Ironing allocations*

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Abstract

I propose a new approach to solving standard screening problems when the monotonicity constraint binds. A simple geometric argument shows that when virtual values are quasi-concave, the optimal allocation can be found by appropriately truncating the solution to the relaxed problem. I provide a simple algorithm for finding this optimal truncation when virtual values are concave.

1 Introduction

This note revisits the problem of finding optimal menus in standard single-agent screening environments. When agents' payoffs are quasi-linear and satisfy single-crossing, the standard approach involves writing the objective as an integral of 'virtual values', which depend only on one type's allocation, and separately choosing each type's allocation to maximize its corresponding virtual value. Incentive compatibility, however, requires that the allocation be increasing in type. If monotonicity binds, however, it is standard to transform virtual values so that after the transformation, point-wise maximization yields and increasing solution. This transformation, known as ironing, was described by Myerson (1981) and subsequently generalized by Toikka (2011). A different approach uses optimal control methods (Guesnerie and Laffont, 1984; Hellwig, 2008; Ruiz del Portal, 2011). These, however, typically require assuming piece-wise differentiability of the control variable, and thereby imposing restrictions on endogenous objects.[1]

I propose an alternative approach that involves solving the relaxed problem without the monotonicity constraint and transforming the resulting allocation to satisfy monotonicity. Theorem 1 says that whenever an optimal allocation rule exists, it can be found by optimally truncating the solution to the relaxed problem. Moreover, the optimal truncation is pinned down by the allocations of types at which the solution to the relaxed problem changes monotonicity. These observations generalize insights about the structure of solutions from the literature—they require no continuity or differentiability assumptions, do not need virtual values to be concave, and do not assume that an agent's allocation is chosen

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[1] Ruiz del Portal (2011) formulates a necessary condition for optimality without this assumption; Hellwig (2008) formulates a version of the maximum principle without it.

from a compact interval. Therefore, in contrast to previous work, my results also apply when the planner can only assign discrete allocations.

I subsequently use these insights to develop a simple algorithm for finding the optimal allocation rule under the additional assumptions that virtual values are concave and that each agent's allocation is chosen from a compact interval.

2 Problem

Agents with types $\theta \in [0,1]$ are allocated objects from a compact set $\mathcal{X} \subset \mathbb{R}$. The planner chooses an increasing allocation rule $x : [0,1] \to \mathcal{X}$ to maximize the following objective:

$$F[x] = \int_0^1 J(x(\theta),\theta)d\theta.$$

I refer to $J : \mathcal{X} \times [0,1] \to \mathbb{R}$ as the **virtual value**.

Assumption 1. *The virtual value satisfies the following properties:*

1. *$J(\cdot,\theta)$ is weakly quasi-concave for every $\theta \in [0,1]$.*
2. *$J(x(\theta),\theta)$ is integrable in θ on $[0,1]$ for every piece-wise monotonic allocation rule $x \in \mathcal{X}^{[0,1]}$.*

Throughout, I will refer to the problem of maximizing $F[x]$ over the space of increasing allocation rules $x \in \mathcal{X}^{[0,1]}$ as the **planner's problem**. I will call the version of this problem where x need not be increasing the **relaxed problem**. I also assume that the relaxed problem has a well-behaved solution:

Assumption 2. *There exists a solution to the relaxed problem, x_R, that is piece-wise monotonic.*

3 Structure of the solution

In this section I present and prove Theorem 1 describing the structure of the solution to the planner's problem. To formulate the result, I first introduce the following definition:

Definition 1. *A **critical point** $i \in [0,1]$ is a point where x_R changes monotonicity, that is, x_R is monotonic on $(i-\epsilon,i)$ for some $\epsilon > 0$ but is not monotonic on $(i-\epsilon,i+\delta)$ for any $\delta > 0$. By convention, I also call 0 and 1 critical points. I use i_n to denote the n-th critical point and $\mathcal{I}$ to denote the set of critical points.*

