RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. (General) Degree in Applied Sciences
First Year Semester I Examination – March 2021
MAP 1301 – LINEAR ALGEBRA

Time: Three (03) hours

Answer All (06) questions

1. a) Let V be the set of all ordered triples (a,b,c) of real numbers. Determine whether the set V is a vector space over the field $\mathbb{R}$ under the following vector addition $\oplus$ and scalar multiplication $\otimes$:

$$(a_1, b_1, c_1) \oplus (a_2, b_2, c_2) = (a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3),$$
$$\alpha \otimes (a, b, c) = (\alpha a, \alpha b, \alpha c), \text{where} \alpha \in \mathbb{R}.$$

(45 Marks)

b) Let W be non-empty subset of a vector space V over the field $\mathbb{F}$. Prove that W is a subspace of V over $\mathbb{F}$ if and only if for all $\alpha, \beta \in \mathbb{F} \text{and} x, y \in W, \alpha x + \beta y \in W$.

(35 Marks)

c) Determine which of the following are subspaces of the given vector spaces:

i. The set of vectors of the form $\{(a, 0,0) | a \in \mathbb{R}\} \subseteq \mathbb{R}^3$, where $\mathbb{R}^3$ is the vector space over $\mathbb{R}$.

ii. The set of matrices of the form $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{N}(\text{Natural numbers})\right\} \subseteq M_{22}$, where M_{22} is the vector space of all 2×2 matrices over $\mathbb{R}$.

(20 Marks)

2. a) Let $S = \{v_1, v_2, v_3, \ldots, v_k\}$ be a subset of a vector space V over a field $\mathbb{F}$.
Explain the following briefly:

i. V is spanned by S.
ii. S is linearly independent.
iii. S is a basis for V.
iv. Dimension of V.

(20 Marks)

b) Determine whether the given set of vectors spans the given vector space:

i. $S = \{(1, -1,2), (1,1,2), (0,0,1)\} \subseteq \mathbb{R}^3$, where $\mathbb{R}^3$ is a vector space over the field $\mathbb{R}$.

ii. $S = \{1 - x, 3 - x^2, x\} \subseteq P_2$. P_2 is the vector space over $\mathbb{R}$ such that $P_2 = \{a_0 + a_1 x + a_2 x^2 | a_0, a_1, a_2 \in \mathbb{R}\}$.

(20 Marks)

c) i. If u_1, u_2, u_3 are linearly independent vectors in a vector space V over the field $\mathbb{F}$, then prove that the set $\{u_1 + u_2, u_2 + u_3, u_3\}$ is also a linearly independent set in V.

ii. Determine the value of q such that the following set of vectors S is linearly independent.
$S = \{(1,1,2,1), (2,1,2,3), (1,4,2,1), (-1,3,5,q)\}$.

(20 Marks)

d) Let W be the subset of P_3 with usual notations, where

$W = \{x^3 - 2x^2 + 4x + 1, 2x^3 - 3x^2 + 9x - 1, x^3 + 6x - 5, 2x^3 - 5x^2 + 7x + 5\}$.

Find a basis for W.

(Hint: Form a matrix then reduced to row echelon form).

(40 Marks)

3. a) Using Gauss-Jordan elimination method, find the inverse of the following matrix A.

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 4 & 1 & 0 \\ 1 & 0 & -2 & 3 \\ 0 & -1 & -2 & 2 \end{bmatrix}.$$

(40 Marks)

b) Discuss the nature of all solutions of the following linear system, depending on constants $b_1, b_2, b_3 \in \mathbb{R}$.

$$\begin{aligned} x - 2y - 2z &= b_1 \\ 2x - 5y - 4z &= b_2 \\ 4x - 9y - 8z &= b_3 \end{aligned}.$$

(60 Marks)

4. a) Define a linear transformation.

Show that the following mappings are linear transformations.

i. $T((x, y)) = (x - y)$, where $T: \mathbb{R}^2 \to \mathbb{R}$.

ii. $T((a, b, c)) = (c, a + b)$, where $T: \mathbb{R}^3 \to \mathbb{R}^2$.

iii. $T((a, b)) = (a + b, a - b, b)$, where $T: \mathbb{R}^2 \to \mathbb{R}^3$.

(40 Marks)

b) Define the kernel and image of a linear transformation.

Let $T : V \to W$ be a linear transformation, where V and W are vector spaces over the field $\mathbb{F}$.

Show that, the kernel of T is a subspace of V and the image of T is a subspace of W.

(40 Marks)

c) Find the kernel and image of the following linear transformations:

i. $T((x, y)) = (x - y)$, where $T: \mathbb{R}^2 \to \mathbb{R}$.

ii. $T((a, b, c)) = (c, a + b)$, where $T: \mathbb{R}^3 \to \mathbb{R}^2$.

(20 Marks)

5. a) State and prove the Rank-Nullity theorem.

(40 Marks)

b) For each of the following linear mappings T, find a basis and the dimension of their Kernel and Image spaces.

i. $T((a, b, c, d)) = (a - b + c + d, a + 2c - d, a + b + 3c - 3d)$, where $T: \mathbb{R}^4 \to \mathbb{R}^3$.

ii. $T((x, y, z)) = (x + y, y + z)$, where $T: \mathbb{R}^3 \to \mathbb{R}^2$.

(60 Marks)

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6. a) Define an inner product space.

Show that $\langle u, v\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ is an inner product on $\mathbb{C}^n$,
where $u = (x_1, x_2, \ldots, x_n), v = (y_1, y_2, \ldots, y_n)$, where $u, v \in \mathbb{C}^n$.

(35 Marks)

b) Let V be an inner product space over the complex field $\mathbb{C}$. Then show that,

i. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

ii. $|\langle x, y\rangle| \leq \|x\| + \|y\|$ for all $x, y \in V$.

(30 Marks)

c) Define an eigenvalue of a matrix of order n.

Let $A = \begin{bmatrix} 6 & 4 \\ -3 & -1 \end{bmatrix}$ be a 2×2 matrix. Find eigenvalues and eigenvectors of A .

(35 Marks)

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