A spanning tree of minimum weight 210



Theorem 3.3.6.

Kruskal's algorithm produces a minimum-weight spanning tree.

Proof. Let G be a connected weighted graph of order n. Let T be the subgraph constructed by Kruskal's algorithm in Step $n-1$. Since the algorithm adds one new edge at each step, the subgraph T has $n-1$ edges. Moreover, we know that the algorithm adds only those edges which do not form cycles, so T has no cycles. Thus, T is a forest, say with m vertices and k connected components. Of course, $m \leq n$ as T is a subgraph of G. By our knowledge on forests, T has exactly $m-k$ edges. Thus,

$$n-1 = m-k.$$

Which implies that $n-m = 1-k$. Since $n-m \geq 0$ and $k \geq 1$, it follows that $k=1$ and $n=m$. So, T has one connected component and so connected. Thus, T is a tree. Moreover, T contains all vertices of G. That is, T is a spanning tree of G.

Therefore, the tree T is the output produced by Kruskal's algorithm. Next, we will show that T is of minimum weight.

Kruskal's algorithm built the spanning tree T edge by edge until a tree containing all the vertices of G is formed. Let e_i be the edge added by the algorithm in the construction at Step i. So the edges of T in order of construction are

$$e_1, e_2, e_3, \ldots, e_{n-1}.$$

Suppose for the sake of contradiction that T is not of minimum weight. Among all spanning trees of minimum weight, choose $\hat{T}$ to be a minimum-weight spanning tree whose edges agree with the edges of T for the longest time (longest initial segment of the sequence above). That is, $\hat{T}$ is a spanning tree of minimum weight with largest k such that

$$\{e_1, e_2, \ldots, e_k\} \subseteq E(\hat{T}).$$

Thus no spanning tree of minimum weight contains all the edges $\{e_1, e_2, \ldots, e_k, e_{k+1}\}$. Notice that if $k = n-1$ then $T = \hat{T}$, but we assumed that T is not of minimum weight and so $k < n-1$. The tree $\hat{T}$ does not contain the edge e_{k+1} and as $\hat{T}$ is a spanning tree it already contains $n-1$ edges and so when we add the edge e_{k+1} to $\hat{T}$ we obtain a connected graph $\hat{T} + e_{k+1}$ which is not a tree, and so it must contain a cycle C. Moreover, the cycle C contains e_{k+1} and it must contain an edge $\hat{e} \in E(\hat{T}) \setminus E(T)$, as otherwise the cycle C will be entirely contained in T contradicting that T is a tree. The edge $\hat{e}$ is in $\hat{T}$ but not in T.

Next we delete $\hat{e}$ from the graph $\hat{T} + e_{k+1}$ to form the graph $\hat{T} + e_{k+1} - \hat{e}$. Since $\hat{e}$ was on a cycle, the graph $\hat{T} + e_{k+1} - \hat{e}$ is a spanning tree. The edge $\hat{e}$ does not form any cycle with the edges $\{e_1, e_2, \ldots, e_k\}$ since if so this cycle will be in the tree $\hat{T}$, a contradiction. This means that at Step $k+1$, the edge $\hat{e}$ must have been available to be chosen, but the algorithm chose edge e_{k+1} over $\hat{e}$ only because $W(e_{k+1}) \leq W(\hat{e})$.

It follows that the total weight of the spanning tree $\hat{T} + e_{k+1} - \hat{e}$ is less than or equal to the total weight of $\hat{T}$, which yields that $\hat{T} + e_{k+1} - \hat{e}$ is a minimum-weight spanning tree. But $\hat{T} + e_{k+1} - \hat{e}$ contains $\{e_1, e_2, \ldots, e_k, e_{k+1}\}$, and this contradicts the choice of $\hat{T}$. Therefore, the tree T produced by Kruskal's algorithm must be a minimum-weight spanning tree. ■

Chapter 4

Euler and Hamilton

Leonhard Euler said *"travel every road exactly once"*. William Hamilton said *"visit every city exactly once"*.

4.1 Eulerian Graphs

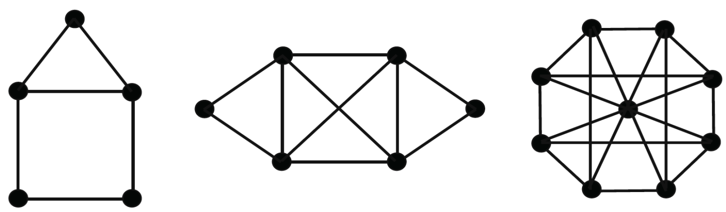
Recall that a walk in a graph is a sequence $(v_0, v_1, v_2, \dots, v_k)$ of vertices where every consecutive vertices are adjacent. A trail is a walk which does not repeat edges. A circuit is a closed trail, that is, a trail which begins and ends at the same vertex.

Definition 4.1.1. (Eulerian Graph)

A graph is called *Eulerian* if it contains a circuit which includes every edge of the graph.

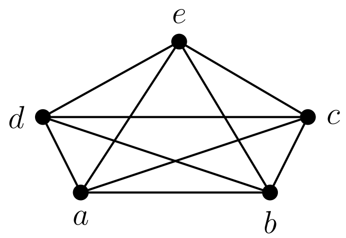
A circuit in a graph which includes every edge is called an *Eulerian circuit.* Similarly, a trail in a graph which includes every edge is called an *Eulerian trail.* A graph being Eulerian means that one can take a walk starting at some vertex, and then travel every edge exactly once, and finally return back to the starting vertex. Clearly, every cycle C_n is an Eulerian graph. While paths P_n for $n \geq 2$ have no circuits at all, and thus they are not Eulerian.

Which of the following graphs are Eulerian?



Among all finite graphs, which ones are Eulerian? There are two famous characterisations of Eulerian graphs. The first involves vertex degrees, and the other requires partitioning the edge set into cycles.

Let us state some observations on the complete graph K_5.



- K_5 is Eulerian. An Eulerian circuit is given by $(a, e, d, a, b, c, e, b, d, c, a)$.
- Every vertex of K_5 is of even degree.
- The edge set $E(K_5)$ can be partitioned into three edge-disjoint cycles where
    1. The first part is $\{ae, ed, da\}$ which is the cycle (a, e, d, a).
    2. The second part is $\{bc, ce, eb\}$ which is the cycle (b, c, e, b).
    3. The third part is $\{ab, bd, dc, ca\}$ which is the cycle (a, b, d, c, a).

  The three sets of edges above are pairwise disjoint and their union is $E(K_5)$.

The following theorem gives a characterisation for Eulerian graphs, it is the result of the work of Euler (1736), Hierholzer (1873), and Veblen (1912).

Theorem 4.1.2. (Euler-Hierholzer-Veblen)

Let G be connected graph. Then the following statements are equivalent.

(i) G is Eulerian.

(ii) Every vertex of G has even degree.

(iii) The edges of G can be partitioned into edge-disjoint cycles.

Proof. We prove the first implies the second, the second implies the third, and the third implies the first.

$(i) \implies (ii)$. Suppose that G is Eulerian, and let Q be an Eulerian circuit of G. Choose any vertex v in G. We may assume without loss of generality that the sequence of the circuit Q does not start with v. Notice that every appearance of v in Q corresponds to two distinct edges incident with v. Since Q is Eulerian, all the edges incident with v must appear in Q. Thus, the degree of v is twice the number of appearances of v in the sequence of Q. Hence, the degree of v is even.

$(ii) \implies (iii)$. Suppose that every vertex in G has even degree. We also may assume that G has no vertices of degree 0. Since G is connected and has no vertices

of degree 1 (no leaves), then G is not a tree and hence it must contain at least one cycle. We prove the implication using induction on the number of cycles of G. For the base case, if G has exactly one cycle, then G itself is a cycle C_n for some n, and so the desired partition of the edge set would constitute just the one cycle itself.
Now, fix some $k \geq 1$, and suppose that if a connected graph whose all of its vertices have even degrees contains at most k cycles, then its edge set can be partitioned into edge-disjoint cycles. Let G be a connected graph where each of its vertices has even degree and suppose it has $k+1$ cycles. Choose any cycle C in G. Let H be $G - E(C)$, that is, H is obtained from G by deleting all the edges of the cycle C. The degree of every vertex on C decreased by 2 in H while the degrees of the other vertices stayed the same. Thus, each vertex in H has even degree. The graph H might be disconnected of course. However, each of its connected components is connected, and all of its vertices are of even degrees, and contains at most k cycles. Thus, by the induction hypothesis, the edge set of each connected component can be partitioned into edge-disjoint cycles. All of these cycles together with the cycle C make up a partition of the edge set of G into cycles. The induction is finished and the implication is proved.

$(iii) \implies (i)$. Suppose that the edge set of G can be partitioned into k many subsets, call them, $S_1, S_2, \ldots, S_k$, and where the edges in each S_i form a cycle. Let Q be a longest circuit in G such that the set of edges of Q is equal to the union of some of these subsets. That is, Q is a longest circuit such that for some subset of indices $I \subseteq \{1, 2, \ldots, k\}$ we have that,

$$E(Q) = \bigcup_{j \in I} S_j.$$

As the subsets $S_1, S_2, \ldots, S_k$ are pairwise disjoint, it follows that for every cycle S_i either all of the edges of S_i are in Q or no edge of S_i is in Q. Now, if Q contains all the edges of G, then Q is Eulerian and we are done. Otherwise, there is some edge e of G that is not an edge of the circuit Q and which is incident with a vertex v on Q. The edge e must belong to some cycle S_l where $l \notin I$. Moreover, no edge of S_l is in Q and the vertex v is on the cycle S_l. Now since there are no edges in common between the circuit Q and the cycle S_l, we can obtain a new circuit $\hat{Q}$ by patching S_l into Q at the vertex v. But the edges of $\hat{Q}$ consist from all the edges of Q together with the edges of S_l, contradicting the maximality of Q. It follows that Q must be an Eulerian circuit and so G is Eulerian. The implication is established. ■

The previous theorem tells us when Eulerian circuits exist. What about Eulerian trails? Certainly, every Eulerian circuit is an Eulerian trail. But are there non-Eulerian graphs with Eulerian trails? The corollary below answers the questions.

Corollary 4.1.3.

A graph G has an Eulerian trail if and only if either each vertex has even degree or exactly two vertices have odd degrees.

4.1.1 Hierholzer's Algorithm

As we now know when Eulerian circuits exist, how do we find them? The German mathematician Carl Hierholzer $(1840 - 1871)$ developed an algorithm for finding Eulerian circuits in Eulerian graphs. Let us comment that the correctness of Hierholzer's algorithm is justified by the proof of the third implication above: if the edge set can be partitioned into edge-disjoint cycles, then the graph is Eulerian.

Algorithm 4.1.4. (Hierholzer's Algorithm)

Given an Eulerian graph G.

(i) Identify any circuit R_1 in G and mark all of its edges.

(ii) Let $i = 1$.

(iii) If R_i contains all the edges of G, then stop.

(iv) Otherwise, if R_i does not contain all edges of G, then let v_i be a vertex on R_i that is incident with an unmarked edge.

(v) Build using the unmarked edges a circuit Q_i starting with v_i.

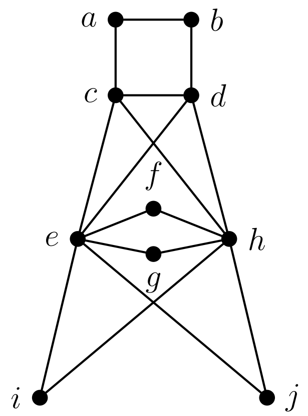
(vi) Mark all the edges of Q_i.

(vii) Create a new circuit R_{i+1} by patching the circuit Q_i into R_i at v_i.

$(viii)$ Increment i by 1 and jump to step (iii).

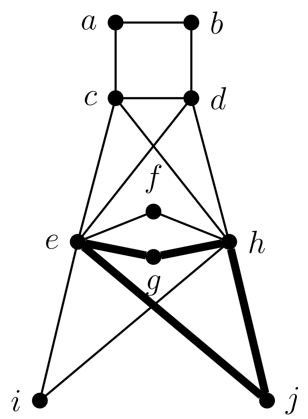
We note that the algorithm succeeds to produce an Eulerian circuit irrespective of the choice of the initial circuit R_1. You may also choose R_1 to be any vertex of the graph since a single vertex is indeed a circuit. Another point is that we need to check that the graph given to the algorithm is Eulerian in the first place. To do this, all what we have to do is to check that the graph is connected and each of its vertices has even degree.

Example 4.1. Apply Hierholzer's Algorithm to find an Eulerian circuit in the graph below.

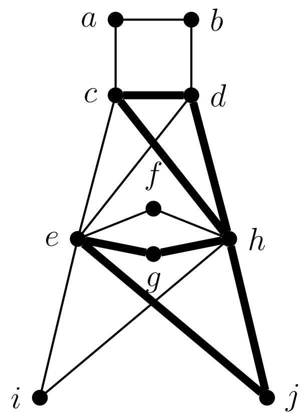


Clearly every vertex has even degree.
Now identify any circuit to start with.
Let us choose (e, g, h, j, e).

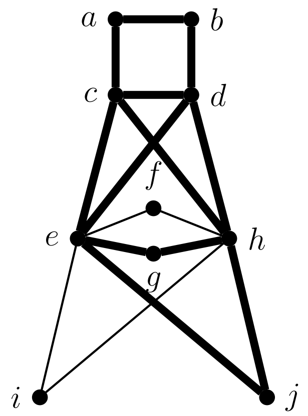
Step 1
$R_1 = (e, g, h, j, e)$
$v_1 = h$ and $Q_1 = (h, d, c, h)$



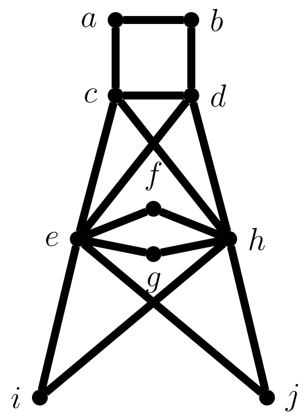
Step 2
$R_2 = R_1 +_h Q_1 = (e, g, h, d, c, h, j, e)$
$v_2 = d$ and $Q_2 = (d, b, a, c, e, d)$



Step 3
$R_3 = R_2 +_d Q_2 = (e, g, h, d, b, a, c, e, d, c, h, j, e)$
$v_3 = h$ and $Q_3 = (h, f, e, i, h)$



Step 4
$R_4 = R_3 +_h Q_3 = (e, g, h, f, e, i, h, d, b, a, c, e, d, c, h, j, e)$ and R_4 contains all 16 edges.



4.2 Hamiltonian Graphs

In the previous section we studied walks which contain all the edges of a given graph without repetitions. Here, we intend to study walks which contain all the vertices of the graph without repetitions. In 1857 the Irish mathematician William Rowan Hamilton invented a puzzle whose goal is to find a cycle in the graph of the dodecahedron (polyhedron with twelve faces) that visits every vertex exactly once. Hamilton dubbed the puzzle the 'Icosian game', as the resulting cycle has exactly twenty ('icosa' in ancient Greek) edges and vertices. The dodecahedron has the interesting property that it looks the same from the point of view of any vertex. In other words, the dodecahedron is vertex-transitive, meaning that any two vertices can be mapped onto each other by an automorphism of the graph. In honour of Hamilton, a cycle that visits every vertex of a graph exactly once is now called a Hamilton cycle.

Definition 4.2.1. (Hamiltonian Graph)

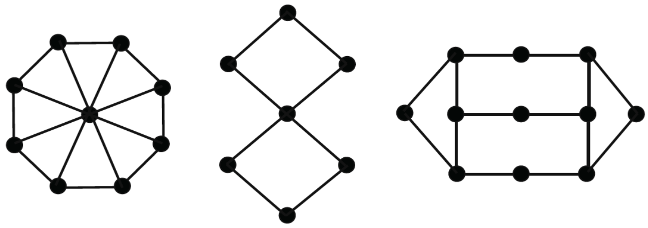
- A graph is called *traceable* if it has a path which contains all the vertices.
- A graph is called *Hamiltonian* if it has a cycle which contains all the vertices.

A path which contains all the vertices in a graph is called a *Hamiltonian path*. Similarly, a cycle which contains all the vertices is called a *Hamiltonian cycle*. Clearly, any Hamiltonian graph is traceable, but not the converse. Every traceable graph is connected. Let G be a graph of order n. Then G is Hamiltonian if and only if G contains the cycle C_n as a subgraph. Also, G is traceable if and only if G contains the path P_n as a subgraph.

Sir William Rowan Hamilton (1805–1865) was an Irish mathematician who worked at Trinity College Dublin. He worked in both pure mathematics and mathematics for physics. He made important contributions to optics, classical mechanics and algebra. Although Hamilton was not a physicist (he regarded himself as a pure mathematician) his work was of major importance to physics, particularly his reformulation of Newtonian mechanics, now called Hamiltonian mechanics. This work has proven central to the modern study of classical field theories such as electromagnetism, and to the development of quantum mechanics. In pure mathematics, he is best known as the inventor of quaternions. Hamilton also invented "icosian calculus", which he used to investigate cycles on an icosahedron that visit each vertex exactly once. An icosahedron is a polyhedron with 20 triangular faces, 12 vertices, and 30 edges.

Observe that no tree is Hamiltonian. Check the following graphs, which one is

Hamiltonian? Which one is traceable? Which one is neither?



The graph G_1 is Hamiltonian and it has vertices of both even and odd degrees. The cycle C_6 is also Hamiltonian and all its vertices have even degrees. The complete graph K_4 is Hamiltonian too, and all of its vertices have odd degrees. Similarly, non-Hamiltonian graphs have all possible scenarios of the parity (even or odd) of vertex degrees. The graph G_2 above is not Hamiltonian and all of its vertices have even degree. The complete bipartite graph $K_{1,3}$ is not Hamiltonian and all vertices have odd degrees. The path P_5 has mixed parity of vertex degree and is not Hamiltonian.

This observation shows that the parity of the degree of the vertices does not have a say on the Hamiltonicity of a graph. It turns out that Hamiltonicity is concerned with the minimum degree of a graph rather than the parity of its vertex degrees. The theorem below from 1952 is due to Gabriel Andrew Dirac.

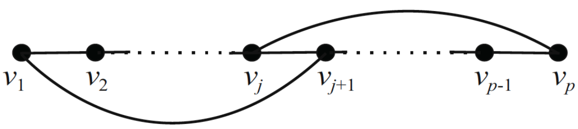
Theorem 4.2.2. (Dirac)

Let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

Proof. Let G be a graph of order $n \geq 3$, and suppose that $\delta(G) \geq \frac{n}{2}$. This implies that G is a connected graph. Let P be a path in G of maximum length. Say,

$$P = (v_1, v_2, \ldots, v_p).$$

Since P is of maximum length, all neighbours of its end vertices v_1 and v_p must be on P. We claim that the picture below must be the case. In other words, there must exist some j such that v_j is a neighbour of v_p and v_{j+1} is a neighbour of v_1.



