Cheat Sheet PG$(3, 2)$

January 16, 2021

The projective space PG$(3, 2)$

$q = 2$
$p = 2$
$e = 1$
$n = 3$
Number of points $= 15$
Number of lines $= 35$
Number of lines on a point $= 7$
Number of points on a line $= 3$

The points of PG$(3, 2)$

PG$(3, 2)$ has 15 points:

$P_0 = (1, 0, 0, 0)$	$P_4 = (1, 1, 1, 1)$	$P_8 = (1, 1, 1, 0)$	$P_{12} = (0, 0, 1, 1)$
$P_1 = (0, 1, 0, 0)$	$P_5 = (1, 1, 0, 0)$	$P_9 = (1, 0, 0, 1)$	$P_{13} = (1, 0, 1, 1)$
$P_2 = (0, 0, 1, 0)$	$P_6 = (1, 0, 1, 0)$	$P_{10} = (0, 1, 0, 1)$	$P_{14} = (0, 1, 1, 1)$
$P_3 = (0, 0, 0, 1)$	$P_7 = (0, 1, 1, 0)$	$P_{11} = (1, 1, 0, 1)$	

Normalized from the left:

$P_0 = (1, 0, 0, 0)$	$P_4 = (1, 1, 1, 1)$	$P_8 = (1, 1, 1, 0)$	$P_{12} = (0, 0, 1, 1)$
$P_1 = (0, 1, 0, 0)$	$P_5 = (1, 1, 0, 0)$	$P_9 = (1, 0, 0, 1)$	$P_{13} = (1, 0, 1, 1)$
$P_2 = (0, 0, 1, 0)$	$P_6 = (1, 0, 1, 0)$	$P_{10} = (0, 1, 0, 1)$	$P_{14} = (0, 1, 1, 1)$
$P_3 = (0, 0, 0, 1)$	$P_7 = (0, 1, 1, 0)$	$P_{11} = (1, 1, 0, 1)$	

The lines of $\mathrm{PG}(3,2)$

$\mathrm{PG}(3,2)$ has 35 1-subspaces:

$L_0 = \begin{bmatrix} 1000 \\ 0100 \end{bmatrix} = \mathbf{Pl}(1,0,0,0,0,0)$

$L_1 = \begin{bmatrix} 1000 \\ 0110 \end{bmatrix} = \mathbf{Pl}(1,0,1,0,0,0)$

$L_2 = \begin{bmatrix} 1000 \\ 0101 \end{bmatrix} = \mathbf{Pl}(1,0,0,0,1,0)$

$L_3 = \begin{bmatrix} 1000 \\ 0111 \end{bmatrix} = \mathbf{Pl}(1,0,1,0,1,0)$

$L_4 = \begin{bmatrix} 1000 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,0,1,0,0,0)$

$L_5 = \begin{bmatrix} 1000 \\ 0011 \end{bmatrix} = \mathbf{Pl}(0,0,1,0,1,0)$

$L_6 = \begin{bmatrix} 1000 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,0,0,0,1,0)$

$L_7 = \begin{bmatrix} 1010 \\ 0100 \end{bmatrix} = \mathbf{Pl}(1,0,0,0,0,1)$

$L_8 = \begin{bmatrix} 1010 \\ 0110 \end{bmatrix} = \mathbf{Pl}(1,0,1,0,0,1)$

$L_9 = \begin{bmatrix} 1010 \\ 0101 \end{bmatrix} = \mathbf{Pl}(1,1,0,0,1,1)$

$L_{10} = \begin{bmatrix} 1010 \\ 0111 \end{bmatrix} = \mathbf{Pl}(1,1,1,0,1,1)$

$L_{11} = \begin{bmatrix} 1100 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,0,1,0,0,1)$

$L_{12} = \begin{bmatrix} 1100 \\ 0011 \end{bmatrix} = \mathbf{Pl}(0,0,1,1,1,1)$

$L_{13} = \begin{bmatrix} 1100 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,0,0,1,1,0)$