Theorem 1. *Let $x_v^* : [0,1] \to \mathcal{X}$ parametrized by $v \in \mathcal{X}^{|\mathcal{I}|-1}$ be such that:*

$$x_v^*(\theta) = \begin{cases} \min\{v_{n+1}, \max\{v_n, x_R(\theta)\}\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing,} \\ v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \end{cases} \tag{1}$$

where $v_{|\mathcal{I}|} = x_v^*(1) = \max_{x\in\mathcal{X}} x$. Then, if the planner's problem has a solution, it has a solution of the form x_v^* for some $v \in \mathcal{X}^{|\mathcal{I}|-1}$. This solution can be recovered by solving:

$$\max_{v\in\mathcal{X}^{|\mathcal{I}|-1}} F[x_v^*] \quad \text{subject to} \quad v_1 \le v_2 \le \cdots \le v_{|\mathcal{I}|-1}. \tag{I}$$

Theorem 1 says that when the planner's problem has a solution, we can find it by optimally truncating the solution to the relaxed problem, x_R. Moreover, the optimal truncation is pinned down by the allocations of types at which x_R changes monotonicity.

The remainder of this section is devoted to proving Theorem 1. I first show three lemmas:

Lemma 1. *If x_1, x_2 are allocation rules s.t. x_2 lies point-wise between x_1 and x_R, then $F[x_2] \geq F[x_1]$.*

Proof. $J(x_R(\theta),\theta) \geq J(x_1(\theta),\theta)$ for almost all θ; else we could strictly improve upon x_R by changing it to x_1 wherever the inequality fails. Now, by quasi-concavity of $J(\cdot,\theta)$:

$$\begin{aligned} F[x_2] = \int_0^1 J(x_2(\theta),\theta)d\theta &\geq \int_0^1 \min\{J(x_1(\theta),\theta), J(x_R(\theta),\theta)\}d\theta \\ &= \int_0^1 J(x_1(\theta),\theta)d\theta = F[x_1]. \end{aligned}$$

□

Lemma 1 tells us that the objective always increases when we move the allocation rule point-wise closer to the first-best one. This property underpins the proofs of Lemmas 2 and 3.

Lemma 2. *Suppose x_R is decreasing on $[a,b)$. Then any increasing $x \in \mathcal{X}^{[0,1]}$ can be weakly improved upon by some increasing $x^* \in \mathcal{X}^{[0,1]}$ that is constant on $[a,b)$ and coincides with x elsewhere.*

Proof. Fix any increasing $x \in \mathcal{X}^{[0,1]}$. Consider $x^* \in \mathcal{X}^{[0,1]}$ that coincides with x on $[0,1] \setminus [a,b)$ and takes the following values for $\theta \in [a,b)$:

$$x^*(\theta) = \begin{cases} \lim_{\theta \to a^+} x(\theta) & \text{if } x_R(\theta) \leq x(\theta) \text{ for all } \theta \in (a,b), \\ x(\sup\{\theta \in (a,b) : x_R(\theta) \geq x(\theta)\}) & \text{if } \sup\{\theta \in (a,b) : x_R(\theta) \geq x(\theta)\} \in (a,b), \\ \lim_{\theta \to b^-} x(\theta) & \text{if } x_R(\theta) \geq x(\theta) \text{ for all } \theta \in (a,b). \end{cases}$$

x^* is increasing and point-wise between x_R and x, so $F[x^*] \geq F[x]$ by Lemma 1. □

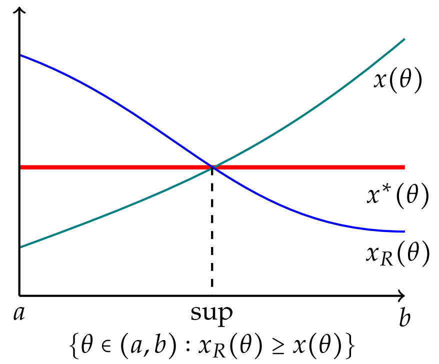
Lemma 3. *Suppose x_R is increasing on $[a,b)$. Then any increasing $x \in \mathcal{X}^{[0,1]}$ can be weakly improved upon by an increasing $x^* \in \mathcal{X}^{[0,1]}$ s.t. $x^*(\theta) = \min\{x(b), \max\{x(a), x_R(\theta)\}\}$ on $[a,b)$ and coincides with x elsewhere.*

