RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

Bachelor of Science in Applied Sciences

Second Year - Semester I Examination - July/August 2023

MAP 2301 - ALGEBRA

Time allowed: **Two and Half ($2\frac{1}{2}$) hours**

Answer ALL (05) questions

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1. a) Let $A, B,$ and C be any three subsets of a given set S. Prove that,
   i. $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$,
   ii. $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

(30 marks)

b) Define an equivalence relation. Prove that the following defined on $\mathbb{Z}$ are equivalence relations:
   i. $R = \{(x, y) | x, y \in \mathbb{Z}, 3x - 5y \text{ is even}\}$,
   ii. $R = \{(x, y) | x, y \in \mathbb{Z}, 4 | (x + 3y)\}$.

(30 marks)

c) Which of the following functions are injective (one to one), surjective (onto), and bijective:
   i. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined by $f(m, n) = 3n - 4m$,
   ii. $f : \{0, 1\} \times \mathbb{N} \to \mathbb{Z}$ is defined by $f(a, b) = (-1)^a b$,
   iii. $f : \mathbb{R} - \{0\} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{x} + 1$.

(40 marks)

2. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 & 8 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 7 & 3 & 2 & 1 & 8 & 4 \end{pmatrix}$ be permutations in S_8.

a) Write α and β as a product of disjoint cycles.

(20 marks)

b) Calculate the orders of α, β and $\alpha\beta$.

(20 marks)

c) Find α^{-1}, α^{2023}, β^{273} and $(\alpha\beta)^{121}$. 29

(20 marks)

d) Find non-trivial permutation σ such that $\alpha^{-1}\beta\sigma\beta^{-1}\alpha = \sigma$.

(20 marks)

e) Does a permutation $\sigma \in S_8$ exist such that $\sigma\alpha\sigma^{-1} = \beta$? Justify your answer.

(20 marks)

3. a) Which of the followings are binary operations defined on the given set:

   i. The operation $*$ defined on $\mathbb{R} - \{0\}$ by $a * b = |a|b$.

   ii. The operation $*$ defined on $\mathbb{R} - \{-1\}$ by $a * b = a + b + ab$.

   iii. The operations $*$ defined on $\mathbb{R}^+ - \{0\}$ by $a * b = a^{\ln b}$.

   iv. The operation $*$ defined on $\mathbb{Q}$ by $a * b = \frac{ab}{3}$.

(20 marks)

b) Define group axioms.

Let $G = \{(a, b) | a \in \mathbb{Z}, b \in \mathbb{Q}\}$. An operation $*$ on G is defined by $(a, b) * (c, d) = (a + c, 2^c b + d)$. Show that $(G, *)$ is a group. Is $(G, *)$ Abelian group? Justify your answer.

(30 marks)

c) Let $G = \{z \in \mathbb{C} | z^n = 1\}$ be the set of all n^{th} roots of unity. Prove that $(G, \cdot)$ is a group under the usual multiplication of complex numbers.

(50 marks)

4. a) Define a subgroup.

Prove that, a non empty subset H of a group G is a subgroup of G if and only if for all $a, b \in H$ implies $ab^{-1} \in H$.

(25 marks)

b) Define coset of a subgroup H of a group G.

Find all distinct right cosets and left cosets of $H = \{-1, +1\}$, where H is a subgroup of the group $G = \{-1, +1, i, -i\}$.

(20 marks)

c) State and prove the Lagrange's Theorem.

(25 marks)

d) Define a normal subgroup H of a group G.

Prove that, a subgroup H of a group G is normal if and only if $gHg^{-1} = H$ for all $g \in G$.

(30 marks)

5. a) Prove that, the linear Diophontine equation $ax + by = c$ has a solution if and only if $d|c$, where $d = gcd(a, b)$. If x_0, y_0 is any particular solution of this equation, then all other solutions are given by $x = x_0 + (\frac{b}{d})t, y = y_0 - (\frac{a}{d})t; t \in \mathbb{Z}$.

(40 marks)

b) Solve the following Diophontine equations:

i. $16x + 54y = 8$,

ii. $19x + 20y = 1909$.

(20 marks)

c) Evaluate $(4655, 12075)$ and express the result as a linear combination of 4655 and 12075 ; that is in the form $4655x + 12075y$.

(20 marks)

d) If n is an integer, show that

i. $3n^2 - 1$ is not a perfect square,

ii. $6|n(n-4)(n-5)$.

(20 marks)

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