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The results from this dissertation are an exact computation of ultrapowers by measures on cardinals $\aleph_n, n \in \omega$, in $L(\mathbb{R})$, and a proof that ordinals in $L(\mathbb{R})$ below $\underset{\sim}{\delta}_5^1$ represented by descriptions and the identity function with respect to sequences of measures are cardinals.

An introduction to the subject with the basic definitions and well known facts is presented in chapter I.

In chapter II, we define a class of measures on the $\aleph_n$, $n \in \omega$, in $L(\mathbb{R})$ and derive a formula for an exact computation of the ultrapowers of cardinals by these measures.

In chapter III, we give the definitions of descriptions and the lowering operator. Then we prove that ordinals represented by descriptions and the identity function are cardinals. This result combined with the fact that every cardinal $< \underset{\sim}{\delta}_5^1$ in $L(\mathbb{R})$ is represented by a description, gives a characterization of cardinals in $L(\mathbb{R})$ below $\underset{\sim}{\delta}_5^1$.

Concrete examples of formal computations are shown in chapter IV.

DESCRIPTIONS AND COMPUTATION OF ULTRAPOWERS IN $L(\mathbb{R})$

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DESCRIPTIONS AND COMPUTATION OF ULTRAPOWERS IN $L(\mathbb{R})$

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CHAPTER 1

INTRODUCTION

The main theme of this dissertation is investigation of the very fine structure of $L(\mathbb{R})$, the universe of sets definable from reals. By "real" we mean an element of the Baire space ω^ω. The importance of understanding the structure of $L(\mathbb{R})$ is based on its relation with descriptive set theory. Descriptive set theory concerns the structure of definable sets of reals, e.g. Borel, analytic, coanalytic, projective sets. In the first half of this century, many natural questions (such as Lebesgue measurability, having the property of Baire, etc.) about sets of reals were answered for the sets in the first level of projective hierarchy. Only many years later was it discovered that the same questions about sets in the higher level of the projective hierarchy were independent of ZFC.

A prevalent belief among set theorists is that the natural extension of ZFC is provided by large cardinal axioms. One of the consequences of the large cardinal axioms is that the Axiom of Determinacy (AD) holds in $L(\mathbb{R})$. AD is the assertion that for any set A, a subset of reals, one of the two players in the game G_A has a winning strategy (or the game G_A is determined). In the game G_A, I and II alternate moves by picking integers infinitely many times, and the player I wins if the real constructed during the game is in A. In 1975, Martin proved that all Borel games are determined [Ma]. It is known, however, that in ZFC one can prove (using the axiom of choice (AC)) existence of a nondetermined game [My]. One of the interesting consequences of AD is the regularity of pointsets — all sets are absolutely measurable, they all have the property of Baire, etc. Moreover, assuming AD in 1967, Solovay showed that $\aleph_1$ is a measurable cardinal [So]. This was one of the first results that focused the attention of "mainstream" set theorists upon studying consequences of AD and the model $L(\mathbb{R})$.

Although AD is false in the real universe, V, — the ground model of ZFC, it turns out that questions about sets in V can be answered working inside of $L(\mathbb{R})$. Then, the results obtained in $L(\mathbb{R})$ can be transferred back into V. More about the theory of AD and its applications to the descriptive set theory can be found in [Mo, Ke].

Throughout the dissertation we will work in $ZF + AD + V = L(\mathbb{R})$. Beginning with the work of Martin, Moschovakis, Solovay in the late 60's, and later others, a structural theory of projective sets of reals was developed assuming projective determinacy — AD restricted to games with projective payoff. This provided a powerful extension of the ZFC theory of analytical ($\underset{\sim}{\Sigma}^1_1$) and coanalytical ($\underset{\sim}{\Pi}^1_1$) sets developed from the 20's to the 40's by the "classical" descriptive set theorists: Novikov, Lusin, Suslin, Sierpinski, etc.

The structural theory was developed largely in terms of the projective ordinals. The projective ordinals are defined by: $\underset{\sim}{\delta}^1_n := \sup\{$ length of the $\underset{\sim}{\Delta}^1_n$ prewellorderings of the reals $\}$. (A prewellordering is a wellordering of equivalence classes.) For example, any set in the n^{th} projective hierarchy can be constructed using $\leq \underset{\sim}{\delta}^1_n$ Borel sets. It was also shown that $\underset{\sim}{\delta}^1_1 = \aleph_1$, $\underset{\sim}{\delta}^1_3 = \aleph_{\omega+1}$ (Martin Solovay), $\underset{\sim}{\delta}^1_{2n+2} = (\underset{\sim}{\delta}^1_{2n+1})^+$ (Kunen Martin). Detailed exposition of this theory can be found in [Mo], [Ke], [Ke1].

and [Ke2].

Kunen (and independently Martin) discovered the concept of a homogeneous tree and proposed a program for computing the $\underset{\sim}{\delta}^1_n$. The problem of computation of $\underset{\sim}{\delta}^1_5$ was reduced by Kunen (see [Ke2]) to the problem of computing ultrapowers by measures, S, that occur in the homogeneous tree construction for a complete Π^1_3 set: $\underset{\sim}{\delta}^1_5 = (\sup_S j_S(\underset{\sim}{\delta}^1_3))^+$. The program stalled, however, and the problem of computing $\underset{\sim}{\delta}^1_5$ remained open. For about ten years, from the mid 70's untill mid 80's, finding the exact place of $\underset{\sim}{\delta}^1_n$, $n \geq 5$, in the sequence of the alephs was one of the most intriguing and difficult problems in descriptive set theory.

In the early 80's, Martin obtained some important results (see [J1]) on the ultrapowers of $\underset{\sim}{\delta}^1_3$ by the normal measures on $\underset{\sim}{\delta}^1_3$. Later, Jackson discovered powerful techniques of applying strong partition relation to functions (we will explain this later). Jackson used these and Martin's results to develop the theory of descriptions in $L(\mathbb{R})$ and successfully completed the program of computing first $\underset{\sim}{\delta}^1_5$ and then $\underset{\sim}{\delta}^1_n$ for all $n \in \omega$ [J2]. He showed that $\underset{\sim}{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$ and $\underset{\sim}{\delta}^1_{2n+1} = (\aleph_{\omega(2n-1)})$, where $\omega(1) = \omega$, and $\omega(n+1) = \omega^{\omega(n)}$ (ordinal exponentiation). Using an absoluteness argument one can derive then that in V, $\underset{\sim}{\delta}^1_5 \leq \aleph_7$ and in general $\underset{\sim}{\delta}^1_n < \aleph_\omega$.

In this dissertation we apply techniques developed in [J1] to investigate ultrapowers by certain measures, S_m, on $\aleph_m, m \in \omega$. As in the case of $\underset{\sim}{\delta}^1_5$, one can consider measures, S_m, that occur in the homogeneous tree construction for a complete Π^1_2 set. Then $\underset{\sim}{\delta}^1_3 = (\sup_{n\in\omega,\, S_m} j_{S_m}(\aleph_n))^+$ [Ke2]. We derive a formula for exact computations of the ultrapowers of cardinals $\aleph_\alpha, \alpha < \omega_1$ by these measures.

Then we investigate descriptions, the objects Jackson introduced in his computation of $\underset{\sim}{\delta}^1_5$. Descriptions proved to be very important in analyzing ultrapowers by measures in $L(\mathbb{R})$. Jackson showed that every cardinal in $L(\mathbb{R})$ below $\underset{\sim}{\delta}^1_5$ can viewed as a description. In this dissertation we show that the converse is also true, i.e., every description can be viewed as a cardinal. Thus we obtain a representation of cardinals less than $\underset{\sim}{\delta}^1_5$ in $L(\mathbb{R})$.

CHAPTER 2

ULTRAPOWERS OF $\aleph_\alpha$ FOR $\alpha < \omega_1$ BY MEASURES ON $\aleph_n$, $n < \omega$

2.1 Introduction

Throughout the dissertation we work in the context of AD inside of $L(\mathbb{R})$. In this section we give basic definitions and facts needed for what follows, then we define *general* measures and state the theorem which gives the formula for ultrapowers. Throughout this chapter c. u. b. set means closed and unbounded subset of $\aleph_1$, *measure*, means a σ-complete ultrafilter, μ is the normal measure on $\aleph_1$, and μ^m is its m-fold product measure on ${\aleph_1}^m$. If η is a measure, then $j_\eta : V \to \mathrm{Ult}_\eta(V)$ is the corresponding (nontrivial) elementary embedding, and a function is η-*nontrivial* if it is not constant η-almost everywhere (a.e.). For every c. u. b. set C, C' stands for the closure points of C, also a c. u. b. set; and $N_C : \aleph_1 \to \aleph_1$ is the function which sends $\alpha \in \aleph_1$ to the ω-th element of C after α. By $Cub(\varphi_1, \ldots, \varphi_n)$ we denote a c. u. b. set closed under the functions $\varphi_1, \ldots, \varphi_n$. If W is a well-order, $|W|$ is its rank, and W^α is the initial segment of W determined by α. We use π_m to denote a partial permutation $(m, i_1, \ldots, i_{m-1})$ of $(1, 2, \ldots, m)$. Given any such permutation π, we define the corresponding well-order, $<^\pi$, on tuples $\bar{\alpha} := (\alpha_1, \ldots, \alpha_m) \in {\aleph_1}^m$, with $\alpha_1 < \cdots < \alpha_m$, by $\bar{\alpha} <^\pi \bar{\beta} \iff \pi(\bar{\alpha}) <^{lex} \pi(\bar{\beta})$, where $<^{lex}$ is the usual lexicographical order.

Definition 2.1.1 *We say $h : <^{\pi_m} \to \aleph_1$ is of* $\mathbf{c}$-correct type *($\mathbf{c}$-c.t.), $0 \leq \mathbf{c} \leq m$, , if:*

1. h is strictly increasing;

2. *h is everywhere discontinuous, i.e.,* $\sup\{h(\bar{\alpha}) \mid \bar{\alpha} <^{\pi_m} \bar{\beta}\} < h(\bar{\beta}),\ \forall \bar{\beta}$;

3. *for all $\bar{\alpha}$, uniformly* $\mathrm{cof}[\ h(\alpha_1, \ldots, \alpha_m)\ ] = \alpha_{\mathbf{c}}$, *where $\alpha_0 = \omega$, i.e., there is a function g* inducing h:

$$h(\alpha_1, \ldots, \alpha_m) = \begin{cases} \text{s[illegible]}\ \bar{\alpha}, n), & \mathbf{c} = 0 \\ \sup_{\beta < \alpha_{\mathbf{c}}} g(\alpha_1, \ldots, \alpha_{\mathbf{c}-1}, \beta, \alpha_{\mathbf{c}}, \ldots, \alpha_m), & \mathbf{c} \neq 0 \end{cases}$$

Sometimes we write c.t. for 0–c.t. Notice that in the absence of the Axiom of Choice the condition (3) is nontrivial.

Let us fix integers m, $0 \leq \mathbf{c} \leq m$, and a permutation π_m. A *basic measure*, $S^{\pi_m, \mathbf{c}}$, on $\aleph_{m+1}$ is defined as follows

Definition 2.1.2 (Basic Measure $S^{\pi_m,\mathbf{c}}$) *If $A \subset \aleph_{m+1}$, then $A \in S^{\pi_m,\mathbf{c}}$ if and only if there is a* c. u. b. *set C such that for every function $h : <^{\pi_m} \rightarrow C$ of the* $\mathbf{c}$*–correct type,* $[h]_{\mu^m} \in A$.

Here we interpret $[h]_{\mu^m}$ as an ordinal in the ultrapower $j_{\mu^m}(\aleph_1) = \aleph_{m+1}$ (see [Kel]). For the functions h as in the definition, we say that the measure $S^{\pi_m,\mathbf{c}}$ *concentrates* on functions h. It is easy to see that $S^{\pi_m,\mathbf{c}}$ is indeed a measure (use the strong partition relation on ω_1).

To each of the measures that we study here we associate a tree. The tree corresponding to $S^{\pi_m,\mathbf{c}}$ is shown in figure 2.1(i) and is rather trivial: just a path of length m with the terminal node $(\pi_m, \mathbf{c})$.

To define a *general measure*, fix a function $\mathcal{F}_m : U \rightarrow \bigcup_{m,\pi_m,0\leq\mathbf{c}\leq m}\{S^{\pi_m,\mathbf{c}}\}$ with finite domain $U \subset \omega^{\leq m}$ such that $\forall u \in U$ with $\mathrm{lh}(u) = k$, $\mathcal{F}_m(u) = S^{\pi_k,\mathbf{c}}$. It is assumed that U contains at least one sequence of length m.

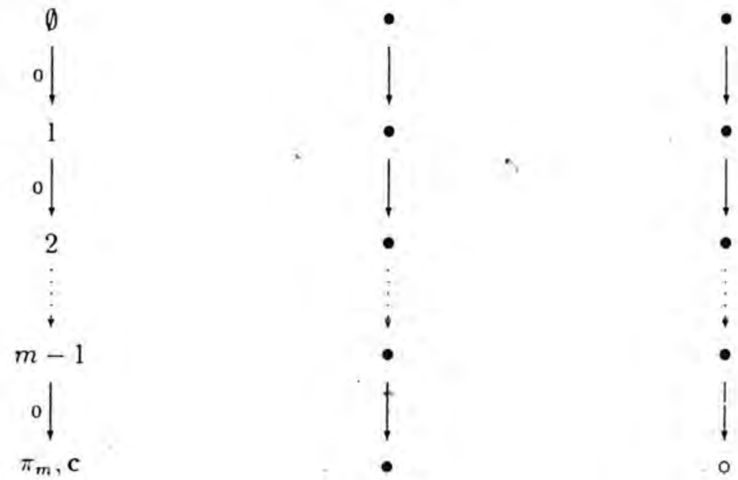

Fig. 2.1(i) Fig. 2.1(ii) Fig. 2.1(iii)

Figure 2.1: (i) Tree corresponding to $S^{\pi_m,\mathbf{c}}$; (ii) simplified representation of the same tree when $\mathbf{c} \neq 0$; (iii) simplified tree $T(S^{\pi_m,\mathbf{c}})$ when $\mathbf{c} = 0$.

On tuples $\langle u, \lambda \rangle \in (\omega \times \aleph_1)^{\leq m}$ where $u \in U$ and $\lambda(1) < \cdots < \lambda(k)$, we define a well-order $<_*$ as follows. Let u and u' be elements of U, and $k = \mathrm{lh}(u)$, $k' = \mathrm{lh}(u')$. Assume $\mathcal{F}_m(u) = S^{\pi,c}$, $\mathcal{F}_m(u') = S^{\pi',c'}$ with $\pi = (k, i_2, \ldots, i_k)$ and $\pi' = (k', j_2, \ldots, j_{k'})$. Then

$$(2.1.2.1) \quad \langle u, \lambda \rangle <_* \langle u', \lambda' \rangle \iff$$

$$(\ \lambda(k),\ u(1),\ \lambda(i_2),\ u(2),\ \ldots,\ \lambda(i_k),\ u(k)\ ) \quad <^{lex}$$

$$(\ \lambda'(k'),\ u'(1),\ \lambda'(j_2),\ u'(2),\ \ldots,\ \lambda'(j_{k'}),\ u'(k')\ )$$

Let $\mathcal{H} :<_* \to \aleph_1$ be order preserving and discontinuous. For each $u \in U$ we define $h_u(\lambda) := \mathcal{H}(\langle u, \lambda \rangle)$. We say $\mathcal{H}$ is of the type *matching with* $\mathcal{F}_m$ if $\forall u \in U$

$$h_u \text{ is of } \mathbf{c}\text{-c.t.} \quad \iff \quad \mathcal{F}_m(u) = S^{\pi_k, \mathbf{c}}.$$

In other words, the function h_u is one of those on which the measure $\mathcal{F}_m(u)$ concentrates.

Let us enumerate $U = \{u_i\}_{i=1}^{p}$ in lexicographically increasing order (here $p = |U|$). We set

$$\delta_i := [h_{u_i}]_{\mu^{k_i}}, \quad \text{for all } u_i \in U,\ k_i = \mathrm{lh}(u_i)$$

$$\delta(\mathcal{H}) := \mathrm{ot}\langle \delta_1, \ldots, \delta_p \rangle \ \in \aleph_{m+1}$$

Notice that given the function $\mathcal{F}_m$ with the domain U, for each function $\mathcal{H} :<_* \to \aleph_1$ of the type matching with $\mathcal{F}_m$, the ordinal $\delta = \delta(\mathcal{H})$ is determined uniquely. One thinks of the measure $\mathfrak{S}_{\mathcal{F}_m}$ being concentrated on such functions $\mathcal{H}$. We summarize all these in the formal

Definition 2.1.3 (General Measure $\mathfrak{S}_{\mathcal{F}_m}$) *Let* $\mathcal{A} \subset \aleph_{m+1}$. *We set* $\mathcal{A} \in \mathfrak{S}_{\mathcal{F}_m} \iff \exists$c. u. b.$C \subset \aleph_1, \forall \mathcal{H} :<_\bullet \to C$ *of the type matching with* $\mathcal{F}_m, \delta(\mathcal{H}) \in A$.

Observe that every basic measure $S^{\pi,c} = \mathfrak{S}_{\mathcal{F}}$ where the domain of $\mathcal{F}$ is a singleton $\{u\}$, and $\mathcal{F}(u) = S^{\pi,c}$.

A measure $\mathfrak{S}_{\mathcal{F}_m}$ (or simply $\mathfrak{S}$) can be represented by a finitely splitting tree of height $m+1$. The root of the tree is the empty set, and every maximal path is an element, u, of the domain of $\mathcal{F}_m$ with the terminal node $(\pi_k, \mathbf{c})$ if and only if $\mathcal{F}_m(u) = S^{\pi_k,\mathbf{c}}$. Moreover, paths are ordered lexicographically, so that the left most maximal path is precisely u_1. An example of such a tree is shown in figure 2.2(i).

It is known that for each general measure $\mathfrak{S}$, $j_{\mathfrak{S}}(\aleph_n) < \aleph_\omega$ [Ke2]. Our goal is to determine the exact values $j_{\mathfrak{S}}(\aleph_n)$. For simplicity of notation we define a *function* $\mathfrak{S} : \omega \to \omega$ as follows

$$\mathfrak{S}(n) = k \iff j_{\mathfrak{S}}(\aleph_n) = \aleph_k. \tag{2.1.2.2}$$

It will be clear from the context whether $\mathfrak{S}$ stands for a measure or for a function.

To state theorem 2.1.1, for each measure $\mathfrak{S}$ we define a *simplified tree*, $T = T(\mathfrak{S})$. If T' is the tree representing $\mathfrak{S}$, we obtain a simplified tree $T(\mathfrak{S})$ by placing black circles to all nonterminal nodes of T'; at the terminal nodes with we place black circles, if $\mathbf{c} \neq 0$, and white circles otherwise. The simplified tree for $S^{\pi,\mathbf{c}}$ is shown in figure 2.1(ii, iii). Figure 2.2(ii) also illustrates a simplified tree obtained from the one shown in figure 2.2(i). By the *the j-th level* of a tree we mean the set of nodes that can be reached from the root in j steps. Thus *the 0-th level* is just the root of a tree. Notice, $T(S^{\pi,\mathbf{c}})$ has $m+1$ levels. Now we define $\mathfrak{S}$ *ultrapower invariants*.

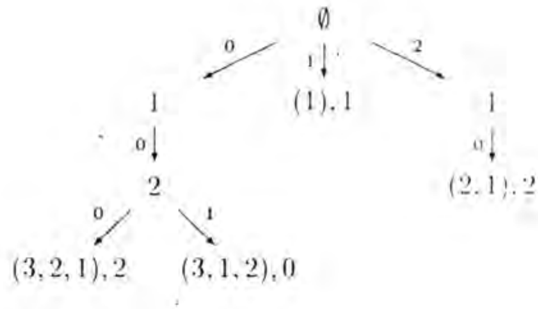

Fig. 2.2(i)

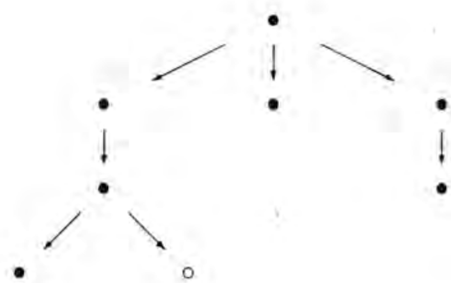

Fig. 2.2(ii)

Figure 2.2: (i) Tree corresponding to $\mathfrak{S}_{\mathcal{F}_3}$. $\mathcal{F}_3$ has domain

$$\{\ (0,0,0),\ (0,0,1),\ (1),\ (2,0)\ \}$$

and range

$$\{S^{(3,2,1),2},\ S^{(3,1,2),0},\ S^{(1),1},\ S^{(2,1),2}\}.$$

(ii) Simplified (version of the same) tree.

$b_j, w_j, \quad 0 \le j \le m+1$:

(2.1.2.3) $\qquad b_j =$ number of black circles in the j-th level

(2.1.2.4) $\qquad w_j =$ number of white circles in the j-th level

Let also $b_{-1} := 0$. Notice that always $b_0 = 1$ and $w_0 = 0$, hence $b_{-1} + b_0 + w_0 = 1$ for all measures.

Theorem 2.1.1 *Let $\mathfrak{S}$ be a general measure on $\aleph_{m+1}$ with the ultrapower invariants $\langle b_j, w_j \rangle_{j=0}^{m+1}$. Then for every integer $n \geq 1$*

$$\mathfrak{S}(n) \; = \; \sum_{j=0}^{m+1} (b_{j-1} + b_j + w_j) \binom{n-1}{j}. \tag{2.1.2.5}$$

If $\omega \le \alpha < \omega_1$, then $\mathfrak{S}(\alpha) = \alpha$ (i.e., $j_{\mathfrak{S}}(\aleph_\alpha) = \aleph_\alpha$).

Thus the ultrapower of $\aleph_n$ by $\mathfrak{S}$ depends on the structure of $\mathfrak{S}$ in a rather weak sense: there are many other measures (except in trivial cases) with the same ultrapower. In particular, notice the following. If $\mathfrak{S}$ is a measure represented by a tree T', and the splitting number of the root of T' is $n > 1$, then $\mathfrak{S}$ is merely the product of n measures corresponding to each subtree starting at the root of T'. Of course, the order of a product is important in determining the measure. However, it is irrelevant for the ultrapower $j_{\mathfrak{S}}$, as follows from the theorem. We state this fact as a

Corollary 2.1.2 *If $\mathfrak{S}$ and $\mathfrak{T}$ are general measures, then $j_{\mathfrak{S}\times\mathfrak{T}}(\aleph_n) = j_{\mathfrak{T}\times\mathfrak{S}}(\aleph_n)$ for all $n < \omega$.*

2.2 Technical Lemmas

Definition 2.2.1 (The strong partition property of κ) *A cardinal κ is said to have the strong partition property, denoted by $\kappa \to (\kappa)^\kappa_2$, if for any partition $G : [\kappa]^\kappa \to \{0,1\}$ (i.e., the domain of G is the set of all increasing functions from κ to κ), there is an unbounded set $A \subset \kappa$ (called homogeneous for G) and there is $i \in \{0,1\}$, so that $G(g) = i$, for all $g \in [A]^\kappa$.*

The following lemma gives a characterization of the strong partition relation on κ, which we will use for $\aleph_1$.

Lemma 2.2.1 *For every uncountable cardinal κ, $\kappa \to (\kappa)^\kappa_2$ is equivalent to the following property of κ. For every partition $F : \{f : \kappa \to \kappa$ of the correct type $\} \to \{0,1\}$, there is a c.u.b. $C \subset \kappa$ and $i \in \{0,1\}$ so that $F(f) = i$, for all $f : \kappa \to C$ of the correct type.*

Proof. Assume $\kappa \to (\kappa)^\kappa_2$ and fix a partition $F : \{f : \kappa \to \kappa$ of the correct type $\} \to \{0,1\}$. Let the partition $G : [\kappa]^\kappa \to \{0,1\}$ be defined as follows. $G(g) = F(f_g)$ where $f_g(\alpha) := \sup_{\beta<\omega(\alpha+1)} g(\beta), \forall \alpha \in \kappa$. Fix $A \subset \kappa$ homogeneous for G, fix i such that $G(g) = i, \forall g \in [A]^\kappa$, and let $C = (A)'$ be the set of closure points of A. Now, for every $f : \kappa \to C$ of the correct type, there is an increasing $g' : \kappa \to \kappa$ which induces f. Let $g : \kappa \to A$ be defined by $g(\beta) =$ the least $\xi \in A$ above $g'(\beta)$. Then for all $\alpha \in \kappa$, $f(\alpha) = \sup_{\beta<\omega(\alpha+1)} g'(\beta) = \sup_{\beta<\omega(\alpha+1)} g(\beta)$, because $f(\alpha)$ is the limit of points in A. Hence $f = f_g$, and $F(f) = F(f_g) = G(g) = i$.

Conversely, assume κ has the property stated in the lemma. Let $G : [\kappa]^\kappa \to \{0,1\}$ be a partition. To find a homogeneous set for G, let F be the restriction of G to functions $f : \kappa \to \kappa$ of the correct type. Fix c. u. b. $C \subset \kappa$ homogeneous for F with the constant i. Let C_ω be the set of ω-limit points of C, and let $A = C_\omega \setminus (C_\omega)'$ be the set of all nonlimit points of C_ω. Then every increasing function $g : \kappa \to A$ must be discontinuous, and of uniform cofinality ω. Hence $G(g) = F(g) = i$. ∎

One of the key facts in our proofs is that we can use the strong partition relation on $\aleph_1$ to answer questions about ordinals in $\underset{\sim}{\delta}^1_3$. For example, to determine whether pairs of ordinals in $\underset{\sim}{\delta}^1_3$ satisfy a certain relation, we can take functions representing the ordinals and make them "nice" in the sense that they form an order preserving map from an order type of cardinality $\aleph_1$ to $\aleph_1$. Thus the strong partition relation on $\aleph_1$ can be applied to these functions. The following lemmas (which justify this approach) are due to Jackson (see [J1]).

