

# Rank-65859 over GF(8)

January 15, 2021

## The equation

The equation of the surface is :

$$X_0^2 X_3 + X_1^2 X_2 + X_0 X_1 X_2 = 0$$

( 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 )

The point rank of the equation over GF(8) is 1244172877

## General information

Number of lines	11
Number of points	81
Number of singular points	9
Number of Eckardt points	4
Number of double points	10
Number of single points	67
Number of points off lines	0
Number of Hesse planes	0
Number of axes	0
Type of points on lines	$9^{11}$
Type of lines on points	$3^4, 2^{10}, 1^{67}$

## Singular Points

The surface has 9 singular points:

$$0 : P_2 = \mathbf{P}(0, 0, 1, 0) = \mathbf{P}(0, 0, 1, 0)$$

$$1 : P_3 = \mathbf{P}(0, 0, 0, 1) = \mathbf{P}(0, 0, 0, 1)$$

$$2 : P_{138} = \mathbf{P}(0, 0, 1, 1) = \mathbf{P}(0, 0, 1, 1)$$

$$3 : P_{201} = \mathbf{P}(0, 0, \gamma, 1) = \mathbf{P}(0, 0, 2, 1)$$

$$4 : P_{265} = \mathbf{P}(0, 0, \gamma^5, 1) = \mathbf{P}(0, 0, 3, 1)$$

$$5 : P_{329} = \mathbf{P}(0, 0, \gamma^2, 1) = \mathbf{P}(0, 0, 4, 1)$$

$$6 : P_{393} = \mathbf{P}(0, 0, \gamma^3, 1) = \mathbf{P}(0, 0, 5, 1)$$

$$7 : P_{457} = \mathbf{P}(0, 0, \gamma^6, 1) = \mathbf{P}(0, 0, 6, 1)$$

$$8 : P_{521} = \mathbf{P}(0, 0, \gamma^4, 1) = \mathbf{P}(0, 0, 7, 1)$$

## The 11 Lines

The lines and their Pluecker coordinates are:

$$\begin{aligned}
\ell_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_0 = \mathbf{Pl}(1, 0, 0, 0, 0, 0)_0 \\
\ell_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{64} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{64} = \mathbf{Pl}(0, 0, 1, 0, 0, 0)_2 \\
\ell_2 &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{137} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{137} = \mathbf{Pl}(0, 0, 1, 0, 0, 1)_{664} \\
\ell_3 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4680} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4680} = \mathbf{Pl}(0, 0, 0, 1, 0, 0)_{17} \\
\ell_4 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4744} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4744} = \mathbf{Pl}(0, 1, 0, 0, 0, 0)_1 \\
\ell_5 &= \begin{bmatrix} 1 & \gamma^6 & 0 & 0 \\ 0 & 0 & 1 & \gamma^3 \end{bmatrix}_{507} = \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}_{507} = \mathbf{Pl}(0, 0, 7, 5, 7, 1)_{4436} \\
\ell_6 &= \begin{bmatrix} 1 & \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & \gamma^5 \end{bmatrix}_{359} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}_{359} = \mathbf{Pl}(0, 0, 4, 3, 5, 1)_{3383} \\
\ell_7 &= \begin{bmatrix} 1 & \gamma^5 & 0 & 0 \\ 0 & 0 & 1 & \gamma^6 \end{bmatrix}_{289} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix}_{289} = \mathbf{Pl}(0, 0, 2, 6, 2, 1)_{1841} \\
\ell_8 &= \begin{bmatrix} 1 & \gamma^4 & 0 & 0 \\ 0 & 0 & 1 & \gamma^3 \end{bmatrix}_{580} = \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}_{580} = \mathbf{Pl}(0, 0, 7, 5, 6, 1)_{3932} \\
\ell_9 &= \begin{bmatrix} 1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & \gamma^6 \end{bmatrix}_{216} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix}_{216} = \mathbf{Pl}(0, 0, 2, 6, 3, 1)_{2345} \\
\ell_{10} &= \begin{bmatrix} 1 & \gamma^3 & 0 & 0 \\ 0 & 0 & 1 & \gamma^5 \end{bmatrix}_{432} = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}_{432} = \mathbf{Pl}(0, 0, 4, 3, 4, 1)_{2879}
\end{aligned}$$

Rank of lines: ( 0, 64, 137, 4680, 4744, 507, 359, 289, 580, 216, 432 )

Rank of points on Klein quadric: ( 0, 2, 664, 17, 1, 4436, 3383, 1841, 3932, 2345, 2879 )

