Targeting Without Transfers*

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Abstract

I study the welfare-maximizing allocation of heterogeneous goods when monetary transfers are prohibited. Agents have private cardinal values, and the designer chooses a non-monetary mechanism subject to incentive compatibility and aggregate supply constraints. I characterize implementable allocations and give sufficient conditions under which the optimum coincides with a competitive equilibrium with equal incomes (CEEI). When these conditions fail, I characterize the optimum for two symmetric goods. I show that when narrow preference margins between goods predict greater need, the designer can sometimes benefit from distorting CEEI by offering a menu containing pure options and bundles.

1 Introduction

When designing mechanisms without transfers, it is often natural to evaluate them using criteria that avoid interpersonal utility comparisons. This approach is especially appealing when the policymaker has explicitly non-welfarist goals (such as fairness) or when participants' cardinal valuations for the allocated goods are plausibly similar. Indeed, the literature on mechanisms without money has largely focused on notions based on Pareto efficiency and ordinal welfare rankings.[1] Nevertheless, criteria agnostic to cardinal values are less fitting for settings like social programs, where policymakers view applicants as differing sharply in terms of need and aim to target those for whom receiving the goods has the greatest social value. For instance, affordable housing programs in many European countries serve a broad population, including families facing eviction as well as middle-class households with stable employment (Whitehead and Scanlon, 2007). In the U.S. context, Cook et al. (2023) find that affordable housing recipients differ substantially in various measures of need, and that this heterogeneity persists even after conditioning on observables.

This paper studies a mechanism design problem without transfers where the designer has a prior over agents' cardinal values for the allocated goods. She possesses a fixed supply of N

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[1]See, among others, Hylland and Zeckhauser (1979); Abdulkadiroğlu and Sönmez (1998); Bogomolnaia and Moulin (2001); Abdulkadiroğlu and Sönmez (2013).

different kinds of goods and aims to distribute them among a unit mass of agents to maximize utilitarian welfare. Importantly, agents' valuations are their private information; this prevents the designer from simply giving the available supply to those who need it most. Indeed, handing out larger allocations to agents who claim to have higher values would incentivize everyone to make such claims.

The designer can, however, elicit agents' *relative* preferences, that is, how much they value some goods compared to others, or how much it matters to them which option they receive. This information can be especially helpful when such preference patterns are correlated with agents' *absolute* level of need. Relationships of that sort are common in the context of social programs. For instance, Cook et al. (2023) find that lower-income households are less selective when applying for affordable housing, that is, they are more willing to trade off assignment to a preferred unit for a higher probability of receiving an offer *somewhere*. I show that optimal mechanisms sometimes exploit such statistical relationships: when participants with higher cardinal valuations tend to have weaker relative preferences, the designer can reward them with larger mixed bundles that "pickier" types are not willing to accept. In other settings, however, the correlation between preference intensities and absolute valuations is likely to be reversed. Consider, for example, school choice environments with specialized curricula such as dual-language immersion. Families who place disproportionate weight on admission to such programs often do so because of the child's idiosyncratic needs, aptitudes, or interests. Thus, intense *relative* preference for a particular option plausibly signals higher *absolute* value for it. When this is the case, offering mixed bundles is likely to be suboptimal.

I prove three main results. First, I characterize which allocations are implementable in the no-transfer environment. After renormalizing types to the simplex of relative values, I show that an indirect utility function is implementable if and only if it is convex and satisfies a new restriction constraining how fast utilities can change in the direction of each of the simplex's vertices.

Second, I consider the mechanism implementing a *competitive equilibrium with equal incomes* (CEEI). In a CEEI, each agent receives the same artificial budget and purchases her preferred bundle at market-clearing prices. Despite the fact that the designer has access to a rich space of mechanisms, I show that the CEEI mechanism is exactly welfare-maximizing for a non-trivial class of distributions. The sufficient conditions for its optimality are stated as a stochastic-dominance comparison on appropriately-constructed signed measures. I also derive a simpler condition in the special case of symmetric goods: the CEEI mechanism is optimal if agents whose cardinal values for their favorite goods are higher tend to be more selective, in a precise stochastic sense. Intuitively, when this is the case, any distortion away from the CEEI, which is the unique implementable Pareto-efficient allocation, reallocates resources toward relatively less-deserving types.

Third, I fully characterize the welfare-maximizing mechanism in the case of two symmetric goods. Here the renormalization effectively makes types one-dimensional, which eliminates the complications of multidimensional screening. I show that in such a setting, the optimal mechanism has an especially simple form: it either offers two "pure" options consisting of one type of good only, or introduces a third option: a larger mixed bundle that combines the two goods in equal proportions. The mixed option screens on the strength of relative preferences: types with narrower margins across goods are more willing to accept mixing and therefore self-select into the larger bundle. This distortion is welfare-improving precisely when weaker margins are sufficiently predictive of higher total value, so that the informational gain from targeting outweighs the allocative inefficiency from mixing.

My paper contributes to the literature on allocating heterogeneous goods without transfers, and connects most directly to the work on pseudo-markets and CEEI. Hylland and Zeckhauser (1979) introduce CEEI as a solution concept for assignment problems. Budish (2011) proposes an approximate CEEI mechanism for combinatorial assignment (such as course schedules), and Budish et al. (2017) document a large-scale implementation. In environments with priorities and related constraints, He et al. (2018) propose a pseudo-market that uses token budgets and priority-dependent prices to produce a fair and constrained-efficient random assignment.

While the literature on allocating heterogeneous goods without transfers has focused mainly on (ex ante and ex post) Pareto efficiency and ordinal efficiency properties, a smaller body of work allows for *cardinal* objectives and looks for mechanisms that maximize them (Miralles, 2012; Chakravarty and Kaplan, 2013; Ashlagi and Shi, 2016; Dogan and Uyanik, 2020; Akyol, 2025). My paper is the closest to Miralles (2012), who studies welfare-maximizing mechanisms with cardinal utilities in a symmetric, two-good setting with finite agents. He shows that while the welfare optimum can deviate from CEEI in finite markets, CEEI becomes optimal in a large-market limit under additional regularity conditions. In this sense, the departures from CEEI in Miralles (2012) are a small-sample phenomenon, and thus arise for reasons logically distinct from those I study. My results are therefore complementary to his: while I focus on large markets, I show that without his regularity condition mechanisms other than CEEI can be optimal for screening reasons.

Finally, my paper relates to work on eliciting preference intensities—information about how strongly agents prefer some options over others. In school choice, Abdulkadiroğlu et al. (2011) observe that the Boston mechanism can elicit the *extent* to which families prefer certain schools—a property that deferred acceptance does not have. In a paper closely related to mine, Ortoleva et al. (2021) consider optimal mechanisms in a setting without transfers where agents have a common ranking over goods but differ in their sensitivity to quality. My paper, by contrast, does not impose such structure and considers heterogeneously differentiated goods. This leads to different and complementary results. Indeed, the authors show that the first-best allocations may offer lotteries between qualities, and that second-best allocations always involve lotteries and may involve free disposal; neither of these results holds in my setting. Similarly to my work, they show that CEEI allocations, despite being Pareto-efficient, do not always maximize weighted welfare.

Finally, my paper builds on methods developed in the multidimensional screening literature. My characterization of implementability extends that of Rochet (1987) to settings without transfers. To derive conditions for the optimality of CEEI, I invoke ideas used in the study of the multi-product monopoly problem (Armstrong, 1996; Rochet and Choné, 1998; Manelli and Vincent, 2006). In particular, my certificate of optimality relies on stochastic dominance and transport arguments related to those in Daskalakis et al. (2013, 2017).

The rest of the paper is structured as follows. Section 2 presents the general model and Section 3 illustrates its core intuitions with simple two-good examples. Then, Section 4 characterizes implementable mechanisms in the general case. The subsequent part of the paper focuses on the mechanism corresponding to a competitive equilibrium with equal incomes (CEEI): Section 6 defines the CEEI mechanism and gives sufficient conditions for its optimality in the N-good case. Section 7 specializes the model to two symmetric goods and fully characterizes the welfare-maximizing mechanism. Finally, Section 8 discusses the implications of the results for market design, with a focus on public housing lotteries.

2 Model

The designer has N different kinds of goods indexed by $i \in \{1, \ldots, N\}$ with $N \geq 2$. She possesses a fixed mass of each, with the supplies given by $s = (s_1, s_2, \ldots, s_N) > 0$. There is a unit mass of agents, each of whom has a profile of values $v = (v_1, v_2, \ldots, v_N)$ for the goods; the values are private information and come from a bounded set $\mathcal{V} \subset \mathbb{R}_+^N$ such that for some $\epsilon > 0$ we have $[0, \epsilon]^N \subset \mathcal{V}$. They are distributed according to a joint distribution F with full support on $\mathcal{V}$. The designer chooses an allocation rule for the goods, $y = (y_1, y_2, \ldots, y_N) : \mathcal{V} \to \mathbb{R}_+^N$, to maximize utilitarian welfare:

$$\int_{\mathcal{V}} v \cdot y(v) dF(v). \tag{O}$$

She faces incentive compatibility and supply constraints:

$$v \cdot y(v) \geq v \cdot y(v') \quad \text{for all } v, v', \tag{IC}$$

$$\int y(v) dF(v) \leq s. \tag{S}$$

An allocation rule $y : \mathcal{V} \to \mathbb{R}_+^N$ that satisfies (IC) is *implementable*. If this allocation rule also satisfies (S), I call it *feasible*.

Remark 1. *One might wonder how to understand agents' cardinal values in a setting where transfers are not permitted. The model allows for multiple interpretations. First, one can still identify v_i with an agent's (latent) willingness to pay for a unit of good i. While these values are not directly elicitable without money, they remain meaningful for the designer's welfare objective. Second, and more generally, one can view them as the designer's subjective conviction about the social value of giving goods to different agents. For instance, the designer may place higher welfare weights on individuals with certain* *characteristics (need, vulnerability, family size, etc.), and believe that these characteristics are correlated with the pattern of preferences agents reveal over the available goods.*

Remark 2. *Some allocation problems without money involve unit demand, in which case an allocation specifies a probability of receiving each good; this is the case e.g. in housing lotteries. While my model does not impose a probability constraint $\sum_i y_i(v) \leq 1$, it nevertheless describes unit-demand environments where supply is sufficiently scarce relative to the population: in such cases, the designer could not afford to offer agents any option with certainty, and so every type's probability constraint would remain slack. This is a plausible approximation in applications such as public housing lotteries where units are extremely scarce relative to applicant numbers.*

3 Examples

To preview the paper's core intuitions, I begin with illustrative examples featuring just two goods. I derive them from Theorems 1 and 2 in Appendix B.

Example 1. *Fix any supplies $s_1, s_2 > 0$ and let values be distributed uniformly on $[0,1]^2$. Then the optimal mechanism offers agents two options:*

$$\left\{q_1 \textit{ of good } 1\right\}, \quad \left\{q_2 \textit{ of good } 2\right\}.$$

The quantities q_1, q_2 are chosen so that the supply constraint holds with equality when all agents pick their preferred option.

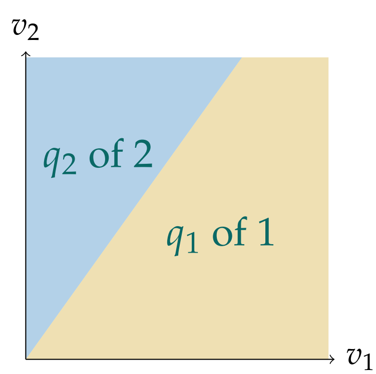

Figure 1: Optimal allocation in Example 1.

Under this mechanism, agents for whom $q_1\, v_1 > q_2\, v_2$ select the former option, while those for whom $q_1\, v_1 < q_2\, v_2$ select the latter. As shown in Figure 1, these two sets of types are separated by a ray from the origin defined by:

$$\frac{v_1}{v_2} = \frac{q_2}{q_1}. \tag{1}$$

Let us note two things about this allocation. First, it can be supported as a *competitive equilibrium with equal incomes*. That is, the designer could implement it by running a procedure where each agent is endowed with a unit amount of token currency that she can use to buy goods at market-clearing prices. In this case, the market-clearing prices will equal $p_1 = 1/q_1$ per unit of good 1 and $p_2 = 1/q_2$ per unit of good 2. Agents below the ray defined by (1) will then spend their budget on q_1 of good 1 while those above it will buy q_2 of good 2.

Second, note that the allocation rule in Example 1 depends only on the *ratio* of agents' values for goods 1 and 2, but not on how large v_1 and v_2 are in absolute terms. This highlights a useful distinction: an agent's *absolute values*, (v_1, v_2), capture the overall intensity of need for the goods, while her *relative values*, $(\frac{v_1}{v_1+v_2}, \frac{v_2}{v_1+v_2})$, capture how strongly she prefers one good over another. Crucially, an incentive-compatible mechanism cannot meaningfully elicit absolute values among agents with the same profile of relative values. Indeed, all agents with the same relative values always rank all offered options the same way. It is thus impossible to give a better bundle to some of them without also giving it to the others.

The designer can, however, elicit *relative* values by offering a menu with different bundles of goods. This motivates the next example:

Example 2. *Let $s_1 = s_2$ and assume values are distributed according to the following density, illustrated in Figure 2a:*

$$f(v_1, v_2) = \begin{cases} 20, & (v_1, v_2) \in [0,1]^2 \text{ and } v_1 + v_2 \leq 0.2 \text{ or } v_1 + v_2 \geq 1.8, \\ \frac{5}{24}, & (v_1, v_2) \in [0,1]^2 \text{ and } 0.2 < v_1 + v_2 < 1.8. \end{cases}$$

Then the optimal mechanism offers three options:

$$\left\{q_L \text{ of good } 1\right\}, \quad \left\{q_L \text{ of good } 2\right\}, \quad \left\{\frac{q_H}{2} \text{ of good } 1 \text{ and } \frac{q_H}{2} \text{ of good } 2\right\},$$

for some $q_L < 2s$ and $q_H > 2s$.

Under this mechanism, each agent can pick between a low amount of their favorite good and a higher amount of an even mixture of the two goods. Agents with strong relative preferences between the two goods pick the pure allocations and agents whose preference margins between goods are narrow choose the mixture.

Here too, all agents with the same relative values $(\frac{v_1}{v_1+v_2}, \frac{v_2}{v_1+v_2})$ receive the same allocation. However, agents whose relative values are close together choose the bundle and thus receive higher total allocations. Crucially, these agents also tend to have higher *absolute* values (v_1, v_2), and so the use of bundles gives the designer an incentive-compatible way of directing more goods to agents in greater need. More generally, doing so can help the designer if relative and absolute values are statistically related. In such cases, she can sometimes proxy for high absolute values by offering more attractive options to agents with certain relative preferences.

Note, however, that the optimal allocation in Example 2 is not Pareto-efficient. Indeed, agents

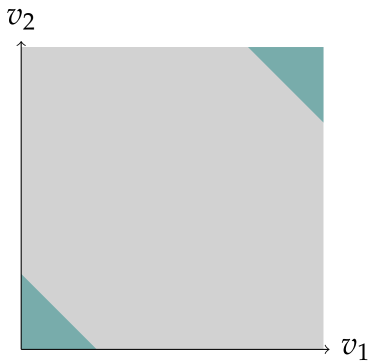

Figure 2a: Value distribution in Example 2.

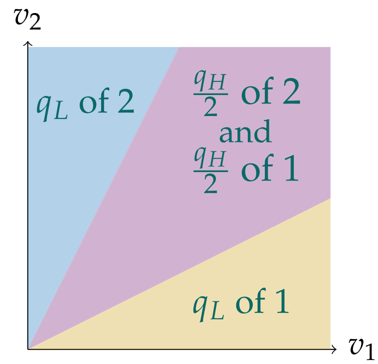

Figure 2b: Optimal allocation in Example 2.

who get the bundle could profitably trade between themselves so that types above and below the 45-degree line in Figure 2b get only the good they prefer.

4 Implementability

I now discuss what mechanisms are implementable. First, however, I transform the type space into a more analytically convenient form. As explained in the previous section, the designer cannot meaningfully elicit absolute values of agents who share the same profile of relative values. It is therefore without loss to identify agents' types with their profile of relative values. Let Γ be the $(N-1)$-simplex containing all possible profiles of relative values:

$$\Gamma := \left\{\theta \in \mathbb{R}_+^N : \sum \theta_i = 1\right\}.$$

Define V as the random variable describing the value vector v of an agent drawn from F and let Θ be the following Γ-valued random variable:[2]

$$\Theta := \frac{V}{\sum_j V_j}.$$

The renormalization thus maps all sets of types that were identical up to scaling to the same renormalized type $\theta \in \Gamma$. The distribution of the random variable Θ will then pin down that over our renormalized types. Denote this distribution by G and note that it is the push-forward of F under the map $v \mapsto v / \sum_j v_j$.