Suppose it is not the case. Then for every neighbour v_i of v_p we have that v_{i+1} is not a neighbour of v_1. But there are at least $\frac{n}{2}$ neighbours of v_p, all on P. This means that at least $\frac{n}{2}$ vertices on P are not neighbours of v_1. Therefore,

$$\deg(v_1) \leq (p-1) - \frac{n}{2} \leq (n-1) - \frac{n}{2} < n - \frac{n}{2} = \frac{n}{2}.$$

So $\deg(v_1) < \frac{n}{2}$. This contradicts that $\deg(v_1) \geq \delta(G) \geq \frac{n}{2}$. Thus, such a j exists, which leads to the existence of a cycle C containing all the vertices on P where

$$C = (v_1, v_2, \dots, v_j, v_p, v_{p-1}, v_{p-2}, \dots, v_{j+1}, v_1).$$

We claim that C is a Hamiltonian cycle. For the contrary, suppose that C (and thus the path P) does not contain all vertices of G. Moreover, as G is connected it follows that there is a vertex w which is not on P but adjacent to some vertex v_i on P. But then the path in G which begins with w, travels to v_i and then travels around the cycle C visiting all its vertices is a longer path than our maximal path P. This is a contradiction. This means that C contains all vertices of G. Therefore, G is Hamiltonian. ■

The lower bound on the minimum degree in the theorem above is the best possible. To see this, consider the complete bipartite graph $K_{r,r+1}$ which has n vertices where $n = 2r+1$ and minimum degree r. This graph is not Hamiltonian, and $\frac{n}{2}-1 < r < \frac{n}{2}$.

One more remark. There are plenty of Hamiltonian graphs whose minimum degree is relatively small. The cycles C_n are Hamiltonian for all n, and with minimum degree 2. Thus, the condition $\delta(G) \geq \frac{|V(G)|}{2}$ is not necessary for G being Hamiltonian. That is, the converse of Dirac's theorem is not always true; G being Hamiltonian does not imply that $\delta(G) \geq \frac{|V(G)|}{2}$.

Dirac's theorem is a corollary to the following result from 1960 by the Norwegian mathematician Øystein Ore whose proof is similar to the proof above.

Theorem 4.2.3.

Let G be a graph of order $n \geq 3$. Suppose that for every pair of nonadjacent vertices u, v we have that $\deg(u) + \deg(v) \geq n$. Then G is Hamiltonian.

4.2.1 Independence Number

Next we discuss another condition which is sufficient for Hamiltonicity. This condition relates the connectivity $\kappa(G)$ of a graph with a new characteristic of the graph.

Definition 4.2.4. (Independence Number)

A set S of vertices in a graph G is said to be an *independent set* if every pair of vertices in S are nonadjacent. The *independence number*, $\alpha(G)$, of G is the largest cardinality of an independent set of vertices from G.

An independent set of G corresponds to a complete subgraph in its complement $\overline{G}$. We have that $\alpha(E_n) = n$ and $\alpha(K_n) = 1$. Consider the complete bipartite graph $K_{m,n}$, then $\alpha(K_{m,n}) = \max(m, n)$.

Example 4.2. Consider the graphs G and H.

G H

The set $\{a, b\}$ is the largest independent set of vertices from G. So $\alpha(G) = 2$. In graph H, there are two independent sets of vertices of size 3, namely, $\{e, c, a\}$ and $\{d, b, f\}$, and there are no independent sets of size 4. Therefore, $\alpha(H) = 3$.

We introduce some useful notation to be used in the coming proof.

Notation. Let P be a path in a graph, and let x, y be vertices on P. We denote by $P[x, y]$ the portion of P that runs from x to y.
Given a cycle C whose vertices are labeled in a clockwise orientation, we denote by $C^+[x, y]$ the portion of C that runs clockwise from x to y. And we denote by $C^-[x, y]$ the portion of C that runs counter-clockwise from x to y.

We are ready to prove a theorem due to Václav Chvátal and Paul Erdős from 1972.

Theorem 4.2.5. (Chvátal-Erdős)

Let G be a connected graph of order $n \geq 3$. Suppose that $\kappa(G) \geq \alpha(G)$. Then G is Hamiltonian.

Proof. Let G be a connected graph of order $n \geq 3$, where $\kappa(G) \geq \alpha(G)$. If $\kappa(G) = 1$, then $\alpha(G) = 1$, meaning that G is a complete graph, and thus G must be K_2, contradicting that $n \geq 3$. So it must be that $\kappa(G) \geq 2$.

It follows that G contains a cycle because we have established before that $\delta(G) \geq \kappa(G)$ and that if $\delta(G) \geq 2$, then G has a cycle. Choose C to be a longest cycle in G. For contradiction suppose that C is not Hamiltonian, and let v be a vertex of G that is not on C.

Let H be the connected component that contains the vertex v in the graph $G-V(C)$. List all vertices of C that are adjacent to some vertex of H. Notice that such vertices exist as G is a connected graph, label these vertices as

$$c_1, c_2, \ldots, c_r$$

where they are labeled in a clockwise direction around the cycle C. Let d_i be the immediate clockwise successor of c_i on C, and let h_i be a vertex in H adjacent to c_i.

First, we claim that no two vertices in the set $\{c_1, c_2, \ldots, c_r\}$ are consecutive on the cycle C. To see this, suppose for some i we have that c_i and c_{i+1} are consecutive vertices on C. Let P be a path in H from h_i to h_{i+1}. Now consider the cycle formed by replacing the edge $c_i c_{i+1}$ on C with the path $c_i, P[h_i, h_{i+1}], c_{i+1}$. That is, we obtain the cycle

$$c_i, P[h_i, h_{i+1}], C^+[c_{i+1}, c_i]$$

in G which is longer than C, a contradiction as C is a cycle of largest length.

Second, it follows that

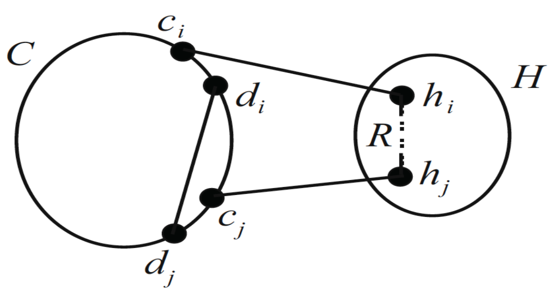
$$\{c_1, c_2, \ldots, c_r\} \cap \{d_1, d_2, \ldots, d_r\} = \emptyset.$$

Suppose not, then $c_i = d_j$ for some i, j. Since d_j is the immediate successor of c_j on C, it follows that c_j and c_i are consecutive on the cycle C, a contradiction.

Third, we deduce that no d_i is adjacent to any vertex in H because if so, then d_i must be equal to some c_j and this is impossible. In particular, as the vertex v is in H, no vertex d_i is adjacent to v.

Fourth, observe that the set $\{c_1, c_2, \ldots, c_r\}$ is a cut set of G. To see this, notice that when the vertices $c_1, c_2, \ldots, c_r$ are deleted from G, there will be no path from v to d_1 as any path in G from v to d_1 must contain some c_i.

Fifth, since $\kappa(G)$ is the size of a smallest cut set, this leads to $\kappa(G) \leq r$, and so $r \geq 2$. Moreover, as $\alpha(G) \leq \kappa(G)$, we have that $\alpha(G) \leq r$. Since $\alpha(G)$ is the size of the largest independent set, the set $\{v, d_1, d_2, \ldots, d_r\}$ which contains $r+1$ vertices is not an independent set and so it must contain at least one edge. We know that v is nonadjacent to any d_i, thus some d_i must be adjacent to some d_j with $i < j$.



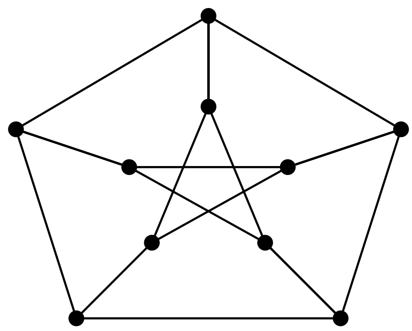
Let R be the path joining h_i with h_j in H. We now have the following cycle

$$c_i, R[h_i, h_j], C^-[c_j, d_i], C^+[d_j, c_i].$$

This is a cycle longer than C because it uses all edges of C except two edges $c_i d_i$ and $c_j d_j$, and additionally it uses at least three new edges, namely, $c_i h_i$, $h_j c_j$, and $d_i d_j$, this is a contradiction. Therefore, our assumption that the cycle C is not Hamiltonian has led to a contradiction. Thus, C contains all vertices in G, and so G is Hamiltonian. ■

Again, the inequality in this theorem is sharp. In other words, graphs in which $\kappa(G) \geq \alpha(G) - 1$ are not necessarily Hamiltonian. For instance, the complete bipartite graph $K_{r,r+1}$ is not Hamiltonian while $\kappa = r$ and $\alpha = r + 1$. Another example is the very famous Petersen graph named after Julius Petersen shown below. It has $\kappa = 3$ and $\alpha = 4$. The Petersen graph is the complement of the line graph of K_5, and it serves as a counterexample to many conjectures in graph theory.

The Petersen Graph



Prove the following fact.

Lemma 4.2.6.

The Petersen graph is not Hamiltonian.

4.2.2 Forbidden Subgraphs

The next theorems relates Hamiltonicity with forbidden graphs.

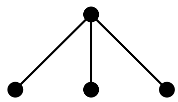
Definition 4.2.7.

Let H and G be graphs. We say that G is H*-free* if H is not an induced subgraph of G.

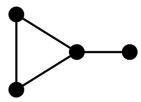
More generally, if S is a collection of graphs, then we say G is S-free if G is H-free for every graph H in S. For example, the Petersen graph is $\{C_3, C_4\}$-free.

Consider the following graphs. We note that the graph $K_{1,3}$ is called a *claw*.

$K_{1,3}$



Z_1



Notice that any Hamiltonian graph must be 2-connected meaning that $\kappa \geq 2$, that is, it has no cut vertices. The theorem below is due to Goodman and Hedetneimi from 1974.

Theorem 4.2.8. (Goodman-Hedetneimi)

Suppose that G is 2-connected and $\{K_{1,3}, Z_1\}$-free. Then G is Hamiltonian.

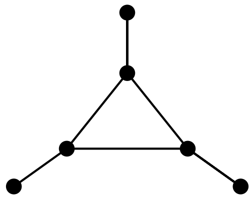
Proof. Suppose that G is 2-connected and $\{K_{1,3}, Z_1\}$-free. Since $\delta(G) \geq \kappa(G) \geq 2$, the graph G must contain a cycle.

Let C be a longest cycle in G, and suppose it is not Hamiltonian. Let z be a vertex in G which is not on C and adjacent to some vertex v on C. Let u be the immediate predecessor of v on C and w be the immediate successor of v.

If wz is an edge, then we can replace the edge vw on C by the path v, z, w obtaining a longer cycle, a contradiction as C is a longest cycle. So w and z are nonadjacent. Similarly, u and z are nonadjacent.

Now if uw is an edge, then the graph Z_1 is an induced subgraph by the vertices $\{z, u, v, w\}$ in G, and this cannot happen since G is Z_1-free. Otherwise, if uw is not an edge, then $K_{1,3}$ is an induced subgraph by the vertices $\{z, u, v, w\}$ in G, again a contradiction as G is $K_{1,3}$-free. This shows that C is a Hamiltonian cycle, and so G is Hamiltonian. ■

Let us state another result due to Duffus, Gould, and Jacobson from 1980 which involves forbidden subgraphs. The claw graph $K_{1,3}$ is still involved here together with another graph shown below. Claw-free graphs are of great importance in graph theory. Let us introduce the graph N below.



N

Theorem 4.2.9.

Let G be a $\{K_{1,3},\, N\}$-free graph.

(i) If G is connected, the G is traceable.

(ii) If G is 2-connected, then G is Hamiltonian.

Chapter 5

Planarity

Draw your graphs elegantly.

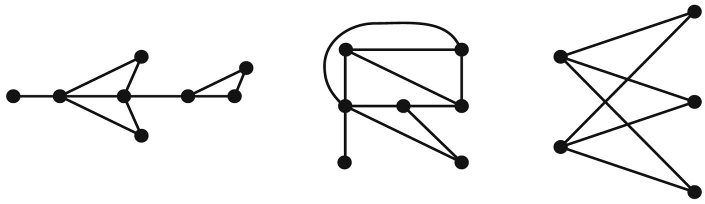
5.1 Planar Graphs

Definition 5.1.1. (Planar Graph)

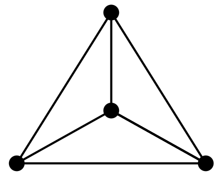
A graph is said to be *planar* if it can be drawn on the plane in such a way that its edges intersect only at their end vertices.

We call such drawing of a graph a *planar representation* or a *planar embedding* of the graph. A graph that has no planar representation is called *nonplanar*.

The following are examples of planar graphs. The third drawing is not a planar representation as edges cross each other, although the graph is planar. You may visualise untangling it to form a planar representation.



To prove that a particular graph is planar all what we need to do is to exhibit a planar representation of the graph. For instance, we show that K_4 is planar by drawing the following diagram of K_4 where edges intersect only at their end vertices.



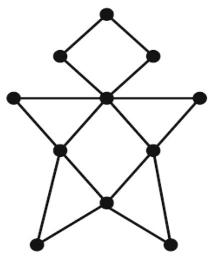
What are your thoughts on K_5? Investigate $K_{2,3}$ and $K_{3,3}$ as well. Are all trees planar?

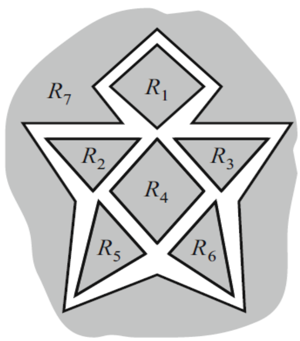
How one could possibly show that a graph is nonplanar? To do so, one needs to show that it is impossible to obtain a planar representation of the graph, in other words, every drawing of the graph has some edges which cross each other at a point different from the vertices. Let us develop some tools towards answering this question.

Definition 5.1.2.

A *region* in a planar representation of a graph is a maximal area of the plane where any two points in this area can be joined by a curve that does not intersect any part of the graph.

The graph below has seven regions. The outer region R_7 is called the *exterior* region. The others: regions $R_1, R_2 \dots, R_6$ are *interior*, and each region of them is bounded completely by the edges of the graph.





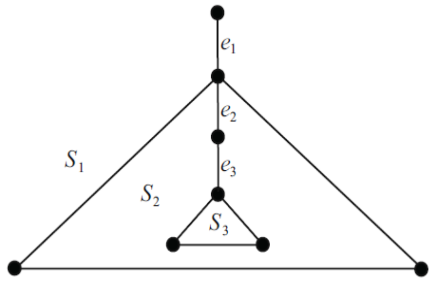
In a planar representation of a graph, a single edge comes into contact with either one or two regions. We say that an edge e *bounds* a region R if the edge e comes into contact with R and with another region different from R. We define the *bound degree* of R to be the number of edges that bound the region R. The bound degree of R is denoted by $b(R)$.

In the example above we have the following bound degrees.

- $b(R_2) = b(R_3) = b(R_5) = b(R_6) = 3$
- $b(R_1) = b(R_4) = 4$
- $b(R_7) = 12$

The graph below has three regions S_1, S_2, S_3. The edge e_1 is in contact with only region S_1, and so it does not bound S_1. The edges e_2 and e_3 are only in contact with region S_2. Each of the six remaining edges is in contact with two regions. Note that e_1, e_2, e_3 do not contribute to the bound degree of any region.

$$b(S_1) = 3 \qquad b(S_2) = 6 \qquad b(S_3) = 3$$



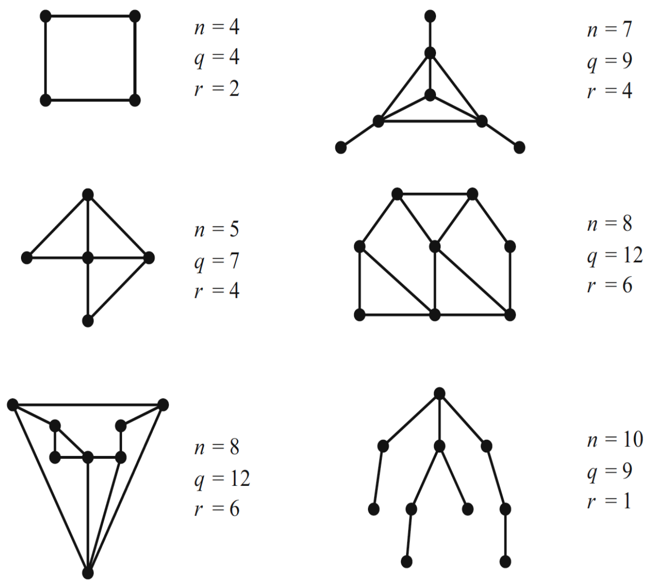
A graph which contains no cycles (a forest) is planar and contains exactly one region of bound degree 0.

Lemma 5.1.3.

Let G be a planar graph which contains a cycle and with r regions: $R_1, R_2, \ldots, R_r$. Then, $r \geq 2$ and

$$3r \leq \sum_{i=1}^{r} b(R_i) \leq 2|E|.$$

Investigate the following planar graphs. For each graph, count the number of vertices n, edges q, and regions r. *What is the relationship between these three numbers?*



To assure you, the numbers are in fact related, their relationship is called *Euler's Formula*.

One more important point to bring up. Two of the drawings above are actually for the same graph. Try yourself to draw different planar representations of the same planar graph. One tends to suggest the following.

All planar representations of the same graph have the same number of regions!

5.2 Euler's Formula

Leonhard Euler was the first to discover the relationship between the number of vertices, edges, and regions in planar graphs. Before we dig into the next theorem, observe that any tree is planar and any planar representation of a tree has exactly one region.

Theorem 5.2.1. (Euler's Formula)

Let G be a connected planar graph with n vertices, q edges, and r regions. Then

$$n - q + r = 2.$$

Proof. We use induction on the number of edges q. When $q = 0$, then as the graph is connected it must be K_1 which has exactly one region. Thus,

$$n - q + r = 1 - 0 + 1 = 2.$$

Now let $q \geq 1$, and suppose that any connected planar graph with fewer than q edges satisfies Euler's Formula. Choose an arbitrary connected planar graph G with q edges. Let n be the number of vertices of G and r be the number of regions in its planar representation. We need to show that $n - q + r = 2$.

Case 1. Suppose that G is a tree. Then G has exactly one region. Moreover, our knowledge on trees tells us that $q = n - 1$. Therefore,

$$n - q + r = n - (n - 1) + 1 = 2,$$

and the result holds in this case.