$L_{14} = \begin{bmatrix} 1001 \\ 0100 \end{bmatrix} = \mathbf{Pl}(1,0,0,1,0,0)$

$L_{15} = \begin{bmatrix} 1001 \\ 0110 \end{bmatrix} = \mathbf{Pl}(1,1,1,1,0,0)$

$L_{16} = \begin{bmatrix} 1001 \\ 0101 \end{bmatrix} = \mathbf{Pl}(1,0,0,1,1,0)$

$L_{17} = \begin{bmatrix} 1001 \\ 0111 \end{bmatrix} = \mathbf{Pl}(1,1,1,1,1,0)$

$L_{18} = \begin{bmatrix} 1001 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,1,1,0,0,0)$

$L_{19} = \begin{bmatrix} 1001 \\ 0011 \end{bmatrix} = \mathbf{Pl}(0,1,1,0,1,0)$

$$L_{20} = \begin{bmatrix} 1010 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,1,0,0,1,0)$$
$$L_{21} = \begin{bmatrix} 1011 \\ 0100 \end{bmatrix} = \mathbf{Pl}(1,0,0,1,0,1)$$
$$L_{22} = \begin{bmatrix} 1011 \\ 0110 \end{bmatrix} = \mathbf{Pl}(1,1,1,1,0,1)$$
$$L_{23} = \begin{bmatrix} 1011 \\ 0101 \end{bmatrix} = \mathbf{Pl}(1,1,0,1,1,1)$$
$$L_{24} = \begin{bmatrix} 1011 \\ 0111 \end{bmatrix} = \mathbf{Pl}(1,0,1,1,1,1)$$
$$L_{25} = \begin{bmatrix} 1101 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,1,1,0,0,1)$$
$$L_{26} = \begin{bmatrix} 1101 \\ 0011 \end{bmatrix} = \mathbf{Pl}(0,1,1,1,1,1)$$
$$L_{27} = \begin{bmatrix} 1110 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,1,0,1,1,0)$$
$$L_{28} = \begin{bmatrix} 0100 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,0,0,0,0,1)$$
$$L_{29} = \begin{bmatrix} 0100 \\ 0011 \end{bmatrix} = \mathbf{Pl}(0,0,0,1,0,1)$$
$$L_{30} = \begin{bmatrix} 0100 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,0,0,1,0,0)$$
$$L_{31} = \begin{bmatrix} 0101 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,1,0,0,0,1)$$
$$L_{32} = \begin{bmatrix} 0101 \\ 0011 \end{bmatrix} = \mathbf{Pl}(0,1,0,1,0,1)$$
$$L_{33} = \begin{bmatrix} 0110 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,1,0,1,0,0)$$
$$L_{34} = \begin{bmatrix} 0010 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,1,0,0,0,0)$$

Lines sorted by Pluecker coordinates

$$0 = \mathbf{Pl}(1,0,0,0,0,0) = L_0 = \begin{bmatrix} 1000 \\ 0100 \end{bmatrix}$$
$$1 = \mathbf{Pl}(0,1,0,0,0,0) = L_{34} = \begin{bmatrix} 0010 \\ 0001 \end{bmatrix}$$
$$2 = \mathbf{Pl}(0,0,1,0,0,0) = L_4 = \begin{bmatrix} 1000 \\ 0010 \end{bmatrix}$$
$$3 = \mathbf{Pl}(0,0,0,1,0,0) = L_{30} = \begin{bmatrix} 0100 \\ 0001 \end{bmatrix}$$
$$4 = \mathbf{Pl}(0,0,0,0,1,0) = L_6 = \begin{bmatrix} 1000 \\ 0001 \end{bmatrix}$$
$$5 = \mathbf{Pl}(0,0,0,0,0,1) = L_{28} = \begin{bmatrix} 0100 \\ 0010 \end{bmatrix}$$