Frequently we want to say that given two functions $f, g : <^\pi \to C$ of c-c.t., $C \subset \aleph_1$ is a c. u. b. set, we can change their values on a $S^{\pi,c}$-measure zero set so that the new functions F and G combined together form a strictly increasing function from $<^\pi$ into C. Then, invoking the strong partition relation, we are able to obtain a desired conclusion. Later we refer to that kind of argument as *the sliding argument* (we "slide" the functions f and g).

Lemma 2.2.2 *Suppose C is a* c. u. b. *subset of $\aleph_1$, $f, g : \aleph_1 \to C$ are of the correct type (c.t.) with $f < g$ μ-a.e., and f is cofinal in $\aleph_1$.*

Then there are $F, G : \aleph_1 \to C$ of c.t. so that $[F]_\mu = [f]_\mu$, $[G]_\mu = [g]_\mu$, and $G(\beta) < F(\alpha) < G(\alpha)$, $\forall \beta < \alpha \in \aleph_1$.

Proof. Let f be cofinal in $\aleph_1$, then $\alpha < f(\alpha), \forall^*_\mu \alpha$. Let $f < g$ μ-a.e., then there is a c.u.b. set $C_1 \subset C$ so that for every $\beta < \alpha \in C_1$, $\alpha < f(\alpha)$ and $g(\beta) < f(\alpha) < g(\alpha)$. We fix such C_1, and define $C_2 := (Cub(g, \psi) \cap C_1)'$, where $\psi(x) := \sup\{ (\omega(2z))^{th}$ element of C_1 after $y \mid y, z < x\}$ is a function from $\aleph_1$ to $\aleph_1$. Let

(2.2.2.1)

$$F(\alpha) := \begin{cases} f(\alpha), & \alpha \in C_2 \\ N_{C_2}(\ \max\{\ \sup_{\beta<\alpha} F(\beta),\ \sup_{\beta<\alpha} G(\beta)\ \}\ ), & \alpha \notin C_2. \end{cases}$$

(2.2.2.2)

$$G(\alpha) := \begin{cases} g(\alpha), & \alpha \in C_2 \\ N_{C_2}(\ \max\{\ F(\alpha),\ \sup_{\beta<\alpha} G(\beta)\ \}\ ), & \alpha \notin C_2. \end{cases}$$

Clearly, both F and G are discontinuous functions into C, have uniform cofinality ω, and $[F]_\mu = [f]_\mu$, $[G]_\mu = [g]_\mu$. It remains to verify that $G(\beta) < F(\alpha) < G(\alpha)$, $\forall \beta < \alpha \in \aleph_1$. Fix $\beta < \alpha \in \aleph_1$. If $\alpha \notin C_2$ the desired inequalities follow from the definitions of F and G. If both $\alpha, \beta \in C_2 \subset C_1$, then $G(\beta) = g(\beta) < F(\alpha) = f(\alpha) < G(\alpha) = g(\alpha)$. Finally, suppose $\alpha \in C_2$ and $\beta \notin C_2$. Let β_0 be the supremum of ordinals in C_2 below β. Then $\beta_0 \in C_2$, so $G(\beta_0) = g(\beta_0) < \alpha \in Cub(g)$. If $\beta_0 < \gamma \leq \beta$, then $\gamma \notin C_2$, hence $F(\gamma), G(\gamma) < \omega(2\gamma))th$ element of C_1 after $g(\beta_0) < \psi(max\{\beta, g(\beta_0)\} + 1) < \alpha \in Cub(\psi)$. So $G(\beta) < \alpha < f(\alpha) = F(\alpha) < G(\alpha)$. ■

Given a permutation π_m and an integer $i < m$, let $\alpha = \pi_m((\alpha_1, \ldots, \alpha_m))$, and $(\alpha)(i) = \pi_m((\alpha_1, \ldots, \hat{\alpha}_{j_1}, \ldots, \hat{\alpha}_{j_{m-i-1}}, \ldots, \alpha_m))$, i.e., the subsequence of α consisting of $i+1$ ordinals which are the most significant in determining the rank of $(\alpha_1, \ldots, \alpha_m$

in the $<^{\pi_m}$ order.

Lemma 2.2.3 *If $f, g : <^{\pi_m} \to C$ are of c.t. with $f < g$, μ^m-a.e., $f(i) < g(i)$, μ^{i+1}-a.e., and $f(i-1) = g(i-1)$ μ^i-almost everywhere. Then there are functions $F, G : <^{\pi_m} \to C$ of c.t. with $[F]_{\mu^m} = [f]_{\mu^m}$, $[G]_{\mu^m} = [g]_{\mu^m}$, and $F(i)(\hat{\beta}) < G(\beta) < G(i)(\hat{\beta}) < F(\alpha)$, for all $\beta <^{\pi_m} \alpha$.*

Proof. For simplisity of notation let us fix $\pi = \pi_m = (1,2,3)$ and $i = 1$, then $f(1) < g(1)$, μ^2-a.e., $f(0) = g(0)$, μ-a.e., $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and $\hat{\bar{\alpha}} = (\alpha_2, \alpha_3)$. Let $\pi' = (1,2)$. It is easy to see that there is a c. u. b. set C_1 such that $f(1)(\alpha_2, \alpha_3) < g(\alpha_1, \alpha_2, \alpha_3)$, $\forall \bar{\alpha} \in [C_1]^3$. Indeed, let C_1 be homogeneous for the partition $\mathcal{P}(\bar{\alpha}) = 1$, if $f(1)(\alpha_2, \alpha_3) < g(\alpha_1, \alpha_2, \alpha_3)$; and 0, otherwise. Assume that $\forall \bar{\alpha} \in [C_1]^3$, $\mathcal{P}(\bar{\alpha}) = 0$. Fix $\alpha_2 < \alpha_3 \in C_1$ and let $\{\beta_n\}$ be an ω-sequence from C_1 converging to α_2. Then $g(1)(\alpha_2, \alpha_3) = \sup_n g(2)(\beta_n, \alpha_2, \alpha_3) \leq f(1)(\alpha_2, \alpha_3)$. Hence $g(1) \leq f(1)$, μ^2-a.e., a contradiction.

Thus we may assume there is a c. u. b. set C and $f, g : <^{(1,2,3)} \to C$ such that $f < g$, $f(1) < g(1)$, $f(0) = g(0)$, f and g are strictly increasing on ordinals from C, and $f(1)(\hat{\alpha}) < g(\alpha)$, $\forall \alpha \in C$. We fix such C and define $\psi : \aleph_1 \to C$, a c. u. b. set H, and functions F and G as follows:

$$\psi(x) := \sup\{(\omega y^4)th \text{ element of } C \text{ after } z \mid y, z < x\},$$

$$H := (Cub(f(0), g(0), \varphi, \psi) \cap C_1 \cap C')'$$

$$F(\bar{\alpha}) := \begin{cases} f(\bar{\alpha}), & \alpha \in [H]^3 \\ (N_C)^2(\max\{\ \sup_{\bar{\beta} <^* \bar{\alpha}} F(\bar{\beta}), \sup_{\bar{\beta} <^{*\prime} \hat{\alpha}} G(1)(\hat{\bar{\beta}})\ \}), & \bar{\alpha} \notin [H]^3 \end{cases}$$

$$G(\bar{\alpha}) := \begin{cases} g(\bar{\alpha}), & \bar{\alpha} \in [H]^3 \\ (N_C)^3(\max\{\ \sup_{\bar{\beta} <^* \bar{\alpha}} G(\bar{\beta}), F(1)(\bar{\alpha})\ \}), & \bar{\alpha} \notin [H]^3 \end{cases}$$

As it can be clearly seen from these definitions, F and G are both discontinuous functions into C with uniform cofinality ω, and represent same ordinals as f and g respectively. Thus it remains to verify that the second assertion of the lemma.

Since $0 \notin H$, $F(0, \beta) > \sup_{\beta' < \beta} G(0)(\beta')$ and $G(0, \beta) > F(0)(\beta)$ for all $\beta \in \aleph_1$, i.e., condition (3) and half of condition (2) are also satisfied.

We define:

$$S_F := \{\ \gamma \mid \exists \delta < \gamma,\ \exists (\alpha, \beta) <^{(2,1)} (\delta, \gamma) \text{ so that } F(\alpha, \beta) \geq F(\delta, \gamma)\ \};$$

$$S_G := \{\ \gamma \mid \exists \delta < \gamma,\ \exists (\alpha, \beta) <^{(2,1)} (\delta, \gamma) \text{ so that } G(\alpha, \beta) \geq G(\delta, \gamma)\ \};$$

$$S_2 := \{\ \beta \mid \text{ condition (2) is not satisfied at } \beta\ \}.$$

If each of these three sets is empty, then the theorem is proven. Assume, towards a contradiction, this is not the case. Then there exists

$$\gamma := \inf(S_F \cup S_G \cup S_2)$$

Suppose $\gamma \in S_F$, i.e., F is not strictly increasing. Let δ be the least ordinal, and let (α, β) be the $<^{(2,1)}$-least pair witnessing that, i.e., $F(\alpha, \beta) \geq F(\delta, \gamma)$. We intend to show $F(\alpha, \beta) < F(\delta, \gamma)$.

Case 1: $(\delta, \gamma) \notin H \times H$. In that case the desired inequality follows from the definition of F and our assumption on (α, β), (δ, γ).

Case 2: $(\delta, \gamma) \in H \times H$, $(\alpha, \beta) \in H \times H$. In that case $F = f$ on these two pairs, and the desired inequality follows because f is strictly increasing.

Case 3: $(\delta, \gamma) \in H \times H$, $(\alpha, \beta) \notin H \times H$.

This is our main case in which $\gamma < f(\delta, \gamma) = F(\delta, \gamma)$, and $\gamma \in Cub(f(0), g(0), \psi)$.

Subcase 3-1: $\beta < \gamma$.

We define $\beta_0 := \sup(H \cap \beta)$, and $\alpha_0 := \begin{cases} \sup(H \cap \alpha), & \text{if } \beta_0 = \beta \\ \sup(H \cap \beta_0), & \text{if } \beta_0 < \beta. \end{cases}$

Here we set $\sup X = 0$ if $X = \emptyset$. Observe that

$$f(0)(\beta_0) < \gamma, \qquad \text{and} \qquad g(0)(\beta_0) < \gamma,$$

because $\beta_0 < \gamma \in Cub(f(0), g(0))$. Next we merely note that

$$m := \max\{\ f(0)(\beta_0), g(0)(\beta_0), \beta\ \}\ <\ \gamma\ \in Cub(\psi),$$

hence $\psi(m) < \gamma$. That tells us that the number of elements of C between m and γ is sufficiently enough to lead us to the desired conclusion:

Subsubcase 3-1-a: $\alpha_0 \notin H$ (i.e., $\alpha_0 = 0$)

In this case, whenever $(0,0) \leq_1^{(2,1)} (\tau, \epsilon) \leq_1^{(2,1)} (\alpha, \beta)$, then

$$F(\tau, \epsilon) = [\ \max\{\ \sup_{\tau' < \tau} F(\tau', \epsilon), \sup_{\epsilon' < \epsilon} G(0)(\epsilon')\ \}\ ]_C^{2-next}\ <\ \psi(m) < \gamma.$$

So, $F(\alpha, \beta) < \gamma < F(\delta, \gamma)$.

Subsubcase 3-1-b: $\alpha_0 = \beta_0$.

In this case $\beta_0 < \beta$, and $G(0)(\beta_0) = g(0)(\beta_0)$ regardless of whether β_0 is zero or a limit point of H. Thus

$$F(0, \beta_0 + 1) = [\sup_{\beta' < \beta_0 + 1} G(0)(\beta')\ ]_C^{2-next} = [\ g(0)(\beta_0)\ ]_C^{2-next} < \psi(m) < \gamma,$$

and the same argument as above goes through.

Subsubcase 3-1-c: $\alpha_0 \in H,\ \alpha_0 < \beta_0$.

Then β_0 is a limit ordinal of H, and

$$F(\alpha_0 + 1, \beta_0) = [\max\{\ F(\alpha_0, \beta_0), \sup_{\beta' < \beta_0} G(0)(\beta')\ \}]_C^{2-next}.$$

Since $(\alpha_0, \beta_0) <^{(2,1)} (\alpha, \beta)$

$$F(\alpha_0, \beta_0) < F(\alpha, \beta),$$

by our assumption on (α, β); and

$$F(\alpha_0, \beta_0) = f(\alpha_0, \beta_0) < f(0)(\beta_0) < \gamma,$$

because $(\alpha_0, \beta_0) \in H \times H$. Also, notice that condition (3) is satisfied at $\beta_0 < \gamma$, i.e.,

$$\sup_{\beta' < \beta_0} G(0)(\beta') \leq F(0, \beta_0) < F(\alpha_0, \beta_0).$$

Combining all these facts we have

$$F(\alpha_0 + 1, \beta_0) = [f(\alpha_0, \beta_0)]_C^{2-next} < \psi(m) < \gamma.$$

In fact, whenever $(\alpha_0, \beta_0) <^{(2,1)} (\tau, \epsilon) \leq_1^{(2,1)} (\alpha, \beta)$, then

$$F(\tau, \epsilon) = [\ \max\{\ \sup_{\tau' < \tau} F(\tau', \epsilon), \sup_{\epsilon' < \epsilon} G(0)(\epsilon')\ \}\ ]_C^{2-next} < \gamma.$$

Thus, $F(\alpha,\beta) < \gamma < F(\delta,\gamma)$.

Subcase 3-2: $\beta = \gamma$.

Notice, $\beta = \gamma$ implies $\beta \in H$, $\alpha \notin H$, $\beta < f(\delta,\beta)$, and $\alpha < \delta$; and our goal is to show $F(\alpha,\beta) < F(\delta,\beta) = f(\delta,\beta)$. Because $\alpha \notin H$, $\alpha_0 := \sup(H \cap \alpha) < \alpha < \beta$. If $\alpha_0 = 0$, then we are in the situation of *Subsubcase 3-1-a* and, therefore, done. Otherwise, $0 \neq \alpha_0 \in H$, and

$$\beta < F(\alpha_0,\beta) = f(\alpha_0,\beta) < f(\delta,\beta).$$

Using $\alpha < \delta \in Cub(\psi)$, we get $\psi(\alpha) < \delta$, i.e., there are (at least) $\omega\alpha^4$ elements of C_1 between α and δ. We write them as a well-ordered sequence $\{x_\xi\}_{\xi<\omega\alpha^2} \subset C_1$. For every x_ξ from that sequence, we obtain $y_\xi := f(x_\xi,\beta) \in C$. Since f is strictly increasing, we have a sequence $\{y_\xi\}_{\xi<\omega\alpha^2}$ of elements of C between $f(\alpha_0,\beta)$ and $f(\delta,\beta)$. This is saying that there are enough elements of C to deduce the desired inequality. More precisely, for every $\alpha_0 < \tau \leq \alpha$, $\tau \notin H$, and

$$F(\tau,\beta) = [\sup_{\tau'<\tau} F(\tau',\beta)]_C^{2-next} < f(\delta,\beta).$$

That shows $F(\alpha,\beta) < f(\delta,\beta) = F(\delta,\beta)$. Hence F is strictly increasing, and $\gamma \notin S_F$.

Similarly one shows that $\gamma \notin S_G$.

If $\gamma \notin S_F \cup S_G$, then F and G are strictly increasing as functions from $<^{(2,1)}\restriction(\delta,\gamma)$ into $\aleph_1$ for all $\delta < \gamma$. Then $G(0,\gamma) < G(0)(\gamma)$, and it follows that condition (2) is satisfied at γ, i.e., $\gamma \notin S_2$. Contradiction. ∎

The next two lemmas give us sufficient conditions for an ordinal to be represented by a function strictly increasing on a cub set.

Lemma 2.2.4 *For every $n \geq 1$, if $\pi = (n, n-1, \ldots, 1)$ and $[h]_{\mu^n} \in (\aleph_n, \aleph_{n+1})$ with $h :<^\pi \to \aleph_1$ 'honestly' depending on n variables (i.e., h is not equal to any function on fewer than n variables), h is strictly increasing on some cub set $C \subset \aleph_1$.*

Proof. Let h be as in the hypothesis. There is a cub set C_1 such that $\forall x_1 < x'_1 < a_2 < \ldots < a_n \in C_1$, $h(x_1, a_2, \ldots, a_n) < h(x'_1, a_2, \ldots, a_n)$. If not, then let C_1 be homogeneous for the contrary side of the corresponding partition. Clearly, if equality holds instead of inequality, then h does not depend of the first variable, contradiction. To see that the reverse inequality is also impossible, fix any infinite increasing sequence $\{x^i\}_{i\in\omega}$ from C_1 with the limit point $a_2 \in C_1$, and observe that $h(x^i, a_2, \ldots, a_n) > h(x^{i+1}, a_2, \ldots, a_n)$ contradicts the well-foundness.

Claim *There is a cub set C_2 such that whenever $x_2 < \ldots < x_n \in C_2$, and $y_1 < y_2 < \ldots < y_n \in C_2$ with $(x_2, \ldots, x_n) <_{n-1} (y_2, \ldots, y_n)$, then $h(n-2)(x_2, \ldots, x_n) < h(y_1, y_2, \ldots, y_n)$.*

Proof of Claim. By considering the partition

$$\mathcal{P}(a_1, \ldots, a_{n-1}, x_n, x'_n) := \begin{cases} 1, & \text{if } h(a_1, \ldots, a_{n-1}, x_n) < h(a_1, \ldots, a_{n-1}, x'_n) \\ 0, & \text{otherwise,} \end{cases}$$

we see that there is a cub set C with the property $\forall a_1 < \ldots < a_{n-1} < x_n < x'_n \in C$,

$$h(a_1, \ldots, a_{n-1}, x_n) < h(a_1, \ldots, a_{n-1}, x'_n).$$

(The reverse inequality is impossible by well-foundness; and if equality holds, then h depends only on first $n-1$ variables.)

Note that there are only finitely many combinations of the natural ordering of ordinals $\{x_2, \dots, x_n, y_2, \dots, y_n\}$ when they satisfy the condition $(x_2, \dots, x_n) <_{n-1} (y_2, \dots, y_n)$. Let us assume $(x_2, \dots, x_n) <_{n-1}^{comb} (y_2, \dots, y_n)$ is the order corresponding to one of such fixed combinations. (For example, $x_2 < y_1 < x_3 < \dots < x_n < y_2 < \dots < y_n$). Let C^{comb} be the cub set, homogeneous for the corresponding partition

$$\mathcal{P}^{comb}(x_2, y_1, x_3, \dots, x_n, y_2, \dots, y_n) := \begin{cases} 1, & h(n-2)(x_2, \dots, x_n) < h(y_1, y_2, \dots, y_n) \\ 0, & \text{otherwise.} \end{cases}$$

Assume for a moment C^{comb} is homogeneous for the zero side of $\mathcal{P}^{comb}$; fix $x_2 < y_1 < x_3 < \dots < x_n < y_2 < \dots < y_{n-1} \in C \cap C^{comb}$. Then $\forall z < z' \in (C \cap C^{comb} - y_{n-1})$ we have $h(y_1, \dots, y_{n-1}, z) < h(y_1, \dots, y_{n-1}, z') \leq h(n-2)(x_2, \dots, x_n)$, which is in fact an embedding of $\aleph_1$ into a smaller ordinal. Contradiction. It is obvious now that the finite intersection of all C^{comb} satisfies the claim. ∎ Claim

Now we set $C = C_1 \cap C_2$. Let $\vec{x}, \vec{y} \in C^n$ be such that $\vec{x} <^\pi \vec{y}$. Then

(i) either $(x_2, \dots, x_n) <_{n-1} (y_2, \dots, y_n)$,

(ii) or $[(x_2, \dots, x_n) = (y_2, \dots, y_n)$ and $x_1 < y_1]$.

Hence $h(\vec{x}) < h(\vec{y})$: use $C \subset C_2$ for (i), and $C \subset C_1$ for (ii). ∎

A slight modification of the argument above enables us to prove the following

Lemma 2.2.5 *Let $h : <^{\pi_n} \to \aleph_1$ be honestly depending on n arguments and assume $\aleph_n < [h]_{\mu^n}$. Then h is strictly increasing on some cub set $C \subset \aleph_1$, i.e., whenever $\vec{x}, \vec{y} \in C^n$ and $\vec{x} <^{\pi_n} \vec{y}$ then $h(\vec{x}) < h(\vec{y})$.*

Proof. Let $i < n$ the position of the least important coordinate of $\vec{x} := (x_1, \ldots, x_n)$ with respect to $<^{\pi_n}$ ordering. There is a cub set C_1 such that

(2.2.2.3) $\forall a_1 < \cdots < a_{i-1} < x_i < x_i' < a_{i+1} < \cdots < a_n \in C_1,$

$$h(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) < h(a_1, \ldots, a_{i-1}, x_i', a_{i+1}, \ldots, a_n).$$

(If not, choose $a_{i+1} \in C_1'$ to get a contradiction.) There is also a cub set C_2 with the property that if $x_1 < \cdots < x_{i-1} < x_{i+1}, \ldots, x_n \in C_2$, $y_1 < \cdots < y_n \in C_2$, are such that $\forall z$,

$$(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) <^{\pi_n} (y_1, \ldots, y_n),$$

then

$$h(n-2)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) < h(y_1, \ldots, y_n).$$

It is easy to see now that $C_1 \cap C_2$ satisfies the lemma. ∎

Lemma 2.2.6 *Let $C \subset \aleph_1$ be a cub set. If $h : \aleph_1 \to C'$ is of 1-c.t., and $g :<^{(2,1)} \to \aleph_1$ of c.t. induces h, then $\forall^*_{\mu^2}(\alpha, \beta)$, $g(\alpha, \beta) \in C$.*

Proof. For every $x \in C$ define

$$f_1(x) := \inf\{\, z \in C \mid z < x \text{ and } \forall(\alpha, \beta), g(\alpha, \beta) \notin (z, x) \,\};$$

$$f_2(x) := \sup\{\, z \in C \mid z < f_1(x) \,\};$$

$$f_3(x) := \inf\{\, g(\alpha, \beta) \mid f_2(x) < g(\alpha, \beta) < f_1(x) \,\}.$$

f_3 is well-defined and pressing down on a cub set, thus $\exists\gamma$ such that $\forall^*_\mu x$, $f_3(x) = \gamma$. Then either g does not induce h or $\forall^*_{\mu^2}(\alpha, \beta)$, $g(\alpha, \beta) \in C$. ∎

Lemma 2.2.7 *Fix integers m, $\mathbf{c}$, n, a c.u.b. set C, and a function $F : \aleph_{m+1} \to \aleph_n$ defined by $[h]_{\mu^m} \mapsto [h']_{\mu^{n-1}}$, where $h :<^{\pi_m,\mathbf{c}} \to \aleph_1$ is of $\mathbf{c}$–c.t. and $h' : {\aleph_1}^{n-1} \to \aleph_1$ is an invariant of h. Suppose a function $H : \aleph_{m+1} \to \aleph_n$ has the property that*

$$H([h]_{\mu^m}) < [N_C \circ h']_{\mu^{n-1}}, \qquad S^{\pi_m,\mathbf{c}}\text{-}a.e.$$

Then there is an embedding $[H]_{S^{\pi_m,\mathbf{c}}} \hookrightarrow [F]_{S^{\pi_m,\mathbf{c}}}$.

Proof. By the theorem of Kunen (see theorem 14.3 in [Ke1]) there is a well-ordering W on $\aleph_1$ representing function N_C. That is to say $N_C(\alpha) = |W \restriction \alpha|, \forall \alpha \in \aleph_1$, and $[N_C]_\mu = |W|$. For a fixed $h :<^{\pi_m} \to \aleph_1$ of $\mathbf{c}$–c.t., let $f : {\aleph_1}^{n-1} \to \aleph_1$ be a function representing $H([h]_{\mu^m})$. Then μ^{n-1}–a.e.

$$f(\vec{x}) < |W \restriction h'(\vec{x})| = (N_C \circ h')(\vec{x}),$$

and there is some $\gamma < h'(\vec{x})$ with $|W^\gamma \restriction h'(x)| = f(x)$. Denote by t a function defined μ^{n-1}–a.e. by $\vec{x} \mapsto \gamma$. Now consider the map

$$U_H := [h]_{\mu^m} \mapsto [t]_{\mu^{n-1}}.$$

It is readily seen that U_H is well defined, and also $[U_H]_{S^{\pi_m,\mathbf{c}}} < [F]_{S^{\pi_m,\mathbf{c}}}$. Finally notice that the map

$$\phi := [K]_{S^{\pi_m,\mathbf{c}}} \mapsto [U_K]_{S^{\pi_m,\mathbf{c}}}.$$

is well defined on ordinals $[K]_{S^{\pi_m,\mathbf{c}}} \le [H]_{S^{\pi_m,\mathbf{c}}}$ and order preserving, and therefore embeds $[H]_{S^{\pi_m,\mathbf{c}}}$ into $[F]_{S^{\pi_m,\mathbf{c}}}$. ∎

2.3 Ultrapowers by Basic Measures

In this section we compute the functions (or ultrapowers by basic measures) S. Recall $S(n) = k \iff j_S(\aleph_n) = \aleph_k$. Fix integers $n, m, \mathbf{c}$, permutation $\pi = \pi_m = (m, i_1, \ldots, i_{m-1})$, and the measure $S = S^{\pi, \mathbf{c}}$. Then S concentrates on functions $h : <^{\pi} \rightarrow \aleph_1$ of $\mathbf{c}$-c.t. We fix such a function h and set $h(m-1) = h$. Then for $0 \leq p \leq m-2$ we define recursively *invariants of* h:

(2.3.2.1)

$$h(p)(x_1, \ldots, x_{i-1}, x_i, \ldots, x_{p+1}) := \sup_{y < x_i} h(p+1)(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{p+1}).$$

(2.3.2.2)

$$\widetilde{h(p+1)}(x_1, \ldots, x_{i-1}, z, x_i, \ldots, x_{p+1}) := \sup_{y < z} h(p+1)(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{p+1}),$$

where y is the least important variable with respect to the order on which $h(p+1)$ is strictly increasing. We denote this order by $<^{\pi(p+1)}$, so that when $p = m-2$ it coincides with $<^{\pi}$. We call these functions "invariants of h" because the maps $[h]_{\mu^m} \mapsto [h(p)]_{\mu^{p+1}}$ and $[h]_{\mu^m} \mapsto [\widetilde{h(p)}]_{\mu^{p+1}}$ are well defined. Observe that (1) the continuous function $\widetilde{h(p+1)}$ is μ^{p+2}-almost everywhere below the discontinuous function $h(p+1)$ and they both induce $h(p)$; (2) $\widetilde{h(p+1)}$ is the least function inducing $h(p)$: any function g which is below $\widetilde{h(p+1)}$ μ^{p+2}-a.e. can not induce $h(p)$; (3) $\widetilde{h(0)}(\alpha) = \alpha$ almost everywhere.

