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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree

Second Year Semester I Examination – February/ March 2013

MAP 2202 - REAL ANALYSIS II

Answer four questions only. **Time allowed: Two Hours**

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(1) (a) Determine the convergence or divergence of each of the following infinite series:

$$(i) \sum_{n=1}^{\infty} e^{1-\frac{1}{n}} \qquad (ii) \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}5^n \qquad (iii) \sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2} 4^n$$

(b) Find the exact sum of the following infinite series:

$$(i) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1/2)^n}{n} \qquad (ii) \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

(2) (a) Define the terms Absolute Convergence and Conditional Convergence.

(b) Show that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{n^2+1}$$

is absolutely convergent. What can be said about the convergence of the series?

(c) Check the convergence of each of the following series:

$$(i) \sum_{n=1}^{\infty} (-1)^n \sin(1/n) \qquad (ii) \sum_{n=1}^{\infty} (-1)^n \frac{e^{-n}}{n^2+1}$$

(3) (a) Prove that if a power series $\sum_{n=1}^{\infty} a_n x^n$ converges for some $x = x_0$, then it is convergent for every x with $|x| < |x_0|$.

(b) Find the exact interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n}$.

(c) Considering a suitable riemann integrable function, evaluate

$$\lim_{n\to\infty} \{ \sum_{k=1}^{n} (\frac{1}{n})[(\frac{k}{n})^2 + 1] \}.$$

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(4) (a) Let $f(x,y) = \begin{cases} \frac{x\sin(x+y)}{x^3+y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Compute the repeated limits, $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$.
What can be said of the simultaneous limit, $\lim_{(x,y)\to(0,0)} f(x,y)$?

(b) Consider the function

$$f(x,y) = (\frac{y-x}{y+x})(\frac{1+x}{1+y})$$

in a neighbourhood of the point $(0,0)$. Show that f is not differentiable at $(0,0)$.

(5) (a) Given $f(x,y) = e^{x^2+y^2}\sin(xy)$, $x = t\cos t$ and $y = t\sin t$, compute $\frac{df}{dt}$ at $t = \pi/2$.

(b) Locate and classify all the critical points of the function $f(x,y) = x^3 + y^2 + 3x^2 + 4xy$.

(c) Find the maximum value of the expression $ax + by + cz$, subject to the constraint $x^2 + y^2 + z^2 = 1$.

(6) (a) Let X be a non-empty set and d be a real valued function such that

$$d : X \times X \to \mathbb{R}.$$

What conditions must d satisfy to become a metric on X?

(b) Let d be the trivial metric on X and $a \in X$. Identify the following sets:
(i) $B_{1/2}(a)$, the open ball with center a and radius $1/2$ units,
(ii) $B_2(a)$, the open ball with center a and radius 2 units .

(c) Let d be a metric on X and $A, B \subset X$. The distance $d(A,B)$ between the subsets A and B of X is defined by

$$d(A,B) = \inf\{d(a,b) | a \in A,\ b \in B\}.$$

Take $A = [1,2)$ and $B = (2,3]$.
Let d_1 be the usual metric on $\mathbb{R}$ and d_2 be the trivial metric on $\mathbb{R}$.
Find $d_1(A,B)$ and $d_2(A,B)$.

End of the test.
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