While the designer cannot screen on absolute values, they are still important for her objective. We will therefore define:

$$\lambda(\theta) := \mathbb{E}\left[\sum_j V_j \;\middle|\; \frac{V_i}{\sum_j V_j} = \theta_i \text{ for all } i\right],$$

[2] Note that we can without loss exclude the **0** type, and so we need not worry about dividing by 0.

which assigns to each renormalized type θ the expected total value of agents whose types v got mapped to θ.[3] Using this object, we can rewrite the designer's problem as follows:

Problem 1. *Choose an allocation rule $x : \Gamma \to \mathbb{R}^N_+$ to maximize weighted expected utility:*

$$\int_\Gamma \lambda(\theta)\, U(\theta)\, dG(\theta), \tag{O′}$$

where $U(\theta) = x(\theta) \cdot \theta$, subject to:

$$\theta \cdot x(\theta) \;\geq\; \theta \cdot x(\theta') \quad \textit{for all} \quad \theta, \theta' \in \Gamma, \tag{IC′}$$

$$\int_\Gamma x(\theta)\, dG(\theta) \;\leq\; s. \tag{S′}$$

Indeed, Problem 1 is equivalent to the designer's original problem in the following sense:

Lemma 1. *For any feasible allocation rule $y : \mathcal{V} \to \mathbb{R}^N_+$, define*

$$x(\theta) \;:=\; \mathbb{E}\left[\, y(V) \mid \Theta = \theta \,\right]. \tag{2}$$

Then x is feasible in Problem 1 and welfare from y equals the (renormalized) welfare from x:

$$\int_{\mathcal{V}} v \cdot y(v)\, dF(v) \;=\; \int_\Gamma \lambda(\theta)\, \theta \cdot x(\theta)\, dG(\theta). \tag{3}$$

Conversely, for any feasible x in Problem 1, the allocation rule $y(v) := x(v/\sum v_i)$ is feasible for the original problem and the two allocation rules satisfy (3).

The above reparametrization will let me formulate my first theorem, which specifies the indirect utility functions $U : \Gamma \to \mathbb{R}_+$ that the designer can implement subject to incentive constraints (IC′). To that end, I introduce the following definition:

Definition 1. *Take $\theta, \theta' \in \Gamma$ with $\theta_i, \theta'_i > 0$. We say θ is closer to vertex e_i than θ', denoted by $\theta \succ_i \theta'$, if for all $k \neq i$:*

$$\frac{\theta_k}{\theta_i} \leq \frac{\theta'_k}{\theta'_i}.$$

Intuitively, $\theta \succ_i \theta'$ means that θ values good i relatively more than does θ', compared to every other good (Figure 3).

Proposition 1. *An indirect utility function $U : \Gamma \to \mathbb{R}$ is implementable if and only if it is convex and satisfies the following condition:*

$$\textit{for every } i \textit{ and every } \theta, \theta' \textit{ in } \Gamma \textit{ such that } \theta \succ_i \theta', \quad \frac{U(\theta')}{\theta'_i} \;\geq\; \frac{U(\theta)}{\theta_i}. \tag{R}$$

[3]The assumption that F had full support over the hypercube $[0, \epsilon]^N$ ensures that G has full support over Γ, and that $\lambda(\theta)$ is well-defined and strictly positive everywhere on it.

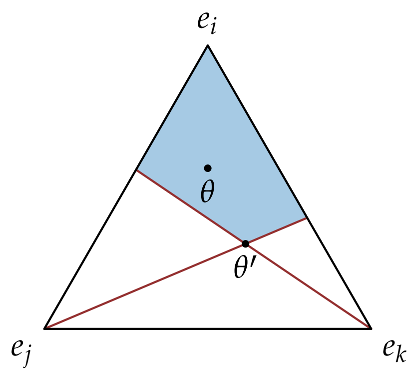

Figure 3: Types in the shaded area are closer to e_i than θ', i.e. $\theta \succ_i \theta'$.

It is not surprising that implementable indirect utility functions U need to be convex, as they are maxima of affine functions of θ:

$$U(\theta) = \max_{\theta' \in \Gamma} \theta \cdot x(\theta').$$

Condition (R) additionally it restricts how fast indirect utility $U(\theta)$ can grow as θ moves towards the vertex e_i. To understand why (R) is necessary for implementability, fix any good i and two types such that $\theta \succ_i \theta'$. Note that normalizing $U(\theta)$ by θ_i gives:

$$\frac{U(\theta)}{\theta_i} = \sum_{k \neq i} \frac{\theta_k}{\theta_i} x_k(\theta) + x_i(\theta).$$

We can then equivalently think of type-θ agents as maximizing their scaled utilities $U(\theta)/\theta_i$. Recall also that by the definition of the $\succ_i$-order, all the ratios θ'_k/θ'_i are higher for θ' than for θ. This implies that type θ' can always guarantee a higher scaled indirect utility than type θ:

$$\frac{U(\theta')}{\theta'_i} = \sum_{k \neq i} \frac{\theta'_k}{\theta'_i} x_k(\theta') + x_i(\theta') \geq \sum_{k \neq i} \frac{\theta_k}{\theta_i} x_k(\theta) + x_i(\theta) = \frac{U(\theta)}{\theta_i}.$$

Indeed, since $\theta'_k/\theta'_i \geq \theta_k/\theta_i$ for all $k \neq i$, type θ' could guarantee $U(\theta')/\theta'_i$ above $U(\theta)/\theta_i$ by simply reporting θ and taking this type's allocation. As it turns out, convexity of $U(\theta)$ and (R) are also sufficient for implementability.

While working with implementability in the simplex representation Γ is more analytically convenient in my setting, one can also characterize it in terms of an indirect utility $\tilde{U} : \mathbb{R}^N_+ \to \mathbb{R}$ defined on unnormalized values v. In a quasilinear model with transfers, $\tilde{U}$ is implementable if and only if it is convex and nondecreasing in each coordinate (Rochet, 1987). Without transfers, incentive compatibility additionally forces $\tilde{U}$ to be positively homogeneous of degree one: for all $v \in \mathbb{R}^N_+$ and $k > 0$, $\tilde{U}(kv) = k\,\tilde{U}(v)$ (Lahr and Niemeyer, 2024).

5 Competitive equilibrium with equal incomes

As shown in Example 1, the optimal mechanism sometimes corresponds to a competitive equilibrium with equal incomes, defined below:

Definition 2. *A **competitive equilibrium with equal incomes (CEEI)** is a vector of prices $p = (p_1, p_2, \dots, p_N) \in \mathbb{R}_+$ and allocations $x : \Gamma \to \mathbb{R}_+^N$ such that the supply constraints (S′) bind for all goods and all types choose utility-maximizing allocations subject to their unit budget constraint:*

$$\textit{for all } \theta \in \Gamma, \quad x(\theta) \in \underset{z \in \mathbb{R}_+^N}{\arg\max} \left\{ \theta \cdot z : \ z \cdot p \le 1 \right\}.$$

Intuitively, a CEEI allocation can arise from the following procedure: give every agent one unit of artificial currency, post per-unit market-clearing prices p, and let everyone buy their favorite bundle z. The resulting aggregate demand for each good will then equal the available supply of it, making the allocation feasible.

In my setting, the CEEI allocation will always take a very simple form:

Fact 1. *A CEEI always exists; the vector of CEEI prices p is unique and strictly positive. The CEEI allocation is unique up to a zero-measure set of agents. Moreover, almost all types θ spend their entire budget on only one kind of good:*

$$x(\theta) = e_i \, \frac{1}{p_i} := e_i \, q_i \quad \textit{for some } i.$$

*We will refer to q_i as the **affordable quantity** of good i.*

This simple structure is a consequence of the linearity of utilities and the lack of a constraint on the total allocation $\sum x_j(\theta)$.[4] In what follows, I refer to the following as the *CEEI mechanism*:

$$x_{\text{CEEI}}(\theta) := \left\{ e_i \, q_i \ : \ \theta_i \, q_i = \max_j \theta_j q_j \right\},$$

Note that the same allocation rule can also be implemented by offering agents a menu of N pure options like those in Example 1:

$$\left\{ q_1 \text{ of good } 1 \right\}, \quad \left\{ q_2 \text{ of good } 2 \right\}, \quad \dots, \quad \left\{ q_N \text{ of good } N \right\}.$$

Let us now describe the set of agents buying each kind of good or, equivalently, picking option

[4] Such a constraint would be present if agents had unit demand and $x_i(\theta)$ represented the probability of getting good i. In those cases, the CEEI allocation could be mixed, which would greatly complicate its structure; see Hylland and Zeckhauser (1979) for a discussion.

i from the above menu. Denote by $\theta^0 \in \Gamma^\circ$ the type who is indifferent among all N options:

$$\theta^0 := \left(\frac{1/q_1}{\sum_{k=1}^N 1/q_k}, \frac{1/q_2}{\sum_{k=1}^N 1/q_k}, \ldots, \frac{1/q_N}{\sum_{k=1}^N 1/q_k} \right).$$

Then, up to breaking indifferences on a null set, the set of agents getting only good i is:

$$\Gamma_i := \left\{ \theta : \theta \succ_i \theta^0 \right\}.$$

The indirect utility of the CEEI mechanism is then given by:

$$U_{\text{CEEI}}(\theta) := \max_j \theta_j \, q_j = \theta_i \, q_i \ \text{ if } \ \theta \in \Gamma_i.$$

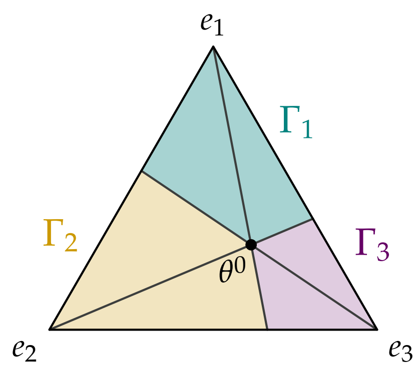

Figure 4: Each region Γ_i contains types who get the affordable quantity q_i of good i under the CEEI mechanism.

6 When is CEEI optimal?

I now present conditions under which the CEEI mechanism is welfare-maximizing. I impose the following integrability condition on the renormalized density g:

Assumption 1. *The renormalized density satisfies $g \in H^1(\Gamma)$, that is, g is square–integrable on Γ and has a first derivative along Γ (in the weak sense) that is also square–integrable.*

To formulate these conditions, however, we must first construct a vector of shadow costs $c \in \mathbb{R}^N_{++}$ which will play the role of multipliers on the supply constraints (S′).

6.1 Shadow costs of supply

First, define:

$$M_i := \int_{\Gamma_i} g(\theta) \, d\theta, \quad A_i := \int_{\Gamma_i} \theta_i g(\theta) \lambda(\theta) \, d\theta,$$

Intuitively, M_i is the mass of agents choosing option i and A_i is the designer's total value of giving each of them a unit of good i. Now, for $i \neq j$, define:

$$T_{ij} := \int_{\Gamma_i \cap \Gamma_j} g(\theta)\, \theta_i \, d\sigma(\theta) \Big/ \sqrt{q_i^2 + q_j^2 - \tfrac{1}{N}(q_i - q_j)^2},$$

where $d\sigma$ denotes $(N-2)$-dimensional Hausdorff measure on $\Gamma_i \cap \Gamma_j$. Intuitively, T_{ij} represents the density of agents who would switch from choosing the affordable quantity q_i to q_j if the latter got increased marginally. Note that for all i and $j \neq i$ we have $M_i, A_i, T_{ij} > 0$.[5] We can now construct the shadow cost vector:

Definition 3. *The vector of **shadow costs** $c = (c_1, c_2, \dots, c_N)$ is given by:*

$$c \;=\; J^{-1} A, \quad \textit{where} \quad A := \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix},$$

and $J \in \mathbb{R}^{N \times N}$ has entries

$$J_{ii} = M_i + q_i \sum_{j \neq i} T_{ij}, \qquad J_{ij} = -\, q_j\, T_{ij} \;\; (i \neq j).$$

Fact 2. *Shadow costs c exist and are strictly positive: $c > 0$.*

Why are these the correct values for the shadow costs? To answer this question, consider an exercise where the designer can allocate any amount of the N goods, but has to pay per-unit costs $c = (c_1, \dots, c_N)$ for them. Consider then the CEEI mechanism for our original problem with its corresponding affordable quantities given by $q = (q_1, \dots, q_N)$ and ask: what would the cost vector have to be so that the designer could not benefit from marginally perturbing these affordable quantities?

Fix any good i and consider the marginal effect of perturbing the offered q_i upwards by ϵ, while keeping the other affordable quantities unchanged. To first order, this perturbation has two effects illustrated in Figure 5. First, agents in Γ_i who chose q_i before continue to do so, but now receive a higher quantity. This improves their utility, but also incurs a cost of $c_i\, \epsilon$ per agent. Second, the perturbation encourages some agents who previously chose q_j, $j \neq i$, to switch to q_i. For every such agent, the designer incurs a cost of $c_i\, q_i$, but saves $c_j\, q_j$ as she no longer has to provide her previous option. However, the welfare effects of such "switchers" are not first-order: this is because both their mass and change in their welfare are of the order of ϵ. Now, as ϵ becomes small, the (per-unit) sum of these two effects converges to:

$$A_i - c_i M_i + \sum_{j \neq i} T_{ij}\,(c_j q_j - c_i q_i).$$

[5] For A_i and M_i, this follows as $\lambda, g > 0$ and each Γ_i has positive measure. For T_{ij}, this is because the surface has a positive $(N-2)$-dimensional Hausdorff measure and because $\theta_i > 0$ on its interior.

Thus, the system $Jc = A$ defining the shadow costs captures precisely the first-order conditions ensuring such perturbations are not beneficial.

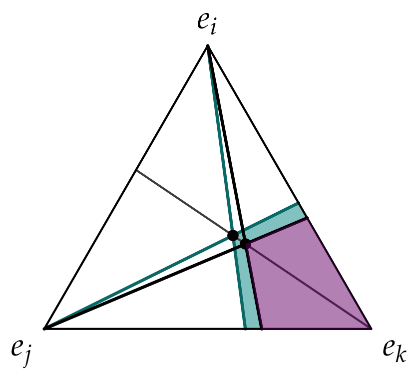

Figure 5: First-order effects of increasing the affordable quantity q_k. Agents in the violet region receive higher quantities of k; agents in the green region switch from other goods to k.

6.2 Sufficient conditions for optimality

To state the main result of this section, I define the following signed measures on Γ_i for each i:

$$\mu_i(A) = \int_{A\cap\Gamma_i} \theta_i \Big[\lambda\, g + \operatorname{div}\Big(\big(c - (\sum_j c_j)\,\theta\big)\, g\Big) - (\sum_j c_j)\, g\Big]\, d\theta - \int_{A\cap\partial\Gamma_i^+} \theta_i\, \big(c - (\sum_j c_j)\,\theta\big)\, g\cdot\nu\, d\sigma, \quad (4)$$

where $\partial\Gamma_i^+ := \partial\Gamma \cap \partial\Gamma_i$ and $\nu(\theta)$ is the outward unit conormal to $\partial\Gamma_i^+$ in Γ_i. The divergence is taken within the hyperplane containing Γ. Also, let μ_i^+ and μ_i^- denote the positive and negative parts of μ_i. Then μ_i is balanced, i.e. $\mu_i^+(\Gamma_i) = \mu_i^-(\Gamma_i)$.

Fact 3. *For all i, $\mu_i(\Gamma_i) = 0$.*

We then get the following result:

Theorem 1. *The CEEI mechanism is optimal if μ_i^+ $\succ_i$-stochastically dominates μ_i^- for every i.*

I now explain the role of the signed measure μ_i. Broadly speaking, it lets us rewrite the designer's objective as a function of indirect utilities. Indeed, for every feasible U, we have:

$$\int_\Gamma \lambda(\theta)\, U(\theta)\, dG(\theta) \;=\; \sum_i \int_{\Gamma_i} \frac{U(\theta)}{\theta_i}\, d\mu_i(\theta) \;+\; \text{const.} \quad (5)$$

In this sense, the measure is similar to a virtual value in a single-dimensional, quasilinear screening problem. The difference, of course, is that while the virtual value multiplies the allocation, my measure μ_i multiplies the (transformed) indirect utility.[6] Indeed, writing the objective as an integral over (weighted) indirect utilities, rather than weighted allocations, is an established practice in the multidimensional screening literature.[7]

This lets us interpret the positive and negative parts of μ_i. Intuitively, μ_i^+ places weight on types whose utility the designer would like to raise, after accounting for how this change affects the objective as it propagates through the local IC constraints. Conversely, the support of μ_i^- consists of types whose utilities the designer would want to decrease. Again, this intuition is similar to that for the role of virtual values. There, they summarize the marginal effect of increasing a type's *allocation* on the objective once the induced local incentive effects are taken into account.

The designer cannot, however, adjust U freely: Proposition 1 tells us that implementable indirect utilities must satisfy certain shape restrictions. In particular, ratio monotonicity (R) bounds how rapidly $U(\theta)$ may increase as θ moves towards the vertices of Γ. Indeed, the CEEI indirect utility U_{CEEI} is exactly the "extremal" one that makes these constraints bind on each region Γ_i. The dominance condition in Theorem 1 then formalizes when this extremal profile is optimal. Intuitively, CEEI is optimal if, for each i, the positive part μ_i^+ lies closer to the vertex e_i than the negative part μ_i^-. When this holds, the best the designer can do is to make $U(\theta)$ increase as rapidly as possible as one moves toward each vertex. This is precisely what the CEEI utility does. The sense in which one measure is closer to e_i than the other is captured by the notion of $\succ_i$-stochastic dominance. While it can be defined in multiple equivalent ways (which are useful in proofs and discussed in Subsection A.1 in the appendix), one definition is as follows:

Definition 4. *Let ρ, τ be measures on some $\Omega \subset \mathbb{R}^N$ with $\rho(\Omega) = \tau(\Omega)$ and let $\succeq$ be a partial order on Ω such that the set $\{(x, y) \in \Omega \times \Omega : \ x \succeq y\}$ is closed in $\Omega \times \Omega$. Then τ **$\succeq$-stochastically dominates** ρ if and only if there exists a **$\succeq$-monotone transport plan** from ρ to τ, that is, a probability measure π on $\Omega \times \Omega$ such that*

$$\pi(A \times \Omega) = \rho(A), \qquad \pi(\Omega \times A) = \tau(A) \quad \textit{for all Borel } A \subseteq \Omega,$$

and π is supported on $\{(x, y) : x \preceq y\}$.

Therefore, the theorem says that the CEEI mechanism is optimal if, for each i, one can transport the negative part onto the positive one by shifting mass only in the direction of the vertex e_i.