Recall that if $\mathbf{c} \neq 0$ there is a function, call it $h(m)$, of $m+1$ variables of the 0 correct type inducing $h = h(m-1)$. Thus, in such a case we obtain one more invariant $\widetilde{h(m)}$ defined exactly as in Equation 2.3.2.2 with $p + 1 = m$. The total number of invariants of h, therefore, is $2m - 1$, if $\mathbf{c} = 0$, and $2m$, otherwise.

For every increasing sequence $\langle k_p, \ldots, k_1 \rangle$, $1 \le p \le m-1$, of elements of $\{1, \ldots, n-2\}$, we define two auxiliary functions on tuples $\vec{x} = (x_1, \ldots, x_{n-1}) \in {\aleph_1}^{n-1}$:

$$(2.3.2.3) \qquad h(p)^{k_p,\ldots,k_1} := \vec{x} \mapsto h(p)(x_{k_p}, \ldots, x_{k_1}, x_{n-1}),$$

$$(2.3.2.4) \qquad \widetilde{h(p)}^{k_p,\ldots,k_1} := \vec{x} \mapsto \widetilde{h(p)}(x_{k_p}, \ldots, x_{k_1}, x_{n-1}).$$

and for $p = 0$

$$(2.3.2.5) \qquad h(0)^{n-1} := \vec{x} \mapsto h(0)(x_{n-1})$$

That defines $1 + 2\sum_{p=1}^{m-1} \binom{n-2}{p}$ auxiliary functions. In the case when $\mathbf{c} \neq 0$ we have additional $\binom{n-2}{m}$ functions defined as in Equation 2.3.2.4 with $p = m$.

Now we are ready to define *canonical functions* from $\aleph_{m+1}$ into $\aleph_n$ (on $S^{\tau,\mathbf{c}}$-measure one set):

$$(2.3.2.6) \qquad \mathrm{CF}(0) := [h]_{\mu^m} \mapsto [h(0)^{n-1}]_{\mu^{n-1}},$$

$$(2.3.2.7) \qquad \mathrm{CF}(k_p, \ldots, k_1) := [h]_{\mu^m} \mapsto [h(p)^{k_p,\ldots,k_1}]_{\mu^{n-1}},$$

$$(2.3.2.8) \qquad \mathrm{CF}_T(k_p, \ldots, k_1) := [h]_{\mu^m} \mapsto [\widetilde{h(p)}^{k_p,\ldots,k_1}]_{\mu^{n-1}}.$$

Clearly, the total number of canonical functions is that of the auxiliary functions and is given by the formula

$$(2.3.2.9) \qquad \mathrm{Card}_{CF} = \begin{cases} \binom{n-2}{m} + 2\left[\sum_{j=1}^{m-1} \binom{n-2}{j}\right] + 1 = 2\sum_{j=1}^{m} \binom{n-1}{j}, & \mathbf{c} \neq 0 \\ 2\left[\sum_{j=1}^{m-1} \binom{n-2}{j}\right] + 1, & \mathbf{c} = 0. \end{cases}$$

In our computation of $S^{\tau,\mathbf{c}}(n)$ we shall make use of the ordinals represented by canonical functions with respect to the measure $S^{\tau,\mathbf{c}}$. We will frequently abuse notation by saying *the* canonical functions meaning the corresponding ordinals. It is straightforward to verify that for given measure $S^{\tau_m,c}$ and cardinal $\aleph$, these ordinals are well defined.

Lemma 2.3.1 *Let* CF(0) *be defined with respect to the measure* $S = S^{\tau,c}$ *on* $\aleph_{m+1}$ *and the cardinal* $\aleph_n$. *Then for arbitrary* $H : \aleph_{m+1} \to \aleph_n$, $[H]_S \hookrightarrow [\mathrm{CF}(0)]_S$.

In particular $j_S(\aleph_n) \leq |[CF(0)]_S|^+$.

Proof. Let C be homogeneous for the partition $\mathcal{P}$ defined as follows

(2.3.2.10)

$$\mathcal{P}(h, g) = 1 \iff [\ H([h]_{\mu^m}) < [g^{n-1}]_{\mu^{n-1}};$$

$$h : <^\tau \to \aleph_1 \text{ is of } \mathbf{c}\text{–c.t.}, \ g : \aleph_1 \to \aleph_1 \text{ is of c.t.};$$

$$\text{and } h(0)(x) < g(x) < h(0, \ldots, 0, x+1)\ ].$$

Here $g^{n-1}(x_1, \ldots, x_{n-1}) := g(x_{n-1})$. We fix a function $h : <^\tau \to C$ of $\mathbf{c}$–c.t. and f with

$$[f]_{\mu^{n-1}} > \max\{\ H([h]_{\mu^m}),\ [h(0)^{n-1}]_{\mu^{n-1}}\ \}.$$

Then $h(0) < f(0)$, and $g := N_C \circ f(0)$ is a function of c.t. into C and $h(0) < g$, μ–a.e. Using the sliding argument we apply the partition to (h, g), and since $H([h]_{\mu^m}) < [g^{n-1}]_{\mu^{n-1}}$ we see that C is homogeneous for the stated side of the partition. Thus

$$H([h]_{\mu^m}) < [N_C \circ h(0)^{n-1}]_{\mu^{n-1}},\ S^{\tau,c} - \text{a.e.},$$

and $[H] \hookrightarrow [\mathrm{CF}(0)]$ by lemma 2.2.7. ∎

Proposition 2.3.2 *Let* $S = S^{\tau_m,c}$, *then* $\mathrm{Crit}(j_S) = \aleph_2$, *and* $j_S(\aleph_2) = \aleph_3$

Proof. Let $\{y_\alpha\}_{\alpha<\aleph_2}$ be a sequence of ordinals cofinal in $\aleph_{m+1}$. Then $\cup_{\alpha<\aleph_2} y_\alpha = \aleph_{m+1} \in S^{\tau,c}$ while each $y_\alpha \notin S^{\tau,c}$. Hence $S^{\tau,c}$ is not $\aleph_2 + 1$ complete. On the other hand, S is $\aleph_2$ complete. To see this fix a disjoint union $\bigsqcup_{\alpha<\aleph_1} X_\alpha \in S^{\tau,c}$. We shall see that for some $\alpha < \aleph_1$, $X_\alpha \in S^{\tau,c}$. Fix a c. u. b. set C (as in the definition of S) so that $\forall f :<^\tau \to C$ of c-c.t., $[f]_{\mu^m} \in \bigsqcup_{\alpha<\aleph_1} X_\alpha$, and define $\eta : \aleph_{m+1} \to \aleph_1 + 1$ by $\eta(\delta) = \alpha$, if $\delta \in X_\alpha$, and $\aleph_1$, otherwise. Let $C_1 \subset C$ be a c. u. b. set homogeneous for the partition

$$\mathcal{P}(f,\beta) = \begin{cases} 1, & (f :<^\tau \to C \text{ is of } \mathbf{c}\text{-c.t.}) \wedge (\eta([f]_{\mu^m}) < \beta) \\ 0, & \text{otherwise.} \end{cases}$$

For any fixed $f :<^\tau \to C_1$ of c-c.t. we can choose $\beta \in C_1$ with $\beta > \eta([f]_{\mu^m})$. That shows C_1 is homogeneous for the stated side of $\mathcal{P}$. Now put $\beta := \inf C_1$ and $A := \{ [f]_{\mu^m} \mid f :<^\tau \to C_1 \text{ is of } \mathbf{c}\text{-c.t.} \} \in S^{\tau,c}$. Note $A \subset \bigsqcup_{\alpha<\beta} X_\alpha$, thus $\bigsqcup_{\alpha<\beta} X_\alpha \in S^{\tau,c}$; and since $S^{\tau,c}$ is σ-complete and $\beta < \aleph_1$, there is $\alpha < \beta$ ($X_\alpha \in S^{\tau,c}$). Thus $\mathrm{Crit}(j_S) = \aleph_2$, and $j_S(\aleph_2) \geq \aleph_3$.

We show the reverse inequality by computing the upper bound of $[\mathrm{CF}(0)]_S$. Here $\mathrm{CF}(0) : \aleph_{m+1} \to \aleph_2$ is defined with respect to the measure S and cardinal $\aleph_2$. Let us fix an arbitrary function $H : \aleph_{m+1} \to \aleph_2$ which is below $\mathrm{CF}(0)$, S-a.e., with the corresponding c. u. b. set C_1. Suppose C is a c. u. b. set homogeneous for the partition

(2.3.2.11)

$$\begin{aligned} \mathcal{P}(h,g) = 1 \quad \iff \quad & [\ H([h]_{\mu^m}) < [g]_\mu; \\ & h :<^{\tau_m} \to \aleph_1 \text{ is of } \mathbf{c}\text{-c.t.}, g : \aleph_1 \to \aleph_1 \text{ is of c.t.}; \\ & \text{and } g(x) < h(0, \ldots, 0, x) < h(0)(x) < g(x+1)\ ]. \end{aligned}$$

We fix functions $h :<^{\tau} \to C_1 \cap Cub(N_C)$ of $\mathbf{c}$-c.t. and $f : \aleph_1 \to \aleph_1$ representing $H([h]_{\mu^m})$. Then $[f]_\mu < [h(0)]_\mu$, and it follows that the function $g := N_C \circ f$ is of c.t. into C and below $h(0)$, μ-a.e., (note $Cub(N_C) \subset C$). By the sliding argument, we can apply $\mathcal{P}$ to the pair of functions (h, g). Then we see that C is homogeneous for the stated side of the partition $\mathcal{P}$ because

$$H([h]_{\mu^m}) \leq [f]_\mu < [g]_\mu.$$

If $\alpha := \inf\{\ [g]_\mu \mid g : \aleph_1 \to C \text{ is of c.t. }\}$, then $\forall^*_{S^*}\ h :<^{\tau} \to C$ of $\mathbf{c}$-c.t., we have $H([h]_{\mu^m}) < \alpha < \aleph_2$. That certainly implies $|[H]| \leq \aleph_1$, but then

$$|[\mathrm{CF}(0)]| \leq |[H]|^+ \leq \aleph_2.$$

By lemma 2.3.1, $j_{S^*}(\aleph_2) \leq |[\mathrm{CF}(0)]|^+$, and the desired inequality follows. ∎

To prove the upper bound inequality for $S(n)$ we make use of the following

Proposition 2.3.3 *Let canonical functions be defined with respect to $S^{\tau_m,\mathbf{c}}$ and $\aleph_n$. Then they represent (with respect to $S^{\tau,\mathbf{c}}$) ordinals of consecutive cardinality starting from $\leq j_{S^{\tau,\mathbf{c}}}(\aleph_{n-1})^+$.*

Proof of Proposition 2.3.3 is given at the end of this section.

Lemma 2.3.4

$$S^{\tau_m,\mathbf{c}}(n) = \begin{cases} 2\left[\sum_{j=1}^{m} \binom{n-1}{j}\right] + 1, & \mathbf{c} = 0, \\ \binom{n-1}{m+1} + 2\left[\sum_{j=1}^{m} \binom{n-1}{j}\right] + 1 = \sum_{j=1}^{m+1} \binom{n}{j}, & \mathbf{c} \neq 1. \end{cases}$$

Proof. We prove the upper bound inequality by induction on n. It certainly holds for $n = 2$ by lemma 2.3.2. Recall that $j_{S^{\tau,\mathbf{c}}}(\aleph_n) \leq |[\mathrm{CF}(0)]|_S{}^+$ by lemma 2.3.1.

Then it follows from the Proposition 2.3.3, $S^{\pi,\mathbf{c}}(n) \leq S^{\pi,\mathbf{c}}(n-1) + \mathrm{Card}_{CF} + 1$. Thus, assuming that the desired inequality holds for $S^{\pi,\mathbf{c}}(n-1)$, we have (for $\mathbf{c} = 0$)

$$\begin{aligned}
&S^{\pi,0}(n-1) + \mathrm{Card}_{CF} \\
\leq &\{2\left[\sum_{j=1}^{m}\binom{n-2}{j}\right] + 1\} \quad + \quad \{2\left[\sum_{j=1}^{m-1}\binom{n-2}{j}\right] + 1\} \\
= &\{2\left[\sum_{j=1}^{m}\binom{n-2}{j}\right] + \binom{n-2}{0}\} \quad + \quad \{2\left[\sum_{j=2}^{m}\binom{n-2}{j-1}\right] + \binom{n-2}{0}\} \\
= &2\sum_{j=1}^{m}\{\binom{n-2}{j} + \binom{n-2}{j-1}\} \quad = \quad 2\sum_{j=1}^{m}\binom{n-1}{j}.
\end{aligned}$$

The upper bound for $S^{\pi,\mathbf{c}}(n)$ when $\mathbf{c} \neq 0$ is proved similarly.

To compute the lower bound we shall construct a well-defined embedding $\phi : \aleph_{q_{m,n}+1} \hookrightarrow j_{S^{\pi,0}}(\aleph_n)$ for some integer $q_{m,n}$. Then we show that $q_{m,n} = 2\sum_{j=1}^{m}\binom{n-1}{j}$ if $\mathbf{c} = 0$, and $q_{m,n} = \sum_{j=1}^{m+1}\binom{n}{j}$ if $\mathbf{c} \neq 0$. For now let us assume $\pi = (m, m-1, \ldots, 2, 1)$ and $\mathbf{c} = 0$. Then we define

$$(2.3.2.12) \qquad \phi([f]_{\mu^{q_{m,n}}}) := [G]_{S^{\pi,0}}$$

where $G : \aleph_{m+1} \to \aleph_n$ is defined on ordinals represented by functions $h :<^{\pi}\to \aleph_1$ of c.t. as follows

(2.3.2.13)

$$G([h]_{\mu^{m+1}}) = [t]_{\mu^{n}} \iff t(x_1, \ldots, x_{n-1}) = f(\iota[1], \ldots, \iota[n-1]), \quad \forall \text{[illegible]}$$

where $e[k_j, \ldots, k_1]$ are blocks of arguments of f:

$$e[k_1] = \langle\, x_{k_1},\ e[1,k_1],\ \ldots,\ e[k_1-1,k_1],\ \widetilde{h(0)}(x_{k_1})\,\rangle, \quad (1 \le k_1 \le n-1)$$

$$\ldots$$

$$e[k_j,\ldots,k_1] = \langle \widetilde{h(j-1)}(x_{k_j},\ldots,x_{k_1}), e[1,k_j,\ldots,k_1],\ldots$$

$$\ldots, e[k_j-1,k_j,\ldots,k_1], h(j-1)(x_{k_j},\ldots,x_{k_1})\rangle,$$

$$(1 \le k_j < \cdots < k_1 \le n-1), (2 \le j \le m-1)$$

$$\ldots$$

$$e[k_m,\ldots,k_1] = \langle \widetilde{h(m-1)}(x_{k_m},\ldots,x_{k_1}), h(x_{k_m},\ldots,x_{k_1})\rangle,$$

$$(1 \le k_m < \cdots < k_1 \le n-1).$$

Observe that the arguments of function f are values of invariants $h(p), \widetilde{h(p)} (0 \le p \le m-1)$, of h (see Definitions 2.3.2.1 and 2.3.2.2) at (some of the) ordinals $x_1, \ldots, x_{n-1}$. Note that if $p = 0$, $\widetilde{h(0)}(x_i) = x_i$ almost everywhere. Thus one can easily verify that the number of arguments, $q_{m,n}$, of f is $2\sum_{j=1}^{m}\binom{n-1}{j}$. Indeed, for any fixed combination of integers $k_j < \cdots < k_1$ from $\{1,2,\ldots,n-1\}$, $1 \le j \le m$, we have two arguments of f, namely the first and the last elements in the block $e[k_j,\ldots,k_1]$. Hence $q_{m,n} \ge 2\sum_{j=1}^{m}\binom{n-1}{j}$. A moment of reflection shows that the reverse inequality is also true.

Remark 2.3.1 *Here we can describe the relation between the simplified tree $T(S)$ (see Fig. 2.1), the arguments of function f, and the statement of theorem 2.1.1: for each node on the pth level $(0 \le p \le m)$ we associate $\binom{n-1}{p}$ values of $h(p-1)$ (let us formally define $h(-1) := 0$) and $\binom{n-1}{p+1}$ values of $\widetilde{h(p)}$. Notice, if $\mathbf{c} = 0$, $\widetilde{h(m)}$ is not defined.*

Then if $\mathbf{c} = 0$, *the total number of ordinals associated with the tree on Fig. 2.1(iii) is* $\Sigma_{p=0}^{m}\binom{n-1}{p} + \Sigma_{p=0}^{m-1}\binom{n-1}{p+1} = 1 + 2\Sigma_{j=1}^{m}\binom{n-1}{j} = \Sigma_{j=0}^{m+1}(\mathrm{b}_{j-1} + \mathrm{b}_j + \mathrm{w}_j)\binom{n-1}{j}$. *The last equality holds because when* $1 \leq j \leq m-1$, $\mathrm{b}_{j-1} = \mathrm{b}_j = 1, \mathrm{w}_j = 0$; *if* $j = 0$, $\mathrm{b}_{j-1} = \mathrm{w}_j = 0, \mathrm{b}_j = 1$; *if* $j = m$, $\mathrm{b}_{j-1} = \mathrm{w}_j = 1, \mathrm{b}_j = 0$; *and if* $j = m+1$, $\mathrm{b}_j = \mathrm{b}_{j-1} = \mathrm{w}_j = 0$. *Similarly if* $\mathbf{c} \neq 0$, *the total number of ordinals associated with the tree on Fig. 2.1(ii) is* $\Sigma_{p=0}^{m}\binom{n-1}{p} + \Sigma_{p=0}^{m}\binom{n-1}{p+1} = \Sigma_{j=0}^{m+1}(\mathrm{b}_{j-1} + \mathrm{b}_j + \mathrm{w}_j)\binom{n-1}{j}$.

Now it only remains to verify that the embedding ϕ is well defined. For simplicity of notation let us assume $m = 1$, and $n = 3$. In that case $q_{1,3} = 2\binom{2}{1} = 4$, and the embedding $\phi : \aleph_{4+1} \hookrightarrow j_{S^{(1),0}}(\aleph_3)$ is defined on functions $f : {\aleph_1}^4 \to \aleph_1$ by $\phi([f]_{\mu^4}) = [G]_{S^{(1),0}}$ with $G : \aleph_2 \to \aleph_3$ being defined on functions $h :<^{(1)}\to \aleph_1$ of c.t. by

$$(2.3.2.14)\ G([h]_\mu) = [t]_{\mu^2} \quad \Longleftrightarrow \quad t(\alpha,\beta) = f(\alpha, h(\alpha), \beta, h(\beta)),\ \forall^*_{\mu^2}(\alpha,\beta)$$

We show that ϕ is well defined in two steps.

Step 1. Fix f and suppose $h = h'$, μ-a.e.. Then $\forall^*_\mu\alpha, \forall^*_\mu\beta$, $f(\alpha, h(\alpha), \beta, h(\beta)) = f(\alpha, h'(\alpha), \beta, h'(\beta))$. Hence $G([h]_\mu) = G([h']_\mu)$.

Step 2. Suppose now $f = f'$, μ^4-a.e., i.e. there is a cub set $C \subset \aleph_1$ such that $f(\gamma, \alpha, \beta, \delta) = f'(\gamma, \alpha, \beta, \delta)$, whenever $\gamma < \alpha < \beta < \delta$ are all in C. To see that $G = G'$, $S^{(1),0}$-a.e., observe that $\{[h]_\mu | h : \aleph_1 \to C \text{ is of c.t.}\}$ has $S^{(1),0}$-measure one, and for any fixed $h : \aleph_1 \to C$ of c.t., $\forall^*_\mu\alpha, \forall^*_\mu\beta$, $\alpha < h(\alpha) < \beta < h(\beta) \in C$, which implies $f(\alpha, h(\alpha), \beta, h(\beta)) = f'(\alpha, h(\alpha), \beta, h(\beta))$. Hence $[t]_{\mu^2} = [t']_{\mu^2}$, μ^2-a.e.. Thus $\forall^*_{S^{(1),0}}[h]_\mu$, $G([h]_\mu) = G'([h]_\mu)$.

Now combining *Step 1* and *Step 2* we see that ϕ is well defined when $m = 1$ and

$n = 3$. The proof that ϕ is well defined for arbitrary fixed m and n goes word by word. Thus the lower bound inequality for $S^{\pi,0}(\aleph_n)$ is established.

Suppose now $\mathbf{c} = 1$. Computation of the lower bound for $S^{\pi,1}(\aleph_1)$ is similar, and the corresponding embedding is defined almost exactly as the one above. The only difference occurs in the definition of the $e[k_m, \ldots, k_1]$. In the case when $\mathbf{c} = 1$ there is an extra invariant of h, $\widetilde{h(m)}$, and therefore we define

$$(2.3.2.15) \quad e[k_m, \ldots, k_1] = \langle \widetilde{h(m-1)}(x_{k_m}, \ldots, x_{k_1}),$$

$$\widetilde{h(m)}(x_1, x_{k_m}, \ldots, x_{k_1}), \; \ldots, \; \widetilde{h(m)}(x_{k_m-1}, x_{k_m}, \ldots, x_{k_1}),$$

$$h(x_{k_m}, \ldots, x_{k_1})\rangle, \quad (1 \leq k_m < \cdots < k_1 \leq n-1).$$

Then the number of arguments of f becomes

$$q_{m,n} = \binom{n-1}{m+1} + 2\left[\sum_{j=1}^{m} \binom{n-1}{j}\right] = \sum_{j=1}^{m+1} \binom{n}{j} - 1,$$

and that establishes the lower bound inequality for the case $\mathbf{c} = 1$.

Finally, if $\pi = (m, i_1, \ldots, i_{m-1})$ and $\mathbf{c}$ are arbitrary, we can construct embeddings as before. Let $\mathbf{c} = 0$. While the order of arguments of the function f depends on π their total number is clearly unchanged. When $\mathbf{c} \neq 0$ we have again an extra invariant of h, therefore the number, $q_{m,n}$, of arguments of f is at least as big as that for $\mathbf{c} = 1$. Well-definedness of the corresponding embeddings is obvious. ∎

Let $h <^{\pi,c} \to \aleph_1$ be a function of the type on which a basic measure S concentrates. In the view of computation of the lower bound of $S(n)$, let us illustrate the notation used in stating theorem 2.1.1. Consider all the invariants of h that were used in defining the function f from equation 2.3.2.13. To every node on level j, $1 \leq j \leq$

$m+1$, of the simplified tree $T(S)$ we can assign the number of invariants of h that depend on j arguments. (To the root (0th level) we assign number 1.) Level 1: we have $n-1$ invariants $h(0)(x_i)$, $1 \leq i \leq n-1$, and same number of variables $x_i = \widetilde{h(0)}(x_i)$. Thus the number assigned to the node 1 is $2(n-1)$. Level 2: there are $\binom{n-1}{2}$ invariants $h(1)(x_i, x_j)$ and same number of their continuous analogues; thus the number assigned here is $2\binom{n-1}{2}$; etc. Note that we always have discontinuous functions $h(j)$ together with $\widetilde{h(j)}$, except the case when $\mathbf{c} \neq 0$ and $j = m$. In the latter case we have only functions $h(m)$ (which induce h). Thus this case is being distinguished from the others in the construction of $T(S)$.

Proof.[of Proposition 2.3.3] Let us first consider the case when $\pi = (m, m-1, \ldots, 2, 1)$ and $\mathbf{c} = 0$. It is easy to see that in this case the canonical functions are ordered (with respect to S) as follows

$$\mathrm{CF}(0) > E[n-2] > \cdots > E[2] > E[1]$$

with $E[1] = \langle \mathrm{CF}(1) \ > \ \mathrm{CF}_T(1) \rangle$

$$E[k_p] = \langle \mathrm{CF}(k_p) > E[k_p - 1, k_p] > \cdots > E[1, k_p] > \mathrm{CF}_T(k_p) \rangle$$

$$(2 \leq k_p \leq n-2)$$

$$\cdots$$

$$E[k_j, \ldots, k_1] = \langle \mathrm{CF}(k_j, \ldots, k_1) > E[k_j - 1, k_j, \ldots, k_1] > \ldots$$

$$\cdots > E[1, k_j, \ldots, k_1] > \mathrm{CF}_T(k_j, \ldots, k_1) \rangle,$$

$$(2 \leq j \leq m-2)$$

$$\cdots$$

$$E[k_{m-1}, \ldots, k_1] = \langle \mathrm{CF}(k_{m-1}, \ldots, k_1) > \mathrm{CF}_T(k_{m-1}, \ldots, k_1) \rangle$$

Let CF() and CF()$'$ be any two canonical functions consecutive in that order (CF() $<_S$ CF()$'$). To prove the Proposition we need to show that if a function $H : \aleph_{m+1} \to \aleph_n$, is S almost everywhere less than CF()$'$, then $[H]_S \hookrightarrow [\mathrm{CF}()]_S$. More specifically, we have to show that

$$\begin{array}{lll}
(1)\ H <_S \mathrm{CF}(0) & \Rightarrow & [H] \hookrightarrow [\mathrm{CF}(n-1)]; \\
(2)\ H <_S \mathrm{CF}(k_p, \ldots, k_1) & \Rightarrow & [H] \hookrightarrow [\mathrm{CF}(k_p - 1, k_p, \ldots, k_1)]; \\
(3)\ H <_S \mathrm{CF}(k_{m-1}, \ldots, k_1) & \Rightarrow & [H] \hookrightarrow [\mathrm{CF}_T(k_{m-1}, \ldots, k_1)]; \\
(4)\ H <_S \mathrm{CF}_T(k_p, k_{p-1}, \ldots, k_1) & \Rightarrow & [H] \hookrightarrow [\mathrm{CF}(k_p - 1, k_{p-1}, \ldots, k_1)]; \\
(5)\ H <_S \mathrm{CF}_T(1, k_p, \ldots, k_1) & \Rightarrow & [H] \hookrightarrow [\mathrm{CF}_T(k_p, \ldots, k_1)]; \\
(6)\ H <_S \mathrm{CF}_T(1) & \Rightarrow & |[H]| \le j_{S^{\pi,0}}(\aleph_{n-1}).
\end{array}$$

Let us prove the implication (2). Suppose a function $H : \aleph_{m+1} \to \aleph_n$ is such that $H \ <_{S^{\pi,0}}\ \mathrm{CF}(k_p, \ldots, k_1)$. Then we shall prove $[H]$ can be embedded into $[\mathrm{CF}(k_p - 1, k_p, \ldots, k_1)]$.