## Eckardt Points

The surface has 4 Eckardt points:

$$\begin{aligned}
0 : P_2 &= \mathbf{P}(0, 0, 1, 0) = \mathbf{P}(0, 0, 1, 0), \\
1 : P_{201} &= \mathbf{P}(0, 0, \gamma, 1) = \mathbf{P}(0, 0, 2, 1), \\
2 : P_{329} &= \mathbf{P}(0, 0, \gamma^2, 1) = \mathbf{P}(0, 0, 4, 1), \\
3 : P_{521} &= \mathbf{P}(0, 0, \gamma^4, 1) = \mathbf{P}(0, 0, 7, 1).
\end{aligned}$$

## Double Points

The surface has 10 Double points:

The double points on the surface are:

$$\begin{aligned}
P_0 &= (1, 0, 0, 0) = \ell_0 \cap \ell_1 & P_6 &= (2, 1, 0, 0) = \ell_0 \cap \ell_5 \\
P_5 &= (1, 1, 0, 0) = \ell_0 \cap \ell_2 & P_7 &= (3, 1, 0, 0) = \ell_0 \cap \ell_6 \\
P_1 &= (0, 1, 0, 0) = \ell_0 \cap \ell_3 & P_8 &= (4, 1, 0, 0) = \ell_0 \cap \ell_7
\end{aligned}$$

$$\begin{aligned}
P_9 &= (5, 1, 0, 0) = \ell_0 \cap \ell_8 \\
P_{10} &= (6, 1, 0, 0) = \ell_0 \cap \ell_9 \\
P_{11} &= (7, 1, 0, 0) = \ell_0 \cap \ell_{10}
\end{aligned}$$

$$P_3 = (0, 0, 0, 1) = \ell_3 \cap \ell_4$$

### Single Points

The surface has 67 single points:  
The single points on the surface are:

- |   |  |
|---|--|
| 0 : $P_{12} = (1, 0, 1, 0)$ lies on line $\ell_1$   | 34 : $P_{262} = (5, 7, 2, 1)$ lies on line $\ell_9$    |
| 1 : $P_{13} = (2, 0, 1, 0)$ lies on line $\ell_1$   | 35 : $P_{263} = (6, 7, 2, 1)$ lies on line $\ell_7$    |
| 2 : $P_{14} = (3, 0, 1, 0)$ lies on line $\ell_1$   | 36 : $P_{265} = (0, 0, 3, 1)$ lies on line $\ell_4$    |
| 3 : $P_{15} = (4, 0, 1, 0)$ lies on line $\ell_1$   | 37 : $P_{340} = (3, 1, 4, 1)$ lies on line $\ell_6$    |
| 4 : $P_{16} = (5, 0, 1, 0)$ lies on line $\ell_1$   | 38 : $P_{344} = (7, 1, 4, 1)$ lies on line $\ell_{10}$ |
| 5 : $P_{17} = (6, 0, 1, 0)$ lies on line $\ell_1$   | 39 : $P_{348} = (3, 2, 4, 1)$ lies on line $\ell_{10}$ |
| 6 : $P_{18} = (7, 0, 1, 0)$ lies on line $\ell_1$   | 40 : $P_{351} = (6, 2, 4, 1)$ lies on line $\ell_6$    |
| 7 : $P_{20} = (1, 1, 1, 0)$ lies on line $\ell_2$   | 41 : $P_{357} = (4, 3, 4, 1)$ lies on line $\ell_{10}$ |
| 8 : $P_{29} = (2, 2, 1, 0)$ lies on line $\ell_2$   | 42 : $P_{358} = (5, 3, 4, 1)$ lies on line $\ell_6$    |
| 9 : $P_{38} = (3, 3, 1, 0)$ lies on line $\ell_2$   | 43 : $P_{362} = (1, 4, 4, 1)$ lies on line $\ell_6$    |
| 10 : $P_{47} = (4, 4, 1, 0)$ lies on line $\ell_2$  | 44 : $P_{367} = (6, 4, 4, 1)$ lies on line $\ell_{10}$ |
| 11 : $P_{56} = (5, 5, 1, 0)$ lies on line $\ell_2$  | 45 : $P_{370} = (1, 5, 4, 1)$ lies on line $\ell_{10}$ |
| 12 : $P_{65} = (6, 6, 1, 0)$ lies on line $\ell_2$  | 46 : $P_{371} = (2, 5, 4, 1)$ lies on line $\ell_6$    |
| 13 : $P_{74} = (7, 7, 1, 0)$ lies on line $\ell_2$  | 47 : $P_{382} = (5, 6, 4, 1)$ lies on line $\ell_{10}$ |
| 14 : $P_{82} = (0, 1, 0, 1)$ lies on line $\ell_3$  | 48 : $P_{384} = (7, 6, 4, 1)$ lies on line $\ell_6$    |
| 15 : $P_{90} = (0, 2, 0, 1)$ lies on line $\ell_3$  | 49 : $P_{387} = (2, 7, 4, 1)$ lies on line $\ell_{10}$ |
| 16 : $P_{98} = (0, 3, 0, 1)$ lies on line $\ell_3$  | 50 : $P_{389} = (4, 7, 4, 1)$ lies on line $\ell_6$    |
| 17 : $P_{106} = (0, 4, 0, 1)$ lies on line $\ell_3$ | 51 : $P_{393} = (0, 0, 5, 1)$ lies on line $\ell_4$    |
| 18 : $P_{114} = (0, 5, 0, 1)$ lies on line $\ell_3$ | 52 : $P_{457} = (0, 0, 6, 1)$ lies on line $\ell_4$    |
| 19 : $P_{122} = (0, 6, 0, 1)$ lies on line $\ell_3$ | 53 : $P_{531} = (2, 1, 7, 1)$ lies on line $\ell_5$    |
| 20 : $P_{130} = (0, 7, 0, 1)$ lies on line $\ell_3$ | 54 : $P_{534} = (5, 1, 7, 1)$ lies on line $\ell_8$    |
| 21 : $P_{138} = (0, 0, 1, 1)$ lies on line $\ell_4$ | 55 : $P_{541} = (4, 2, 7, 1)$ lies on line $\ell_5$    |
| 22 : $P_{213} = (4, 1, 2, 1)$ lies on line $\ell_7$ | 56 : $P_{544} = (7, 2, 7, 1)$ lies on line $\ell_8$    |
| 23 : $P_{215} = (6, 1, 2, 1)$ lies on line $\ell_9$ | 57 : $P_{547} = (2, 3, 7, 1)$ lies on line $\ell_8$    |
| 24 : $P_{218} = (1, 2, 2, 1)$ lies on line $\ell_9$ | 58 : $P_{551} = (6, 3, 7, 1)$ lies on line $\ell_5$    |
| 25 : $P_{222} = (5, 2, 2, 1)$ lies on line $\ell_7$ | 59 : $P_{556} = (3, 4, 7, 1)$ lies on line $\ell_8$    |
| 26 : $P_{226} = (1, 3, 2, 1)$ lies on line $\ell_7$ | 60 : $P_{558} = (5, 4, 7, 1)$ lies on line $\ell_5$    |
| 27 : $P_{232} = (7, 3, 2, 1)$ lies on line $\ell_9$ | 61 : $P_{567} = (6, 5, 7, 1)$ lies on line $\ell_8$    |
| 28 : $P_{235} = (2, 4, 2, 1)$ lies on line $\ell_9$ | 62 : $P_{568} = (7, 5, 7, 1)$ lies on line $\ell_5$    |
| 29 : $P_{240} = (7, 4, 2, 1)$ lies on line $\ell_7$ | 63 : $P_{570} = (1, 6, 7, 1)$ lies on line $\ell_5$    |
| 30 : $P_{244} = (3, 5, 2, 1)$ lies on line $\ell_7$ | 64 : $P_{573} = (4, 6, 7, 1)$ lies on line $\ell_8$    |
| 31 : $P_{245} = (4, 5, 2, 1)$ lies on line $\ell_9$ | 65 : $P_{578} = (1, 7, 7, 1)$ lies on line $\ell_8$    |
| 32 : $P_{251} = (2, 6, 2, 1)$ lies on line $\ell_7$ | 66 : $P_{580} = (3, 7, 7, 1)$ lies on line $\ell_5$    |
| 33 : $P_{252} = (3, 6, 2, 1)$ lies on line $\ell_9$ |  |

The single points on the surface are:

### Points on surface but on no line

The surface has 0 points not on any line:  
The points on the surface but not on lines are:

## Line Intersection Graph

	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	0	1	1	1	1	1	1
1	1	0	1	0	1	0	0	0	0	0	0
2	1	1	0	0	1	0	0	0	0	0	0
3	1	0	0	0	1	0	0	0	0	0	0
4	0	1	1	1	0	1	1	1	1	1	1
5	1	0	0	0	1	0	0	0	1	0	0
6	1	0	0	0	1	0	0	0	0	0	1
7	1	0	0	0	1	0	0	0	0	1	0
8	1	0	0	0	1	1	0	0	0	0	0
9	1	0	0	0	1	0	0	1	0	0	0
10	1	0	0	0	1	0	1	0	0	0	0

Neighbor sets in the line intersection graph:

