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RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES - DEPARTMENT OF PHYSICAL SCIENCES

B.Sc. (General) Degree

First Year – Semester II Examination – September 2013

MAP 1301 – LINEAR ALGEBRA

Proper Candidates (who want this year's Mid-Semester Marks counted); **Time:** Two Hours

Answer **FOUR QUESTIONS**, selecting two questions from Section A and two from Section B

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All Other Candidates: Answer **ALL SIX QUESTIONS** **Time:** Three Hours

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Section A

1. Row reduce the matrix $A = [\underline{\mathbf{a}}_1\ \underline{\mathbf{a}}_2\ \underline{\mathbf{a}}_3\ \underline{\mathbf{a}}_4\ \underline{\mathbf{a}}_5] = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 4 & 3 & 3 & 4 & 7 \\ 1 & 3 & 2 & 8 & 3 \\ 1 & 4 & -1 & 4 & 0 \end{bmatrix}$ to obtain an echelon form B **with all pivot elements equal** to 1, and express B in the reduced row echelon form $C = [\underline{\mathbf{c}}_1\ \underline{\mathbf{c}}_2\ \underline{\mathbf{c}}_3\ \underline{\mathbf{c}}_4\ \underline{\mathbf{c}}_5]$.

   Hence (i) identify the pivot columns of C and A and

   (ii) find bases for the row space and the column space of A.

P.T.O.

By solving the system of equations $C\underline{\mathbf{x}} = \underline{0}$, where x = $\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix}$ and $\underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, which is equivalent

to the system $A\,\underline{\mathbf{x}} = \underline{0}$, or otherwise, find a basis for the null space of the matrix $\boldsymbol{A}$.

Express $\underline{\mathbf{c}}_4$ as a linear combination of $\underline{\mathbf{c}}_1, \underline{\mathbf{c}}_2, \underline{\mathbf{c}}_3$ and hence shoe that $-2\underline{\mathbf{a}}_1 + 2\,\underline{\mathbf{a}}_2 + 2\,\underline{\mathbf{a}}_3 - \underline{\mathbf{a}}_4 = \underline{0}$.

2. Find the product matrix $M = LU$ of the lower and upper **triangular matrices** given below:

$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$. With the column vector $\underline{\mathbf{b}} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$,

solve the matrix equation $L\underline{\mathbf{y}} = \underline{\mathbf{b}}$ for unknown vector $\underline{\mathbf{y}}$ and use this vector to solve the matrix equation $U\underline{\mathbf{x}} = \underline{\mathbf{y}}$, for unknown vector $\underline{\mathbf{x}}$.

Also, solve the equation $\boldsymbol{M}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$, using row reduction method.

Explain briefly why

(i) the solution vector $\underline{\mathbf{x}}$ is unique;

(ii) the equation $\boldsymbol{M}\,\underline{\mathbf{x}} = \underline{0}$ has only the trivial solution.

If a linear transformation T from $R^3 \to R^3$ is defined by $T(\underline{\mathbf{x}}) = A\underline{\mathbf{x}}$, test whether

T is **onto** and **one-to one.**

3. Show that any 2×2 real symmetric matrix of the form $\boldsymbol{M} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, belongs to a subspace V of the vector space of all 2×2 real matrices, and that the matrix M can be expressed as a linear combination of the three matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Hence find the dimension and a basis for the vector space V.

Given further that $a > 0$ and $ad - b^2 \neq 0$ solve the matrix equation $\boldsymbol{M}\,\boldsymbol{X} = \boldsymbol{I}$ for the unknown matrix $\boldsymbol{X}$, by applying elementary row operations on the augmented matrix:

$$(M:I) = \begin{pmatrix} a & b & 1 & 0 \\ b & d & 0 & 1 \end{pmatrix}.$$

Establish the uniqueness of the solution, and verify that $\boldsymbol{X}\boldsymbol{M} = \boldsymbol{I}$.

What further condition should be satisfied by a, b and d so that the given matrx $\boldsymbol{M}$ is positive definite?

If $ad - b^2 = 0$ solve the equation $M\underline{\mathbf{x}} = \underline{\mathbf{0}}$, for the unknown vector $\underline{\mathbf{x}}$.

Section B

4. By row replacement method, or otherwise, evaluate $|P|$, where $\boldsymbol{P} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

Write down the transpose $\boldsymbol{P}^T$ of the matrix $\boldsymbol{P}$, and show that the product $\boldsymbol{P}\boldsymbol{P}^T$ is a scalar matrix of the form $k\boldsymbol{I}$, and find the value of k; Here $\boldsymbol{I}$ is the identity matrix of order three.

Construct an orthogonal matrix $\boldsymbol{Q}$ in the form $\boldsymbol{Q} = c\,\boldsymbol{P}$, where the scalar c is to be determined.

Using the above results

(i) Show that $|P^T| = |P|$,

(ii) find the inverse of $\boldsymbol{P}$, and evaluate its determinant;

(iii) Show that $adjP = 3\,P^T$, and evaluate its determinant.

Verify the formula for $adjP$ in (iii) by finding $adjP$, independently.

5. Find the characteristic equation of the matrix $A = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$.

Using Cayley - Hamilton theorem,

(i) write down a polynomial equation satisfied by the matrix **A** and

(ii) show that $|A^2 - 26A + 169I| = 36^2$.

P.T.O.

Find the eigenvalues λ_1 , λ_2 and the corresponding unit eigenvectors $\underline{u}_1$, $\underline{u}_2$ of the matrix $\boldsymbol{A}$.

Construct an **orthogonal matrix** $\boldsymbol{P}$ and show that $A\ P = \frac{1}{\sqrt{5}}\left[A\underline{u}_1 \quad A\underline{u}_2\right] = \frac{1}{\sqrt{5}}\begin{bmatrix} -4 & 18 \\ 8 & 9 \end{bmatrix}$.

Hence show that the product $\boldsymbol{P}^{\mathrm{T}} \boldsymbol{A}\, \boldsymbol{P}$ is a diagonal matrix $\mathbf{D}$, and find another diagonal matrix $\mathbf{C}$ whose elements are both positive and such that $\mathbf{C}^2 = \mathbf{D}$.

Show that the matrix product $\boldsymbol{B} = PCP^T$ satisfies the equation $B^2 = A$.

6. Find the symmetric matrix $\boldsymbol{A}$ such that the quadratic form $f(x, y, z)$ can be expressed as

$$f(x, y, z) = 2x^2 + 4y^2 + 5z^2 - 4xz \equiv \begin{bmatrix} x & y & z \end{bmatrix} \mathrm{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Obtain the eigen values of the matrix $\boldsymbol{A}$, and the corresponding unit eigenvectors.

Hence find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{\mathrm{T}} \boldsymbol{A}\, \boldsymbol{P}$ is a diagonal matrix $\mathbf{D}$, whose elements along the principal diagonal are the eigenvalues of $\boldsymbol{A}$.

Show further that the quadric surface $f(x, y, z) = 12$ is the ellipsoid whose equation is $\frac{X^2}{12} + \frac{Y^2}{3} + \frac{Z^2}{2} = 1$, and find the Cartesian equations of the principal axes of this ellipsoid.