Let us denote the order $<^{(p+2,p+1,\ldots,2,1)}$ by $<^{p+2}$. Let also $\bar{\alpha} = \langle \alpha_1, \ldots, \alpha_{p+1} \rangle$. Every function $g :<^p \to \aleph_1$ of c.t. induces a function $g^{k_{p-1},\ldots,k_1} : {\aleph_1}^{n-1} \to \aleph_1$. We fix a c. u. b. set C_1 such that for every $h :<^\pi \to \aleph_1$ of c.t.

$$H([h]_{\mu^{n-1}}) \ < \ \mathrm{CF}(k_p - 1, k_p, \ldots, k_1)([h]_{\mu^{n-1}}).$$

Under AD , $\aleph_1$ satisfies the strong partition relation. Thus we can fix a c. u. b. set C homogeneous for the partition

$$\begin{aligned}
\mathcal{P}(h, g) = 1 \quad \Longleftrightarrow \quad & [\ H([h]_{\mu^m}) < [g^{k_p-1,k_p,\ldots,k_1}]_{\mu^{n-1}}; \\
& h :<^\pi \to \aleph_1,\ g :<^{p+2} \to \aleph_1 \text{ are of c.t.}; \\
& \text{and } h(p+1)(\beta, \bar{\alpha}) < g(\beta, \alpha) < h(0, \ldots, 0, \beta + 1, \alpha)\ ].
\end{aligned}$$

Notice, in particular, the last conjunction asserts that $g(> h(p+1))$ induces $h(p)$.

We now fix $h :<^{\tau} \to Cub(N_C) \cap C_1$ of c.t., and let

$$[f]_{\mu^{n-1}} = \max\{H([h]_{\mu^m}), [h(p+1)^{k_p-1,k_p,\ldots,k_1}]_{\mu^{n-1}}\} \;<\; [h(p)^{k_p,\ldots,k_1}]_{\mu^{n-1}},$$

so that $\forall^*_{\mu^{n-1}} \vec{x}$,

$$\begin{aligned} h(p+1)(x_{k_p-1}, x_{k_p}, \ldots, x_{k_1}, x_{n-1}) \quad &\leq \quad f(\vec{x}) \\ < \quad h(p)(x_{k_p}, \ldots, x_{k_1}, x_{n-1}) \\ = \sup_{y < x_{k_p}} h(p+1)(y, x_{k_p}, \ldots, x_{k_1}, x_{n-1}). \end{aligned}$$

Then $\forall^*_\mu x_1, \ldots, x_{k_p-1}$, $\exists y = y(x_1, \ldots, x_{k_p-1})$, $\forall^*_\mu x_{k_p}, \ldots, x_{n-1}$, so that

$$\begin{aligned} f(\vec{x}) \quad &< \quad h(p+1)(\; y(0)(x_{k_p-1}), x_{k_p}, \ldots, x_{k_1}, x_{n-1} \;) \\ &< \quad h(p)(x_{k_p}, \ldots, x_{k_1}, x_{n-1}) \quad \in Cub(N_C). \end{aligned}$$

We set

$$\zeta(\alpha_1, \ldots, \alpha_{p+2}) := h(p+1)(y(0)(\alpha_1), \alpha_2, \ldots, \alpha_{p+2}).$$

Then

$$h(p+1)^{k_p-1,k_p,\ldots,k_1} < \zeta^{k_p-1,k_p,\ldots,k_1} < h(p)^{k_p,\ldots,k_1}$$

and

$$\zeta(p) = h(p),$$

which implies ζ is strictly increasing as a function from $<^{p+2}$, and, therefore, $g := N_C \circ z :<^{p+2} \to C$ is of c.t., g induces $h(p)$, and $g < h(p)$.

Now (using sliding argument) we apply the partition to (h, g), and since $H([h]_{\mu^m}) < [g^{k_p-1,k_p,\ldots,k_1}]_{\mu^{n-1}}$, we see that C is homogeneous for the stated side of the partition. Thus

$$H([h]_{\mu^m}) < [(N_C \circ h(p+1))^{k_p-1,k_p,\ldots,k_1}]_{\mu^{n-1}}, \quad S^{\tau,0} - \text{a.e.},$$

and $[H] \hookrightarrow [\mathrm{CF}(k_p - 1, k_p, \ldots, k_1)]$, as it follows by the standard argument du[e to] Kunen (see [Ke1]). Thus, the implication (2) is proven. Other implications can be proven by similar partition argument.

Then it follows that when $\mathbf{c} = 0$,

$$j_{S^{\pi,0}}(\aleph_n) \leq |[\mathrm{CF}(0)]|^{+} \leq |[\mathrm{CF}(n-2)]|^{++} \leq \ldots$$

$$\ldots \leq |[\mathrm{CF}_T(1)]|^{(\mathrm{Card}_{CF})+} \leq j_{S^{\pi,0}}(\aleph_{n-1})^{(\mathrm{Card}_{CF}+1)+}$$

Hence $S^{\pi,0}(n) \leq S^{\pi,0}(n-1) + \mathrm{Card}_{CF} + 1$.

In the case when $\mathbf{c} = 1$ we have additional canonical functions, $\mathrm{CF}_T(k_m, k_{m-1}, \ldots, k_1)$, $1 \leq k_m < \cdots < k_1 \leq n-2$; ordered (with respect to $S^{\pi,1}$) as shown

$$(2.3.2.16)\quad \mathrm{CF}(k_{m-1}, \ldots, k_1) >$$

$$\mathrm{CF}_T(k_{m-1} - 1, k_{m-1}, \ldots, k_1) > \cdots > \mathrm{CF}_T(1, k_{m-1}, \ldots, k_1)$$

$$> \mathrm{CF}_T(k_{m-1}, \ldots, k_1)$$

and the proof of the proposition is exactly the same as in the case $\mathbf{c} = 0$.

The reader can verify that the Proposition holds also in the case when $\pi = (m, i_1, \ldots, i_{m-1})$ and $\mathbf{c}$ are arbitrary. Notice that the order of canonical functions is determined by π, however their number is the same. The partition arguments similar to the one above can be carried over to finish the proof. ■

To illustrate the proof of lemma 2.3.4 we consider an example $S^{\pi_4,3}(\aleph_{10})$ with $\pi_4 = (4, 1, 2, 3)$ and list some of the canonical functions. $S_1^{(4,1,3,2),3}$ is the measure (on $\aleph_5$) concentrated on discontinuous strictly increasing functions $h : \omega_1^{(4,1,3,2)} \to \aleph_1$

with cof[$h(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$] $= \alpha_3$ uniformly. There is a continuous strictly increasing function of five arguments

$$\widetilde{h(4)} : <^{(5,1,4,2,3)} \to \aleph_1$$

inducing h:

$$h(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sup_{y<\alpha_3} \widetilde{h(4)}(\alpha_1, \alpha_2, y, \alpha_3, \alpha_4).$$

There are eight invariants of h: $h(0), h(1), \widetilde{h(1)}, h(2), \widetilde{h(2)}, h(3), \widetilde{h(3)}, \widetilde{h(4)}$. They induce $2[\binom{9}{1} + \binom{9}{2} + \binom{9}{3}]$ functions from ${\aleph_1}^9 \to \aleph_1$, and the same number of corresponding to them canonical functions. For example, if $\vec{x} = (x_1, \ldots, x_9)$, CF(4, 8, 9) corresponds to

$$h(2)^{4,8,9} := \vec{x} \mapsto h(2)(x_4, x_8, x_9) = \sup_{y<x_8} h(3)(x_4, y, x_8, x_9)$$

Recall, in this example $h(3) = h$. Some of the canonical functions are listed below in decreasing order along with corresponding values of invariants of h:

CF(0)	$h(0)(x_9)$
…	…
CF(4, 8)	$h(2)(x_4, \ x_8, x_9)$
CF(4,7, 8)	$h(3)(x_4, x_7, \ x_8, x_9)$
$\mathrm{CF}_T(4,7, \ 8)$	$\widetilde{h(3)}(x_4, x_7, \ x_8, x_9)$
CF(4,6, 8)	$h(3)(x_4, x_6, \ x_8, x_9)$
$\mathrm{CF}_T(4,6,7,8)$	$\widetilde{h(4)}(x_4, x_6, x_7, x_8, x_9)$
$\mathrm{CF}_T(4,6, \ 8)$	$\widetilde{h(3)}(x_4, x_6, \ x_8, x_9)$
CF(4,5, 8)	$h(3)(x_4, x_5, \ x_8, x_9)$
$\mathrm{CF}_T(4,5,7,8)$	$\widetilde{h(4)}(x_4, x_5, x_7, x_8, x_9)$
$\mathrm{CF}_T(4,5,6,8)$	$\widetilde{h(4)}(x_4, x_5, x_6, x_8, x_9)$
$\mathrm{CF}_T(4,5, \ ,8)$	$\widetilde{h(3)}(x_4, x_5, \ x_8, x_9)$
$\mathrm{CF}_T(4, \ ,8)$	$\widetilde{h(2)}(x_4, \ x_8, x_9)$
…	…
$\mathrm{CF}_T(1)$	$\widetilde{h(1)}(x_1, x_9)$.

Corollary 2.3.5 *For every measure $S^{\tau_{m,\varepsilon}}$, if $0 \leq n \leq m+1$, then*

$$S_1^{\tau_{m,\varepsilon}}(\aleph_n) = 2^n - 1.$$

Proof. Fix $0 \le n \le m+1$. If $j > n$, then $\binom{n}{j} = 0$. Thus for the case $\mathbf{c} \neq 0$, we have

$$\sum_{j=1}^{m+1} \binom{n}{j} = \sum_{j=0}^{n} \binom{n}{j} - 1 = 2^n - 1.$$

For the case $\mathbf{c} = 0$ we have

$$\begin{aligned} 2[\sum_{j=1}^{m} \binom{n-1}{j}] + 1 &= 2[\sum_{j=1}^{n-1} \binom{n-1}{j}] + 1 \\ &= 2[\sum_{j=0}^{n-1} \binom{n-1}{j} - 1] + 1 = 2[2^{n-1} - 1] + 1 = 2^n - 1. \end{aligned}$$

∎

Next we prove theorem 2.1.1 which was stated in introduction. Recall that a general measure $\mathfrak{S}$ was determined by a function $\mathcal{F}_m : U \to \bigcup_{m,\pi_m,0\leq c\leq m}\{S^{\pi_m,c}\}$ with the finite domain $U \subset \omega^{\leq m}$, and $\mathfrak{S}$ was concentrating on functions $\mathcal{H}$ of the type matching with $\mathcal{F}_m$. Each such $\mathcal{H}$ was coding $p = |U|$ many functions h_u, $u \in U$ (see definition 2.1.3). Each $u \in U$ determines a path through the tree $T(\mathfrak{S})$. As we did in remark 2.2.1, for each $u \in U$, we assign invariants of h_u to every node in the path u: to a node at level j we assign invariants $h_u(j-1)$ and $\widetilde{h_u(j)}$. If two path $u, v \in U$ split at level k, then $h_u(j) = h_v(j)$ for all $j \leq k$. The lower bound inequality for $\mathfrak{S}(n)$ can be obtained via constructing an embedding similar to the one given by equations 2.3.2.12 and 2.3.2.13 with the arguments of function f being the invariants $\{h_u(j), \widetilde{h_u(j)}\}_{u,j}$. Note that these invariants are well ordered according to $<_*$ defined on page 3.

Proof.[of theorem 2.1.1] We give a proof for a simple general measure. Generalization to an arbitrary measure is immediate. Let $\mathfrak{S} = \mathfrak{S}_{\mathcal{F}_m}$ be the general measure on $\aleph_{m+1}$ with $\mathcal{F}_m : \{u, v\} \to \mathcal{S}$ being defined at $u = \langle 0,0,0,0,\ldots,0\rangle$ and $v = \langle 0,1,0,0,\ldots,0\rangle$ by $\mathcal{F}_m(u) = S^{\pi,0}$ and $\mathcal{F}_m(v) = S^{\pi,1}$, where m is a fixed integer, both u and v have length m, and $\pi = (m, m-1, \ldots, 2, 1)$. As it follows from definition 2.1.3, $\mathfrak{S}$ is concentrated on order preserving functions $\mathcal{H} :<_* \to \aleph_1$ inducing functions $h_s(\alpha) := \mathcal{H}(s, \alpha)$, $(s = u, v)$, where $h_u :<^\pi \to \aleph_1$ is of 0-c.t., and $h_v :<^\pi \to \aleph_1$ is of 1-c.t.

In other words, a set $\mathcal{A}$ has $\mathfrak{S}$-measure one when there is a cub set C with the property that for each function $\mathcal{H}$ as above mapping into C, $\delta(\mathcal{H}) := \mathrm{ot}\langle\delta_u, \delta_v\rangle \in \mathcal{A}$.

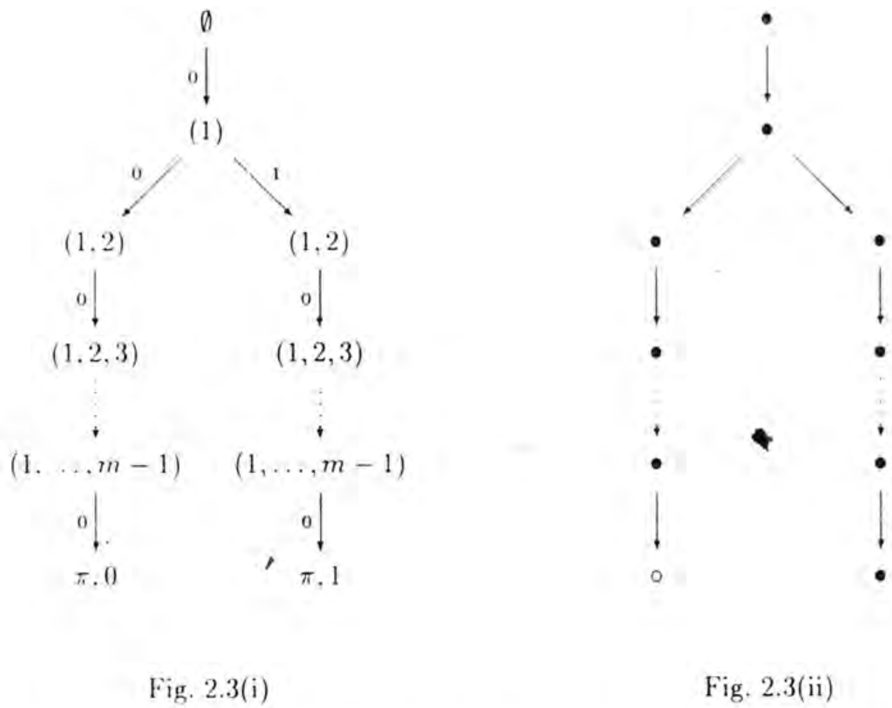

Figure 2.3: (i) Tree corresponding to the measure $\mathfrak{S}$. (ii) The simplified tree of $\mathfrak{S}$.

where $\delta_u := [h_u]_{\mathfrak{S}^{-9}}$ and $\delta_v := [h_v]_{\mathfrak{S}^{-1}}$. Since $\mathcal{H}$ preserves the $<_*$-order (see equation 2.1.2.1),

$$h_u(1)(\alpha,\beta) \; < \; h_v(1)(\alpha,\beta) \; < \; h_u(1)(\alpha+1,\beta), \qquad \forall\alpha, \forall\beta.$$

It follows that $h_u(0) = h_v(0)$, and $\widetilde{h_u(1)} = \widetilde{h_v(1)}$.

The tree representing $\mathfrak{S}$ is shown in figure 2.3. As it can be easily seen (from the simplified tree), the ultrapower invariants of $\mathfrak{S}$ are: $\mathrm{b}_0 = \mathrm{b}_1 = 1$; $\mathrm{b}_j = 2$, $2 \le j \le m-1$; $\mathrm{b}_m = 1$; $\mathrm{w}_m = 1$; and all other $\mathrm{b}_j, \mathrm{w}_k = 0$. Thus theorem 2.1.1 suggests the following

Claim

(2.4.2.1)

$$\mathfrak{S}(n) \; = \; 1 \; + \; 2(n-1) \; + \; 3\binom{n-1}{2} \; + \; 4[\sum_{j=3}^{m}\binom{n-1}{j}] \; + \; \binom{n-1}{m+1}.$$

Proof of Claim. The lower bound inequality follows easily from the observation that the total number of distinct values of invariants of h_u and h_v at ordinals $x_1, \ldots, x_n$ is $\varphi(n,m) = 2(n-1) + 3\binom{n-1}{2} + 4[\sum_{j=3}^{m}\binom{n-1}{j}] + \binom{n-1}{m+1}$. In particular, there are $n-1$ ordinals x_i; $n-1$ ordinals $h_u(0)(x_i) = h_v(0)(x_i)$; $3\binom{n-1}{2}$ values of functions $h_u(1)$, $h_v(1)$, $\widetilde{h_u(1)} = \widetilde{h_v(1)}$; $4\binom{n-1}{3}$ values of functions $h_s(2), \widetilde{h_s(2)}, s = u, v$; etc. Thus we can construct an embedding $\phi : \aleph_{\varphi(m,n)+1} \hookrightarrow j_{\mathfrak{S}}(\aleph_n)$ by $\phi([f]_{\mu^{\varphi(m,n)}}) = [G]_{\mathfrak{S}}$ with $G : \aleph_{m+1} \to \aleph_n$ being defined (on functions $\mathcal{H}$ of the type matching $\mathcal{F}_m$) by

(2.4.2.2) $G(\delta(\mathcal{H})) = [t]_{\mu^{n-1}} \iff$

$$t(x_1, \ldots, x_{n-1}) = f(\iota[1], \iota[2], \ldots, \iota[n-1]), \quad \forall \vec{x}.$$

where f has $\varphi(n, m)$ arguments:

$$e[k] = \langle\ x_k,\ e[1,k],\ e[2,k],\ \ldots,\ e[k-1,k],\ h_u(0)(x_k) = h_v(0)(x_k)\rangle$$

$$1 \leq k \leq n-1$$

$$e[k_2, k_1] = \langle\ \widetilde{h_u(1)}(x_{k_2}, x_{k_1}) = \widetilde{h_v(1)}(x_{k_2}, x_{k_1}),\ e_u[1, k_2, k_1],\ e_v[1, k_2, k_3],\ \ldots,$$

$$e_u[k_2 - 1, k_2, k_1],\ e_v[k_2 - 1, k_2, k_1],\ h_u(1)(x_{k_2}, x_{k_1}),\ h_v(1)(x_{k_2}, x_{k_1})\ \rangle$$

$$1 \leq k_2 < k_1 \leq n-1$$

and $e_s[k_j, \ldots, k_1]$, $s = u, v$; (blocks of entries of function $f : \aleph_1{}^{\varphi(m,n)} \to \aleph_1$) are defined recursively by

$$e_s[k_j, \ldots, k_1] = \langle \widetilde{h_s(j-1)}(x_{k_j}, \ldots, x_{k_1}), e_s[1, k_j, \ldots, k_1], \ldots,$$

$$\ldots, e_s[k_j - 1, k_j, \ldots, k_1], h_s(j-1)(x_{k_j}, \ldots, x_{k_1})\rangle,$$

$$(1 \leq k_j < \cdots < k_1 \leq n-1), \quad (3 \leq j \leq m-2), \quad s = u, v.$$

$$e_u[k_m, \ldots, k_1] = \langle \widetilde{h_u(m-1)}(x_{k_m}, \ldots, x_{k_1}), h_u(x_{k_m}, \ldots, x_{k_1})\rangle,$$

$$1 \leq k_m < \cdots < k_1 \leq n-1.$$

$$e_v[k_m, \ldots, k_1] = \langle \widetilde{h_v(m-1)}(x_{k_m}, \ldots, x_{k_1}),$$

$$\widetilde{h_v(m)}(x_1, x_{k_m}, \ldots, x_{k_1}),\ \ldots,\ \widetilde{h_v(m)}(x_{k_m - 1}, x_{k_m}, \ldots, x_{k_1}),$$

$$h_v(x_{k_m}, \ldots, x_{k_1})\rangle, \qquad 2 \leq k_m < \cdots < k_1 \leq n-1.$$

That establishes the lower bound inequality.

A natural extension of a definition of canonical functions yields two groups of canonical functions for $\Im(n)$ distinguished by their superscript and ordered as shown (here $<$ denotes $<_{\mathfrak{S}}$):

$$\mathrm{CF}(0)^v \;=\; \mathrm{CF}(0)^u > \epsilon\,[n-2] > \epsilon\,[n-3] > \cdots > \epsilon\,[1]$$

with

$$\begin{aligned}\epsilon[k] \;=\langle\; &\mathrm{CF}(k)^v \;>\; \epsilon_v[k-1,k] \;>\; \epsilon_v[k-2,k] \;> \cdots >\; \epsilon_v[1,k]\\ &> \mathrm{CF}(k)^u \;>\; \epsilon_u[k-1,k] \;>\; \epsilon_u[k-2,k] \;> \cdots >\; \epsilon_u[1,k]\\ &> \mathrm{CF}_T(k)^v = \mathrm{CF}_T(k)^u \;\rangle, \qquad 1 \le k \le n-2,\end{aligned}$$

and $\epsilon_s[k_p,\ldots,k_1]$, $s = u, v$, being defined recursively by

$$\begin{aligned}\epsilon_s[k_p,\ldots,k_1] = \langle\; &\mathrm{CF}(k_p,\ldots,k_1)^s \;>\; \epsilon_s[k_p-1,k_p,\ldots,k_1]\\ &> \cdots >\; \epsilon_s[1,k_p,\ldots,k_1] \;>\; \mathrm{CF}_T(k_p,\ldots,k_1)^s \;\rangle,\\ &1 \le k_p < \cdots < k_1 \le n-2, \qquad 2 \le p \le m-2.\end{aligned}$$

$$\begin{aligned}\epsilon_u[k_{m-1},\ldots,k_1] = \langle\; &\mathrm{CF}(k_{m-1},\ldots,k_1)^u \;>\; \mathrm{CF}_T(k_{m-1},\ldots,k_1)^u \;\rangle\\ &1 \le k_{m-1} < \cdots < k_1 \le n-2.\end{aligned}$$

$$\begin{aligned}\epsilon_v[k_{m-1},\ldots,k_1] = \langle\; &\mathrm{CF}(k_{m-1},\ldots,k_1)^v \;>\\ &\mathrm{CF}_T(k_{m-1}-1,k_{m-1},\ldots,k_1)^v \;> \cdots >\; \mathrm{CF}_T(1,k_{m-1},\ldots,k_1)^v\\ &> \;\mathrm{CF}_T(k_{m-1},\ldots,k_1)^v \;\rangle. \qquad 2 \le k_{m-1} < \cdots < k_1 \le n-2.\end{aligned}$$

Therefore, the number of distinct canonical functions is

$$\mathrm{Card}\,\epsilon_F \;=\; 1 + 3(n-2) + 4\left[\sum_{j=1}^{m-1}\binom{n-2}{j}\right] + \binom{n-2}{m}.$$

and can be computed as the sum of numbers of canonical functions for $S^{\tau,0}(n)$ and $S^{\tau,1}(n)$ minus number of canonical functions that are dublicated, i.e., CF(0) and $\mathrm{CF}_T(k), 1 \leq k \leq n-2$.

When we write, for example, $\mathrm{CF}(0)^v > \mathrm{CF}(n-2)^v$ we mean the is a cub set C so that for every function $\mathcal{H} :<_{\bullet} \to \aleph_1$ of the type matching with $\mathcal{F}_m$, if $\delta = \mathrm{ot} < [h_u]_{S^{\tau,0}}, [h_v]_{S^{\tau,1}} >$, then

$$\mathrm{CF}(0)^v(\delta) := [h_v(0)^{n-1}]_{\mu^{n-1}} \quad > \quad \mathrm{CF}(n-2)^v(\delta) := [h_v(1)^{n-2}]_{\mu^{n-1}}$$

In other words, a canonical function with the superscript v, say $\mathrm{CF}(n-2)^v$ to be specific, ignores the h_u–component of its argument δ and can be viewed simply as the function $\mathrm{CF}(n-2)$ corresponding to $S^{\tau,1}(n)$. Other canonical functions are defined likewise. Hence, by proposition 2.3.3, within each block $\epsilon_s[k_p, \ldots, k_1]$ canonical functions represent ordinals of consequitive cardinality. Then it follows from the claim below that all canonical functions represent ordinals of consequitive cardinality starting from

$$|[\mathrm{CF}_T(1)^u]_{\mathfrak{S}}| \leq j_{\mathfrak{S}}(\aleph_{n-1})^+.$$

Claim *Fix $H : \aleph_{m+1} \to \aleph_n$, and $1 \leq k \leq n-2$. If $H <_{\mathfrak{S}} \mathrm{CF}_T(1,k)^v$, then $[H]_{\mathfrak{S}} \hookrightarrow [\mathrm{CF}(k)^u]_{\mathfrak{S}}$.*

Proof of Claim. We give the standard argument again. Fix C_1, a cub set corresponding to the inequality $H <_{\mathfrak{S}} \mathrm{CF}_T(1,k)^v$, and let C be a cub set homogeneous for the partition

$$\mathcal{P}(\mathcal{H},g) = 1 \iff [\ H(\delta(\mathcal{H})) < [g^k]_{\mu^{n-1}};$$

$$\mathcal{H} \text{ is of the type matching with } \mathcal{F}_m;$$

$$g : {\aleph_1}^{(2,1)} \to \aleph_1 \text{ is of c.t.};$$

$$h_u(1)(\alpha,\beta) < g(\alpha,\beta) < h_v(2)(0,\alpha,\beta)$$

$$< h_v(1)(\alpha,\beta) < h_u(0,\ldots,0,\alpha+1,\beta)].$$

Here $g^k(x_1,\ldots,x_{n-1}) := g(x_k,x_{n-1})$.

