RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree
Second Year – Semester II Examination– October/ November 2017

MAP 2204– COMPLEX CALCULUS

Time: Two (02) hours

Answer **all questions.**

1.

a) State and prove the **Inverse triangle inequality** for complex numbers.

Use the Inverse triangle inequality to show that, $\left|\dfrac{1}{z^4-4z^2+3}\right| \leq \dfrac{1}{3}$ when $|z|=2$.

(40 marks)

b) Write down **De Moivre's theorem.**

Find all complex roots of the equation $z^3-8=0$.

(30 marks)

c) Consider the following sets and determine which are open? Closed? Neither open nor closed?

i. $E_1 = \{z \in \mathbb{C} : |Re\, z| \leq 1\}$

ii. $E_2 = \{z \in \mathbb{C} : Re\, z = Im\, z\}$

iii. $E_3 = \{z \in \mathbb{C} : z = in\, ; n = 1,2,3\}$

(30 marks)

2.

a) Let $z=(x+iy)$ and $f(z)=u(x,y)+iv(x,y)$, be a complex valued function with u and v are real-valued functions. Write down the **Cauchy Riemann equations** for u and v.

i. Use Cauchy Riemann equations to prove that function $f(z)=\mathrm{Re}(z)+\mathrm{Im}(z)$ for $z \in \mathbb{C}$ is nowhere analytic.

ii. Let $f(z)$ be defined by, $f(z)=\begin{cases} \dfrac{e^x\left[x^3+y^3+i(y^3-x^3)\right]}{x^2+y^2}; & z\neq 0 \\ 0 \quad ; & z=0 \end{cases}$

Show that Cauchy-Riemann equations are satisfied at $z=0$.

(50 marks)

b) Let $u(x,y)=ax^2+bxy+cy^2$, where a, b and c are constants.

i. Show that $u(x,y)$ is harmonic in $\mathbb{C}$ if and only if $a=-c$.

ii. Find a real valued function $v(x,y)$ such that $u(x,y)+iv(x,y)$ is analytic in $\mathbb{C}$.

(50 marks)

3.

a) State **Cauchy's Integral formula** for derivatives.

Hence evaluate $\int_C \frac{\cos 3z}{z^2(z-1)^2}dz$ where C is the circle,

i. $|z-1|=\frac{1}{2}$

ii. $|z|=2$

(50 marks)

b) Let $f(z)$ be a complex valued function such that $|f(z)|\leq M, \forall z\in C$ where $M>0$ and L is the length of the curve c then show that $\left|\int_C f(z)dz\right|\leq ML$.

Hence show that $\left|\int_C \frac{z^2}{(2z^2-1)(3z^2-4)}dz\right|\leq \frac{\pi R^3}{(2R^2-1)(3R^2-4)}$, where c is the semi-circle $z=Re^{i\theta}; 0\leq\theta\leq\pi$.

(50 marks)

4.

a) Write down **Taylor series expansion** for a function $f(z)$ if it is analytic throughout an open disk $|z-z_0|<R$.

i. Find the Taylor series expansion of the function $f(z)=\frac{\cos z-1}{z^2}; z\neq 0$, given that f is analytic in $|z|<\infty$.

ii. Find the Laurent's series expansion of $f(z) = \dfrac{1}{(3z+1)(z-1)}$ in the annular region $\dfrac{1}{3} < |z| < 1$.

(50 marks)

b)

i. Define **Cauchy's Residue** theorem.

ii. Show that,

$$\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \frac{2\pi}{\sqrt{3}} \text{ for } |z| < 1$$

Hint: $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$

(50 marks)

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