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ON CURVES OF MINIMAL LENGTH WITH A CONSTRAINT ON AVERAGE CURVATURE, AND WITH PRESCRIBED INITIAL AND TERMINAL POSITIONS AND TANGENTS.*[1]

By L. E. Dubins.

1. Introduction and summary. Let a particle pursue a continuously differentiable path from an initial point u to a terminal point v. Suppose that its speed is unity and suppose that its velocity vectors at u and v are U and V respectively. We are interested in a path of minimal length for the particle. It is easy to see that there exist u, U, v and V for which no path of minimal length exists. We need some further reasonable restriction. At first, it seems natural to require that the path possess a curvature everywhere, and to prescribe that its radius of curvature be everywhere greater than or equal to a fixed number R. But again there exist (u, U, v, V, R) for which no path of minimal length exists (Proposition 14). The difficulty is that we have imposed too severe a restriction. In order to arrive at the correct restriction to impose, we observe that if X is a curve in real n-dimensional Euclidean space, parameterized by arc length, for which $X''(s)$ exists everywhere, then the curvature, $\| X''(s) \|$, is less than or equal to R^{-1} everywhere, if and only if,

$$(1) \qquad \| X'(s_1) - X'(s_2) \| \leqq R^{-1} | s_1 - s_2 |,$$

for all s_1 and s_2 in the interval of definition of X. By the *average curvature* of X in the interval $[s_1, s_2]$ we mean the left side of (1) divided by $| s_1 - s_2 |$. We say that a curve X in real Euclidean n-space parameterized by arc length has *average curvature always less than or equal to* R^{-1} provided that its first derivative X' exists everywhere and satisfies the Lipschitz condition (1). We inquire, for fixed vectors u, U, v, V in real n-dimensional Euclidean space, E_n, and a fixed positive number R, as to the existence and nature of a path of minimal length among the curves in E_n, of average curvature everywhere less than or equal to R^{-1}. Now we find that paths of minimal length necessarily exist. We call such a path an *R-geodesic.* The purpose of this paper

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is to prove Theorem 1, which implies that for $n=2$, an R-geodesic is necessarily a continuously differentiable curve which consists of not more than three pieces, each of which is either a straight line segment or an arc of a circle of radius R. Furthermore, the corollary to Theorem 1 implies that three is the least integer for which this is true. The nature of R-geodesics for $n \geqq 3$ is open.

2. Existence of *R*-geodesics. Let u, v, U and V be vectors in real n-dimensional Euclidean space, E_n. Let $\| U \| = \| V \| = 1$ and let $R > 0$. Let $C = C(n, u, U, v, V, R)$ be the collection of all curves X defined on a closed interval $[0, L]$, where $L = L(X)$ varies with X, such that $X(s) \in E_n$ for $0 \leqq s \leqq L$; $\| X'(s) \| \equiv 1$; the average curvature of X is everywhere less than or equal to R^{-1}; $X(0) = u$, $X'(0) = U$, $X(L) = v$ and $X'(L) = V$.

PROPOSITION 1. *For any n, u, U, v, V, and R, there exists an X in $C = C(n, u, U, v, V, R)$ of minimal length.*

Proof. We omit the verification that C is non empty. Let $X_1 \in C$. Let d_1 be the length of X_1 and let d be the infimum of lengths of all curves in C. Clearly $d \leqq d_1$. There exists a sequence $X_n \in C$ such that the length d_n of X_n is monotonely decreasing to d. Since $\| X_n'(s) \| \equiv 1$, it follows that the X_n' are a uniformly bounded family of functions. Since $\| X_n'(s_1) - X_n'(s_2) \| \leqq R^{-1} | s_1 - s_2 |$ for all s_1 and s_2 in the interval $[0, d]$, it follows that the X_n' also form an equicontinuous family on $[0, d]$. Therefore by Ascoli's theorem, [1], there is a subsequence of the X_n whose derivatives converge uniformly on $[0, d]$ to a function Y. For convenience, we assume that X_n is itself such a sequence. It is easy to see that $Y(0) = U$ and $\| Y(s_1) - Y(s_2) \| \leqq R^{-1} | s_1 - s_2 |$ for all s_1 and s_2 in $[0, d]$. Since

$$X_n(s) = X_n(0) + \int_0^s X_n'(t)\, dt = u + \int_0^s X_n'(t)\, dt$$

it follows that for $0 \leqq s \leqq d$,

$$\| X_n(s) - X_m(s) \| \leqq \int_0^s \| X_n'(t) - X_m'(t) \| \, dt \leqq d \cdot \sup \| X_n'(t) - X_m'(t) \|,$$

where the sup is taken over t in $[0, d]$. Therefore X_n converges uniformly in $[0, d]$ to a function X. Since X_n' converges uniformly to Y and X_n converges to X it follows that $X' = (\lim X_n)' = \lim X_n' = Y$. It is elementary to complete the proof of the theorem by showing that $X(d) = v$ and $X'(d) = V$. Namely,

$$\| X(d) - v \| \leqq \| X(d) - X_n(d) \| + \| X_n(d) - v \|$$
$$= \| X(d) - X_n(d) \| + \| X_n(d) - X_n(d_n) \| \leqq \| X(d) - X_n(d) \| + d_n - d$$

which converges to zero as n approaches infinity. Also,

$$\| X'(d) - V \| \leqq \| X'(d) - X_n'(d) \| + \| X_n'(d) - V \|$$
$$= \| X'(d) - X_n'(d) \| + \| X_n'(d) - X_n'(d_n) \| \leqq \| X'(d) - X_n'(d) \| + R^{-1} \mid d_n - d \mid$$

which also converges to zero as n approaches ∞. Thus $X(d) = v$, $X'(d) = V$, and therefore $X \in C$, and X is of minimal length.

3. Some preliminary propositions. The purpose of this section is to prove Proposition 6. We begin by borrowing ideas from E. Schmidt's proof, [2], of A. Schur's Lemma and thereby prove:

PROPOSITION 2. *Let X be a curve of average curvature everywhere less than or equal to R^{-1} and Let Z be a semicircle of radius R. Then*

$$\| X'(s) - X'(t) \| \leqq \| Z'(s) - Z'(t) \|, \tag{2}$$

and

$$(X'(s), X'(t)) \geqq (Z'(s), Z'(t)) \tag{3}$$

for all s and t in $[0, \pi R]$. Furthermore, for any fixed s and t, $s < t$, equality holds in (2) if and only if equality holds in (3), and equality holds in (3) if and only if $X(r)$ is an arc of a circle of radius R for $s \leqq r \leqq t$.