Suppose now $\mathcal{H} :\lessdot_{\{u,v\}} \to Cub(N_C) \cap C_1$ is of the type matching with $\mathcal{F}_m$, and let

$$[f]_{\mu^{n-1}} = \max\{H(\delta(\mathcal{H})), [h_u(1)^k]_{\mu^{n-1}}\} < [\widetilde{h_v(2)}^{1,k}]_{\mu^{n-1}},$$

so that $\forall^*_{\mu^{n-1}}\vec{x}$,

$$h_u(1)(x_k,x_{n-1}) \leq f(\vec{x}) < \widetilde{h_v(2)}(x_1,x_k,x_{n-1}) = \sup_{y<x_1} \widetilde{h_v(2)}(y,x_k,x_{n-1}).$$

Then $\exists y$, so that $\forall^*_{\mu^{n-1}}(x_1,\ldots,x_{n-1})$, $y < x_1$ and

$$f(\vec{x}) < \widetilde{h_v(2)}(y,x_k,x_{n-1}) < \widetilde{h_v(2)}(x_1,x_k,x_{n-1}) \in Cub(N_C).$$

We set $\zeta(\alpha,\beta) := \widetilde{h_v(2)}(y,\alpha,\beta)$. Notice that $h_u(1) < \zeta < h_v(1)$, μ^2-a.e., and thus ζ preserves the ${\aleph_1}^{(2,1)}$-order μ^2-a.e. Then $g := N_C \circ \zeta : {\aleph_1}^{(2,1)} \to C$ is of c.t., and

$$h_u(1)(\alpha,\beta) < g(\alpha,\beta) < h_v(2)(0,\alpha,\beta),\ \forall^*_{\mu^2}(\alpha,\beta)$$

Now using sliding argument we apply the partition to $(\mathcal{H},g)$. Since $H(\delta(\mathcal{H})) < [g^k]_{\mu^{n-1}}$, we see that C is homogeneous for the stated side of the partition. That implies existence of the desired embedding. ■ Claim

It follows now that $\mathfrak{S}(n) \leq \mathfrak{S}(n-1) + \mathrm{Card}_{CF} + 1$. It can be shown directly as in proposition 2.3.2 that $\mathfrak{S}(2) = 4$. Thus we may assume equation 2.4.2.1 holds for $\mathfrak{S}(n-1)$. Then

$$\begin{aligned}
&\mathfrak{S}(n-1) + \mathrm{Card}_{CF} = \\
&\qquad 1 \;+\; (n-2) \;+ (n-2) + 3\binom{n-2}{2} + 4[\sum_{j=3}^{m}\binom{n-2}{j}] + \binom{n-2}{m+1} \\
&+ \qquad\qquad\qquad 1 \quad + 3\binom{n-2}{1} + 4[\sum_{j=3}^{m}\binom{n-2}{j-1}] + \binom{n-2}{m} \\
&= \quad (n-1) \qquad + (n-1) + 3\binom{n-1}{2} + 4[\sum_{j=3}^{m}\binom{n-1}{j}] + \binom{n-1}{m+1}.
\end{aligned}$$

(Here the upper line is $\mathfrak{S}(n-1)$, the second line is Card_{CF}, and we add expressions in each column.) This finishes the proof for the simple general measure. ∎ Claim

For an arbitrary general measure similar argument can be carried over, or else one can do induction on the number of terminal nodes of $T(\mathfrak{S})$.

Second part of the theorem is a corollary of lemma 2.4.1. ∎

Lemma 2.4.1 *Let S be a (countably additive) measure on $\aleph_m$ with $j_S(\aleph_n) < \aleph_\omega$ for all $n \in \omega$, and let $\omega \leq \alpha \leq \omega_1$. Then $j_S(\aleph_\alpha) = \aleph_\alpha$.*

Proof. It suffices to show $j_S(\aleph_\alpha) \leq \aleph_\alpha$.

If α is a limit ordinal, then it is countable, hence $\mathrm{cof}(\aleph_\alpha) = \omega$. Let $\{\lambda_n\}_{n\in\omega}$ be a sequence of cardinals cofinal in $\aleph_\alpha$. By countable additivity of S, for every function $f : \aleph_m \to \aleph_\alpha$, there is n so that $f(\xi) < \lambda_n$ for S-almost all $\xi \in \aleph_m$. Thus $\|f\|_S < j_S(\lambda_n)$. Now, if $\aleph_\alpha = \aleph_\omega$, then $j_S(\lambda_n) = j_S(\aleph_n) < \aleph_\omega$. If $\aleph_\alpha > \aleph_\omega$, the let

$j_S(\lambda_n) = \lambda_n$ be our induction hypothesis. Hence, in either case $[f]_S < j_S(\lambda_n) \leq \aleph_\alpha$, and therefore $j_S(\aleph_\alpha) = \sup_f [f]_S \leq \aleph_\alpha$.

Consider now $\alpha + 1$. We may assume $j_S(\aleph_\alpha) = \aleph_\alpha$ to be our induction hypothesis (we just showed it for $\alpha = \omega$). Fix a function $f : \aleph_m \rightarrow \aleph_{\alpha+1}$ and put $\beta_f = \sup_{\xi < \aleph_m} f(\xi)$. We claim that $\beta_f < \aleph_{\alpha+1}$. (If $\aleph_{\alpha+1} \geq \aleph_{\omega+2}$, recall the theorem of Kechris and Woodin [KeW] which says that $\mathrm{cof}(\aleph_{\alpha+1}) \geq \underset{\sim}{\delta}^1_{2n}$ whenever $\aleph_{\alpha+1} \geq \underset{\sim}{\delta}^1_{2n}$ and $n \geq 1$. Hence $\mathrm{cof}(\aleph_{\alpha+1}) \geq \aleph_{\omega+2}$ which implies $\beta_f < \aleph_{\alpha+1}$. Otherwise $\aleph_{\alpha+1} = \aleph_{\omega+1}$, and we use the regularity of $\aleph_{\omega+1}$.) Because the ordinal β is between $\aleph_\alpha$ and $\aleph_{\alpha+1}$, $Card(j_S(\beta)) = Card(j_S(\aleph_\alpha)) = \aleph_\alpha$. Now $j_S(\aleph_{\alpha+1}) = \sup_f [f]_S \leq \sup_f j_S(\beta_f) \leq \sup_f Card(j_S(\beta_f))^+ = \aleph_{\alpha+1}$. ∎

CHAPTER 3

DESCRIPTIONS AND REPRESENTATION OF CARDINALS BELOW $\underset{\sim}{\delta}^1_5$

Intuitively, a description is an object "describing" how to build an ordinal (an equivalence class of functions $f : \underset{\sim}{\delta}^1_3 \to \underset{\sim}{\delta}^1_3$) with respect to the measures S^m_3. Descriptions were introduced by Jackson first in his computation of $\underset{\sim}{\delta}^1_5$. Here we give rigorous definitions of *descriptions*, *interpretation function*, and *the lowering operator*. These definition and results related to the theory of descriptions (which we did not include here) can be found in the original paper [J1]. It follows from the results in [J1], that every cardinal in $L(\mathbb{R})$ below $\underset{\sim}{\delta}^1_5$ can be represented by a description and an identity function with respect to certain measures. In the last section of this chapter we will prove that the converse is also true, i.e., every ordinal represented by a description and the identity function with respect to a sequence of measures is a cardinal below $\underset{\sim}{\delta}^1_5$.

We will slightly abuse notation by referring to "ordinals represented by descriptions and the identity function with respect to sequences of measures" simply as "descriptions". We will also make a few changes in the original notation of [J1] for the sake of simplicity.

3.1 Descriptions and their interpretation

Descriptions have indices associated with them. An index is of the form (f_m) or $()$, and written as a superscript of the description. Descriptions indexed as $d^{(f_m)}$ will be called *type-0* descriptions, and those of the form $d^{()}$, *type-1* descriptions. Later we will suppress writing the index when it is understood or irrelevant. The descriptions defined directly will be also referred at as *basic* descriptions, and the ones defined in terms of the other descriptions will be called *non-basic*.

Fix $m, t \in \omega$, let $m(i) \in \omega$ and $K_i = S_1^{m(i)}$ or $W_1^{m(i)}$ for all $i = 1, \dots, t$. A set of descriptions, $\mathcal{D}_m = \mathcal{D}_m(K_1, \dots, K_t)$, is defined relative to this sequence of measures. Along with D_m is also defined a numerical function $k : \mathcal{D} \to \{1, \dots, t\} \cup \{\infty\}$.

Definition 3.1.1 (Descriptions) *Elements of $\mathcal{D}_m(K_1, \dots, K_t)$ and $k : \mathcal{D} \to \{1, \dots, t\} \cup \{\infty\}$ are defined by reverse induction on $k(d)$ through the following cases:*

Basic Type-1:

$d^{()} := (k; p)^{()}$ *where* $1 \le k \le t$, $K_k = W_1^r$, *and* $1 \le p \le r$. $k(d) := k$.

Basic Type-0:

1. $d^{(f_m)} := (k; p)^{(f_m)}$ *where* $1 \le k \le t$, $K_k = W_1^r$, *and* $1 \le p \le r$. $k(d) := k$.

2. $d^{(f_m)} := (p)^{(f_m)}$ *where* $1 \le p \le m$. $k(d) := \infty$.

Non-Basic Descriptions:

1. $d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \dots, d_l^{(f_m)})^{(f_m)}$ *where* $1 \le k \le t$, $K_k = S_1^r$, $l \le r-1$, *and* $k(d_1), \dots, k(d_l), k(d_r) > k$. $k(d) := k$.

2. $d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \dots, d_l^{(f_m)})^{s(f_m)}$ *(Here s stands for "sup"), where* $r \ge 2$, $1 \le k \le t$, $K_k = S_1^r$, $l \le r-1$, *and* $k(d_1), \dots, k(d_l), k(d_r) > k$. $k(d) := k$.

3. *Same as 1. with $()$ replacing (f_m) everywhere.*

4. Same as 2. with $(.)$ replacing (f_m) everywhere.

Now let $\mathcal{D}(K_1, \ldots, K_t) := \cup_m \mathcal{D}_m(K_1, \ldots, K_t)$ to be the set of descriptions relative to $K_1, \ldots, K_t$. We will suppress the background sequence of measures simply writing $\mathcal{D}$ or $\mathcal{D}_m$. We call $\mathcal{D}_m$ the set of m-descriptions.

Next we give the definition of the function h which interprets descriptions. Fix $d \in \mathcal{D}$, let $h_1, \ldots, h_t$ be functions of the type $K_1, \ldots, K_t$, i.e., if $K_i = W_1^r$, then $h_i : r \to \aleph_1$, and if $K_i = S_1^r$, then $h :<_r \to \aleph_q$ of c.t. We define the ordinal $h(d; \bar{h}) = h(d; h_1, \ldots, h_t)$ through cases by reverse induction on $k(d)$. If $d = d^{()}$ then $h(d; h_1, \ldots, h_t) < \aleph_1$ and if $d = d^{(f_m)}$ then $h(d; h_1, \ldots, h_t) < \aleph_{m+1}$ and is represented with respect to W_1^m by a function which is also denoted by $h(d; h_1, \ldots, h_t)(\alpha_1, \ldots, \alpha_m)$.

Definition 3.1.2 (Interpretation of Descriptions)

<u>*Basic Type-1:*</u> *If $d^{()} = (k; p)$, then $h(d; \bar{h}) = h_k(p)$.*

<u>*Basic Type-0:*</u>

1. If $d^{(f_m)} = (k; p)$, then $h(d; \bar{h})(\alpha_1, \ldots, \alpha_m) = h_k(p)$.

2. If $d^{(f_m)} = (p), 1 \leq p \leq m$, then $h(d; \bar{h})(\alpha_1, \ldots, \alpha_m) = \alpha_p$.

<u>*Non-Basic:*</u>

1. $d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \ldots, d_l^{(f_m)})^{(f_m)}$ where $1 \leq k \leq t$, $K_k = S_1^r$, $l \leq r-1$, and $k(d_1), \ldots, k(d_l), k(d_r) > k$.

a. If $l = r-1$, then $h(d; \bar{h})(\alpha) := h_k(\, h(d_1; \bar{h})(\alpha), \ldots, h(d_r; \bar{h})(\alpha)\,)$

b. If $l < r-1$, then $h(d;\bar{h})(\bar{\alpha}) := \sup_{\beta_{l+1}<\cdots<\beta_{r-1}<h(d_r;\bar{h})(\bar{\alpha})} h_k(\ h(d_1;\bar{h})(\bar{\alpha}), \ldots, h(d_l;\bar{h})(\bar{\alpha}),\ \beta_{l+1}, \ldots, \beta_{r-1}, h(d_r;\bar{h})(\bar{\alpha})\ )$

2. *Let $d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \ldots, d_l^{(f_m)})^{s(f_m)}$ where $1 \leq k \leq t$, $K_k = S_1^r$, $l \leq r-1$, and $k(d_1), \ldots, k(d_l), k(d_r) > k$. Then $h(d;\bar{h})(\bar{\alpha}) := \sup_{\beta_l<h(d_l;\bar{h})(\bar{\alpha}), \beta_{l+1}<\cdots<\beta_{r-1}<h(d_r;\bar{h})(\bar{\alpha})} h_k(\ h(d_1;\bar{h})(\bar{\alpha}), \ldots, h(d_{l-1};\bar{h})(\bar{\alpha}), \beta_l, \beta_{l+1}, \ldots, \beta_{r-1}, h(d_r;\bar{h})(\bar{\alpha})\ )$*

3. *Same as 1., except now $h(d;\bar{h})$ is a single ordinal $< \aleph_1$.*

4. *Same as 2., except now $h(d;\bar{h})$ is a single ordinal $< \aleph_1$.*

Next we put an ordering $<$ on $\mathcal{D} = \mathcal{D}(K_1, \ldots, K_t)$ as follows.

Definition 3.1.3 (Order $<$ on $\mathcal{D}(K_1, \ldots, K_t)$)

If $d_1, d_2 \in \mathcal{D}$ have the same index, then $d_1 < d_2$ iff for almost all $h_1, \ldots, h_t$, $h(d_1, \bar{h}) < h(d_2, \bar{h})$.

The following two definitions give conditions which descriptions must satisfy in order to be well defined.

Definition 3.1.4 (Condition C) *Inductively, we say $d \in \mathcal{D}$ satisfies condition C if either d is basic or else d is non-basic, say of the form $d = (k; d_r, d_1, \ldots, d_l)^s$, where s may or may not appear, and $d_1 < d_2 < \cdots < d_l < d_r$, and $d_1, \ldots, d_l, d_r$ satisfy codition C.*

Let $\mathcal{D}$ be the set of the objects of the form (d) or $(d)^s$, where $d \in \mathcal{D}$, along with the distinguished object $()^s$.

Definition 3.1.5 (Condition D) *Let $x \in \mathcal{D}$, $x = (d)^s$, where s may or may not appear. Suppose $d \doteq d^{(J_m)} \in \mathcal{D}_m^*$. If $x = ()^s$, it satisfies condition D. Otherwise, we say $x = (d)^s$ satisfies condition D provided d satisfies condition C and*

1. *If $x = (d)$ (i.e., s does not appear), then for almost all $h_1, \ldots, h_t$, $h(d; h) : <_m \to \aleph_1$ is of the correct type W_1^m almost everywhere.*

2. *If $x = (d)^s$, then for almost all $h_1, \ldots, h_t$, $h(d; h)$ is the supremum of ordinals representable by functions $h : \aleph_1{}^m \to \aleph_1$ of the correct type almost everywhere.*

The following lemma demonstrates the significance of condition C.

Lemma 3.1.1 (Jackson) *Suppose d satisfies C. Then for a.a. h_1, if $h_1 = h_1'$ a.e., then for a.a. h_2, if $h_2 = h_2'$ a.e., ..., for a.a. h_t, if $h_t = h_t'$, then $h(d; \bar{h}) = h(d; \bar{h}')$.*

Proof. See [J1]. ∎

Next, let us show how to use descriptions to generate equivalence classes of functions from $\underset{\sim}{\delta}_3^1$ to $\underset{\sim}{\delta}_3^1$ with respect to the measures W_3^m and K. The measures W_3^m are the ones induced by the weak partition relation on $\underset{\sim}{\delta}_3^1$. In [J1] only the measures defined by

Definition 3.1.6 (A measure on $\underset{\sim}{\delta}_3^1$)

$$A \in W_3^m \iff \exists \text{c.u.b.} C \subset \underset{\sim}{\delta}_3^1, \ \forall f : \aleph_{m+1} \to C \text{ of the correct type}, \ [f]_{S^{\pi_m,0}} \in A$$

were considered. Here $\pi_m = (m, 1, 2, \ldots, m-1)$.

Remark 3.1.1 *In the last section of this chapter, when we prove that descriptions represent cardinals, we will consider measures defined similarly, except that the func-*

tion f is chosen to be continuous. One can easily verify, that the results holds when f is of the correct type as well.

Definition 3.1.7 (Ordinal represented by description) *Fix $m \in \omega$, and let $(d)^s \in \mathcal{D}$ satisfy condition D where s may or may not appear. Let $g : \underset{\sim}{\delta}^1_3 \to \underset{\sim}{\delta}^1_3$ be given.*

- *We define an ordinal $(g;(d)^s;K_1,\dots,K_t)(W_3^m)$, where $d \in \mathcal{D}(K_1,\dots,K_t)$. This ordinal is represented w.r.t. W_3^m by the function which assigns to $[f]_{S_1^m}$ the ordinal $(g;f;(d)^s;\bar{K})$, where $f : \aleph_{m+1} \to \underset{\sim}{\delta}^1_3$ is continuous. (Note: in [J1] function f has been chosen to be of the correct type.)*

- *$(g;f;(d)^s;K)$ is represented w.r.t. K_1 by the function which assigns to $[h_1]$ the ordinal $(g;(d)^s;h_1,K_2,\dots,K_t)$.*

- *In general, $(g;(d)^s;h_1,\dots,h_{i-1},K_i,\dots,K_t)$ is represented w.r.t. K_i by the function which assigns to $[h_i]$ the ordinal $(g;(d)^s;h_1,\dots,h_{i-1},h_i,K_{i+1},\dots,K_t)$.*

- *Finally, $(g;(d)^s;h_1,\dots,h_t)$ is defined by cases as follows*

  1. *if s does not appear, then $(g;(d)^s;h_1,\dots,h_t) = g(f(h(d;h_1,\dots,h_t)))$.*

  2. *if s appears, then $(g;(d)^s;h_1,\dots,h_t) = g(\sup_{\beta<h(d;h_1,\dots,h_t)} f(\beta))$.*

  3. *if $(d)^s = ()^s$, then $(g;()^s;h_1,\dots h_t) = g(\sup_{\beta<\aleph_{m+1}} f(\beta))$.*

Remark 3.1.2 *If d satisfies condition D, then $(g;(d)^s;K)$ is well defined.*

In our later discussion the function g from the definition above will be the identity function id $:= \alpha \mapsto \alpha$ on $\underset{\sim}{\delta}^1_3$. If $g \neq id$, then the corresponding ordinal will not be a cardinal.

3.2 The Lowering Operator

Next we need the lowering operator $\mathcal{L}$ on $\mathcal{D}$. For every description $d \in \mathcal{D}$, $\mathcal{L}$ applied to d gives the next description below d. First, given measures $K_1, \ldots, K_t$ and an integer k ($1 \leq k \leq t$ or $k = \infty$), an operator $\mathcal{L}^k$ is defined on those d satisfying $k(d) \geq k$, except for a unique $d = d(k)$ which is called the *minimal* description with respect to $\mathcal{L}^k$. Then $\mathcal{L} := \mathcal{L}^1$. $\mathcal{L}^k$ is defined by reverse induction on k as follows:

Definition 3.2.1 (Operator $\mathcal{L}^k$)

I. $k = \infty$. *So, d is basic type-0 with $d = d^{(J_m)} = (i)$ for $1 \leq i \leq m$. If $i > 1$, then $\mathcal{L}^\infty := (i-1)$. If $i = 1$, d is minimal with respect to $\mathcal{L}^\infty$.*

II. $1 \leq k \leq t$.

1. $k = k(d)$

a. d is basic type-1, so $d = (k; p)$. If $p > 1$, then $\mathcal{L}^k := (k; p-1)$. If $p = 1$, d is minimal.

b. $d = d^{(J_m)} = (k; d_r, d_1, \ldots, d_l)$, with $l = r - 1$ and $K_k = S_1^r$. If $l \geq 1$, then $\mathcal{L}^k := (k; d_r, d_1, \ldots, d_l)^s$. If $d = (k; d_r)$, then $\mathcal{L}^k(d) := d_r$.

c. d is non-basic as in b., but $l < r - 1$ and s does not appear. If $\mathcal{L}^{k+1}(d_r)$ is defined and also $> d_l$ in case $l \geq 1$, then $\mathcal{L}^k(d) := (k; d_r, d_1, \ldots, d_l, \mathcal{L}^{k+1}(d_r))$. If $\mathcal{L}^{k+1}(d_r)$ is not defined or [is $\leq d_l$ and $l \geq 1$], then $\mathcal{L}^k(d) := (k; d_r, d_1, \ldots, d_l)^s$, if $l \geq 1$; otherwise $\mathcal{L}^k(d) := d_r$.

d. d is non-basic and s *appears. Then* $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_{l-1}, \mathcal{L}^{k+1}(d_l))$ *if* $\mathcal{L}^{k+1}(d_l)$ *is defined and satisfies* $\mathcal{L}^{k+1}(d_l) \geq d_{l-1}$ *if* $l \geq 2$. *Otherwise, set* $\mathcal{L}^k(d) := (k; d_r, d_1, \ldots, d_{l-1})^*$ *if* $l \geq 2$, *and for* $l = 1$, $\mathcal{L}^k(d) := d_r$.

2. $k < k(d)$, $K_k = W_1^{m(k)}$.

   a. d is not minimal with respect to $\mathcal{L}^{k+1}$. *Then* $\mathcal{L}^k(d) := \mathcal{L}^{k+1}(d)$.

   b. d is minimal with respect to $\mathcal{L}^{k+1}$. *Then* $\mathcal{L}^k(d) := (k; m(k))$.

3. $k < k(d)$, $K_k = S_1^{m(k)}$.

   a. d is not minimal with respect to $\mathcal{L}^{k+1}$. *Then* $\mathcal{L}^k(d) := (k; \mathcal{L}^{k+1}(d))$.

   b. d is minimal with respect to $\mathcal{L}^{k+1}$. *Then d is minimal with respect to* $\mathcal{L}^k$.

3.3 Representation of cardinals below $\underset{\sim}{\delta}^1_5$

In this section we prove that every ordinal represented by a description and the identity function with respect to a sequence of measures is a cardinal. Let $\pi'_z = (z, i_1, \ldots, i_{z-1})$ denote a permutation of $(1, \ldots, z)$ for some integer z. We define a measure W^z on $\underset{\sim}{\delta}^1_3$ as follows:

Definition 3.3.1 (A measure on $\underset{\sim}{\delta}^1_3$)

$$A \in W^z \iff \exists \text{c.u.b.}\, C \subset \underset{\sim}{\delta}^1_3,\ \forall \text{ continuous } f_z : \aleph_{z+1} \to C\ \ (\ldots, [f_z]_{S^{\pi'_z},0}, \ldots) \in A$$

Here $(\ldots, [f_z]_{S^{\pi'_z},0}, \ldots)$ denotes a $(z-1)!$-ary tuple of ordinals. The size of the tuple is determined by the number of possible permutations π'_z.

Note the difference in the definitions 3.3.1 and 3.1.6. This difference affects some of the definitions (for example, the lowering operator $\mathcal{L}$) from the previous two sections, however it is irrelevant to our computation. Indeed, if $f' : \aleph_{z+1} \to \underset{\sim}{\delta}^1_3$ is of the correct type, then $f(\xi) := \sup_{\eta<\xi} f(\eta)$ is continuous everywhere, and therefore all the definitions given in the case of f' are still valid for f, but those that correspond to the description d with the superscript s become now redundant. With our definition 3.3.1, the analog of condition D simply becomes $d > (z)^{f_z}$, because for every such d the interpretation function $h(d, \bar{h})$ (see definition 3.1.2) is order preserving from one of the orders $<^{\bar{r}_z}$ or a supremum of ordinals represented by order preserving functions.

We fix a description d, a measure $W = W^z$, and a sequence of measures $\bar{S}$. Every measure S_i (or μ^{r_i}) from $\bar{S}$ is a measure on $\aleph_{m_i+1}$ (or ${\aleph_1}^{r_i}$) and it concentrates on ordinals represented by functions $h :<^{\bar{r}m_i} \to \aleph_1$ of c.t. with respect to μ^i (or $h : r_i \to \aleph_1$). We say that the function h has type S_i (or μ^{r_i}). As was stated earlier, a description d is a finitary object which describes how to construct an ordinal, $(\mathrm{id}; d; \bar{S})(W)$ (definition 3.1.7), from the given sequence of measures $\bar{S}$. We make a few simplifications of the notation introduced in section 3.1. Let $d := (d, \bar{S})$ denote a description d defined relative to $\bar{S}$. For $1 \leq i \leq z$, we use $\cdot_i$ to denote the description $(i)^{(f_z)}$ which corresponds to the function $(\alpha_1, \ldots, \alpha_z) \mapsto \alpha_i$. We refer to $\cdot_i$ as the i^{th} *dot variable*.