[6] One could also integrate the objective by parts to obtain a representation involving the allocation rule $x(\theta)$. However, because x is a vector field, such a representation is not unique: it depends on a choice of vector-valued "flows" which, intuitively, correspond to sets of paths in the type space Γ along which one integrates by parts. Then, when optimizing over x to maximize such an expression, one implicitly accounts only for the effects of perturbing x that propagate through local IC constraints along these paths. In general, this can lose important information about effects propagating through other local IC constraints.

Representing the designer's objective in terms of $U(\theta)$ avoids this issue: since U is a scalar potential, the objective can be rewritten in terms of $U(\theta)$ without having to select paths along which indirect utility is integrated. As a result, this representation encodes information about effects propagating through *all* local IC constraints.

[7] See, for instance, Armstrong (1996); Rochet and Choné (1998); Manelli and Vincent (2006); Daskalakis et al. (2013, 2017).

Importantly, this condition depends only on the *relative placement* of μ_i^+ and μ_i^- in Γ_i, not on their total masses; the particular choice of shadow costs c ensures that μ_i is always balanced.

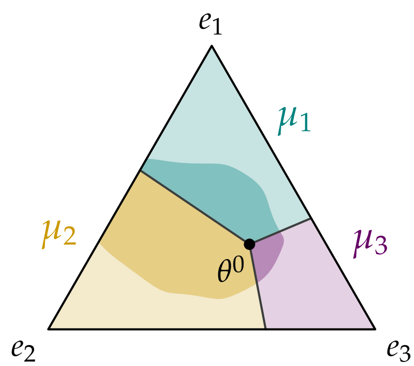

Figure 6: An example where each μ_i^+ $\succ_i$-stochastically dominates μ_i^-. The supports of the negative parts are marked by darker colors; the supports of positive parts are marked by lighter ones.

Remark 3. *The condition in Theorem 1 resembles the stochastic-dominance certificates developed in Daskalakis et al. (2013, 2017) for the problem of a multi-good monopolist. In particular, Daskalakis et al. (2013) provide a dominance condition for the optimality of grand bundling that is phrased in terms of a signed measure similar to mine. Our approaches are closely related: I rewrite the objective as an integral against a signed measure and certify optimality of an "extremal" indirect-utility profile through a stochastic-dominance comparison. However, several features of my environment require a different construction. First, my types live on a simplex and the planner maximizes weighted welfare rather than revenue. Second, feasibility is governed by aggregate supply constraints rather than per-agent quantity caps, so the relevant signed measures must incorporate the shadow costs of supply, and they are naturally defined separately on each region Γ_i induced by the CEEI menu. Most importantly, the constraints that make the candidate solution extremal are different. In Daskalakis et al. (2013), extremality is driven by unit caps on allocations. Here, it is due to the ratio monotonicity condition (R) which bounds how fast $U(\theta)$ can grow as θ approaches a vertex. This is why the objective representation in (5) involves the transformed term $U(\theta)/\theta_i$, rather than $U(\theta)$ alone.*

When are the stochastic dominance conditions in Theorem 1 satisfied? To provide intuition for this, I give a simple sufficient condition in the special case of symmetric supplies and exchangeable value distributions. In this benchmark, the condition can be stated directly in terms of the joint distribution of the *unnormalized* values $V = (V_1, \ldots, V_N)$. To phrase it, I first introduce a notion of stochastic monotonicity.

Definition 5. *Let X be an $\mathcal{X}$-valued random variable and Y be a real-valued random variable. Let $\geq$ be a partial order on $\mathcal{X}$. Fix an event E with $\mathbb{P}(E) > 0$. For any t with $\mathbb{P}(Y \geq t,\ E) > 0$, let $\mathcal{L}(X \mid Y \geq t,\ E)$ denote the conditional law of X given $\{Y \geq t\} \cap E$.*

*Then X is $\geq$-**stochastically decreasing in** Y **conditional on** E if for all such t, t' for which $t > t'$:*

$$\mathcal{L}(X \mid Y \geq t',\ E) \quad \geq\text{-stochastically dominates} \quad \mathcal{L}(X \mid Y \geq t,\ E).$$

Corollary 1. *Assume $s_1 = \cdots = s_N$ and let the unnormalized density f be exchangeable. Then the CEEI mechanism is optimal if the random vector*

$$\left(\frac{V_1}{V_i}, \ldots, \frac{V_N}{V_i}\right)$$

is $\geq$-stochastically decreasing in V_i conditional on $V_i > V_j$ for all $j \neq i$.

In particular, suppose $V_1, V_2, \ldots, V_N$ are distributed i.i.d. according to F_M with support on $[0, \bar{v}]$ and Lipschitz density f_M. Suppose also that

$$x \, \frac{f_M(x)}{F_M(x)} \quad \text{is non-increasing on } [0, \bar{v}]. \tag{6}$$

Then the above $>_i$-stochastic monotonicity condition holds.

The stochastic monotonicity requirement in Corollary 1 is stronger than necessary but provides a clean condition. Intuitively, it says that CEEI is optimal if agents with higher values for their favorite good tend to be *more picky*: conditional on i being the favorite good, higher realizations of V_i are associated with smaller ratios $(V_j/V_i)_{j \neq i}$ in the sense of $\leq$-stochastic dominance. This echoes the intuition from Example 2. There, distorting the CEEI menu by introducing mixtures was beneficial precisely because *less* picky agents had higher cardinal values. Under the condition in Corollary 1 the opposite is true, and such distortions are counterproductive.

The results of this section may raise the question: why is a mechanism as specific as CEEI exactly optimal in a rich class of cases? Indeed, the CEEI mechanism might at first seem knife-edge. After all, the designer possesses many seemingly powerful tools: she could, for instance, try to screen agents by distorting the competitive price vector, or by offering a menu of personalized budgets and price schedules. Still, for a non-trivial class of primitives, none of these distortions are helpful: the optimal mechanism still gives everyone the same budget and lets agents spend it at competitive prices.

To understand why this is the case, note that the CEEI allocation is in fact *the only* allocation that is both Pareto-efficient and satisfies IC constraints. It therefore remains to ask: when can departing from Pareto efficiency, and so from CEEI, benefit the designer? The intuition presented above for Corollary 1 provides a partial answer to this question: distortions away from CEEI produce mixed allocations, which are preferred by agents whose relative values for the mixed items are closer together.[8] Thus, any distortion away from CEEI—and hence Pareto efficiency—will necessarily reward agents who are *less picky*, at least among the subset of goods that is being

[8] The only other possible distortion includes discarding some of the supply. This can never be optimal, however, as the designer could then do better by simply allocating this supply evenly across all agents.

mixed. This in turn lets us explain why CEEI is optimal for one special class of distributions: ones where being less picky is always a sign of *lower* cardinal values. In such cases, all such distortions will redistribute rents to less-worthy recipients, and thus the designer will do better by simply sticking to the Pareto-efficient outcome.

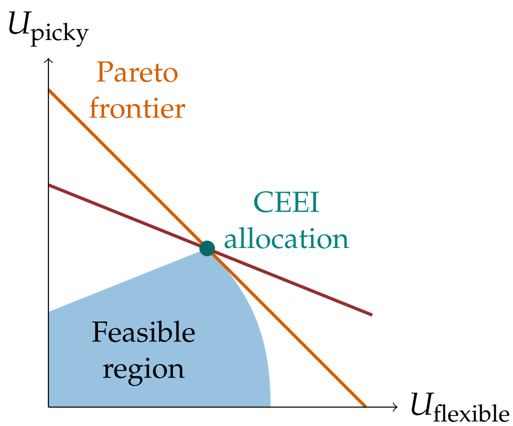

Figure 7a: The designer's Pareto weights skew towards picky agents.

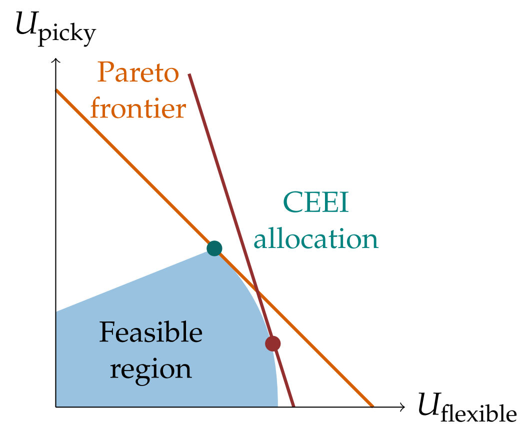

Figure 7b: The designer's Pareto weights skew towards flexible agents

Figure 7: A heuristic illustration showing that distorting away from the only implementable Pareto-efficient allocation can reward flexible agents, but never picky ones. Suppose the designer puts Pareto weights on two kinds of agents: flexible and picky. Then if the designer's Pareto weights are tilted towards picky agents, she always prefers the Pareto-efficient CEEI allocation. If they are tilted towards flexible ones, she might want to distort the CEEI.

Still, the designer might not only want to target agents based on the strength of their relative preferences, but also based on *which specific goods* they like. Nevertheless, as discussed above, any potentially beneficial distortion will still inevitably involve mixing and thus the intuition discussed here will remain relevant. Consequently, if the association between the strength of relative preference and cardinal values is strong, CEEI is likely to remain optimal even when strong preferences for some goods correlate with high cardinal values.

6.3 Proof of Theorem 1

I now present the key steps in the proof of the theorem; the facts and lemmas invoked here are shown in the appendix.

6.3.1 Bounding program. We begin by finding a different program whose value puts an upper bound on Problem 1 faced by the designer:

Problem 2. *Choose $U : \Gamma \to \mathbb{R}_+$ to maximize:*

$$\int_\Gamma U\left[\lambda\, g + \operatorname{div}\left[\left(c - (\textstyle\sum c_j)\,\theta\right) g\right] - (\textstyle\sum c_j)\, g\right] d\theta - \int_{\partial\Gamma} g\, U\left(c - (\textstyle\sum c_j)\theta\right)\cdot \nu\, d\sigma + s\cdot c, \tag{7}$$

subject to:

$$\text{for all} \ \ \theta, \theta' \in \Gamma_i \ \ \text{such that} \ \ \theta' \succ_i \theta, \qquad \frac{U(\theta')}{\theta'_i} \le \frac{U(\theta)}{\theta_i}. \tag{8}$$

Lemma 2. *The value of Problem 2 is weakly higher than that of Problem 1.*

I show that U_{CEEI}—the indirect utility function of the CEEI mechanism—solves this bounding program. Since the CEEI mechanism is also feasible in the designer's original problem, this will imply its optimality in both problems.

Let us comment on the choice of this bounding program. First, as noted in the discussion following the theorem, the objective is written in terms of the indirect utility function; this is accomplished using a version of the divergence theorem on the hyperplane containing the simplex Γ. Second, Problem 2 relaxes certain constraints required for implementability. Indeed, it imposes ratio monotonicity (R) in direction i only on the region Γ_i, that is for types receiving good i under the CEEI allocation. It also drops the requirement that indirect utility functions be convex (although this property is invoked earlier, as it allows us to write the objective in the form (7)). Finally, the problem incorporates the supply constraints (S′) into the objective through the use of shadow costs constructed in Subsection 6.1.

6.3.2 Measure formulation. We subsequently rewrite Problem 2 in a different form (and drop the constant $s \cdot c$ from the objective):

Problem 3. *Choose $Y : \Gamma \to \mathbb{R}_+$ to maximize:*

$$\sum_i \int_{\Gamma_i} Y(\theta)\, d\mu_i(\theta), \tag{9}$$

where the measure μ_i is defined as:

$$\mu_i(A) = \int_{A \cap \Gamma_i} \theta_i \Big[\lambda\, g + \operatorname{div}\Big(\big(c - (\textstyle\sum c_j)\, \theta\big)\, g \Big) - (\textstyle\sum c_j)\, g \Big]\, d\theta - \int_{A \cap \partial \Gamma_i^+} \theta_i\, g \left(c - (\textstyle\sum c_j)\, \theta\right) \cdot \nu\, d\sigma. \tag{10}$$

subject to:

$$\forall\, \theta, \theta' \in \Gamma_i \ \ \text{such that} \ \ \theta' \succ_i \theta, \qquad Y(\theta') \le Y(\theta). \tag{11}$$

The problem is written in terms of transformed variables:

$$Y(\theta) \ := \ \frac{U(\theta)}{\theta_i} \qquad \text{for } \theta \in \Gamma_i.$$

This lets us express implementability constraint (11) in a simpler form. It also rephrases the objective in terms of integrals of $Y(\theta)$ with respect to a measure capturing the benefits of increasing or decreasing this transformed variable for particular types.

Note Y_{CEEI}, which corresponds to the CEEI mechanism, is feasible in (3) as it is given by:

$$Y_{\text{CEEI}}(\theta) = q_i \quad \text{if } \theta \in \Gamma_i^\circ.$$

In fact, this choice of Y makes constraints (11) bind on each region Γ_i.

6.3.3 Monotone transport. It remains to show that the $\succ_i$-stochastic dominance condition of the theorem guarantees that Y_{CEEI} solves Problem 3. Fix any i and recall that the $\succ_i$-stochastic dominance condition implies the existence of a $\succ_i$-monotone transport plan π_i from μ_i^- to μ_i^+. Thus, for every Y satisfying (8), we have:

$$\int_{\Gamma_i} Y \, d\mu_i = \int_{\Gamma_i \times \Gamma_i} \big(Y(\theta) - Y(\theta')\big) \, d\pi_i(\theta, \theta').$$

Since π has support only on pairs (θ, θ') satisfying $\theta \succ_i \theta'$, the constraint (8) implies that:

$$\int_{\Gamma_i} Y \, d\mu_i \ \le \ 0,$$

for all admissible Y. Since $Y_{\text{CEEI}} \equiv q_i$ attains this upper bound of 0, it is optimal.

7 The symmetric two-good case

So far I have focused on understanding when and why the CEEI mechanism is optimal. In this section, I provide a full characterization of the optimal mechanism in the limited case with two goods with symmetric supplies and exchangeable value distributions. The assumption of symmetry is not crucial: while the general two-good case can be handled with a similar approach, the simplifications coming from symmetry make the underlying intuitions clearer. The restriction to two goods is, however, important for overcoming the general intractability of the multidimensional screening problem. As I explain below, with two goods, the reparametrization from Section 4 effectively makes types one-dimensional.

While the reparametrization of types is useful analytically, the main result of this section is phrased in the language of unnormalized values:

Theorem 2. *Let the distribution over renormalized types G have a density g. Define:*

$$\zeta(z) \ := \ z - (2z-1)\, \mathbb{P}\big[V_2 - z(V_1 + V_2) \ge 0\big],$$

and let:

$$z^* \ \in \ \operatorname*{arg\,max}_{z \in [1/2, 1]} \frac{1}{\zeta(z)} \Big(z \, \mathbb{E}[V_1 + V_2] \ + \ 2\, \mathbb{E}\big[\big(V_2 - z(V_1 + V_2)\big)_+\big]\Big). \tag{12}$$

If there exists $z^* = \frac{1}{2}$, then the optimal mechanism offers two options:

$$\left\{2s \text{ of good } 1\right\}, \quad \left\{2s \text{ of good } 2\right\}. \tag{13}$$

Otherwise, the two-option mechanism is not optimal. Then $z^* \in (1/2, 1)$ and the optimal mechanism offers three options:

$$\left\{\frac{s}{\zeta(z^*)} \text{ of good } 1\right\}, \quad \left\{\frac{s}{\zeta(z^*)} \text{ of good } 2\right\}, \quad \left\{\frac{s}{\zeta(z^*)}z^* \text{ of good } 1 \text{ and } \frac{s}{\zeta(z^*)}z^* \text{ of good } 2\right\}. \tag{14}$$

Thus, the optimal mechanism can take one of two forms. In the first form, it offers equal quantities of the two goods and lets agents choose their favorite; this is a special case of the CEEI mechanism discussed in the previous section. In the latter form, the mechanism has the structure discussed in Example 2: it offers two small, "pure" options and a larger equal mixture of the two goods.

While the proof is in the appendix, I explain its core logic as well as the reason for the simple structure of the optimal mechanism. In the first step of the proof, I show that the symmetry of the setting lets us restrict attention to symmetric mechanisms, that is, ones where permuting an agent's value profile permutes her allocation of goods in the same way. Moreover, the symmetry of the allocation tells us that all agents will get weakly more of their preferred good than of the other one. Indeed, suppose some type θ with $\theta_i \geq \theta_j$ received $x_i(\theta) < x_j(\theta)$. Such an agent could then profitably deviate to the "mirrored" version of her type whose allocations of the two goods are flipped. This observation greatly simplifies the analysis, as it guarantees that we need only be concerned with IC constraints between types preferring the same goods. To see this, consider some type $\theta = (1-t, t)$ with $t < 1/2$. Suppose such a type considered reporting $(1-t', t')$ with $t' > 1/2$ (see Figure 8). By the above, after such a deviation, she would be receiving more of good 2 than she would of good 1, which is her preferred. At the same time, the "reflection" of type $(1-t', t')$, $(t', 1-t')$, has a flipped version of this allocation with more of good 1 than good 2. Since type $(1-t, t)$ prefers good 1, she would therefore prefer to imitate this mirrored type on "her side" of the simplex Γ.

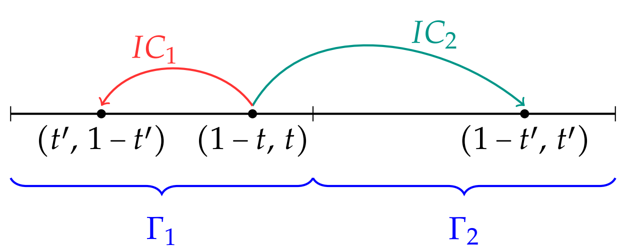

Figure 8: IC_1 is redundant, as the deviation along IC_2 is always more tempting.

