RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES - DEPARTMENT OF PHYSICAL SCIENCES

B.Sc. Four Year Degree in Industrial Mathematics

Fourth Year – Semester II Examination – September / October 2013

MAT 4311 – FLUID DYNAMICS - II

Time: **THREE HOURS.** Answer **FIVE** questions only, **including Q.4 and Q.6**

---

1. State clearly, without proof, Milne-Thomson's circle theorem.

   A straight, infinite, rectilinear vortex filament of strength κ per unit length, is placed parallel to the axis of a rigid circular cylinder with cross-section C, $|z| = a$, and this filament meets the complex z - plane at the point representing number $z_1 = r_1 e^{i\theta}$, $r_1 > a$. Using the circle theorem of Milne- Thomson, show that the complex potential w of the resulting fluid motion is given by

   $$w = i\kappa \log(z - z_1) - i\kappa \log\left(z - \frac{a^2}{\bar{z}_1}\right) + i\kappa \log z + \text{constant}.$$

   Identify the image system and obtain the complex velocity, $Q = -\dfrac{dw}{dz}$.

   (i) Show that the integral $\oint_C Q\,dz = 0$, and by considering its real and imaginary parts, give a physical interpretation of this result.

   (ii) Evaluate the integral $\oint_C \left(-\dfrac{\rho}{2}\right) Q^2\, dz = X - iY$, where ρ denotes the density and interpret the real number pair (X, Y). Hence determine the magnitude and direction of the resultant fluid thrust $\underline{\mathbf{F}}$ on the cylinder, calculated per unit length of the cylinder.

   (iii) Derive the velocity induced on vortex filament, and discuss its motion.

[P.T.O.

2. An inviscid liquid of constant density flows past a fixed cylinder of cross-section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The **velocity at infinity** has constant components $(-U\cos\alpha,\ -U\sin\alpha)$, where U is positive and α is acute. The pressure at infinity is p_∞. Using the conformal transformation $Z=\left(z+\sqrt{z^2-c^2}\right)/(a+b)$, **or** its inverse, where $c^2=a^2-b^2>0$, map the above ellipse, Γ, into the circle C, $|Z|=1$, points outside Γ into points outside C and the stream at infinity in the z-plane, into a stream $(-V\cos\beta,\ -V\sin\beta)$ in the Z-plane, where V and β are to be determined, in terms of given constants. Hence obtain the complex potential in the form $w=F(Z)$, and show that when $Z=re^{i\theta}$, $r\geq a$, the stream function in either of the two planes is $\psi=\frac{U(a+b)}{2}\left(r-\frac{a^2}{r}\right)\sin(\theta-\alpha)$.

Sketch the dividing streamline and two streamlines on either side of it, indicating the directions of flow, in the z-plane.

Derive the complex velocity and deduce that the fluid speed q at any point $(a\cos\theta,\ b\sin\theta)$ on Γ is given by $\frac{q}{U}=\frac{(a+b)\sin(\theta-\alpha)}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$, $0\leq\theta\leq\pi$.

Identify the points of maximum pressure p_0, find the value of p_0 and calculate the moment of the couple experienced by the cylinder, per unit length.

3. An incompressible inviscid fluid moves in the z-plane, within the region $-a\leq x\leq a,\ y\geq 0$.

Using Schwarz-Christoffel theorem, map the flow region into the upper-half of a certain $\zeta-plane$ in such a way that the two corners correspond to the points $\zeta=\mp 1$, and derive the relation $\zeta=\sin\left(\frac{\pi z}{2a}\right)$.

If the motion is due to a doublet of strength μ placed at the point $z_0=ib$ with its axis pointing towards the origin, show that the complex potential is $w=\frac{(\pi\mu/2a)\sinh(\pi b/a)}{\sin^2(\pi z/2a)+\sinh^2(\pi b/2a)}$.

Derive the complex velocity and locate the points of stagnation.

Find the fluid velocity at any point

(i) on the line segment: $-a < x < a,\ y = 0,$ and (ii) on the axis of the doublet $x = 0,\ y \neq b$.

Sketch the streamlines and mark the directions of motion at all points on the boundaries.

Find the points of maximum fluid speed q_0 on the boundaries, and the value of q_0.