Case 2. Suppose that G is not a tree. As G is connected, G must contain at least one cycle C. Choose some edge e on the cycle C, and consider the graph $\hat{G} = G - e$ which is still connected. Compared to G, the graph $\hat{G}$ is planar and has the same number of vertices, one edge fewer, one region fewer since the two regions that were bounded by e in G became one region in $\hat{G}$. Now $\hat{G}$ has $q - 1$ edges and so by the induction hypothesis it satisfies Euler's Formula. Let $\hat{n}$, $\hat{q}$, $\hat{r}$ be the number of vertices, edges, and regions, respectively, in $\hat{G}$. Thus, by induction hypothesis, we have that

$$\begin{aligned} \hat{n} - \hat{q} + \hat{r} = 2 &\Longrightarrow n - (q - 1) + (r - 1) = 2 \\ &\Longrightarrow n - q + r = 2. \end{aligned}$$

Therefore, the result holds in this case too, and the induction is complete. ■

We can use our work above to find Euler's Formula for any (not necessarily connected) planar graph.

Corollary 5.2.2.

[Generalised Euler's Formula] Let G be a planar graph with n vertices, q edges, r regions, and k connected components. Then

$$n - q + r = k + 1.$$

5.2.1 Nonplanar Graphs

Let us use Euler's Formula to answer one of our previous questions.

Theorem 5.2.3.

The graph $K_{3,3}$ is nonplanar.

Proof. Suppose for the sake of contradiction that $K_{3,3}$ is planar, and so we have a planar representation of $K_{3,3}$. By Euler's Formula such representation must have 5 regions since

$$r = 2 - n + q = 2 - 6 + 9 = 5.$$

Let S be the sum of the bound degrees of all the regions R_1, R_2, R_3, R_4, R_5 in the planar representation of the graph. That is,

$$S = \sum_{i=1}^{5} b(R_i).$$

On one hand, as each edge could contribute to the bound degree of at most two regions, it must be that $S \leq 2q = 2(9) = 18$. On the other hand, as $K_{3,3}$ is bipartite, it does not contain odd cycles (such as C_3), and so each region is bounded by at least 4 edges, that is, $b(R_i) \geq 4$ for all $1 \leq i \leq 5$. Thus, $S \geq 5(4) = 20$. We have reached a contradiction. Therefore, $K_{3,3}$ is nonplanar. ■

In general, a graph with n vertices may have at most $\frac{1}{2}n(n-1)$ edges. For example, the complete graph K_n achieves this upper bound. Nevertheless, when the graph is planar we have more restriction on the number of edges.

Theorem 5.2.4.

Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then,

$$|E| \leq 3|V| - 6.$$

Furthermore, if equality holds, then every region is bounded by three edges.

Proof. Suppose that G has n vertices, q edges, and r regions called $R_1, R_2, \ldots, R_r$. If $r = 1$, we have only one region, and G is a forest with, say, k connected components. Thus,

$$q = n - k \leq n - 1 \leq 3n - 6.$$

(Prove the last inequality by induction where $n \geq 3$.)

Otherwise, $r \geq 2$, and so there are at least two regions and G has at least one cycle. It follows that the bound degree $b(R_i)$ of each region is at least 3. Consider the sum

S of bound degrees of all regions.

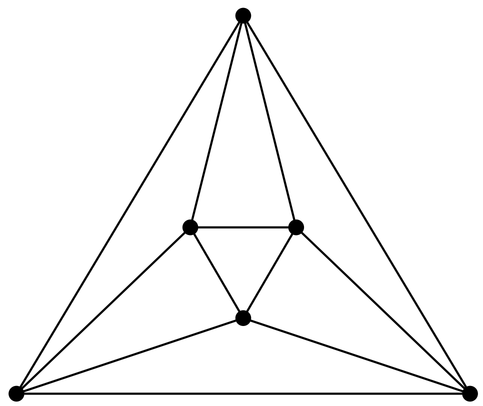
$$S = \sum_{i=1}^{r} b(R_i).$$

We know that $S \geq 3r$. Moreover, as each edge contributes to the bound degree of at most two regions, it must be that $S \leq 2q$. By the Generalised Euler's Formula we know that $r = k + 1 - n + q \geq 2 - n + q$. Putting all of these conditions together we get that

$$\begin{aligned} 3r \leq S \leq 2q &\iff 3r \leq 2q \\ &\iff 3(2 - n + q) \leq 3r \leq 2q \\ &\iff 6 - 3n + 3q \leq 2q \\ &\iff q \leq 3n - 6. \end{aligned}$$

Finally, when equality holds, that is, $q = 3n - 6$, then $3r = 2q$. Thus, $S = 3r$ which means that $b(R_i) = 3$ for each region R_i. ■

Example 5.1. Below is a planar graph with $n = 6$, $q = 12$, and $r = 8$. Since $q = 3n - 6$, every region is bounded by three edges.



Remark. The converse is not necessarily true. There are planar graphs where every region is bounded by 3 edges but $q < 3n - 6$.

We use the above theorem to establish another nonplanarity result.

Theorem 5.2.5.

The complete graph K_5 is nonplanar.

Proof. There are 5 vertices and 10 edges in K_5. By the contrapositive of the previous theorem, since $q = 10 > 9 = 3n - 6$, we infer that K_5 is nonplanar. ■

Theorem 5.2.6.

If G is a planar graph, then $\delta(G) \leq 5$.

Proof. Suppose that G is planar with n vertices and q edges. If $n \leq 6$, then all vertices have degree less than or equal to 5. So suppose that $n > 6$. Towards a contradiction, suppose $\delta(G) \geq 6$. This means that every vertex has degree at least 6. Let D be the sum of the degrees of all vertices. Using the Handshaking Lemma, we obtain the following.

$$\begin{aligned} 6n \leq D = 2q &\Longrightarrow 6n \leq 2q \\ &\Longrightarrow 3n \leq q \leq 3n - 6 \\ &\Longrightarrow n \leq n - 2 \\ &\Longrightarrow 0 \leq -2. \end{aligned}$$

This is impossible. Therefore it must be that $\delta(G) \leq 5$. ■

Let us collect our knowledge on planar graphs in one place.

Theorem 5.2.7.

Let G be a planar graph with n vertices, q edges, r regions, and k connected components. Then the following holds.

(i) $n - q + r = k + 1$.

(ii) $q \leq 3n - 6$.

(iii) $\delta(G) \leq 5$.

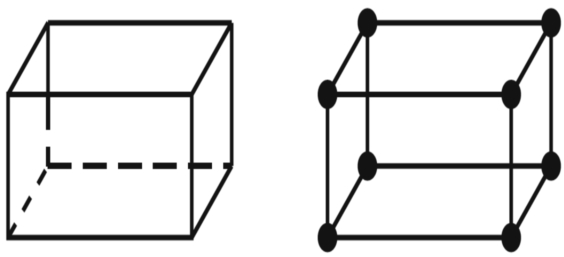
(iv) *G is a forest if and only if $r = 1$.*

(v) *G contains at least $r - 1$ cycles.*

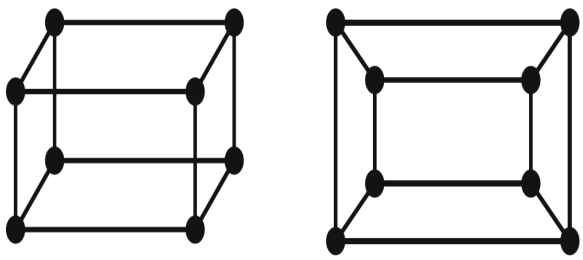
5.3 Regular Polyhedra

A *polygon* is a closed shape in the plane made of straight line segments. In other words, a polygon is a flat surface determined by straight edges and vertices. The triangle △, and the square □ are examples of polygons. A *polyhedron* is a closed solid in the three-dimensional space that is bounded by polygons. We think of these polygons as the faces of the polyhedron. The Great Pyramid of Giza (Khufu), cubes, bricks are all examples of polyhedra.

Given a polyhedron, we can associate to it a graph in a natural way. We simply think of the corners of the polyhedron as vertices, and the boundaries of its faces as edges. The resultant graph is called the *skeleton* of the polyhedron. For example, consider a cube which is a polyhedron with 6 faces. As shown below, the skeleton graph of a cube has 8 vertices, and 12 edges.



The skeleton graph of the cube is planar. To see this, imagine taking hold of the top face and stretching it and translating it downward to the same level as the bottom face so that its edges form the boundary of the exterior region of the graph. The skeleton graph of the cube and its planar representation are demonstrated below.



What we did for the cube works for every polyhedron. It follows that

The skeleton graph of any polyhedron is a planar graph.

The faces of a polyhedron directly correspond to the regions of the planar representation of its skeleton graph. Moreover, notice that any region in the skeleton graph has a bound degree of at least 3 edges, and any vertex (or corner) has at least 3 neighbours.

The fact that any polyhedron has an associated planar graph allows us to utilize our knowledge on planar graphs in studying polyhedra.

The following result is a direct consequence of Euler's Formula for planar graphs.

Theorem 5.3.1.

Given a polyhedron with V many vertices, E many edges, and F many faces, then

$$V - E + F = 2.$$

Given a polyhedron P, define $\rho(P)$ to be the minimum bound degree of all the regions in a planar representation of the skeleton graph of P. In other words,

$$\rho(P) = \min\{b(R) \mid R \text{ is a region of } P\}.$$

We know that every vertex in P has degree at least 3, and so $\delta(P) \geq 3$. Also every region (face) is bounded by at least 3 edges, and so $\rho(P) \geq 3$. Moreover, every edge in P comes in contact with exactly 2 regions. This means that every edge contributes to the bound degree of the two regions it is in contact with. Thus, the sum of all the bound degrees of regions is equal to twice the number of edges. By the Handshaking Lemma we obtain that the sum of all vertex degrees must be equal to the sum of the bound degrees of all regions.

Lemma 5.3.2.

Let P be a polyhedron with E many edges. Then $\delta(P) \geq 3$ and $\rho(P) \geq 3$. Moreover,

$$\sum_{R \text{ region}} b(R) = 2E = \sum_{v \text{ vertex}} \deg(v).$$

The next theorem says that any polyhedron must contain at least one face which is bounded by at most 5 edges.

Theorem 5.3.3.

Let P be a polyhedron. Then

$$3 \leq \rho(P) \leq 5.$$

Proof. Any face (region) of a polyhedron is formed from at least 3 edges (one or two edges cannot form a closed region in the plane). It follows that $3 \leq \rho(P)$. It remains to show that $\rho(P) \leq 5$. We think of the polyhedron P as its planar skeleton graph.

Suppose that P has V many vertices, E many edges, and F many faces. Towards a contradiction, suppose that $\rho(P) \geq 6$, in other words, every face of the F many faces is bounded by 6 edges or more. Hence,

$$2E = \sum_{R \text{ region}} b(R) = b(R_1) + b(R_2) + \ldots + b(R_F) \geq 6F.$$

This yields that $E \geq 3F$. On the other hand, since $\delta(P) \geq 3$ we get that

$$2E = \sum_{u \text{ vertex}} \deg(u) = \deg(u_1) + \deg(u_2) + \ldots + \deg(u_V) \geq 3V.$$

Thus, $2E \geq 3V$. Using these two inequalities and Euler's Formula for Polyhedra we get that

$$E = V + F - 2 \leq \frac{2}{3}E + \frac{1}{3}E - 2 = E - 2,$$

and this is clearly a contradiction. Therefore, there must be a face in P which is bounded by fewer than 6 edges. This shows that $\rho(P) \leq 5$. ■

5.3.1 Platonic Solids

A *regular polygon* is a polygon which is equilateral and equiangular. In other words, a polygon is regular if all of its sides are of equal length, and all of its angles are of equal measure. An equilateral triangle and a square are both regular polygons. Regular polygons are used in the construction of regular polyhedra.

Definition 5.3.4. (Regular Polyhedron)

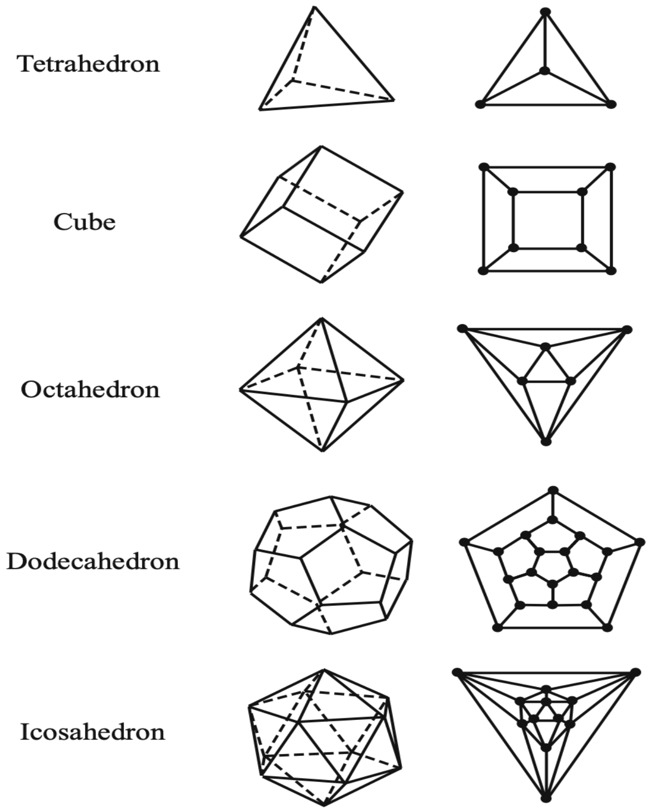
A *regular polyhedron* is a polyhedron such that

(i) each of its faces is identical to the same regular polygon, and

(ii) every vertex meets the same number of faces as every other vertex.

The cube is a regular polyhedron; all of its faces are squares and each vertex meets three faces. What are the other regular polyhedra? The answer to this question is a geometric fact that has been known to the ancient Greeks since 2300 years ago. The Greeks knew that there are only five regular polyhedra! They are called the *Platonic Solids*. The tetrahedron (4 faces), the hexahedron or cube (6 faces), the octahedron (8 faces), the dodecahedron (12 faces), and the icosahedron (20 faces).

The Platonic Solids



The ancient Greeks studied the Platonic solids extensively. The Greek philosopher Plato wrote about these solids in the dialogue *Timaeus* (360 BC) where he described the formation of the universe and explained its beauty and order. He associated each of the four classical elements of matter with a regular polyhedron: the tetrahedron with fire, the octahedron with air, the icosahedron with water, and the cube with earth. The remaining Platonic solid, the dodecahedron, represents the universe as a whole. The Greek mathematician Theaetetus of Athens ($417 - 369$ BC) proved that the Platonic solids are the only regular polyhedra. Euclid of Alexandria in Book XIII of his treatise the *Elements* (300 BC) provided a mathematical study describing the construction and properties of the five Platonic solids.

We will prove the fact that the Platonic solids are the only regular polyhedra using graph-theoretic techniques. We start by an easy consequence of Lemma 5.3.2 applied to a regular polyhedron.

Corollary 5.3.5.

Let P be a regular polyhedron with V vertices, E edges, and F faces. Suppose that every vertex has degree r and every face has bound degree k. Then,

$$rV = 2E = kF.$$

Theorem 5.3.6.

There are exactly five regular polyhedra.

Proof. Let P be a regular (convex) polyhedron. We may think of P as its associated planar skeleton graph. Suppose that P has V vertices, E edges, and F faces. Since all faces are the same, each of them is bounded by the same number of edges, say k edges. This means that $\rho(P) = k$, and so by Theorem 5.3.3, we know that $3 \leq k \leq 5$. This means that either all the faces of P are equilateral triangles, or all are squares, or all are pentagons (5-sided regular polygons). Suppose that each vertex in P meets r faces, so P is a regular graph of degree r. Thus $rV = 2E = kF$.

Using Euler's Formula, $V - E + F = 2$, we get that

$$\begin{aligned}
8 &= 4V - 4E + 4F \\
&= 4V - 2E + 4F - 2E \\
&= 4V - rV + 4F - kF \\
&= (4 - r)V + (4 - k)F.
\end{aligned}$$

Let us collect what we know in one place.

- $V, F \in \mathbb{Z}^+$ and $3 \leq k \leq 5$ and $r \geq 3$.
- $rV = kF$.
- $(4 - r)V + (4 - k)F = 8$.

Let us examine which pairs (r, k) work. Suppose that $k = 3$. Then $rV = 3F$ and $(4 - r)V + F = 8$. Thus,

$$\begin{aligned}
(4 - r)V + F = 8 &\implies 3(4 - r)V + 3F = 24 \\
&\implies 3(4 - r)V + rV = 24 \\
&\implies 12V - 3rV + rV = 24 \\
&\implies 12V - 2rV = 24 \\
&\implies (6 - r)V = 12.
\end{aligned}$$

Since V is a positive integer, the last equation implies that $r < 6$. This shows that when $k = 3$, then $r \leq 5$. We leave it to the reader to show that $k = 4$ implies that $r = 3$, and $k = 5$ implies that $r = 3$. Therefore, the analysis above produces only five possible cases.

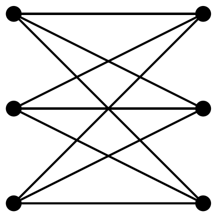
	r	k	V	F	Polyhedron
Case I	3	3	4	4	Tetrahedron
Case II	3	4	8	6	Cube
Case III	4	3	6	8	Octahedron
Case IV	3	5	20	12	Dodecahedron
Case V	5	3	12	20	Icosahedron

The fact that the tetrahedron is the only regular polyhedron with $V = 4 = F$ requires a geometrical argument that we will omit. This applies to the remaining cases as well. ■

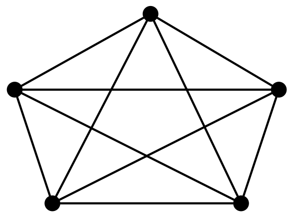
5.4 Kuratowski's Theorem

The Polish mathematician Kazimierz Kuratowski (1896 – 1980) found a clever way to know which graphs are planar and which are not. Let us start searching for nonplanar graphs. In Theorem 5.2.3 and Theorem 5.2.5 we established that both graphs $K_{3,3}$ and K_5 are nonplanar. These two graphs are infamously known as the real enemies of planarity.

$K_{3,3}$



K_5

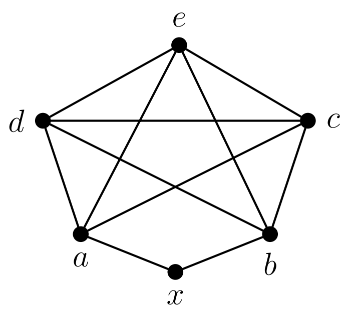


Next, suppose that G is a graph which contains K_5 as a subgraph, then one can see that G is nonplanar too. For if G were planar, then a planar representation of G would give us a planar representation of K_5, a contradiction. It follows that we can add to our collection of nonplanar graphs every graph which contains $K_{3,3}$ or K_5 as a subgraph.

