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RAJARATA UNIVERSITY OF SRI LANKA - SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. Four Year Degree in Industrial Mathematics

Fourth Year Semester II Examination- September/October 2013

MAT 4305 – Stochastic Processes

Answer **Six(06)** questions only Time: Three (3) hrs.

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1. A communication system transmits the two digits 0 and 1, each of them passing through several stages. Let X_n be the digit leaving the n^{th} stage of the system and X_0 be the digit entering the first stage (leaving the 0^{th} stage). At each stage the probability that a 0 is received as a 1 is α and the probability that a 1 is received as a 0 is β.

   (i) Show that $\{X_n : n = 0, 1, \ldots\}$ is a homogeneous two-state Markov chain.

   (ii) Obtain its one-step transition probability matrix.

   (iii) Show that the chain is ergodic.

   Hence, show that $\begin{pmatrix}\pi_0 & \pi_1\end{pmatrix} = \begin{pmatrix}\dfrac{\beta}{\alpha+\beta} & \dfrac{\alpha}{\alpha+\beta}\end{pmatrix}$, where $\begin{pmatrix}\pi_0 & \pi_1\end{pmatrix}$ is the limiting distribution of the chain.

   (iv) Let $f_{00}^{(n)}$ be the probability distribution of the recurrence time of the state 0.

   Show that $$f_{00}^{(n)} = \begin{cases} 1-\alpha, & \text{if } n = 1, \\ \alpha\beta(1-\beta)^{n-2}, & \text{if } n = 2, 3, \cdots \end{cases}$$

   (v) Calculate μ_{00} , the mean recurrence time of the state 0.

   Verify that $\pi_0 = \dfrac{1}{\mu_{00}}$.

2. A toothpaste manufacturing company is considering introducing a new brand to the market. The research department of company decides to produce it in small scale and then to do a study how the customers respond on the new brand. Data collected over a considerable period of time reveals that a customer buying existing brand one week would buy the new brand next week with probability α and a customer buying new brand one week would buy the existing brand next week with probability β. Initially $C_0(0)$ of the customers bought existing brand and $C_1(0)$ of the customers bought new brand. Use your knowledge in the theory of Markov Chain to prepare a report for the company on behalf of the research department.

3. Consider the operation of an automatic loom use to weave cloth. Normally, the loom will operate without human intervention. Occasionally, however, a thread breaks and the shuttle may jam. There is an attendant standing by whose sole responsibility is to unjam the shuttle, tie threads, and put the loom back into operation. Ignoring shift changes, lunch hours and tea breaks the system will always be in one of the two states: 0 – the loom is shut off and the man is working to repair it; 1 – the loom is in operation and the man is idle. Suppose that the lengths of operating period and the period under repair are independent random variables having Negative Exponential distributions with means $\frac{1}{\mu}$ and $\frac{1}{\lambda}$ $(\lambda, \mu > 0)$ respectively. Explain how you would study the evolution of the system in any one of the two methods given below:

Method *A*: through limiting arguments on the Two-state Markov Chain.

Method *B*: through an independent approach based on certain postulate necessary for the Two-state Markov Process.

4. (*a*) Define what is meant by each of the following as used in homogeneous Markov chains:
   (i) A transient state,
   (ii) A positive recurrent state,
   (iii) A null recurrent state,
   (iv) An aperiodic state,
   (v) A periodic state with period *t*.

(*b*) Let $\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$ be a one-step transition probability matrix of a six-state Markov chain with sates 0, 1, 2, 3, 4 and 5.

   (i) Draw the circle-arrow directed (transition) diagram and identify the equivalence classes of the states of the chain.
   (ii) Show that the chain is non-irreducible and aperiodic.
   (iii) Classify the states of the chain.
   (vi) Obtain the limiting distribution of the chain.

5. (*a*) Show that a state *i* is recurrent or transient according as $\sum_{n=1}^{\infty} P_{ii}^{(n)}$ is divergent or convergent, where $P_{ii}^{(n)} = P[X_n = i \mid X_0 = i]$.

(*b*) Suppose that a state *i* and a state *j* of a Markov chain communicate each other. Show that
(*i*) *i* and *j* are either both transient, positive recurrent or both null recurrent,
(*ii*) *i* and *j* both have the same period.

6. Consider a single server queuing process in which customers arrive in a Poisson process and the service times are independent and identically distributed. Let X_n be the size of the queue left behind by the n^{th} customer at the instant when he completes his service. Let ρ be the mean number of arrivals during one service time.
   (i) Show that $\{X_n\}$ is an irreducible aperiodic infinite Markov Chain.

   (ii) Prove that, if $\rho < 1$ then the Markov Chain $\{X_n\}$ is ergodic and the limiting distribution of X_n has probability generating function $\pi(z) = (1-\rho)\dfrac{G(z)(z-1)}{z-G(z)}$, where *G(z)* is the probability generating function of the distribution of the number of arrivals in one service time.

   (iii) Show that if $\rho = 1$ then the Markov Chain $\{X_n\}$ is null recurrent.

   (iv) Prove that when the service time has an Exponential distribution then $\pi(z) = \dfrac{1-\rho}{1-\rho z}$.

7. Consider a branching process which satisfies the following conditions:
   (i) initially there is only one individual in the system,
   (ii) the numbers of offspring of individuals (irrespective of their generations) are independent and identically distributed random variables with probability generating function *G(z)*,
   (iii) the individuals of each generation are replaced by their offspring, which then forms the succeeding generation.

   If $G_n(z)$ is the probability generating function of the number of individuals in the n^{th} generation, show that $G_{n+1}(z) = G_n(G(z))$ for $n = 1, 2, \ldots$.

   Prove that the mean number of individuals in the n^{th} generation is μ^n, where μ is the mean of the progeny distribution.

   Determine the mean number of individuals who ever lived distinguishing between the cases $\mu < 1$ and $\mu \geq 1$.

8. In a population of bacteria the probability that a bacterium divides into two smaller bacteria in $(t, t+\delta t)$ is $\lambda\delta t + o(\delta t)$ and the probability that it dies in $(t, t+\delta t)$ is $\mu\delta t + o(\delta t)$; where λ and μ are positive constants. If $N(t)$ is the population size of the bacteria at time t prove that $\Pi(z,t)$, the probability generating function of $N(t)$ satisfies the difference-differential equation

$$\frac{\partial \Pi(z,t)}{\partial t} - (\lambda z - \mu)(z-1)\frac{\partial \Pi(z,t)}{\partial z} = 0.$$

If $N(0) = n_0$ find $\Pi(z,t)$ and hence deduce that the mean number of bacteria that alive at time t is $n_0 e^{(\lambda-\mu)t}$ and the ultimate extinction of the population is certain if $\lambda \le \mu$.

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