Proof. Fix any increasing $x \in \mathcal{X}^{[0,1]}$ and consider:

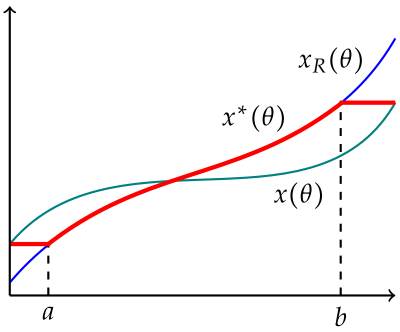
$$x^*(\theta) = \begin{cases} \min\{x(b), \max\{x(a), x_R(\theta)\}\} & \text{if } \theta \in [a,b), \\ x(\theta) & \text{elsewhere.} \end{cases}$$

x^* is point-wise between x and x_R, so $F[x^*] \geq F[x]$ by Lemma 1. Moreover, x^* is increasing on $[a,b)$ as x_R is increasing on $[a,b)$ by assumption. □

Observations similar to Lemmas 2 and 3 were independently made by Sandmann (2022) who studies the optimality of sparse menus in a price discrimination problem.



Constructing an improvement in the proof of Lemma 2.



Constructing an improvement in the proof of Lemma 3.

Given the above lemmas, the proof of Theorem 1 is straightforward. By Assumption 2, $[0,1)$ can be partitioned into finitely many intervals $[i_n, i_{n+1})$ where x_R is monotonic. Then, by Lemmas 2 and 3, if the planner's problem has a solution, it has one of the form in (1) with $v_1 \leq v_2 \leq \cdots \leq v_{|\mathcal{I}|-1}$. Solving problem (I) recovers this solution.

4 Solution algorithm

I now provide a simple algorithm that solves planner's problem under the following additional assumption:

Assumption 3. *The virtual value satisfies the following properties:*

1. *The planner chooses allocations from a closed interval:* $\mathcal{X} = [l, h]$.
2. $J(\cdot, \theta)$ *is weakly concave for every* $\theta \in [0,1]$.
3. $J(x, \theta)$ *is uniformly bounded across all* $x \in [l, h]$ *and* $\theta \in [0,1]$.

The algorithm uses the following transformation $T : [l,h]^{[0,1]} \times [l,h] \times \mathbb{N} \to [l,h]^{[0,1]}$:

$$\mathrm{T}[x, v, n](\theta) := \begin{cases} \min\{v, x(\theta)\} & \text{if } \theta < i_n, \\ v & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \max\{v, x(\theta)\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing,} \\ h & \text{if } \theta = i_{n+1}, \\ x(\theta) & \text{if } \theta > i_{n+1}. \end{cases}$$

When $T[\cdot\,, v, n]$ is applied to x, the allocation rule is truncated from above by v before the nth critical point and set equal to v or truncated by it from below between the nth and $n+1$st critical points. It is also set to h at the $n+1$st critical point and remains unchanged beyond it. Note that if $x(\theta)$ was increasing on the interval $[0, i_n)$, then $\mathrm{T}[x, v, n](\theta)$ will be increasing on $[0, i_{n+1})$. By iteratively applying this transformation at subsequent critical points, the algorithm expands the interval on which the resulting allocation rule is increasing.

