§1. Ultrapowers of $\aleph_\alpha$ for $\alpha < \omega_1$ by measures on $\aleph_n, n < \omega$.

1.1. Introduction. We use π_m to denote a partial permutation $(m, i_1, \ldots, i_{m-1})$ of $(1, 2, \ldots, m)$. Given any such permutation π, we define the corresponding wellorder, $<^\pi$, on tuples $\bar\alpha := (\alpha_1, \ldots, \alpha_m) \in \aleph_1^m$, with $\alpha_1 < \cdots < \alpha_m$, by

$$\bar\alpha <^\pi \bar\beta \iff \pi(\bar\alpha) <^{lex} \pi(\bar\beta)$$

Definition 1.1. $h :<^{\pi_m} \to \aleph_1$ is of **c-correct type (c-c.t.)**, $0 \le c \le m$, if (1) h is strictly increasing, (2) everywhere discontinuous, and (3) $\forall \bar\alpha$, uniformly cof $h(\bar\alpha) = \alpha_c$, where $\alpha_0 = \omega$.

Sometimes we write c.t. for 0-c.t.

Definition 1.2 (Basic Measure $S^{\pi_m,c}$). If $A \subseteq \aleph_{m+1}$, then

$$A \in S^{\pi_m,c} \iff \exists \text{ c.u.b. } C \subset \aleph_1 \text{ s.t. } \forall h :<^{\pi_m} \to C \text{ of c.t. } , [h]_{\mu^m} \in A$$

To define a *general measure*, fix a function $\mathcal{F}_m : U \to \cup_{m,\pi_m,0\le c\le m}\{S^{\pi_m,c}\}$ with finite domain $U \subset \omega^{\le m}$ such that $\forall u \in U$ with $\mathrm{lh}(u) = k$, $\mathcal{F}_m(u) = S^{\pi_k,c}$. c!
It is assumed that U contains at least one sequence of length m.

On tuples $<u, \lambda> \in (\omega \times \aleph_1)^{\le m}$, where $u \in U$ and $\lambda(1) < \cdots < \lambda(k)$, we define a well-order $<_*$ as follows. Let $u, u' \in U$, $k = \mathrm{lh}(u)$, $k' = \mathrm{lh}(u')$; assume $\mathcal{F}_m(u) = S^{\pi_k,c}$, $\mathcal{F}_m(u') = S^{\pi_{k'},c'}$ with $\pi = (k, i_2, \ldots, i_k)$, $\pi' = (k', j_2, \ldots, j_k')$. Then

$$\begin{aligned}<u,\lambda> \ <_* <u',\lambda'> \quad &\iff \\ &(\lambda(k), u(1), \lambda(i_2), u(2), \ldots, \lambda(i_k), u(k)) <^{lex} \\ &\quad (\lambda'(k'), u'(1), \lambda'(j_2), u'(2), \ldots, \lambda'(j_k), u'(k))\end{aligned}$$

Let $\mathcal{H} :<_* \to \aleph_1$ be order preserving and discontinuous. For each $u \in U$ we define $h_u(\lambda) := \mathcal{H}(u, \lambda)$. We say $\mathcal{H}$ is of the type *matching with* $\mathcal{F}_m$ if $\forall u \in U$

$$h_u \text{ is of } c\text{-c.t.} \qquad \iff \qquad \mathcal{F}_m(u) = S^{\pi_k,c}$$

In other words, the function h_u is one of those on which the measure $\mathcal{F}_m$ concentrates.

Let us enumerate $U = \{u_i\}_{i=1}^p$ in lexicographically increasing order (here $p = |U|$). We set

$$\delta_i := [h_{u_i}]_{\mu^{k_i}}, \quad \text{for all } u_i \in U,\ k_i = \mathrm{lh}(u_i)$$
$$\delta(\mathcal{H}) := \mathrm{ot}(\delta_1, \ldots, \delta_p) \in \aleph_{m+1}$$

Notice that given the function $\mathcal{F}_m$ with the domain U, for each function $\mathcal{H} :<_* \to \aleph_1$ of the type matching with $\mathcal{F}_m$, the ordinal $\delta = \delta(\mathcal{H})$ is determined uniquely. One thinks of the measure $\mathcal{G}_{\mathcal{F}_m}$ being concentrated on such functions $\mathcal{H}$. We summarize all these in the formal

Definition 1.3 (General Measure $\mathcal{G}_{\mathcal{F}_m}$). Let $\mathcal{A} \subseteq \aleph_{m+1}$. We set $\mathcal{A} \in \mathcal{G}_{\mathcal{F}_m}$ iff $\exists$ **c.u.b.** $C \subset \aleph_1$, $\forall \mathcal{H} :<_* \to C$ of the type matching with $\mathcal{F}_m, \delta(\mathcal{H}) \in \mathcal{A}$

Observe that every basic measure $S^{\pi,c} = \mathcal{G}_{\mathcal{F}}$, where the domain of $\mathcal{F}$ is a singleton $\{u\}$ and $\mathcal{F}(u) = S^{\pi,c}$.