Proof. Condition (1) implies that $X''(s)$ exists almost everywhere and is a measurable function of s bounded by R^{-1}. Therefore

$$\left| \int_s^t \| X''(r) \| \, dr \right| \leqq R^{-1} \mid s - t \mid.$$

Since X' is absolutely continuous, we see that the arc length of the curve $X'(r)$ for $s \leqq r \leqq t$ is less than or equal to $R^{-1} \mid s - t \mid$. Sincs $X'(r)$ is a curve on the surface, S_n, of the unit sphere in E_n, it follows that the length of a geodesic on this surface which connects $X'(s)$ and $X'(t)$ is certainly less than or equal to $R^{-1} \mid s - t \mid$. That is, the great circle on S_n containing both $X'(s)$ and $X'(t)$ is divided into two arcs by $X'(s)$ and $X'(t)$. The length of the smaller of these is $\leqq R^{-1} \mid s - t \mid$. It is well known that if two great circular arcs on the unit sphere in E_n are each of length less than or equal to π, then the length of the chord subtended by the smaller arc is less than or equal to the length of the chord subtended by the larger arc.

Therefore, for $R^{-1}|s-t| \leqq \pi$, $\|X'(s)-X'(t)\|$ is less than or equal to the length of the chord subtended by a great circular arc of length $R^{-1}|t-s|$. It is easy to see that for $|t-s| \leqq \pi R$, $\|Z'(s)-Z'(t)\|$ is the length of such a chord. Therefore $\|X'(s)-X'(t)\| \leqq \|Z'(s)-Z'(t)\|$ for $|s-t| \leqq R\pi$. This completes the proof of the first part of the proposition.

It is trivial that if $X(r)$ is an arc of a circle of radius R for $s \leqq r \leqq t$, then $\|X'(s)-X'(t)\| = \|Z'(s)-Z'(t)\|$. Assume, therefore, for some s and t, $0 \leqq s < t \leqq \pi R$, that $\|X'(s)-X'(t)\| = \|Z'(s)-Z'(t)\|$. It follows that the length of the smaller great circular arc between $X'(s)$ and $X'(t)$ equals $R^{-1}|s-t|$. Therefore the arc length of the curve X', for $s \leqq r \leqq t$, is $\geqq R^{-1}|s-t|$. Since it was previously shown to be $\leqq R^{-1}|s-t|$, we conclude that it equals $R^{-1}|s-t|$. Therefore X' is a geodesic on S_n connecting $X'(s)$ with $X'(t)$. Therefore $X'(r)$ is an arc of a unit circle of length $R^{-1}|s-t|$ for $s \leqq r \leqq t$. It now easily follows that $X(r)$ is an arc of a circle of radius R for $s \leqq r \leqq t$. The remainder of the proof of the proposition is trivial.

PROPOSITION 3. *Let X be a curve of average curvature everywhere less than or equal to R^{-1} and let Z be a semicircle of radius R. Let γ be the vector of length R determined by the condition that $Z(0)+\gamma$ is the center of the semicircle Z. Let λ be any vector of length R orthogonal to $X'(0)$. Then*

$$(1) \qquad (Z'(s),\gamma) \geqq (X'(s),\lambda)$$

for $0 \leqq s \leqq \frac{1}{2}\pi R$. Furthermore equality holds for some s in this interval if and only if $X(r)$ is an arc of a circle of radius R for $0 \leqq r \leqq s$, and λ is the vector of length R determined by the condition that $X(0)+\lambda$ is the center of the circle determined by X.

Proof. It is easy to see that $Z'(s)$ is a linear combination of the two unit orthogonal vectors $Z'(0)$ and γ/R. Therefore

$$(2) \qquad 1 = (Z'(s),Z'(s)) = (Z'(s),Z'(0))^2 + (Z'(s),\gamma/R)^2.$$

By Proposition 2, we have,

$$(3) \qquad (X'(s),X'(0)) \geqq (Z'(s),Z'(0)).$$

Since $(Z'(s),Z'(0)) \geqq 0$ for $0 \leqq s \leqq \frac{1}{2}\pi R$, we see that (3) implies

$$(4) \qquad (X'(s),X'(0))^2 \geqq (Z'(s),Z'(0))^2.$$

Combining (2) and (4) we get

$$(5) \qquad 1 \leqq (X'(s),X'(0))^2 + (Z'(s),\gamma/R)^2.$$

Since λ/R and $X'(0)$ are unit orthogonal vectors, it follows that

$$(6) \qquad 1 = (X'(s), X'(s)) \geqq (X'(s), X'(0))^2 + (X'(s), \lambda/R)^2.$$

From (5) and (6) we immediately obtain

$$(7) \qquad (Z'(s), \gamma/R)^2 \geqq (X'(s), \lambda/R)^2.$$

Since $(Z'(s), \gamma/R) \geqq 0$ for $0 \leqq s \leqq \frac{1}{2}\pi R$, we conclude from (7) that

$$(8) \qquad (Z'(s), \gamma) \geqq (X'(s), \lambda)$$

for $0 \leqq s \leqq \frac{1}{2}\pi R$. This proves the first part of the proposition.

Assume now, for some s, $0 \leqq s \leqq \frac{1}{2}\pi \mathrm{R}$, that equality holds in (8). It is easy to see that, therefore, equality must hold in (7). This implies that equality must hold in both (5) and (6) for this particular s. From the equality in (5) we obtain equality in (4) and hence, equality in (3). Therefore, by Proposition 2, $X(r)$ is an arc of a circle of radius R for $0 \leqq r \leqq s$. Equality in (6) implies that $X'(s)$ is spanned by $X'(0)$ and λ/R. Since λ is orthogonal to $X'(0)$ it is either the vector determined by the condition that $X(0) + \lambda$ is the center of the semicircle X or the negative of this vector. For $0 < s \leqq \frac{1}{2}\pi R$, it is not the negative of this vector since $0 < (Z'(s), \gamma) = (X'(s), \lambda)$.

PROPOSITION 4. *Let X be a curve with average everywhere less than or equal to R^{-1} and let λ be any vector of length R orthogonal to $X'(0)$. Then*

$$(X(s) - X(0) - \lambda, X'(s)) \geqq 0$$

for $0 \leqq s \leqq \frac{1}{2}\pi R$. Furthermore, equality holds for some s in this interval if and only if $X(r)$ is an arc of a circle of radius R for $0 \leqq r \leqq s$ and λ is the vector of length R determined by the condition that $X(0) + \lambda$ is the center of the circle determined by X.

