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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. Four Year Degree in Industrial Mathematics
Fourth Year - Semester II Examination - May 2016

MAT 4305 - STOCHASTIC PROCESSES

Time Allowed: **Two Hours** Answer **All** Questions.

Calculators will be provided.

1. Suppose that the probability of rain today depends on weather conditions from the previous three days. If it has rained for the past three days, then it will rain today with probability 0.7; if it did not rain for any of the past three days, then it will rain today with probability 0.1; if it rained each of the past two days but not three days ago, it will rain with probability 0.8; and, in any other case, the weather today will match yesterday's weather with probability 0.6.

   (a) Describe this process using a Markov chain, i.e., define a state space and the corresponding transition probability matrix for the process.
   (b) Suppose you know that it rained on days one, two, and three. What is the probability that it will rain on day seven?

2. A DNA nucleotide has any of four values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1 - 3\alpha$, and if it does change then it is equally likely to change to any of the other three values, for some $0 < \alpha < \frac{1}{3}$.

   (a) Show that $P^{n}_{1,1} = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n}$.
   (b) What is the long-run proportion of time the chain is in each state?

3. In a simple model, the state of a car dynamo at the beginning of year n is given by a random variable $\{X_n : n = 0, 1, 2, ...\}$, which takes values between 0 and 6. When $X_n = 0$, the dynamo is 'as new', when $X_n = 6$, it is in 'very bad' condition. Depending on the state i of the dynamo at the beginning of a year, it breaks down during this year with probability $i/6$. After a breakdown, the dynamo is completely revised, so that it is as new at the start of the next year. If a dynamo does not break down during a year, the state is increased by one at the end of the year.

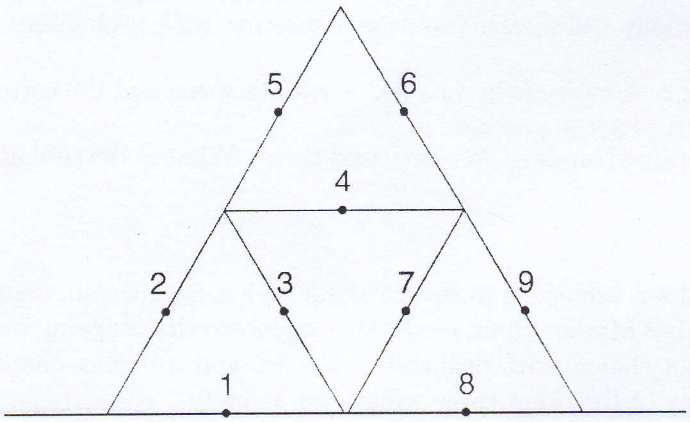
   (a) Show that $\{X_n : n \geq 0\}$ is a discrete time Markov chain and determine whether the chain is: (i) irreducible, (ii) aperiodic, (iii) transient.
   (b) Find the long run probability that the dynamo is 'as new'.
   (c) Find the long run probability that the dynamo is 'as new', and was in state 2 the year before, i.e. determine $lim_{n\to\infty} P(X_{n-1} = 2,\ X_n = 0)$.

4. Suppose that customers arrive at a single server service station in accordance with a Poisson process with rate λ. Upon arrival each customer goes directly into service if the server is free; if not, then the customer joins the queue(that is, he/she waits in line). When the server finishes serving a customer, the customer leaves the system and the next customer in line, if there are any waiting, enters the service. The successive service times are assumed to be independent exponential random variables with mean $1/\mu$. This is a special case of the birth and death processes. Assume $\lambda < \mu$. Compute P_n, $n = 0, 1, \ldots, k$, where P_n being the long-run proportion of time that the system has n customers.

5. (a) Define the term structure function of a system.

   (b) State the structure functions for a series system and a parallel system.

   (c) Answer the following questions based on the structure below.



(i) Find the minimal path and minimal cut sets. Hint: Threre are 12 minimal path sets and 8 minimal cut sets.
(ii) Write the structure function using minimal path sets.
(iii) Give the reliability function of the structure.

Formula Sheet

1. **Chapman-Kolmogorov equation**
Consider a discrete Markov Chain X_n. Let $P_{ij}^n = P\{X_{n+k} = j | X_k = i\}$, $n \geq 0$, $i, j \geq 0$. Then

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \quad \text{for all } n, m \geq 0, \quad \text{all } i, j$$

If we let $\boldsymbol{P}^{(n)}$ denote the matrix of n-step transition probabilities P_{ij}^n, then

$$\boldsymbol{P}^{(n+m)} = \boldsymbol{P}^{(n)} . \boldsymbol{P}^{(m)}$$

2. **Limiting probabilities**
For an irreducible ergodic Markov chain $\lim_{n\to\infty} P_{ij}^n$ exists and is independent of i. Furthermore, letting

$$\pi_j = \lim_{n\to\infty} P_{ij}^n \,, \; j \geq 0$$

then π_j is the unique non-negative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \,, \; j \geq 0, \qquad \sum_{j=0}^{\infty} \pi_j = 1.$$

3. **Limiting probabilities and the balance equations**
Let $P_j = lim_{t\to\infty} P_{ij}(t)$. Then

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \text{ for all states } j.$$

Note that in the balance equations, $v_j P_j$ is the rate at which the process leaves state j, and $\sum_{k\neq j} q_{kj} P_k$ is the rate at which the process enters state j.

4. **Limiting probabilities for a birth and death process**

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}}$$

$$P_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}\right)}, \quad n \geq 1$$