A measure $\mathcal{G}_{\mathcal{F}_m}$ (or simply $\mathcal{G}$) can be represented by a finitely splitting tree of height $m+1$. The root of the tree is the empty set, and every maximal path is an element, u, of the domain of $\mathcal{F}_m$ with the terminal node (π_k, c) if and only if $\mathcal{F}(u) = S^{\pi_k, c}$. Moreover, paths are ordered lexicographically, so that the left most maximal path is precisely u_1. An example of such a tree is shown in figure.

1.2. Technical Lemmas.

1.3. Ultrapowers by Basic Measures.

1.4. Ultrapowers by General Measures.

§2. Descriptions and Representation of Cardinals Below $\boldsymbol{\delta}^1_5$.

2.1. Descriptions and their interpretation.

2.2. The Lowering Operator.

2.3. Representation of Cardinals below $\boldsymbol{\delta}^1_5$.

§3. Basic Definitions.

Definition 3.1. A linear ordering $<$ of P is a **wellordering** if every subset of P has a $<$-least element.

Definition 3.2. A **filter** over a set S is a collection F of subsets of S s.t.

1) $S \in F$;
2) if $X \in F$ and $Y \in F$, then $X \cap Y \in F$;
3) if $X \subset Y \subset S$ and $X \in F$, then $Y \in F$.

A filter F is a **proper filter** if

4) $\emptyset \notin F$;

Examples:

(e1) A trivial filter $F = \{S\}$
(e2) The Fréchet filter $F = \{X \subseteq S : S - X \text{ is finite }\}$, where S is infinite.

Definition 3.3. F is a **principle filter** if $\exists X_0 \subset S$ s.t. $\forall X \in F, X_0 \subset X$.

Definition 3.4. A proper filter U over S is an **ultrafilter** if $\forall X \subseteq S$, either $X \in U$ or $S - X \in U$. (But not both since $X \cap (S - X) = \emptyset \notin U$)

Theorem 3.5 (Tarski). *(AC) Every filter can be extended to an ultrafilter.*

Definition 3.6. A relation $\preceq$ on A is a **prewellordering** (of A) iff it is reflexive, transitive, and connected. (I.e., $\forall a, b, c \in A, \quad (a \preceq a); \quad (a \preceq b) \wedge (b \preceq c) \to (a \preceq c); \quad (a \preceq b) \vee (b \preceq a)$.)

Example: If U is an ultrafilter over S, and $\forall f, g : S \to \mathrm{ON}$ we define

$$f \preceq_U g \iff \{i \in S : f(i) \leq g(i)\} \in U,$$

then $\preceq_U$ is a prewellordering.

Definition 3.7. A **prewellordering** is a wellordering of equivalence classes.

Definition 3.8. A set $H \subseteq A$ is **homogeneous for a partition** $\{X_i : i \in I\}$ of A^n, if $\exists i, H^n \subseteq X_i$.

Definition 3.9. $k \to (\lambda)^n_m$ means every $F : k^n \to m$ is constant on H^n for some $H \subseteq k$ with $|H| = \lambda$.

Theorem 3.10 (Ramsey). $\omega \to (\omega)^n_k, \ \forall n, k \in \omega$.
I.e., $\forall F : \omega^n \to \{1, \dots, k\}$, $\exists H \subseteq \omega$ s.t. $|H| = \omega$ and F is constant on H^n.
Equivalently, every partition $\{X_1, \dots, X_k\}$ of ω^n into K pieces has an infinite homogeneous set.

Lemma 3.11. *For all κ and λ,*

(a) $2^\kappa \not\to (\lambda)^2_\kappa$
(b) $2^\kappa \not\to (\kappa^+)^2_2$
(c) $\aleph_1 \not\to (\aleph_1)^2_2$

§4. Definitions and the general picture.

Definition 4.1. For each $n \geq 1$ we define a projective ordinal $\boldsymbol{\delta}^1_n$ as follows

$$\boldsymbol{\delta}^1_n := \sup\{\, \xi \,:\, \xi \text{ is the length of } \boldsymbol{\Delta}^1_n \text{ prewellordering of } \mathbb{R}(=\omega^\omega)\}$$

§5. For all n, $\boldsymbol{\delta}^1_n$ is a cardinal.

§6. For all n, $\boldsymbol{\delta}^1_n$ is a successor cardinal.

a) $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$
b) $\boldsymbol{\delta}^1_{2n+1} = (\kappa_{2n+1})^+$, where κ_{2n+1} is a cardinal of cofinality ω
c) $\boldsymbol{\delta}^1_{2n+1} = \aleph_{e(2n-1)+1}$, where $e(1) = \omega$ and $e(n+1) = \omega^{e(n)}$ (ordinal exponentiation)

Concretely,

$$\begin{array}{ll} \boldsymbol{\delta}^1_1 = \aleph_1 & \boldsymbol{\delta}^1_2 = \aleph_2 \\ \boldsymbol{\delta}^1_3 = \aleph_{\omega+1} & \boldsymbol{\delta}^1_4 = \aleph_{\omega+2} \\ \boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1} & \boldsymbol{\delta}^1_6 = \aleph_{\omega^{\omega^\omega}+2} \\ \boldsymbol{\delta}^1_7 = \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+1} & \boldsymbol{\delta}^1_8 = \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+2} \end{array}$$

§7. For all n, $\boldsymbol{\delta}^1_n$ is a regular cardinal.

