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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Science
Second Year - Semester II Examination – September/October. 2020

PHY 2106 – ATOMIC AND NUCLEAR PHYSICS

Time: One (01) hours

Answer ALL Questions

The symbols have their usual meanings. Provide detailed solutions to ensure total points.

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1. The complete general solution $\psi(\vec{r}, t)$, of the hydrogen atom time-dependent

Schrodinger equation is given by,

$$\psi(\vec{r}, t)^{m}_{n,l} = A\, e^{-r/na_0} \left(\frac{2r}{na_0}\right)^{l} L^{2l+1}_{n-l-1}\left(\frac{2r}{na_0}\right)\, P_l^m(\cos\theta)\; e^{im\phi}\, e^{-i\omega\, t},$$

where A is a normalization constant. The functions denoted by $L^{2l+1}_{n-l-1}\left(\frac{2r}{n\,a_0}\right)$ are the Laguerre polynomials in $\frac{2r}{n\,a_0}$, and $a_0 = \frac{\hbar^2}{m_e k e^2}$ is the *Bohr Radius*. The functions denoted by $P_l^m(\cos\theta)$ are associated Legendre polynomials. The associated Laguerre polynomials $L_j^p(x)$ have the form,

$$L_j^p(x) = (-1)^p \frac{d^p}{dx^p} L_{j+p}(x)$$

where the Laguerre polynomials $L_j(x)$ generated from,

$$L_j(x) = e^x \frac{d^j}{dx^j} e^{-x} x^j.$$

The associated Legendre polynomials $P_l^m(\cos\theta)$, generated from the Rodriguez's formula given by,

$$P_l^m(x) = (-1)^{|m|}(1 - x^2)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} P_l(x), \quad m = 0, 1, 2...,$$

where,

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l, \quad l = 0, 1, 2, \ldots$$

a) Note that the general solution $\psi(\vec{r}, t)$, requires three indices n, l, m to specify a particular hydrogen atom wavefunction. List the rules used to determine possible values for n, l, m indices that label the complete hydrogen atom position dependent solution. **(6 marks)**

b) Prepare a table listing all the possible values for n, l, m, up to $n = 2$. **(6 marks)**

c) The constant A can determine through normalizing $\psi(\vec{r}, t)$. Which physical purpose is served through the normalization operation? **(8 marks)**

d) Starting from the general normalization condition for a quantum mechanical wavefunction $\psi(\vec{r}, t)$, establish the particular normalization condition applicable to the solution of the hydrogen atom Schrodinger equation. **(10 marks)**

e) Evaluate the complete hydrogen wave-function for $n = 1$ state and determine the normalization constant A for this state. Present the normalized hydrogen atom wavefunction for $n=1$ state. **(20 marks)**

2.

a) Briefly explain the classical purpose served by quantum mechanical operators **(5 marks)**

b) Suppose a quantum mechanical operator $\hat{Q}$ operating on a quantum state ψ follows the eigenvalue equation, $\hat{Q}\psi = \lambda\,\psi$. Provide the classical meanings of each term in the eigenvalue equation. **(5 marks)**

c) The hydrogen atom Hamiltonian operator is given by,

$$\hat{H} = -\frac{\hbar^2}{2m_e}\vec{\nabla}^2 - k\frac{e^2}{r},$$

Where in the Spherical polar coordinates system, the Laplacian operator $\vec{\nabla}^2$ has the form,

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Show that, for the ground state ($n = 1$) of the hydrogen atom system (use the result from problem #1), the Hamiltonian operator <u>strictly</u> measures the ground state energy E_1, such that,

$$E_1 = -\frac{m_e k^2 e^4}{2\hbar^2} = -\frac{\hbar^2}{2m_e a_0^2}.$$

(10 marks)

d) In the three-dimensional Cartesian coordinate system, the angular momentum operator $\hat{L}$, has the form,

$$\hat{L} = \hat{x}\hat{L}_x + \hat{y}\hat{L}_y + \hat{z}\hat{L}_z.$$

where, $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ are the Cartesian x, y, z components of $\hat{L}$. In spherical polar coordinates system, the quantum mechanical angular momentum operator has the form,

$$\hat{L} = -i\hbar\left[-\hat{\theta}\left(\frac{1}{sin\theta}\frac{\partial}{\partial\phi}\right) + \hat{\phi}\frac{\partial}{\partial\theta}\right]$$

Show that Cartesian x, y, z components of the angular momentum operator are given by

$$\hat{L}_x = i\hbar\left(sin\,\phi\frac{\partial}{\partial\theta} + cot\,\theta\,cos\,\phi\,\frac{\partial}{\partial\,\phi}\right),$$

$$\hat{L}_y = i\hbar\left(-cos\,\phi\frac{\partial}{\partial\theta} + cot\,\theta\,sin\,\phi\,\frac{\partial}{\partial\,\phi}\right), \text{ and}$$

$$\hat{L}_z = -i\hbar\,\frac{\partial}{\partial\,\phi}.$$

(20 marks)

e) Fully normalized angular portion (spherical harmonics) of the hydrogen atom wave function, $Y_{l,m}(\theta,\phi)$, is given by,

$$Y_{l,m}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(cos\theta)e^{im\phi}, \text{ where } m \geq 0.$$

Show that $Y_{l,m}(\theta,\phi)$ are eigenstates of $\hat{L}_z$, and indicate the corresponding eigenvalue.

(10 marks)

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