$$6 = \mathbf{Pl}(1,0,1,0,0,0) = L_1 = \begin{bmatrix} 1000 \\ 0110 \end{bmatrix}$$
$$7 = \mathbf{Pl}(0,1,1,0,0,0) = L_{18} = \begin{bmatrix} 1001 \\ 0010 \end{bmatrix}$$
$$8 = \mathbf{Pl}(1,0,0,1,0,0) = L_{14} = \begin{bmatrix} 1001 \\ 0100 \end{bmatrix}$$
$$9 = \mathbf{Pl}(0,1,0,1,0,0) = L_{33} = \begin{bmatrix} 0110 \\ 0001 \end{bmatrix}$$
$$10 = \mathbf{Pl}(1,1,1,1,0,0) = L_{15} = \begin{bmatrix} 1001 \\ 0110 \end{bmatrix}$$
$$11 = \mathbf{Pl}(1,0,0,0,1,0) = L_2 = \begin{bmatrix} 1000 \\ 0101 \end{bmatrix}$$
$$12 = \mathbf{Pl}(0,1,0,0,1,0) = L_{20} = \begin{bmatrix} 1010 \\ 0001 \end{bmatrix}$$
$$13 = \mathbf{Pl}(0,0,1,0,1,0) = L_5 = \begin{bmatrix} 1000 \\ 0011 \end{bmatrix}$$
$$14 = \mathbf{Pl}(1,0,1,0,1,0) = L_3 = \begin{bmatrix} 1000 \\ 0111 \end{bmatrix}$$
$$15 = \mathbf{Pl}(0,1,1,0,1,0) = L_{19} = \begin{bmatrix} 1001 \\ 0011 \end{bmatrix}$$
$$16 = \mathbf{Pl}(0,0,0,1,1,0) = L_{13} = \begin{bmatrix} 1100 \\ 0001 \end{bmatrix}$$
$$17 = \mathbf{Pl}(1,0,0,1,1,0) = L_{16} = \begin{bmatrix} 1001 \\ 0101 \end{bmatrix}$$
$$18 = \mathbf{Pl}(0,1,0,1,1,0) = L_{27} = \begin{bmatrix} 1110 \\ 0001 \end{bmatrix}$$
$$19 = \mathbf{Pl}(1,1,1,1,1,0) = L_{17} = \begin{bmatrix} 1001 \\ 0111 \end{bmatrix}$$
$$20 = \mathbf{Pl}(1,0,0,0,0,1) = L_7 = \begin{bmatrix} 1010 \\ 0100 \end{bmatrix}$$
$$21 = \mathbf{Pl}(0,1,0,0,0,1) = L_{31} = \begin{bmatrix} 0101 \\ 0010 \end{bmatrix}$$
$$22 = \mathbf{Pl}(0,0,1,0,0,1) = L_{11} = \begin{bmatrix} 1100 \\ 0010 \end{bmatrix}$$
$$23 = \mathbf{Pl}(1,0,1,0,0,1) = L_8 = \begin{bmatrix} 1010 \\ 0110 \end{bmatrix}$$
$$24 = \mathbf{Pl}(0,1,1,0,0,1) = L_{25} = \begin{bmatrix} 1101 \\ 0010 \end{bmatrix}$$
$$25 = \mathbf{Pl}(0,0,0,1,0,1) = L_{29} = \begin{bmatrix} 0100 \\ 0011 \end{bmatrix}$$
$$26 = \mathbf{Pl}(1,0,0,1,0,1) = L_{21} = \begin{bmatrix} 1011 \\ 0100 \end{bmatrix}$$

$27 = \mathbf{Pl}(0,1,0,1,0,1) = L_{32} = \begin{bmatrix} 0101 \\ 0011 \end{bmatrix}$

$28 = \mathbf{Pl}(1,1,1,1,0,1) = L_{22} = \begin{bmatrix} 1011 \\ 0110 \end{bmatrix}$

