Ironing allocations*

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Abstract

I propose a new approach to solving standard screening problems when the monotocnitity constraint binds. A simple geometric argument shows that when virtual values are quasi-concave, the optimal allocation can be found by approperiately truncating the solution to the relaxed problem. I provide a simple algorithm for finding this optimal truncation when virtual values are concave.

1 Introduction

This paper revisits the problem of finding optimal menus in standard single-agent screening environments. When agents' payoffs are quasi-linear and satisfy single crossing, the standard approach involves writing the objective as an integral of 'virtual values' which depend only on one type's allocation, and separately choosing each type's allocation to maximize its corresponding virtual value. Incentive compatibility, however, requires that the allocation be increasing in type.[1] If point-wise maximization happens to yield an increasing solution, this constraint is slack—this is known as the 'regular case'. If monotonicity binds, however, it is standard to transform virtual values so that after the transformation, point-wise maximization yields and increasing solution. This transformation, known as ironing, was described by Myerson (1981) and subsequently generalized by Toikka (2011). A different approach uses optimal control methods (Guesnerie and Laffont (1984), Hellwig (2008), Ruiz del Portal (2011)). These, however, typically require assuming piece-wise differentiability of the control variable, and thereby imposing restrictions on endogenous objects.[2]

I propose an alternative approach which involves solving the relaxed problem without the monotonicity constraint and directly transforming the resulting allocation to satisfy monotonicity. Theorem 1 says that whenever an optimal allocation rule exists, it can be found by optimally truncating the solution to the relaxed problem. Moreover, the optimal truncation is pinned down by the allocations of types at which the solution to the relaxed problem changes monotonicity. These observations generalize insights about the structure of solutions from the literature—they require no continuity or differentiability assumptions, do not need virtual values to be concave, and do not assume that an agent's allocation is chosen from a compact interval. Therefore, in contrast to previous work, my results also apply

*I am grateful to Ilya Segal, Piotr Dworczak and Ben Brooks for helpful comments.

[1]Throughout, I use 'increasing' to mean weakly increaing, and 'monotone' to mean either weakly increasing or weakly decreasing.

[2]Ruiz del Portal (2011) formulates a necessary condition for optimality without this assumption; Hellwig (2008) formulates a version of the maximum principle without it.

when the planner can only assign discrete allocations. I use these insights to develop a simple algorithm for finding the optimal allocation rule under the additional assumptions that virtual values are concave and that each agent's allocation is chosen from a compact interval.

2 Problem

Agents with types $\theta \in [0,1]$ are allocated objects from a compact set $\mathcal{X} \subset \mathbb{R}$. The planner chooses an increasing allocation rule $x : [0,1] \to \mathcal{X}$ to maximize the following objective:

$$F[x] = \int_0^1 J(x(\theta), \theta) d\theta.$$

I refer to $J : \mathcal{X} \times [0,1] \to \mathbb{R}$ as the **virtual value** and assume it has the following properties:

Assumption 1. *$J(x(\theta), \theta)$ is measurable for every increasing $x \in \mathcal{X}^{[0,1]}$. Moreover, $J(\cdot, \theta)$ is weakly quasi-concave for every $\theta \in [0,1]$.*

Throughout, I will refer to the problem of maximizing $F[x]$ over the space of increasing $x \in \mathcal{X}^{[0,1]}$ as the **planner's problem**. I will call the version of this problem where x need not be increasing the **relaxed problem**.

Assumption 2. *There exists a solution to the relaxed problem, x_R, that is piece-wise monotonic.*

3 Structure of the solution

The objective I consider has the following property, which is central my approach:

Fact 1. *If x_1, x_2 are allocation rules s.t. x_2 lies point-wise between x_1 and x_R, then $F[x_2] \geq F[x_1]$.*

Proof. $J(x_R(\theta), \theta) \geq J(x_1(\theta), \theta)$ for almost all θ; else we could strictly improve upon x_R by changing it to x_1 wherever the inequality failed. Now, by quasi-concavity of $J(\cdot, \theta)$:

$$\begin{aligned} F[x_2] = \int_0^1 J(x_2(\tau), \tau) d\tau &\geq \int_0^1 \min\{J(x_1(\tau), \tau), J(x_R(\tau), \tau)\} d\tau \\ &= \int_0^1 J(x_1(\tau), \tau) d\tau = F[x_1]. \end{aligned}$$

□

Fact 1 tells us that the objective always increases when we move the allocation rule point-wise closer to the first-best one. This property underpins Lemmas 1 and 2.

