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RAJARATA UNIVERSITY OF SRILANKA
FACULTY OF APPLIED SCIENCES
B.Sc. (General) Degree

First Year-Semester II Examination-March/April 2014
MAP 1203 - REAL ANALYSIS I

Answer **FOUR** Questions Only — Time Allowed: **Two hours**

1.

a. Show that $\sqrt{2}$ is irrational.

b. Show that $\sqrt{p}$ is irrational if p is prime.

c. Find the supremum and infimum of each S. State whether they are in S.

i. S=$\{x|x^2 < 9\}$

ii. S=$\{x||2x+1| < 5\}$

iii. S=$\{x|x = rational\ and\ x^2 \leq 7\}$

d. Which of the following sets are bounded above, bounded below or otherwise? Also find the supremum and infimum, if they exist.

(i) $\left\{(-1)^n \frac{1}{n} : n \in \mathrm{N}\right\}$

(ii) $\left\{\frac{(4n+3)}{n} : n \in \mathrm{N}\right\}$

(iii) $\left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, \cdots, 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}, \cdots\right\}$

2.

a) Suppose $f(x)$ is a function satisfying the conditions:

i. $f(0)=2,\ f(1)=1$

ii. f has a minimum value at $x=5/2$

iii. $f'(x)=2ax+b$ for all x.

Determine the constants a, b and the function $f(x)$.

b) Find

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i. $\displaystyle \lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$

ii. $\displaystyle \lim_{x\to 0} [x]$, where $[x]$ denotes the greatest integer not greater than *x*.

iii.

$$\lim_{x\to a} f(x), \text{ where } f(x)=\begin{cases} (x^2/a)-a, & \text{for } 0<x<a \\ 0, & \text{for } x=a \\ a-(a^3/x^2), & \text{for } x>a \end{cases}$$

c) Determine the value of constants A an B so that the following function f is continuous at every real number:

$$f(x)=\begin{cases} \dfrac{\sin x}{x}; & \text{if } x<0 \\ Ax+B; & \text{if } 0\le x\le 2 \\ \dfrac{x^2-4}{x-2}; & \text{if } 2<x \end{cases}$$

3.

I. Using the definition of limit, prove that $\displaystyle \lim_{x\to 0} x\sin\frac{1}{x}=0$. Hence show that if $f(x)=x^2\sin\left(\frac{1}{x}\right)$ when $x\neq 0$ and $f(0)=0$, show that f is derivable for every value of *x* but the derivative is not continuous for *x=0*.

II. Prove that continuity is a necessary condition for the existence of a finite derivative.

i. Examine the following function for continuity and differentiability at *x=0* and *x*=1.

$$y=\begin{cases} x^2, & \text{for } x\le 0 \\ 1, & \text{for } 0<x\le 1 \\ 1/x, & \text{for } x>1 \end{cases}$$

ii. The function f defined by $f(x)=\begin{cases} x^2+3x+a, & \text{if } x\le 1 \\ bx+2, & \text{if } x>1 \end{cases}$ is given to be differentiable for every *x*. Find *a* and *b*.

4.

I. State carefully the Cauchy Mean Value Theorem.

Show that if f is continuous on [a, b] and differentiable on (a, b) and $f'(x) \neq 0$ for all x ∈ (a, b) then there exists c ∈ (a, b) such that

f (b) = f (a) + $e^{f(b)-f(c)} - e^{f(a)-f(c)}$.

(Hint . apply the Cauchy Mean Value Theorem with ln g (x) = f (x).)

II. Let f : [1, 2] → R, $x \mapsto x^2 + x + 1$ and, for every n ≥ 1, define the partition

$P_n = \{1 + \frac{i}{n} : 0 \le i \le n\}$ of [1, 2]. Show that the Upper Sum is

$U(P_n, f) = \frac{29n^2+12n+1}{6n^2}$

Find a similar expression for $L(P_n, f)$.

(You may assume that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.)

5.

A. State and prove the Rolle's theorem for derivatives.Using Rolle's theorem, derive " First mean value theorem" for derivatives. Show that

$$\left(1 - \frac{x}{y}\right) < \log_e\left(\frac{y}{x}\right) < \left(\frac{y}{x} - 1\right), \text{where } y > x > 0$$

B. Consider the limit

$$\lim_{x\to 0} \frac{\sin x^3 - x^3}{x^9}$$

Compute the above limit using **L'Hopital's Rule**. Be sure to justify each step.

6.

i. Show that $\lim_{n\to\infty} \frac{n^2+3n+5}{2n^2+5n+7} = \frac{1}{2}$

ii. Show that the sequence $\{f_n\}$, where $f_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}$ converges. Find $\lim_{n\to\infty} f_n$.

iii. Show that the sequence $\{f_n\}$ where $f_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$ is not a **Cauchy** sequence.

iv. Let $S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ for all $n \in \mathrm{N}$. Show that $\{S_n\}$

a. is a monotonic sequence
b. is bounded
c.and that $2 \leq \lim_{n\to\infty} S_n \leq 3$

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