10

RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (Four Year) Degree in Applied Sciences
Fourth Year - Semester II Examination – July 2020

PHY 4203 – CLASSICAL MECHANICS

Time: 02 hours

Answer All Questions

In each problem, you are expected to work-out detailed answers that involve clearly labeled force diagrams and clear applications of appropriate laws of physics. You will lose a large portion of points if your solutions would simply substitute given information to equations you memorized

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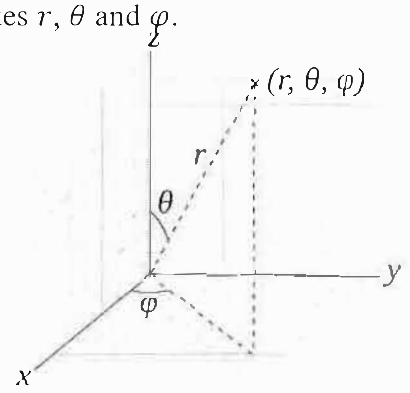
1. In the Cartesian coordinate system, the position vector of a point P, is given by,

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}, \text{ such that } |\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

The Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ are chosen along the direction of increasing x, y, and z. In spherical polar coordinates system, the position of an object may specify in spherical polar coordinates r, θ and φ.



At a certain instant the position of an object specified in terms of the Cartesian coordinates x, y, z, and r, θ, φ are the corresponding spherical coordinates.

a) Show that, in terms of r, θ, φ, the position vector $\vec{r}$ becomes,
$\vec{r} = r\ sin\theta\ cos\phi\ \hat{x} + r\ sin\theta\ sin\phi\ \hat{y} + r\ cos\theta\ \hat{z}$

b) Show that the position vector $\vec{r}$ given above in terms of r, θ, φ, follows $|\vec{r}| = r$.

(10 marks)

2. The gravitational central force field, $\vec{E}_r$ represented by,

$$\vec{E}_r = k\frac{1}{r^2}\hat{r},$$

where k is a constant that absorbs the information about the gravitational mass that produces the gravitational force field (such as, the Sun, the Earth), and r is the magnitude of the radial position vector of a point with respect to the center of the force field. The gravitational potential $V(r)$ at this position relates to the filed through,

$$|\vec{E}_r| = -\frac{d}{dr}V(r).$$

Then, the gravitational potential energy $U(r)$ of the object of mass m at this position is given by, $U(r) = m\,V(r)$.

a) Starting from the given central field $\vec{E}_r$, derive expressions for $V(r)$ and $U(r)$. (Pay attention to signs given in the definitions.)

b) In terms of the Cartesian coordinates, write an expression for the kinetic energy, T, of the system.

c) Using the relationship between Cartesian coordinates and spherical coordinates, write T in the spherical coordinate notation.

d) The Lagrangian L is a system function of coordinates and velocities that has the form, $L = T - V$ where T is the kinetic energy and V is the potential energy of the system. Show that the Lagrangian of an object moving in the central potential becomes,

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 + \frac{km}{r}\right).$$

Use dot notation for time derivatives ($\dot{r} = \frac{dr}{dt}$).

e) Derive the Euler-Lagrangian equations of motion of the system. (You are expected to simplify these equations as much as possible, but not to solve them.)

f) Rewrite the Lagrangian equations of motion, if the object is confined to move on the surface of a sphere of radius R. Clearly indicate your assumptions.

g) Rewrite the Lagrangian equations of motion, if the object is <u>confined to move along the equatorial line</u> of the sphere of radius R. Clearly indicate your assumptions.

h) Rewrite the Lagrangian equations of motion if the object confined to move (under the central potential) on the equatorial plane of the sphere but without limiting to move at a fixed distance from the center. Clearly indicate your assumptions.

(30 marks)

3. Consider a mass m on the end of a spring of natural length l and spring constant k. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume the mass can only move up and down in the vertical direction.

a) Obtain the Lagrangian of this system.

b) Derive the Euler-Lagrangian equations of motion of this system

c) Solve the equation of motion and obtain the general solution $y(t)$

(20 marks)

4. Consider an object moving in the central force field as described in the Problem #2.

   a) Starting from the general Lagrangian for the central force field, obtain the generalized momenta, p_r, p_θ, and p_ϕ of this system.

   b) Show that, in terms of the generalized momenta, the Lagrangian transforms into the form,

   $$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\ \dot{\phi}^2 + \frac{km}{r}\right).$$

   c) Hamiltonian (H) of a system may derive from the Lagrangian (L) through the transformation,

   $$H = \sum_i p_i\dot{q}_i - L.$$

   where p_i, and q_i respectively are the generalized momenta and the generalized coordinates.

      i. Use this transformation to obtain the Hamiltonian of the object in the central force field,

      ii. and show that this Hamiltonian has the form, $H = T + V$, where, T is the kinetic energy and V is the potential energy.

   d) Obtain the Hamiltonian canonical equations for the system. (You are expected to simplify these equations as much as possible, but not to solve them.)

(20 marks)

5. The Hamiltonian of a particle of mass m confined to move in single dimension is given by,

   $$H = \frac{q^2p^2}{2m} + \frac{\lambda}{q^2}.$$

   where p, and q respectively are the generalized momentum and the coordinates of the particle, and λ is a constant.

   a) Obtain the Hamiltonian canonical equations for this system. (Simplify the equations but do not solve them.)

   b) Using the definition of generalized momenta and the transformation between Lagrangian and the Hamiltonian, obtain the Lagrangian of this system.

   c) Obtain the Euler-Lagrangian equations of motion of this system. (Simplify the equations but do not solve them.)

(20 marks)

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