Line 0 intersects

Line	$\ell_1$	$\ell_2$	$\ell_3$	$\ell_5$	$\ell_6$	$\ell_7$	$\ell_8$	$\ell_9$	$\ell_{10}$
in point	$P_0$	$P_5$	$P_1$	$P_6$	$P_7$	$P_8$	$P_9$	$P_{10}$	$P_{11}$

Line 1 intersects

Line	$\ell_0$	$\ell_2$	$\ell_4$
in point	$P_0$	$P_2$	$P_2$

Line 2 intersects

Line	$\ell_0$	$\ell_1$	$\ell_4$
in point	$P_5$	$P_2$	$P_2$

Line 3 intersects

Line	$\ell_0$	$\ell_4$
in point	$P_1$	$P_3$

Line 4 intersects

Line	$\ell_1$	$\ell_2$	$\ell_3$	$\ell_5$	$\ell_6$	$\ell_7$	$\ell_8$	$\ell_9$	$\ell_{10}$
in point	$P_2$	$P_2$	$P_3$	$P_{521}$	$P_{329}$	$P_{201}$	$P_{521}$	$P_{201}$	$P_{329}$

Line 5 intersects

Line	$\ell_0$	$\ell_4$	$\ell_8$
in point	$P_6$	$P_{521}$	$P_{521}$

Line 6 intersects

Line	$\ell_0$	$\ell_4$	$\ell_{10}$
in point	$P_7$	$P_{329}$	$P_{329}$

Line 7 intersects

Line	$\ell_0$	$\ell_4$	$\ell_9$
in point	$P_8$	$P_{201}$	$P_{201}$

Line 8 intersects

Line	$\ell_0$	$\ell_4$	$\ell_5$
in point	$P_9$	$P_{521}$	$P_{521}$

Line 9 intersects

Line	$\ell_0$	$\ell_4$	$\ell_7$
in point	$P_{10}$	$P_{201}$	$P_{201}$

Line 10 intersects

Line	$\ell_0$	$\ell_4$	$\ell_6$
in point	$P_{11}$	$P_{329}$	$P_{329}$

The surface has 81 points:

The points on the surface are:

0 : $P_0 = (1, 0, 0, 0)$	28 : $P_{106} = (0, 4, 0, 1)$	56 : $P_{362} = (1, 4, 4, 1)$
1 : $P_1 = (0, 1, 0, 0)$	29 : $P_{114} = (0, 5, 0, 1)$	57 : $P_{367} = (6, 4, 4, 1)$
2 : $P_2 = (0, 0, 1, 0)$	30 : $P_{122} = (0, 6, 0, 1)$	58 : $P_{370} = (1, 5, 4, 1)$
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4 : $P_5 = (1, 1, 0, 0)$	32 : $P_{138} = (0, 0, 1, 1)$	60 : $P_{382} = (5, 6, 4, 1)$
5 : $P_6 = (2, 1, 0, 0)$	33 : $P_{201} = (0, 0, 2, 1)$	61 : $P_{384} = (7, 6, 4, 1)$
6 : $P_7 = (3, 1, 0, 0)$	34 : $P_{213} = (4, 1, 2, 1)$	62 : $P_{387} = (2, 7, 4, 1)$
7 : $P_8 = (4, 1, 0, 0)$	35 : $P_{215} = (6, 1, 2, 1)$	63 : $P_{389} = (4, 7, 4, 1)$
8 : $P_9 = (5, 1, 0, 0)$	36 : $P_{218} = (1, 2, 2, 1)$	64 : $P_{393} = (0, 0, 5, 1)$
9 : $P_{10} = (6, 1, 0, 0)$	37 : $P_{222} = (5, 2, 2, 1)$	65 : $P_{457} = (0, 0, 6, 1)$
10 : $P_{11} = (7, 1, 0, 0)$	38 : $P_{226} = (1, 3, 2, 1)$	66 : $P_{521} = (0, 0, 7, 1)$
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16 : $P_{17} = (6, 0, 1, 0)$	44 : $P_{251} = (2, 6, 2, 1)$	72 : $P_{551} = (6, 3, 7, 1)$
17 : $P_{18} = (7, 0, 1, 0)$	45 : $P_{252} = (3, 6, 2, 1)$	73 : $P_{556} = (3, 4, 7, 1)$
18 : $P_{20} = (1, 1, 1, 0)$	46 : $P_{262} = (5, 7, 2, 1)$	74 : $P_{558} = (5, 4, 7, 1)$
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25 : $P_{82} = (0, 1, 0, 1)$	53 : $P_{351} = (6, 2, 4, 1)$	
26 : $P_{90} = (0, 2, 0, 1)$	54 : $P_{357} = (4, 3, 4, 1)$	
27 : $P_{98} = (0, 3, 0, 1)$	55 : $P_{358} = (5, 3, 4, 1)$	