Rank-20 over GF(8)

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The equation

The equation of the surface is :

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_2 + X_0^2X_3 + X_0X_1^2 + X_1^2X_2 + X_1^2X_3$$
$$+X_0X_2^2 + X_1X_2^2 + X_2^2X_3 + X_0X_3^2 + X_1X_3^2 + X_2X_3^2 + X_0X_1X_2 + X_0X_1X_3 + X_0X_2X_3 + X_1X_2X_3 = 0$$

(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
The point rank of the equation over GF(8) is 20

General information

Number of lines	9
Number of points	89
Number of singular points	4
Number of Eckardt points	5
Number of double points	6
Number of single points	54
Number of points off lines	24
Number of Hesse planes	0
Number of axes	0
Type of points on lines	9^9
Type of lines on points	3^5, 2^6, 1^{54}, 0^{24}

Singular Points

The surface has 4 singular points:

0 : $P_{20} = \mathbf{P}(1,1,1,0) = \mathbf{P}(1,1,1,0)$
1 : $P_{83} = \mathbf{P}(1,1,0,1) = \mathbf{P}(1,1,0,1)$
2 : $P_{139} = \mathbf{P}(1,0,1,1) = \mathbf{P}(1,0,1,1)$
3 : $P_{146} = \mathbf{P}(0,1,1,1) = \mathbf{P}(0,1,1,1)$

The 9 Lines

The lines and their Pluecker coordinates are:

$$\ell_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{138} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{138} = \mathbf{Pl}(0,0,1,1,1,1)_{1322}$$

$$\ell_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{666} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{666} = \mathbf{Pl}(1,0,1,1,1,1)_{1323}$$

$$\ell_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{81} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{81} = \mathbf{Pl}(1,1,0,0,1,1)_{1217}$$

$$\ell_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{82} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{82} = \mathbf{Pl}(1,1,1,0,1,1)_{1224}$$

$$\ell_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{585} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{585} = \mathbf{Pl}(1,1,1,1,0,0)_{32}$$

$$\ell_5 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{658} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{658} = \mathbf{Pl}(1,1,1,1,0,1)_{874}$$

$$\ell_6 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{593} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{593} = \mathbf{Pl}(1,1,1,1,1,0)_{306}$$

$$\ell_7 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{665} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{665} = \mathbf{Pl}(1,1,0,1,1,1)_{1273}$$

$$\ell_8 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{722} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{722} = \mathbf{Pl}(0,1,1,1,1,1)_{1330}$$

Rank of lines: (138, 666, 81, 82, 585, 658, 593, 665, 722)
Rank of points on Klein quadric: (1322, 1323, 1217, 1224, 32, 874, 306, 1273, 1330)

Eckardt Points

The surface has 5 Eckardt points:
$0 : P_4 = \mathbf{P}(1,1,1,1) = \mathbf{P}(1,1,1,1)$,
$1 : P_{20} = \mathbf{P}(1,1,1,0) = \mathbf{P}(1,1,1,0)$,
$2 : P_{83} = \mathbf{P}(1,1,0,1) = \mathbf{P}(1,1,0,1)$,
$3 : P_{139} = \mathbf{P}(1,0,1,1) = \mathbf{P}(1,0,1,1)$,
$4 : P_{146} = \mathbf{P}(0,1,1,1) = \mathbf{P}(0,1,1,1)$.

Double Points

The surface has 6 Double points:
The double points on the surface are:

$P_5 = (1,1,0,0) = \ell_0 \cap \ell_1$
$P_{138} = (0,0,1,1) = \ell_0 \cap \ell_8$
$P_{12} = (1,0,1,0) = \ell_2 \cap \ell_3$
$P_{82} = (0,1,0,1) = \ell_2 \cap \ell_7$
$P_{19} = (0,1,1,0) = \ell_4 \cap \ell_5$
$P_{75} = (1,0,0,1) = \ell_4 \cap \ell_6$

Single Points

The surface has 54 single points:
The single points on the surface are:

0 : $P_{155} = (2,2,1,1)$ lies on line ℓ_0
1 : $P_{156} = (3,2,1,1)$ lies on line ℓ_1
2 : $P_{163} = (2,3,1,1)$ lies on line ℓ_1
3 : $P_{164} = (3,3,1,1)$ lies on line ℓ_0
4 : $P_{173} = (4,4,1,1)$ lies on line ℓ_0
5 : $P_{174} = (5,4,1,1)$ lies on line ℓ_1
6 : $P_{181} = (4,5,1,1)$ lies on line ℓ_1
7 : $P_{182} = (5,5,1,1)$ lies on line ℓ_0
8 : $P_{191} = (6,6,1,1)$ lies on line ℓ_0
9 : $P_{192} = (7,6,1,1)$ lies on line ℓ_1
10 : $P_{199} = (6,7,1,1)$ lies on line ℓ_1
11 : $P_{200} = (7,7,1,1)$ lies on line ℓ_0
12 : $P_{211} = (2,1,2,1)$ lies on line ℓ_2
13 : $P_{212} = (3,1,2,1)$ lies on line ℓ_3
14 : $P_{218} = (1,2,2,1)$ lies on line ℓ_4
15 : $P_{220} = (3,2,2,1)$ lies on line ℓ_6
16 : $P_{226} = (1,3,2,1)$ lies on line ℓ_5
17 : $P_{227} = (2,3,2,1)$ lies on line ℓ_7
18 : $P_{228} = (3,3,2,1)$ lies on line ℓ_8
19 : $P_{275} = (2,1,3,1)$ lies on line ℓ_3
20 : $P_{276} = (3,1,3,1)$ lies on line ℓ_2
21 : $P_{282} = (1,2,3,1)$ lies on line ℓ_5
22 : $P_{283} = (2,2,3,1)$ lies on line ℓ_8
23 : $P_{284} = (3,2,3,1)$ lies on line ℓ_7
24 : $P_{290} = (1,3,3,1)$ lies on line ℓ_4
25 : $P_{291} = (2,3,3,1)$ lies on line ℓ_6
26 : $P_{341} = (4,1,4,1)$ lies on line ℓ_2
27 : $P_{342} = (5,1,4,1)$ lies on line ℓ_3
28 : $P_{362} = (1,4,4,1)$ lies on line ℓ_4
29 : $P_{366} = (5,4,4,1)$ lies on line ℓ_6
30 : $P_{370} = (1,5,4,1)$ lies on line ℓ_5
31 : $P_{373} = (4,5,4,1)$ lies on line ℓ_7
32 : $P_{374} = (5,5,4,1)$ lies on line ℓ_8
33 : $P_{405} = (4,1,5,1)$ lies on line ℓ_3
34 : $P_{406} = (5,1,5,1)$ lies on line ℓ_2
35 : $P_{426} = (1,4,5,1)$ lies on line ℓ_5
36 : $P_{429} = (4,4,5,1)$ lies on line ℓ_8
37 : $P_{430} = (5,4,5,1)$ lies on line ℓ_7
38 : $P_{434} = (1,5,5,1)$ lies on line ℓ_4
39 : $P_{437} = (4,5,5,1)$ lies on line ℓ_6
40 : $P_{471} = (6,1,6,1)$ lies on line ℓ_2
41 : $P_{472} = (7,1,6,1)$ lies on line ℓ_3
42 : $P_{506} = (1,6,6,1)$ lies on line ℓ_4
43 : $P_{512} = (7,6,6,1)$ lies on line ℓ_6
44 : $P_{514} = (1,7,6,1)$ lies on line ℓ_5
45 : $P_{519} = (6,7,6,1)$ lies on line ℓ_7
46 : $P_{520} = (7,7,6,1)$ lies on line ℓ_8
47 : $P_{535} = (6,1,7,1)$ lies on line ℓ_3
48 : $P_{536} = (7,1,7,1)$ lies on line ℓ_2
49 : $P_{570} = (1,6,7,1)$ lies on line ℓ_5
50 : $P_{575} = (6,6,7,1)$ lies on line ℓ_8
51 : $P_{576} = (7,6,7,1)$ lies on line ℓ_7
52 : $P_{578} = (1,7,7,1)$ lies on line ℓ_4
53 : $P_{583} = (6,7,7,1)$ lies on line ℓ_6

The single points on the surface are:

Points on surface but on no line

The surface has 24 points not on any line:
The points on the surface but not on lines are:

0 : $P_{32} = (5,2,1,0)$
1 : $P_{42} = (7,3,1,0)$
2 : $P_{49} = (6,4,1,0)$
3 : $P_{53} = (2,5,1,0)$
4 : $P_{63} = (4,6,1,0)$
5 : $P_{70} = (3,7,1,0)$
6 : $P_{95} = (5,2,0,1)$
7 : $P_{105} = (7,3,0,1)$
8 : $P_{112} = (6,4,0,1)$
9 : $P_{116} = (2,5,0,1)$
10 : $P_{126} = (4,6,0,1)$
11 : $P_{133} = (3,7,0,1)$
12 : $P_{206} = (5,0,2,1)$
13 : $P_{241} = (0,5,2,1)$
14 : $P_{272} = (7,0,3,1)$
15 : $P_{321} = (0,7,3,1)$
16 : $P_{335} = (6,0,4,1)$
17 : $P_{377} = (0,6,4,1)$
18 : $P_{395} = (2,0,5,1)$
19 : $P_{409} = (0,2,5,1)$
20 : $P_{461} = (4,0,6,1)$
21 : $P_{489} = (0,4,6,1)$
22 : $P_{524} = (3,0,7,1)$
23 : $P_{545} = (0,3,7,1)$

Line Intersection Graph

	0 1 2 3 4 5 6 7 8
0	0 1 1 0 1 0 0 0 1
1	1 0 0 1 0 1 1 1 0
2	1 0 0 1 1 0 0 1 0
3	0 1 1 0 0 1 1 0 1
4	1 0 1 0 0 1 1 0 0
5	0 1 0 1 1 0 0 1 1
6	0 1 0 1 1 0 0 1 1
7	0 1 1 0 0 1 1 0 1
8	1 0 0 1 0 1 1 1 0

Neighbor sets in the line intersection graph:
Line 0 intersects

Line	ℓ_1	ℓ_2	ℓ_4	ℓ_8
in point	P_5	P_4	P_4	P_{138}

Line 1 intersects

Line	ℓ_0	ℓ_3	ℓ_5	ℓ_6	ℓ_7
in point	P_5	P_{146}	P_{139}	P_{146}	P_{139}

Line 2 intersects

Line	ℓ_0	ℓ_3	ℓ_4	ℓ_7
in point	P_4	P_{12}	P_4	P_{82}

Line 3 intersects

Line	ℓ_1	ℓ_2	ℓ_5	ℓ_6	ℓ_8
in point	P_{146}	P_{12}	P_{83}	P_{146}	P_{83}

Line 4 intersects

Line	ℓ_0	ℓ_2	ℓ_5	ℓ_6
in point	P_4	P_4	P_{19}	P_{75}

Line 5 intersects

Line	ℓ_1	ℓ_3	ℓ_4	ℓ_7	ℓ_8
in point	P_{139}	P_{83}	P_{19}	P_{139}	P_{83}

Line 6 intersects

Line	ℓ_1	ℓ_3	ℓ_4	ℓ_7	ℓ_8
in point	P_{146}	P_{146}	P_{75}	P_{20}	P_{20}

Line 7 intersects

Line	ℓ_1	ℓ_2	ℓ_5	ℓ_6	ℓ_8
in point	P_{139}	P_{82}	P_{139}	P_{20}	P_{20}

Line 8 intersects

Line	ℓ_0	ℓ_3	ℓ_5	ℓ_6	ℓ_7
in point	P_{138}	P_{83}	P_{83}	P_{20}	P_{20}

The surface has 89 points:
The points on the surface are:

0 : $P_4 = (1, 1, 1, 1)$
1 : $P_5 = (1, 1, 0, 0)$
2 : $P_{12} = (1, 0, 1, 0)$
3 : $P_{19} = (0, 1, 1, 0)$
4 : $P_{20} = (1, 1, 1, 0)$
5 : $P_{32} = (5, 2, 1, 0)$
6 : $P_{42} = (7, 3, 1, 0)$
7 : $P_{49} = (6, 4, 1, 0)$
8 : $P_{53} = (2, 5, 1, 0)$
9 : $P_{63} = (4, 6, 1, 0)$
10 : $P_{70} = (3, 7, 1, 0)$
11 : $P_{75} = (1, 0, 0, 1)$
12 : $P_{82} = (0, 1, 0, 1)$
13 : $P_{83} = (1, 1, 0, 1)$
14 : $P_{95} = (5, 2, 0, 1)$
15 : $P_{105} = (7, 3, 0, 1)$
16 : $P_{112} = (6, 4, 0, 1)$
17 : $P_{116} = (2, 5, 0, 1)$
18 : $P_{126} = (4, 6, 0, 1)$
19 : $P_{133} = (3, 7, 0, 1)$
20 : $P_{138} = (0, 0, 1, 1)$