The following algorithm takes in the solution to the relaxed problem, x_R, and the set of critical points $i \in \mathcal{I}$ as inputs and produces the solution to planner's problem:

Algorithm 1

$r \leftarrow x_R$
$n \leftarrow 1$
while $n \leq |\mathcal{I}| - 1$ **do**
  $v^* \leftarrow \text{argmax}_{v \in [l,h]} \int_0^{i_{n+1}} J(\mathrm{T}[r,v,n](\theta))d\theta$
  $r \leftarrow \mathrm{T}[x, v^*, n]$
  $n \leftarrow n+1$
end while
return r

Intuitively, each iteration of Algorithm 1 starts with the optimum subject to monotonicity from 0 until the nth critical point and 'irons out' this partial solution further to produce a solution subject to monotonicity until the $n+1$st critical point (Figure 2).

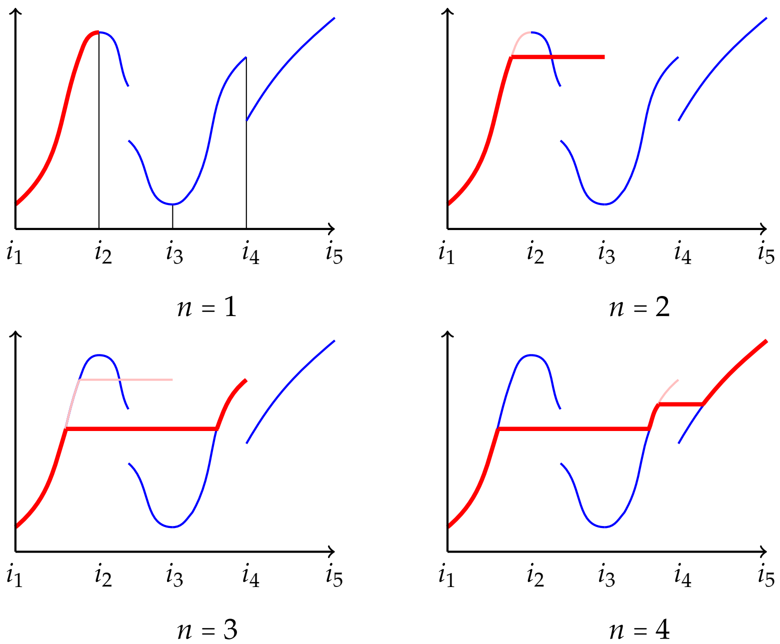


Figure 2: Algorithm 1 recursively transforming x_R (blue) into subsequent $\mathrm{T}[x, v^*, n]$ (red).

Theorem 2. *Under Assumption 3, the output of Algorithm 1 solves the planner's problem.*

The remainder of this section is devoted to proving this result. We begin by defining some useful objects and establishing their properties. Let F_a be the value of the objective evaluated only until type a, that is:

$$F_a[x] := \int_0^a J(x(\theta), \theta)d\theta.$$

Since $J(\cdot, \theta)$ is concave for every θ, $F_a[x]$ is concave in the allocation rule x for every $a \in [0,1]$.

Now, consider the set of allocation rules that are increasing on $[0,a)$ and below k everywhere on that interval. Let $\mathcal{O}_a^k$ contain the allocation rules that maximize F_a over this set:

$$\mathcal{O}_a^k := \operatorname{argmax}\left\{F_a[x] : \ x \in [l,h]^{[0,1]} \text{ is increasing on } [0,a) \text{ and } x(\theta) \le k \text{ for all } \theta \in [0,a)\right\}.$$

I now show that $\mathcal{O}_a^k$ is non-empty for any $a \in (0,1]$ and $k \in [l,h]$. Let $\mathcal{A}$ be the set of functions $x \in [l,k]^{[0,a]}$ that are increasing. It suffices to show that F_a attains a maximum on $\mathcal{A}$. $\mathcal{A}$ is homeomorphic to a Helly space and so is compact. Hence, by Weierstrass' theorem, it suffices to show that F_a is continuous on this space. Since $\mathcal{A}$ is homeomorphic to a Helly space, it is first-countable. Thus, F_a is continuous on it if for any sequence in $\mathcal{A}$ such that $x_n \to x$ we have $F_a[x_n] \to F_a[x]$. Fix any such x_n. Then $x_n(\theta) \to x(\theta)$ for every $\theta \in [0,a]$. Since $J(\cdot,\theta)$ is concave, it is continuous, which implies $J(x_n(\theta),\theta) \to J(x(\theta),\theta)$ for every $\theta \in [0,a]$. The Dominated Convergence Theorem then implies that $F_a[x_n] \to F_a[x]$.

