RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES - DEPARTMENT OF PHYSICAL SCIENCES

B.Sc. (General) Degree

Second Year – Semester II Examination – September 2013

MAA 2201 – Mathematical Methods II

Proper Candidates (who want this year's Mid-Semester Marks counted); **Time:** 1 & ½ hours.

Answer **THREE QUESTIONS**, selecting one question from Section A and two from Section B

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All Other Candidates; **Time:** 2 & ½ hours. Answer **ALL FIVE QUESTIONS**

Section A

1. A system of cylindrical polar coordinates (R,φ,z) is defined by the **position vector equation** $\underline{\mathbf{r}} = \underline{\mathbf{i}}\,(R\cos\varphi)+\underline{\mathbf{j}}\,(R\sin\varphi)+\underline{\mathbf{k}}\,(z)$. Find the scale factors associated with the unit base vectors $\underline{\mathbf{e}}_R, \underline{\mathbf{e}}_\varphi, \underline{\mathbf{e}}_z$, (**to be expressed** in terms of $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$), and show that they form a right-handed orthogonal triad.

(i) Write down Laplace's equation, $\nabla^2\Psi = div(grad\Psi) = 0$, and find possible values of n such that the function $\Psi = R^n \cos 2\varphi$ satisfies this equation.

(ii) A smooth wire in the shape of a curve C whose position vector equation is $\underline{\mathbf{r}} = a\,\underline{\mathbf{e}}_R + c\varphi\,\underline{\mathbf{e}}_z$,

where a and c are positive constants, is fixed with the positive Oz-axis pointing **vertically downwards.** A small bead P free to move along the wire is released from rest at the point where the parameter $\varphi = 0$. If P moves under gravity, use energy conservation to show that when P has fallen a vertical distance z, its velocity is of magnitude $v = \sqrt{2gz}$ and is directed along the unit tangent vector (**to be derived as**) $\underline{\mathbf{T}} = \dfrac{a\,\underline{\mathbf{e}}_\varphi + c\,\underline{\mathbf{e}}_z}{\sqrt{a^2+c^2}}$. Hence show that, the particle will be at the point where $\varphi = \dfrac{gct^2}{2(a^2+c^2)}$ after time t.

2. Assuming Gauss' divergence theorem $\int_S \underline{\mathbf{F}}.\,\underline{\mathbf{n}}\,dS = \int_V (div\underline{\mathbf{F}})dV$, with the usual notation, for a vector field $\underline{\mathbf{F}}$ establish the indicated results, by taking $\underline{\mathbf{F}}$ as instructed:

(i) With $\underline{\mathbf{F}} = r\,\underline{\mathbf{e}}_r$ where $\underline{\mathbf{e}}_r$ is a unit vector in the radially outward direction, show that the position vector $\hat{\underline{\mathbf{r}}}$ of the centre of mass of a uniform solid body of constant density ρ, bounded by a surface S is given by the formula $\left(\frac{\rho}{3}\int_S (r\underline{\mathbf{e}}_r).\underline{\mathbf{n}}\,dS\right)\hat{\underline{\mathbf{r}}} = \int_V (r\underline{\mathbf{e}}_r)\,\rho dV$.

Using this formula, or otherwise, show that the position vector of centre of mass of a uniform solid **sector** of a sphere, defined in terms of spherical polar coordinates (r,θ,φ) by the inequalities $0 \le r \le a,\ 0 \le \theta \le \alpha,\ 0 \le \varphi < 2\pi$ is given by $\hat{\underline{\mathbf{r}}} = \frac{3a}{8}\underline{\mathbf{k}}(1+\cos\alpha)$.

(ii) With $\underline{\mathbf{F}} = \frac{\underline{\mathbf{e}}_r}{r^2}$, show that the surface integral $J = \int_S \left(\frac{\underline{\mathbf{e}}_r}{r^2}\right).\underline{\mathbf{n}}\,dS = 4\pi$, provided that the origin O is inside the **spherical surface** S, having equation $r = a$. Show further that, if the spherical surface S in the integral formula for J above is **replaced by the spherical cap** S_0 where $r = a$, $0 \le \theta \le \alpha,\ 0 \le \varphi < 2\pi$, then J = $2\pi(1-\cos\alpha)$. Deduce the solid angle subtended at the origin O by the spherical cap S_1 where $r = a,\ \alpha \le \theta \le \pi,\ 0 \le \varphi < 2\pi$.

Section B

3. (a) (i) Find the Laplace transform of $\sin^2 t$ and hence find the Laplace transform of $(\sin^2 t)/t$, stating any theorem you may use.

(ii) Using the identity $\frac{8}{(s^2+1)(s^2+9)} \equiv \frac{1}{s^2+1} - \frac{1}{s^2+9}$, and finding the inverse Laplace transform of the right side, or otherwise, show that the Laplace transform of $\sin^3 t$ is $\frac{6}{(s^2+1)(s^2+9)}$. Hence show that $\int_0^\infty e^{-\sqrt{3}t}\left(\frac{\sin^3 t}{t}\right)dt = \frac{\pi}{24}$.

(b) Show that $L\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, p > -1$, where the gamma function: $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx,\ r > 0$.

By considering the Laplace transform of the function $H(t) = \int_0^t u^{m-1}(t-u)^{n-1}du$, and applying the convolution theorem show that $H(t) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$. Hence find the integrals $\int_0^1 u^{m-1}(1-u)^{n-1}du$ and $\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$, in terms of the Γ function defined above, and evaluate $\Gamma(1/2)$.

Using the above results, evaluate: (*i*) $\int_0^{\pi/2} \sin^4\theta \cos^6\theta \, d\theta$, *and* (*ii*) $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$, $0 < a < b$.

4. (a) With the usual notation, if the Laplace transform of $Y(t)$ is denoted by $L\{Y(t)\} = \bar{y}(s)$, show that

(*i*) $L\{Y'(t)\} = s\,\bar{y}(s) - Y(0)$
(*ii*) $L\{Y''(t)\} = s^2\bar{y}(s) - Y(0) - Y'(0)$

(b) Using Laplace transform method

(i) solve the differential equation $Y''(t) + Y(t) = e^{-t} + \cos t$, for the unknown function $Y(t)$, subject to the conditions: $Y(0) = 0$ and $Y'(0) = 0$;

(ii) solve the simultaneous equations

$$X'(t) + Y(t) = e^{t}, \quad X(t) - Y'(t) = t,$$

For the unknown functions $X(t)$ and $Y(t)$, subject to the conditions: $X(0) = 1$, $Y(0) = 2$.

5. Defining finite Fourier sine transform $\overline{F}(n,t)$ of a function $F(x,t)$, $0 < x < l$, by the integral

$$\overline{F}(n,t) = \int_0^l F(x,t)\sin\left(\frac{n\pi x}{l}\right)dx,$$ for positive integers n,

obtain the corresponding inversion formula $F(x,t) = \frac{2}{l}\sum_{n=1}^{\infty}\overline{F}(n,t)\sin\left(\frac{n\pi x}{l}\right)$.

Applying finite Fourier sine transform find the solution to the differential equation $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions: $u(0, t) = 0$, $u(l, t) = 0$, for all $t > 0$ and the initial conditions:

$$u(x,0) = \begin{cases} x, & 0 \le x \le \frac{l}{2} \\ l - x, & \frac{l}{2} \le x \le l \end{cases}$$