Recall the final part of definition 3.1.7 which defines ordinal $(\mathrm{id}; d; h_1, \ldots, h_t)$. Let $(\mathrm{id}; d; h_1, \ldots, h_t)^*$ be defined as $(\mathrm{id}; d; h_1, \ldots, h_t)$ from which we omit id and f. functions. We will view each description d defined relative to measures $\bar{S}$ as a formal object $(\mathrm{id}; d; h_1, \ldots, h_t)^*$ in the language with terms: $h_i(j)$ and $\widetilde{h_i(j)}$, α_j, and $\cdot_1, \ldots, \cdot_{z+1}$. Informally, we will call these terms "functions", "ordinals", and "dots". We refer to such representation of d (with $\bar{S}$) as *functional representation of $\bar{d}$* (or simply *functional representation of d*, if $\bar{S}$ is fixed). For instance, if the ordinal $(\mathrm{id}; d; h_1, \ldots, h_t) = \mathrm{id} \circ f_z([h_1(2)(\alpha_{5,3},\ h_2(0)(\alpha_{6,1}),\ h_3(0)(\cdot_m)]_{\mu^m})$, we will refer to the description d defined with respect to measures $\bar{S}$ simply by writing $h_1(2)(\alpha_{5,3},\ h_2(0)(\alpha_{6,1}),\ h_3(0)(\cdot_m)\ )$ whenever confusion is impossible.

Remark 3.3.1 *The arguments of function f_z are ordinals $< \aleph_{z+1}$, and f_z is continuous. Thus $\mathrm{cof}(\mathrm{id}; d; h_1, \ldots, h_t) = \omega_0, \omega_1$, or ω_2. If $\mathrm{cof}(\mathrm{id}; d; h_1, \ldots, h_t) = \omega_i$, we will say* cofinality of $\bar{d}$ *is ω_i and write $\mathrm{cof}\, \bar{d} = \omega_i$ (and $\mathrm{cof}\, d = \omega_i$ will mean the same thing, when $\bar{S}$ is fixed).*

Our goal is to show that each $(\mathrm{id}; d; \bar{S})(W)$ is a cardinal. The strategy of our proof is as follows. First we will define a tree T_d corresponding to $(d, \bar{S})$. The tree T_d will have infinitely many nodes, which we will partition into equivalence classes; these equivalence classes will in turn be partitioned into finitely many blocks. For each such block we will assign an ordinal. Being added in a proper way these ordinals will give an ordinal ξ_d. Then we will show that $(\mathrm{id}; d; \bar{S})(W) = \aleph_{\omega+\xi_d}$.

We define $(T_d, <)$ as the transitive closure of $<'$, where $\bar{q} <' \bar{p} \iff [\bar{p} = (p; \bar{S})$ and $\bar{q} = ((\mathcal{L}p); \bar{S}, K)]$, and we say $\bar{d}$ is the root of T_d. Intuitively, T_d is constructed by repeatedly applying the lowering operator $\mathcal{L}$ to $\bar{d}$. By doing so, we obtain new descriptions q together with new measures $M_q = \langle \ldots, S_{n_i}, \mu^{n_i}, \ldots \rangle$. We are adding at most one measure at a time. In the definition of $<'$ above, the type of the new measure K depends on $\mathrm{cof}(\mathcal{L}p)$: if it is ω_0, then no measure is added; if ω_1, then $K = \mu^{\omega_1}$;

and if ω_2, then $K = S_1$. Adding other measures is irrelevant to our computation (see [J1]). As in [J1], we define the rank function on the nodes of the tree T_d by $|\bar{q}| := (\sup_{\bar{p}<'\bar{q}} |\bar{p}|) + 1$, and $|T_d| = |\bar{d}|$.

We refer to each tuple $\bar{q} = (q; \bar{S}, \bar{M}_q)$ as a *node* in T_d. In T_d, many nodes share the same description, but each node $\bar{q}$ has its own unique sequence of measures $\bar{M}_q$. In the *functional representation* of $\bar{q}$, in addition to symbols $h_i(l), \widetilde{h_i(l)}$, $\alpha_{j,i}$, and $\gamma_1, \ldots, \gamma_s$, we could have $\gamma_{j,i}$ corresponding to measures $\mu^{n_i} \in \bar{M}_q$, and $k_i(l)$ or $\widetilde{k_i(l)}$ corresponding to $S_{n_i} \in \bar{M}_q$.

Next, for every node $\bar{q}$ we define a sequence $\mathrm{oseq}_d(\bar{q})$ which will consist of symbols representing "ordinals" and "functions" corresponding to both types of measures from $\bar{S}$ and $\bar{M}_q$. We shall refer to functions (ordinals) corresponding to $\bar{S}$ measures as h-functions (α-ordinals), and to the ones related to $\bar{M}_q$ measures as k-functions (γ-ordinals).

Definition 3.3.2 (The o–sequence of $\bar{q}$, $\mathrm{oseq}_d(\bar{q})$)

Given $\bar{q}$, let $g(d_1, d_2, \ldots, d_l, d_0)$ be the functional representation of $\bar{q}$. Here g stands for an invariant of some h or some k function. We have numbered the arguments of g according to their significance in determining the size of $h(q, \bar{S})$. (Each d_i can be viewed as a subdescription with the same sequence of measures $\bar{S}$.) We define *recursively the o-sequence of q as follows*

$$\operatorname{oseq}_d(\bar{q}) := \begin{cases} [\operatorname{oseq}_d(d_0)^\frown \operatorname{oseq}_d(d_1)^\frown \dots {}^\frown \operatorname{oseq}_d(d_l)]' & , \text{if } g = h(j) \text{ or } \widetilde{h(j)} \\ \operatorname{oseq}_d(d_0) & , \text{if } (g = k(j) \text{ or } \widetilde{k(j)}) \text{ and } d_0 \neq \cdot_\tau \\ k(\cdot_\tau) & , \text{if } g = k(j) \text{ and } d_0 = \cdot_\tau \\ k(\cdot_\tau) & , \text{if } \widetilde{k(j)} (\text{ with } j \geq 1) \text{ and } d_0 = \cdot_\tau \\ \alpha \quad (\text{ or } \gamma) & , \text{if } g(\dots) = \alpha \quad (\text{ or } \gamma) \\ \emptyset & , \text{if } g(\dots) = \cdot_\tau \end{cases}$$

Here $'$ denotes the operation which eliminates repetition of ordinals and functions: we concatenate all $\operatorname{oseq}_d(d_i)$*, and then if a symbol α, γ, or $k(\cdot_\tau)$ appears in the resulting sequence more than once, we keep it only in the position where it appears the first time and remove from the other places.*

Note that $\operatorname{oseq}_d(\bar{q})$ is uniquely determined by the functional representation of $\bar{d}$. While the measures $\bar{S}$ are fixed, the other measures, $\bar{M}_q$, vary as we range over all possible nodes. The fact that the k-functions and γ-ordinals from $\operatorname{oseq}_d(\bar{q})$ are in some sense arbitrary is important in our computation. Now we split $\operatorname{oseq}_d(\bar{q})$ in two parts.

Definition 3.3.3 (Head and Tail) *Let $\bar{d} = (d; \bar{S})$ be fixed. For a description $q < d$ defined relative to measures $\bar{S}, \bar{M}_q$, we define* $\operatorname{head}_d(\bar{q})$ *:= the maximal proper initial subsequence of* $\operatorname{oseq}_d(\bar{q})$ *with no γ-ordinals or k-functions in it. We also define* $\operatorname{tail}_d(\bar{q})$ *to be the remaining part of* $\operatorname{oseq}_d(\bar{q})$*, so that* $\operatorname{oseq}_d(\bar{q}) = \operatorname{head}_d(\bar{q})^\frown \operatorname{tail}_d(\bar{q})$.

Definition 3.3.4 (Supremum of a description) *If q is a description defined relative to measures $K_1, \dots, K_t$, and $1 \leq m \leq t$, then by the* supremum of a description *q with respect to $K_m, \dots, K_t$, $\sup_{K_m, \dots, K_t} q$, we mean a description q' defined relative* *to* $K_1, \ldots, K_{m-1}$ *such that* $\forall^*_{K_1} h'_1$, *if we fix* h'_1, *then* $\forall^*_{K_2} h'_2$, *if we fix* h'_2, *then* $\ldots$, $\forall^*_{K_{m-1}} h'_{m-1}$, *if we fix* h'_{m-1}, *then for every ordinal* α, *if* $\alpha < h(q'; h')$, *then* $\forall^*_{K_m} h_m$, *if we fix* h_m, *then* $\forall^*_{K_{m+1}} h_{m+1}$, *if we fix* h_{m+1}, *then* $\ldots$, $\forall^*_{K_t} h_t$, *if we fix* h_t, *then* $\alpha < h(q; \bar{h})$.

To see that such q' exists, let $q = h_m(n)(d_1, \ldots, d_l, d_r)$ and assume $d'_i = \sup_{K_m, \ldots, K_t} d_i$ exists for all $1 \leq i \leq r$. Then if, for example, $d'_r = d_r, d'_1 = d_1$, and $d'_2 > d_2$, then $q' = \widetilde{h_m(2)}(d'_1, d'_2, d'_r)$, if $d'_2 < d'_r$, or else $d'_2 = d'_r$, in which case $q' = h_m(1)(d'_1, d'_r)$. If $d'_r > d_r$, then $q' = d'_r$.

We define $\sup_{\mathrm{tail}_d(\bar{q})} q := \sup_{\bar{M}_q} q$.

Proposition 3.3.1 *Suppose the description* p *is defined relative to measures* $\bar{S}$ *and* $\bar{M}$. *Let* $\bar{p} = (p; \bar{S}, \bar{M}, K)$ *be a node in* T_d *below* $\bar{d} = (d; \bar{S})$ *with* K *being a new measure. Let* $p^- := \mathcal{L}(\bar{p})$. *If* $\operatorname{cof} p = \omega_2$, *then* K *is a basic measure on* $\aleph_n$, *for some* $n > 1$, *and* k, *which represents the function corresponding to* K, *occurs in the functional interpretation of* p^-. *If* $\operatorname{cof} p = \omega_1$, *then* $K = \mu^n$, *for some* $n \geq 1$, *and* γ_n, *representing the largest ordinal corresponding to* K, *occurs in the functional interpretation of* p^-.

Proof. By induction on complexity of p. Let us identify p with its functional representation. Suppose $\operatorname{cof} p = \omega_2$. Then the measure K is a basic measure on $\aleph_n$, for some $n > 1$, by construction of T_d. It remains to consider the following three cases.

- $p = h_i(l)(\ldots, \cdot_r)$. Because $\operatorname{cof} p = \omega_2$, we have $r > 1$ and $h_i(l)$ is a proper invariant of h_i. Then $k(0)(\cdot_{r-1})$ is greater than all arguments of $h_i(l)$ to the left from $\cdot_r$. Thus $p^- = h_i(l+1)(\ldots, k(0)(\cdot_{r-1}), \cdot_r)$.

- Suppose $p = \widetilde{h_i(l)}(\ldots, p', s)$, where p' and s are some descriptions defined relative to $\bar{S}, \bar{M}$. It follows that p^- is of the form $h_i(l)(\ldots, (p')^-, s)$. Since $\operatorname{cof} p = \omega_2$, we have $\operatorname{cof} p' = \omega_2$. Then, by induction, k appears in the functional representation of $(p')^-$, and we are done.

- Finally, if $p = h_i(l)(\ldots, p')$, then $p^- = h_i(l)(\ldots, (p')^-, p')$. As above, $\operatorname{cof} p = \omega_2$ implies $\operatorname{cof} p' = \omega_2$, hence, by induction, k appears in the functional representation of $(p')^-$. To see that p^- is well-defined, observe that k does not appear in p, hence k does not appear in the arguments of $h_i(l)$. Thus $(p')^-$ is bigger than all the arguments of $h_i(l)$ except p'.

The proof when $\operatorname{cof} p = \omega_1$ is similar. ∎

Proposition 3.3.2 *If $q \leq \mathcal{L}(\bar{d})$ is a description below d, $\bar{d} = (d; \bar{S})$, and q is defined relative to measures $\bar{S}$ only, then there is a node $\bar{q}$ in T_d with description q.*

Proof. By induction on $|T_d|$. Let $p = \mathcal{L}(\bar{d})$. Then $\bar{p} = (\mathcal{L}(\bar{d}); \bar{S}, K)$. If $q = p$, we are done since $\bar{p} = \bar{q}$ is in the tree by definition. If $q < p$, then by induction there is a node $\bar{q}$ in $T_{\bar{p}}$ with the description q. However $T_{\bar{p}} \subset T_d$, hence we are done. ∎

Proposition 3.3.3 *For any node $\bar{q}$ in T_d, $p := \sup_{\mathrm{tail}_d(\bar{q})} q$ is another description which appears in some node $\bar{p}$ in the tree T_d, and $\mathrm{head}_d(\bar{q}) = \mathrm{oseq}_d(\bar{p})$.*

Proof. $\bar{p}$ exists by the argument following definition 3.3.4. Clearly, $p \leq d$. Thus, by proposition 3.3.2, to prove that p appears somewhere in the tree T_d it suffices to show that p is a description defined relative to the measures $\bar{S}$. Let $f = h(n-1)$

or $\widehat{h(n-1)}$, and let $k()$ be the first element in the sequence $\mathrm{tail}_d(q)$. We verify this and that $\mathrm{head}_d(\bar{q}) = \mathrm{oseq}_d(\bar{p})$ by induction on the complexity of q. The ground case is trivial: if $q = f(\ldots, p_i, k(), \ldots, \gamma_\tau)$, then $p = h(i)(\ldots, p_i, \gamma_\tau)$ is a description defined relative to S, and $\mathrm{head}_d(q) = \mathrm{oseq}_d(p)$. Next we consider the general case $q = f(q_1, \ldots, q_i, \ldots, q_n)$. If $k()$ appears in $\mathrm{oseq}_d(q_n)$, then $\mathrm{head}_d(q) = \mathrm{head}_d(q_n)$ and $p = \sup_{\mathrm{tail}_d(q_n)} q_n$. Hence by induction $\mathrm{head}_d(q_n) = \mathrm{oseq}_d(p)$, and we are done in this case.

If $k()$ does not appear in q_n, we fix $i < n$ so that $k()$ appears in q_i, and i is the least such integer. Then

$$\mathrm{head}_d(q) = [\mathrm{oseq}_d(q_n)^\frown \mathrm{oseq}_d(q_1)^\frown \ldots {}^\frown \mathrm{oseq}_d(q_{i-1})^\frown \mathrm{head}_d(q_i)]'$$

If $\widehat{q}_i := \sup_{\mathrm{tail}_d(q_i)} q_i < q_n$, then $p = \widetilde{h(i)}(q_1, \ldots, \widehat{q}_i, q_n)$ and

$$\mathrm{oseq}_d(p) = [\mathrm{oseq}_d(q_n)^\frown \mathrm{oseq}_d(q_1)^\frown \ldots {}^\frown \mathrm{oseq}_d(q_{i-1})^\frown \mathrm{oseq}_d(\widehat{q}_i)]'.$$

However, by induction $\mathrm{oseq}_d(\widehat{q}_i) = \mathrm{head}_d(q_i)$, hence we are done in this case too. Finally, if $\widehat{q}_i = q_n$, then $\mathrm{head}_d(q_i) = \mathrm{oseq}_d(\widehat{q}_i) = \mathrm{oseq}_d(q_n)$, $p = h(i-1)(q_1, \ldots, q_{i-1}, q_n)$, and

$$\begin{aligned}
\mathrm{oseq}_d(p) &= [\mathrm{oseq}_d(q_n)^\frown \mathrm{oseq}_d(q_1)^\frown \ldots {}^\frown \mathrm{oseq}_d(q_{i-1})]' \\
&= [\mathrm{oseq}_d(q_n)^\frown \mathrm{oseq}_d(q_1)^\frown \ldots {}^\frown \mathrm{oseq}_d(q_{i-1})^\frown \mathrm{oseq}_d(q_n)]' \\
&= [\mathrm{oseq}_d(q_n)^\frown \mathrm{oseq}_d(q_1)^\frown \ldots {}^\frown \mathrm{oseq}_d(q_{i-1})^\frown \mathrm{head}_d(q_i)]' = \mathrm{head}_d(q)
\end{aligned}$$

∎

Example: $q = h_0(1)(h_1(2)(\alpha_2, k(\gamma_1, \gamma_2), \gamma_3), h_1(1)(\alpha_2, \gamma_3))$. Then $\mathrm{oseq}_d(q) = [\langle \alpha_2, \alpha_2, k(\gamma_1, \gamma_2)\rangle]' = \langle \alpha_2, k(\gamma_1, \gamma_2)\rangle$, $\mathrm{head}_d(q) = \langle \alpha_2 \rangle$, $\mathrm{tail}_d(q) = \langle k(\gamma_1, \gamma_2)\rangle$. Note that

$p = \sup_{k(\cdot_1,\cdot_2)} q = h_0(0)(h_1(1)(\alpha_2, \cdot_3))$, hence $\mathrm{head}_d(q) = \mathrm{oseq}_d(p)$.

Definition 3.3.5 (Level of $\bar{q}$) *Let $\bar{q}$ be in T_d. We put $u = \mathrm{tail}_d(\bar{q})$ and define a linear order $<_u$ on elements of the sequence u as follows*

1. $\alpha_i <_u \alpha_j \iff i < j$;
2. $\alpha_i < \gamma_j$ *for all* α_i *and all* γ_j;
3. $\gamma_i <_u \gamma_j \iff i < j$;
4. $\gamma_i <_u k_j(\cdot_r)$; *for any ordinal* γ_i *and any function* $k_j(\cdot_r)$;
5. $k_i(\cdot_n) <_u k_j(\cdot_m) \iff n < m$ *or* $(n = m$ *and* $i < j)$.

Next define a subsequence w of u as follows: $w(0) = u(0)$. Assume that $w(i)$ has been defined for all $i = 0, \ldots, l$, and $w(l) = u(r)$. If there is $r' > r$ such that $u(r) <_u u(r')$, then let r'' be the least such, and we put $w(l+1) = u(r'')$. If there is no such r', we stop. Let $\#k(\cdot_n) := n$ and let $\#\gamma := 0$, for all ordinals γ. Then we set

$$lev_d(\bar{q}) := \Sigma^{0}_{i=|w|-1} \omega^{\#w(i)}.$$

Example: if $q = h_0(h_1(\gamma, \cdot_1), h_1(\gamma', k(\cdot_1)), \cdot_2)$, then $u = \mathrm{tail}_d(q) = \langle \gamma, k(\cdot_1), \gamma' \rangle$, and $w = \langle \gamma, k(\cdot_1) \rangle$. So, $lev_d(q) = \omega^{\#k(\cdot_1)} + \omega^{\#\gamma} = \omega + 1$.

Lemma 3.3.4 *The set $\{lev_d(\bar{q}) \mid \bar{q} \in T_d\}$ is finite.*

Proof. Consider a node $\bar{q}$ with the functional representation $g(f_1, \ldots, f_l, f_0)$. Let us temporarily call the description $g(f_1, \ldots, f_l, f_0)$ of rank one. We refer to each f_i as a subdescription of rank two, to subdescriptions of f_i, of rank three, and so on. Without loss of generality assume $g = h_i(j)$. Then all the arguments of g are considered in computing $lev_d(\bar{q})$. Because the S measures are fixed (hence there are only finitely many $h_i(j)$, α_i) there is $v < \omega$, such that all of the subdescription of $\bar{q}$ that do not start with $k_i(j)$, for some i, j, have rank less than v. If a (sub)description starts with $k_i(j)$, then by the definition of level, only its largest dot variable, $\cdot_r$, is relevant for computation of $lev_d(\bar{q})$. However there are only finitely many possible dot variables. Finally observe, that although there are infinitely many γ–ordinals, the number of places where they could occur and be considered in computing $lev_d(\bar{q})$ (that is the case when γ is not inside $k()$-function) is finite. ∎

Now we partition nodes of T_d into equivalence classes. We want two nodes to be equivalent if their functional representations are the same except for the indices of the $k()$-functions and the indices of the γ–ordinals. Moreover, we want the indices of corresponding elements of two equivalent functional representations to be ordered in the same way.

In other words, two nodes are equivalent if the functional representation of one of them can be obtained from that of the other just by replacing indices of $k()$ and γ in the corresponding order. Formally we define the equivalence relation, $\sim$, on the nodes of T_d as follows.

Definition 3.3.6 (Equivalence relation on the nodes of T_d)

Let $\bar{p} = (p; S, \bar{M}_p)$ and $\bar{q} = (q; \bar{S}, \bar{M}_q)$ be in T_d. Let $g(f_1, \ldots, f_l, f_0)$ be the functional representation of p, where each f_i is a description defined relative to (S, M_p). Let $g'(f'_1, \ldots, f'_l, f'_0)$ be functional representation of q, where each f'_i is a description *defined relative to* (S, M_q). *We define recursively* $p \sim q \iff$

1. *there are two partial functions,* ϕ *and* φ, *on integers with the properties*

   a. ϕ *and* φ *are order preserving bijections;*

   b. *domain of* ϕ *is the set of indices of* γ*-ordinals from the functional representation of* p *and its range is the set of indices of* γ*-ordinals from the functional representation of* q;

   c. *domain of* φ *is the set of indices of* $k()$*-functions from the functional representation of* p, *its range is the set of indices of* $k()$*-functions from the functional representation of* q;

and 2. $[\exists r = 1, \dots, z\ ,\ p = \cdot_r = q]$ *or* $[\exists r,\ p = \alpha_r = q]$ *or* $[\exists i,\ (p = \gamma_i \ \&\ q = \gamma_{\phi(i)})\ ]$

or $[\exists i, j,\ (g = h_i(j) = g')\ \&\ l = n\ \&\ \forall i = 0, \dots, l\ ,\ (\bar{f}_i \sim \bar{f}'_i)]$

or $[\exists i, j,\ (\ g = k_i(j)\ \&\ g' = k_{\varphi(i)}(j)\ )\ \&\ l = n\ \&\ \forall i = 0, \dots, l\ ,\ (f_i \sim f'_i)]$

It follows from definition 3.3.6 that if $\bar{p} \sim q$, then $lev_d(\bar{p}) = lev_d(\bar{q})$, $\sup_{\mathrm{tail}_d(\bar{p})} p = \sup_{\mathrm{tail}_d(\bar{q})} q$, $\mathrm{head}_d(\bar{p}) = \mathrm{head}_d(\bar{q})$, and $\mathrm{tail}_d(\bar{p}) =^* \mathrm{tail}_d(\bar{q})$, where $u =^* u'$ means [a.] sequences u and u' have the same length, [b.] $u(i) = \alpha_j \iff u'(i) = \alpha_j$, [c.] $u(i) = \gamma \iff u'(i) = \gamma'$, and [d.] $u(i) = k(\cdot_r) \iff u'(i) = k'(\cdot_r)$, for some r.

We group the equivalence classes of nodes into blocks.

Definition 3.3.7 (Block $\mathrm{B}_d(q)$, Depth of a block $\mathrm{depth}(\mathrm{B}_d(q))$) *For every node* q *with* $\mathrm{tail}_d(\bar{q}) = \emptyset$ *we define a block,* $\mathrm{B}_d(q)$, *as the set of all equivalence classes* $[q']$ *with* $\sup_{\mathrm{tail}_d(q')} q' = q$. *We also define the depth of a block by* $\mathrm{depth}(\mathrm{B}_d(q)) := \max\{lev_d(q') \mid [q'] \in \mathrm{B}_d(q)\}$.

Observe that the number of blocks is determined by the number of descriptions q with $\mathrm{tail}_d(q) = \emptyset$, that is to say, q is defined relative to the measures S only. Clearly, the are only finitely many such descriptions. Let us enumerate them in decreasing order: $q_1 > q_2 > \cdots > q_n$. Therefore the number of blocks is also finite and it is n. Note $q_1 = d$. Although there may be infinitely many nodes q_i with the same description q_i, they are all elements of the same equivalence class $[\bar{q}_i] = [\hat{q}_i]$.

It follows from propositions 3.3.3 and 3.3.2 that every node $\bar{q}$ in T_d represents an equivalence class $[q]$ which is in one of these blocks. Now we define the ordinal

$$\xi_d := \omega^{\mathrm{depth}(\mathrm{B}_d(q_n))} + \cdots + \omega^{\mathrm{depth}(\mathrm{B}_d(q_2))} + \omega^{\mathrm{depth}(\mathrm{B}_d(q_1))}$$

which as we shall see determines the cardinality of $(\mathrm{id}; d; \bar{S})(W)$. We will also write $\xi_d = \Sigma_{\bar{q} \leq \bar{d},\ \mathrm{tail}_d(q) = \emptyset} \omega^{\mathrm{depth}(\mathrm{B}_d(q))}$, meaning that if $q > q'$ then in the sum above $\omega^{\mathrm{depth}(\mathrm{B}_d(q))}$ is being added after $\omega^{\mathrm{depth}(\mathrm{B}_d(q'))}$.

Remark 3.3.2 *The last summand in the definition of ξ_d is always 1. That is because the functional interpretation of $\mathcal{L}d$ is defined relative to $\bar{S}$, and therefore* $\mathrm{B}_d(q_1) = \mathrm{B}_d(d) = \{[d]\}$. *Consequently,* $\mathrm{depth}(\mathrm{B}_{q_1}(d)) = 0$ *and* $\omega^{\mathrm{depth}(\mathrm{B}_{q_1}(d))} = 1$.

Proposition 3.3.5 *Fix nodes $\bar{d} = (d; \bar{S})$ and $\bar{p} = (p; \bar{S}, S^*)$ in T_d, with $p = \mathcal{L}(\bar{d})$. Suppose $\bar{q}$ is a node in $T_{\bar{p}}$, a proper subtree of T_d. Then $lev_p(\bar{q}) \leq lev_d(\bar{q})$. Moreover, if* $\mathrm{tail}_d(\bar{q})$ *starts with the function induced by S^* measure, then strict inequality holds, and if otherwise, then* $\sup_{\mathrm{tail}_d(\bar{q})} q = \sup_{\mathrm{tail}_p(\bar{q})} q$.