Finally, the following proposition is shown in the appendix. It says that when we have an x in $\mathcal{O}_a^h$ and impose the additional constraint that x be below k on $[0,a)$, we need not resolve the problem, but can simply truncate the solution without this constraint by k.

Proposition 1. *If $x_a \in \mathcal{O}_a^h$, then* $\max\{k, x_a\} \in \mathcal{O}_a^k$.

I now present an inductive proof of Theorem 2. Recall that $|\mathcal{I}| \ge 2$ since $0,1 \in \mathcal{I}$. The base case demonstrates that the first iteration of the algorithm produces $r \in \mathcal{O}_{i_2}^h$. The step shows that when the nth iteration starts with $r \in \mathcal{O}_{i_n}^h$, it produces $r \in \mathcal{O}_{i_{n+1}}^h$. The base and the step thus imply that the algorithm will produce $r \in \mathcal{O}_{i_{|\mathcal{I}|}}^h$ solving the planner's problem in $|\mathcal{I}| - 1$ steps.

Base. Since $\mathcal{O}_{i_2}^h \neq \varnothing$, Lemmas 2 and 3 tell us there exists $x^* \in \mathcal{O}_{i_2}^h$ for which:

$$x^*(\theta) := \begin{cases} v_1 & \text{if } \theta \in [i_1, i_2) \text{ where } x_R \text{ is decreasing,} \\ \min\left\{v_2, \max\left\{v_1, x_R(\theta)\right\}\right\} & \text{if } \theta \in [i_1, i_2) \text{ where } x_R \text{ is increasing.} \end{cases}$$

Moreover, by Lemma 1 we can without loss set $v_2 = h$. The first iteration of the algorithm will recover such an x^* when optimizing over v.

Step. Since $\mathcal{O}_{i_{n+1}}^h \neq \varnothing$, Lemmas 2 and 3 tell us there exists $x^* \in \mathcal{O}_{i_{n+1}}^h$ for which:

$$x^*(\theta) := \begin{cases} v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \min\left\{v_{n+1}, \max\left\{v_n, x_R(\theta)\right\}\right\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing.} \end{cases}$$

Moreover, by Lemma 1 we can without loss set $v_{n+1} = h$. Let $r \in \mathcal{O}_{i_n}^h$ be the allocation rule produced by the $n-1$st iteration of the algorithm. By Proposition 1, $\max\{v_n, r\} \in \mathcal{O}_{i_n}^{v_n}$. Thus, $F_{i_n}[\max\{v_n, r\}] \ge F_{i_n}[x^*]$ and so the following allocation rule weakly improves upon x^*:

$$x^{**}(\theta) := \begin{cases} \min\{v_n, r(\theta)\} & \text{if } \theta < i_n, \\ v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \max\left\{v_n, x_R(\theta)\right\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing.} \end{cases}$$

Moreover, x^{**} is increasing and weakly below k everywhere on $[0,a)$, so x^{**} belongs to $\mathcal{O}_{i_{n+1}}^h$. The nth iteration of the algorithm will recover such an x^{**} when optimizing over v.

5 Appendix

5.1 Proof of Proposition 1.

Proof. If $x_a(\theta) \leq k$ for all $\theta \in [0, a)$, the result is trivial as the additional constraint does not bind, so consider the case where $x_a(\theta) > k$ for some $\theta \in [0, a)$. Let b be the earliest point where x_a crosses k, so $b := \inf\{\theta : x_a(\theta) \geq k\}$. We can assume without loss that $x_a(b) = k$, as this does not violate monotonicity on $[0, a)$ or alter the value of $F_a[x_a]$.