Proof. It is easy to see that

$$(9) \qquad (X(s) - X(0) - \lambda, X'(s)) = \int_0^s (X'(t), X'(s))\,dt - (\lambda, X'(s)),$$

for

$$\begin{aligned}(X(s) - X(0) - \lambda, X'(s)) &= (X(s) - X(0), X'(s)) - (\lambda, X'(s)) \\ (10) \qquad &= \left(\int_0^s X'(t)\,dt, X'(s)\right) - (\lambda, X'(s)) \\ &= \int_0^s (X'(t), X'(s))\,dt - (\lambda, X'(s)).\end{aligned}$$

Let $Z(s)$ and γ be as in the hypothesis of Proposition 3. Then Propositions 2 and 3 imply that (10) is greater than or equal to

$$\int_0^s (Z'(t), Z'(s))\,dt - (\gamma, Z'(s)). \tag{11}$$

It is easy to see that (11) is equal to

$$(Z(s) - Z(0) - \gamma, Z'(s)) \tag{12}$$

for the argument is the same as the one which established equality in (9). We now observe that (12) equals one half of

$$d(Z(s) - Z(0) - \gamma, Z(s) - Z(0) - \gamma)/ds, \tag{13}$$

which in turn is equal to

$$d \,\| Z(s) - Z(0) - \gamma \|^2/ds = dR^2/ds = 0. \tag{14}$$

We now prove the second part of the theorem. Assume that for some s, $0 < s \leqq \frac{1}{2}\pi R$,

$$(X(s) - X(0) - \lambda, X'(s)) = 0. \tag{15}$$

Thus the inequalities from (9) through (14) become equalities. In particular, (10) equals (11). But in virtue of Propositions 2 and 3, equality in (10) and (11) imply:

$$\int_0^s (X'(t), X'(s))\,dt = \int_0^s (Z'(t), Z'(s))\,dt; \tag{16}$$

and

$$(\lambda, X'(s)) = (\gamma, Z'(s)). \tag{17}$$

Thus if (15) holds, (17) necessarily also holds. We now invoke the second part of Proposition 3 to complete the proof of the non-trivial part of the present theorem.

Our previous propositions are valid for curves X in any Euclidean space. However, our next two propositions deal only with planar curves. At each point of a differentiable planar curve X there are two tangent circles of radius R. The curve X induces on each of these circles an orientation, so that one of these circles is oriented clockwise, the other counterclockwise. *Let Z_s and Y_s be, respectively, the counterclockwise and clockwise oriented circles of radius R, tangent to the curve X at the point $X(s)$.*

PROPOSITION 5. *Let X be a planar curve with average curvature everywhere less than or equal R^{-1}. Let $D(s)$ be the distance between the center* *of the circle Y_s and the center of the circle Z_0. Then $D(s)$ is a monotone non-decreasing function of s for $0 \leqq s \leqq \frac{1}{2}\pi R$. Furthermore, $D(s) = D(0)$ for some s in this interval if and only if $X(r)$ is a continuously differentiable curve in $[0, s]$ such that, for some r_0 in the closed interval $[0, s]$, $X(r)$ is a counterclockwise oriented arc of a circle of radius R for $0 \leqq r \leqq r_0$ and $X(r)$ is a clockwise oriented arc of a circle of radius R for $r_0 \leqq r \leqq s$.*

Proof. Let $f(s)$ equal $D^2(s)$. We show that f is non-decreasing. Let T be a rotation through an angle of $\frac{1}{2}\pi$ in the counterclockwise direction. Clearly

$$D(s) = \| X(s) - RT(X'(s)) - X(0) - RT(X'(0)) \|. \tag{18}$$

Since both X and X' are absolutely continuous and the product of two absolutely continuous functions is likewise absolutely continuous, it follows that $f(s)$ is absolutely continuous. Furthermore

$$f'(s) = 2(X(s) - RT(X'(s)) - X(0) - RT(X'(0)), X'(s) - RTX''(s)), \tag{19}$$

Since $T(X'(s))$ is orthogonal to both $X'(s)$ and $T(X''(s))$, it follows that

$$f'(s) = 2(X(s) - X(0) - RTX'(0), X'(s) - RTX''(s)) \tag{20}$$

for all s for which $X''(s)$ exists. Furthermore, $-T(X''(s))$ is some scalar multiple of $X'(s)$, say, $-T(X''(s)) = k(s)X'(s)$. Therefore

$$f'(s) = 2(X(s) - X(0) - RTX'(0), X'(s) + Rk(s)X'(s)) \tag{21}$$

or equivalently,

$$f'(s) = (1 + Rk(s))(X(s) - X(0) - RTX'(0), X'(s)) \tag{22}$$

for all s such that $k(s)$ exists. Furthermore, since $\| X''(s) \| \leqq R^{-1}$,

$$1 + Rk(s) \geqq 0. \tag{23}$$

We easily conclude from (22), (23) and Proposition 4 that $f'(s) \geqq 0$ almost everywhere. Since we already showed that $f(s)$ is absolutely continuous it follows that $f(s)$ is montone non-decreasing. Hence, so is $D(s)$.

This completes the proof of the first part of the theorem. Now assume that $D(s) = D(0)$ for some s with $0 < s \leqq \frac{1}{2}\pi R$. Since D is monotone it follows that $D(r) = D(0)$ for all r with $0 \leqq r \leqq s$. Therefore, $f(r) = D^2(r)$ is constant and consequently $f'(r) = 0$ for $0 \leqq r \leqq s$. It follows from (22) that for every r for which $k(r)$ exists, either

(24) $\quad 1 + Rk(r) = 0$ or (25) $\quad (X(r) - X(0) - RTX'(0), X'(r)) = 0.$

Let r_0 be the least upper bound of the set of r in $[0, s]$ for which (25) holds. By continuity of X and X', it is easy to show that (25) also holds for $r = r_0$. By the second part of Proposition 4, we conclude that $X(r)$ is an arc of a circle of radius R for $0 \leqq r \leqq r_0$. Again by (25), we conclude that this arc is counterclockwise oriented. Suppose $r_0 < s$. Then for all r in the half-closed interval $(r_0, s]$ for which $k(r)$ exists, (24) holds. Thus $k(r) = -R^{-1}$ for all r in $(r_0, s]$ for which $k(r)$ exists. Since X' is absolutely continuous, it is easy to prove, therefore, that $X(r)$ is an arc of a circle of radius R for $(r_0, s]$. Since $k(r)$ is negative this arc is clockwise oriented. Since X and X' are continuous at $r = r_0$, the proposition is proven.

As an immediate corollary to Proposition 5 we obtain the main result of this section:

Proposition 6. *Let X be a planar curve with average curvature everywhere less than or equal to R^{-1} and let s be any point in the closed interval $[0, \frac{1}{2}R\pi]$. Then the circle Y_s is either disjoint from, or tangent to, the circle Z_0. Furthermore Y_s is tangent to Z_0 if and only if $X(r)$ is a continuously differentiable curve for $0 \leqq r \leqq s$ such that, for some r_0 in the closed interval $[0, s]$, $X(r)$ is a counterclockwise oriented arc of a circle of radius R for $0 \leqq r \leqq r_0$, and $X(r)$ is a clockwise oriented arc of a circle of radius R for $r_0 \leqq r \leqq s$.*

It is of course obvious that there is a proposition similar to Proposition 6 which is concerned with the circles Z_s and Y_0, rather than Z_0 and Y_s. We do not state this theorem but will also refer to it as Proposition 6 in the sequel.