Therefore, IC constraints do not bind across sets of types who prefer different goods; we can thus solve the problem in both such sets separately, with symmetry guaranteeing that the solutions in those sets will be the same, up to the labelling of goods. Let us then relax such "across-Γ_i" constraints and consider the problem within the set of agents preferring good i to good j. Fix such a type and note we can rewrite her utility as:

$$\begin{aligned}
U(\theta) &= \theta_i \; x_i(\theta) + \theta_j \; x_j(\theta) \\
&= \theta_i \; \Big(x_i(\theta) - x_j(\theta)\Big) + \Big(\theta_j + \theta_i\Big) \; x_j(\theta) \\
&= \theta_i \; \underbrace{\Big(x_i(\theta) - x_j(\theta)\Big)}_{=\Delta x(\theta)} + x_j(\theta).
\end{aligned}$$

This reparametrization has a linear structure which will let us apply Myersonian methods (Myerson, 1981). Indeed, we can think of these agents as trading off Δx, i.e. how much more she gets of her favorite good than her less-favorite good, against allocation of the less-favorite good x_j. By Myerson's lemma, IC constraints on Γ_i permit all and only increasing Δx. We can then implement any such "allocation" of Δx by using $x_j(\theta)$ as a payment rule.

However, even with this observation, there are three differences relative to the standard Myersonian problem. First, there are two supply constraints, one for each good. Nevertheless, a symmetric mechanism will allocate equal amounts of both goods, and so we can without loss merge the supply constraints into a single supply constraint on $x_1 + x_2$.

The second difference comes from the positivity constraint on the "payment rule", $x_j(\theta)$. Note, however, that IC requires $x_j(\theta)$ to be decreasing in θ_i, and thus the positivity constraint will only bind at the highest type: $\theta_i = 1$. I show this requirement can be subsumed into the supply constraint. Intuitively, we can always make this type's x_j positive by giving everyone a sufficiently large lump-sum allocation of their less-preferred good. The positivity requirement then boils down to the supply constraint holding even with such a lump-sum allocation.[9]

Finally, unlike in the Myersonian problem, the allocation rule is not exogenously bounded from above. This turns out to greatly simplify the solution. While maximizing over increasing allocation rules into $[0,1]$ subject to a single linear constraint would sometimes produce ironed regions, the lack of an upper bound means that bang-bang allocation rules are always optimal. Thus, optimal allocation rules Δx are always step functions. This guarantees the simple structure of the optimal menu in the theorem.

Let us now discuss conditions under which introducing the mixed option is optimal. To that end, consider the following corollary:

[9] This step is also complicated by the fact that the "payment" $x_j(\theta)$ also enters the supply constraint.

Corollary 2. *The mechanism offering the two options in* (13) *is optimal if and only if:*

$$\text{for every } k \in [0,1], \quad \mathbb{E}\left[V_{(1)} - V_{(2)}\,k \;\middle|\; \frac{V_{(1)}}{V_{(2)}} \geq k\right] \leq \mathbb{E}[V_{(2)}]\,(1-k), \tag{15}$$

with $V_{(2)} = \max\{V_1, V_2\}$, $V_{(1)} = \min\{V_1, V_2\}$. *In particular, this is the case if:*

$$\mathbb{E}\left[V_1 + V_2 \;\middle|\; \frac{V_{(1)}}{V_{(2)}} = r\right] \quad \text{is non-increasing in } r. \tag{16}$$

To understand the result, consider first the mechanism with the two options in (13) and order all agents by the ratios of their lowest to highest value: $v_{(1)}/v_{(2)}$. Note that agents for whom the ratio is closer to 1, i.e. those who have weaker preferences over which good they get, are more willing to accept mixtures of goods. Now, consider a perturbation to the mechanism under which all agents with $v_{(1)}/v_{(2)} > k$ get some of their less-preferred good alongside their favorite one, and the allocations of all agents' preferred goods are reduced. To maintain incentive compatibility, these changes have to be calibrated to keep the types with $v_{(1)}/v_{(2)} = k$ indifferent between the pure and mixed options. Also, the reduction in all types' favorite good allocation is chosen so that the perturbation does not violate the supply constraint. The difference between the left- and right-hand sides of (15) then captures the welfare effects of such a perturbation. If it is welfare-improving for some k, the two-option mechanism clearly cannot be optimal. Since Theorem 2 lets us restrict attention to mechanism with one symmetric mixed option, the absence of such a profitable perturbation is also sufficient for optimality.

It is then intuitive that introducing such a mixed option would not be beneficial under condition (16). Echoing the intuitions from Example 2 and Section 6, offering the mixed option serves to direct rewards to less picky agents. If such agents tend to have lower cardinal values, doing so is counterproductive. Importantly, however, the opposite monotonicity of $\mathbb{E}[V_1 + V_2 \mid V_{(1)}/V_{(2)} = r]$ is *not* sufficient to conclude that the designer should introduce the mixed option. This is because mixing goods is an intrinsically distortionary screening device: to direct rents toward less picky types, the mechanism must give them some of the good they value less, and must finance this by reducing other agents' allocations of their preferred good to satisfy the supply constraint. Thus, even if less picky types tend to have higher total values, this correlation must be strong enough to compensate for the resulting inefficiency.

8 Discussion

This paper studies how a welfarist designer can target high-need recipients in settings without transfers. Although incentive constraints prevent the designer from eliciting agents' absolute values directly, she can sometimes use *relative* preferences—how strongly an agent favors one option over another—as a proxy for them. Indeed, when weaker preference margins are predictive of higher need, the designer can sometimes improve welfare by offering menus that include bundled or mixed options. These options are disproportionately attractive to "less picky" agents and thereby provide an incentive-compatible way to direct larger total allocations toward types that are more likely to have high absolute values. Such distortions are beneficial when the informational gain from screening on relative preferences outweighs the allocative inefficiency created by mixing. This logic fails, however, when the correlation runs in the opposite direction, that is when higher-value agents tend to be *more* selective. When the type distribution also satisfies certain stochastic-dominance conditions, the optimal mechanism implements the competitive equilibrium with equal incomes (CEEI).

These observations speak to market design questions in settings such as public housing allocation. Housing authorities commonly use a variant of choice-based lottery systems, where applicants may list up to N developments, and units within each development are allocated by lottery among those who selected them. For example, the Amsterdam housing lottery allows applicants to enter two draws per week.[10] Such mechanisms can be mapped into my model by interpreting developments as different goods and equilibrium winning probabilities as allocations.[11] Indeed, the special case where each applicant is allowed to enter one lottery corresponds to the CEEI mechanism: in equilibrium, the resulting winning probability for each good equals the affordable quantity q_i in the CEEI menu. While previous work on public housing design has considered the trade-offs between allowing for choice and targeting (Arnosti and Shi, 2020; Waldinger, 2021), it has focused on extreme mechanisms giving agents no choice, or letting them choose a specific development. My results suggest that moving beyond these extremes can be welfare-improving: the designer may benefit from offering *both* limited and full-choice options within the same mechanism, leveraging self-selection to improve targeting while preserving choice for applicants who value it most.

References

ABDULKADIROĞLU, A. AND T. SÖNMEZ (1998): "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, 66, 689–701.

——— (2013): "Matching markets: Theory and practice," *Advances in economics and econometrics*, 1, 3–47.

ABDULKADIROĞLU, A., Y.-K. CHE, AND Y. YASUDA (2011): "Resolving Conflicting Preferences in School Choice: The "Boston Mechanism" Reconsidered," *American Economic Review*, 101, 399–410.

AKYOL, E. (2025): "Allocation without transfers: a welfare-maximizing mechanism under incomplete information: E. Akyol," *Social choice and welfare*, 64, 603–632.

[10] https://www.wooninfo.nl/nieuws/2013/04/nieuw-een-woning-via-loting/

[11] The complication arising from the probability constraint $\sum x_j(\theta) \leq 1$ is discussed in Remark 2.

ARMSTRONG, M. (1996): "Multiproduct nonlinear pricing," *Econometrica: Journal of the Econometric Society*, 51–75.

ARNOSTI, N. AND P. SHI (2020): "Design of lotteries and wait-lists for affordable housing allocation," *Management Science*, 66, 2291–2307.

ASHLAGI, I. AND P. SHI (2016): "Optimal allocation without money: An engineering approach," *Management Science*, 62, 1078–1097.

BOGOMOLNAIA, A. AND H. MOULIN (2001): "A new solution to the random assignment problem," *Journal of Economic theory*, 100, 295–328.

BUDISH, E. (2011): "The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes," *Journal of Political Economy*, 119, 1061–1103.

BUDISH, E., G. P. CACHON, J. B. KESSLER, AND A. OTHMAN (2017): "Course match: A large-scale implementation of approximate competitive equilibrium from equal incomes for combinatorial allocation," *Operations Research*, 65, 314–336.

CHAKRAVARTY, S. AND T. R. KAPLAN (2013): "Optimal allocation without transfer payments," *Games and Economic Behavior*, 77, 1–20.

COOK, C., P. Z. LI, AND A. J. BINDER (2023): *Where to build affordable housing?: Evaluating the tradeoffs of location*, US Census Bureau, Center for Economic Studies Rochester, NY.

DASKALAKIS, C., A. DECKELBAUM, AND C. TZAMOS (2013): "Mechanism design via optimal transport," in *Proceedings of the fourteenth ACM conference on Electronic commerce*, 269–286.

——— (2017): "Strong Duality for a Multiple-Good Monopolist," *Econometrica*, 85, 735–767.

DOGAN, M. AND M. UYANIK (2020): "Welfare maximizing allocation without transfers," *Economics Letters*, 195, 109437.

EDWARDS, D. A. (1978): "On the existence of probability measures with given marginals," in *Annales de l'institut Fourier*, vol. 28, 53–78.

FRITZ, T. (2018): "Antisymmetry of the stochastic order on all ordered topological spaces," *arXiv preprint arXiv:1810.06771*.

GUL, F. AND E. STACCHETTI (1999): "Walrasian equilibrium with gross substitutes," *Journal of Economic theory*, 87, 95–124.

HE, Y., A. MIRALLES, M. PYCIA, AND J. YAN (2018): "A pseudo-market approach to allocation with priorities," *American Economic Journal: Microeconomics*, 10, 272–314.

HYLLAND, A. AND R. ZECKHAUSER (1979): "The efficient allocation of individuals to positions," *Journal of Political economy*, 87, 293–314.

KELLERER, H. G. (1984): "Duality theorems for marginal problems," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 67, 399–432.

KELSO JR, A. S. AND V. P. CRAWFORD (1982): "Job matching, coalition formation, and gross substitutes," *Econometrica: Journal of the Econometric Society*, 1483–1504.

LAHR, P. AND A. NIEMEYER (2024): "Extreme Points in Multi-Dimensional Screening," *arXiv preprint* arXiv:2412.00649.

MANELLI, A. M. AND D. R. VINCENT (2006): "Bundling as an optimal selling mechanism for a multiple-good monopolist," *Journal of Economic Theory*, 127, 1–35.

MIRALLES, A. (2012): "Cardinal Bayesian allocation mechanisms without transfers," *Journal of Economic Theory*, 147, 179–206.

MÜLLER, A. AND D. STOYAN (2002): *Comparison Methods for Stochastic Models and Risks*, Wiley Series in Probability and Statistics, Wiley.

MYERSON, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

ORTOLEVA, P., E. SAFONOV, AND L. YARIV (2021): "Who cares more? Allocation with diverse preference intensities," Tech. rep., National Bureau of Economic Research.

ROCHET, J.-C. (1987): "A necessary and sufficient condition for rationalizability in a quasilinear context," *Journal of Mathematical Economics*, 16, 191–200.

ROCHET, J.-C. AND P. CHONÉ (1998): "Ironing, sweeping, and multidimensional screening," *Econometrica*, 783–826.

RODRIGUES, J.-F. (1987): *Obstacle problems in mathematical physics*, vol. 134, Elsevier.

STRASSEN, V. (1965): "The existence of probability measures with given marginals," *The Annals of Mathematical Statistics*, 36, 423–439.

WALDINGER, D. (2021): "Targeting in-kind transfers through market design: A revealed preference analysis of public housing allocation," *American Economic Review*, 111, 2660–2696.

WHITEHEAD, C. AND K. J. SCANLON (2007): *Social housing in Europe*, London School of Economics and Political Science.

A Omitted proofs

A.1 Strassen's theorem

Definition 6. *Let $\geq$ be a partial order on Ω. A set $C \subseteq \Omega$ is an $\geq$-**upper set** if $\theta \in C$, $\theta \geq \theta'$ implies $\theta' \in C$. A function $\eta : \Omega \to \mathbb{R}$ is $\geq$-**increasing** if $\theta \leq \theta'$ implies $\eta(\theta) \geq \eta(\theta')$.*

The following is a special case of Strassen's theorem stated in Fritz (2018):

Theorem 3 (Strassen (1965); Kellerer (1984); Edwards (1978))**.** *Let ρ, τ be measures on some $\Omega \subset \mathbb{R}^N$ with $\rho(\Omega) = \tau(\Omega)$ and let $\geq$ be a partial order on Ω such that the set $\{(x, y) \in \Omega \times \Omega : \; x \geq y\}$ is closed in $\Omega \times \Omega$. Then τ $\geq$-**stochastically dominates** ρ if and only if any of the following conditions holds:*

1. *$\rho(C) \leq \tau(C)$ for every closed $\geq$-upper set $C \subseteq \Omega$.*
2. *For every bounded, lower semicontinuous, $\geq$-increasing $\eta : \Omega \to \mathbb{R}$,*
$$\int_\Omega \eta \, d\rho \;\leq\; \int_\Omega \eta \, d\tau.$$
3. *There exists a $\geq$-monotone transport plan from ρ to τ.*

A.2 Differential geometry facts

Let H denote the $(N-1)$-dimensional hyperplane containing the simplex Γ:

$$H := \left\{\theta \in \mathbb{R}^N : \; \sum \theta_i = 1\right\}.$$

Note that for every $\theta \in H$, the tangent space to H at any θ is:

$$TH := \left\{v \in \mathbb{R}^N : \sum v_i = 0\right\}.$$

Let us also define the intrinsic gradient for this surface:

Definition 7. *Let $\eta : H \to \mathbb{R}$ and fix $\theta \in H$. The **intrinsic gradient** $\nabla_H \eta(\theta) \in TH$ is the unique vector such that:*

$$D_v \eta(\theta) = \nabla_H \eta(\theta) \cdot v \qquad \text{for all } v \in TH.$$

I now introduce a version of the divergence theorem on the surface H. This result is a direct application of Green's formula in $\mathbb{R}^{N-1}$ (see e.g. Rodrigues (1987)).

Theorem 4. *Let $\Omega \subset H$ be a bounded, open set such that $\partial\Omega$ is Lipschitz. Let $\eta : \overline{\Omega} \to \mathbb{R}$ be Lipschitz. Fix a tangent vector field $X : \Omega \to \mathbb{R}^N$, $X(\theta) \in TH$, such that $X \in H^1\big(\overline{\Omega}; TH\big)$. Then:*

$$\int_\Omega \nabla_H \eta(\theta) \cdot X(\theta) \, dV_H(\theta) + \int_\Omega \eta(\theta) \; \operatorname{div} X(\theta) \, dV_H(\theta) = \int_{\partial\Omega} \eta(\theta) \, X(\theta) \cdot \nu(\theta) \, dS_{\partial\Omega}(\theta), \tag{17}$$

where dV_H denotes the $(N-1)$–dimensional surface measure on H, $dS_{\partial\Omega}$ denotes the $(N-2)$–dimensional surface measure on $\partial\Omega$, and ν is the outward unit conormal along $\partial\Omega$. Finally, $\operatorname{div} X(\theta)$ *is the divergence taken in the $(N-1)$-dimensional subsurface H.*

A.3 Properties of feasible indirect utility functions U

Let us first find the intrinsic gradient of U in H:

Fact 4. $\nabla_H U = x - \mathbf{1}\,\frac{1}{N}\left(\sum x_i\right)$.