Recall that $\mathcal{O}_a^k$ is non-empty for all $a \in (0, 1]$ and all $k \in [l, h]$, and that all allocation rules in $\mathcal{O}_a^k$ are increasing and below k on $[0, a)$. Let c be the earliest point in $[b, a]$ at which some allocation rule x belonging to $\mathcal{O}_a^k$ takes the value k:

$$c = \inf_{\theta \geq b,\ x \in \mathcal{O}_a^k} \{\theta : x(\theta) = k\}.$$

Notice that $c \leq a$. To see this, take any $x^* \in \mathcal{O}_a^k$ and notice that setting $x^*(a) = k$ does not affect the value of F_a or violate the constraints, as these are only imposed on $[0, a)$. Thus, there always exists some $x^* \in \mathcal{O}_a^k$ such that $x^*(a) = k$.

I now show that this infimum is attained, i.e. that there exists an allocation rule $x \in \mathcal{O}_a^k$ such that $x(c) = k$. I already showed this for the case of $c = a$, so suppose $c < a$. The following lemma will be useful:

Lemma 4. *Fix $z \in [b, a)$ such that some sequence of allocation rules $f_n \in \mathcal{O}_a^k$ converges point-wise to k on (z, a). Take any $x_z \in \mathcal{O}_z^k$. Then any allocation rule x^* satisfying (2) belongs to $\mathcal{O}_a^k$.*

$$x^*(\theta) = \begin{cases} x_z(\theta) & \text{if } \theta < z, \\ k & \text{if } \theta \in [z, a). \end{cases} \tag{2}$$

Proof. The allocation rule x^* is increasing and below k on $[0, a)$, so it suffices to show that it guarantees a value of F_a at least as large as some other allocation rule in $\mathcal{O}_a^k$.

By assumption, $f_n(\theta) \to k$ for every $\theta \in (z, a)$. $J(\cdot, \theta)$ is concave and hence continuous, so $J(f_n(\theta), \theta) \to J(k, \theta)$ for all $\theta \in (z, a)$. By the Dominated Convergence Theorem, we have:

$$\int_z^a J(f_n(\theta), \theta) d\theta \to \int_z^a J(k, \theta) d\theta. \tag{3}$$

Take any $\epsilon > 0$. Then there exists n^* for which:

$$\int_z^a J(k, \theta) d\theta + \epsilon > \int_z^a J(f_{n^*}(\theta), \theta) d\theta. \tag{4}$$

Note that $f_n(z) \leq k$, so allocation rules in $\mathcal{O}_z^k$ guarantee a weakly higher F_z than those in $\mathcal{O}_z^{f_n(z)}$. Hence:

$$\int_0^z J(x_z(\theta), \theta) d\theta \geq \int_0^z J(f_{n^*}(\theta), \theta) d\theta. \tag{5}$$

Adding together (4) and (5) gives:

$$\int_0^a J(x^*(\theta), \theta) d\theta + \epsilon > \int_0^a J(f_{n^*}(\theta), \theta) d\theta.$$

Recall that ϵ was arbitrary, so $F_a[x^*] \geq F_a[f_{n^*}]$. Since $f_{n^*} \in \mathcal{O}_a^k$, we get that $x^* \in \mathcal{O}_a^k$ too. $\square$

Return to the case of $c < a$. By monotonicity and the definition of c, there exists a sequence of allocation rules $f_n \in \mathcal{O}_n^k$ converging point-wise to k on (c, a). Hence, Lemma 4 tells us there exists $x^* \in \mathcal{O}_a^k$ for which $x^*(c) = k$.

Now, let $u := \sup\{x(b) : x \in \mathcal{O}_a^k \text{ and } x(c) = k\}$. In what follows, I show that $u = k$.

Suppose towards a contradiction that $u < k$. Then it must be that $b < c$. By Assumption 1, x_R is monotonic on $[c-\epsilon, c)$ for ϵ small enough. Fix such an ϵ small enough that $b < c - \epsilon$.