Definition 1. *A **critical point** $i \in [0,1]$ is a point where x_R changes monotonicity, that is, x_R is monotonic on $(i - \epsilon, i)$ for some $\epsilon > 0$ but is not monotonic on $(i - \epsilon, i + \delta)$ for any $\delta > 0$. By convention, I also call 0 and 1 critical points. I use i_n to denote the n-th critical point and $\mathcal{I}$ to denote the set of critical points.*

Lemma 1. *Suppose x_R is decreasing on $[a,b)$. Then any increasing $x \in \mathcal{X}^{[0,1]}$ can be weakly improved upon by some increasing $x^* \in \mathcal{X}^{[0,1]}$ that is constant on $[a,b)$ and coincides with x elsewhere.*

Proof. Fix any increasing $x \in \mathcal{X}^{[0,1]}$. Consider $x^* \in \mathcal{X}^{[0,1]}$ which coincides with x on $[0,1] \setminus [a,b)$ and takes the following value for $\theta \in [a,b)$:

$$x^*(\theta) = \begin{cases} \lim_{\theta \to a^+} x(\theta) & \text{if } x_R(\theta) \le x(\theta) \text{ for all } \theta \in (a,b), \\ x(\sup\{\theta \in (a,b) : x_R(\theta) \ge x(\theta)\}) & \text{if } \sup\{\theta \in (a,b) : x_R(\theta) \ge x(\theta)\} \in (a,b), \\ \lim_{\theta \to b^-} x(\theta) & \text{if } x_R(\theta) \ge x(\theta) \text{ for all } \theta \in (a,b). \end{cases}$$

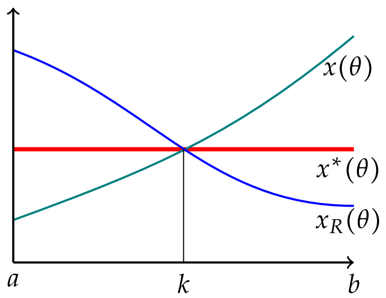
x^* is increasing and point-wise between x_R and x, so $F[x^*] \ge F[x]$ by Fact 1. □

Lemma 2. *Suppose x_R is increasing on $[a,b)$. Then any increasing $x \in \mathcal{X}^{[0,1]}$ can be weakly improved upon by an increasing $x^* \in \mathcal{X}^{[0,1]}$ s.t. $x^*(\theta) = \min\{x(b), \max\{x(a), x_R(\theta)\}\}$ on $[a,b)$ and coincides with x elsewhere.*

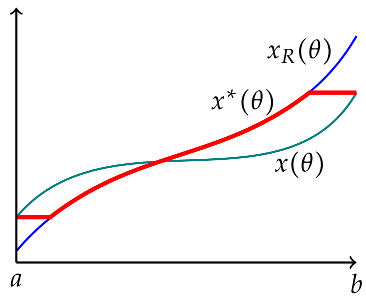
Proof. Fix any increasing $x \in \mathcal{X}^{[0,1]}$ and consider

$$x^*(\theta) = \begin{cases} \min\{x(b), \max\{x(a), x_R(\theta)\}\} & \text{if } \theta \in [a,b), \\ x(\theta) & \text{elsewhere.} \end{cases}$$

x^* is point-wise between x and x_R, so $F[x^*] \ge F[x]$ by Fact 1. Moreover, x^* is increasing on $[a,b)$ as x_R is increasing on $[a,b)$ by assumption. □



Proof of Lemma 1



Proof of Lemma 2

Observations similar to Lemmas 1 and 2 were independently made by Sandmann (2022) who studies the optimality of sparse menus in a price discrimination problem.