$29 = \mathbf{Pl}(1,1,0,0,1,1) = L_{9} = \begin{bmatrix} 1010 \\ 0101 \end{bmatrix}$

$30 = \mathbf{Pl}(1,1,1,0,1,1) = L_{10} = \begin{bmatrix} 1010 \\ 0111 \end{bmatrix}$

$31 = \mathbf{Pl}(1,1,0,1,1,1) = L_{23} = \begin{bmatrix} 1011 \\ 0101 \end{bmatrix}$

$32 = \mathbf{Pl}(0,0,1,1,1,1) = L_{12} = \begin{bmatrix} 1100 \\ 0011 \end{bmatrix}$

$33 = \mathbf{Pl}(1,0,1,1,1,1) = L_{24} = \begin{bmatrix} 1011 \\ 0111 \end{bmatrix}$

$34 = \mathbf{Pl}(0,1,1,1,1,1) = L_{26} = \begin{bmatrix} 1101 \\ 0011 \end{bmatrix}$

PG$(3,2)$ has the following low weight Pluecker lines:

$L_0 = \begin{bmatrix} 1000 \\ 0100 \end{bmatrix} = \mathbf{Pl}(1,0,0,0,0,0)$

$L_4 = \begin{bmatrix} 1000 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,0,1,0,0,0)$

$L_6 = \begin{bmatrix} 1000 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,0,0,0,1,0)$

$L_{28} = \begin{bmatrix} 0100 \\ 0010 \end{bmatrix} = \mathbf{Pl}(0,0,0,0,0,1)$

$L_{30} = \begin{bmatrix} 0100 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,0,0,1,0,0)$

$L_{34} = \begin{bmatrix} 0010 \\ 0001 \end{bmatrix} = \mathbf{Pl}(0,1,0,0,0,0)$

The planes of $\mathrm{PG}(3,2)$

$\mathrm{PG}(3,2)$ has 15 2-subspaces:

$$L_0 = \begin{bmatrix} 1000 \\ 0100 \\ 0010 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1000 \\ 0100 \\ 0011 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1000 \\ 0100 \\ 0001 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1000 \\ 0101 \\ 0010 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 1000 \\ 0101 \\ 0011 \end{bmatrix}$$

$$L_5 = \begin{bmatrix} 1000 \\ 0110 \\ 0001 \end{bmatrix}$$

$$L_6 = \begin{bmatrix} 1000 \\ 0010 \\ 0001 \end{bmatrix}$$

$$L_7 = \begin{bmatrix} 1001 \\ 0100 \\ 0010 \end{bmatrix}$$

$$L_8 = \begin{bmatrix} 1001 \\ 0100 \\ 0011 \end{bmatrix}$$

$$L_9 = \begin{bmatrix} 1010 \\ 0100 \\ 0001 \end{bmatrix}$$

$$L_{10} = \begin{bmatrix} 1001 \\ 0101 \\ 0010 \end{bmatrix}$$

$$L_{11} = \begin{bmatrix} 1001 \\ 0101 \\ 0011 \end{bmatrix}$$

$$L_{12} = \begin{bmatrix} 1010 \\ 0110 \\ 0001 \end{bmatrix}$$

$$L_{13} = \begin{bmatrix} 1100 \\ 0010 \\ 0001 \end{bmatrix}$$

$$L_{14} = \begin{bmatrix} 0100 \\ 0010 \\ 0001 \end{bmatrix}$$

The polynomial rings associated with $\mathrm{PG}(3,2)$

h	monomial	vector
0	X_0	$(1,0,0,0)$
1	X_1	$(0,1,0,0)$
2	X_2	$(0,0,1,0)$
3	X_3	$(0,0,0,1)$