4. Certain curves are R-geodesics. The purpose of this section is to prove Proposition 9 which states that certain curves, composed of arcs of circles of radius R and straight line segments, are R-geodesics.

Let $X(s)$ be a convex arc defined for $a \leqq s \leqq b$. Let m be the line determined by the two points $X(a)$ and $X(b)$ and let T be the perpendicular projection onto the line m. We will say that a *point p is above the curve X* provided the line segment whose end points are p and $T(p)$ contains a point of the arc $X(s)$. And we will say that a *curve $Y(s)$* defined for $c \leqq s \leqq d$ *lies above the curve $X(s)$* provided that the image under T of all point of Y which lie above the curve X is the segment whose end points are $X(a)$ and $X(b)$.

We state without proof the following proposition which is geometrically obvious.

Proposition 7. *Let X and Y be planar curves defined on the intervals*

$[a,b]$ and $[c,d]$ respectively. Suppose that X is a convex arc and that $X(a)=Y(c)$ and $X(b)=Y(d)$. Then if Y lies above X the arc length of Y is not less than the arc length of X, and equality holds if and only if Y is a one-one parameterization of the range of X.

As an immediate corollary we have:

PROPOSITION 8. (See fig. 1.) *Let d be the length of the smaller arc of a circle Z determined by two points A and B on Z. Let m be the half*

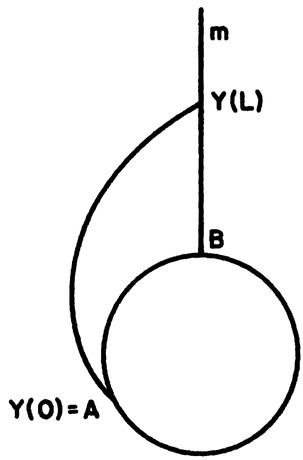

Figure 1.

ray perpendicular to Z at B which does not meet the interior of Z. Suppose $Y(s)$ is a continuous curve parameterized by arc length for $0 \leqq s \leqq L$ which has no point in the interior of Z. Suppose that $Y(0)=A$ and $Y(L)$ is on m. Then $L \geqq d$ and equality holds if and only if $Y(s)$ is a 1-1 parameterization of the smaller arc AB.

We will say that a continuously differentiable curve Y, with arc length s as parameter, defined for $a \leqq s \leqq d$, is of *type ALA* (Arc, Line, Arc) provided there exist b and c with $a \leqq b \leqq c \leqq d$ such that Y restricted to $[a,b]$ is a 1-1 parameterization of an arc of a circle of radius R of length less than or equal to $\frac{1}{2}R\pi$, and such that Y restricted to $[b,c]$ is a line segment, and such that Y restricted to $[c,d]$ is also an arc of a circle of radius R of length less than or equal to $\frac{1}{2}R\pi$.

PROPOSITION 9. *Let u, v, U, and V be vectors in Euclidean 2 space with $\| U \| = \| V \| = 1$ and let R be a positive real number. Suppose Y is* *of type ALA defined for* $0 \leqq s \leqq d$, *and suppose* $Y(0) = u$, $Y'(0) = U$, $Y(d) = v$ *and* $Y'(d) = V$. *Then* Y *is the unique R-geodesic in the collection* $C = C(2, u, U, v, V, R)$.

Proof. Suppose that $0 \leqq d_1 \leqq d_1 + d_2 \leqq d$ and that Y restricted to $[0, d_1]$ is an arc of a circle, Z_1, of radius R and length $\leqq \frac{1}{2}R\pi$, and that Y restricted to $[d_1, d_1 + d_2]$ is a line segment, and Y restricted to $[d_1 + d_2, d]$ is an arc of a circle, Z_2, of radius R and length less than or equal to $\frac{1}{2}R\pi$. Let m_1 be the line passing through $Y(d_1)$ and perpendicular to Y at $Y(d_1)$ and let m_2 be the line parallel to m_1 which passes through $Y(d_1 + d_2)$. The plane is divided by m_1 and m_2 into three strips. Now let X be any curve in C. By a connectedness argument, it is easy to see that there is a least positive number s_1 such that $X(s_1)$ is on m_1. We claim $s_1 \geqq d_1$. For either, for all s, $0 \leqq s \leqq s_1$, $X(s)$ is not in the interior of the circle Z_1, or, for some $s_0 < s_1$, $X(s_0)$ is in the interior of Z_1. In the first case, the preceding proposition implies that $s_1 \geqq d_1$. In the second case, Projosition 6 implies that $s_0 > \frac{1}{2}\pi R$. Since $s_1 > s_0 > \frac{1}{2}\pi R \geqq d_1$, it follows that $s_1 > d_1$. Again by a connectedness argument, there is a smallest number s_2 such that $X(s_2)$ is on m_2. Since m_1 and m_2 are a distance d_2 apart, it follows that $s_2 \geqq s_1 + d_2$. Suppose L is any number such that $X(L) = Y(d)$. An argument similar to the one which showed $s_1 \geqq d_1$ will prove that $L - s_2 \geqq d - (d_1 + d_2)$. Therefore $L \geqq d$. Thus we've shown that Y is an R-geodesic in C. Furthermore the conditions for equality in the preceding proposition imply that Y is the unique R-geodesic in C. This completes the proof.

5. Another preliminary proposition. It is the purpose of this section to prove Proposition 13 which states that an R-geodesic consists of pieces, each of which is either a straight line segment or an arc of a circle of radius R. But first we need a few preliminary definitions and propositions.

Let $L(s)$ be a parameterized straight line segment for $a \leqq s \leqq b$, where s is arc length for L. We will say that the line segment *leaves a curve* $X(s)$ *at the point* $X(s_0)$, provided that $L(a) = X(s_0)$ and $L'(a) = X'(s_0)$. Similarly, we say that the parameterized line segment *arrives at the curve* $X(s)$ *at the point* $X(s_0)$, provided that $L(b) = X(s_0)$ and $L'(b) = X'(s_0)$.

We state without proof the following obvious geometric fact.