21 : $P_{139} = (1, 0, 1, 1)$
22 : $P_{146} = (0, 1, 1, 1)$
23 : $P_{155} = (2, 2, 1, 1)$
24 : $P_{156} = (3, 2, 1, 1)$
25 : $P_{163} = (2, 3, 1, 1)$
26 : $P_{164} = (3, 3, 1, 1)$
27 : $P_{173} = (4, 4, 1, 1)$
28 : $P_{174} = (5, 4, 1, 1)$
29 : $P_{181} = (4, 5, 1, 1)$
30 : $P_{182} = (5, 5, 1, 1)$
31 : $P_{191} = (6, 6, 1, 1)$
32 : $P_{192} = (7, 6, 1, 1)$
33 : $P_{199} = (6, 7, 1, 1)$
34 : $P_{200} = (7, 7, 1, 1)$
35 : $P_{206} = (5, 0, 2, 1)$
36 : $P_{211} = (2, 1, 2, 1)$
37 : $P_{212} = (3, 1, 2, 1)$
38 : $P_{218} = (1, 2, 2, 1)$
39 : $P_{220} = (3, 2, 2, 1)$
40 : $P_{226} = (1, 3, 2, 1)$
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42 : $P_{228} = (3, 3, 2, 1)$
43 : $P_{241} = (0, 5, 2, 1)$
44 : $P_{272} = (7, 0, 3, 1)$
45 : $P_{275} = (2, 1, 3, 1)$
46 : $P_{276} = (3, 1, 3, 1)$
47 : $P_{282} = (1, 2, 3, 1)$
48 : $P_{283} = (2, 2, 3, 1)$
49 : $P_{284} = (3, 2, 3, 1)$
50 : $P_{290} = (1, 3, 3, 1)$
51 : $P_{291} = (2, 3, 3, 1)$
52 : $P_{321} = (0, 7, 3, 1)$
53 : $P_{335} = (6, 0, 4, 1)$
54 : $P_{341} = (4, 1, 4, 1)$
55 : $P_{342} = (5, 1, 4, 1)$
56 : $P_{362} = (1, 4, 4, 1)$
57 : $P_{366} = (5, 4, 4, 1)$
58 : $P_{370} = (1, 5, 4, 1)$
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61 : $P_{377} = (0, 6, 4, 1)$
62 : $P_{395} = (2, 0, 5, 1)$
63 : $P_{405} = (4, 1, 5, 1)$
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66 : $P_{426} = (1, 4, 5, 1)$
67 : $P_{429} = (4, 4, 5, 1)$
68 : $P_{430} = (5, 4, 5, 1)$
69 : $P_{434} = (1, 5, 5, 1)$
70 : $P_{437} = (4, 5, 5, 1)$
71 : $P_{461} = (4, 0, 6, 1)$
72 : $P_{471} = (6, 1, 6, 1)$
73 : $P_{472} = (7, 1, 6, 1)$
74 : $P_{489} = (0, 4, 6, 1)$
75 : $P_{506} = (1, 6, 6, 1)$
76 : $P_{512} = (7, 6, 6, 1)$
77 : $P_{514} = (1, 7, 6, 1)$
78 : $P_{519} = (6, 7, 6, 1)$
79 : $P_{520} = (7, 7, 6, 1)$
80 : $P_{524} = (3, 0, 7, 1)$
81 : $P_{535} = (6, 1, 7, 1)$
82 : $P_{536} = (7, 1, 7, 1)$
83 : $P_{545} = (0, 3, 7, 1)$
84 : $P_{570} = (1, 6, 7, 1)$
85 : $P_{575} = (6, 6, 7, 1)$
86 : $P_{576} = (7, 6, 7, 1)$
87 : $P_{578} = (1, 7, 7, 1)$
88 : $P_{583} = (6, 7, 7, 1)$