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RAJARATA UNIVERSITYOF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. Four Year Degree in Industrial Mathematics
Fourth Year - Semester II Examination-October / Nov. 2015

MAT 4305 – STOCHASTIC PROCESSES

Answer all questions **Time: 2 hours only**

Calculators will be provided.

1. The discrete-time Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ has a transition probability matrix

$$P = \begin{Vmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{5} & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} \end{Vmatrix}.$$

At time 0 the Markov chain starts with probability 1 in state 1. Each visit to state 1 will cost \$ 40 per time unit, to state 2 it will cost \$ 35, to state 3 it will cost \$ 60, to state 4 it will cost \$ 25, to state 5 it will cost \$ 70.

(a) Specify the classes of the above Markov chain, and determine whether they are transient or recurrent.
(b) Determine the limiting distribution of the Markov chain.
(c) Find the total expected time the process will be in the transient states.
(d) What is the long-run expected costs per time unit?

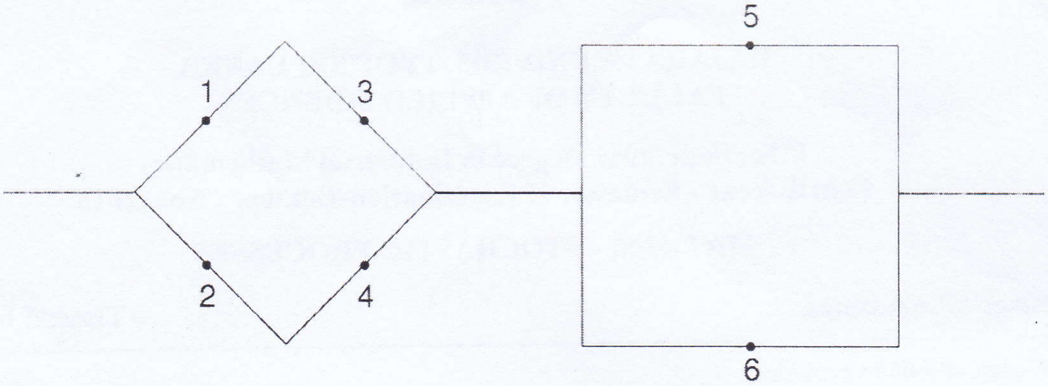
2. (i) Consider a branching process. Let X_n be the population of the n^{th} generation, and let μ be the expected number of offspring produced by an individual in this population. Let us assume $X_0 = 1$.
   (a) Compute $E[X_n]$.
   Hint: Represent $X_n = \sum_{i=1}^{X_{n-1}} Z_i$ where Z_i is the number of offspring of the i^{th} individual of the $(n-1)^{th}$ generation and $E[Z_i] = \mu$.
   (b) Show that ,
   $E[X_m X_n] = \mu^{n-m} E[X_m^2]$ for $m \leq n$.
   Hint: First consider $E[X_n|X_m]$ and then consider $E[X_m X_n|X_m]$.

   (ii) Consider a branching process $\{X_n : n = 0, 1, 2, ...\}$, starting with one particle: $X_0 = 1$. The number of offsprings Z of one particle has distribution

$$P(Z = i) = \begin{cases} 0.1, & \text{for } i = 0, \\ 0.7, & \text{for } i = 1, \\ 0.2, & \text{for } i = 2. \end{cases}$$

   (a) Calculate the probability that the population eventually dies out and show that for $n \geq 1$, $E[X_n] = (1.1)^n$ and $Var[X_n] = (0.29)\left[(1.1)^{n-1} + (1.1)^n + ... + (1.1)^{2n-2}\right]$.
   (b) Now assume that $X_0 = k$, for some arbitrary positive integer $k > 1$, instead of $X_0 = 1$. In this case, give again $E(X_n)$, $Var[X_n]$ and the probability that the population eventually dies out.

3. (i) (a) Find the minimal path sets for the given structure.



(b) What are the minimal cut sets?
(c) Write the structure function.
(d) Give the reliability function of the structure.

(ii) Find the mean lifetime of a series system of two components when the component lifetimes are respectively uniform on $(0, 1)$ and uniform on $(0, 2)$. Repeat for a parallel system.

4. Consider a production process that has two states namely "good state" (state 1) and "poor state" (state 2). If the process is in state 1 during a period then, independent of the past, with probability 0.9 it will be in state 1 during the next period and with probability 0.1 it will be in state 2. Once in state 2, it remains in that state forever. Suppose that a single item is produced each period and that each item produced when the process is in state 1 is of acceptable quality with probability 0.99, while each item produced when the process is in state 2 is of acceptable quality with probability 0.96. The signal is the status of the item produced, and has value either a or u, depending on whether the item is acceptable or unacceptable.

(i) Explain why this process is a hidden Markov chain.
(ii) Find the transition probabilities of the underlying Markov chain.
(iii) Suppose that $P\{X_1 = 1\} = 0.8$. It is given that the successive conditions of the first three items produced are a, u, a.

(a) What is the probability that the process was in its good state when the third item was produced?
(b) What is the probability that X_4 is 1?
(c) What is the probability that the next item produced is acceptable?

Hint:

$P\{X_n = j/S^n = s_n\} = \frac{F_n(j)}{\sum_i F_n(i)}$
$F_n(j) = p(s_n/j) \sum_i F_{n-1}(i) P_{i,j}$
$P\{X_n = j, S_n = s_n/X_{n-1} = i\} = P_{i,j}\, p(s_n/j)$