PROPOSITION 10. *If* $C(s)$ *and* $B(s)$ *are any two distinct similarly oriented parameterized circles of radius* R *in a plane, then there exists a unique parameterized straight line segment which leaves* $C(s)$ *and arrives at* $B(s)$. *Furthermore if* $C(s)$ *and* $B(s)$ *are oppositely oriented then there* *exists a parameterized straight line segment which leaves $C(s)$ and arrives at $B(s)$ if and only if no point of $B(s)$ is in the interior of $C(s)$. If such a segment exists, then it is unique.*

PROPOSITION 11. *Let X be a planar curve with average curvature everywhere less than or equal to R^{-1} defined for $0 \leqq s \leqq d \leqq \pi R/8$. Then the collection $C = C(2, X(0), X'(0), X(d), X'(d), R)$ contains a curve W of type ALA.*

Proof. It is convenient to assume that the plane π which contains the curve X is the collection of all ordered couples of real numbers. It is clearly no loss of generality to assume that $X(0) = (0,0)$ and that $X'(0) = (1,0)$. We consider the four circles Z_0, Z_d, Y_0, Y_d. Clearly Z_0 and Y_0 are the circles of radius R with centers respectively at $(0, R)$ and $(0, -R)$. It is an easy consequence of Proposition 6 that if $X(d)$ is on Y_0 or Z_0, then X is an arc of Y_0 or Z_0 respectively. In this event, X is itself of type ALA. Therefore we can assume that $X(d)$ is on neither Z_0 nor Y_0. An application of the techniques of Section 3 shows that the first coordinates of the centers of Z_d and Y_d are strictly positive. There are only three cases to consider: (1) every point on Z_d has a non-negative second coordinate; (2) every point on Y_d has a non-positive second coordinate; (3) some, but not all, points on Z_d have negative second coordinates, and some, but not all, points on Y_d have positive second coordinates. We first consider case (1). Proposition 10 assures the existence of a unique parameterized line segment m which leaves Z_0 and arrives at Z_d. There are now two subcases: (a), the slope of m is less than or equal to the slope of $X'(d)$; and (b), the slope of m is greater than the slope of $X'(\mathrm{d})$. Suppose (a) holds. Let A_1 be the smaller arc of Z_0 whose end points are $(0,0)$ and the point of tangency of m with Z_0. Let A_2 be the smaller arc of Z_d whose end points are $X(d)$ and the point of tangency of m with Z_d. Then one sees that the curve, W, which consists of A_1, followed by m, followed by A_2, satisfies the conclusion of the proposition. Suppose now that subcase (b) holds. Proposition 6 implies that Z_0 and Y_d do not intersect. Therefore Proposition 10 assures the existence of a unique parameterized line segment n which leaves Z_0 and arrives at Y_d. This time we let A_1 be the smaller arc of Z_0 whose end points are $(0,0)$ and the point of tangency of n with Z_0. Similarly, we let A_2 be the smaller arc of Y_d whose end points are $X(d)$ and the point of tangency of n with Y_d. This time one sees that the curve, W, which consists of A_1, followed by n, followed by A_2, satisfies the conclusion of the theorem.

Case (2) is treated similarly. We proceed to case (3). It is clear that the first coordinates of the centers of Z_d and Y_d are unequal. We suppose the first coordinate of the center of Z_d to be less than that of Y_d. The other case can be treated similarly. Propositions 6 and 10 assure the existence of a unique directed line segment m which leaves Y_0 and arrives at Z_d. Let A_1 be the smaller arc of Y_0 whose end points are $(0,0)$ and the point of tangency of m with Y_0. Let A_2 be the smaller arc of Z_d whose end points are $X(d)$ and the point of tangency of m with Z_d. Then one sees that the curve, W, which consists of A_1, followed by m, followed by A_2, satisfies the conclusion of the proposition. This completes the proof.

As an immediate corollary to Propositions 9 and 11 we have:

PROPOSITION 12. *Let X be a planar curve of length less than or equal to $\pi R/8$. Then X is an R-geodesic if and only if X is of type ALA.*

We can now easily establish the main result of this section.

PROPOSITION 13. *Let X be a planar curve defined on a closed finite interval $[0, d]$ parameterized by arc length. Then if X is an R-geodesic, it is a continuously differentiable curve which consists of a finite number of pieces, each of which is either a straight line segment, or an arc of a circle of radius R.*

Proof. Since X' satisfies a Lipschitz condition, X is continuously differentiable. The interval $[0, d]$ can be partitioned into a finite number of subintervals each of length less than or equal to $\pi R/8$. Clearly, X restricted to any of these subintervals is also an R-geodesic. Therefore by the preceding proposition, X, so restricted, is of type ALA. This completes the proof.

6. Principal result. We wish to show that a planar R-geodesic is necessarily a very special curve, i.e. a continuously differentiable curve which consists of at most *three* pieces, each of which is either a straight line segment or an arc of a circle of radius R. Errett Bishop pointed out to the author that in view of Proposition 13 it is sufficient to show that no curve which consists of four such pieces can be an R-geodesic.

Let us designate a continuously differentiable curve by $CCCC$, provided that it consists of precisely four arcs of circles of radius R. Similarly let us designate by $CLCL$ a continuously differentiable curve which consists of precisely four pieces, the first of which is an arc of a circle of radius R, the second a line segment, the third an arc of a circle of radius R, and the last a line segment. Similarly any differentiable curve which consists of precisely n arcs of circles of radius R and line segments can be represented by an n-tuple of symbols, each of which is either a C or an L.

We already know that there exist planar R-geodesics of type CLC. We will show that there exist planar R-geodesics of type CCC. It is easy to see that any subpath of an R-geodesic is an R-geodesic. We will show that

* *every R-geodesic is necessarily a subpath of a path of type CLC or of type CCC.*

In order to prove that every R-geodesic is necessarily a subpath of a path consisting of three arcs and line segments, it is sufficient to show that no path consisting of four arcs and line segments can be an R-geodesic. There are eight paths of this type, namely $CCCC$, $CCCL$, $CCLC$, $CLCC$, $CLCL$, $LCCC$, $LCCL$ and $LCLC$. If a curve is an R-geodesic, then so is the curve obtained by traversing the path in the opposite direction. Therefore, if we can show that no curve of type $CCLC$ is an R-geodesic, we will also have shown that no curve of type $CLCC$ is an R-geodesic. Thus we wish to show that no curves of type $CCCC$, $CCCL$, $CCLC$, $CLCL$ and $LCCL$, are R-geodesic. Also, if a curve is an R-geodesic, then so is every subpath of the curve. Thus if we can show that no curve of type CCL is an R-geodesic it will follow that neither are curves of type $CCCL$, $CCLC$ and $LCCL$. Likewise, if we can show that no curve of type LCL is an R-geodesic, it will follow that neither is one of type $CLCL$. *Thus in order to show * it is sufficient to show that none of the following three types of curves is an R-geodesic: CCCC, CCL, LCL.* We begin with a proof that no curve of type LCL is an R-geodesic. The proof that no curve of type CCL is an R-geodesic is rather similar, and is therefore omitted. We then show that no curve of type $CCCC$ is an R-geodesic.

LEMMA 1. *If X is a curve of type LCL defined for $0 \leqq s \leqq d$, then it is not an R-geodesic.*

Proof. Let $P_1P_2P_3P_4$ be a curve of type LCL where: (1) P_1P_2 is a line segment, (2) P_2P_3 is an arc of circle of radius R, (3) P_3P_4 is a line segment, (4) P_1P_2 is tangent to P_2P_3 at P_2 and (5) P_3P_4 is tangent to P_2P_3 at P_4.