Proof. The envelope theorem tells us that for every $\theta \in \Gamma^\circ$ and direction $v \in TH$ in which U is differentiable, we have:

$$D_v U(\theta) = v \cdot x(\theta).$$

We can use it to verify that for all such v we have:

$$v \cdot \nabla_H U = v \cdot \left(x - \mathbf{1}\tfrac{1}{N}\left(\sum x_i\right)\right) = v \cdot x - \underbrace{\left(\sum v_i\right)}_{=0} \tfrac{1}{N} \sum x_i = D_v U.$$

Moreover, $x - \mathbf{1}\,\frac{1}{N}(\sum x_i) \in TH$ as $\sum_i (x_i - \frac{1}{N}\sum_k x_k) = 0$. □

The following fact will let us apply Theorem 4 to indirect utility functions:

Lemma 3. *Every feasible indirect utility U is Lipschitz.*

Proof. Fix any feasible U and let x be the allocation rule that implements it. Recall that U is convex and so to show it is Lipschitz it suffices to prove that its gradient is uniformly bounded, wherever it exists. By Fact 4, we have:

$$\nabla_H U = x - \mathbf{1}\,\tfrac{1}{N}\left(\sum x_i\right).$$

Since $x \geq 0$, it suffices to show that $x_i(\theta)$ it is uniformly bounded across i and θ. I prove that in what follows. Fix i. Let

$$Z_i := \int_{\{\theta \in \Gamma:\ \theta_i \geq \frac{1}{2}\}} g(\theta)\, d\theta.$$

Recall g has full support on Γ, so we have $Z_i > 0$. Now, for $k \geq 0$, define

$$m(k) := \int_{\{\theta \in \Gamma:\ \sum x_j \geq k\}} g(\theta)\, d\theta.$$

Since x is feasible, it satisfies the supply constraint (S′):

$$\int_\Gamma \sum x_j(\theta)\, g(\theta)\, d\theta \leq \sum s_j,$$

so $m(k) \leq \frac{1}{k}\sum s_j$, implying $m(k) \to 0$ as $k \to \infty$. We can therefore pick $\tilde{k}$ such that $m(\tilde{k}) < Z_i/2$. Then the set

$$S := \left\{\theta \in \Gamma :\ \theta_i \geq \tfrac{1}{2},\ \sum x_j(\theta) \leq \tilde{k}\right\}$$

has mass at least $Z_i - m(\tilde{k}) > Z_i/2 > 0$. Moreover, we can bound the utility of the agents with $\theta \in S$ as follows:

$$U(\theta) = \theta \cdot x(\theta) \leq \sum x_j(\theta) \leq \tilde{k}.$$

Notice that if there were some type θ' with $x_i(\theta') > 2\tilde{k}$, then every $\theta \in S$ would profitably deviate:

$$\theta \cdot x(\theta') \geq \theta_i\, x_i(\theta') > \tfrac{1}{2} \cdot 2\tilde{k} = \tilde{k} \geq U(\theta),$$

contradicting (IC′). Hence $x_i(\theta) \leq 2\tilde{k}$ for all θ. Since i was arbitrary and N is finite, the overall supremum is finite. $\square$

A.4 Proof of Lemma 1

Consider any feasible allocation rule $y : \mathcal{V} \to \mathbb{R}_+^N$ in the original problem and construct $x : \Gamma \to \mathbb{R}_+^N$ as in (2). Fix $\theta \in \Gamma$ and choose $t, t' > 0$ such that $t\theta \in \mathcal{V}$ and $t'\theta \in \mathcal{V}$. Such t, t' exist since $[0, \epsilon]^N \setminus \mathbf{0} \subset \mathcal{V}$. Then, by (IC), we have:

$$t\,\theta \cdot y(t\theta) \;\geq\; t\,\theta \cdot y(t'\theta), \qquad t'\,\theta \cdot y(t'\theta) \;\geq\; t'\,\theta \cdot y(t\theta),$$

which implies:

$$\theta \cdot y(t\theta) = \theta \cdot y(t'\theta).$$

We can therefore define $\tilde{U} : \Gamma \to \mathbb{R}_+$ such that for every θ:

$$\tilde{U}(\theta) \;=\; \theta \cdot y(t\theta) \;\text{ for any } t > 0 \text{ such that } t\theta \in \mathcal{V}.$$

Moreover, note that:

$$\Theta \cdot y(V) = \tilde{U}(\Theta) \quad \text{almost surely.} \tag{18}$$

I now show x satisfies (IC′). Fix $\theta, \theta' \in \Gamma$ and choose any $t > 0$ with $t\theta \in \mathcal{V}$. For any $v' \in \mathcal{V}$, (IC) applied to $v = t\theta$ gives

$$t\,\theta \cdot y(t\theta) \;\geq\; t\,\theta \cdot y(v') \quad \Rightarrow \quad \tilde{U}(\theta) \;\geq\; \theta \cdot y(v').$$

In particular, this holds for all v' such that $v'/(\sum_i v'_i) = \theta'$. Taking the conditional expectation over V given $\Theta = \theta'$ yields

$$\tilde{U}(\theta) \;\geq\; \theta \cdot \mathbb{E}\left[y(V) \mid \Theta = \theta'\right] = \theta \cdot x(\theta'). \tag{19}$$

Also, by definition of x and (18),

$$\theta \cdot x(\theta) = \theta \cdot \mathbb{E}\left[y(V) \mid \Theta = \theta\right] = \mathbb{E}\left[\Theta \cdot y(V) \mid \Theta = \theta\right] = \mathbb{E}\left[\tilde{U}(\Theta) \mid \Theta = \theta\right] = \tilde{U}(\theta). \tag{20}$$

Combining (19) and (20) gives:

$$\theta \cdot x(\theta) \geq \theta \cdot x(\theta').$$

Let us now show that x satisfies (S′). By the tower property:

$$\int_\Gamma x(\theta)\, dG(\theta) = \mathbb{E}\big[\mathbb{E}[y(V) \mid \Theta]\big] = \mathbb{E}[y(V)] = \int_{\mathcal{V}} y(v)\, dF(v) \leq s,$$

where the last inequality follows from (S).

Finally, let us show (3). Using $V = (\sum V_i)\,\Theta$ and (18), we get:

$$\begin{aligned}\int_{\mathcal{V}} v\cdot y(v)\,dF(v) &= \mathbb{E}\left[V\cdot y(V)\right]\\ &= \mathbb{E}\Big[\mathbb{E}\left[\textstyle\sum V_i \mid \Theta\right]\ \tilde{U}(\Theta)\Big]\\ &= \int_\Gamma \lambda(\theta)\,\tilde{U}(\theta)\,dG(\theta) \;=\; \int_\Gamma \lambda(\theta)\,\theta\cdot x(\theta)\,dG(\theta),\end{aligned}$$

where the last equality follows from (20).

Now, fix x that is feasible in Problem 1 and let $y(v) := x(v/\sum_j v_j)$. I show y satisfies (IC). Fix any $v, v' \in \mathcal{V}$; then there exist $\theta,\theta' \in \Gamma$ such that $\theta = v/(\sum v_i)$ and $\theta' = v'/(\sum v_i')$. (IC′) then implies that:

$$\theta\cdot x(\theta) \;\geq\; \theta\cdot x(\theta') \quad\Rightarrow\quad (\textstyle\sum v_i)\,\theta\cdot x(\theta) \;\geq\; (\textstyle\sum v_i)\,\theta\cdot x(\theta') \quad\Rightarrow\quad v\cdot y(v) \;\geq\; v\cdot y(v').$$

It also satisfies (S) because:

$$\int_{\mathcal{V}} y(v)\,dF(v) = \mathbb{E}\left[y(V)\right] = \mathbb{E}\left[x(\Theta)\right] = \int_\Gamma x(\theta)\,dG(\theta) \leq s,$$

where the last inequality follows from (S′). Note (3) follows because:

$$\int_{\mathcal{V}} v\cdot y(v)\,dF(v) = \mathbb{E}\left[V\cdot y(V)\right] = \mathbb{E}\Big[\mathbb{E}\left[\textstyle\sum V_i \mid \Theta\right]\ \Theta\cdot x(\Theta)\Big] \;=\; \int_\Gamma \lambda(\theta)\,\big(\theta\cdot x(\theta)\big)\,dG(\theta).$$

A.5 Proof of Proposition 1

Necessity has been shown in the main body. Let us then show sufficiency. Assume U is convex and satisfies (R). I construct an allocation rule $x:\Gamma\to\mathbb{R}^N$ that implements U. At every θ where $\nabla_H U(\theta)$ exists (which is the case a.e. by convexity), define $x(\theta)$ as follows:

$$x(\theta) := \nabla_H U(\theta) + \big(U(\theta) - \theta\cdot\nabla_H U(\theta)\big)\mathbf{1}.$$

At points where $\nabla_H U$ does not exist, choose any $p(\theta)\in\partial_H U(\theta)$ and let

$$x(\theta) := p(\theta) + \big(U(\theta) - \theta\cdot p(\theta)\big)\mathbf{1}.$$

We then get:

$$U(\theta) = \theta\cdot x(\theta) \qquad\text{and}\qquad x(\theta) - \mathbf{1}\,\tfrac{1}{N}\textstyle\sum x_j(\theta) \in \partial_H U(\theta). \tag{21}$$

Incentive compatibility. Fix $\theta,\theta'\in\Gamma$. Then (IC′) requires that for all θ,θ' we have:

$$\begin{aligned}U(\theta) &\geq x(\theta')\cdot\theta\\ &= U(\theta') + x(\theta')\cdot(\theta-\theta')\\ &= U(\theta') + \Big(x(\theta') - \mathbf{1}\,\tfrac{1}{N}(\textstyle\sum x_i(\theta'))\Big)\cdot(\theta-\theta'),\end{aligned}$$

where the last line follows because $\mathbf{1}\,\frac{1}{N}(\sum x_i(\theta')) \cdot (\theta - \theta') = 0$ since $\theta, \theta' \in H$. This, however, holds by convexity of $U(\theta)$ and the fact that $x(\theta') - \mathbf{1}\,\frac{1}{N}(\sum x_i(\theta'))$ belongs to its subgradient.

Nonnegativity of x. Fix any $\theta \in \Gamma$ such that $\nabla_H U(\theta)$ exists (by convexity of U, this is the case a.e.) and fix a coordinate k. We will show $x_k(\theta) \geq 0$. Fix any $i \neq k$; since the gradient exists only in the interior of Γ, we know that $\theta_i > 0$.

Now, for $0 < \epsilon < 1$, define

$$\theta^\epsilon := \theta + \epsilon(e_k - \theta) = (1-\epsilon)\theta + \epsilon e_k \in \Gamma.$$

Since $i \neq k$, we have $\theta_i^\epsilon = (1-\epsilon)\theta_i > 0$. Moreover, $\theta \succ_i \theta^\epsilon$ because for any $l \neq i$,

$$\frac{\theta_l^\epsilon}{\theta_i^\epsilon} = \begin{cases} \frac{(1-\epsilon)\theta_l}{(1-\epsilon)\theta_i} = \frac{\theta_l}{\theta_i}, & l \neq k, \\ \frac{(1-\epsilon)\theta_k + \epsilon}{(1-\epsilon)\theta_i} = \frac{\theta_k}{\theta_i} + \frac{\epsilon}{(1-\epsilon)\theta_i} > \frac{\theta_k}{\theta_i}, & l = k. \end{cases}$$

Since $\theta \succ_i \theta^\epsilon$, (R) then implies that for all $\epsilon \in [0,1]$:

$$\frac{U(\theta^\epsilon)}{\theta_i^\epsilon} \geq \frac{U(\theta)}{\theta_i}.$$

A limiting argument therefore gives:

$$D_{e_k - \theta}\left(\frac{U(\theta)}{\theta_i}\right) = \frac{\theta_i\, D_{e_k-\theta} U(\theta) - U(\theta)\, D_{e_k-\theta}\theta_i}{\theta_i^2} \geq 0. \tag{22}$$

Note that:

$$D_{e_k-\theta}\theta_i = (e_k)_i - \theta_i = -\theta_i,$$

and, since $e_k - \theta \in TH$,

$$D_{e_k-\theta}U(\theta) = \nabla_H U(\theta) \cdot (e_k - \theta) = \left(x(\theta) - \mathbf{1}\,\frac{1}{N}\sum x_j(\theta)\right) \cdot (e_k - \theta) = x_k(\theta) - \theta \cdot x(\theta) = x_k(\theta) - U(\theta).$$

Substituting into (22) gives:

$$D_{e_k-\theta}\left(\frac{U(\theta)}{\theta_i}\right) = \frac{x_k(\theta)}{\theta_i} \geq 0.$$

Since the gradient existed at θ, it must have been in the interior of Γ, and thus $\theta_i > 0$. Consequently, $x(\theta) \geq 0$ where the gradient $\nabla_\Gamma U$ exists, which is the case a.e.; the positivity of $x(\theta)$ elsewhere is guaranteed by taking limits from nearby differentiability points.

A.6 Proof of Fact 1

Existence and uniqueness of the CEEI price vector p follows from the fact that linear utilities satisfy the gross substitutes condition (Kelso Jr and Crawford, 1982; Gul and Stacchetti, 1999).

I now show that $p > 0$. If $p_i = 0$ for some good i, then any type with $\theta_i > 0$ (of whom there is a unit measure) would demand infinite amounts of good i, contradicting market clearing.

Now, recall each type θ solves

$$\max_{z \geq 0} \{\theta \cdot z : \; p \cdot z \leq 1\}. \tag{23}$$

For almost all types θ there exists $i(\theta)$ such that:

$$\frac{\theta_{i(\theta)}}{p_{i(\theta)}} > \frac{\theta_j}{p_j} \quad \text{for all } j \neq i(\theta).$$

For such types, the unique solution to (23) is a corner solution involving spending the entire budget for the highest "bang-per-buck" good $i(\theta)$.

A.7 Proof of Fact 2

First, to show they exist we must show that J is indeed invertible. For this purpose, define $k_i := c_i q_i$, so $c_i = k_i / q_i$. We can write $J\, c = A$ as:

$$Hk = A,$$

where

$$H_{ii} = \frac{M_i}{q_i} + \sum_{j \neq i} T_{ij}, \qquad H_{ij} = -T_{ij} \leq 0 \quad (i \neq j).$$

Moreover, for each row i,

$$H_{ii} - \sum_{j \neq i} |H_{ij}| = \frac{M_i}{q_i} > \; 0,$$

so H is a strictly diagonally dominant Z-matrix. Hence H is nonsingular and is a nonsingular M-matrix, so

$$H^{-1} \geq 0 \qquad \text{entrywise.}$$

Moreover, $A > 0$ and therefore $c = H^{-1} A \geq 0$. In fact $c > 0$: since $H^{-1} \geq 0$ and H^{-1} is invertible, each row of H^{-1} contains at least one strictly positive entry, and because $A > 0$ we get $k_i > 0$ for all i. Finally, $q_i > 0$ yields

$$c_i = \frac{k_i}{q_i} > 0.$$

A.8 Proof of Fact 3

Note $(c - (\sum c_j)\,\theta)\, g(\theta) \in TH$ as $(c - (\sum c_j)\,\theta) \cdot \mathbf{1} = 0$. Thus, we can apply Theorem 4 with $\Omega = \Gamma_i^\circ$, $\eta(\theta) = \theta_i$ and $X(\theta) = (c - (\sum c_j)\,\theta)\, g(\theta)$ to get:

$$\int_{\Gamma_i} \theta_i \operatorname{div}\left[(c - (\textstyle\sum c_j)\,\theta)\, g\right] d\theta + \int_{\Gamma_i} \nabla_H \theta_i \cdot (c - (\textstyle\sum c_j)\,\theta)\, g \, d\theta = \int_{\partial \Gamma_i} \theta_i\, (c - (\textstyle\sum c_j)\,\theta)\, g \cdot \nu \, d\sigma.$$

Substitute this into the definition of $\mu_i(\Gamma_i)$ to obtain:

$$\mu_i(\Gamma_i) = A_i - \int_{\Gamma_i} \nabla_H \theta_i \cdot \left(c - (\textstyle\sum c_j)\theta\right) g \, d\theta - (\textstyle\sum c_j) \int_{\Gamma_i} \theta_i \, g \, d\theta$$
$$+ \int_{\partial\Gamma_i} \theta_i \left(c - (\textstyle\sum c_j)\theta\right) g \cdot \nu \, d\sigma - \int_{\partial\Gamma_i^+} \theta_i \left(c - (\textstyle\sum c_j)\theta\right) g \cdot \nu \, d\sigma. \tag{24}$$

Note that $\nabla_H \theta_i = e_i - \frac{1}{N}\mathbf{1}$, and hence:

$$-\int_{\Gamma_i} \nabla_H \theta_i \cdot \left(c - (\textstyle\sum c_j)\theta\right) g \, d\theta = -\int_{\Gamma_i} (c_i - (\textstyle\sum c_j)\theta_i) \, g \, d\theta = -c_i M_i + (\textstyle\sum c_j) \int_{\Gamma_i} \theta_i g \, d\theta.$$

Substituting into (24) and simplifying gives:

$$\mu_i(\Gamma_i) = A_i - c_i M_i + \int_{\partial\Gamma_i} \theta_i \left(c - (\textstyle\sum c_j)\theta\right) g \cdot \nu \, d\sigma - \int_{\partial\Gamma_i^+} \theta_i \left(c - (\textstyle\sum c_j)\theta\right) g \cdot \nu \, d\sigma.$$

We can further combine the boundary terms to get:

$$\mu_i(\Gamma_i) = A_i - c_i M_i + \int_{\partial\Gamma_i \setminus \partial\Gamma_i^+} \theta_i \left(c - (\textstyle\sum c_j)\theta\right) g \cdot \nu \, d\sigma.$$

Note that, up to lower-dimensional edges, we have $\partial\Gamma_i \setminus \partial\Gamma_i^+ \; = \; \bigcup_{k \neq i} \Gamma_i \cap \Gamma_k$, giving:

$$\mu_i(\Gamma_i) = A_i - c_i M_i + \sum_{k \neq i} \int_{\Gamma_i \cap \Gamma_k} \theta_i \left(c - (\textstyle\sum c_j)\theta\right) g \cdot \nu_{ik}^{(i)} \, d\sigma, \tag{25}$$

where $\nu_{ik}^{(i)}$ is the outward unit conormal from Γ_i into Γ_k along $\Gamma_i \cap \Gamma_k$. Now, fix $k \neq i$ and note $\Gamma_i \cap \Gamma_k$ is the level set of $q_i\theta_i - q_k\theta_k$, with $q_i\theta_i - q_k\theta_k = 0$ on $\Gamma_i \cap \Gamma_k$, and:

$$\nu_{ik}^{(i)} = -\frac{\nabla_H(q_i\theta_i - q_k\theta_k)}{\|\nabla_H(q_i\theta_i - q_k\theta_k)\|}.$$

Thus, the integrand in the last term of (25) becomes:

$$\begin{aligned}
\theta_i \left(c - (\textstyle\sum c_j)\theta\right) g \cdot \nu_{ik}^{(i)} \; &= \; -\left(c - (\textstyle\sum c_j)\theta\right) \cdot \nabla_H(q_i\theta_i - q_k\theta_k) \, g \, \frac{\theta_i}{\|\nabla_H(q_i\theta_i - q_k\theta_k)\|} \\
&= -\left(c - (\textstyle\sum c_j)\theta\right) \cdot \left(q_i(e_i - \tfrac{1}{N}\mathbf{1}) - q_k(e_k - \tfrac{1}{N}\mathbf{1})\right) g \, \frac{\theta_i}{\|\nabla_H(q_i\theta_i - q_k\theta_k)\|} \\
&= -\left[q_i(c_i - (\textstyle\sum c_j)\theta_i) - q_k(c_k - (\textstyle\sum c_j)\theta_k)\right] g \, \frac{\theta_i}{\|\nabla_H(q_i\theta_i - q_k\theta_k)\|} \\
&= (q_k c_k - q_i c_i) \, g \, \frac{\theta_i}{\|\nabla_H(q_i\theta_i - q_k\theta_k)\|},
\end{aligned}$$

where the last line follows because on $\Gamma_i \cap \Gamma_k$ we have $q_i\theta_i = q_k\theta_k$, so the $(\sum c_j)$-terms cancel. Since $q_kc_k - q_ic_i$ is constant along $\Gamma_i \cap \Gamma_k$, substituting into (25) gives:

$$\mu_i(\Gamma_i) = A_i - c_iM_i + \sum_{k\neq i} (q_kc_k - q_ic_i) \int_{\Gamma_i\cap\Gamma_k} g \frac{\theta_i}{\|\nabla_H(q_i\theta_i - q_k\theta_k)\|}\, d\sigma. \tag{26}$$

Moreover:

$$\nabla_H\big(q_i\theta_i - q_k\theta_k\big) = q_i(e_i - \tfrac{1}{N}\mathbf{1}) - q_k(e_k - \tfrac{1}{N}\mathbf{1}) \quad \Longrightarrow \quad \big\|\nabla_H\big(q_i\theta_i - q_k\theta_k\big)\big\|^2 = (q_i^2 + q_k^2) - \frac{(q_i - q_k)^2}{N},$$

and therefore:

$$\mu_i(\Gamma_i) = A_i - c_iM_i + \sum_{k\neq i}(q_kc_k - q_ic_i)\, T_{ik}.$$

Finally, the ith row of the system $Jc = A$ gives exactly:

$$A_i - c_iM_i + \sum_{k\neq i}(q_kc_k - q_ic_i)\, T_{ik} = 0,$$

so $\mu_i(\Gamma_i) = 0$ by the construction of the cost vector c.