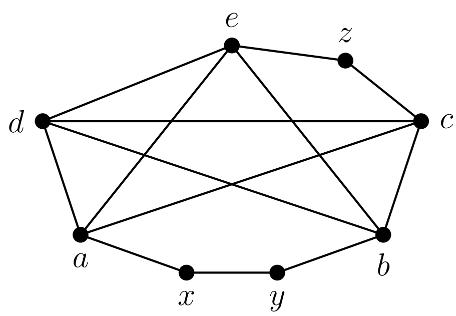
Lemma 5.4.1.

A graph G is planar if and only if every subgraph of G is planar.

What else should we add to our collection of nonplanar graphs? Below is a graph which does not contain $K_{3,3}$ nor K_5 as a subgraph. Is it planar though?



The graph above is obtained from K_5 by replacing the original edge ab by a new path (a, x, b) of length two. Clearly, if this new graph has a planar representation, then so does K_5. Thus, we add this graph to our collection of nonplanar graphs. Following the same line of thinking, we can even add more graphs to the collection. We can add any graph which can be obtained from $K_{3,3}$ or K_5 by replacing edges by paths of any length (essentially, sticking new vertices on edges), like this graph.



This experimentation motivates the following definition.

Definition 5.4.2. (Subdivision)

Let G be a graph.

- A *subdivision* of an edge e in G is the replacement of e by a new path starting at one of the end vertices of e and ending at the other one.
- A graph H is called a *subdivision* of G if H can be obtained from G by applying finitely many subdivisions of edges.

For example, the last two graphs above are subdivisions of K_5. The path graph P_7 is a subdivision of P_3. The cycle C_6 is a subdivision of C_4. We may think of the subdivision of an edge by taking the edge and dividing it into pieces by sticking new vertices on it.

Lemma 5.4.3.

A graph G is planar if and only if every subdivision of G is planar.

From all the discussion above, we conclude that any graph which contains a subdivision of $K_{3,3}$ or K_5 as a subgraph is nonplanar. Kuratowski's theorem tells us that these graphs make up the complete collection of all nonplanar graphs!

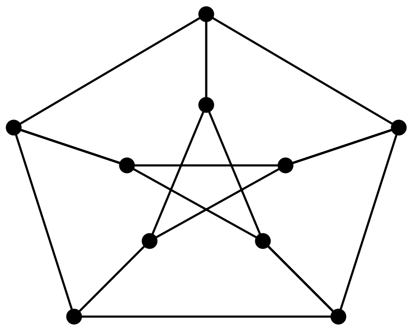
Theorem 5.4.4. (Kuratowski's Theorem)

A graph G is planar if and only if G does not contain any subdivision of $K_{3,3}$ or K_5 as a subgraph.

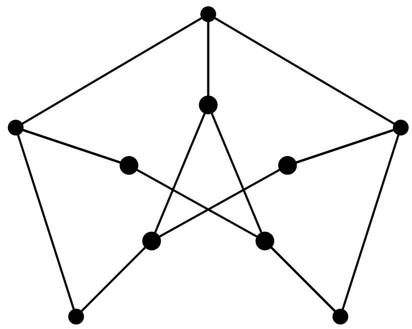
Kuratowski who worked in topology proved his beautiful theorem in 1930. It is worth mentioning that it is believed that the Russian mathematician Lev Pontryagin (1908 – 1988), who was blind his entire adult life, proved this result sometime earlier but he did not publish his proof in any refereed journal. That is why, in Russia the result is sometimes referred to as the Pontryagin-Kuratowski's theorem. To sprinkle more suspense, it turned out that later in 1930, after Kuratowski submitted his paper, there were two American mathematicians, Orrin Frink (1901 – 1988) and Paul Smith (1900 – 1980), who submitted together a paper proving the same theorem as well. They later withdrew their submission after they became aware that Kuratowski had preceded them. At the end, the theorem is generally credited to Kuratowski alone.

Example 5.2. By Kuratowski's Theorem, it follows that the Petersen graph is not planar since it contains a subgraph H which is a subdivision of $K_{3,3}$.

Petersen Graph



A subgraph H



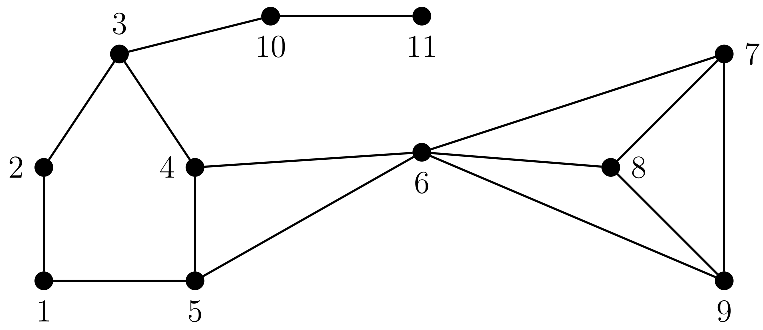
We have already proved the forward direction of Kuratowski's Theorem. More precisely, we proved its contrapositive; if a graph G contains a subdivision of $K_{3,3}$ or K_5 as a subgraph then G is nonplanar. We need to work much harder to establish the other direction, that is, if a graph G does not contain any subdivision of $K_{3,3}$ or K_5, then G is planar. This direction says that $K_{3,3}$ or K_5 are the only real obstacles for planarity. A starting point towards proving Kuratowski's Theorem is to learn what a block in a graph is.

Definition 5.4.5. (Block)

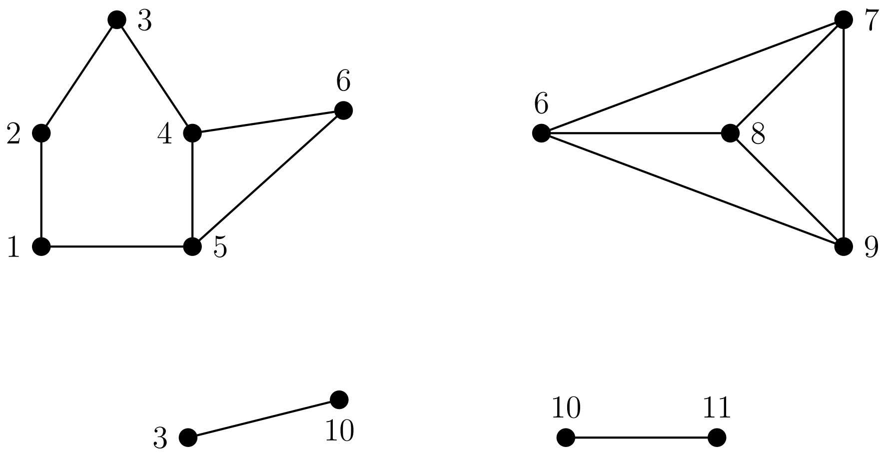
A *block* in G is a maximal connected induced subgraph with no cut vertex.

This means that a block B is a connected induced subgraph with no cut vertex and if we add any vertex to B, then a cut vertex will arise in B. Notice that a block is a maximal 2-connected induced subgraph.

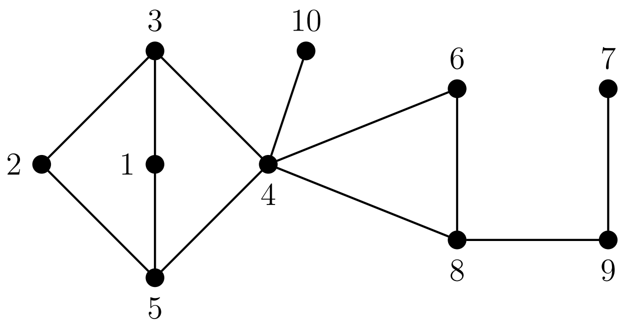
Example 5.3. Find all the blocks of the graph below.



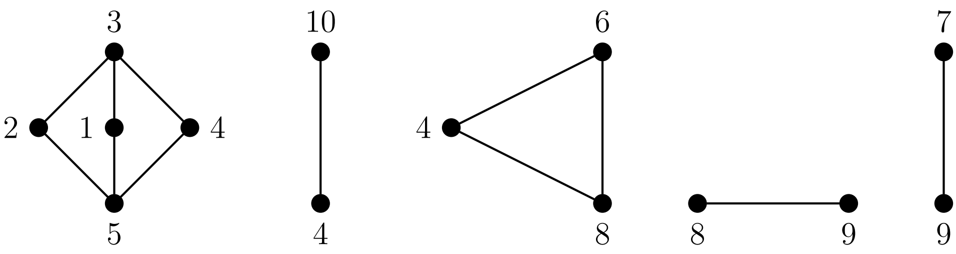
This graph has four blocks shown below.



Example 5.4. Find the blocks of the graph below.



The graph above has 5 blocks.



Lemma 5.4.6.

Let G be a connected graph.

- *Any two distinct blocks of G have at most one vertex in common.*
- *If a vertex v belongs to two distinct blocks of G, then v is a cut vertex of G.*
- *Let B_1 and B_2 be two blocks of G containing the cut vertex v. If $x \in B_1$ and $y \in B_2$ and x, y are distinct from v, then every xy-path in G passes through v.*

Chapter 6

Graph Colouring

Colour the world, neighbouring countries coloured differently.

6.1 Chromatic Number

We may think of positive integers $1, 2, 3, 4, 5, \ldots$ as colours. For instance, we could associate 1 with red, 2 with blue, 3 with green, 4 with white, 5 with black, 6 with yellow, 7 with purple, 8 with orange, and so on.

Definition 6.1.1. (Graph Colouring)

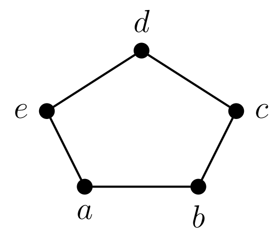
A *k-colouring* of a graph G is a function $K : V_G \to \{1, 2, 3, \ldots, k\}$ such that whenever u, v are adjacent vertices, then $K(u) \neq K(v)$.

A colouring of a graph assigns to every vertex one colour bearing in mind that adjacent vertices are coloured differently. Sometimes we say "proper colouring" to emphasize that adjacent vertices are coloured differently. If a k-colouring exists for a graph G, we say that G is *k-colourable*. In other words, G is k-colourable if k many colours are enough to colour the vertices of G such that adjacent vertices have different colours. Notice that if G is k-colourable, then G is t-colourable for any integer $t \geq k$.

Remark. Suppose that $K : V_G \to \{1, 2, \ldots, k\}$ is a colouring of a graph G. Choose a colour $r \in \{1, 2, \ldots, k\}$. Let $V_r = \{v \in V_G \mid K(v) = r\}$. So V_r is the set of all vertices coloured with colour r. Since no adjacent vertices have the same colour, it follows that no two vertices in V_r are adjacent. Thus, V_r is an independent subset of vertices. (Note that $V_r = K^{-1}(r)$.)

Any graph G of order n is n-colourable. We simply colour every vertex with a unique colour (in this case, the colouring function K will be injective). Consequently, the challenge is to colour the graph with less than n colours.

Let us colour the cycle C_5 with vertices labeled a, b, c, d, e.



The cycle C_5 is 3-colourable. One colouring of C_5 is the function

$$K : \{a, b, c, d, e\} \to \{1, 2, 3\}$$

where $K(a) = 1$, $K(b) = 2$, $K(c) = 1$, $K(d) = 2$, $K(e) = 3$.

Can you colour C_5 using only 2 colours? An interesting question is: what is the least number of colours needed to colour a given graph?

Definition 6.1.2. (Chromatic Number)

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the least number of colours needed to colour the vertices of G such that adjacent vertices are coloured differently.

In other words, $\chi(G)$ is the smallest positive integer k such that the graph G is k-colourable. If $\chi(G) = k$, it means that k colours are sufficient to properly colour G, and it is impossible to colour G with less than k colours.

Since any two vertices in K_n are adjacent, it must be that $\chi(K_n) = n$. As in the empty graph E_n no pair of vertices are adjacent, we may colour all vertices with the same colour, so $\chi(E_n) = 1$. For any graph G which has at least one edge we have that $\chi(G) \geq 2$. Check that it is impossible to colour C_5 with two colours. Consequently, as C_5 is 3-colourable, it follows that $\chi(C_5) = 3$. Here are more examples.

Graph	Chromatic number
K_n	n
E_n	1
$K_{m,n}$	2
P_n for $n \geq 2$	2
C_n for even n	2
C_n for odd n	3

Remark. A graph G is k-colourable if and only if $\chi(G) \leq k$.

Lemma 6.1.3.

Let G be a graph of order n. Then $\chi(G) \leq n$.

Proof. Label the vertices of G as $v_1, v_2, \ldots, v_n$. The function $K : V_G \to \{1, 2, \ldots, n\}$ given by $K(v_i) = i$ is an n-colouring of G. Therefore, n many colours, or possibly less, are enough to colour G. Thus, the chromatic number is no more than n. ■

One can see that if H is a subgraph of G, then any colouring of G induces a colouring of H. So any collection of colours which properly colour a graph G suffice to colour any subgraph of G. This observation gives the following result.

Lemma 6.1.4.

Suppose that H is a subgraph of G. Then $\chi(H) \leq \chi(G)$.

The *complete multipartite graph* $K_{n_1,n_2,\ldots,n_k}$ is the graph whose vertex set is the union of k many pairwise disjoint sets $V_1, V_2, \ldots, V_k$ where $|V_i| = n_i$ for each i. And the edge relation is defined by the rule:

$$u \in V_i \text{ and } v \in V_j \text{ are adjacent if and only if } i \neq j.$$

As no two vertices in the same partite set V_i of $K_{n_1,n_2,\ldots,n_k}$ are adjacent, we can colour all vertices in V_i with same colour. Moreover, any two different partite sets must have different colours. It follows that at least we need k many colours to colour $K_{n_1,n_2,\ldots,n_k}$. That is,

$$\chi(K_{n_1,n_2,\ldots,n_k}) = k.$$

Furthermore, we have the following result.

Lemma 6.1.5.

Let G be a graph and $k \geq 1$. Then $\chi(G) \leq k$ if and only if G is a subgraph of a multipartite graph $K_{n_1,n_2,\ldots,n_k}$.

6.1.1 Greedy Algorithm

Let us introduce one of the graph colouring algorithms.

Algorithm 6.1.6. (Greedy Algorithm)

Given a graph G of order n.

(i) Label the vertices as $v_1, v_2, v_3, \ldots, v_n$.

(ii) The available colours are $1, 2, 3, \ldots, n$.

(iii) Let $i = 1$.

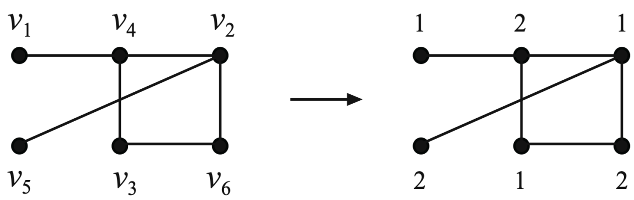
(iv) Colour v_i with the first available colour that has not been used in colouring any of the previously coloured neighbours of v_i.
This means, colour v_i with the least integer in the following set,

$$\{1, 2, \ldots, n\} \setminus \{K(v_j) \mid v_j v_i \in E_G \text{ and } 1 \leq j < i\},$$

where $K(v_j)$ is the colour of v_j.

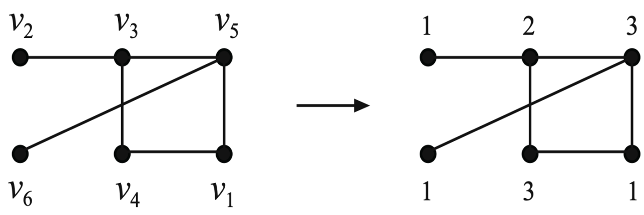
(v) If $i = n$, then stop. Otherwise, increment i by 1 and jump to step (iv).

For example, here is an application of the greedy algorithm to colour the graph on the left where the colours 1 and 2 were only used.



1. Colour v_1 with colour 1.
2. Colour v_2 with colour 1 since v_1 is nonadjacent to v_2.
3. Colour v_3 with colour 1 since v_1, v_2 are nonadjacent to v_3.
4. Colour v_4 with colour 2 since v_1, v_2, v_3 are adjacent to v_4 and they are all coloured with colour 1.
5. Colour v_5 with colour 2 since v_2 is the only previously coloured neighbour of v_5 and it has colour 1.
6. Colour v_6 with colour 2 since both v_2, v_3 are the only previously coloured neighbours of v_6 and they are coloured with colour 1.

We remark that the colouring produced by the greedy algorithm depends heavily on the vertex labelling in the very beginning. Different labellings may produce different colourings. For example, when we change the labelling of the vertices in the graph above, we obtain a different colouring where the colours $1, 2, 3$ were used by the greedy algorithm as shown below.



Lemma 6.1.7.

Let G be a graph and let K be a colouring function produced by the greedy algorithm. Then for each vertex v of G, we have that

$$K(v) \le \deg(v) + 1.$$

Proof. Label the vertices of G as $v_1, v_2, \dots, v_n$ and run the greedy algorithm to obtain a colouring function K. Choose any vertex v_i. Let $d = \deg(v_i)$. So v_i has exactly d neighbours in G. It follows that when the algorithm was in the stage of colouring v_i at least one colour from the set $\{1, 2, \dots, d, d+1\}$ was not used to colour the previously coloured neighbours of v_i. Since if all of these colours were used, it means that v_i has at least $d+1$ neighbours, contradicting that it has exactly d neighbours. Thus, the greedy algorithm will colour v_i with the smallest integer in $\{1, 2, \dots, d, d+1\}$ which was not used in colouring the previously coloured neighbours of v_i. So $K(v_i) \in \{1, 2, \dots, d, d+1\}$, meaning that $K(v_i) \le \deg(v) + 1$. ■

Recall that $\Delta(G)$ is the maximum vertex degree of a graph G.

Corollary 6.1.8.

Let G be a graph. Then $\chi(G) \le \Delta(G) + 1$.

Proof. Apply the greedy algorithm to obtain a colouring function K of G. By Lemma 6.1.7, we get that for every vertex v, we have that

$$K(v) \le \deg(v) + 1 \le \Delta(G) + 1.$$

This means that the algorithm coloured the graph using at most $\Delta(G) + 1$ colours. Thus, it follows that $\chi(G) \le \Delta(G) + 1$. ■

Exercise. Prove Corollary 6.1.8 by induction on the order of the graph.

We conclude this section by emphasising that the greedy algorithm produces a colouring of the graph where adjacent vertices are coloured differently, nevertheless, there is no guarantee that it uses the minimum number of colours needed to colour the graph.

6.2 Brooks's Theorem

In general, finding the chromatic number of a graph is a computationally hard problem. Therefore, finding bounds on the chromatic number would be of great value. We have already seen one bound on the chromatic number. Namely, for any graph G, we know that $\chi(G) \leq \Delta(G) + 1$.

Can we even improve this upper bound? Notice that in this bound equality holds for complete graphs and odd cycles. That is, $\chi(K_n) = \Delta(K_n)+1$ and $\chi(C_m) = \Delta(C_m)+1$ where m is odd. The English mathematician Leonard Brooks $(1916 - 1993)$ showed that these are the only graphs where equality holds! The proof below of Brooks's Theorem is due to the Hungarian mathematician László Lovász.