It is obvious that we may assume P_2P_3 counterclockwise oriented and we do so. We now consider two cases.

Case 1. (See fig. 2.) The length of the arc P_2P_3 is greater than zero but not greater than πR. It is easy to see, by considering subpaths, that we may assume that P_1P_2 and P_3P_4 have the same length. Let S_1 be the counterclockwise oriented circle of radius R, tangent to P_2P_1 at P_1, which is on the same side of P_1P_2 as is the arc P_2P_3. Similarly, let S_2 be the counterclockwise oriented circle of radius R, tangent to P_3P_4 at P_4 which is on the same side of P_3P_4 as is the arc P_2P_3. There exists a unique line segment Q_1Q_2 which leaves S_1 and arrives at S_2. Since the arc length of

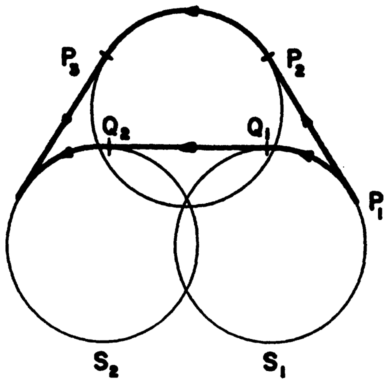

Figure 2.

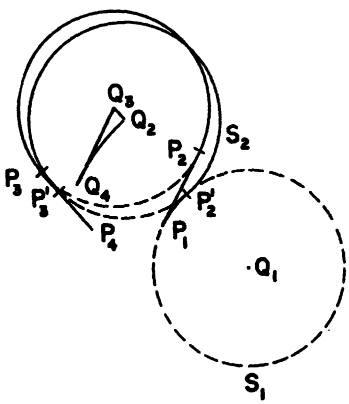

Figure 3.

P_2P_3 does not exceed πR, the curve $P_1Q_1Q_2P_4$ is of type ALA, and hence, by Proposition 9, it is the unique R-geodesic with initial and final positions $X(0)$ *and* $X(d)$ respectively, and initial and final tangent vectors $X'(0)$ and $X'(d)$ respectively.

Case 2. (See fig. 3.) The length of the arc P_2P_3 is strictly between πR and $2\pi R$. It is clear that by considering subpaths we may assume that the segments P_1P_2 and $P_3\ P_4$ do not intersect. Let S_1 *be* the clockwise oriented circle of radius R tangent to P_1P_2 at P_1 which is on the opposite side of P_1P_2 as is the arc P_2P_3. Let S_2 be the counterclockwise oriented circle of radius R which is tangent to S_1 and to P_3P_4. Let P_2' and P_3' be the points of tangency of S_2 with S_1 and P_3P_4 respectively. We will complete our proof by showing that the path $P_1P_2'P_3'P_4$ has length strictly less than the length of X, where: (a) P_1P_2' is the shorter arc of S_1 with end points P_1 and P_2'; and (b) $P_2'P_3'$ is the longer arc of S_2 with end points P_2' and P_3'; and (c) $P_3'P_4$ is the line segment with end points P_3' and P_4. It is sufficient to show that $P_1P_2'P_3'$ is shorter than $P_1P_2P_3P_3'$, where P_1P_2 and P_2P_3 were as defined above and where P_3P_3' is the line segment with end points P_3 and P_3'. There should be no ambiguity in the remainder of this proof if we use the symbol P_2P_3 to represent the length of the arc P_2P_3 as well as the arc itself, similarly for other arcs and line segments. Thus we will complete our proof by showing

$$(1) \qquad P_1P_2' + P_2'P_3' < P_1P_2 + P_2P_3 + P_3P_3'.$$

Let u_2 and u_2' be defined by $Ru_2 = P_2P_3$ and $Ru_2' = P_2'P_3'$. Let u_1' be the angle between the tangent vectors to S_1 at P_1 and P_2'. It is easy to see that (1) is equivalent to

$$(2) \qquad P_1P_2' + Ru_2' < P_1P_2 + Ru_2 + P_3P_3'.$$

Furthermore, since $u_2 = u_2' - u_1'$ we see that (2) is equivalent to

$$(3) \qquad P_1P_2' + Ru_1' < P_1P_2 + P_3P_3'.$$

For $i = 1$ and 2, let Q_i be the center of S_i. P_2P_3 is an arc of a circle whose center we designate by Q_3. Let Q_4Q_2 be the arc of the circle of radius $2R$ with center Q_1, which is concentric with the arc P_1P_2'. Clearly, the arc Q_4Q_2 has a length equal to the left side of (3), whereas the segments Q_4Q_3 and Q_3Q_2 have lengths equal respectively to P_1P_2 and P_3P_3'. Thus (3) is equivalent to

$$(4) \qquad Q_4Q_2 < Q_4Q_3 + Q_3Q_2.$$

Since the arc $Q_4Q_3Q_2$ lies above the convex arc Q_4Q_2, Proposition 7 implies (4). This proves the lemma.

We are indebted to Errett Bishop for the main ideas in the proof of the following lemma. We also wish to thank Horace Moore for pointing out an error in an eariler version.

LEMMA 2. *No path of type CCCC is an R-geodesic.*

Proof. (See fig. 4.) Assume otherwise and let X be a path of type $CCCC$ which is an R-geodesic. Let P_0 and P_4 be the initial and terminal points of X. X consists of four arcs of circles. Let the first arc of X be an arc of the circle S_1 with center Q_1 and let the last arc of X be an arc of the circle S_4 with center Q_4. It is no loss of generality to assume that $R=1$, i.e., the radius of S_1 is 1. Let $2d$ be the distance between Q_1 and Q_4. Clearly $0 \leqq 2d \leqq 6$. It is no loss of generality to assume that S_1 is oriented clockwise. We may also assume that the plane has rectangular coordinates with the origin at Q_1. We may further assume that Q_4 has coordinates

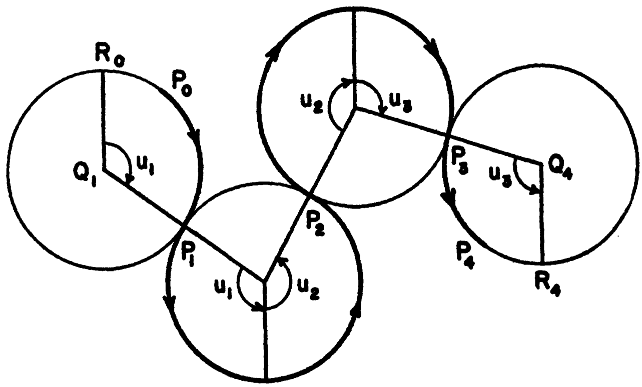

Figure 4.