A.9 Proof of Lemma 2

Recall that $c > 0$ by Fact 2 and so, for any allocation rule x satisfying the supply constraint (S′):

$$c \cdot \left(\int_\Gamma x(\theta)\, g(\theta)\, d\theta - s\right) \;\leq\; 0.$$

Therefore:

$$\int_\Gamma \lambda\, U\, g\, d\theta \leq \int_\Gamma \lambda\, U\, g\, d\theta - c \cdot \left(\int_\Gamma x\, g\, d\theta - s\right) = c \cdot s \;+\; \int_\Gamma \lambda\, U\, g\, d\theta - \int_\Gamma c \cdot x\, g\, d\theta. \tag{27}$$

Let us now rewrite the term involving x. Recall that by Fact 4 we have:

$$\nabla_H U = x - \mathbf{1}\, \tfrac{1}{N}\left(\sum x_i\right).$$

Moreover, note that:

$$\nabla_H U - \mathbf{1}(\nabla_H U \cdot \theta - U) = x - \mathbf{1}\, \tfrac{1}{N} \sum x_i - \mathbf{1}\Big(x \cdot \theta - \tfrac{1}{N} \sum x_i(\mathbf{1} \cdot \theta) - x \cdot \theta\Big) = x.$$

Thus, we have:

$$\begin{aligned}\int_\Gamma x\cdot c\, g\, d\theta &= \int_\Gamma \big(\nabla_H U - \mathbf{1}(\nabla_H U\cdot\theta - U)\big)\cdot c\, g\, d\theta\\ &= \int_\Gamma \big(\nabla_H U - (\nabla_H U\cdot\theta)\,\mathbf{1}\big)\cdot c\, g\, d\theta + (\textstyle\sum c_j)\displaystyle\int_\Gamma U\, g\, d\theta\\ &= \int_\Gamma \big(c - (\textstyle\sum c_j)\,\theta\big)\, g\cdot\nabla_H U\, d\theta \;+\; (\textstyle\sum c_j)\displaystyle\int_\Gamma U\, g\, d\theta.\end{aligned}$$

Now, $c - (\sum c_j)\,\theta \in TH$ because $(c_i - (\sum c_j)\,\theta_i)\cdot\mathbf{1} = 0$; also, U is Lipschitz by Lemma 3. We can therefore apply Theorem 4 to the former integral on the RHS above. This gives:

$$\int_\Gamma \big(c - (\textstyle\sum c_j)\,\theta\big)\, g\cdot\nabla_H U\, d\theta = -\displaystyle\int_\Gamma U\,\mathrm{div}\big[\big(c - (\textstyle\sum c_j)\,\theta\big)\, g\big]\, d\theta + \displaystyle\int_{\partial\Gamma} U\big(c - (\textstyle\sum c_j)\,\theta\big)\, g\cdot\nu\, d\sigma.$$

We therefore get:

$$\int_\Gamma x\cdot c\, g\, d\theta = \int_\Gamma U\,\Big((\textstyle\sum c_j)\, g - \mathrm{div}\big[\big(c - (\textstyle\sum c_j)\,\theta\big)\, g\big]\Big)\, d\theta + \displaystyle\int_{\partial\Gamma} U\big(c - (\textstyle\sum c_j)\theta\big)\, g\cdot\nu\, d\sigma.$$

Plugging back into (27) and collecting terms gives:

$$\int_\Gamma \lambda\, U\, g\, d\theta \le \int_\Gamma U\Big[\lambda\, g + \mathrm{div}\big[\big(c - (\textstyle\sum c_j)\,\theta\big)\, g\big] - (\textstyle\sum c_j)\, g\Big]\, d\theta - \displaystyle\int_{\partial\Gamma} U\big(c - (\textstyle\sum c_j)\theta\big)\, g\cdot\nu\, d\sigma \;+\; c\cdot s.$$

It therefore suffices to show that the constraints in Problem 2 are relaxed versions of those in the original one. This is because the supply constraint (S′) is dropped and the constraint (8) is weaker than (R).

A.10 Proof of Corollary 1

First, note that exchangeability and $s_1 = \cdots = s_N$ guarantees $q_1 = \cdots = q_N$ and so:

$$\theta^0 := \tfrac{1}{N}\mathbf{1}.$$

Recall also that the random vector $\big(V_i/V_1, \dots, V_i/V_N\big)$ a.s. coincides with

$$R_i(\Theta) := \left(\frac{\Theta_i}{\Theta_1}, \dots, \frac{\Theta_i}{\Theta_N}\right).$$

Similarly, we have:

$$V_i = \lambda(\Theta)\,\Theta_i,$$

and so:

$$\{V_i > V_j\ \forall j\} = \{V_i q_i > V_j q_j\ \forall j\} = \{\Theta_i q_i > \Theta_j q_i\ \forall j\} = \{\Theta \in \Gamma_i\}.$$

with the last inequality holding up to a measure-zero set. Thus, the hypothesis of the corollary says that R_i is $\geq$-stochastically decreasing in $\lambda(\Theta)\,\Theta_i$ conditional on $\Theta \in \Gamma_i$.

We now prove the fact which uses the hypothesis about stochastic monotonicity. Note that in the symmetric case all A_i are equal, and so we can denote them by $\bar{A}$.

Fact 5. *For every $\succ_i$-upper set $C \subseteq \Gamma_i$,*

$$\int_C \lambda\, \theta_i\, g\, d\theta \;\geq\; N\bar{A} \int_C g\, d\theta. \tag{28}$$

Proof. Because $\succ_i$ is the coordinatewise order on the ratio vector $R_i(\Theta)$, an $\succ_i$-upper set $C \subseteq \Gamma_i$ can be written as

$$C = \{\theta \in \Gamma_i : \; R_i(\theta) \in B\} \tag{29}$$

for some $\geq$-lower set $B \subseteq \mathbb{R}_+^N$. Since B is a $\geq$-lower set, $\mathbb{R}_+^N \setminus B$ is an $\geq$-upper set and thus stochastic monotonicity and Theorem 3 tell us that for any $t \geq 0$:

$$\begin{aligned}\mathbb{P}\left[R_i(\Theta) \in \mathbb{R}_+^N \setminus B \;\middle|\; \lambda(\Theta)\Theta_i \geq t,\, \Theta \in \Gamma_i\right] &\leq \mathbb{P}\left[R_i(\Theta) \in \mathbb{R}_+^N \setminus B \;\middle|\; \lambda(\Theta)\Theta_i \geq 0,\, \Theta \in \Gamma_i\right] \\ &= \mathbb{P}\left[R_i(\Theta) \in \mathbb{R}_+^N \setminus B \;\middle|\; \Theta \in \Gamma_i\right].\end{aligned}$$

Taking complements gives:

$$\mathbb{P}\left[R_i(\Theta) \in B \;\middle|\; \lambda(\Theta)\Theta_i \geq t,\, \Theta \in \Gamma_i\right] \geq \mathbb{P}\left[R_i(\Theta) \in B \;\middle|\; \Theta \in \Gamma_i\right]. \tag{30}$$

Then, by (29), we can rewrite (30) as:

$$\mathbb{P}[\Theta \in C \mid \lambda(\Theta)\Theta_i \geq t,\, \Theta \in \Gamma_i] \geq \mathbb{P}[\Theta \in C \mid \Theta \in \Gamma_i]. \tag{31}$$

Now, note that:

$$\begin{aligned}\mathbb{E}\left[\lambda(\Theta)\Theta_i \mathbf{1}_{\Theta \in C} \mid \Theta \in \Gamma_i\right] &= \int_0^\infty \mathbb{P}\left[\lambda(\Theta)\Theta_i \mathbf{1}_{\Theta \in C} \geq t \mid \Theta \in \Gamma_i\right] dt \\ &= \int_0^\infty \mathbb{P}[\Theta \in C \mid \lambda(\Theta)\Theta_i \geq t,\, \Theta \in \Gamma_i]\, \mathbb{P}[\lambda(\Theta)\Theta_i \geq t \mid \Theta \in \Gamma_i]\, dt.\end{aligned}$$

By (31) we then have:

$$\begin{aligned}\mathbb{E}\left[\lambda(\Theta)\Theta_i \mathbf{1}_{\{\Theta \in C\}} \,\middle|\, \Theta \in \Gamma_i\right] &\geq \mathbb{P}\left[\Theta \in C \,\middle|\, \Theta \in \Gamma_i\right] \int_0^\infty \mathbb{P}\left[\lambda(\Theta)\Theta_i \geq t \,\middle|\, \Theta \in \Gamma_i\right] dt \\ &= \mathbb{P}\left[\Theta \in C \,\middle|\, \Theta \in \Gamma_i\right] \mathbb{E}\left[\lambda(\Theta)\Theta_i \,\middle|\, \Theta \in \Gamma_i\right].\end{aligned}$$

which is equivalent to:

$$\int_C \lambda\, \theta_i\, g\, d\theta \;\geq\; \frac{\int_{\Gamma_i} \lambda\, \theta_i\, g\, d\theta}{\int_{\Gamma_i} g\; d\theta} \int_C g\; d\theta. \tag{32}$$

Under exchangeability, $\int_{\Gamma_i} g\, d\theta = \frac{1}{N}$ and $\int_{\Gamma_i} \lambda\, \theta_i\, g\, d\theta = \bar{A}$, so (32) reduces to (28). $\square$

By Theorem 1, it suffices to show μ_i^+ $\succ_i$-stochastically dominates μ_i^-. I do so by showing condition 1. of Theorem 3 holds, i.e. that for every closed $\succ_i$-upper set C we have $\mu_i^+(C) \geq \mu_i^-(C)$, which is equivalent to:

$$\mu_i(C) \geq 0.$$

Now, note that in the exchangeable case, the shadow costs reduce to:

$$c = N\bar{A}\,\mathbf{1}, \qquad \left(\sum c_j\right) = N^2\bar{A}.$$

Thus, for any Borel set $\Omega \subseteq \Gamma_i$,

$$\mu_i(\Omega) = \int_\Omega \lambda\,\theta_i\,g\,d\theta - N^2\bar{A}\int_\Omega \theta_i\big[\operatorname{div}\big((\theta-\theta^0)g\big)+g\big]\,d\theta + N^2\bar{A}\int_{\Omega\cap\partial\Gamma_i^+}\theta_i\,g\,(\theta-\theta^0)\cdot\nu\,d\sigma. \tag{33}$$

I now show that $\mu_i(C) \geq 0$ for well-behaved $\succ_i$-upper sets C. I then extend this logic to other sets through an approximation argument.

Fact 6. *Let C be an $\succ_i$-upper set with a Lipschitz boundary ∂C. Then $\mu_i(C) \geq 0$.*

Proof. Note $(\theta-\theta^0)g \in TH$, and so Theorem 4 yields:

$$\int_C \theta_i\,\operatorname{div}\big((\theta-\theta^0)g\big)\,d\theta + \int_C (\theta-\theta^0)\,g\cdot\nabla_H\theta_i\,d\theta = \int_{\partial C}\theta_i\,(\theta-\theta^0)\,g\cdot\nu_C\,d\sigma, \tag{34}$$

where ν_C is the outward unit conormal to the boundary of C. Since $\nabla_H\theta_i = e_i - \frac{1}{N}\mathbf{1}$, we have:

$$(\theta-\theta^0)\cdot\nabla_H\theta_i = (\theta-\theta^0)\cdot(e_i - \tfrac{1}{N}\mathbf{1}) = \theta_i - \tfrac{1}{N}\sum\theta_i - \theta^0\cdot e_i + \tfrac{1}{N}\sum\theta^0 = \theta_i - \tfrac{1}{N}.$$

Substituting into (34) we get:

$$\int_C \theta_i\,\operatorname{div}\big((\theta-\theta^0)g\big)\,d\theta + \int_C g\,\theta_i\,d\theta - \frac{1}{N}\int_C g\,d\theta = \int_{\partial C}\theta_i\;(\theta-\theta^0)\cdot\nu_C\,g\,d\sigma.$$

$$\int_C \theta_i\Big[\operatorname{div}\big((\theta-\theta^0)g\big)+g\Big]\,d\theta = \int_{\partial C}\theta_i\,(\theta-\theta^0)\,g\cdot\nu_C\,d\sigma \;+\; \tfrac{1}{N}\int_C g\,d\theta.$$

Plugging back into (33) and simplifying the boundary integrals gives:

$$\mu_i(C) = \int_C \lambda\,\theta_i\,g\,d\theta \;-\; N\bar{A}\int_C g\,d\theta \;-\; N^2\bar{A}\int_{\partial C\cap\Gamma^\circ}\theta_i\,g\,(\theta-\theta^0)\cdot\nu_C\,d\sigma\,.$$

By Fact 5, the sum of the first two terms is positive. Thus, it suffices to show that:

$$(\theta-\theta^0)\cdot\nu_C \;\leq 0 \quad \text{for a.e. } \theta\in\partial C\cap\Gamma^\circ.$$

To that end, I first show that for any $\theta\in\Gamma_i\cap\Gamma^\circ$ and all $t>0$, we have $\theta + t(\theta-\theta_0) \succ_i \theta$. Indeed:

$$\frac{\theta_k + t(\theta_k-\theta_k^0)}{\theta_i + t(\theta_i-\theta_i^0)} \leq \frac{\theta_k}{\theta_i} \quad\Leftrightarrow\quad \frac{\theta_k^0}{\theta_i^0} \geq \frac{\theta_k}{\theta_i} \quad\Leftrightarrow\quad \theta \succ_i \theta^0,$$

which follows by $\theta\in\Gamma_0$. Now, fix $\theta\in\partial C\cap\Gamma^\circ$. C is an $\succ_i$-upper set, so $\theta + t(\theta-\theta^0)\in C$ for all small $t>0$, meaning $\theta-\theta^0$ cannot point outward. Thus, $(\theta-\theta^0)\cdot\nu_C \leq 0$ a.e. on $\partial C\cap\Gamma^\circ$. □

We now extend this logic to all closed $>_i$-upper sets using the following lemma:

Lemma 4. *Fix i. Let $C \subseteq \Gamma_i$ be a closed $>_i$-upper set. Then there exists a decreasing sequence $(K_m)_{m\geq 1}$ of closed $>_i$-upper sets such that*

$$K_{m+1} \subseteq K_m, \qquad \bigcap_{m\geq 1} K_m = C,$$

where each K_m is a finite union of polytopes in H defined by finitely many inequalities $\theta_k \leq a\,\theta_i$ ($k \neq i$).

Proof. Define:

$$Q_i : \Gamma_i \to \mathbb{R}_+^{N-1}, \qquad Q_i(\theta) := \left(\frac{\theta_k}{\theta_i}\right)_{k\neq i}.$$

Note Q_i is injective on Γ_i. Moreover,

$$\theta' \succeq_i \theta \quad \Longleftrightarrow \quad Q_i(\theta') \leq Q_i(\theta).$$

Now, note $Q_i(\Gamma_i) \subset \mathbb{R}_+^{N-1}$ and is compact. Also, notice C is $>_i$-upper if and only if $Q_i(C)$ is a $\geq$-lower set. Moreover, note that $Q_i(C)$ is compact.

Fix $m \geq 1$. Now define the finite union of lower boxes:

$$D_m := \bigcup\left\{\left[0, b + \tfrac{1}{m}\mathbf{1}\right] : \; b \in \tfrac{1}{m}\mathbb{Z}^{N-1},\ [0,b] \subseteq Q_i(C)\right\} \quad \text{where} \quad [0,b] := \{r \in \mathbb{R}^{N-1} : \; 0 \leq r \leq b\}.$$

Then D_m is closed, lower, and a finite union of boxes.