Proof. Assume $\bar{q} = (q; S, S^*, M_q)$, for some sequence of measures M_q. If $\bar{q} \in T_p \subset T_d$, then we may consider ω sequences of q defined relative to T_p and T_d. Let us fix them:

$u_p := \mathrm{oseq}_{\bar{p}}(q)$ and $u_d := \mathrm{oseq}_d(q)$. We want to analyze the relationship between these two sequences. Let us denote the function induced by S^* in the functional representation of q by k^*. Recall the definition of the o–sequence. In that definition we concatenated recursively o–sequences of the corresponding subdescriptions. We can repeat the same constructions with the only difference that we stop when the subdescription is $k^*(j)(\dots)$, for some j. Suppose that happens t times. Then

$$u_d = [u_1{}^\frown \mathrm{oseq}_d(k^*(j_1)(\dots))^\frown \dots {}^\frown u_2{}^\frown \mathrm{oseq}_d(k^*(j_t)(\dots))^\frown u_{t+1}]'$$

$$u_p = [u_1{}^\frown \mathrm{oseq}_p(k^*(j_1)(\dots))^\frown \dots {}^\frown u_2{}^\frown \mathrm{oseq}_p(k^*(j_t)(\dots))^\frown u_{t+1}]'$$

In other words, the difference between u_d and u_p is determined only by the o–sequences of the subdescriptions starting with an invariant of k^*. Let us fix such a subdescription $s_m = k^*(j_m)(f_1, \dots, f_l, f_0)$, for some $1 \leq m \leq t$. Note that every f_i either starts with an invariant of some k-function (different from k^*), or it is a γ–ordinal, or it is $\cdot_r$, for some r. We will argue that $lev_p(s_m) \leq lev_d(s_m)$.

Suppose $f_0 = \cdot_r$. Then $\mathrm{oseq}_d(s_m) = k^*(\cdot_r)$, hence $lev_d(s_m) = \omega^r$, and $\mathrm{oseq}_p(s_m) = [\mathrm{oseq}_p(f_1)^\frown \dots {}^\frown \mathrm{oseq}_p(f_l)]'$. Because for each $1 \leq i \leq l$, $f_i < \cdot_r$, f_i can not have k-functions with dot variables $\geq \cdot_r$. Thus $lev_p(f_i) < \omega^r$, and hence $lev_p(s_m) < lev_d(s_m)$.

Suppose now f_0 begins with some k–function and has the highest dot variable $\cdot_r$, for some r. Then $\mathrm{oseq}_d(s_m) = k'(\cdot_r)$ and $\mathrm{oseq}_p(s_m) = k'(\cdot_r)^\frown \mathrm{oseq}_p(f_1) \dots {}^\frown \mathrm{oseq}_p(f_l)$. Note that for all $1 \leq i \leq l$, f_i description can not have a k–function with a dot variable higher than $\cdot_r$. If $\mathrm{oseq}_p(f_i)$ contains some $k(\cdot_r)$, then $k \leq k'$ almost everywhere, because $f_i < f_0$. Thus $k(\cdot_r)$ will be canceled when we compute $lev_p(s_m)$. Therefore, $lev_p(s_m) < \omega^r \leq lev_d(s_m)$. Similarly $lev_p(s_m) = lev_d(s_m)$, when f_0 is an ordinal. It follows now that $lev_p(q) \leq lev_d(q)$.

Finally, suppose $\text{tail}_d(\bar{q})$ begins with the symbol b, which is some k function or γ-ordinal. If $b = k^*$, we must have $s_1 = k^*(j_1)(\ldots, \gamma)$. Then, as we argued above, $lev_p(s_1) < lev_d(s_1)$, and therefore $lev_p(q) < lev_d(q)$. If $b \neq k^*$, then both the p-tail and d-tail of $\bar{q}$ begin with b, which corresponds to the most important subdescription in determining the rank of $\bar{q}$. Clearly, $\text{head}_p(\bar{q}) = \text{head}_d(\bar{q})$. Let $f(g_1, \ldots, g_l, g_0)$ be the functional representation of $\bar{q}$. Let i be the least integer so that a subdescription with b appears in g_i. Then $\text{tail}_p(g_i)$ and $\text{tail}_d(g_i)$ both begin with b. By induction we may assume $\sup_{\text{tail}_p(g_i)} g_i = \sup_{\text{tail}_d(g_i)} g_i$, which implies $\sup_{\text{tail}_p(\bar{q})} q = \sup_{\text{tail}_d(\bar{q})} q$. ∎

Lemma 3.3.6 *For a fixed node $\bar{d} = (d, S)$ with all the definitions as above, if $\bar{p}$ is a node in T_d below $\bar{d}$, then $\xi_p \leq \xi_d - 1$.*

Proof. By induction on the rank of $\bar{d}$. Let us fix two nodes $\bar{d} = (d, \bar{S})$ and $\bar{p}$ with description $p = \mathcal{L}(\bar{d})$. Assume that the lemma holds for every $\bar{q}$ extending $\bar{d}$. In particular, it holds for $\bar{p}$. If $\text{cof}\, \bar{p} = \omega$, i.e., the tree T_d does not split at the node d, then the proof is trivial. We consider the case when tree T_d splits at $\bar{d}$. Say, $\text{cof}\, \bar{p} = \omega_2$. We fix two trees: T_d and its proper subtree $T_p = T_{\bar{p}}$. Note that $\bar{p} = (p; \bar{S}, S^*)$: a new measure S^* has been added since $\text{cof}\, \bar{p} = \omega_2$. We shall refer to the function induced by the S^* measure as the k^*-function. A node $\bar{s}$ whose d-tail, $\text{tail}_d(\bar{s})$, begins with k^*-function will be called a *star* node. Otherwise $\bar{s}$ is a *nonstar* node.

Let $B_d(q_1), \ldots, B_d(q_n)$ be all the blocks of T_d where $[q_1] = [d] > [q_2] = [p] > [q_3] > \cdots > [q_n]$ and $\text{tail}_d(q_i) = \emptyset$. Note all the equivalence classes $[q_i]$ with $i > 1$ have representatives in T_p as well.

It is a trivial observation that $\mathrm{tail}_d(s) = \emptyset \Rightarrow \mathrm{tail}_p(s) = \emptyset$. The converse, however, is not true; there could a node s with $\mathrm{tail}_p(\bar{s}) = \emptyset$ while $\mathrm{tail}_d(\bar{s}) \neq \emptyset$. If we fix a d-block, $\mathrm{B}_d(q_i)$ with $i > 1$, then some of the nodes $\bar{q}$ with $[q] \in \mathrm{B}_d(q_i)$ may be such that $\mathrm{tail}_p(q) = \emptyset$, whence a d-block may split into several p-blocks. The idea of the proof then is to show that $\omega^{\mathrm{depth}(\mathrm{B}_d(q_i))}$ is no less than the sum of the ordinals assigned to the corresponding p-blocks.

So let fix q_i for some $i = 2, \ldots, n$, and let $\bar{s_1} \leq \bar{p}$ be a node with $s_1 = q_i$, i.e., $\mathrm{tail}_d(\bar{s_1}) = \emptyset$. Fix a path through the tree T_d containing $\bar{p}$ and $\bar{s_1}$, and let $L = \langle \bar{s_1}, \ldots, \bar{s}_m \rangle$ be the list of all the nodes $\bar{s}$ in that path with the property that $[\bar{s}] \in \mathrm{B}_d(s_1)$. We have enumerated them in decreasing order. The same nodes are present in T_p tree. For every node $\bar{s} \neq \bar{s_1}$ in L, $\mathrm{tail}_d(s) \neq \emptyset$ and $\sup_{\mathrm{tail}_d(s)} s = s_1$ by the definition of a block. For the rest of the proof we identify every node $\bar{s}$ with its descriptions s.

As was pointed out earlier, the number of d-blocks is (strictly) less than the number of p-blocks. Let $s_{i_1}, s_{i_2}, \ldots, s_{i_t}$ be the subsequence of L consisting of nodes with $\mathrm{tail}_p(s_{i_j}) = \emptyset$. For each s_{i_j} there is a p-block corresponding to it. Note that $s_{i_1} = s_1$.

Claim *In the definitions as above*

1. $\Sigma_{j=t}^{1} \omega^{\mathrm{depth}(\mathrm{B}_p(s_{i_j}))} \leq \omega^{\mathrm{depth}(\mathrm{B}_d(s_1))}$.

2. *If* $s_1 = p$, *then* $\Sigma_{j=t}^{1} \omega^{\mathrm{depth}(\mathrm{B}_p(s_{i_j}))} < \omega^{\mathrm{depth}(\mathrm{B}_d(s_1))}$.

Proof of Claim. In proposition 3.3.5, we proved that $lev_p(s) < lev_d(s)$, for every star node s, and if s is a nonstar node in T_p, then $\sup_{\mathrm{tail}_d(s)} s = \sup_{\mathrm{tail}_p(s)} s$. Thus

$\mathrm{depth}(\mathrm{B}_p(s_{i_1})) \leq \mathrm{depth}(\mathrm{B}_d(s_{i_1})) = \mathrm{depth}(\mathrm{B}_d(s_1))$

Now fix any $j = 2, \ldots, t$, and $s \in L$ with $[s] \in \mathrm{B}_p(s_{i_j})$. Then s must be a star node, because otherwise $s_{i_1} = q_i = \sup_{\mathrm{tail}_d(s)} s = \sup_{\mathrm{tail}_p(s)} s$, hence $[s] \in \mathrm{B}_p(s_{i_1})$, a contradiction. So, for every s with $[s] \in \mathrm{B}_p(s_{i_j})$, $lev_p(s) < lev_d(s)$. Consequently, $\mathrm{depth}(\mathrm{B}_p(s_{i_j})) < \mathrm{depth}(\mathrm{B}_d(s_1))$, for all $j = 2, \ldots, t$. The first part of the claim is proven.

Now if $s_1 = p$, then $L = \langle s_1 = p, s_2 = \mathcal{L}p, s_3, \ldots, s_n \rangle$. Clearly, $\mathrm{tail}_d(\bar{p}) = \emptyset$, so p is a nonstar node. Since $\mathrm{cof}\, p = \omega_2$, the node $\bar{p}$ has a new measure, S^*, which by proposition 3.3.1 induces k^* function in the functional representation of $\mathcal{L}p$. Thus $\mathcal{L}p$ is a star node. It follows that $\mathrm{depth}(\mathrm{B}_d(p)) > 0$, and $\mathrm{B}_p(p) = \{[p]\}$. So $\mathrm{depth}(\mathrm{B}_p(p)) = 0$. Hence $\mathrm{depth}(\mathrm{B}_p(p = s_1)) < \mathrm{depth}(\mathrm{B}_d(p = s_1))$, and the second part of the claim is proven as well. ∎ Claim

The result stated in lemma is an immediate consequence of the last claim:

$$\xi_p = \Sigma_{i=n}^{2}[\Sigma_{j=t_i}^{1} \omega^{\mathrm{depth}(\mathrm{B}_p(s_{i_j}))}] \leq \Sigma_{i=n}^{2} \omega^{\mathrm{depth}(\mathrm{B}_d(q_i))} = \xi_d - 1.$$

The proof of the case when $\mathrm{cof}\, p = \omega_1$ is similar but simpler. ∎

Corollary 3.3.7 $(\mathrm{id}; d; \bar{S})(W) \leq \aleph_{\omega+\xi_d}$.

Proof. By lemma 3.3.6, $\sup_{\bar{p}<\bar{d}} \xi_p \leq \xi_d - 1 < \xi_d$. Hence $|T_d| \leq \xi_d$. By the results of [J1], $(\mathrm{id}; d; S)(W) \leq \aleph_{\omega+|T_d|}$. So $(\mathrm{id}; d; \bar{S})(W) \leq \aleph_{\omega+\xi_d}$. ∎

To show that the lower bound of d is also $\aleph_{\omega+\xi_d}$, we recall the following fact.

Theorem 3.3.8 (Martin) *Assume $\kappa \to \kappa^{\kappa}$. Then for any measure ν on κ, the ultrapower $j_\nu(\kappa)$ is a cardinal.*

Proof. See [J1]. ∎

Our stategy for the rest of the proof is to embed the ultrapower of $\underset{\sim}{\delta}^1_3$ by the measure corresponding to ξ_d (made precise below) into $(\mathrm{id}; \bar{d}; S)(W^m)$. We require first some lemmas. The following definition and related results on strong embeddability are due to Jackson.

Definition 3.3.8 (Strong embedding) *Let $(D_i, <_{D_i}), (E_i, <_{E_i})$, $1 \le i \le n$ be well-orderings of length $< \underset{\sim}{\delta}^1_3$, and M_i, N_i measures on D_i, E_i. Let $D = D_1 \oplus \cdots \oplus D_l$, $E = E_1 \oplus \cdots \oplus E_l$, the sum of the order types. We say $(D, \{M_i\})$ strongly embeds into $(E, \{N_i\})$ if there is a measure μ on $\kappa < \underset{\sim}{\delta}^1_3$, and a function H with the following properties:*

1. *$\forall^*_\mu \theta \; H(\theta) = ([\phi_1]_{M_1}, \ldots, [\phi_l]_{M_l})$, where $\phi_i : D_i \to E_i$ is order-preserving.*

2. *For all $A_i \subseteq E_i$, $1 \le i \le n$, of N_i measure 1, $\forall^*_\mu \theta \; \forall i \; \forall^*_{M_i} \alpha \in D_i \; \phi_i(\alpha) \in A_i$.*

If (D_i, M_i) strongly embeds into (E_i, N_i) for all $1 \le i \le n$, then $D = \oplus D_i$ strongly embeds into $E = \oplus E_i$.

Given the ordering $D = D_1 \oplus \cdots \oplus D_l$ and measures M_i, let ν_D denote the measure on l-tuples from $\underset{\sim}{\delta}^1_3$ induced by the weak partition relation on $\underset{\sim}{\delta}^1_3$, functions $f : D \to \underset{\sim}{\delta}^1_3$ of the correct type, and the M_i.

Proposition 3.3.9 (Jackson) *If $(D, \{M_i\})$, $1 \le i \le n$, strongly embeds into $(E, \{N_i\})$, then $j_{\nu_D}(\underset{\sim}{\delta}^1_3) \le j_{\nu_E}(\underset{\sim}{\delta}^1_3)$.*

Proof. Let μ, H witness the strong embeddability. We define an embedding π from $j_{\nu_D}(\underset{\sim}{\delta}^1_3)$ to $j_{\nu_E}(\underset{\sim}{\delta}^1_3)$. Define $\pi([F]_{\nu_D}) = [G]_{\nu_E}$, where for $g = (g_1 \oplus \cdots \oplus g_l) : E \to \underset{\sim}{\delta}^1_3$

of the correct type, $G([g_1]_{E_1}, \ldots, [g_l]_{E_l}) = [\theta \to F([g_1 \circ \phi_1]_{M_1}, \ldots, [g_l \circ \phi_l]_{M_l})]_\mu$, where $H(\theta) = ([\phi_1]_{M_1}, \ldots, [\phi_l]_{M_l})$. Using the properties of H, this is easily well-defined and an embedding. ∎

In the following discussion we will use S_1^i to denote basic measures S_i on $\aleph_{i+1}$.

Proposition 3.3.10 (Jackson) *Let $\mathcal{O}$ be an order type of length $< \delta_3^1$, and ν a measure on $\mathcal{O}$. Let $0 \leq k < l$, $m > 0$. Let D be lexicographic order on $(\alpha_1, \ldots, \alpha_m, \gamma)$ where $\alpha_i < \aleph_{k+1}$, $\gamma \in \mathcal{O}$, and let M be the product measure $M = S_1^k \times \cdots \times S_1^k \times \nu$, or $= W_1^1 \times \cdots \times W_1^1 \times \nu$ if $k = 0$. Let E be lexicographic order on (β, γ), where $\beta < \aleph_{l+1}$ and $\gamma \in \mathcal{O}$, and N the product measure $S_1^l \times \nu$ on E. Then (D, M) strongly embeds into (E, N). Similarly if D is the sum of m copies of $\mathcal{O}$, and $l = 0$ (with measure $W_1^1 \times \nu$).*

Proof. We prove the result for $k \geq 1$, the other cases being similar. Let $\mu = S_1^{l+m}$. Define $H([h]_{W_1^{l+m}}) = [\phi]_M$, where $\phi : D \to E$ is defined as follows. $\phi([f_1]_{W_1^k}, \ldots, [f_m]_{W_1^k}, \gamma) = ([g]_{W_1^l}, \gamma)$, where $g(\delta_1, \ldots, \delta_l) = h(\delta_1, \ldots, \delta_k, f_1(\delta_1, \ldots, \delta_k), \ldots, f_m(\delta_1, \ldots, \delta_k), \delta_{k+1}, \ldots, \delta_l)$. This is easily well-defined, and gives a strong embedding. ∎

By a *basic order type*, we mean $D = D_1 \oplus \cdots \oplus D_l$, where for all $1 \leq i \leq l$, either $D_i = 1$ (i.e., the order type of a single point), or $D_i = \aleph_{k'_m+1} \otimes \aleph_{k'_{m-1}+1} \cdots \otimes \aleph_{k'_1+1}$ (i.e., lexicographic ordering on tuples $(\alpha_1, \ldots, \alpha_m)$ where $\alpha_j < \aleph_{k'_j+1}$, and m depends on i). Let M_i be the product measure $M_i = S_1^{k'_1} \times \cdots \times S^{k'_m}$. We refer to such a D_i as a *sub-basic* order type. To each such D_i we associate an ordinal $c(D)$ as follows.

If $D_i = 1$, $c(D_i) = 1$. If $D_i = \aleph_{k_m+1} \otimes \cdots \otimes \aleph_{k_1+1}$, then $c(D_i) = \omega^{\omega^{k_m}} \cdots \omega^{\omega^{k_2}} \cdot \omega^{\omega^{k_1}}$. Finally, $c(D) = c(D_1) + \cdots + c(D_l)$.

Lemma 3.3.11 *For D a basic order type with corresponding measure ν_D, $j_{\nu_D}(\underset{\sim}{\delta}^1_3) \geq \aleph_{\omega+c(D)+1}$.*

Proof. An easy induction on the length of D, $|D|$, using proposition 3.3.10. For example, the inductive step at $D = \aleph_3$ would be: $j_{\nu_{\aleph_3}}(\underset{\sim}{\delta}^1_3) \geq \sup_n j_{\nu_{(\aleph_2)^n}}(\underset{\sim}{\delta}^1_3) \geq \sup_n \aleph_{\omega+\omega^{\omega \cdot n}+1} = \aleph_{\omega^{\omega^2}}$. Since $\operatorname{cof} j_\nu(\underset{\sim}{\delta}^1_3) > \omega$ for any measure ν, we then have $j_{\nu_{\aleph_3}}(\underset{\sim}{\delta}^1_3) \geq \aleph_{\omega^{\omega^2}+1} = \aleph_{\omega+\omega^{\omega^2}+1}$. ∎

Suppose now $M = M_1 \times \cdots \times M_k = M^0_1 \times \cdots \times M^0_{a_0} \times \cdots \times M^n_1 \times \cdots \times M^n_{a_n}$ is a product measure, where $M^i_j = W^T_1$ if $i = 0$, and $M^i_j = S^i_j$ for $i > 0$. Let $\pi = (p_1, \ldots, p_k)$ be a permutation of k. Let D be the M measure one set of $(\alpha_1, \ldots, \alpha_k) = (\alpha^0_1, \ldots, \alpha^0_{a_0}, \ldots, \alpha^n_1, \ldots, \alpha^n_{a_n})$ such that $\alpha^0_1 < \cdots < \alpha^0_{a_0}$, $\alpha^i_j > \aleph_i$, and $\alpha_i(0) < \alpha_j(0)$ for $i < j$ and $\alpha_i > \aleph_1$. Let $<_D$ be the ordering of D defined by: $(\alpha_1, \ldots, \alpha_k) <_D (\beta_1, \ldots, \beta_k)$ iff $(\alpha_{p_1}, \ldots, \alpha_{p_k}) <^{lex} (\beta_{p_1}, \ldots, \beta_{p_k})$.

We define the *canonical subsequence* π^* of π as follows. $\pi^* = (q_1, \ldots, q_l) = (p_{s_1}, \ldots, p_{s_l})$, where $s_1 = 1$, and $s_{i+1} > s_i$ is least such that $p_{s_{i+1}} > p_{s_i}$. Note that $q_l = k$. To fix notation, let $M_i = M^{r(i)}_{u(i)}$ for $1 \leq i \leq k$. Define N to be the product measure $N = M_{q_1} \times \cdots \times M_{q_l}$, and let E be lexicographic ordering on tuples $(\beta_1, \ldots, \beta_l)$ with $\beta_i < \aleph_{r(q_i)+1}$.

Notice that $(E, <_E)$ is a basic order type.

Lemma 3.3.12 (Jackson) *With $(D, <_D)$, $(E, <_E)$ as above, $(E, <_E)$ strongly embeds into $(D, <_D)$.*

Proof. Let $\mu = M_1 \times \cdots \times M_{q_1-1} \times \prod_{j=q_1}^{k} M_j^+$, where $(W_1^1)^+ = S_1^1$, and $(S_1^r)^+ = S_1^{r+1}$. Fix $\theta = (\theta_1, \ldots, \theta_k)$, and let $h_i :<_{r(i)+1} \rightarrow \aleph_1$ represent θ_i if $r(i) > 0$ and $i \geq q_1$. Set $H(\theta) = [\phi]_N$, where $\phi(\alpha_1, \ldots, \alpha_l) = (\beta_1, \ldots, \beta_k)$ is defined as follows. First, $\beta_1, \ldots, \beta_{q_1-1} = \theta_1, \ldots, \theta_{q_1-1}$. Next, suppose $q_i \leq j < q_{i+1}$. If $r(j) = 0$, set $\beta_j = h_j(\alpha_{q_i})$. If $r(j) > 0$ and $r(q_i) = 0$, set $\beta_j = [g_j]$, where $g_j(\gamma_1, \ldots, \gamma_{r(j)}) = h_j(\alpha_{q_i}, \gamma_1, \ldots, \gamma_{r(j)})$. If $r(q_i) > 0$, set $\beta_j = [g_j]$, where $g_j(\gamma_1, \ldots, \gamma_{r(j)}) = h_j(\gamma_1, \ldots, \gamma_{r(q_i)}, f_i(\gamma_1, \ldots, \gamma_{r(q_i)}), \gamma_{r(q_i)+1}, \ldots, f_i(0)(\gamma_{r(j)}))$, where $[f_i] = \alpha_{q_i}$, and the argument $\gamma_{r(i)}$ of h_j is omitted if $r(q_i) = r(j)$ (this is just to give the correct number of arguments). This is easily checked to be well-defined and a strong embedding. ∎

Remark 3.3.3 *The proof of lemma 3.3.12 also shows if π' is any subsequence of the canonical sequence π^* of π, and E', N' the corresponding order and product measure, then (E', N') strongly embeds into (D, M).*

Lemma 3.3.13 $(\mathrm{id}; d; \tilde{S})(W) \geq \aleph_{\omega+\xi_d}$.

Proof. The idea of the proof is as follows. We will define two measures μ and ν on $\underset{\sim}{\delta}^1_3$, and we will show that $(\mathrm{id}; d; \tilde{S})(W) \geq j_\nu(\underset{\sim}{\delta}^1_3) \geq j_\mu(\underset{\sim}{\delta}^1_3) \geq \aleph_{\omega+\xi_d}$.

We enumerate equivalence classes of nodes $\bar{q}$ with $\mathrm{tail}(q) = \emptyset$: $[q_1], [q_2], \ldots, [q_n]$ in decreasing order, i.e., $q_i > q_j, \forall i < j$. Then the number of d-blocks is also n. Recall that $q_1 = d$, and $\mathrm{B}_d(q_1) = \{[d]\}$.

Claim *For every block $\mathrm{B}_d(q_i)$, $1 \leq i \leq n$, with $\mathrm{depth}(\mathrm{B}_d(q_i)) > 0$, there is a node p_i, with description p_i so that $[p_i] \in \mathrm{B}_d(q_i)$, $\mathrm{lev}_d(p_i) = \mathrm{depth}(\mathrm{B}_d(q_i))$, and p_i has* *functional representation $p_i = h_k(r)(f_1, \ldots, f_r, f_0)$ where $h_k = h_k(r)$. (hence, in particular, p_i has maximal possible length, h_k is the uniform cofinality ω, and* $\operatorname{cof} p_i =$ *ω).*

Proof of Claim. Suppose q_i has functional representation $h_k(l)(f_1, \ldots, f_l, f_0)$ with $h_k = h_k(r)$. Since $\operatorname{depth}(\mathrm{B}_d(q_i)) > 0$, $l < r$ and $\operatorname{cof} q_i = \operatorname{cof} f_0 > \omega$. By lemma 3.3.4, the set $\{lev_d(\bar{p}) \mid [\bar{p}] \in \mathrm{B}_d(q_i)\}$ is finite. Hence there is a node $\bar{p}$ with $[\bar{p}] \in \mathrm{B}_d(q_i)$, and $lev_d(\bar{p}) = \operatorname{depth}(\mathrm{B}_d(q_i))$. Fix such p and let $g(\ldots, f_0)$ be the functional representation of p. Without loss of generality $g = h_k(j)$. If $j = r$ we are done, otherwise, g is a proper invariant of h_k. Then the node $\bar{p'}$ below $\bar{p}$, with $p' = \mathcal{L}(\bar{p})$, has the functional representation $h_k(j+1)(\ldots, \mathcal{L}\bar{f}_0, f_0)$. Clearly $[\bar{p'}] \in \mathrm{B}_d(q_i)$, and $lev_d(\bar{p'}) \geq lev_d(\bar{p}) = \operatorname{depth}(\mathrm{B}_d(q_i))$. If $h_k(j+1)$ is still a proper invariant h_k, we continue lowering p' until we encounter a desired description.