$(2d,0)$. Let R_0 be the point $(0,1)$. R_0 is on S_1. Let R_4 be the point $(2d,-1)$. R_4 is on S_4. Now let W be the collection of all paths Y of the type $CCCC$ with initial point P_0 and terminal point P_4 and where the first arc of Y is a clockwise oriented arc of the circle S_1 and where the last arc of Y is a counterclockwise oriented arc of the circle S_4. $Y=P_0P_1P_2P_3P_4$, where $P_{i-1}P_i$ is an oriented arc with length L_i of an oriented circle S_i with center Q_i for $i=1,2,3,4$ and where S_i is tangent to S_{i+1} at the point P_i for $i=1,2,3$. Here S_1, S_4, P_0 and P_4 are fixed whereas S_2, S_3, P_1, P_2, P_3, L_1, L_2, L_3 and L_4 vary with Y. Let u_1 be the arc length of the clockwise oriented arc R_0P_1 of the circle S_1. Let u_2 and u_3 be defined by $u_1+u_2=L_2$ and $u_2+u_3=L_3$. Clearly

$$
\begin{aligned}
L &= L_1+L_2+L_3+L_4 \\
&= (L_1-u_1+u_1)+(u_1+u_2)+(u_2+u_3)+(u_3+L_4-u_3) \\
&= L_1-u_1+2(u_1+u_2+u_3)+L_4-u_3.
\end{aligned}
$$

Furthermore, $L_1 - u_1$ and $L_4 - u_3$ are constants independent of Y in W. Hence, a necessary and sufficient condition for $L(Y)$ to attain its minimum at X is that $u_1 + u_2 + u_3$ attain its minimum at X. Clearly (u_1, u_2, u_3) is an admissible triad if and only if it satisfies the two constraints:

(1) $\sin u_1 + \sin u_2 + \sin u_3 = \mathrm{d}$ and (2) $\cos u_1 - \cos u_2 + \cos u_3 = 0$.

There are now two possibilities for any point, (u_1, u_2, u_3), at which a minimum of $u_1 + u_2 + u_3$, subject to the two constraints, occurs: Either the cross product of the two gradient vectors

$$w_1 = (\cos u_1, \cos u_2, \cos u_3) \quad \text{and} \quad w_2 = (-\sin u_1, \sin u_2, -\sin u_3)$$

is the zero vector or $w_1 \times w_2 \neq 0$. Clearly

$$w_1 \times w_2 = (-\sin(u_2 + u_3), \sin(u_3 - u_1), \sin(u_1 + u_2)).$$

Suppose first that $w_1 \times w_2 = 0$. Then $\sin(u_1 + u_2) = 0$. Then $u_1 + u_2$ is a multiple of π. That is, L_2, the arc length of P_1P_2, is a multiple of π. Since X is assumed to be an R-geodesic of type $CCCC$, it is easy to see that L_2 is neither the zero multiple of π nor can L_2 be as large as 2π. Hence $L_2 = \pi$. Then part of X, namely $P_0P_1P_2P_3$ is a curve of type CCC, where the middle arc is of length π. But we have the following

SUBLEMMA. *No curve of type CCC, where the middle arc is of length greater than zero but not greater than πR, can be an R-geodesic.*

We omit the proof of this sublemma for it is quite similar to the proof of Case 1 of Lemma 1. This sublemma implies that $P_0P_1P_2P_3$ is not an R-geodesic. Hence $X = P_0P_1P_2P_3P_4$ could not be an R-geodesic. Therefore, we need only consider the case $w_1 \times w_2 \neq 0$. Hence, we may assume that the two constraints (1) and (2) determine a curve in the neighborhood of the point (u_1, u_2, u_3) at which the minimum of $u_1 + u_2 + u_3$ is attained, and that $w_1 \times w_2$ is tangent to the curve. Let $m =$ minimum of $u_1 + u_2 + u_3$ subject to the constraints. Then the plane $u_1 + u_2 + u_3 = m$ is tangent to the constraint curve at the point at which the minimum is attained. That is, this plane contains the tangent vector $w_1 \times w_2$ at this point. Equivalently, the vector $(1, 1, 1)$ is orthogonal to $w_1 \times w_2$. Hence

$$-\sin(u_2 + u_3) + \sin(u_3 - u_1) + \sin(u_1 + u_2) = 0.$$

It is not difficult to show that if $\sin(A + B) = \sin A + \sin B$, then either A or B or $A + B$ is a multiple of 2π. Thus, either $u_2 + u_3$ or $u_3 - u_1$ or $u_1 + u_2$ is a multiple of 2π. But $u_1 + u_2 = L_2$, the arc length of P_1P_2.

Hence $0 < u_1 + u_2 < 2\pi$. Therefore $u_1 + u_2$ is not a multiple of 2π. Similarly $u_2 + u_3$ is not. Hence $u_3 - u_1$ is a multiple of 2π. Thus at the critical point, $w_1 \times w_2 = (\sin(v_2 + v_1), 0, \sin(v_2 + v_1))$. Since $w_1 \times w_2 \neq 0$, the first component of the tangent $w_1 \times w_2$ is unequal to zero. Therefore, the constraint curve can be parametrized by u_1. at least in a neighborhood of the critical point. In the sequel, all differentiations will be with respect to u_1 in a neighborhood of the critical point. Differentiating the two constraints we get:

$$u_1' \cos u_1 + u_2' \cos u_2 + u_3' \cos u_3 = 0, \qquad -u_1' \sin u_1 + u_2' \sin u_2 - u_3' \sin u_3 = 0$$

Furthermore, at the critical point, $u_1 + u_2 + u_3$ is a minimum, and hence $u_1' + u_2' + u_3' = 0$. These three equations imply that at a critical point $u_2' \sin(u_1 + u_2) = 0$. Since $\sin(u_1 + u_2) \neq 0$, we conclude that $u_2' = 0$. Hence $u_3' = -u_1' = -1$. We next observe that a necessary condition for $u_1 + u_2 + u_3$ to have a minimum at a regular point of the constraint curve is that the inner product of $r = (1, 1, 1)$ with the curvature vector be non-negative. Let $Z(s) = (u_1(s), u_2(s), u_3(s))$ be a parametrization of the constraint curve in a neighborhood of the critical point, where s is arc length for the curve. Clearly dz/ds is the tangent vector. We may assume that $w_1 \times w_2(u_1) = b(s) dz/ds$, where $b(s) > 0$ and where $s = s(u_1)$ is a function of u_1. Hence $(w_1 \times w_2)' = b(s)(d^2z/ds^2)s' + (db/ds)s'(dz/ds)$. At the critical point, dz/ds is perpendicular to r. Hence $(r, (w_1 \times w_2)')$ $= b(s)s'(r, d^2z/ds^2)$. Since $b(s) > 0$, we conclude that a necessary condition for $u_1 + u_2 + u_3$ to have a minimum is that $s'(r, (w_1 \times w_2)') \geqq 0$. Clearly

$$\begin{aligned}&(w_1 \times w_2)' \\ &= (-\cos(u_2 + u_3)(u_2' + u_3'), \cos(u_3 - u_1)(u_3' - u_1'), \cos(u_1 + u_2)(u_1' + u_2')).\end{aligned}$$