Also $Q_i(C) \subseteq D_m$. To see why, fix any $r \in Q_i(C)$; then, since $Q_i(C)$ is a lower set, $[0,r] \subset Q_i(C)$. Moreover, there exists some $b \in \frac{1}{m}\mathbb{Z}^{N-1}$ such that $b \leq r \leq b + \frac{1}{m}$. Since $[0,b] \subseteq Q_i(C)$, it follows that $r \in \left[0, b + \frac{1}{m}\mathbf{1}\right] \subseteq D_m$.

We want to show that $Q_i(C) = \bigcap_{m\geq 1} D_m$. Since we already know that $Q_i(C) \subseteq D_m$ for every m, it suffices to show that $\bigcap_{m\geq 1} D_m \subseteq Q_i(C)$. To that end, take any $r \in \bigcap_m D_m$. For each m choose $b_m \in \frac{1}{m}\mathbb{Z}^{N-1} \cap Q_i(C)$ such that:

$$r \leq b_m + \tfrac{1}{m}\mathbf{1}.$$

Since $Q_i(C)$ is compact, there exists a convergent subsequence of $\{b_m\}_m$ and thus a point $\tilde{b} \in Q_i(C)$ such that $r \leq \tilde{b}$. Finally, since $Q_i(C)$ is a lower set, $\tilde{b} \in Q_i(C)$ implies that $r \in Q_i(C)$.

Now, set $C_m := Q_i^{-1}(D_m)$ and define the decreasing sequence

$$K_m := \bigcap_{n=1}^{m} C_n.$$

Each K_m is closed and i-upper, and

$$\bigcap_{m\geq 1} K_m \;=\; \bigcap_{m\geq 1} C_m \;=\; Q_i^{-1}\!\left(\bigcap_{m\geq 1} D_m\right) \;=\; Q_i^{-1}(Q_i(C)) \;=\; C.$$

Moreover, each C_m is a finite union of sets $\{\theta \in \Gamma_i : \; \theta_k/\theta_i \leq b_k \;\forall k \neq i\} = \{\theta \in \Gamma_i : \; \theta_k \leq b_k\,\theta_i \;\forall k \neq i\}$, i.e. finite unions of polytopes in H. Finite intersections of finite unions of polytopes are again finite unions of polytopes, so the same holds for K_m. $\square$

Thus, for any closed $>_i$-upper set C we can construct such a sequence of upper sets K_m with a Lipschitz boundary. Applying Fact 6 then tells us that $\mu_i(K_m) \geq 0$ for every such set. Since μ is a finite measure, taking limits yields $\mu_i(C) \geq 0$.

I now show the latter part of the result providing a sufficient condition for stochastic monotonicity in the i.i.d. case. A simple change of variable shows that the induced normalized density g lies in $H^1(\Gamma)$. Let us then show $\left(\frac{V_1}{V_i}, \ldots, \frac{V_N}{V_i}\right)$ is $\geq$-stochastically decreasing in V_i conditional on $V_i\, q_i > V_j\, q_j$ for all $j \neq i$. By independence, conditional on $\{V_i = v\}$ and $\{V_j < V_i\ \forall j \neq i\}$ the coordinates $\{V_j\}_{j \neq i}$ remain independent. Now, let $V_i(k)$ be distributed like V_i conditional on $V_i < k$. Note that the cdf of $V_i(k)$ is zero above k and below k it is:

$$\frac{F_M(x)}{F_M(k)}.$$

I now show that for $j \neq i$, $V_j(V_i)/V_i$ is $\geq$-stochastically decreasing in V_i. It suffices to show that:

$$\mathbb{P}\left[\frac{V_j(V_i)}{V_i} \geq t \,\middle|\, V_i = k\right] \quad \text{is non-increasing in } k \text{ for all } t.$$

Note this probability is zero for $t \geq 1$ and one for $t = 0$. For $t \in (0,1)$, we have:

$$\mathbb{P}\left[\frac{V_j(V_i)}{V_i} \geq t \,\middle|\, V_i = k\right] = \mathbb{P}\left[V_j(V_i) \geq t\, k\right] = 1 - \frac{F_M(t\, k)}{F_M(k)}.$$

It therefore suffices to show that $\frac{F_M(t\, k)}{F_M(k)}$ is non-decreasing in k. Indeed, note that:

$$\frac{\partial}{\partial k}\frac{F_M(t\, k)}{F_M(k)} = \frac{F_M(tk)}{F_M(k)}\left[t\,\frac{f_M(tk)}{F_M(tk)} - \frac{f_M(k)}{F_M(k)}\right].$$

However, $\frac{F_M(tk)}{F_M(k)} \geq 0$ and (6) gives $t\,\frac{f_M(tk)}{F_M(tk)} - \frac{f_M(k)}{F_M(k)} \geq 0$.

Now, define:

$$V(V_i) := \left(\frac{V_1(V_i)}{V_i}, \ldots, \frac{V_{i-1}(V_i)}{V_i},\ 1\ , \frac{V_{i+1}(V_i)}{V_i}, \ldots, \frac{V_N(V_i)}{V_i}\right).$$

It suffices to show that for every $k_1 \leq k_2$, the law of $V(k_1)$ $\geq$-stochastically dominates that of $V(k_2)$ when $V_i > 0$. However, since $V_j(V_i)/V_i$ is $\geq$-stochastically increasing in V_i and $V_j(V_i)/V_i$ are independent for $j \neq i$, this follows from Theorem 3.3.10. on p. 94 of Müller and Stoyan (2002).

A.11 Proof of Theorem 2

I first show we can without loss restrict attention to symmetric mechanisms, that is, ones where:

$$\text{for every } \theta, \quad x_1(\theta_1, \theta_2) = x_2(1-\theta_1, 1-\theta_2). \tag{35}$$

Suppose (x_1, x_2) is the optimal mechanism. Then, by symmetry the mechanism $\tilde{x}_1, \tilde{x}_2$ such that $\tilde{x}_1(a,b) = x_2(b,a)$ and $\tilde{x}_2(a,b) = x_1(b,a)$ is also feasible and gives the same objective value. Since the objective and constraints are linear in the allocation, the symmetric mechanism $(\frac{x_1+\tilde{x}_1}{2}, \frac{x_2+\tilde{x}_2}{2})$ is also feasible and optimal.

Now, note symmetry implies that:

$$x_1(1/2, 1/2) = x_2(1/2, 1/2).$$

We now show that for all implementable mechanisms we have the following:

$$\text{for every } \theta \text{ such that } \theta_i \geq 1/2, \quad x_i(\theta) \geq x_j(\theta). \tag{36}$$

Fix $t \in [1/2, 1]$ and write $\theta = (1-t, t)$ and $\tilde{\theta} = (t, 1-t)$. By (IC′) we have:

$$t\, x_2(\theta) + (1-t)\, x_1(\theta) \;\geq\; t\, x_2(\tilde{\theta}) + (1-t)\, x_1(\tilde{\theta}).$$

By symmetry, $x_2(\tilde{\theta}) = x_1(\theta)$ and $x_1(\tilde{\theta}) = x_2(\theta)$, and hence:

$$t\, x_2(\theta) + (1-t)\, x_1(\theta) \;\geq\; t\, x_1(\theta) + (1-t)\, x_2(\theta),$$

so $(2t-1)\big(x_2(\theta) - x_1(\theta)\big) \geq 0$. Therefore, for all $t \in [1/2, 1]$ we have $x_2(1-t, t) \geq x_1(1-t, t)$.

Incentive constraints. I will now show we can relax (IC′) to the following subsets of IC constraints:

$$\text{for all } \theta, \theta' \text{ such that } \theta_1 \geq 1/2, \quad x(\theta)\cdot\theta \geq x(\theta')\cdot\theta, \tag{IC1}$$

$$\text{for all } \theta, \theta' \text{ such that } \theta_2 \geq 1/2, \quad x(\theta)\cdot\theta \geq x(\theta')\cdot\theta. \tag{IC2}$$

Indeed, I show that, together with properties (35) and (36), they imply all other IC constraints. To that end, fix any θ such that $\theta_1 \geq 1/2$ and θ' such that $\theta_2 \geq 1/2$ (the other case is symmetric). I now show:

$$x(\theta)\cdot\theta \geq x(\theta')\cdot\theta.$$

Sequentially applying (35) and (36), and (IC1), we get:

$$\begin{aligned}
x(\theta')\cdot\theta &= \theta_1\, x_1(\theta'_1, \theta'_2) + \theta_2\, x_2(\theta'_1, \theta'_2) \\
&= \theta_1\, x_2(1-\theta'_1,\, 1-\theta'_2) + \theta_2\, x_1(1-\theta'_1,\, 1-\theta'_2) \\
&\leq \theta_2\, x_2(1-\theta'_1,\, 1-\theta'_2) + \theta_1\, x_1(1-\theta'_1,\, 1-\theta'_2) \\
&\leq \theta\cdot x(\theta).
\end{aligned}$$

Now, for $t \in [1/2, 1]$ define:

$$\Delta x(t) := x_2(1-t, t) - x_1(1-t, t).$$

Furthermore, note we can rewrite the utility of types with $\theta_2 \geq 1/2$ as:

$$\begin{aligned} U(1-\theta_2,\theta_2) &= \theta_2\ x_2(\theta) + \theta_1\ x_1(\theta) \\ &= \theta_2\ \Big(x_2(\theta) - x_j(\theta)\Big) + (\theta_1+\theta_2)\ x_j(\theta) \\ &= \theta_2\ \underbrace{\Big(x_2(\theta) - x_1(\theta)\Big)}_{=\Delta x(\theta)} + x_1(1-\theta_2,\theta_2). \end{aligned}$$

Moreover, symmetry and property (36) guarantee that $\Delta x \geq 0$ and $\Delta(1/2) = 0$. Thus, the envelope formula tells us that:

$$U(1-t,t)\ =\ x_2(0,1) - \int_t^1 \Delta x(z)\,dz\ =\ x_1(0,1) + \Delta x(1) - \int_t^1 \Delta x(z)\,dz. \tag{37}$$

We can further use it to recover the "payment rule", i.e. the allocation of x_1:

$$x_1(1-t,t) = x_1(0,1) + \Delta x(1) - \int_t^1 \Delta x(z) dz - t\ \Delta x(t). \tag{38}$$

We can then invoke Myerson's lemma (Myerson, 1981) to conclude that x_1, x_2 satisfy (IC2) if and only if Δx is non-decreasing and x_1 satisfies (38). Moreover, when those conditions hold, (IC1) is satisfied by the symmetry of the mechanism.

Welfare. We will now transform the expression for welfare. Using the fact that the primitives and the mechanism are symmetric, as well as (37), we get:

$$\begin{aligned} &\int_0^1 U(1-t,t)\ g(1-t,t)\ \lambda(1-t,t)dt\ = \\ &\qquad = \ 2\int_{1/2}^1 U(1-t,t)\ g(1-t,t)\ \lambda(1-t,t)dt \\ &\qquad = \ 2\int_{1/2}^1 \left(x_1(0,1) + \Delta x(1) - \int_t^1 \Delta x(z)\,dz\right)\ g(1-t,t)\ \lambda(1-t,t)dt \\ &\qquad = \mathbb{E}[\lambda(\Theta)](x_1(0,1) + \Delta x(1)) - 2\int_{1/2}^1 \int_t^1 \Delta x(z) dz\ g(1-t,t)\ \lambda(1-t,t)dt \\ &\qquad = \mathbb{E}[\lambda(\Theta)](x_1(0,1) + \Delta x(1)) - 2\int_{1/2}^1 \Delta x(t)\ \int_{1/2}^t g(1-z,z)\lambda(1-z,z)dz\,dt. \end{aligned}$$

Supply constraints. The type distribution is symmetric, so for all symmetric mechanisms:

$$\int_0^1 x_1(1-t,t)\ g(1-t,t)dt = \int_0^1 x_2(1-t,t)\ g(1-t,t)dt.$$

Moreover, since $s_1 = s_2$, we can reduce both goods' supply constraints to a single total supply constraint:

$$2s \geq \int_0^1 \left(x_1(1-t,t) + x_2(1-t,t)\right) g(1-t,t)dt$$

Exploiting the symmetry of the distribution and the mechanism, we can rewrite it as:

$$2s \geq 2\int_{1/2}^1 \left(x_1(1-t,t) + x_2(1-t,t)\right) g(1-t,t)dt. \tag{39}$$

Now, note that:

$$\Delta x(\theta) + 2x_1 = x_2 - x_1 + 2x_1 = x_2 + x_1.$$

Exploiting this identity and the "payment rule" condition (38), I transform (39) as follows:

$$\begin{aligned} s &\geq \int_{1/2}^1 \left(x_1(1-t,t) + x_2(1-t,t)\right) g(1-t,t)dt \\ &= \int_{1/2}^1 \left(\Delta x(t) + 2x_1(1-t,t)\right) g(1-t,t)\, dt \\ &= x_1(0,1) + \Delta x(1) + \int_{1/2}^1 \Delta x(t)\, g(1-t,t)dt - 2\int_{1/2}^1 \left(\int_t^1 \Delta x(z)dz + t\,\Delta x(t)\right) g(1-t,t)dt \\ &= x_1(0,1) + \Delta x(1) - \int_{1/2}^1 \Delta x(t) \left[2\int_{1/2}^t g(1-z,z)dz + g(1-t,t)(2t-1)\right] dt. \end{aligned}$$

Transformed problem. We have now showed that the designer's problem is equivalent to the following one:

Problem 4. *Choose positive $x_1(\theta), x_2(\theta)$ for θ such that $\theta_2 \geq 1/2$ to maximize:*

$$\mathbb{E}[\lambda(\Theta)](x_1(0,1) + \Delta x(1)) - 2\int_{1/2}^1 \Delta x(t) \int_{1/2}^t g(1-z,z)\lambda(1-z,z)dz\, dt, \tag{40}$$

subject to:

$$x_1(0,1) + \Delta x(1) - \int_{1/2}^1 \Delta x(t)\left[2\int_{1/2}^t g(1-z,z)dz + g(1-t,t)(2t-1)\right] dt \;\leq\; s, \tag{41}$$

$$x_1(1-t,t) = x_1(0,1) + \Delta x(1) - \int_t^1 \Delta x(z)dz - t\,\Delta x(t) \quad \textit{for} \quad t \in [1/2,1], \tag{42}$$

and:

$$\Delta x(1/2) = 0 \;\; \textit{and} \;\; \Delta x(t) \;\; \textit{non-decreasing}.$$

Indeed, the values of x_1, x_2 for types θ for whom $\theta_2 < 1/2$ are pinned down by (35). Let us now further transform this problem to simplify the positivity constraints on x_1 and x_2.

Note that by (42), $x_1(1-t,t)$ is non-increasing for $t \in [1/2,1]$. The positivity constraint on x_1 thus reduces to:

$$x_1(0,1) \geq 0. \tag{43}$$

Note also that since Δx is non-decreasing and $\Delta x(1/2) = 0$, the positivity of x_2 is guaranteed.

Now, I show that we can without loss assume (43) binds. Indeed, fix any symmetric x_1, x_2 satisfying the constraints of Problem 4. We can then construct symmetric $\tilde{x}_1, \tilde{x}_2$ such that:

$$\tilde{x}_1(\theta) = x_1(\theta),\ \tilde{x}_2(\theta) = x_2(\theta) \quad \text{for } \theta \text{ such that } \theta_2 \in (1/2,1),$$

$$\tilde{x}_1(0,1) = 0, \quad \tilde{x}_2(0,1) = x_2(0,1).$$

Indeed, note that $\tilde{x}_1, \tilde{x}_2$ give the same value of (40), do not affect (41) and (42), while also relaxing the monotonicity requirement on Δx. By this observation, we can without loss reduce the designer's problem to the following one:

Problem 5. *Let $\Delta x(1/2) = 0$. Choose a non-decreasing $\Delta x : (1/2,1] \to \mathbb{R}_+$ to maximize:*

$$\mathbb{E}[\lambda(\Theta)]\, \Delta x(1) - 2\int_{1/2}^{1} \Delta x(t) \int_{1/2}^{t} g(1-z,z)\lambda(1-z,z)dz\, dt, \tag{44}$$

subject to:

$$\Delta x(1) - \int_{1/2}^{1} \Delta x(t) \left[2 \int_{1/2}^{t} g(1-z,z)dz + g(1-t,t)(2t-1)\right] dt \ \leq\ s. \tag{45}$$

In fact, we can show that the solution to Problem 5 takes a very simple form:

Lemma 5. *Define z^* as in (12). Then the following Δx^* solves Problem 5:*

$$\Delta x^*(t) = \frac{s}{\zeta(z^*)}\ \mathbf{1}_{t \geq z^*} \qquad \text{for all } t \in (1/2,1].$$

Proof. Since Δx is non-decreasing and right-continuous up to modification on a null set, there exists a unique finite Borel measure ν on $[1/2,1]$ such that

$$\nu(\{1/2\}) := \lim_{t \downarrow 1/2} \Delta x(t), \quad \nu((a,b]) = \Delta x(b) - \Delta x(a) \ \text{ for } 1/2 < a < b \leq 1.$$

In particular, $\nu([1/2,t]) = \Delta x(t)$ for all $t \in (1/2,1]$, so

$$\Delta x(t) = \nu([1/2,t]) = \int_{[1/2,1]} \mathbf{1}_{z \leq t}\, d\nu(z). \tag{46}$$

Then we can rewrite (44) as:

$$\int_{[1/2,1]} \left(\mathbb{E}[\lambda(\Theta)] - 2\int_{z}^{1} \left[\int_{1/2}^{t} g(1-u,u)\lambda(1-u,u)\, du \right] dt \right) d\nu(z).$$

Similarly, we can rewrite (45) as:

$$\int_{[1/2,1]} \left(1 - \int_z^1 \left[2\int_{1/2}^t g(1-u,u)\,du + g(1-t,t)(2t-1)\right]dt\right) d\nu(z) \;\le\; s.$$

Unnormalizing types lets us then reduce the designer's problem to the following one:

Problem 6. *Choose a finite, non-negative measure ν over $[1/2,1]$ to maximize:*

$$\int_{[1/2,1]} \left(z\,\mathbb{E}[V_1+V_2] \;+\; 2\,\mathbb{E}\big[\big(V_2 - z(V_1+V_2)\big)_+\big]\right) d\nu(z). \tag{47}$$

subject to:

$$\int_{[1/2,1]} \left(z - (2z-1)\,\mathbb{P}\big[(1-z)V_2 \ge zV_1\big]\right) d\nu(z) \;\le\; s. \tag{48}$$

Note that the integrands in (47) and (48) are strictly positive for every $z \in [1/2,1]$. This in turn implies that the constraint (48) always binds.