4. In a two-dimensional motion under no body forces, liquid issues symmetrically from a very large vessel through a slit, $-a \leq x \leq a$, in one of the straight walls $y = 0$. The ultimate width of the issuing jet is $2b$ as $y \to -\infty$, and the uniform velocity there has components $(0, -U)$, where U is equal to the constant speed on either of the free streamlines.

Choosing the stream function $\psi = 0$ on the central streamline $x = 0$, and the velocity potential $\phi = 0$ at the two edges $(\mp a, 0)$ of the slit, sketch the liquid boundary in the w - plane, where $w = \phi + i\psi$.

Defining Kirchoff's function $\Omega = \log_e(U/Q)$, where Q is the complex velocity, map both the w-plane and the Ω-plane into the same ζ - plane in such a way that the two points corresponding to the edges $(\mp a, 0)$ map into the points $\zeta = \mp 1$, respectively. Hence express ζ in terms of w, and in terms of Ω.

Show that $x = -a + \dfrac{2b}{\pi}(1 - \cos\theta)$ on the free streamline on which $Q = Ue^{-i\theta},\ 0 \geq \theta > -\dfrac{\pi}{2}$.

Deduce that the coefficient of contraction of the issuing jet is $\pi/(\pi + 2)$.

5. Define Stokes' stream function $\psi(r,\theta)$ for axi-symmetric motion of an incompressible fluid, and write down the components of the velocity vector $\underline{\mathbf{q}} = q_r\underline{\mathbf{e}}_r + q_\theta\underline{\mathbf{e}}_\theta + q_\omega\underline{\mathbf{e}}_\omega$, in terms of spherical polar coordinates (r, θ, ω) and partial derivatives of ψ.

Show that, if the motion is irrotational as well, then $\psi(r,\theta)$ satisfies the partial differential equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0.$$

Find the stream function due to each of the following flows, and establish irrotationality, in each case:

(i) A uniform stream U parallel to the axis of symmetry $\theta = 0$.

(ii) A spherically symmetric radial flow issuing from a point source of strength m at the origin.

[P.T.O.

The equation $r\sin(\theta/2) = a$ represents the surface S of a rigid cylinder, symmetric about the axis $\theta = 0$. An inviscid incompressible fluid flows irrotationally past this cylinder, velocity far from the cylinder being U parallel to the axis of the cylinder. Show the sum of the stream functions in (i) and (ii) above, with $m = Ua^2$, may be used as the stream function Ψ to represent this flow, so that S is the stream surface $\Psi = C$, through the point of stagnation, where C is a constant.

Find the components of velocity $\underline{\mathbf{q}}$ at any point P (r,θ,ω) and verify that this vector is normal to the surface $\Psi =$ constant through point P. Show further that the pressure on the surface of the cylinder is $p = p_\infty - \rho U^2\left(1 + 2\cos\theta - 3\cos^2\theta\right)/8$.

6. A rigid sphere $r = a$ is fixed in a steady stream of liquid of constant density ρ and constant viscosity μ, whose velocity at infinity is $U(-\underline{\mathbf{i}})$. Assuming the liquid to be Newtonian in a motion with small Reynold's number $\left(R = \dfrac{a\rho U}{\mu} << 1\right)$, derive an approximate equation for the liquid velocity $\underline{\mathbf{q}}$ in the form *curl curl curl* $\underline{\mathbf{q}} = \underline{\mathbf{0}}$, stating clearly, the boundary conditions satisfied by the velocity $\underline{\mathbf{q}}$.

Using a Stokes stream function Ψ, show that the vorticity vector takes the form $\underline{\zeta} = \dfrac{\mathbf{e}_\omega}{r\sin\theta}D^2\Psi$, where $D^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right)$, and (r,θ,ω) denote spherical polar coordinates.

Hence derive an equivalent differential equation satisfied by Ψ in the form $D^2\left(D^2\Psi\right) = 0$.

Stating clearly, the boundary conditions satisfied by Ψ, find possible values of n such that $\Psi = r^n\sin^2\theta$ satisfies the above partial differential equation and all the boundary conditions.

Deduce that the vorticity vector $\underline{\zeta}$ is of magnitude $\zeta = \left(\dfrac{3aU}{2r^2}\right)\sin^2\theta$, and find its direction.

Assuming the formula $W = \mu\int_V \zeta^2 dV$ for the rate W of dissipation (loss) of energy due to viscosity, where V denotes the entire volume of fluid outside the sphere, show that the viscous drag on the sphere is $6\pi\mu aU$.