Recall that a graph G is k-connected if the connectivity $\kappa(G) \geq k$. This means that the smallest cut set in G has cardinality at least k, and so deleting strictly less than k vertices does not disconnect the graph. Notice that 1-connected exactly means connected. And 2-connected means that G is connected with no cut vertex. Also if G is 3-connected then G is 2- connected and 1-connected as well. The larger the k, the more connected the graph is.

Theorem 6.2.1. (Brooks's Theorem; 1941)

Suppose that G is a connected graph which is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof. Let G be a connected graph of order n that is neither K_n nor an odd cycle. Let $\Delta = \Delta(G)$. We know that $\Delta \neq 0$ and $\Delta \neq 1$, since otherwise G is either K_1 or K_2, respectively. If $\Delta = 2$, then G is either an even cycle or a path. In this case, $\chi(G) = \Delta(G)$. So from now on, assume that $\Delta \geq 3$.

We will divide our argument into three cases. Our strategy is to choose a specific labelling of the vertices of G in the form $v_1, v_2, \ldots, v_n$. We will then apply the greedy algorithm to colour G with no more than Δ colours.

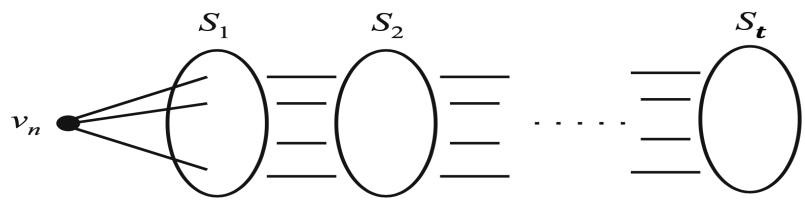
Case 1. Suppose that G is not regular.
Then there exists a vertex, label it v_n, such that $\deg(v_n) < \Delta$. Remember, $N(v_n)$ denotes the neighbourhood of v_n. Define the following sets of vertices for each i.

$$\begin{aligned}
S_0 &= \{v_n\} \\
S_1 &= N(v_n) \\
S_2 &= N(S_1) \setminus S_0 \cup S_1 \\
S_3 &= N(S_2) \setminus S_1 \cup S_2 \\
&\vdots \\
S_i &= N(S_{i-1}) \setminus S_{i-2} \cup S_{i-1}
\end{aligned}$$

Since G is connected and finite, there exists a t such that $S_t \neq \emptyset$ but $S_r = \emptyset$ for all $r > t$. Notice that the sets S_i partition the vertex set. That is,

$$V_G = S_0 \cup S_1 \cup \ldots \cup S_t$$

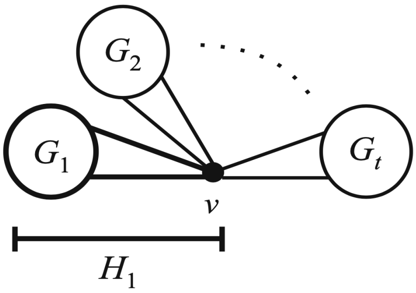
and $S_i \cap S_j = \emptyset$ when $i \neq j$. One can also show that $S_i = \{x \in V_G \mid d(v_n, x) = i\}$ and that $\mathrm{ecc}(v_n) = t$. Here is a picture of the sets S_i.



Let $s_i = |S_i|$. We will now start labelling the vertices in the reverse order starting with the last set S_t. So we label the vertices in S_t with labels $v_1, v_2, \ldots, v_{s_t}$. Then label the vertices in S_{t-1} with labels $v_{s_t+1}, v_{s_t+2}, \ldots, v_{s_t+s_{t-1}}$. We continue labelling in this fashion until all vertices of G have been labelled. For instance, vertices in S_1 are labelled with labels $v_{n-1}, v_{n-2}, \ldots, v_{n-s_1}$. And vertices of S_2 with labels $v_{n-s_1-1}, v_{n-s_1-2}, \ldots, v_{n-s_1-s_2}$.

Now we run the greedy algorithm with this labelling and we obtain a colouring function K of G. Notice that the algorithm starts by colouring vertices of S_t, and then S_{t-1}, and so on, and finally stops when the last vertex $v_n \in S_0$ is coloured. Suppose that u is a vertex different from v_n. Say $u \in S_i$ for some $1 \leq i \leq t$. By construction of these sets, there is a vertex $w \in S_{i-1}$ which is adjacent to u. So the label of w is larger than that of u, meaning that when the algorithm coloured u, its neighbour w was not a previously coloured neighbour. Thus, there were at most $\Delta - 1$ previously coloured neighbours of u at this stage. It follows that one of the colours $1, 2, 3, \ldots, \Delta$ was available to colour u. So $K(u) \leq \Delta$. Now, it remains to examine the colour of the last vertex v_n whose all of its neighbours have been coloured at its turn. Since by choice, $\deg(v_n) < \Delta$, one of the colours $1, 2, 3, \ldots, \Delta$ was available for the greedy algorithm to colour v_n. Thus, $K(v_n) \leq \Delta$ too. Therefore, the greedy algorithm used at most Δ colours to colour G. So G is Δ-colourable, and $\chi(G) \leq \Delta$. Case 1 is complete.

Case 2. Suppose that G is regular and contains a cut vertex v.
As G is regular, every vertex has degree Δ. Since v is a cut vertex, $G - v$ is a disconnected graph whose connected components are $G_1, G_2, \ldots, G_t$ where $t \geq 2$. Notice that v must be adjacent to at least one vertex in each G_i. Consider the induced subgraphs, $H_i = G_i \cup \{v\}$.



Each graph H_i is connected, and $\deg(v) < \Delta$ computed in H_i because we have at least two connected components in $G - v$. Also in H_i, the degree of any vertex from G_i is Δ. Therefore, each H_i is not regular, and so using the technique of Case 1, each H_i can be coloured using the colours $1, 2, 3, \ldots, \Delta$. By permuting the colours in each H_i, if needed, we can obtain colourings of H_i where the vertex v has the same colour in all of them. These colourings together compose a colouring of G which uses the colours $1, 2, 3, \ldots, \Delta$. Thus, $\chi(G) \leq \Delta$ and Case 2 is complete.

Case 3. Suppose that G is regular and 2-connected.
As G is regular, every vertex has degree Δ. First, we will show in this case that there must be three vertices v, v_1, v_2 such that v is adjacent to both v_1, v_2 where v_1, v_2 are nonadjacent, and $G - \{v_1, v_2\}$ is connected. To show this case we will divide this case into two subcases.

Case 3a. Suppose that G is regular and 3-connected.
Since G is regular and not complete, it follows that $\Delta < n - 1$. Let v_1 be any vertex of G and let A be the set of all vertices which are nonadjacent to v_1. As $\deg(v_1) = \Delta < n - 1$ there must be a vertex nonadjacent to v_1, and so $A \neq \emptyset$. Suppose for the moment that no neighbour of v_1 is adjacent to some vertex in A. But then there will be no path in G from v_1 to any vertex in A contradicting that G is connected. Therefore, there exists some neighbour v of v_1 which is adjacent to some vertex $v_2 \in A$. As $\kappa(G) \geq 3$, we know that $G - \{v_1, v_2\}$ is connected. This completes Case 3a.

Case 3b. Suppose that G is regular and $\kappa(G) = 2$. (This means G is 2-connected but not 3-connected.)
This means that there are two vertices v, w which form a cut set. So $G - \{v, w\}$ is disconnected. Let $G_1, G_2, \ldots, G_t$ where $t \geq 2$ be the connected components of $G - \{v, w\}$. Since $\Delta \geq 3$, each G_i must contain at least 2 vertices. We will establish two facts.
Claim 1. The vertex v has at least one neighbour in each G_i.
To see this, observe that w is not a cut vertex of G since $\kappa(G) = 2$. So $G - w$ is connected. Suppose that v is not adjacent to any vertex in G_i for some i. Let x be a vertex in G_i, as $G - w$ is connected, there is vx-path P in $G - w$. Since v is nonadjacent to all vertices in G_i, the successor of v in P must be some vertex z from G_j where $j \neq i$. But then $P[z, x]$ is a path which does not use either v or w.

This shows that G_i and G_j are connected in $G - \{v, w\}$, a contradiction. Thus, the vertex v must be adjacent to at least one vertex in each G_i as required.
Claim 2. The vertex v has a neighbour v_1 in G_1 that is not a cut vertex of $G - v$. To see this, by Claim 1 the vertex v has some neighbour in G_1. For the contrary, suppose that all neighbours of v in G_1 are cut vertices of $G - v$. Among all such neighbours choose a neighbour u of maximum distance $d(u, w)$. Let P be a shortest uw-path, so P has length $d(u, w)$ (such path is called geodesic), say

$$P = (u = u_0, u_1, u_2, u_3, \ldots, u_{k-1}, u_k = w).$$

In fact, it is possible that $u_1 = w$ meaning that u, w are adjacent. Note that $u_0, u_1, \ldots, u_{k-1}$ are all in G_1 because G_1 is a connected component of $G - \{v, w\}$. As $\deg_G(u) = \Delta \geq 3$, it must be that the degree of u in $G - v$ is at least 2, meaning that u has neighbours other than u_1. Moreover, since u is a cut vertex of $G - v$ there must exist a neighbour y of u such that $y \neq u_1$ and every path in $G - v$ from y to u_1 passes through u. This implies that

$$d(y, w) = d(y, u) + d(u, w) = 1 + d(u, w).$$

Since $\kappa(G) = 2$, we have that $G - u$ is connected, and so it must be that y is a neighbour of v. But $d(y, w) > d(u, w)$ which contradicts the choice of the vertex u. Thus, there exists a neighbour of v in G_1 which is not a a cut vertex of $G - v$.
By Claim 2, the vertex v has a neighbour v_1 in G_1 that is not a cut vertex of $G - v$. By a similar argument, v also has a neighbour v_2 in G_2 that is not a cut vertex of $G - v$ as well. As they lie in different connected components of the graph $G - \{v, w\}$, vertices v_1 and v_2 are nonadjacent, and moreover, it must be that the graph $G - \{v_1, v_2\}$ is connected. This finishes Case 3b.

Therefore, in either subcase of Case 3 we identified three vertices v, v_1, v_2 in G where v is adjacent to both v_1, v_2, but v_1, v_2 are nonadjacent, and also $G - \{v_1, v_2\}$ is connected. We now proceed to label the vertices of G in order to run the greedy algorithm and colour G.

Let v_1 and v_2 be labelled as they are. We now proceed in labelling the vertices of the graph $G - \{v_1, v_2\}$ using the labels $v_3, v_4, \ldots, v_n$. Label the vertex v as v_n. As $G - \{v_1, v_2\}$ is connected, there is a path from v_n to any vertex in $G - \{v_1, v_2\}$. Thus we can apply the labelling technique we used in Case 1. That is, let $S_0 = \{v_n\}$, and in general $S_i = \{x \in G - \{v_1, v_2\} \mid d(v_n, x) = i\}$. Say we get the sets $S_0, S_1, S_2, \ldots, S_t$ whose union is the set of all vertices of $G - \{v_1, v_2\}$. Observe that S_1 contains all neighbours of v_n except v_1 and v_2. We now start labelling in the reverse order. So choose a vertex in the last set S_t and label it as v_3, and the next vertex in S_t, if it exists, label it as v_4, and so on, until we label all vertices in S_t. Then we move to label vertices of S_{t-1}, and then S_{t-2}, and so on, until we finish by labelling all vertices of S_1. For instance S_1 will contain the vertex v_{n-1} which is a neighbour of v_n (different from v_1 and v_2).

Finally we are ready to run the greedy algorithm on G with this labelling. Vertex v_1 gets the colour 1. Vertex v_2 will also get colour 1 since v_2 is nonadjacent to v_1. Vertex v_3 in S_t will be coloured either with colour 1 or 2 depending on whether it is adjacent to either of its predecessors v_1 and v_2. In general, consider a vertex labelled v_i where $3 \le i < n$. Such vertex v_i belongs to S_k for some $1 \le k \le t$. By construction of these sets, there is a vertex $v_j \in S_{k-1}$ which is adjacent to v_i. Since we labelled in the reverse order, $j > i$, and so when the algorithm colours v_i, its neighbour v_j has not been coloured yet. As $\deg(v_i) = \Delta$, at most there are $\Delta - 1$ already coloured predecessors which are adjacent to v_i. It follows that one of the colours $1, 2, 3, \ldots, \Delta$ is available to colour v_i. It remains to colour the last vertex v_n which has Δ many neighbours already coloured. However, two of these neighbours are v_1 and v_2 which were coloured by the same colour and so at most $\Delta - 1$ colours were used in colouring all the neighbours of v_n. Therefore, one of the colours $1, 2, 3, \ldots, \Delta$ is available to colour v_n. We established that the greedy algorithm used the colours $1, 2, 3, \ldots, \Delta$ to colour all vertices of G. So $\chi(G) \le \Delta$, and Case 3 is complete. ■

Corollary 6.2.2.

Let G be a connected graph. Then $\chi(G) = \Delta(G) + 1$ if and only if G is either a complete graph or an odd cycle.

We will give two more bounds on the chromatic number. The first involves the independence number $\alpha(G)$. Recall that the independence number is the cardinality of a largest subset of vertices which are pairwise nonadjacent.

Lemma 6.2.3.

Let G be a graph of order n. Then

$$\frac{n}{\alpha(G)} \le \chi(G) \le n + 1 - \alpha(G).$$

The second bound on the chromatic number of a graph involves the following characteristic.

6.2.1 Clique Number

Definition 6.2.4. (Clique Number)

The *clique number* of a graph G, denoted by $\omega(G)$, is the largest m such that the complete graph K_m is a subgraph of G.

Consider graphs G and H below. The clique number of G is $\omega(G) = 3$. The clique number of H is $\omega(H) = 4$.

G H

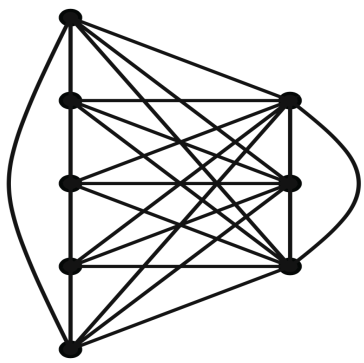
Observe that $\omega(G) = \alpha(\overline{G})$ for any graph G. Furthermore, it should be clear that the chromatic number is at least the clique number.

Lemma 6.2.5.

Let G be a graph. Then $\omega(G) \leq \chi(G)$.

Proof. Let G be a graph with clique number $\omega(G) = m$. This implies that K_m is a subgraph of G, and so any two vertices of K_m are adjacent in G. It follows we need m different colours to colour just the vertices of K_m, and so at least m colours are needed to colour the vertices of G. Thus, $\chi(G) \geq m = \omega(G)$. ■

One may wonder if it is always the case that $\omega(G)$ many colours are enough to colour G, that is, whether $\chi(G) = \omega(G)$. The answer is no! For example, $\omega(C_5) = 2$ while $\chi(C_5) = 3$. Here is another example. Consider the graph H below which consists of a copy of C_5 and a copy of C_3 and an edge between every vertex in C_5 and every vertex in C_3. The chromatic number of H is strictly bigger its clique number. Check that $\omega(H) = 5$ while $\chi(H) = 6$.



The following theorem sums up the bounds on the chromatic number $\chi(G)$.

Theorem 6.2.6.

Let G be a graph. Then the following holds.

(i) $\chi(G) \leq n$.

(ii) $\chi(G) \leq \Delta(G) + 1$.

(iii) If G is connected and neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

(iv) $\omega(G) \leq \chi(G)$.

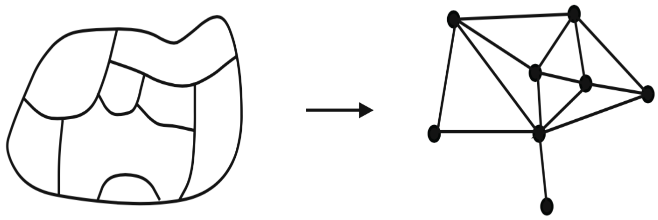
(v) $\chi(G) \leq n + 1 - \alpha(G)$.

(vi) $n \leq \chi(G)\alpha(G)$.

6.3 Four Colour Theorem

The Four Colour Problem. Is it possible to colour any given map in a plane with at most four colours in such a way that neighbouring countries are coloured differently?

We can express the Four Colour Problem using the language of graph theory. Any given map may be represented by a planar graph. Represent each country on the map by a vertex, and two vertices are adjacent if and only if their corresponding countries share a border. Here is an example.



Any proper colouring of the vertices of this graph translates to a colouring of the countries on the map in such a way that neighbouring countries have different colours. Therefore, a given map may be coloured with k colours if and only if its corresponding graph is k-colourable.

Here is the plot of the story of this famous problem.

- The Four Colour Problem was first introduced by the student Francis Guthrie to his professor Augustus De Morgan in 1852.
- In the same year, De Morgan sent a written letter to his friend William Rowan Hamilton describing the problem.
- In the following years, many of the world's brilliant mathematical minds attempted to attack the problem; either to prove or disprove the conjecture. Such minds include Arthur Cayley and Benjamin Peirce.
- In 1879, Alfred Kempe announced that he had found a proof.
- Kempe's proof stood for 10 years until 1890 when Percy Heawood discovered a major mistake in the proof.
- Heawood used Kempe's ideas known as "Kempe Chains" to make a significant breakthrough where he proved that five colours suffice to colour a given map as required.
- The Four Colour Theorem remained unsolved for more than 100 years.

- In 1976, Kenneth Appel and Wolfgang Haken who collaborated with John Koch, announced that they had found a proof.
- Part of their proof involved checking thousands of cases using computers.
- Due to the role of computers in the Appel-Haken proof mathematicians are still searching for alternative proofs.
- Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas found another simpler and cleverer proof. However the proof still requires computer assistance.
- In their 1996 paper titled "A New Proof of the Four-Colour Theorem", the abstract says the following.

  "*The four-colour theorem, that every loopless planar graph admits a vertex-colouring with at most four different colours, was proved in 1976 by Appel and Haken, using a computer. Here we announce another proof, still using a computer, but simpler than Appel and Haken's in several respects*".

Theorem 6.3.1. (Four Colour Theorem)

Every planar graph is 4*-colourable.*

We now present Heawood's proof of the Five Colour Theorem in 1890. Heawood used ideas from Kempe's attempt to attack the Four Colour Problem.

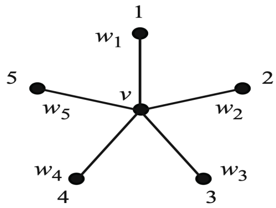
Theorem 6.3.2. (Five Colour Theorem)

Every planar graph is 5*-colourable.*

Proof. Let G be a planar graph of order n. We will prove the theorem by induction on n. Any graph of order 5 or less is 5-colourable. So suppose G has order $n \geq 6$. Moreover, suppose that any planar graph of order $n-1$ is 5-colourable. As G is planar, by Theorem 5.2.6, we know that $\delta(G) \leq 5$.