But at the critical point, $u_1 = u_3$, $u_2' = 0$, $u_1' = 1$, $u_3' = -1$. Hence

$$(w_1 \times w_2)' = (\cos(u_2 + u_1), -2, \cos(u_1 + u_2)).$$

Therefore $(r, (w_1 \times w_2)') = 2\cos(u_1 + u_2) - 1$. Hence, a necessary condition for a minimum is that $2s'(\cos(u_1 + u_2) - 1) \geqq 0$. Hence, a necessary condition for a minimum is either $s' \leqq 0$ or $u_1 + u_2$ a multiple of 2π. Now we observe that the first component of $w_1 \times w_2$ equals the first component of $b(s) dz/ds$. Hence $-\sin(u_2 + u_3) = b(s) du_1/ds$. Since $b(s) > 0$, we have $\sin(u_2 + u_3) \geqq 0$ if and only if $du_1/ds \leqq 0$. Furthermore, since $w_1 \times w_2 \neq 0$ at a regular point, $\sin(u_2 + u_3) \neq 0$ there. Hence

$$\sin(u_2 + u_3) > 0 \iff du_1/ds \leqq 0 \iff ds/du_1 = s' \leqq 0.$$

We again recall that $u_3 = u_1$ at a critical point. Hence, a necessary condition for a minimum of $u_1 + u_2 + u_3$ subject to the two constraints is that $0 \leqq u_1 + u_2 \leqq \pi$. Now $X = P_0P_1P_2P_3P_4$, where L, the arc length of P_1P_2, is $u_1 + u_2$. By the sublemma, $P_0P_1P_2P_3$ is not an R-geodesic. Hence, neither is X. This completes the proof of the lemma.

We have now established our main result:

THEOREM I. *Every planar R-geodesic is necessarily a continuously differentiable curve which is either* (1) *an arc of a circle of radius R, followed by a line segment, followed by an arc of a circle of radius R; or* (2) *a sequence of three arcs of circles of radius R; or* (3) *a subpath of a path of type* (1) *or* (2).

COROLLARY. *There exists an R-geodesic of type CCC.*

Proof. Consider the problem of making a U-turn. That is let X be an R-geodesic whose initial and terminal positions are the same, and whose terminal tangent vector is the negative of its initial tangent vector. Let P be the finite set which consists of the paths of types (1), (2), and (3) referred to in Theorem 1 with the prescribed boundary conditions. In P there are two paths of type CCC. It is at most an elementary calculation to show that no path in P has a length less than either of these. Hence, by Theorem 1, these two paths are R-geodesics. This completes the proof of the corollary.

7. Non-existence of paths of minimal length. Let u, v, U and V be vectors in real n-dimensional Euclidean space, E_n. Let $\| U \| = \| V \| = 1$ and let $R > 0$. Let $C^* = C^*(n, u, U, v, V, R)$ be the collection of all curves X defined on a closed interval $[0, L]$, where $L = L(X)$ varies with X such that: $X(s) \in E_n$ for $0 \leqq s \leqq L$; $\| X'(s) \| = 1$; $X''(s)$ exists everywhere and $\| X''(s) \| \leqq R^{-1}$ for $0 \leqq s \leqq L$; $X(0) = u$, $X'(0) = U$, $X(L) = v$, and $X'(L) = V$.

PROPOSITION 14. *There exist u, U, v, V and R such that the infimum of the length of the curves X in $C^* = C^*(2, u, U, v, V, R)$ is not attained.*

Proof. Let $u = (0,0)$, $U = (0,1)$, $v = (5,0)$, $V = (0,-1)$ and $R = 1$. It is easy to see that there exists a curve Y of type ALA of length $\pi + 3$ such that $Y \in C = C(2, u, U, v, V, R)$. Proposition 9 implies that Y is the unique curve of minimal length in C. It is clear that C^* is a subset of C and that Y is not an element of C^*. To complete the proof of the proposition it is sufficient to show that for any $\epsilon > 0$, there exists $X \in C^*$ of length less than $\pi + 3 + 4\epsilon$. We will define X uniquely on an interval $[0, L_\epsilon]$ by specifying L_ϵ, $X(0)$, $X'(0)$ and the oriented curvature $k(s)$ for $0 \leqq s \leqq L_\epsilon$. Let $X(0) = u$, $X'(0) = U$, and $k(s) = 1$ for $0 \leqq s \leqq \frac{1}{2}\pi - \epsilon$. Let $k(s)$ be linear for s between $\frac{1}{2}\pi - \epsilon$ and $\frac{1}{2}\pi + \epsilon$, and let $k(\frac{1}{2}\pi + \epsilon) = 0$. Now let d be the first coordinate of the point $X(\frac{1}{2}\pi + \epsilon)$. We can now continue to define k. Let $k(s) = 0$ for $\frac{1}{2}\pi + \epsilon \leqq s \leqq 5 + \frac{1}{2}\pi + \epsilon - 2d$. Let $k(s)$ be linear for s between $5 + \frac{1}{2}\pi + \epsilon - 2d$ and $5 + \frac{1}{2}\pi + 3\epsilon - 2d$. Lastly, let $k(s) = 1$ for $5 + \frac{1}{2}\pi + 3\epsilon - 2d \leqq L_\epsilon$, where $L_\epsilon = 5 + \pi + 2\epsilon - 2d$. Thus X is uniquely defined. Since both the curvature k and its antiderivative can easily be integrated by elementary means, it is at most an elementary calculation to establish that X indeed is in C^*, and that the length L_ϵ of X is indeed less than $\pi + 3 + 4\epsilon$. We omit the details. The intuitive idea is that X was constructed so as to be a twice differentiable curve which approximates Y in an appropriate sense. This completes the proof.

In the course of the proof, we showed that the particular R-geodesic Y in $C(2, u, U, v, V, R)$ had the property that given any $\epsilon > 0$, there exists an X in $C^*(2, u, U, v, V, R)$ whose length differs from the length of Y by a quantity $f(\epsilon)$ which goes to zero with ϵ. It might be conjectured that every R-geodesic has this property. The following is a counterexample to this conjecture. Let Y be a continuously differentiable curve which consists of an arc of length $\frac{1}{2}\pi$ of a counterclockwise oriented circle of radius 1, followed by an arc of length $\frac{1}{2}\pi$ of a clockwise oriented circle of radius 1. We omit the proof that such a Y is indeed a counterexample.

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