Theorem 1. *Let $x^* : [0,1] \times \mathcal{X}^{|\mathcal{I}|-1} \to \mathcal{X}$ be given by:*

$$x^*(\theta; v) = \begin{cases} \min\{v_{n+1}, \max\{v_n, x_R(\theta)\}\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing,} \\ v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing.} \end{cases} \tag{1}$$

Then, if the planner's problem has a solution, it has a solution of the form $x^*(\theta; v)$. This solution can be recovered by solving:

$$\max_{v \in \mathcal{X}^{|\mathcal{I}|-1}} F[x^*(\cdot\,; v)] \quad \text{subject to} \quad v_1 \le v_2 \le \cdots \le v_{|\mathcal{I}|-1}. \tag{I}$$

Proof. By Assumption 2, $[0,1)$ can be partitioned into finitely many intervals $[i_n, i_{n+1})$ where x_R is monotonic. Then, by Lemmas 1 and 2, if the planner's problem has a solution, it has one of the form in (1) with $v_1 \leq v_2 \leq \cdots \leq v_{|\mathcal{I}|-1}$. Solving problem (I) recovers this solution. □

4 Solution algorithm

I now provide a simple recursive algorithm that solves planner's problem under the following additional assumption:

Assumption 3. *Let $\mathcal{X} = [l,h]$, $J(\cdot,\theta)$ be weakly concave for every $\theta \in [0,1]$ and $J(x,\theta)$ be uniformly bounded across all $x \in [l,h]$ and $\theta \in [0,1]$.*

The algorithm uses the following transformation $T : [l,h]^{[0,1]} \times [l,h] \times \mathbb{N} \to [l,h]^{[0,1]}$:

$$\mathrm{T}[x,v,n](\theta) := \begin{cases} \min\{v, x(\theta)\} & \text{if } \theta < i_n, \\ v & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \max\{v, x_R(\theta)\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing,} \\ x(\theta) & \text{if } \theta > i_{n+1}. \end{cases}$$

When $T[\cdot, v, n]$ is applied to x, the allocation rule is truncated from above to v before the kth critical point, and set equal to v or truncated to it from below between the kth and $k+1$st critical points.

Algorithm 1

$r \leftarrow x_R$
$n \leftarrow 1$
while $n \leq |\mathcal{I}| - 1$ **do**
  $v^* \leftarrow \max_{v \in [l,h]} \int_0^{i_n} J(\mathrm{T}[r,v,n](\theta)) d\theta$
  $r \leftarrow \mathrm{T}[x, v^*, n]$
  $n \leftarrow n + 1$
end while
return r

Algorithm 1 takes the solution to the relaxed problem x_R and critical points $i \in \mathcal{I}$ as primitives. Then each iteration of the algorithm starts with the optimum subject to monotonicity from 0 until the nth critical point, and 'irons out' this partial solution further to produce the solution subject to monotonicity until the $n+1$st critical point (Figure 2).

Before proving Algorithm 1 solves the planner's problem, I present lemmas illustrating the role of concavity of $J(\cdot,\theta)$ for this result.

Lemma 3. *Let $F_a[x] := \int_0^a J(x(\theta),\theta) d\theta$. $F_a[x]$ is concave in x for every $a \in [0,1]$.*

Proof. Fix $\alpha \in [0,1]$ and $x_1, x_2 \in [l,h]^{[0,1]}$. $J(\cdot,\theta)$ is concave, so for all $\theta \in [0,1]$:

$$J(\alpha x_1(\theta) + (1-\alpha)x_2(\theta), \theta) \geq \alpha J(x_1(\theta),\theta) + (1-\alpha) J(x_2(\theta),\theta).$$