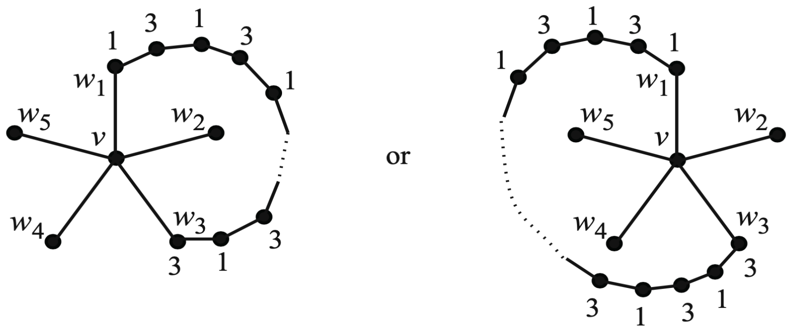
Let v be a vertex in G with $\deg(v) \leq 5$. The graph $G - v$ has order $n-1$ and is planar as well. By the induction hypothesis we have that $G - v$ is 5-colourable. Say $G - v$ is coloured using the colours $1, 2, 3, 4, 5$. It remains to properly colour the vertex v to obtain a colouring of G. Clearly, all the neighbours of v belong to the graph $G - v$ and so they all have been coloured. Now, if not all of the colours $1, 2, 3, 4, 5$ have been used in colouring the neighbours of v in $G - v$, then we may colour v with any of the unused colours and thus obtain a 5-colouring of G itself.

Otherwise, suppose that all the colours $1, 2, 3, 4, 5$ were used in colouring the neighbours of v in $G - v$. This implies that $\deg(v) = 5$. The strategy now is to permute the colours of $G - v$ in such a way that one of the colours $1, 2, 3, 4, 5$ will be available to properly colour v. Let w_i be the neighbour of v which was assigned the colour i. Say, without loss of generality, we have the following diagram in the planar representation of G.



Case 1. Suppose that in $G - v$ there is no w_1w_3-path where all the vertices on the path are coloured with colours 1 or 3. Let H be the union of all paths in $G - v$ which start from w_1 and whose vertices are all coloured with colours 1 or 3. Notice that $w_1 \in H$ while $w_3 \notin H$ because there is no such w_1w_3-path. We now swap the two colours in H, that is, every vertex in H coloured with colour 1, its new colour becomes 3, and every vertex in H which was assigned colour 3 will be coloured with colour 1. The resulting new colouring of $G - v$ is still a proper colouring, meaning that no two adjacent vertices in $G - v$ have the same colour. To see this, observe that no neighbour of w_3 belongs to H, as otherwise there will be a w_1w_3-path whose vertices are coloured with 1 or 3. Furthermore, any vertex not in H which is adjacent to some vertex in H has a colour different from 1 and 3. As $w_1 \in H$, its new colour is 3. At this point, we have a 5-colouring of $G - v$ where the neighbours of v are coloured with colours $2, 3, 4, 5$. We now colour v with colour 1 obtaining a colouring of G with 5 colours. Thus, G is 5-colourable.

Case 2. Suppose that in $G - v$ there is a w_1w_3-path P where all the vertices on the path are coloured with colours 1 or 3. In the planar representation shown above, the path P either lies to the right of vertex v or to its left. See the drawings below. The path P together with the two edges vw_1 and vw_3 form a cycle C which encloses either vertex w_2 or w_4.



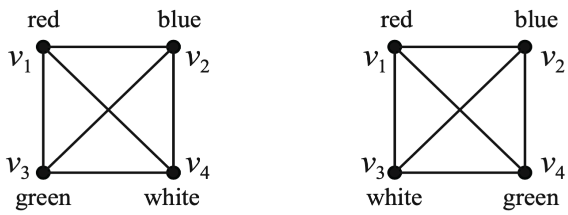
As we are working with a planar representation of G it must be that every w_2w_4-path must pass through some vertex on C. This implies that there is no w_2w_4-path in $G - v$ whose all of its vertices are coloured with colours 2 or 4. Now apply the method of Case 1 to vertices w_2 and w_4 and change the colour of w_2 from 2 to 4, hence, making colour 2 available for the vertex v. Thus, G is 5-colourable. ■

Exercise. Where does this proof fail if only 4 colours are available?

6.4 Chromatic Polynomials

George David Birkhoff, a preeminent American mathematician, developed chromatic polynomials in 1912 when he was working on the Four Colour Problem. A chromatic polynomial is a device which counts the number of different colourings of a given graph. Such polynomials became important objects of study in algebraic graph theory; a branch of mathematics where algebraic methods such as linear algebra and group theory are applied to study graphs.

Suppose that K and K' are colouring functions of a graph G. Then the colouring K is *different* from K' if there is one vertex v in G such that $K(v) \neq K'(v)$. So two colourings are considered different if they assign different colours to the same vertex. For instance the two colourings below of the complete graph K_4 are different because the first colouring assigns green to v_3 while the second assigns white to v_3.



Restricting ourselves to the four colours $1, 2, 3, 4$, how many different colourings of K_4 are there? Let us start colouring the vertices one by one. We may choose any of the four colours for v_1. For v_2 we may choose any colour except that of v_1 because v_2 is adjacent to v_1, so we have three choices for v_2. Two colours remain to choose from for v_3 because v_3 is adjacent to both v_1 and v_2. Finally, we are forced to colour v_4 with the last remaining colour since v_4 is adjacent to all v_1, v_2, v_3. Therefore, there are $4 \times 3 \times 2 \times 1$ different colourings of K_4 using the four colours.

There will be more different colourings of K_4 when the six colours $1, 2, 3, 4, 5, 6$ are available. Precisely, there are $6 \times 5 \times 4 \times 3$ different colourings of K_4 using these six colours. On the other hand if only two colours are available, there would be no proper colourings of K_4, since K_4 is not 2-colourable.

Definition 6.4.1. (Chromatic Polynomial)

Let $P_G(k)$ be the number of different colourings of a graph G using the colours $1, 2, 3, \ldots, k$.

In other words, $P_G(k)$ is the number of different k-colourings of G. From the discussion above, we have $P_{K_4}(4) = 4! = 24$, and $P_{K_4}(6) = 360$, and $P_{K_4}(2) = 0$. In general, for the complete graph K_n we have the following.

$$P_{K_n}(k) = \begin{cases} \frac{k!}{(k-n)!} & \text{if } k \geq n, \\ 0 & \text{if } k < n. \end{cases}$$

For the empty graph E_n, we have $P_{E_n}(k) = k^n$ because when we colour E_n, we are free to choose any of the k colours to colour each vertex. Let us state a simple but useful property of $P_G(k)$.

Lemma 6.4.2.

Let G be a graph. Then the following are equivalent.

(i) $P_G(k) > 0$.

(ii) *G is k-colourable.*

(iii) $\chi(G) \leq k$.

It turns out that $P_G(k)$ is a polynomial in the variable k of degree n. It is called the *chromatic polynomial* of the graph G. The Four Colour Theorem can be proved by showing that for every planar graph G we have that $P_G(4) \geq 1$. From this perspective, Birkhoff hoped to apply the powerful tools of analysis and algebra in studying the roots of chromatic polynomials. He hoped to find a chromatic polynomial of a planar graph G which has 4 as a root, that is, $P_G(4) = 0$. If he had found one, then the corresponding planar graph G of such polynomial has no colourings with 4 colours meaning that it is not 4-colourable, and hence such G would serve as a counterexample for the Four Colour Conjecture. However, his attempts were unsuccessful and the conjecture was later proved to be true.

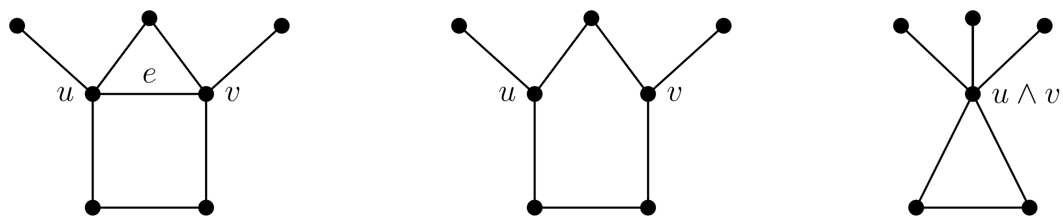
We will define a new operation on graphs which is useful in computing $P_G(k)$.

Definition 6.4.3. (Edge Contraction)

Let G be a graph and e be an edge of G where $e = uv$. The graph G/e is the graph obtained from G by

1. Deleting the edge e.
2. Identifying its end vertices u, v into one vertex denoted by $u \wedge v$.
3. Leaving only one copy of any resulting multiple edges with common neighbours of u and v.

Here is an example of contracting an edge.



G $\qquad$ $G-e$ $\qquad$ G/e

In general, calculating the value $P_G(k)$ is hard. However, Birkhoff and D.C. Lewis in 1946 managed to reduce this hard problem to an easier one.

Theorem 6.4.4. (Birkhoff-Lewis)

Given a graph G and any edge e of G. Then

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

Proof. Let G be a graph and e be an edge of G with end vertices u and v. Notice that it is possible for the vertices u, v to have the same colour in $G-e$ since they are nonadjacent in $G-e$.

Claim 1. $P_{G/e}(k)$ is equal to the number of different k-colourings of $G-e$ where the vertices u and v are assigned the same colour.
To see this, let $P_{G/e}(k) = s$ and let t be the number of k-colourings of $G-e$ where the vertices u and v are assigned the same colour. We will show that $s = t$. Now, let K be a k-colouring of G/e. Then K induces a k-colouring $\hat{K}$ on $G-e$ where u and v have the same colour. To see this, define $\hat{K}(u) = \hat{K}(v) = K(u \wedge v)$, and for any vertex w in $G-e$ distinct from u and v define $\hat{K}(w) = K(w)$. Thus, every colouring of G/e can be thought of as a k-colouring of $G-e$ where u, v have the same colour. Thus, $s \leq t$. On the other hand, suppose that F is a k-colouring of $G-e$ such that $F(u) = F(v)$. Then F induces a k-colouring $\hat{F}$ of G/e where $\hat{F}(u \wedge v) = F(u)$, and for any vertex x in G/e distinct from $u \wedge v$ define $\hat{F}(x) = F(x)$. This shows that any k-colouring of $G-e$ which assigns the same colour for u and v can be thought of as a colouring of G/e. Thus $t \leq s$, and therefore $s = t$.

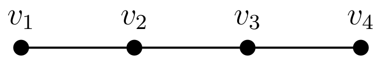
Claim 2. $P_G(k)$ is equal to the number of k-colourings of $G-e$ where the vertices u and v are assigned different colours.
To see this, first notice that $G-e$ is a subgraph of G. So if K is a k-colouring of G, then K is also a k-colouring of $G-e$. Moreover, K must assign different colours to u and v since they are adjacent in G, so K itself is a colouring of $G-e$ which assigns different colours to u and v. Conversely, if F is a k-colouring of $G-e$ which assigns different colours to u and v, then F itself is also a colouring of G.

The total number of k-colourings of $G-e$ is equal to the number of k-colourings of $G-e$ which assign u, v the same colour plus the number of those colourings which assign u, v different colours. Therefore,

$$P_{G-e}(k) = P_{G/e}(k) + P_G(k).$$

This finishes the proof. ■

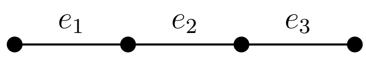
Recall that P_4 is the path graph on four vertices, say, v_1, v_2, v_3, v_4.



How many colourings of P_4 are there using the colours $1, 2, \ldots, k$? Let us colour the vertices one by one. The vertex v_1 can be coloured with any of the k colours, v_2 can be coloured with any colour different from the colour of v_1, so there are $k-1$ choices for colouring v_2. Vertex v_3 can have any colour except that of v_2, so we also have $k-1$ choices. Similarly, there are $k-1$ colours available for v_4. Thus, the chromatic polynomial of P_4 is

$$P_{P_4}(k) = k(k-1)^3 = k^4 - 3k^3 + 3k^2 - k.$$

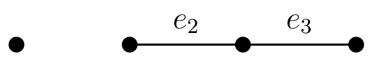
For instance, $P_{P_4}(5) = 5 \cdot 4^3 = 320$, meaning that there are 320 different ways to properly colour P_4 using 5 colours. Now let us successively apply the Birkhoff-Lewis reduction process to compute the chromatic polynomial of P_4.



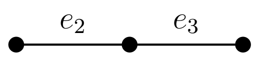
Theorem 6.4.4 implies that

$$P_{P_4}(k) = P_{P_4 - e_1}(k) - P_{P_4/e_1}(k)$$

For clarity of writing, let $G = P_4$ and $G_0 = P_4 - e_1$ and $G_1 = P_4/e_1$. (In general, we will add an index of 0 for deleting an edge and an index 1 for contracting it.)



G_0

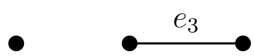


G_1

Now apply the reduction to G_0. Let $G_{00} = G_0 - e_2$ and $G_{01} = G_0/e_2$. Thus,

$$P_{G_0}(k) = P_{G_{00}}(k) - P_{G_{01}}(k).$$

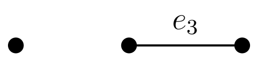
G_{00}



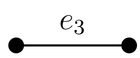
G_{01}

Now apply the reduction to G_1. Let $G_{10} = G_1 - e_2$ and $G_{11} = G_1/e_2$. Thus,

$$P_{G_1}(k) = P_{G_{10}}(k) - P_{G_{11}}(k).$$



G_{10}



G_{11}

Now lets apply the reduction process for $G_{00}, G_{01}, G_{10}, G_{11}$.

$$
\begin{aligned}
G_{000} &= G_{00} - e_3 = E_4 \\
G_{001} &= G_{00}/e_3 = E_3 \\
G_{010} &= G_{01} - e_3 = E_3 \\
G_{011} &= G_{01}/e_3 = E_2 \\
G_{100} &= G_{10} - e_3 = E_3 \\
G_{101} &= G_{10}/e_3 = E_2 \\
G_{110} &= G_{11} - e_3 = E_2 \\
G_{111} &= G_{11}/e_3 = E_1
\end{aligned}
$$

Finally, we know all the information needed to compute $P_{P_4}(k)$.

$$
\begin{aligned}
P_{P_4}(k) &= P_{G_0}(k) - P_{G_1}(k) \\
&= [P_{G_{00}}(k) - P_{G_{01}}(k)] - [P_{G_{10}}(k) - P_{G_{11}}(k)] \\
&= P_{G_{00}}(k) - P_{G_{01}}(k) - P_{G_{10}}(k) + P_{G_{11}}(k) \\
&= [P_{G_{000}}(k) - P_{G_{001}}(k)] - [P_{G_{010}}(k) - P_{G_{011}}(k)] \\
&\quad - [P_{G_{100}}(k) - P_{G_{101}}(k)] + [P_{G_{110}}(k) - P_{G_{111}}(k)] \\
&= P_{E_4}(k) - P_{E_3}(k) - P_{E_3}(k) + P_{E_2}(k) \\
&\quad - P_{E_3}(k) + P_{E_2}(k) + P_{E_2}(k) - P_{E_1}(k) \\
&= P_{E_4}(k) - 3P_{E_3}(k) + 3P_{E_2}(k) - P_{E_1}(k) \\
&= k^4 - 3k^3 + 3k^2 - k.
\end{aligned}
$$

Clearly we cannot colour P_4 using only colour 1. So the number of colourings of P_4 with one colour is

$$P_{P_4}(1) = 1^4 - 3(1)^3 + 3(1)^2 - 1 = 0.$$

The number of colourings of P_4 with two colours is

$$P_{P_4}(2) = 2^4 - 3(2)^3 + 3(2)^2 - 2 = 2.$$

So there are 2 different colourings of P_4 using the colours 1 and 2. Here they are.

The number of colourings of P_4 with three colours is

$$P_{P_4}(3) = 3^4 - 3(3)^3 + 3(3)^2 - 3 = 24.$$

It is worth mentioning that one can stop the reduction process once it arrives to graphs whose chromatic polynomials are known, and so we do not always need to work all the way down to empty graphs as we did above.

Let us now present some properties of the chromatic polynomial $P_G(k)$.

Theorem 6.4.5.

Let G be a graph of order n. The following statements are true.

(i) $P_G(k)$ *is a polynomial in the variable* k *of degree* n.

(ii) The leading coefficient of $P_G(k)$ *is always 1.*

(iii) The constant term of $P_G(k)$ *is always 0.*

(iv) The coefficients of $P_G(k)$ *alternate in sign.*

(v) The coefficient of k^{n-1} *in* $P_G(k)$ *is negative the number of edges in* G.

The theorem may be proved by induction on the number of edges in G, where in the induction step we apply the Birkhoff-Lewis reduction theorem to G to obtain two graphs, namely $G - e$ and G/e, with less number of edges.

Chapter 7

Matchings

Match every woman with a man she would happily marry.

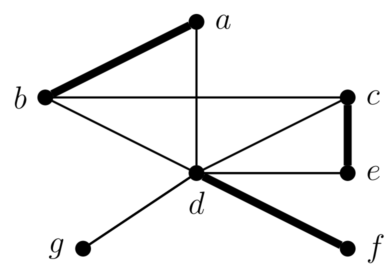
7.1 Types of Matchings

Definition 7.1.1. (Matching)

A *matching* in a graph is a set of pairwise disjoint edges.

A matching is a set of edges where no two edges share a common vertex. A matching is also called an *independent* set of edges. Given a matching M in a graph G, any end vertex of an edge in M is called *M-saturated* or *saturated* by M. So a vertex v is M-saturated if there exists an edge $e \in M$ such that $v \in e$. A vertex which does not belong to any of the edges in M is called *M-unsaturated*.

Example 7.1. Consider the graph shown below. The set of edges $M = \{ab, ce, df\}$ forms a matching. Vertices a, b, c, d, e, f are M-saturated, while vertex g is M-unsaturated. Another matching is the set $M' = \{bc, df\}$. The vertices b, c, d, f are M'-saturated, while a, e, g are M'-unsaturated. Clearly, the set $M'' = \{gd\}$ is another matching.



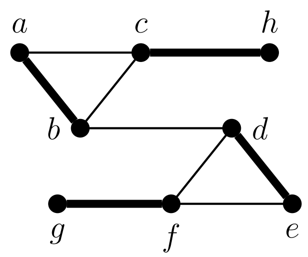
Definition 7.1.2. (Maximal; Maximum; and Perfect Matchings)

Let M be a matching in a graph G.