h	monomial	vector
0	X_0^2	$(2,0,0,0)$
1	X_1^2	$(0,2,0,0)$
2	X_2^2	$(0,0,2,0)$
3	X_3^2	$(0,0,0,2)$
4	X_0X_1	$(1,1,0,0)$
5	X_0X_2	$(1,0,1,0)$
6	X_0X_3	$(1,0,0,1)$
7	X_1X_2	$(0,1,1,0)$
8	X_1X_3	$(0,1,0,1)$
9	X_2X_3	$(0,0,1,1)$

h	monomial	vector
0	X_0^3	$(3,0,0,0)$
1	X_1^3	$(0,3,0,0)$
2	X_2^3	$(0,0,3,0)$
3	X_3^3	$(0,0,0,3)$
4	$X_0^2X_1$	$(2,1,0,0)$
5	$X_0^2X_2$	$(2,0,1,0)$
6	$X_0^2X_3$	$(2,0,0,1)$
7	$X_0X_1^2$	$(1,2,0,0)$
8	$X_1^2X_2$	$(0,2,1,0)$
9	$X_1^2X_3$	$(0,2,0,1)$
10	$X_0X_2^2$	$(1,0,2,0)$
11	$X_1X_2^2$	$(0,1,2,0)$
12	$X_2^2X_3$	$(0,0,2,1)$
13	$X_0X_3^2$	$(1,0,0,2)$
14	$X_1X_3^2$	$(0,1,0,2)$
15	$X_2X_3^2$	$(0,0,1,2)$
16	$X_0X_1X_2$	$(1,1,1,0)$
17	$X_0X_1X_3$	$(1,1,0,1)$
18	$X_0X_2X_3$	$(1,0,1,1)$
19	$X_1X_2X_3$	$(0,1,1,1)$

h	monomial	vector
0	X_0^4	$(4,0,0,0)$
1	X_1^4	$(0,4,0,0)$
2	X_2^4	$(0,0,4,0)$
3	X_3^4	$(0,0,0,4)$
4	$X_0^3X_1$	$(3,1,0,0)$
5	$X_0^3X_2$	$(3,0,1,0)$
6	$X_0^3X_3$	$(3,0,0,1)$
7	$X_0X_1^3$	$(1,3,0,0)$
8	$X_1^3X_2$	$(0,3,1,0)$
9	$X_1^3X_3$	$(0,3,0,1)$
10	$X_0X_2^3$	$(1,0,3,0)$
11	$X_1X_2^3$	$(0,1,3,0)$
12	$X_2^3X_3$	$(0,0,3,1)$
13	$X_0X_3^3$	$(1,0,0,3)$
14	$X_1X_3^3$	$(0,1,0,3)$
15	$X_2X_3^3$	$(0,0,1,3)$
16	$X_0^2X_1^2$	$(2,2,0,0)$
17	$X_0^2X_2^2$	$(2,0,2,0)$
18	$X_0^2X_3^2$	$(2,0,0,2)$
19	$X_1^2X_2^2$	$(0,2,2,0)$
20	$X_1^2X_3^2$	$(0,2,0,2)$
21	$X_2^2X_3^2$	$(0,0,2,2)$
22	$X_0^2X_1X_2$	$(2,1,1,0)$
23	$X_0^2X_1X_3$	$(2,1,0,1)$
24	$X_0^2X_2X_3$	$(2,0,1,1)$

h	monomial	vector
25	$X_0X_1^2X_2$	$(1,2,1,0)$
26	$X_0X_1^2X_3$	$(1,2,0,1)$
27	$X_1^2X_2X_3$	$(0,2,1,1)$
28	$X_0X_1X_2^2$	$(1,1,2,0)$
29	$X_0X_2^2X_3$	$(1,0,2,1)$
30	$X_1X_2^2X_3$	$(0,1,2,1)$
31	$X_0X_1X_3^2$	$(1,1,0,2)$
32	$X_0X_2X_3^2$	$(1,0,1,2)$
33	$X_1X_2X_3^2$	$(0,1,1,2)$
34	$X_0X_1X_2X_3$	$(1,1,1,1)$