I now show that a Dirac measure is optimal in Problem 6. To that end, define:

$$r(z) := \frac{1}{\zeta(z)}\left(z\,\mathbb{E}[V_1+V_2] \;+\; 2\,\mathbb{E}\big[\big(V_2 - z(V_1+V_2)\big)_+\big]\right). \tag{49}$$

Fix any non-negative measure ν for which (48) holds with equality and notice that:

$$\begin{aligned}\int_{[1/2,1]} \left(z\,\mathbb{E}[V_1+V_2] \;+\; 2\,\mathbb{E}\big[\max\{0,(1-z)V_2 - zV_1\}\big]\right) d\nu(z) \;&=\; \int_{[1/2,1]} r(z)\,\zeta(z)\,\nu(dz) \\ &\le\; s \max_{z\in[1/2,1]} r(z).\end{aligned} \tag{50}$$

Choose z^* attaining the maximum and define the positive Dirac measure:

$$\nu^* := \frac{s}{\zeta(z^*)}\,\delta_{z^*}.$$

Then, by construction it attains the upper bound on the objective in (50) and satisfies (48). Thus, for any feasible ν, there exists a feasible Dirac ν^* with a weakly larger objective value.

Now, let Δx^* be the Δx corresponding to ν^* in Problem 5. By (46), we have:

$$\Delta x^*(t) = \int \mathbf{1}_{z\le t}\,\nu^*(dz) = \frac{s}{\zeta(z^*)}\,\mathbf{1}_{t\ge z^*} \quad \text{for all } t > 1/2.$$

□

Let Δx be the solution to this problem. Then the following expressions for the optimal x_i, x_j can be recovered through the definition of Δx, equation (42), and symmetry:

$$\text{when } \theta_i \ge \theta_j, \quad x_j(\theta_1,\theta_2) = \frac{s}{\zeta(z^*)}\,z^*\,\mathbf{1}_{\theta_i<z^*}, \qquad x_i(\theta_1,\theta_2) = \frac{s}{\zeta(z^*)}\left(z^*\mathbf{1}_{\theta_i<z^*} + \mathbf{1}_{\theta_i\ge z^*}\right).$$

This in turn pins down the quantities offered in the optimal mechanism, as written in (14). Moreover, when z^* can equal 1/2, we get two options of size $2s$ as $\zeta(1/2) = 1/2$. Finally, note that z^* can never equal 1, as:

$$r(1/2) = \mathbb{E}[V_1 + V_2] \;+\; 2\,\mathbb{E}\left[(V_2 - V_1)_+\right] \;>\; \mathbb{E}[V_1 + V_2] = r(1).$$

A.12 Proof of Corollary 2

For $z \in [1/2, 1]$, define $r(z)$ as in (49). By Theorem 2, mechanism letting agents choose between $2s$ of goods 1 and 2 is optimal if and only if:

$$r(1/2) \geq r(z) \quad \text{for all } z \in [1/2, 1]. \tag{51}$$

Changing variables from $z \in [1/2, 1]$ to $k = \frac{1-z}{z}$ in $[0, 1]$ reduces (51) to:

$$\mathbb{E}[V_1 + V_2] + 2\,\mathbb{E}\left[(V_2 - V_1)_+\right] \;\geq\; \frac{\mathbb{E}[V_1 + V_2] + 2\,\mathbb{E}\left[(kV_2 - V_1)_+\right]}{1 - (1-k)\,\mathbb{P}(kV_2 \geq V_1)} \quad \text{for all } k \in [0, 1].$$

Since $\mathbb{E}[V_1 + V_2] + 2\,\mathbb{E}[(V_2 - V_1)_+] = 2\mathbb{E}[V_{(2)}]$, this is equivalent to

$$2\mathbb{E}[V_{(2)}]\Big(1 - (1-k)\mathbb{P}(kV_2 \geq V_1)\Big) \;\geq\; \mathbb{E}[V_1 + V_2] + 2\,\mathbb{E}\left[(kV_2 - V_1)_+\right] \quad \text{for all } k \in [0, 1]. \tag{52}$$

Fix any such k. By exchangeability, on the event $\{kV_2 \geq V_1\}$ we must have $V_2 \geq V_1$ and hence $(V_1, V_2) = (V_{(1)}, V_{(2)})$; moreover, conditional on $(V_{(1)}, V_{(2)})$, each index is the maximum with probability 1/2. Therefore

$$\mathbb{P}(kV_2 \geq V_1) = \frac{1}{2}\,\mathbb{P}\left(\frac{V_{(1)}}{V_{(2)}} \leq k\right), \qquad \mathbb{E}\left[(kV_2 - V_1)_+\right] = \frac{1}{2}\,\mathbb{E}\left[(kV_{(2)} - V_{(1)})\mathbf{1}_{V_{(1)}/V_{(2)} \leq k}\right].$$

Also $\mathbb{E}[V_1 + V_2] = \mathbb{E}[V_{(1)} + V_{(2)}]$. Substituting into (52) gives:

$$\mathbb{E}\left[(V_{(1)} - kV_{(2)})\mathbf{1}_{V_{(1)}/V_{(2)} \geq k}\right] \;\leq\; (1-k)\mathbb{E}[V_{(2)}]\,\mathbb{P}\left(\frac{V_{(1)}}{V_{(2)}} \geq k\right),$$

which (when the event has positive probability) is equivalent to

$$\mathbb{E}\left[V_{(1)} - kV_{(2)} \;\middle|\; \frac{V_{(1)}}{V_{(2)}} \geq k\right] \;\leq\; (1-k)\mathbb{E}[V_{(2)}].$$

If $\mathbb{P}\left(\frac{V_{(1)}}{V_{(2)}} \geq k\right) = 0$, the conditional inequality is vacuous.

Let us then prove the sufficiency of (16). For $r \in (0, 1]$, on the event $\{V_{(1)}/V_{(2)} = r\}$ we have

$V_{(1)} = rV_{(2)}$. Hence:

$$\mathbb{E}\left[V_1 + V_2 \;\middle|\; \frac{V_{(1)}}{V_{(2)}} = r\right] = \mathbb{E}\left[V_{(1)} + V_{(2)} \;\middle|\; \frac{V_{(1)}}{V_{(2)}} = r\right] = (1+r)\,\mathbb{E}\left[V_{(2)} \;\middle|\; \frac{V_{(1)}}{V_{(2)}} = r\right]. \tag{53}$$

Now, fix $k \in [0,1]$. On $\{V_{(1)}/V_{(2)} \geq k\}$ we have $0 \leq V_{(1)}/V_{(2)} - k \leq 1-k$, so:

$$\mathbb{E}\left[V_{(1)} - kV_{(2)} \;\middle|\; \frac{V_{(1)}}{V_{(2)}} \geq k\right] = \mathbb{E}\left[\left(\frac{V_{(1)}}{V_{(2)}} - k\right)V_{(2)} \;\middle|\; \frac{V_{(1)}}{V_{(2)}} \geq k\right] \leq (1-k)\,\mathbb{E}\left[V_{(2)} \;\middle|\; \frac{V_{(1)}}{V_{(2)}} \geq k\right]. \tag{54}$$

By (53) and (16), $\mathbb{E}\left[V_{(2)} \mid V_{(1)}/V_{(2)} = r\right]$ is non-increasing in r. Thus, conditioning on $\{V_{(1)}/V_{(2)} \geq k\}$ can only decrease its average:

$$\mathbb{E}\left[V_{(2)} \;\middle|\; \frac{V_{(1)}}{V_{(2)}} \geq k\right] \leq \mathbb{E}[V_{(2)}]. \tag{55}$$

Combining (54) and (55) gives (15).

B Deriving examples

B.1 Example 1

For convenience, identify Γ with $[0,1]$ via $\theta = (t, 1-t)$. The induced density g of $t = \Theta_1$ and the weight $\lambda(t)$ are:

$$g(t) = \begin{cases} \dfrac{1}{2(1-t)^2}, & 0 < t \leq \frac{1}{2}, \\ \dfrac{1}{2t^2}, & \frac{1}{2} \leq t < 1, \end{cases} \qquad \lambda(t) = \begin{cases} \dfrac{2}{3(1-t)}, & 0 < t \leq \frac{1}{2}, \\ \dfrac{2}{3t}, & \frac{1}{2} \leq t < 1. \end{cases}$$

Fix any supplies and let q_1, q_2 be the corresponding affordable quantities. We then get that $\Gamma_1 = [t_0, 1]$ and $\Gamma_2 = [0, t_0]$ for $t_0 \in (0,1)$ given by:

$$t_0 := \frac{q_2}{q_1 + q_2}.$$

We then compute the measures μ_i and get:

$$\mu_1(A) = \int_{A \cap [t_0,1]} t\, b(t)\, dt \;+\; \frac{c_2}{\sqrt{2}} \mathbf{1}\{1 \in A\}, \quad \mu_2(A) = \int_{A \cap [0,t_0]} (1-t)\, b(t)\, dt \;+\; \frac{c_1}{\sqrt{2}} \mathbf{1}\{0 \in A\},$$

where

$$b(t) := \begin{cases} \dfrac{\frac{1}{3} - c_2}{(1-t)^3}, & 0 < t \le \frac{1}{2}, \\ \dfrac{\frac{1}{3} - c_1}{t^3}, & \frac{1}{2} \le t < 1. \end{cases}$$

Now, by Theorem 1 it suffices to show that μ_i^+ $>_i$-stochastically dominates μ_i^- for $i \in \{1,2\}$. Indeed, since $\mu_i^+(C) \ge \mu_i^-(C)$ is equivalent to $\mu_i(C) \ge 0$, Strassen's Theorem (in the form of Theorem 3) tells us it suffices to show the following:

$$\text{for } i \in \{1,2\} \text{ and every } >_i\text{-upper set } C, \quad \mu_i(C) \ge 0. \tag{56}$$

Note also that $>_i$-upper sets for 1 take the form $[a_1, 1]$ for $a_1 \ge t_0$. For 2, they take the form $[0, a_2]$ for $a_2 \le t_0$. Moreover, Theorem 1 tells us that $\mu_1([t_0, 1]) = \mu_2([0, t_0]) = 0$. Thus, to show (56), it suffices to prove that $c_1, c_2 \ge 1/3$. I do this in what follows. Note we can without loss show it for the case where $t_0 \in [1/2, 1)$; the other case is symmetric.

Let us then find c_1, c_2 by inverting the system $Jc = A$. To that end, we first obtain:

$$M_1 = \int_{t_0}^{1} \frac{1}{2t^2}\, dt = \frac{1-t_0}{2t_0}, \qquad A_1 = \int_{t_0}^{1} t \cdot \frac{1}{2t^2} \cdot \frac{2}{3t}\, dt = \frac{1-t_0}{3t_0}.$$

For $\Gamma_2 = [0, t_0]$ we split at 1/2 and obtain:

$$M_2 = \int_0^{1/2} \frac{1}{2(1-t)^2}\, dt + \int_{1/2}^{t_0} \frac{1}{2t^2}\, dt = \frac{3t_0 - 1}{2t_0},$$

$$A_2 = \int_0^{1/2} (1-t) \cdot \frac{1}{2(1-t)^2} \cdot \frac{2}{3(1-t)}\, dt + \int_{1/2}^{t_0} (1-t) \cdot \frac{1}{2t^2} \cdot \frac{2}{3t}\, dt = \frac{2t_0^2 + 2t_0 - 1}{6t_0^2}.$$

Recall the matrix J has the form:

$$J_{11} = M_1 + q_1 T_{12}, \quad J_{12} = -q_2 T_{12}, \qquad J_{22} = M_2 + q_2 T_{21}, \quad J_{21} = -q_1 T_{21}.$$

We then get:

$$q_2 T_{12} = \sqrt{2}\, g(t_0)\, t_0^2, \qquad q_1 T_{21} = \sqrt{2}\, g(t_0)\, (1-t_0)^2, \qquad q_1 T_{12} = q_2 T_{21} = \sqrt{2}\, g(t_0)\, t_0(1-t_0).$$

Plugging in $g(t_0) = 1/(2t_0^2)$ gives:

$$q_2 T_{12} = \frac{\sqrt{2}}{2}, \qquad q_1 T_{21} = \frac{\sqrt{2}}{2} \frac{(1-t_0)^2}{t_0^2}, \qquad q_1 T_{12} = q_2 T_{21} = \frac{\sqrt{2}}{2} \frac{1-t_0}{t_0}.$$

Therefore J is:

$$J = \begin{pmatrix} \frac{1-t_0}{2t_0} + \frac{\sqrt{2}}{2} \frac{1-t_0}{t_0} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \frac{(1-t_0)^2}{t_0^2} & \frac{3t_0-1}{2t_0} + \frac{\sqrt{2}}{2} \frac{1-t_0}{t_0} \end{pmatrix}. \tag{57}$$

Inverting the system $Jc = A$ yields:

$$c_2(t_0) = \frac{(2+4\sqrt{2})t_0^2 + (2-2\sqrt{2})t_0 + (\sqrt{2}-1)}{3t_0\big((3+2\sqrt{2})t_0 - 1\big)}, \quad c_1(t_0) = \frac{(2-2\sqrt{2})t_0^2 + (4+6\sqrt{2})t_0 - \sqrt{2}}{3(1-t_0)\big((7+5\sqrt{2})t_0 - (1+\sqrt{2})\big)}.$$

We now show $c_2(t_0) > \frac{1}{3}$. Note:

$$c_2(t_0) - \frac{1}{3} = \frac{(2\sqrt{2}-1)t_0^2 + (3-2\sqrt{2})t_0 + (\sqrt{2}-1)}{3t_0\big((3+2\sqrt{2})t_0 - 1\big)}.$$

For $t_0 \in [1/2, 1)$ the denominator is > 0, and the numerator is > 0 since $2\sqrt{2}-1 > 0$, $3-2\sqrt{2} > 0$, and $\sqrt{2}-1 > 0$. Hence $c_2(t_0) > \frac{1}{3}$.

Finally, we show $c_1(t_0) > \frac{1}{3}$. Note:

$$c_1(t_0) - \frac{1}{3} = \frac{(9+3\sqrt{2})t_0^2 - 4t_0 + 1}{3(1-t_0)\big((7+5\sqrt{2})t_0 - (1+\sqrt{2})\big)}.$$

For $t_0 \in [1/2, 1)$ the denominator is > 0. The numerator is the convex quadratic $Q(t) := (9 + 3\sqrt{2})t^2 - 4t + 1$ whose minimizer $t^* = \frac{2}{9+3\sqrt{2}} < \frac{1}{2}$; thus Q is increasing on $[1/2, 1)$ and

$$Q(t_0) \geq Q(1/2) = \frac{5+3\sqrt{2}}{4} > 0.$$

Therefore $c_1(t_0) > \frac{1}{3}$.

B.2 Example 2

For $z \in [1/2, 1]$, define $r(z)$ as in (49). By symmetry of f under $(v_1, v_2) \mapsto (v_2, v_1)$, we have:

$$\mathbb{E}[V_1] = \mathbb{E}[V_2].$$

Moreover, f is symmetric under $(v_1, v_2) \mapsto (1-v_1, 1-v_2)$ so $\mathbb{E}[V_1] = 1 - \mathbb{E}[V_1]$ and hence:

$$\mathbb{E}[V_1] = \mathbb{E}[V_2] = \frac{1}{2}, \qquad \mathbb{E}[V_1 + V_2] = 1.$$

Also note that:

$$V_2 - z(V_1 + V_2) \geq 0 \quad \Longleftrightarrow \quad V_2 \geq \frac{z}{1-z} V_1.$$

We can therefore define:

$$R_z := \Big\{(v_1, v_2) \in [0,1]^2 : \; v_2 \geq \tfrac{z}{1-z} v_1\Big\},$$

And write:

$$r(z) = \frac{z \; + \; 2 \, \iint_{R_z} \big((1-z) v_2 - z v_1 \big) \, f(v_1, v_2) \, dv_1 \, dv_2}{z - (2z-1) \; \iint_{R_z} f(v_1, v_2) \, dv_1 \, dv_2}.$$

Computing the integrals yields:

$$r(z) = \begin{cases} \dfrac{1729z^3 - 2929z^2 + 1607z - 300}{30\left(95z^3 - 155z^2 + 83z - 15\right)}, & \dfrac{1}{2} \le z \le \dfrac{5}{9}, \\ \dfrac{2\left(19z^3 + 347z^2 - 31z + 25\right)}{15\left(38z^3 + z^2 + 4z + 5\right)}, & \dfrac{5}{9} \le z \le 1. \end{cases}$$

Checking first- and second-order conditions in both regions reveals that the unique maximizer solves:

$$4389z^4 - 836z^3 + 382z^2 - 1140z + 85 = 0,$$

giving $z^* \approx 0.63$. Thus, by Theorem 2, the optimal mechanism lets agents choose between q_L of good 1, q_L of good 2, and a mass q_H of an equal mixture of the two goods, where $q_L < q_H$.