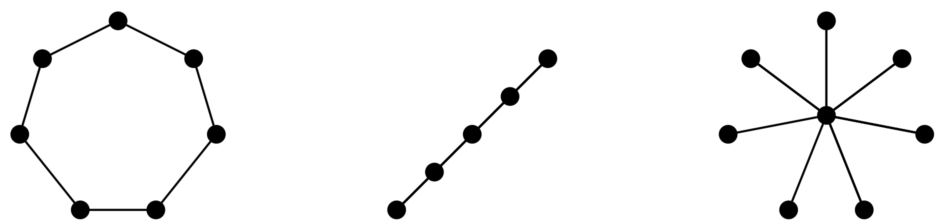
- M is a *maximal matching* if $M \cup \{e\}$ is not a matching for every edge $e \notin M$.
- M is a *maximum matching* if M has largest cardinality among all matchings.
- M is called a *perfect matching* if every vertex in G is M-saturated.

Let us elaborate more on the notions above. A maximal matching is a matching which is not a proper subset of any other matching. So a matching M' is maximal if every edge in the graph has a nonempty intersection with at least one edge in M'. Any maximum matching is a maximal matching. But a maximal matching is not necessarily a maximum matching. In a graph, there could be several maximum matchings. Given a perfect matching M of a graph G, then every vertex in the graph is incident with exactly one edge in M. Thus, $|V_G| = 2|M|$. Moreover, every perfect matching is maximum, and hence maximal. To conclude, every perfect matching is a maximum matching, and every maximum matching is a maximal matching.

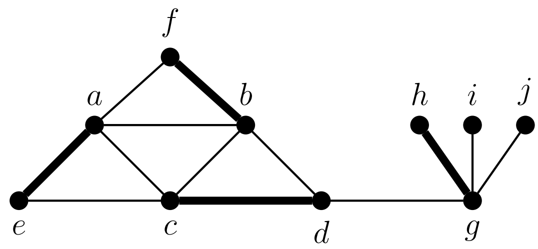
Example 7.2. In the graph below, the matching $\{ab, ch, de, fg\}$ is a perfect matching.



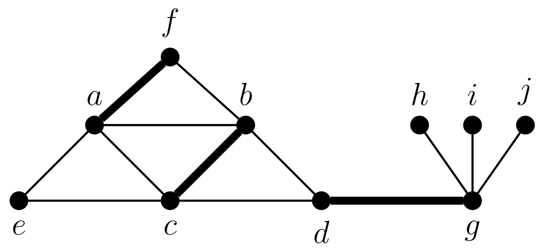
Example 7.3. None of the three graphs C_7, P_5, and $K_{1,7}$ has a perfect matching.



Example 7.4. In the graph below, the set $M = \{ae, bf, cd, gh\}$ is a maximum matching.



The matching $M' = \{af, bc, dg\}$ is a maximal matching, but not maximum.



7.2 Berge's Theorem

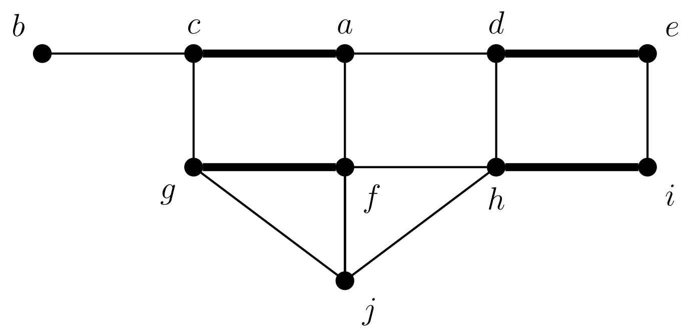
In this and the following sections we discuss important classic results on matchings, namely, Berge's Theorem, Hall's Theorem, and König-Egerváry Theorem.

Definition 7.2.1. (Alternating and Augmenting Paths)

Let G be a graph and M be a matching of G.

- A path in G is called *M-alternating* if its edges alternate between edges in M and edges not in M.
- A path in G is called *M-augmenting* if it is M-alternating and its first and last vertices are M-unsaturated.

Every M-augmenting path must be of odd length. In the graph below, consider the matching $M = \{ca, gf, de, hi\}$. The paths $P = (d, e, i, h, j)$ and $Q = (g, f, a, c, b)$ are M-alternating but not M-augmenting. The paths $S = (j, g, f, a, c, b)$ and $R = (b, c, a, d, e, i, h, j)$ are M-augmenting as the vertices b, j are M-unsaturated.

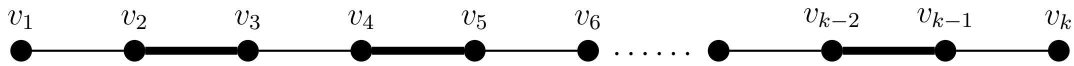


The matching M is not maximum. A matching with a bigger cardinality is given by $N = \{bc, ad, ei, hf, gj\}$. The following theorem tells us the relationship between maximum matchings and augmenting paths. It is due to the French mathematician Claude Berge $(1926-2002)$ who worked at the French National Centre for Scientific Research (CNRS), and became a Professor at the University of Paris. Berge was interested in graph theory, combinatorics, topology, and game theory, and also in literature and art.

Theorem 7.2.2. (Berge's Theorem)

Let M be a matching in a graph G. Then M is a maximum matching if and only if G has no M-augmenting paths.

Proof. Let us show first the forward direction. Assume that M is a maximum matching, and for contradiction, suppose that $P = (v_1, v_2, \dots, v_{k-1}, v_k)$ is an M-augmenting path. Since the edges in P alternate between being in M and not, and as v_1 and v_k are M-unsaturated, it follows that the edges $v_1v_2, v_3v_4, v_5v_6, \ldots, v_{k-1}v_k$ are not in M, while the edges $v_2v_3, v_4v_5, \ldots, v_{k-2}v_{k-1}$ are in M. Moreover, P has odd many edges and so k is an even integer. Here is a picture of the path P.



Let A be the set of edges of P which are in M, so $A = \{v_2v_3, v_4v_5, \ldots, v_{k-2}v_{k-1}\}$, and let B be the set of edges of P not in M, that is, $B = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{k-1}v_k\}$. Note that $A \cap B = \emptyset$ and $|B| = |A| + 1$. More precisely, $|A| = (k-2)/2$, while $|B| = k/2$. Consider the symmetric difference $\hat{M}$ of M and the set of all edges of P.

$$\hat{M} = M \bigtriangleup E_P = (M \setminus A) \cup B.$$

Since v_1, v_k are M-unsaturated, they are not incident with any edge in M, and so $\hat{M}$ is another matching of G with cardinality $|\hat{M}| = |M| - |A| + |B| = |M| + 1$. This means that $\hat{M}$ is a matching with more edges than the maximum matching M, a clear contradiction. Therefore, G cannot have any M-augmenting paths.

For the converse, assume that M is a matching of G and there is no M-augmenting paths in G. We will show that M must be a maximum matching. Suppose that there is a matching N such that $|N| > |M|$. Define a subgraph $H \subseteq G$ as follows.

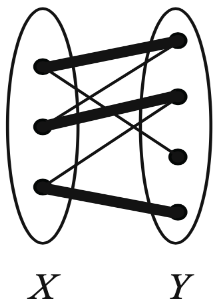
$$\begin{aligned} V_H &= V_G; \\ E_H &= M \bigtriangleup N = (M \setminus N) \cup (N \setminus M). \end{aligned}$$

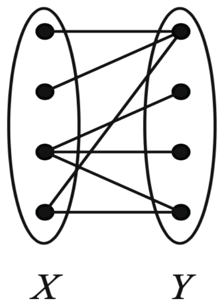
So the subgraph H has all the vertices of G and its edges are precisely those edges which are exactly in one of the matchings M and N. Now let v be any vertex in G. Since M is a matching, v is incident with at most one edge from M. Similarly, v is incident with at most one edge from N. This implies that $\deg_H(v) \leq 2$ for every v in H. Therefore, $\Delta(H) \leq 2$, and so every connected component of H is either a path or a cycle. Since $|N| > |M|$, we have that $|N \setminus M| > |M \setminus N|$ meaning that there are more edges from N in H than there are from M. When a connected component of H is a cycle, its edges must alternate between edges from M and N, and thus it is an even cycle containing equal number of edges from M and N. Similarly, every path in H must be M-alternating. Since H has more edges from N, there must be at least one connected component of H which is a path that begins and ends with edges from N. This particular path is an M-augmenting path, contradicting our initial assumption. Therefore, no matching of G has greater cardinality than that of M, implying that M is a maximum matching. ■

7.3 Hall's Marriage Theorem

Matchings are of particular interest in bipartite graphs. Given a bipartite graph G with partite sets X and Y, we say that X is *matched into* Y if there exists a matching M in G where every vertex in X is M-saturated.

Check below the bipartite graph on the left, the set X in this graph is matched into Y by the bold edges. However, it is impossible to match X into Y in the bipartite graph on the right. What is the obstacle?





The obstacle has to do with the neighbourhoods of subsets of X. In 1935, the English mathematician Philip Hall (1904 – 1982) discovered when a partite set can be matched into the other one in bipartite graphs. His result is known as Hall's Marriage Theorem. We remark that the major work of Hall was on group theory, notably on finite groups and solvable groups.

Recall that the neighbourhood $N(v)$ of a vertex v is the set of all vertices adjacent to v, and the neighbourhood $N(S)$ of a set S of vertices is the union of all the neighbourhoods of the vertices in S.

Theorem 7.3.1. (Hall's Marriage Theorem)

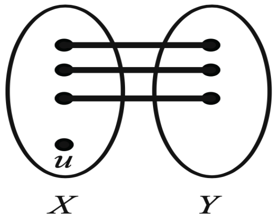
Let G be a bipartite graph with partite sets X and Y. Then X can be matched into Y if and only if for every subset $S \subseteq X$ we have that $|S| \leq |N(S)|$.

Proof. For the forward direction, suppose that X can be matched into Y, say, via a matching M. Then for every $x \in X$, there is a unique edge $e_x \in M$ such that $x \in e_x$. Now, let $S \subseteq X$ and define a function $f : S \to N(S)$ by setting $f(x) = y$ where $x \in S$ and y is adjacent to x via the edge e_x. Since M is a matching, whenever $x, z \in S$ are distinct then $e_x \cap e_z = \emptyset$, and thus $f(x) \neq f(z)$. This means that f is an injective function, and so $|S| \leq |N(S)|$.

For the converse, suppose that for every subset $S \subseteq X$, we have that $|S| \leq |N(S)|$. We will find some matching M which matches X into Y. Take M to be any maximum matching in G, and for the contrary, assume there is some vertex $u \in X$ which is M-unsaturated. Define the following set of vertices of G.

$$A = \{w \in V_G \mid \text{there is an } M\text{-alternating } uw\text{-path}\}$$

Observe that the first edge of any uw-path does not belong to M because u is M-unsaturated. Let $S = X \cap A$ and $T = Y \cap A$. Clearly, $u \in S$. Due to the alternating nature of the paths, every vertex in S other than u is M-saturated. Furthermore, since M is a maximum matching, by Berge's Theorem G does not contain any M-augmenting path, and thus every vertex in T is also M-saturated, as otherwise we will have an M-augmenting path. In summary, every vertex in A except u is M-saturated. Here is a picture.



Claim I. $|S| - 1 = |T|$.
First, we show that $|S \setminus \{u\}| = |T|$. We start by defining a function $f : S \setminus \{u\} \to T$. Let $x \in S \setminus \{u\}$. Since $x \in S$, there is an M-alternating ux-path, call it P. Due to the alternating nature of P, and x being in X, and u being M-unsaturated, the last edge e of P which is incident with x must be in M. As M is a matching, the edge e is the only edge in M containing x (so any M-alternating ux-path must end with the edge e). Let y be the other end vertex of e. So y is the predecessor of x on the path P. Since x is adjacent to y and $x \in X$ and G is bipartite, y must be in Y. Moreover, as $P[u, y]$ is an M-alternating uy-path, it follows that $y \in A$. Thus, $y \in T$. Define $f(x) = y$. It is clear to see that f defined this way must be an injective function. It remains to show that f is surjective. Let t be a vertex in the codomain T of f. As $t \in A$, there is an M-alternating ut-path R. Due to the alternating nature of R, its last edge is not in M. As t is M-saturated, there is a unique edge $e' \in M$ such that $t \in e'$. Let s be the other end vertex of e', then adding e' at the end of R we obtain an M-alternating us-path, this shows that $s \in S$ and $f(s) = t$. Thus f is surjective. It follows that f is a bijection and so $|S \setminus \{u\}| = |T|$. The claim follows since $|S \setminus \{u\}| = |S| - 1$.

Claim II. $N(S) = T$.
Let $w \in N(S)$. Then there is $s \in S$ such that s is adjacent to w and so $w \in Y$. Let P be an M-alternating us-path. As w is adjacent to s, using P we obtain an M-alternating uw-path and so $w \in T$. This shows that $N(S) \subseteq T$. Conversely, let $t \in T$. Let $s \in S$ be the preimage under the function f in Claim I. Since s is adjacent to t, we have that $t \in N(S)$. Thus $T \subseteq N(S)$. Therefore, $N(S) = T$.

Using Claim I and II, we get the following,

$$|S| = |T| + 1 = |N(S)| + 1 > |N(S)|.$$

This is a contradiction since $S = A \cap X \subseteq X$. Thus, such vertex u cannot exist, meaning that all vertices of X are M-saturated, and so X is matched into Y via the matching M. ■

Next, we will present two applications of Hall's Theorem.

Application to Marriage

Imagine we have a collection X of women and a collection Y of men. Each woman in X has chosen a group of special men from Y, any one of which she would happily marry. On the other hand, any man is happy to marry a woman who is happy to marry him. In such situation, is it possible to make every woman happily married?

Think of the bipartite graph whose partite sets are X and Y, and in which a women $x \in X$ is adjacent to a man $y \in Y$ if and only if y is one of those men whom x is happy to marry. By Hall's Marriage Theorem, X can be matched into Y, or equivalently, every woman can happily marry a man if and only if for any chosen group of k women from X, the number of men whom these women are happy to marry is at least k. Note that the forward direction is obvious since if all women are happily married, then for any collection of k women, there are at least k many men that these women are happy to marry, namely their k husbands. What is more interesting is the reverse direction.

Systems of Distinct Representatives

Definition 7.3.2.

Let $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ be a family of nonempty sets. A *system of distinct representatives* for $\mathcal{F}$ is a set $\{x_1, x_2, \ldots, x_k\}$ such that $x_i \in S_i$ and $x_i \neq x_j$ whenever $i \neq j$.

To find a system of distinct representatives, we choose a distinct element x_i from each set S_i in the family, and we think of x_i as the representative S_i. Consider the following sets,

$$S_1 = \{2, 8\},\ S_2 = \{8\},\ S_3 = \{5, 7\},\ S_4 = \{2, 4, 8\},\ S_5 = \{2, 4\}.$$

And consider the family of sets given by $\mathcal{F} = \{S_1, S_2, S_3, S_4\}$. A system of distinct representatives for $\mathcal{F}$ is given by $R = \{2, 8, 7, 4\}$. So here the element 8 must be the representative of S_2, and so 2 must represent S_1, and so 4 must represent S_4, and finally we chose 7 to represent S_3.

Do we always succeed in finding systems of distinct representatives of a given family? The answer is no! For instance, it is impossible to find such system for the family $\mathcal{F}' = \{S_1, S_2, S_3, S_4, S_5\}$. So when such systems exist?

Theorem 7.3.3.

Let $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ be a family of finite nonempty sets. Then $\mathcal{F}$ has a system of distinct representatives if and only if for every $m \in \{1, 2, \ldots, k\}$, the union of any m of the sets in $\mathcal{F}$ contains at least m elements.

Proof. We want prepare the setting to apply Hall's Theorem by constructing a bipartite graph G with partite sets X and Y. Let $X = \mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ and let $Y = S_1 \cup S_2 \cup \cdots \cup S_k$. Note that Y is finite since it is a finite union of finite sets. Define the edge relation in G by setting: $S_i \in X$ is adjacent to $x \in Y$ if and only if $x \in S_i$. Observe that the neighbourhood of the vertex labelled S_i is the set S_i. Add to this Hall's Theorem and the definitions involved to get the following.

The family $\mathcal{F}$ has a system of distinct representatives $\iff$ there exists some set $\{x_1, x_2, \ldots, x_k\}$ where $x_i \in S_i$ and $x_i \neq x_j$ if $i \neq j$ $\iff$ there exists a matching in G of the form $\big\{\{S_i, x_i\} \in E_G \mid 1 \leq i \leq k\big\}$ $\iff$ X can be matched into Y $\iff$ for every subset $T \subseteq X$ we have that $|T| \leq |N(T)|$ $\iff$ for every subset $T \subseteq X$ we have that $|T| \leq |\bigcup\limits_{S_i \in T} S_i|$ $\iff$ for every subset $I \subseteq \{1, 2, \ldots, k\}$ we have that $|I| \leq |\bigcup\limits_{i \in I} S_i|$ $\iff$ for every $m \in \{1, 2, \ldots, k\}$, the union of any m of the sets in $\mathcal{F}$ contains at least m elements. ■

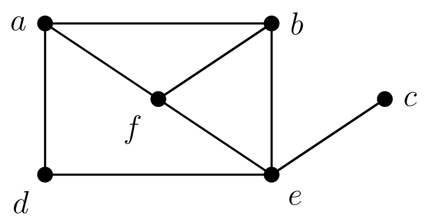
7.4 König-Egerváry Theorem

As previously mentioned, matchings play an important role in the world of bipartite graphs. Next, we will study another fact of bipartite graphs related to matchings.

Definition 7.4.1. (Vertex Cover)

A set C of vertices in a graph G is called a *vertex cover* if every edge of G is incident with at least one vertex from C.

Obviously, the whole vertex set V_G is always a vertex cover of a graph G. A vertex cover of smallest cardinality is called a *minimum vertex cover*. In the graph below, the set $\{a, b, d, e\}$ is a vertex cover, as well as the set $\{a, f, e\}$. The latter set is a minimum vertex cover. Check that no subset of two vertices is a vertex cover. A maximum matching of the graph below is given by $\{ad, ce, bf\}$. Note that both a maximum matching and a minimum vertex cover have cardinality 3.



Consider the cycle C_5, it has a minimum vertex cover of cardinality 3, but a maximum matching of cardinality 2. In general, we have the following.

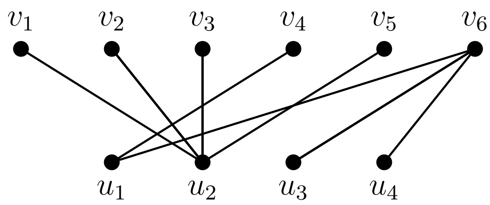
Lemma 7.4.2.

Let G be any graph, and M be any matching of G and C be any vertex cover of G. Then $|M| \leq |C|$.

