99

RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences
Second Year - Semester II Examination – November/December 2016

MAP 2202 – REAL ANALYSIS II

Time: Two (02) hours

Answer **four** questions only.

1.

a) Explain the following clearly, with suitable examples.

i. Convergent Sequence

ii. Infinite Series

iii. Monotone Sequence

(30 marks)

b) State whether each of the following statement is **true** or **false**. Justify your answer.

i. $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}$ is absolutely convergent.

ii. If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ is convergent.

iii. Root test can be used to determine the convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

iv. The interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(x-6)^n}{n^n}$ is $(-\infty,\infty)$.

v. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent.

(70 marks)

$I_n = \int \frac{x^n}{}$ $\quad (n-1)\, x^{n-2} (1+x^2)^{\frac{1}{2}}$

$+ x^{n-1}\, \frac{1}{2}(1+x^2)^{-\frac{1}{2}}\, 2x$

100

$(n-1)x^{n-2}(1+x^2)^{\frac{1}{2}} + x(1+x^2)^{-\frac{1}{2}}$

2.

a)

i. Find the **reduction formula** of $\int \cos^n x dx$.

ii. Hence, evaluate $\int \cos^5(x) dx$.

$I_n = \int x^{n-1} \frac{d\left[(1+x^2)^{\frac{1}{2}}\right]}{dx} dx$

(40 marks)

b) Given that $I_n = \int \frac{x^n}{\sqrt{1+x^2}} dx$; $n \in N$

i. Find an expression for $\frac{d}{dx}\left[x^{n-1}(x^2+1)^{\frac{1}{2}}\right]$ $= x^{n-1}(1+x^2)^{\frac{1}{2}} + \int (n-1)x^{n-2}(1+x^2)^{\frac{1}{2}} dx$

ii. By using part **b)i** ,show that $nI_n + (n-1)I_{n-2} = x^{n-1}\sqrt{x^2+1}$; $n \geq 2$.

$x^{n-1}(1+x^2)^{\frac{1}{2}}$

(60 marks)

3.

a) State **Cauchy's integral test** and **ratio test** for infinite series.

Discuss the convergence or divergence of,

i. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

ii. $\sum_{n=1}^{\infty} (-1)^n \frac{1.3.5\cdots(2n-1)}{n!}$

$y = x^{n-1}(x^2+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = (n-1)x^{n-2}(x^2+1)^{\frac{1}{2}} + \frac{1}{2}x^{n-1}(x^2+1)^{-\frac{1}{2}} \cdot 2x$

$= (n-1)x^{n-2}(x^2+1)^{\frac{1}{2}} + x^{n}(x^2+1)^{-\frac{1}{2}}$

(40 marks)

$\sum \frac{1}{x^2+1}$

b) Let $A = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$

Does this series **converge conditionally** or **converge absolutely**? Justify your answer.

If given that, $B = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{6}$.

$I_n = \int (1+x^2)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) dx$

Find the value of A.

$(1+x^2)^{-\frac{1}{2}}\, x^{n+1}$

{**Hint:**Use the expression of $(B-A)$.}

(40 marks)

$\frac{x^{n-1}\, d(1+x^2)^{\frac{1}{2}}}{dx} \cdot x$

$x^{n-1} \cdot x(1+x^2)^{\frac{1}{2}} - \int \cancel{x(1+x^2)^{\frac{1}{2}} \cdot x\, (n-1)x^{n-2}} dx$

101 $\frac{d[x^{n-1}(1+x^2)^{1/2}]}{dx} = x^{n-1}(1+x^2)^{-1/2}\cdot 2x + (n-1)x^{n-2}(1+x^2)^{1/2}$

$x^{n-1}(1+x^2)^{1/2} = I_n + (n-1)\int x^{n-2}(1+x^2)^{1/2}\,dx$

c) If $a_n = \dfrac{1}{\sqrt{n}}$ for $n = 1,2,3,\cdots$. Using the definition prove that, $\lim_{n\to\infty} a_n = 0$.

(20 marks)

$+(n-1)\int (1+x^2)^{1/2}\, d\left(\frac{x^{n-1}}{n-1}\right) dx$

4.

a)

i. Find the Maclaurin series of $f(x) = \cos x$.

$(1+x^2)^{1/2} x^{n-1} - \int \frac{x^{n-1}}{n-1} \times \frac{1}{2}(1+x^2)^{-1/2}\cdot 2x$

ii. Hence, evaluate $\lim_{x\to 0} \dfrac{\cos x - 1 + \dfrac{x^2}{2}}{x^4}$

$(1+x^2)^{1/2}\cdot x^{n-1} - \frac{1}{n-1}\int x^n (1+x^2)^{-1/2}$

(50 marks)

b) Using **part i** and the identity $\sin^2 x = \dfrac{1}{2}(1-\cos(2x))$, show that

$$\sin^2 x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}$$

$x^{n-1}(1+x^2)^{1/2} \pm I_n$

(50 marks)

5.

a) Define a **metric space** (X,d).

Let $d : \Re^2 \times \Re^2 \to \Re$ by, $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$; $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Show that d is a metric on $\Re^2$.

(40 marks)

b) Define the **Gamma function** and **Beta function**.

Using above functions evaluate,

$I_n = \int \frac{x^2}{\sqrt{1+x^2}}\,dx$

i. $\displaystyle\int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$

ii. $\displaystyle\int_0^1 (x \ln x)^3\, dx$

$\frac{d[x^{n-1}(1+x^2)^{1/2}]}{dx} = (n-1)x^{n-2}(1+x$

iii. $\displaystyle\int_0^1 (1 - x^{2/3})^{3/2}\, dx$

(60 marks)

6.

a) Let $f(x)=x^3$ for $x\in[0,1]$ and $P_n=\left\{0,\frac{1}{n},\frac{2}{n},\cdots,\frac{n-1}{n},1\right\}$ for $n\in\mathbb{N}$.

i. Find the lower sum $L(f;P_n)$ and the upper sum $U(f;P_n)$.

ii. Find the lower integral $L(f)$ and the upper integral $U(f)$.

iii. Hence, show that $f(x)$ is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)dx=\frac{1}{4}$.

(50 marks)

b) Write down the **comparison test** for improper integrals.

i. Determine if the following integral is convergent.

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$$

ii. Determine if the following integral is divergent.

$$\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$$

(30 marks)

c) By using a suitable substitution discuss the convergence of, $\int_{-\infty}^{\infty}\frac{1}{e^x+e^{-x}}\,dx.$

(20 marks)

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