Suppose now $g = \widetilde{h_k(j)}(f_1, \ldots, f_j, f_0)$, and $h_j = h_j(r)$. Since $\operatorname{depth}(\mathrm{B}_d(q_i)) > 0$, $\operatorname{cof} f_j > \omega$. As above, we fix $\bar{p} = (p; \bar{S}, \bar{K}) \in \mathrm{B}_d(q_i)$ with maximum possible level. Then $p = h_k(l)(f_1, \ldots, f_{j-1}, s_j, \ldots, s_l, f_0)$ for some $l \leq r$. Let $u \geq j$ be the largest such that s_j involves one of the $\bar{K}$ measures. We may assume p is chosen to maximize u (subject to having maximum level). If $u = r$, we are done, so assume $u < r$. Let $s' = \sup_{\bar{K}} s_u$. Thus $\operatorname{cof} s' > \omega$. If $s' = f_0$, then we are finished as before. Otherwise, consider $p' = \widetilde{h_k(u+1)}(f_1, \ldots, f_{j-1}, s_j, \ldots, s_u, s', f_0)$. But then $(h_k(u)(f_1, \ldots, f_{j-1}, s_j, \ldots, s_u, f_0); \bar{S}, \bar{L}) \in T_d$, for some sequence of measures $\bar{L}$, and, by proposition 3.3.2, $(p'; \bar{S}, \bar{L}, \bar{M}) \in T_d$ for some $\bar{M}$. Since $\operatorname{cof} p' = \operatorname{cof} s' > \omega$, $\bar{M} \neq \emptyset$. Then $s'' = \mathcal{L}(s'; \bar{S}, \bar{L}, \bar{M}) > s_u$, as it involves a measure from $\bar{M}$, by proposition

3.3.1. Thus $\mathcal{L}(p'; S, L, M) = h_k(u+1)(f_1, \ldots, f_{j-1}, s_j, \ldots, s_u, s'', f_0)$ is in $\mathrm{B}_d(q_i)$, which violates the maximality of u. Contradiction. ∎ Claim

For each block $\mathrm{B}_d(q_i)$, we fix $\bar{p}_i$ as in the claim. For the rest of the argument let us identify each node $\bar{p}_i$ with its description p_i.

Recall that the ordinal $lev_d(p_i)$ was derived from w_i, a subsequence of $\mathrm{tail}_q(p_i)$ (see definition 3.3.5). Let $t_i = \mathrm{tail}_d(p_i)$, $v_i = \mathrm{lh}(t_i) - 1$, and $l_i = \mathrm{lh}(w_i) - 1$. We put $E_i := \aleph_{1+\#w_i(l_i)} \cdot \ldots \cdot \aleph_{1+\#w_i(0)}$ and fix the measure μ_i on $\underset{\sim}{\delta}^1_3$ which concentrates on functions $g' : E_i \to \underset{\sim}{\delta}^1_3$ of the correct type. Notice that the measure μ_i is of the same type as measures considered in lemma 3.3.11, and $\omega^{\#w_i(l_i)} + \omega^{\#w_i(l_i-1)} + \cdots + \omega^{\#w_i(0)} = lev_d(p_i)$. Therefore, by lemma 3.3.11, $j_{\mu_i}(\underset{\sim}{\delta}^1_3) \geq \aleph_{\omega+\omega^{lev_d(p_i)}+1}$. Then we put $D_i := \aleph_{1+\#t_i(v_i)} \cdot \ldots \cdot \aleph_{1+\#t_i(0)}$ and fix the measure ν_i on $\underset{\sim}{\delta}^1_3$ which concentrates on functions $g : D_i \to \underset{\sim}{\delta}^1_3$ of the correct type. By lemma 3.3.12, $j_{\mu_i}(\underset{\sim}{\delta}^1_3)$ can be strongly embedded into $j_{\nu_i}(\underset{\sim}{\delta}^1_3)$. Hence

$$j_{\nu_i}(\underset{\sim}{\delta}^1_3) \;\geq\; j_{\mu_i}(\underset{\sim}{\delta}^1_3) \;\geq\; \aleph_{\omega+\omega^{lev_d(p_i)}+1} \;=\; \aleph_{\omega+\omega^{\mathrm{depth}(\mathrm{B}_d(q_i))}+1}.$$

Now we consider measures $\nu = \nu_n \oplus \cdots \oplus \nu_2$ and $\mu = \mu_n \oplus \cdots \oplus \mu_2$ on $\underset{\sim}{\delta}^1_3$ that concentrate on functions

$$g : D_n \oplus D_{n-1} \oplus \cdots \oplus D_2 \to \underset{\sim}{\delta}^1_3 \quad \text{of the correct type}$$

$$\text{and} \quad f : E_n \oplus E_{n-1} \oplus \cdots \oplus E_2 \to \underset{\sim}{\delta}^1_3 \quad \text{of the correct type},$$

respectively. Using the argument above and induction on n, one can easily show that

$$j_\nu(\underset{\sim}{\delta}^1_3) \geq j_\mu(\underset{\sim}{\delta}^1_3) \geq \aleph_{\omega+\omega^{\mathrm{depth}(\mathrm{B}_d(q_n))}+\cdots+\omega^{\mathrm{depth}(\mathrm{B}_d(q_2))}+1} = \aleph_{\omega+\xi_d}.$$

Finally, we show that there is an embedding $j_\nu(\underset{\sim}{\delta}^1_3) \hookrightarrow (\mathrm{id}; d; S)(W)$. Suppose each p_i is defined with respect to the measures $S = (S_1, \dots, S_{r'})$ and $K_{i_1}, \dots, K_{i_r}$, and let h_j, k_{i_l} be a function of S_j, K_{i_l} type, respectively, for all $1 \le j \le r'$ and all $1 \le l \le r$. We put $\bar{h} = (h_1, \dots, h_{r'})$ and $\bar{k}_i = (k_{i_1}, \dots, k_{i_r})$. Consider the map $\phi := [G]_\nu \mapsto [\theta_G]_W$ where function $G : \underset{\sim}{\delta}^1_3 \to \underset{\sim}{\delta}^1_3$ is defined on ν-measure one set, and ordinal $[\theta_G]_W$ is represented by function

$$\theta_G(f, \bar{h}) \quad = \quad G([g]) \qquad \Longleftrightarrow$$

g is defined by the following cases (for each $i = n, \dots, 2$):

$$g(\bar{k}_n) = (\mathrm{id}; f; p_n; \bar{h}, \bar{k}_n)^*$$

$$\dots$$

$$g(\bar{k}_2) = (\mathrm{id}; f; p_2; \bar{h}, \bar{k}_2)^*$$

Here $(\mathrm{id}; f; p_n; \bar{h}, \bar{k}_n)^*$ stands for the ordinal defined as follows. If $g(f_1, \dots, f_l, f_0)$ is the functional representation of p, then $(\mathrm{id}; f; p_n; \bar{h}, \bar{k}_n)^* := f(g(f'_1, \dots, f'_l, f_0))$ where $f'_i = \mathrm{oseq}_d(f_i)$, if f_l starts with a k function, and $f'_i = f_i$, otherwise. Note that in the definition $(\mathrm{id}; f; p_n; \bar{h}, \bar{k}_n)^*$ no two k functions are ever composed, and only the k functions which are in $\mathrm{tail}_d(p_n)$ are being used.

By our choice of descriptions p_i and because f is continuous, g is uniformly of cofinality ω. Clearly, g is order preserving. For every c. u. b.$C \subset \underset{\sim}{\delta}^1_3$, W–almost all ordinals are represented by functions f into C. Hence $\forall^* f, \bar{h}, \bar{k}_i$, $g(\bar{k}_i) \in C$. So, $[\theta_G]_W$ depends only on $[G]_\nu$. Hence ϕ is well-defined. It is easy to see that ϕ is order preserving: if $G < G'$ ν a.e., there is a c. u. b. set $C \subset \underset{\sim}{\delta}^1_3$ such that $\forall g$ into C, $G([g]_\nu) < G'([g]_\nu)$, hence almost everywhere $\theta_G(\dots) < \theta_{G'}(\dots)$. Finally note that

$\theta_G < (\mathrm{id}; d; S)(W)$, as $p_n < \cdots < p_2 < d$.

It follows that $(\mathrm{id}; d; S)(W) \geq \aleph_{\omega+\xi_4}$. ∎

Thus we have proved

Theorem 3.3.14 *Every ordinal represented by a description and the identity function with respect to a sequence of measures is a cardinal.*

Corollary 3.3.15 *Cardinals below $\underset{\sim}{\delta}^1_5$ in $L(\mathbb{R})$ are exactly the ordinals represented by descriptions and the identity function with respect to sequences of measures.*

Proof. Use theorem 3.3.14 and [J1]. ∎

CHAPTER 4

EXAMPLES

In order to illustrate the formal computation that was done in the previous chapter, in this chapter, we will consider two concrete examples. We will compute the ordinals represented by specific descriptions and the identity function with respect to some measures.

Let us fix the measure W^1 on $\underset{\sim}{\delta}^1_3$ that concentrates on continuous functions $f : \aleph_2 \to \underset{\sim}{\delta}^1_3$. Note, W^1 is the ω_2-cofinal normal measure on $\underset{\sim}{\delta}^1_3$. Let $\bar{S} = (S^{(2,1),0};\ S^{(3,1,2),0};\ \mu^5)$. We fix a description d with the functional representation $f([h_0(0)(h_1(0)(\cdot))]_\mu)$ (we will omit f from functional representations of descriptions below d). Consider the ordinal $\lambda = (\mathrm{id}; d; \bar{S})(W^1)$. Here h_0 and h_1 are functions of the $S^{(2,1),0}$ and $S^{(3,1,2),0}$-types respectively. We will show that $\lambda = \aleph_{\omega^3+\omega(1+5)+1}$.

Repeatedly lowering d, we obtain T_d. Let $(\alpha_1, \ldots, \alpha_5)$ be ordinals representing a function corresponding to μ^5 measure. Then $\mathcal{L}d = h_0(h_1(1)(\alpha_5, \cdot), h_1(0)(\cdot))$ has cofinality ω, and $\mathcal{L}^2 d = \widetilde{h_0}(h_1(1)(\alpha_5, \cdot),\ h_1(0)(\cdot))$ is of cofinality ω_1. Thus, the tree T_d splits at the node $\mathcal{L}d$. Every node $\overline{\mathcal{L}^2 d}$, below $\overline{\mathcal{L}d}$, will have a new measure $K = \mu^n$, for some n. Let $\gamma_1, \ldots, \gamma_n$ denote ordinals corresponding to K. A typical path through the tree T_d has the following sequence of descriptions.

B_1 : $d = h_0(0)(h_1(0)(\cdot))$ $\qquad\qquad \alpha_1, \ldots, \alpha_5$

$$
\begin{array}{lll}
B_2: \quad \mathcal{L}d & h_0(h_1(1)(\alpha_5,\cdot),\ h_1(0)(\cdot)) & \\
B_3: \quad \mathcal{L}^2 d & \widetilde{h_0}(h_1(1)(\alpha_5,\cdot),\ h_1(0)(\cdot)) & \gamma_1,\ldots,\gamma_n \\
4n & \left\{ \begin{array}{l}
h_0(h_1(\alpha_5,\gamma_n,\cdot),\ h_1(0)(\cdot)) \\
\widetilde{h_0}(h_1(\alpha_5,\gamma_n,\cdot),\ h_1(0)(\cdot)) \\
h_0(\widetilde{h_1}(\alpha_5,\gamma_n,\cdot),\ h_1(0)(\cdot)) \\
\widetilde{h_0}(\widetilde{h_1}(\alpha_5,\gamma_n,\cdot),\ h_1(0)(\cdot)) \\
\ldots \\
\widetilde{h_0}(\widetilde{h_1}(\alpha_5,\gamma_1,\cdot),\ h_1(0)(\cdot))
\end{array} \right. & \\
B_4: & h_0(\widetilde{h_1(1)}(\alpha_5,\cdot),\ h_1(0)(\cdot)) & \\
B_5: & \widetilde{h_0}(\widetilde{h_1(1)}(\alpha_5,\cdot),\ h_1(0)(\cdot)) & \\
B_6: & h_0(h_1(1)(\alpha_4,\cdot),\ h_1(0)(\cdot)) & \\
B_7: & \widetilde{h_0}(h_1(1)(\alpha_4,\cdot),\ h_1(0)(\cdot)) & \gamma_{n+1},\ldots,\gamma_{n_2} \\
 & \left\{ \begin{array}{l}
h_0(h_1(\alpha_4,\gamma_{n_2},\cdot),\ h_1(0)(\cdot)) \\
\ldots
\end{array} \right. & \\
 & \ldots & \\
B_M: & \widetilde{h_0}(\alpha_1,\cdot) &
\end{array}
$$

Here $B_i, 1 \leq i \leq M$, marks the beginning of the i^{th} block. There are only finitely many blocks. Say, M many. If $\text{rank}(B_i)$ is the maximum rank of a subtree corresponding to descriptions in B_i, then $\lambda \leq \aleph_{\omega+\text{rank}(B_M)+\cdots+\text{rank}(B_2)+\text{rank}(B_1)}$. Block B_j, $1 < j \leq M$, with the property $\text{rank}B_j + \text{rank}(B_i) = \text{rank}(B_i)$, for some $i < j$, is called *insignificant*. Otherwise, we say B_j is a *significant* block.

We see that the ranks of the blocks B_1, B_2, B_4, B_5, and B_6 are all 1. There are

$4n+1$ many descriptions from the block B_3 in the sequence of descriptions presented above. Since n is arbitrary, the rank of B_3 is $(\sup_{n\in\omega} n) + 1 = \omega + 1$. Similarly, the rank of B_7 is also $\omega + 1$. It follows that blocks B_4, B_5, B_6, and B_M are insignificant for computation of λ. Clearly, it suffices to consider only significant blocks. The list of descriptions corresponding to significant blocks is presented on the next diagram. In this diagram, for each block B_i with $\mathrm{depth}(B_i) > 0$, we fix its representative, a description p_i with $lev(p_i) = \mathrm{depth}(B_i)$.

$$
\begin{array}{lll}
B_1: & \alpha_1, \ldots, \alpha_5 & h_0(0)(h_1(0)(\cdot)) \\
B_2: & & h_0(h_1(1)(\alpha_5, \cdot),\ h_1(0)(\cdot)) \\
B_3: & \gamma_1, \ldots, \gamma_n & \widetilde{h_0}(h_1(1)(\alpha_5, \cdot),\ h_1(0)(\cdot)) \\
 & & \quad h_0(h_1(\alpha_5, \gamma_n, \cdot),\ h_1(0)(\cdot)) \quad p_3 \\
 & & \ldots \\
B_7: & \gamma_{n+1}, \ldots, \gamma_{n_2} & \widetilde{h_0}(h_1(1)(\alpha_4, \cdot),\ h_1(0)(\cdot)) \\
 & & \quad h_0(h_1(\alpha_4, \gamma_{n_2}, \cdot),\ h_1(0)(\cdot)) \quad p_7 \\
 & & \ldots \\
B_{i_1}: & \gamma_{m_1}, \ldots, \gamma_{m_2} & \widetilde{h_0}(h_1(1)(\alpha_3, \cdot),\ h_1(0)(\cdot)) \\
 & & \quad h_0(h_1(\alpha_3, \gamma_{m_2}, \cdot),\ h_1(0)(\cdot)) \quad p_{i_1} \\
 & & \ldots \\
B_{i_2}: & \gamma_{m_3}, \ldots, \gamma_{m_4} & \widetilde{h_0}(h_1(1)(\alpha_2, \cdot),\ h_1(0)(\cdot)) \\
 & & \quad h_0(h_1(\alpha_2, \gamma_{m_4}, \cdot),\ h_1(0)(\cdot)) \quad p_{i_2}
\end{array}
$$

$$
\begin{array}{lll}
B_{i_3}: \quad \gamma_{m_5}, \dots, \gamma_{m_6} & \widetilde{h_0}(h_1(1)(\alpha_1, \cdot),\ h_1(0)(\cdot)) & \\
 & \quad h_0(h_1(\alpha_1, \gamma_{m_6}, \cdot),\ h_1(0)(\cdot)) & p_{i_3} \\
 & \dots & \\
B_{i_4}: \quad \gamma_{m_7}, \dots, \gamma_{m_8} & \widetilde{h_0}(\cdot, h_1(0)(\cdot)) & \\
 & \quad h_0(h_1(1)(\gamma_{m_8}, \cdot),\ h_1(0)(\cdot)) & p_{i_4} \\
 & \dots & \\
B_{i_5}: \quad \gamma_{m_9}, \dots, \gamma_{m_{10}} & h_1(0)(\cdot) & \\
\qquad\quad \gamma_{m_{10}+1}, \dots, \gamma_{m_{11}} & \quad h_0(0)(h_1(1)(\gamma_{m_{10}}, \cdot)) & \\
 & \qquad h_0(h_1(\gamma_{m_{10}}, \gamma_{m_{11}}, \cdot), h_1(1)(\gamma_{m_{11}}, \cdot)) & \\
 & \quad \dots & \\
\qquad\quad \gamma_{m_{12}}, \dots, \gamma_{m_{13}} & \quad h_1(1)(\gamma_{m_{10}}, \ ) & \\
 & \qquad \dots & \\
 & \qquad h_0(\gamma_{m_{13}},\ h_1(\gamma_{m_{10}}, \gamma_{m_{11}}, \cdot)) & p_{i_5}
\end{array}
$$

The reader can verify directly that all other blocks are insignificant. Blocks $B_3, B_7, B_{i_1}, B_{i_2}, B_{i_3}$, and B_{i_4} have the same rank $\omega + 1$. Notice the difference between B_{i_4} and the other five blocks, which are similar in their dependence on ordinal α_1. Obviously, if we had measure μ^k instead of μ^5 in $\bar{S}$, then the number of significant blocks with the rank $\omega + 1$ would have been $k + 1$.

It can be easily seen that the rank of the block B_{i_5} is $\omega^3 + 1$. Hence $\lambda \leq \omega^3 + \omega(1 + 5) + 1 + 1$.

To see that the lower bound for λ is the same, consider the measure ν on $\underset{\sim}{\delta}^1_3$ concentrating on functions: $g : \aleph_1^3 \oplus \aleph_1(1 + 5) \oplus 1 \to \underset{\sim}{\delta}^1_3$ of the uniform cofinality ω.

There is an embedding of $j_\nu(\underset{\sim}{\delta}^1_3) = \aleph_{\omega^3+\omega(1+5)+1+1}$ into λ defined by

$$\theta_G(f; h_0; h_1; \alpha_1, \ldots, \alpha_5) \quad = \quad G([g])$$

$$\Longleftrightarrow$$

g is defined by the following cases:

$$g(\gamma_1, \gamma_2, \gamma_3) = f([h_0(\gamma_3,\ h_1(\gamma_1, \gamma_2, \cdot))]_\mu)$$

$$g(\gamma, 0) = f([h_0(h_1(1)(\gamma, \cdot),\ h_1(0)(\cdot))]_\mu)$$

$$g(\gamma, k) = f([h_0(h_1(1)(\alpha_k, \gamma, \cdot), h_1(0)(\cdot))]_\mu), \quad 1 \le k \le 5$$

$$g(1) = f([h_0(h_1(1)(\alpha_5, \cdot), h_1(0)(\cdot))]_\mu)$$

Hence $\lambda = \aleph_{\omega^3+\omega 6+2}$. As was pointed out earlier, if we had μ^k instead of μ^5 in S, then $\lambda = \aleph_{\omega^3+\omega(1+k)+2}$.

Next we fix the measure W^2 on $\underset{\sim}{\delta}^1_3$ that concentrates on continuous functions $f : \aleph_3 \to \underset{\sim}{\delta}^1_3$. We fix now $\bar{S} = (S^{(3,1,2),0}, S^{(2,1),0})$, description $d = f([h_0(0)(\cdot_2)]_{\mu^2})$, and $\lambda = (\mathrm{id}; d; S)(W^2)$. We will show $\lambda = \aleph_{\omega^{\omega+1}2+1}$. As before, we assume that $h_0 :<^{(3,1,2)}\to \aleph_1$ and $h_1 :<^{(2,1)}\to \aleph_1$ are function of the correct type corresponding to $S^{(3,1,2),0}$ and $S^{(2,1),0}$, respectively.

Repeatedly applying the lowering operator to d, one obtains a tree T_1, where a typical path through T_d contains the following descriptions:

$$B_1: \quad d = \quad h_0(0)(\cdot_2) \qquad\qquad h_1 :<^{(2,1)} \to \aleph_1$$

$$B_2: \quad \mathcal{L}d = \quad h_0(1)(h_1(0)(\cdot_1), \cdot_2) \qquad\qquad k :<^{\pi m_1} \to \aleph_1$$

$$h_0(h_1(0)(\cdot_1),\ h_1(0)(k(0)(\cdot_1)),\ \cdot_2)$$

$$\widetilde{h_0}(h_1(0)(\cdot_1),\ h_1(0)(k(0)(\cdot_1)),\ \cdot_2) \qquad\qquad \gamma_1, \ldots, \gamma_{n_1}$$

$$h_0(h_1(0)(\cdot_1),\ h_1(k(\gamma_{n_1}, \cdot_1), k(0)(\cdot_1)),\ \cdot_2)$$

$$h_0(h_1(0)(\cdot_1),\ h_1(\gamma_{n_1}, k(0)(\cdot_1)),\ \cdot_2)$$

$$\ldots$$

$$B_{i_3}: \quad \widetilde{h_0(1)}(h_1(0)(\cdot_1), \cdot_2) \qquad\qquad \gamma_{n_1+1}, \ldots, \gamma_{n_2}$$

$$h_0(1)(h_1(\gamma_{n_2}, \cdot_1),\ \cdot_2) \qquad\qquad k_2 :<^{\pi m_2} \to \aleph_1$$

$$h_0(h_1(\gamma_{20}, \cdot_1),\ h_1(0)(k_2(\cdot_1)),\ \cdot_2)$$

$$h_0(h_1(\gamma_{20}, \cdot_1),\ h_1(\gamma_{30}, k_2(\cdot_1)),\ \cdot_2)$$

$$\ldots$$

$$B_{i_4}: \quad h_0(1)(\cdot_1, \cdot_2) \qquad\qquad k_3 :<^{\pi m_3} \to \aleph_1$$

$$h_0(1)(\cdot_1,\ k_3(0)(\cdot_1),\ \cdot_2)$$

$$\ldots$$

$$B_{i_5}: \quad h_0(1)(\cdot_1,\ h_1(0)(\cdot_1),\ \cdot_2)$$

$$h_0(1)(\cdot_1,\ k_1(1)(\gamma_{90}, \cdot_1),\ \cdot_2)$$

$$\ldots$$

$$B_{i_6}: \quad \widetilde{h_0(1)}(\cdot_1,\ h_1(0)(\cdot_1),\ \cdot_2)$$

$$\widetilde{h_0(1)}(\cdot_1,\ k_1(1)(\gamma_{100}, \cdot_1),\ \cdot_2)$$

$$\ldots$$

$B_{i_7}: \quad \widetilde{h_0(1)}(\cdot_1, \cdot_2)$ $\qquad\qquad k_4 :<^{\tau} m_1 \to \aleph_1$

$\qquad h_0(h_1(\gamma_{120}, \gamma_{140}),\ h_1(\gamma_{150}, k_4(\cdot_1)),\ \cdot_2)$

$\qquad h_0(1)(\gamma_{140},\ k_4(0)(\cdot_1),\ \cdot_2)$

$\qquad \ldots$

$B_M: \quad \widetilde{h_0(1)}(\gamma_1, \cdot_2)$

Note that in this diagram, if a description q is followed by s it does not necessarily mean that $s = \mathcal{L}\bar{q}$. However, all descriptions are listed in consecutive order. Out of eight blocks shown on the diagram only four are significant: B_1, B_2, B_{i_3} and B_{i_7}. Below we specify ranks of these blocks and fix a representative for each block with rank > 1.

$B_1: \quad h_0(0)(\cdot_2)$

$B_2: \quad h_0(1)(h_1(0)(\cdot_1),\ \cdot_2)$

$\qquad h_0(h_1(0)(\cdot_1),\ k(0)(\cdot_1),\ \cdot_2) \qquad = p_2$

$\qquad$ block depth is $\omega^1 = \omega$ and rank ω^ω

$B_{i_3}: \quad \widetilde{h_0(1)}(h_1(0)(\cdot_1), \cdot_2)$

$\qquad h_0(h_1(\gamma_{20}, \cdot_1),\ k_2(\cdot_1),\ \cdot_2) \qquad = p_{i_3}$

$\qquad$ block depth is $\omega^1 + \omega^0 = \omega + 1$ and rank $\omega^{\omega+1}$

$B_{i_7}: \quad \widetilde{h_0(1)}(\cdot_1, \cdot_2)$

$\qquad h_0(1)(\gamma_{140},\ k_4(0)(\cdot_1),\ \cdot_2) \qquad = p_{i_7}$

$\qquad$ block depth is $\omega^1 + \omega^0 = \omega + 1$ and rank $\omega^{\omega+1}$

The reader can verify directly that there are no more significant blocks. Hence $\lambda \leq \aleph_{\omega+1+\omega^{\omega+1}+\omega+\omega+\omega^{\omega}+\omega^{\omega+1}+\omega^{\omega}+1} = \aleph_{\omega^{\omega+1}2+\omega^{\omega}+1}$.

Finally, let ν be the measure on $\underset{\sim}{\delta}^1_1$ which concentrates on functions $q : \aleph_2\aleph_1 2$

$\aleph_2 \to \underset{\sim}{\delta}^1_3$ of the correct type. Then $\aleph_{\omega^{\omega+1}2+\omega^\omega+1} \leq j_\nu(\underset{\sim}{\delta}^1_3)$, and $j_\nu(\underset{\sim}{\delta}^1_3)$ can be embedded into $\mathcal{L}d$ via

$$\theta_G(f, h_0, h_1) = G([g]) \iff$$

$$g([k]_\mu, \gamma, 0) = f([h_0(\ \gamma,\ k(\cdot_1),\ \cdot_2\ )]_{\mu^2})$$

$$g([k]_\mu, \gamma, 1) = f([h_0(\ h_1(\gamma, \cdot_1),\ k(\cdot_1),\ \cdot_2\ )]_{\mu^2})$$

$$g([k]_\mu) = f([h_0(\ h_1(0)(\cdot_1),\ k(\cdot_1),\ \cdot_2\ )]_{\mu^2})$$

It can be easily verified that the map $[G]_{W^2} \mapsto \theta_G$ is well defined, order preserving, and into $(\mathrm{id}; d; S)(W^2)$. Hence $\lambda = \aleph_{\omega^{\omega+1}2+\omega^\omega+1}$.

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