Proof. Let C be a vertex cover, and M be a matching and let $m = |M|$. Say $M = \{e_1, e_2, \dots, e_m\}$. Since C is a vertex cover, we have that $C \cap e_i \neq \emptyset$ for every edge $e_i \in M$. Let $c_i \in C \cap e_i$ for every $1 \leq i \leq m$. This means that C contains at least one end vertex c_i from each edge $e_i \in M$. Moreover, since M is a matching, distinct edges in M have no common end vertices. This implies that $c_1, c_2, \dots, c_m$ are pairwise distinct. Since $\{c_1, c_2, \dots, c_m\} \subseteq C$, it follows that C contains at least m many distinct elements. Thus $|M| \leq |C|$. ■

From the lemma above, we know that any vertex cover, in particular, any minimum vertex cover has size at least the size of a maximum matching. Furthermore, any vertex cover of size equal to the size of a maximum matching must be a minimum vertex cover.

Consider the bipartite graph below. Each one of the sets $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\{u_1, u_2, v_6\}$ is a vertex cover. Check that the latter is a minimum vertex cover, and that $\{v_1u_2, u_1v_4, u_4v_6\}$ is a maximum matching. Both of these sets contain the same number of elements; this is not a coincidence.



So when is it possible for minimum vertex covers to attain the size of maximum matchings? Two Hungarian mathematicians Dénes Kőnig and Jenő Egerváry found independently in 1931 that such phenomenon always happen in bipartite graphs!

Theorem 7.4.3. (König-Egerváry Theorem)

The cardinality of a maximum matching in any bipartite graph G equals the cardinality of a minimum vertex cover of G.

Proof. Let X and Y be some partite sets of G. Let M be a maximum matching of G. As G is bipartite, the partite set X is a vertex cover of G. If every vertex in X is M-saturated, then $|X| = |M|$. And as M is a maximum matching, by Lemma 7.4.2, X must be a vertex cover of minimum cardinality, and so we are done. Otherwise, let W be the set of all M-unsaturated vertices of X. So $|M| + |W| = |X|$. As in the proof of Hall's Theorem, define the following set of vertices of G.

$$A = \{v \in V_G \mid \text{there are } w \in W \text{ and an } M\text{-alternating } wv\text{-path}\}.$$

Let $S = A \cap X$ and $T = A \cap Y$. Note that $W \subseteq S$. Since M is a maximum matching, by Berge's Theorem it follows that G has no M-augmenting paths. This implies that every vertex in $(S \setminus W) \cup T$ is M-saturated. Moreover, one can show as we did in the proof of Hall's Theorem that $|S \setminus W| = |T|$ and that $N(S) = T$. Thus, $|S| - |W| = |T|$.

Now let $e = xy$ be any edge in G. Since G is bipartite, we may assume without loss of generality that $x \in X$ and $y \in Y$. Moreover, if $x \in S$, then as $N(S) = T$ we have that $y \in T$. So either e intersects $X \setminus S$ or e intersects T. This shows that

$$C = (X \setminus S) \cup T$$

is a vertex cover of G. Moreover,

$$|C| = |X \setminus S| + |T| = |X| - |S| + |T| = |X| - (|S| - |T|) = |X| - |W| = |M|.$$

By Lemma 7.4.2, we know that the size of any vertex cover in G can be no smaller than $|M|$. Since $|C| = |M|$, it means that C is a minimum vertex cover. Thus, we proved that the maximum cardinality of a matching equals the minimum cardinality of a vertex cover in bipartite graphs. ■

Chapter 8

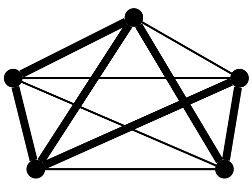
Ramsey Theory

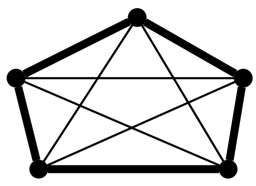
Among any group of six people, either there are three mutual friends or three mutual strangers. Ramsey theory shows that complete disorder is an impossibility.

Frank Ramsey was a British mathematician, philosopher, and economist who died in 1930 at a very young age of 26 years old. He is considered one of the brilliant minds of the twentieth century. The mathematical field known as Ramsey Theory was initiated by a paper of his own titled “On a problem of formal logic”. Economists study Ramsey pricing. Philosophers discuss Ramsey sentences, Ramseyfication, and the Ramsey test. Mathematicians are questing for Ramsey numbers.

8.1 Ramsey Numbers

A *k-colouring of the edges* of a graph G is a function $K : E_G \to \{1, 2, \dots, k\}$. So we just assign to every edge some colour with no restrictions on the assignment of colours. We will usually use only two colours, red and blue. By a *red graph*, we mean a graph whose all of its edges are coloured red. Similarly, for a blue graph. Below are two different 2-colourings of the edges of K_5 where **thick** edges are **red** edges and thin edges are blue edges. Notice that the edge colouring on the left contains a red K_3 as a subgraph. In this case, we say that such K_3 is *monochromatic*, since all of its edges are coloured with the same colour. On the other hand, the edge colouring on the right neither contains a red K_3 nor a blue K_3.





We will use such edge colouring to associate to every pair of positive integers a unique positive integer called their Ramsey number.

Definition 8.1.1. (Ramsey Number)

The *Ramsey number* $R(a, b)$ of positive integers a and b is the smallest integer n such that every 2-colouring of the edges of K_n contains either a red K_a or a blue K_b as a subgraph.

Read the definition again and again. So far, there is no reason to believe that the Ramsey number $R(a, b)$ exists for every integers a and b. Let us start computing some Ramsey numbers.

Let m be a positive integer, then $R(1, m)$ is the smallest integer n such that any 2-colouring of the edges of K_n contains either a red K_1 or a blue K_m. As K_1 has no edges, we may think of any single vertex as a red K_1. It follows that any 2-colouring of the edges of any K_n contains a red K_1. So any complete graph will suffice, and so we have to choose the smallest complete graph K_1. In other words, any 2-colouring of the edges of K_1 (there are none) contains a red K_1. Thus $R(1, m) = 1$. Similarly, $R(m, 1) = 1$ for every positive integer m.

Let us compute $R(2, 2)$. This is the smallest integer n such that any 2-colouring of the edges of K_n contains either a red K_2 or a blue K_2. Clearly, K_2 does the job. There is only one edge in K_2, if this edge is coloured red, then we get a red K_2, and if it is coloured blue then we get a blue K_2. Moreover, as K_1 does not contain K_2 as a subgraph, the smallest such n is 2. Thus, $R(2, 2) = 2$.

Let us compute $R(2, 3)$. This is the smallest integer n such that any 2-colouring of the edges of K_n contains either a red K_2 or a blue K_3. Let us colour the edges of K_3 using red and blue in any way you like. If all the edges were blue, then we have a blue K_3. Otherwise, at least one edge is red, and so there is a red K_2. This shows that K_3 does the job. Moreover, K_2 is not enough. To see this, think of a K_2 whose edge is coloured with blue, then there is no red K_2 nor a blue K_3. So the smallest required n is 3. Thus, $R(2, 3) = 3$. Similarly, $R(3, 2) = 3$.

Lemma 8.1.2.

If $r \geq R(a, b)$, then any 2-colouring of the edges of K_r either contains a red K_a or a blue K_b as a subgraph.

Proof. Let $n = R(a, b)$, and suppose that $r \geq n$. Since K_n is a subgraph of K_r, given any 2-colouring of the edges of K_r, we obtain a 2-colouring of the edges of K_n. As $n = R(a, b)$, such colouring of K_n either contains a red K_a or a blue K_b as a subgraph of K_n and so as a subgraph of K_r. ∎

Lemma 8.1.3.

Let $m \geq 2$. Then $R(2, m) = m$.

Proof. We need to prove two facts. First, that any 2-colouring of the edges of K_m either contains a red K_2 or a blue K_m as a subgraph, and second, that m is the least integer with such property.

We establish the first point. Suppose that the edges of K_m are coloured with red and blue in any fashion. If all edges are blue, then we have a blue K_m. Otherwise, there is a red edge, and so we have a red K_2. So, no matter the colouring of the edges of K_m, there will always be a red K_2 or a blue K_m.

For the second point, we will show that K_{m-1} does not do the job. That is, there exists a 2-colouring of the edges of K_{m-1} which contains neither a red K_2 nor a blue K_m. Just think of the graph K_{m-1} whose all of its edges are blue. Such colouring of the edges of K_{m-1} does not contain a red K_2, and certainly does not contain a blue K_m since K_m cannot be a subgraph of K_{m-1} in the first place. This proves that $R(2, m) = m$. ∎

A similar proof to the above shows that $R(m, 2) = m$.

Lemma 8.1.4.

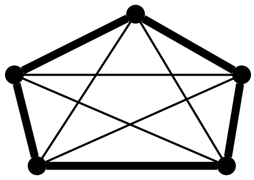
Let a, b be positive integers. Then $R(a, b) = R(b, a)$.

Proof. Let $n = R(a, b)$. We will show that $R(b, a) \leq n$. Let G be a copy of K_n and colour the edges of G using red and blue in any fashion. We will show that G either contains a red K_b or a blue K_a. Take another copy H of K_n and colour its edges by flipping the colours in G, that is, an edge in H is coloured red if its counterpart in G is blue, and it is coloured blue if its counterpart in G is red. Since n is the Ramsey number of (a, b), such colouring of H must either contain a red K_a or a blue K_b as a subgraph of H. This translates to having in G either a blue K_a or a red K_b as required. Since $R(b, a)$ is the least integer l such that any 2-colouring of K_l either contains a red K_b or a blue K_a, it follows that $R(b, a) \leq n = R(a, b)$. Similarly, $R(a, b) \leq R(b, a)$. Therefore, $R(a, b) = R(b, a)$. ∎

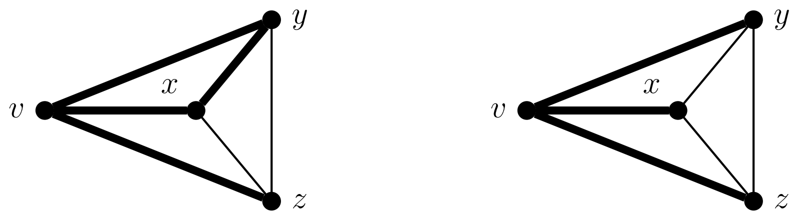
Lemma 8.1.5.

$R(3, 3) = 6$.

Proof. We need to prove that K_6 is the smallest complete graph such that any 2-colouring of its edges must contain a monochromatic K_3. We need to show two points. First, K_6 has this property, and second, K_5 does not. We start with latter one. The following colouring of the edges K_5 where thick edges are red and thin edges are blue does not not contain any red or blue K_3. Thus $R(3,3) > 5$.



We now show that any 2-colouring of K_6 contains a monochromatic K_3. Suppose the edges of K_6 are coloured using red and blue in any fashion. Choose any vertex v in K_6. There are five edges incident with v each is either red or blue. By the pigeonhole principle, at least three of these edges must be of the same colour. Without loss of generality, suppose there are three red edges incident with v. Say these three red edges are vx, vy, vz where x, y, z are distinct vertices in K_6. One of the following two cases must happen.



To see this, consider the edges xy, yz, zx. If any of these edges is red, say xy is red, then the induced subgraph on $\{v, x, y\}$ is a red K_3 as in the diagram on the left. Otherwise all of them are blue, and so the subgraph induced on $\{x, y, z\}$ is a blue K_3 as shown on the right. Therefore, any 2-colouring of the edges of K_6 contains either a red K_3 or a blue K_3. This shows that $R(3,3) = 6$. ■

Corollary 8.1.6.

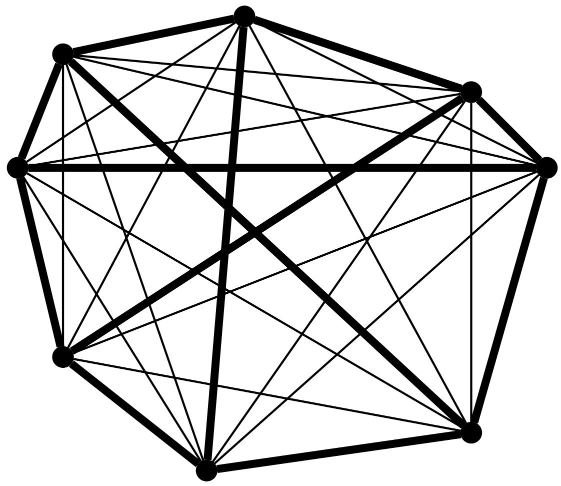
Any group of six people must contain either three mutual friends or three mutual strangers.

Proof. Given any group of six people, think of them as the vertices of K_6. For any two of them x and y, colour their edge xy with red if x and y are friends, otherwise if they are strangers to each other, colour their edge xy with blue. This produces an edge colouring of K_6. Since $R(3,3) = 6$, such colouring must contain either a red K_3 and hence 3 mutual friends, or a blue K_3 and hence 3 mutual strangers. ■

Lemma 8.1.7.

$R(3,4) = 9$.

Proof. First, we show that there exists a 2-colouring of the edges of K_8 which neither contains a red K_3 nor a blue K_4. Such edge colouring is given below where thick edges are red and thin edges are blue. This proves that $R(3,4) > 8$.



Second, we will show that any 2-colouring of the edges of K_9 either contains a red K_3 or a blue K_4. So suppose we are given any colouring of the edges of K_9 using colours red and blue.

Claim. There exists a vertex in the given coloured K_9 which is either incident with at least 4 red edges or at least six blue edges.
To prove the claim, assume for the sake of contradiction that it is not the case. So every vertex in the K_9 is incident with at most 3 red edges and at most 5 blue edges. Since in K_9 the degree of every vertex is 8, it must be that every vertex is incident with exactly 3 red edges and exactly 5 blue edges. Now consider the subgraph H of K_9 that is made of all the nine vertices and all the red edges. Every vertex in H has degree 3. Thus, the number of vertices of odd degree in H is odd, this contradicts the Handshaking Lemma. And so the claim is established.

We are now left with two cases. In each case we will show that there is either a red K_3 or a blue K_4 in the given coloured K_9.

Case 1. Suppose that there exists a vertex v that is incident with 4 red edges. Say these red edges are va, vb, vc, vd where a, b, c, d are distinct vertices in K_9. The subgraph S induced on $\{a, b, c, d\}$ is K_4 whose edges are already coloured with red and blue. If all of the six edges in S are blue, then we have a blue K_4 in the given K_9 as desired. Otherwise, there is at least one red edge in S, say ab is red, in such case we have a red K_3 given by the induced subgraph on $\{v, a, b\}$.

Case 2. Suppose that there exists a vertex v that is incident with 6 blue edges. Say these blue edges are va, vb, vc, vd, ve, vf where a, b, c, d, e, f are distinct vertices in K_9. The subgraph T induced on $\{a, b, c, d, e, f\}$ is K_6 whose edges are already coloured with red and blue. Since $R(3,3) = 6$, either T contains a red K_3 or a blue K_3. If the former case holds, then we are done as we have the desired red K_3 in the given K_9. Otherwise, there is in T a blue K_3, say $\{a, b, c\}$ is a blue K_3, but then the subgraph induced on $\{v, a, b, c\}$ is a blue K_4 within the given K_9.

We have therefore shown that any 2-colouring of the edges of K_9 produces either a red K_3 or a blue K_4. This shows that $R(3, 4) = 9$. ■

Here is a list collecting known Ramsey numbers.

- $R(1, k) = 1$.
- $R(2, k) = k$.
- $R(3, 3) = 6$.
- $R(3, 4) = 9$.
- $R(3, 5) = 14$.
- $R(3, 6) = 18$.
- $R(3, 7) = 23$.
- $R(3, 8) = 28$.
- $R(3, 9) = 36$.
- $R(4, 4) = 18$.
- $R(4, 5) = 25$.
- $R(5, 5) = ??$

8.2 Ramsey Theorem

In a paper titled "On a problem of formal logic" that was presented to the London Mathematical Society in 1930, Frank Ramsey proved a minor lemma that he used to solve a problem in formal logic. Forty years later, this little lemma became a famous result known as Ramsey Theorem and it led to the establishment of the mathematical area known as Ramsey Theory.

Theorem 8.2.1. (Ramsey Theorem)

For every positive integers a and b, there exists a positive integer n such that every 2-colouring of the edges of K_n contains a red K_a or a blue K_b as a subgraph.

In other words, Ramsey Theorem states the following.

Corollary 8.2.2.

Ramsey numbers always exist.

Bounds on Ramsey Numbers

Although Ramsey numbers exist, it is extremely difficult to determine their exact value. The Ramsey numbers given in the previous section are the only known Ramsey numbers! Mathematicians are still working on finding new Ramsey numbers. However, bounds on Ramsey numbers have been found. The following result is due to the Hungarian mathematicians Paul Erdős and George Szekeres from 1987 who were two major pioneers in the development of Ramsey Theory.

Theorem 8.2.3.

Let a, b be positive integers. Then

$$R(a,b) \leq \frac{(a+b-2)!}{(a-1)!\,(b-1)!}\ .$$

We state below other known bounds.

Theorem 8.2.4.

Let $a \geq 2$ and $b \geq 2$. Then

$$R(a,b) \leq R(a-1,b) + R(a,b-1).$$

Theorem 8.2.5.

Let $m \geq 3$. *Then*

$$R(3, m) \leq \frac{m^2 + 3}{2} .$$

Theorem 8.2.6. (Erdős)

Let $m \geq 3$. *Then*

$$\lfloor 2^{m/2} \rfloor < R(m, m).$$

Here is a table of some bounds on Ramsey numbers.

Lower bound	Ramsey number	Upper bound
35	$R(4,6)$	41
43	$R(5,5)$	49
58	$R(5,6)$	87
102	$R(6,6)$	165
205	$R(7,7)$	540
282	$R(8,8)$	1870
565	$R(9,9)$	6588
798	$R(10,10)$	23556

The exact value of a Ramsey number is extremely difficult to compute even with the use of modern computers. Below is a famous quote of Paul Erdős describing how hard it is to compute Ramsey numbers, even small Ramsey numbers.

"Suppose an evil alien would tell mankind "Either you compute the value of $R(5, 5)$ *or else I will exterminate the human race." It would be best in this case if all mathematicians and computers start working on the answer. If, however, he orders us to compute* $R(6, 6)$, *then we had better think about how to destroy him before he destroys us."*

We next define the Ramsey number of k many positive integers $m_1, m_2, \ldots, m_k$.

Definition 8.2.7. (Ramsey Number)

The *Ramsey number* $R(m_1, m_2, \ldots, m_k)$ is the smallest integer n such that any k-colouring of the edges of K_n contains a subgraph K_{m_i} for some integer $i \in \{1, 2, \ldots, k\}$ such that every edge of K_{m_i} is coloured with colour i.

We finish by stating the infinite version of Ramsey Theorem. Let X be a set and n a nonnegative integer. By $[X]^n$ we denote the set of all subsets of X containing exactly n elements. Notice that if G is a graph, then $E_G \subseteq [V_G]^2$.

Theorem 8.2.8. (Infinite Ramsey Theorem)

Let X be any infinite set. Then for any colouring of the elements of $[X]^n$ in k different colours, there exists an infinite subset $M \subseteq X$ such that all the elements of $[M]^n$ have the same colour.