§8. For all n, $\boldsymbol{\delta}^1_n$ is a measurable cardinal.

§9. Calculating $\boldsymbol{\delta}^1_n$ for $n \leq 4$.

§10. The closed unbounded measure on $\aleph_1$.

§11. Uniform indiscernibles and the $\aleph_n$'s for $n \leq \omega$.

§12. Back to the real world. What was developed so far was in the context of ZF + DC + AD. Now we review the results under ZF + AC and **Projective Determinacy** only.

Theorem 12.1. *Under ZF + AC and* ***Projective Determinacy*** *only,*

1. $\boldsymbol{\delta}^1_1 = \aleph_1$
2. $\boldsymbol{\delta}^1_2 \leq \aleph_2$
   $[\forall\alpha(\alpha^\#$ *exists* $)]\, \boldsymbol{\delta}^1_2 = u_2$

§13. Infinite exponent partition relations and the singular measures μ_λ.

Definition 13.1. If α, β, γ are ordinals with $\gamma \leq \beta \leq \alpha$, we write $\alpha \to \beta^\gamma$ iff $\forall X \subset \alpha^\gamma \uparrow, \exists H \subset \alpha$ with $o.t.(H) = \beta$ such that either $H^\gamma \uparrow\subset X$ or $H^\gamma \uparrow\subset (\alpha - X)$

Remark 13.2. Assuming ZFC, there is no κ such that $\kappa \to \omega^\omega$

Definition 13.3. $A \subseteq \kappa$ is λ-closed if every increasing λ-sequence from A has its limit in A.

Definition 13.4. If λ, κ are regular cardinals, and $\lambda < \kappa$. μ_λ is defined to be the set of all subsets of κ which contain a λ-closed unbounded set.

Theorem 13.5. *Let λ, κ be regular cardinals, κ uncountable, $\lambda < \kappa$, and $\kappa \to \kappa^{\lambda+\lambda}$. Then μ_λ is a normal measure.*

§14. Countable exponent partition relations for $\boldsymbol{\delta}^1_n$, n odd.

§15. $\omega_1 \to \omega_1^{\omega_1}$.

§16. The Martin-Paris Theorem.

§17. The measure μ_ω on $\boldsymbol{\delta}^1_n$, n odd.

§18. The measure μ_λ, with $\lambda > \omega$, on $\boldsymbol{\delta}^1_n$, n odd.

§19. Countable exponent partition relations on $\boldsymbol{\delta}^1_n$, n even.

§20. The measure μ_ω on $\boldsymbol{\delta}^1_n$, n even.

§21. Some singular cardinals.

§22. Stuff. Throughout the dissertation we work in the context of AD inside of $L(\mathbb{R})$. In this section we give basic definitions and facts needed for what follows, then we define *general* measures and state the theorem which gives the formula for ultrapowers. Throughout this chapter *c.u.b. set* means closed and unbounded subset of $\aleph_1$; *measure* means a σ-complete ultrafilter; μ is the normal measure on $\aleph_1$; and μ^m is its m-fold product measure on $\aleph_1^m$.

§23. Infinity. "The Mathematics of Infinity" by Theodore G. Faticoni (QA 248.F29 2006)

"Roads to Infinity" by John Stillwell. (QA 248.S778 2010)

1. The Diagonal Argument
2. Ordinals
3. Computability and Proof
4. Logic
5. Arithmetic
6. Natural Unprovable Sentences
7. Axioms of Infinity

   "By the 1930s, mathematicians were contemplating sets large enough to model the universe itself: the co-called *inaccessible* and *measurable* cardinals."

   "a better understanding of finite objects depends on a better understanding of infinity"

   (a) Set Theory without Infinity
   (b) Inaccessible Cardinals

      Let $V_0 = \emptyset$ is the empty set, $V_1 = \{\emptyset\}$, V_α is consists of sets built by $< \alpha$ applications of the power set operation. V_λ, for limit λ, is the union of the V_α for $\alpha < \lambda$.

      The least α for which a set X is a subset of V_α is called the *rank* of X. The axiom of foundation implies that every set has a rank.

      A set V_α that is closed under the power set and replacement operations is necessarily of the form V_κ for some cardinal κ. The cardinal κ is then called *inaccessible*, because in some sense V_κ cannot be reached from below. Note, in ZF we cannot prove the existence of an inaccessible $\kappa > \omega$. Otherwise we could use V_κ to model all the axioms of ZF, and hence prove in ZF the sentence Con(ZF), which is impossible by Godel's second incompleteness theorem (ZF includes PA).

      Thus "inaccessible $\kappa > \omega$ exists" is a new kind of axiom of infinity.

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