Lemma 5. *x_R is decreasing on $[c-\epsilon, c)$.*

Proof. Suppose towards a contradiction that x_R is increasing on $[c-\epsilon, c)$ and take $x^* \in \mathcal{O}_a^k$ such that $x^*(c) = k$ (by the above, such an x^* exists). Consider two cases:

Case 1: there exists $\epsilon' \in (0, \epsilon)$ such that $x^R(c-\epsilon') \geq k$. Then $x_R(\theta) \geq k$ on $[c-\epsilon', c)$. By Lemma 1, we can improve upon x^* by setting it to k on $[c-\epsilon', c)$. Thus, there is some allocation rule $x^{**} \in \mathcal{O}_a^k$ equal to k at $c - \epsilon' < c$. This contradicts the definition of c.

Case 2: $x_R(\theta) < k$ everywhere on $[c-\epsilon, c)$. By the definition of c, we have $x^*(c-\epsilon) < k$. Since $b < c - \epsilon$, we also know that $x_a(c-\epsilon) \geq k$. Hence, there exists $\alpha \in (0,1]$ for which $\alpha x_a(c-\epsilon) + (1-\alpha)x^*(c-\epsilon) = k$. Consider the following allocation rule:

$$x^{**}(\theta) = \begin{cases} \alpha x_a(\theta) + (1-\alpha)x^*(\theta) & \text{if } \theta < c - \epsilon, \\ k & \text{if } \theta \in [c-\epsilon, a). \end{cases}$$

I will show that x^{**} belongs to $\mathcal{O}_a^k$. The allocation rules x_a and x^* are increasing on $[0, a)$, so their convex combination is too. It is also weakly below k on $[0, c-\epsilon)$ and so x^{**} is increasing and below k on $[0, a)$. Therefore, it suffices to show that x^{**} guarantees a weakly higher value of F_a than some other allocation rule in $\mathcal{O}_a^k$. I will show that $F_a[x^{**}] \geq F_a[x^*]$. Since $x_a \in \mathcal{O}_a^h$ solves a problem with laxer constraints than does $x^* \in \mathcal{O}_a^k$, we have $F_a[x_a] \geq F_a[x^*]$. Hence, by concavity of F_a:

$$F_a[\alpha x_a + (1-\alpha)x^*] \geq F_a[x^*].$$

Notice x^{**} and $\alpha x_a + (1-\alpha)x^*$ coincide everywhere on $[0, a)$ except for the interval $[c-\epsilon, c)$, where $x^{**}(\theta) \leq \alpha x_a(\theta) + (1-\alpha)x^*(\theta)$. However, recall that $x_R < k$ on $[c-\epsilon, c)$, and so $F_a[x^{**}] > F_a[\alpha x_a + (1-\alpha)x^*]$ by Lemma 1. Therefore, $F_a[x^{**}] \geq F_a[x^*]$ and so $x^{**} \in \mathcal{O}_a^k$. However, $x^{**}(c-\epsilon) = k$, which contradicts the definition of c. $\square$

In what follows, I obtain a contradiction by constructing an allocation rule in $\mathcal{O}_a^k$ that attains a value above u at b. Fix some strictly positive sequence $\{\delta_n\}_n$ such that $\delta_n \to 0$. By the definition of u, for any n there exists $g_n \in \mathcal{O}_a^k$ such that $g_n(c) = k$ and $g_n(b) + \delta_n > u$.

Moreover, I show that for any n there exists an allocation rule $g_n^* \in \mathcal{O}_n^k$ such that $g_n^*(b) + \delta_n > u$, $g_n^*(c) = k$ and $g_n^*(\theta)$ is constant on $[c-\epsilon, c)$. Lemma 5 tells us that x_R is decreasing on $[c-\epsilon, c)$ and so, by Lemma 2, for every n there exists $g_n^* \in \mathcal{O}_a^k$ that is constant on $[c-\epsilon, c)$ and coincides with g_n elsewhere. Since $b, c \notin [c-\epsilon, c)$, for any n we still have $g_n^*(b) + \delta_n > u$ and $g_n^*(c) = k$.

