RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences
First Year – Semester II Examination – October / November 2017

MAP1302 - DIFFERENTIAL EQUATIONS I

Answer ALL Questions . **Time: Three Hours.**

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01. (a) A population model is given by the following differential equation:

$$\frac{dP}{dt} = P(aP - b);$$ where a and b are positive constants.

Discuss what happens to the population P, as time t increases. **[20marks]**

(b) Consider the following differential equation:

$m\frac{dv}{dt} = mg - mkv$, where k is a positive constant and g is the acceleration due to the gravity as a model for the velocity v of a body of mass m that is falling under the influence of gravity and a resistance. Since the expression $-kv$ represents air resistance per unit mass, the velocity of the body falling from a great height increases and approaches a limiting value V, as time t increases. Find this **terminal velocity**, V, of the body explaining your reasons.

[20marks]

(c) When certain amounts of two chemicals are combined, the rate at which a new compound is formed is modeled by the following differential equation, where $X(t)$ denotes the number of grams of the new compound formed in time t:

Turn over

$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$, where k (>0) is a constant of proportionality and constants, α *and* β are such that $\beta > \alpha > 0$. Use the above differential equation to predict the behavior of $X(t)$ as $t \to \infty$. **[20marks]**

(i) Consider the case when $\alpha = \beta$. Use the above differential equation to predict the behavior of $X(t)$ as $t \to \infty$, when $X(0) = \alpha$. **[20marks]**

(ii) Verify that a solution of the differential equation in the case when $k = 1$ and $\alpha = \beta$ is $X(t) = \frac{\alpha - 1}{t + c}$. where c is an arbitrary constant. **[20marks]**

02. (a). Show that, if $n \neq 0, 1$, a change of the dependent variable obtained by the substitution $z = y^{1-n}$ transforms the equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$ into a linear differential equation in z. **[25marks]**

(b) If M and N are functions of x and y, then the equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact when there exists a solution of the form $f(x,y) = C$, where C is an arbitrary constant, such that $df(x,y) = M(x,y)dx + N(x,y)dy$. Show that a necessary and sufficient condition for the differential equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. **[25marks]**

(c) Solve the following equations:

(i) $\frac{dy}{dx} - y\tan x = -y^2 \sec x$ **[25marks]**

(ii) $(\cos x \sin x - xy^2)dx + y(1 - x^2)dy = 0$ **[25marks]**

03 (a) Let $F(D) = p_n D^n + p_{n-1} D^{n-1} + p_{n-2} D^{n-2} + \ldots + p_1 D + p_0$, where $p_0, p_1, \ldots, p_n$ are constants and the operator $D = \frac{d}{dx}$.

Show that

(i). $F(D^2)\{\sin ax\} = F(-a^2)\sin ax$ **[15marks]**

(ii). $F(D)\{e^{ax}\} = F(a)e^{ax}$ **[10marks]**

(iii). $\frac{1}{D-a} f(x) = e^{-ax} \int f(x) e^{ax} dx$, **[20marks]**

where a is a constant and $f(x)$ is a polynomial in x.

(b) Find the general solution of each of the following differential equations:

(i) $(D^2 + 9)y = \sec 3x$ **[25marks]**

(ii) $(D^3 + 1)y = 5e^{2x} + e^{-x} + 3$. **[30marks]**

04. (a) Discuss a method of finding the general solution of Clairaut's equation $y = px + f(p)$, where $f(p)$ is a function of $p = \frac{dy}{dx}$. **[30marks]**

(b) Reduce the equation $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form by using

the substitutions $y - x = v$ *and* $x^2 = u$, and hence solve the equation.

[30marks]

(c) Solve the following simultaneous linear differential equations, where ω is a constant:

$$\frac{dx}{dt} = -\omega y \text{ and } \frac{dy}{dt} = \omega x$$

Also show that any point (x, y) satisfying the two equations lies on a circle.

[40marks]

05 (i) Obtain a method of finding the general solution of Riccati's equation,

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2.$$

[30marks]

(ii) Show that, $y = x^2$ is a particular integral of the following Ricaati's equation,

$x(1 - x^3)y_1 = x^2 + y - 2xy^2$ and hence obtain its general solution. **[30marks]**

(iii) Solve the equation, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0$

[40marks]

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