RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

BSc in Applied Sciences

Second Year - Semester I Examination – June/July 2022

MAA 2302 – Probability and Statistics II

Time: Three (03) hours

Answer **all** questions.

You should use only probability axioms to provide the answers.

01. a) A certain family has 6 children, consisting of 3 boys and 3 girls. Assuming that all birth orders are likely to happen equally, what is the probability that the 3 eldest children to be 3 girls?

(7.5 marks)

b) Show that for any event A and B,

$P(A) + P(B) - 1 \leq P(A \cap B) \leq P(A \cup B) \leq P(A) + P(B)$.

(7.5 marks)

c) Show that if A and B are independent, then

$P(A \cup B) = 1 - P(A^c)P(B^c)$.

(5 marks)

d) There are 100 passengers lined up to board an aeroplane with 100 seats (with each seat assigned to one of the passengers). The first passenger in line crazily decides to sit in a randomly chosen seat (with all seats equally likely). Each subsequent passenger takes their assigned seat if available, and otherwise sits in a random available seat. What is the probability that the last passenger in line gets to sit in his/her assigned seat?

(5 marks)

02. a) In the definition of the independence of two events, you were given three equalities to check.

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \cap B) = P(A)P(B)$

If any one of these equalities holds, A and B are independent. Show that if any of these equalities hold, the other two also hold.

(5 marks)

b) If $P(A) > 0, P(B) > 0$, and $P(A) < P(A|B)$, show that $P(B) < P(B|A)$.

(5 marks)

c) A crime is committed by one of two suspects, A and B. Initially, there is equal evidence against both of them. Further investigation at the crime scene, it is found that the guilty party had a blood type found in 10% of the population. Suspect A does match this blood type, whereas the blood type of Suspect B is unknown.

i. Given this information, what is the probability that A is the guilty party?

ii. Given this new information, what is the probability that B's blood type matches that found at the crime scene?

(10 marks)

d) A sequence of n independent experiments is performed. Each experiment is a success with probability p, and a failure with probability $q = 1 - p$. Show that conditional on the number of successes, all valid possibilities for the list of outcomes of the experiment are equally likely.

(5 marks)

03. a) Benford's law states that in a very large variety of real-life data sets, the first digit approximately follows a particular distribution with about a 30% chance of a 1, an 18% chance of a 2, and in general $P(D = j) = log_{10}\left(\frac{j+1}{j}\right)$, for $j \in \{1,2,3, \dots, 9\}$, where D is the first digit of a randomly chosen element. Check that this is a valid probability mass function (*pmf*) or not.

(5 marks)

b) Let $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ be independent of X. Show that $X - Y$ is not Binomial.

(2.5 marks)

c) Suppose that a lot of 5000 electrical fuses contain 5% defectives. If a sample of 5 fuses is tested, find the probability of identifying at least one defective.

(5 marks)

d) Suppose the engine malfunction probability during any one-hour period is $p = 0.02$. Find the probability that a given engine will survive two hours.

(5 marks)

e) Find the moment generating function $m(t)$ for a Poisson distributed random variable with mean λ.

(10 marks)

04. Let's consider the joint probability density function (*pdf*),

$$f(x,y) = \begin{cases} k(1-y)\,; 0 \le x \le y \le 1 \\ 0\;; \quad elsewhere \end{cases}$$

a) What could be the possible value k can take where $f(x,y)$ to be a joint *pdf.*

(5 marks)

b) Show that X and Y are not independent.

(10 marks)

c) Find the conditional density function of X given $Y = y$.

(5 marks)

d) Find $E(X|Y = y)$

(5 marks)

05. Consider X and Y are continuous, uniformly distributed, independent random variables on the interval (0,1).

a) Write down the joint distribution of X and Y.

(2.5 marks)

b) Sketch the support of function of random variables $U = X + Y$.

(2.5 marks)

c) Find the probability density function (*pdf*) of U, using the distribution function method.

(20 marks)

06. a) Let X and Y have a joint density function given by,

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0 \le x,\ 0 \le y \\ 0, & elsewhere \end{cases}$$

Use the method of transformation and find the density function for $U = X + Y$.

(10 marks)

b) Using the moment generating function method, show that standard normal random variable Z has a normal distribution with mean 0 and variance 1.

(5 marks)

c) Let $X_1, X_2, \dots, X_n$ be independent, uniformly distributed random variables on the interval $[0, \theta]$. Find the,

i. probability distribution function of $X_{(n)} = \max(X_1, X_2, \dots, X_n)$.

ii. density function of $X_{(n)}$.

iii. mean and variance of $X_{(n)}$.

(10 marks)

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