Furthermore, the sequence of allocation rules $\{g_n^*\}_n$ has to be uniformly bounded away from k on $[0, c)$. By the above, for any n there exists $\gamma_n \in \mathbb{R}$ s.t. $g_n^*(\theta) = \gamma_n$ for $\theta \in [c-\epsilon, c)$. Suppose towards a contradiction that $\sup \gamma_n = k$. Then there exists a subsequence of $\{g_n^*\}_n$

that converges point-wise to k on $[c-\epsilon, c)$. Lemma 4 then tells us there exists an allocation rule $x^* \in \mathcal{O}_a^k$ equal to k on $[c-\epsilon, c)$. This contradicts the definition of c.

Thanks to this uniform bound, we know there exists $\alpha \in (0,1)$ such that for all n and $\theta \in [0, c)$ we have $\alpha x_a(\theta) + (1-\alpha) g_n^*(\theta) \leq k$. Let us then consider g_n^{**} defined as follows:

$$g_n^{**}(\theta) = \begin{cases} \alpha x_a(\theta) + (1-\alpha) g_n^*(\theta) & \text{if } \theta < c, \\ k & \text{if } \theta \geq c. \end{cases}$$

I now show g_n^{**} belongs to $\mathcal{O}_a^k$ for any n. Notice $x_a(\theta) + (1-\alpha) g_n^*(\theta)$ is increasing on $[0, c)$ as g_n^* and x_a were increasing on that interval. By the choice of α, $x_a(\theta) + (1-\alpha) g_n^*(\theta)$ is also below k on $[0, c)$. Thus, to show that g_n^{**} belongs to $\mathcal{O}_a^k$, it suffices to show that it guarantees a value of F_a as high as another allocation rule in that set. I will show $F_a[g_n^{**}] \geq F_a[g_n^*]$.

First, note that $F_c[x_a] \geq F_c[g_n^*]$; otherwise $F_c[x_a]$ could be increased by replacing x_a for g_n^* on $[0, c)$. This replacement would preserve monotonicity and produce an allocation rule that is below k on $[0, a)$. To see that, recall that x_a equals to k on $[b, a)$ and $c > b$, so $x_a(c) = k$. Since $g_n^*(c) \leq k$ and g_n^* is increasing on $[0, c)$, both constraints hold after the replacement.

By concavity of F_c and the fact that $F_c[x_a] \geq F_c[g_n^*]$, we have:

$$F_c[\alpha x_a(\theta) + (1-\alpha) g_n^*] \geq F_c[g_n^*]. \tag{6}$$

Both g_n^* and g_n^{**} equal to k on $[c, a)$, so (6) implies that $F_a[g_n^{**}] \geq F_a[g_n^*]$.

Now, recall that $b < c$ and $x_a(b) = k$. Thus, for every n:

$$g_n^{**}(b) = \alpha k + (1-\alpha) g_n^*(b) > \alpha k + (1-\alpha)(u - \delta_n),$$

where the latter inequality holds because $g_n^*(b) + \delta_n > u$ for every n. Since we assumed that $u < k$, for sufficiently large n we have $g_n^{**}(b) > u$. Since $g_n^{**} \in \mathcal{O}_a^k$ for any n, this contradicts the definition of u.

Thus, we have shown by contradiction that $u = k$. I will now apply Lemma 4 to show that $\max\{x_a, k\} \in \mathcal{O}_a^k$. First, note that by the definition of u there exists a sequence of allocation rules $f_n \in \mathcal{O}_a^k$ converging point-wise to k on (b, a). Second, observe that $x_a \in \mathcal{O}_b^k$; otherwise we could improve $F_a[x_a]$ by replacing x_a for some $x^* \in \mathcal{O}_b^k$ on $[0, b)$. Since $x_a(b) = k$, this replacement would not violate either constraint.

Finally, x_a is below k on $[0, b)$ and above k on $[b, a)$, so Lemma 4 tells us $\max\{x_a, k\} \in \mathcal{O}_a^k$. □

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