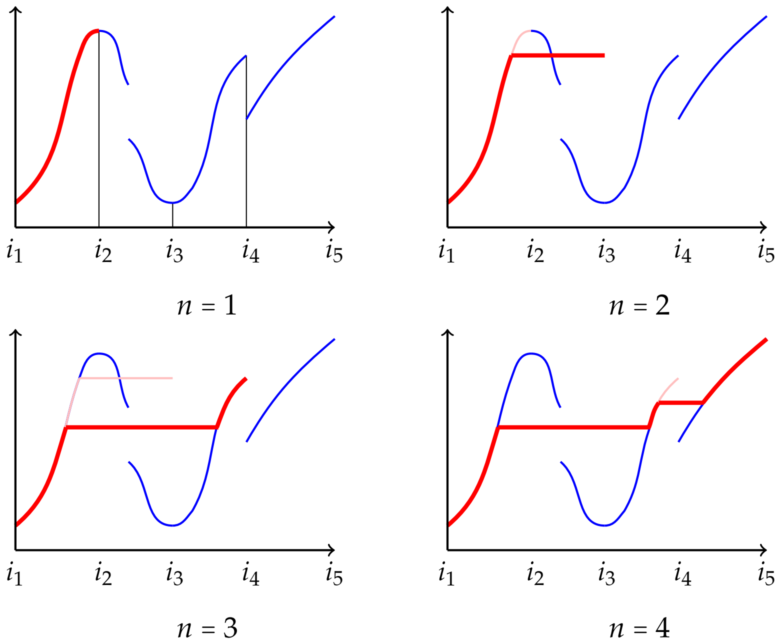


Figure 2: Algorithm 1 recursively transforming x_R (blue) into subsequent $\mathrm{T}[x, v^*, n]$ (red).

Integrating the inequality from 0 to a gives the lemma. □

Lemma 4. *The set $\mathcal{O}_a^k$ defined below is non-empty for any $a \in [0,1]$ and $k \in [l,h]$:*

$$\mathcal{O}_a^k := \operatorname{argmax}\left\{F_a[x] : \ x \in [l,h]^{[0,1]} \text{ is increasing and below } k \text{ on } [0,a)\right\}.$$

Proof. Since the values of x on $(a,1]$ do not affect $F_a[x]$, we can look for maximizers in $[l,h]^{[0,a]}$. Let $\mathcal{J}$ be the set of $x \in [l,k]^{[0,a]}$ that are increasing. I will show F_a attains a maximum on $\mathcal{J}$. $\mathcal{J}$ is homeomorphic to a Helly space and so is compact. Hence, by Weierstrass' theorem, it suffices to show F_a is continuous on this space. Since $\mathcal{J}$ is a Helly space, it is first-countable so F_a is continuous on it if for any sequence in $\mathcal{J}$ such that $x_n \to x$ we have $F_a[x_n] \to F_a[x]$. Fix any such x_n. Then $x_n(\theta) \to x(\theta)$ for every $\theta \in [0,a]$. Since $J(\cdot,\theta)$ is concave, it is continuous, which implies $J(x_n(\theta),\theta) \to J(x(\theta),\theta)$ for every $\theta \in [0,a]$. Hence, the Dominated Convergence Theorem implies $F_a[x_n] \to F_a[x]$. □

The following Proposition is shown in the Appendix. It says that when the additional constraint that x be below k on $[0,a)$ is imposed, we need not resolve for the optimum but can simply truncate the solution without this constraint to k.

Proposition 1. *If $x_a \in \mathcal{O}_a^h$, then $\max\{k, x_a\} \in \mathcal{O}_a^k$.*

Theorem 2. *Under Assumption 3, the output of Algorithm 1 solves the planner's problem.*

Proof. I proceed by induction. Recall that $|\mathcal{I}| \geq 2$ since $0,1 \in \mathcal{I}$. The base case demonstrates that the first iteration produces $r \in \mathcal{O}_{i_2}^h$. The step shows that when the nth iteration starts with $r \in \mathcal{O}_{i_n}^h$, it outputs $r \in \mathcal{O}_{i_{n+1}}^h$. Hence, the algorithm will produce $r \in \mathcal{O}_{i_{|\mathcal{I}|}}^h$ solving the planner's problem in $|\mathcal{I}| - 1$ steps.

