RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

Bachelor of Science in Applied Sciences

First Year - Semester I Examination - July/August 2023

MAP 1203 - REAL ANALYSIS I

Time allowed: Two (2) hours

Answer ALL (04) questions

1. a) i. Define an upper bound and a lower bound of a non-empty set.

   ii. State the completeness axiom of a subset of real numbers.

(10 marks)

b) Let A and B be non-empty bounded subsets of $\mathbb{R}$. Show that the set $S = \{a + b \,|a \in A, b \in$ is bounded above and that $\sup(A+B) = \sup A + \sup B$.

(30 marks)

c) Consider the set $A = \{\frac{(-1)^n}{n} |n \in \mathbb{N}\}$.

   i. Show that A is bounded above, then find the supremum of A. Is this supremum the maximum of A ?

   ii. Show that A is bounded below, then find the infimum of A. Is this infimum the minimum of A ?

(30 marks)

d) Find the Supremum and Infimum of the following sets, if they exist:

   i. $\left\{1 + \frac{(-1)^n}{n} \,|n \in \mathbb{N}\right\}$.

   ii. $\{-\frac{n+1}{n} \,|n \in \mathbb{N}\}$.

   iii. $\left\{\frac{(-1)}{e^x} \,|0 < x \leq 1\right\}$.

(15 marks)

e) Prove that, the set $\mathbb{N}$ of natural numbers is not bounded above.

(15 marks)

2. a) Using the $\epsilon - N$ definition, show the following:

   i. $\lim\limits_{n\to\infty} \left(\frac{1+n^2}{2+3n^2}\right) = \frac{1}{3}$

ii. $\lim_{n \to \infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2$

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(30 marks)

b) Prove that, the every convergent sequence of real numbers has unique limit.

(25 marks)

c) Let $\{S_n\}$ be a bounded sequence of real numbers such that $S_1 = a > 0$, $S_{n+1} = \sqrt{\frac{ab^2+S_n{}^2}{a+1}}$, $b > a$, for all $n \geq 1$. Then using mathematical induction prove that $a \leq S_n \leq b$ and that S_n is increasing for all $n \in \mathbb{N}$. Hence, discuss the convergence of S_n and find the limit of the sequence.

(45 marks)

3. a) Using $\epsilon - \delta$ definition prove the following limits:

i. $\lim_{x \to 2} \frac{x^2+1}{x} = \frac{5}{2}$

ii. $\lim_{x \to 0} x \cos \frac{1}{x} = 0$

(30 marks)

b) Let $f(x)$ be a real valued function defined on some interval I containing a, except possibly at a. If $\lim_{x \to a} f(x) = l_1$ and $\lim_{x \to a} f(x) = l_2$, then prove that $l_1 = l_2$.

(20 marks)

c) Determine the constants a and b so that the function $f(x)$ defined below is everywhere continuous,

$$f(x) = \begin{cases} 2x+1, & \text{if } x \leq 1 \\ ax^2+b, & \text{if } 1 < x < 3 \\ 5x+2a, & \text{if } x \geq 3 \end{cases}.$$

(20 marks)

d) Using the definition prove that,

i. $f(x) = \frac{1}{x^2+1}$ is continuous at $x = -1$.

ii. $f(x) = x^2$ is continuous on $\mathbb{R}$.

(30 marks)

4. a) Prove that, the every differentiable function is continuous. Is every continuous function differentiable? Justify your answer.

(15 marks)

b) Show that the following function $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$,

$$\text{where } f(x) = \begin{cases} x\frac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(30 marks)

c) Show that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$, if $0 < a < b$.
Deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

(35 marks)

d) State the L'Hospital Rule. .Evaluate $\lim_{x \to 0} \dfrac{x \cot x - 1}{x^2}$.

(20 marks)

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