Base. By Lemma 4, $\mathcal{O}^h_{i_2} \neq \varnothing$. Then by Lemmas 1 and 2 there exists $x^* \in \mathcal{O}^h_{i_2}$ for which:

$$x^*(\theta) := \begin{cases} v_1 & \text{if } \theta \in [i_1, i_2) \text{ where } x_R \text{ is decreasing,} \\ \min\{v_1, \max\{v_2, x_R(\theta)\}\} & \text{if } \theta \in [i_1, i_2) \text{ where } x_R \text{ is increasing.} \end{cases}$$

Moreover, Fact 1 tells us that in the latter case there exists $x^* \in \mathcal{O}^k_{i_2}$ for which $v_2 = h$. In either case, the algorithm will recover such an x^* when optimizing over v.

Step. By Lemma 4, $\mathcal{O}^h_{i_{n+1}} \neq \varnothing$. Then by Lemmas 1 and 2 there exists $x^* \in \mathcal{O}^h_{i_2}$ for which:

$$x^*(\theta) := \begin{cases} v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \min\{v_n, \max\{v_{n+1}, x_R(\theta)\}\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing.} \end{cases}$$

Moreover, Fact 1 tells us that in the latter case there exists $x^* \in \mathcal{O}^k_{i_{n+1}}$ for which $v_{n+1} = h$. Now, let $r \in \mathcal{O}^h_{i_n}$ be the function produced by the previous iteration of the algorithm. By Proposition 1, $\max\{v_n, r\} \in \mathcal{O}^{v_n}_{i_n}$. Hence, there exists $x^* \in \mathcal{O}^h_{i_{n+1}}$ satisfying:

$$x^*(\theta) := \begin{cases} \min\{v_n, r(\theta)\} & \text{if } \theta < i_n, \\ v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \max\{v_n, x(\theta)\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing.} \end{cases}$$

The nth iteration of the algorithm will recover such an x^* when optimizing over v. □

5 Appendix

5.1 Proof of Proposition 1.

Proof. If $x_a(\theta) \leq k$ for $\theta \in [0, a)$, the result is trivial as the additional constraint does not bind. Consider the case where $x_a(\theta) > k$ for some $\theta \in [0, a)$ and let $b := \inf\{\theta : x_a(\theta) \geq k\}$. Notice we can assume $x^a(b) = k$ as this does not violate monotonicity or alter the value of $F_a[x_a]$.

Lemma 5. *Fix $z \in [b, a)$ such that some sequence of functions $f_n \in \mathcal{O}^k_a$ converges point-wise to k on (z, a). Take any $x_z \in \mathcal{O}^k_z$ (it exists by Lemma 4). Then any x^* satisfying (2) belongs to $\mathcal{O}^k_a$.*

$$x^*(\theta) = \begin{cases} x_z(\theta) & \text{if } \theta < z, \\ k & \text{if } \theta \in [z, a). \end{cases} \tag{2}$$

Proof. x^* is increasing and below k on $[0, a)$ so it suffices to show that $F_a[x^*] \geq F_a[f_n]$ for some n. By assumption, $f_n(\theta) \to k$ for every $\theta \in (z, a)$. $J(\cdot, \theta)$ is concave and hence continuous, so $J(f_n(\theta), \theta) \to J(k, \theta)$ for all $\theta \in (z, a)$. Then the Dominated Convergence Theorem gives:

$$\int_z^a J(f_n(\theta), \theta) d\theta \to \int_z^a J(k, \theta) d\theta. \tag{3}$$

Take any $\epsilon > 0$. Then there exists n^* for which:

$$\int_z^a J(k, \theta) d\theta + \epsilon > \int_z^a J(f_{n^*}(\theta), \theta) d\theta. \tag{4}$$

Note $f_n(z) \le k$, so functions in $\mathcal{O}_z^k$ guarantee a weakly higher F_z than those in $\mathcal{O}_z^{f_n(z)}$. Hence:

$$\int_0^z J(x_z(\theta),\theta)d\theta \ge \int_0^z J(f_{n^*}(\theta),\theta)d\theta. \tag{5}$$

Adding together (4) and (5) gives:

$$\int_0^a J(x^*(\theta),\theta)d\theta + \epsilon > \int_0^a J(f_{n^*}(\theta),\theta)d\theta.$$

Since ϵ was arbitrary and $f_{n^*} \in \mathcal{O}_a^k$, we get $x^* \in \mathcal{O}_a^k$. □

Let c be the earliest θ (across all $x \in \mathcal{O}_a^k$) where x takes value k:

$$c = \inf_{\theta\in[b,a],\ x\in\mathcal{O}_a^k} \{\theta : x(\theta) = k\}.$$

By monotonicity and the definition of c, there exists a sequence of $f_n \in \mathcal{O}_n^k$ converging pointwise to k on (c,a). Hence, Lemma 5 tells us there exists $x \in \mathcal{O}_a^k$ for which $x(c) = k$.

Let $u := \sup\{x(b) : x \in \mathcal{O}_a^k \text{ and } x(c) = k\}$ and fix some positive sequence $\{\delta_n\}_n$ such that $\delta_n \to 0$. Then, for any n, there exists $g_n \in \mathcal{O}_a^k$ such that $g_n(c) = k$ and $g_n(b) + \delta_n > u$. Suppose towards a contradiction that $u < k$. Since there are $x \in \mathcal{O}_a^k$ s.t. $x(c) = k$, it follows that $b < c$. Now, by Assumption 2 x_R is monotonic on $[c-\epsilon, c)$ for ϵ small enough. Choose such an ϵ small enough that $b < c - \epsilon$.

Lemma 6. *x_R is decreasing on $[c-\epsilon, c)$.*

Proof. Let x_R be increasing on $[c-\epsilon,c)$ and take $x^* \in \mathcal{O}_a^k$ such that $x^*(c) = k$. Suppose there exists $\epsilon' \in (0,\epsilon)$ such that $x^R(c-\epsilon') \ge k$. Then $x_R(\theta) \ge k$ on $[c-\epsilon', c)$. By Fact 1, we can improve upon x^* by setting it to k on $[c-\epsilon', c)$. This contradicts the definition of c. Consider the opposite case where $x_R(\theta) < k$ everywhere on $[c-\epsilon, c)$. By the definition of c, we have $x^*(c-\epsilon) < k$. Since $b < c-\epsilon$, we also know that $k \le x_a(c-\epsilon)$. Hence, there exists $\alpha \in (0,1)$ for which $\alpha x_a(c-\epsilon) + (1-\alpha)x^*(c-\epsilon) = k$. Now consider:

$$x^{**}(\theta) = \begin{cases} \alpha x_a(\theta) + (1-\alpha)x^*(\theta) & \text{if } \theta < c-\epsilon, \\ k & \text{if } \theta \in [c-\epsilon, a). \end{cases}$$

I will show $x^{**} \in \mathcal{O}_a^k$. x_a and x^* are increasing on $[0,a)$, so their convex combination is too. It is also weakly below k on $[0, c-\epsilon)$ and so x^{**} is increasing and below k on $[0,a)$. It therefore suffices to show that $F_a[x^{**}] \ge F_a[x^*]$. Recall $x_a \in \mathcal{O}_a^h$ solves a problem with laxer constraints than $x^* \in \mathcal{O}_a^k$, so $F_a[x_a] \ge F_a[x^*]$. Hence, by Lemma 3:

$$F_a[\alpha x_a + (1-\alpha)x^*] \ge F_a[x^*].$$

Notice x^{**} and $\alpha x_a + (1-\alpha)x^*$ coincide on $[0,a)$ except on $[c-\epsilon, c)$, where $x^{**}(\theta) \le \alpha x_a(\theta) + (1-\alpha)x^*(\theta)$. However, recall that $x_R < k$ on $[c-\epsilon, c)$, and so $F_a[x^{**}] > F_a[\alpha x_a + (1-\alpha)x^*]$ by Fact 1. Therefore $x^{**} \in \mathcal{O}_a^k$. Since $x^{**}(c-\epsilon) = k$, this contradicts the definition of c. □

I now show that for any n, there exists $g_n^* \in \mathcal{O}_n^k$ such that $g_n^*(b) + \delta_n > u$, $g_n^*(\theta)$ is constant on $[c-\epsilon, c)$, and $g_n^*(c) = k$. Lemma 6 tells us that x_R is decreasing on $[c-\epsilon, c)$ and so, by

Lemma 1, for every n there exists $g_n^* \in \mathcal{O}_a^k$ that is constant on $[c-\epsilon, c)$ and coincides with to g_n elsewhere. Since $b, c \notin [c-\epsilon, c)$, for any n we also have $g_n^*(b) + \delta_n > u$ and $g_n^*(c) = k$.

Moreover, the sequence of functions $\{g_n^*\}_n$ has to be uniformly bounded away from k on $[0, c)$. By the construction of g_n^*, for any n there exists $\gamma_n \in \mathbb{R}$ s.t. $g_n^*(\theta) = \gamma_n$ for $\theta \in [c-\epsilon, c)$. Suppose towards a contradiction that $\sup \gamma_n = k$. Then there exists a subsequence of $\{g_n^*\}_n$ that converges point-wise to k on $[c-\epsilon, c)$. Lemma 5 then tells us there exists an $x^* \in \mathcal{O}_a^k$ equal to k on $[c-\epsilon, c)$. This contradicts the definition of c.

Thanks to this uniform bound, we know there exists $\alpha \in (0,1)$ such that for all n and $\theta \in [0, c)$ we have $\alpha x_a(\theta) + (1-\alpha) g_n^*(\theta) \le k$. I now prove the following lemma:

Lemma 7. *The function g_n^{**} defined below belongs to $\mathcal{O}_a^k$ for any n.*

$$g_n^{**}(\theta) = \begin{cases} \alpha x_a(\theta) + (1-\alpha) g_n^*(\theta) & \text{if } \theta < c, \\ k & \text{if } \theta \ge c. \end{cases}$$

Proof. $x_a(\theta) + (1-\alpha) g_n^*(\theta)$ is increasing on $[0, c)$ as g_n^* and x^a both were. By the choice of α, $x_a(\theta) + (1-\alpha) g_n^*(\theta)$ is also below k on $[0, c)$. Hence g_n^{**} is increasing and below k on $[0, a)$. It suffices to show that $F_a[g_n^{**}] \ge F_a[g_n^*]$. Note that $F_c[x_a] \ge F_c[g_n^*]$; else $x_a \in \mathcal{O}_a^k$ could be improved by replacing it for g_n^* on $[0, c)$ (recall $g_n^*(c) \le k = x_a(c)$ so this would not violate monotonicity). $F_c[x_a] \ge F_c[g_n^*]$ and Lemma 3 give:

$$F_c[x_a(\theta) + (1-\alpha) g_n^*] \ge F_c[g_n^*]. \tag{6}$$

Since $g_n^*(c) = k$, g_n^* coincides with g_n^{**} on $[c, a)$. Hence, (6) implies that $F_a[g_n^{**}] \ge F_a[g_n^*]$. □

Since $b < c$ and $x_a(b) = k$, we have:

$$g_n^{**}(b) = \alpha k + (1-\alpha) g_n^*(b) > \alpha k + (1-\alpha)(u - \delta_n),$$

where the latter inequality holds because $g_n^*(b) + \delta_n > u$. Since $u < k$, for sufficiently large n we have $g_n^{**}(b) > u$ which contradicts the definition of u. Hence, $u = k$. Notice also that since $x_a(\theta) = k$ for $\theta \in [b, a)$, we have $x_a \in \mathcal{O}_b^k$; else x_a could be feasibly improved upon by changing it for some $x^* \in \mathcal{O}_b^k$ on $[0, b)$. Applying Lemma 5 completes the proof. □

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