RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. General Degree in Applied Sciences
First Year - Semester II Examination – October/November 2017

MAP 1203 – REAL ANALYSIS I

Time: Two (02) hours

Answer all (04) questions

1. a) Define a bounded set. Let $x \in (a,b)$, prove that $\left| x - \frac{a+b}{2} \right| < \frac{b-a}{2}$.

   b) Define the set of rational numbers. Show that $\sqrt{p} + \sqrt{q}$, where p and q are odd prime numbers, is not a rational number

   c) State the completeness axiom. Find the *Supremum* , *Infimum*, *Maximum* and *Minimum* of the following sets, if they exist.

   i. $\left\{ \frac{1}{5n} \middle| n \in Z, n \neq 0 \right\}$.

   ii. $\{(-1)^n | n \in N\}$.

   iii. $\{2^n | n \in N\}$.

   d) Let $A, B\, and\, C$ be bounded subsets of real numbers. Prove that $Sup(A+B+C) = SupA + SupB + SupC$.

2. a) Define a limit of a sequence. Show that "every convergent sequence has a unique limit".

   b) Using the definition of a limit of a sequence, prove that

   i. $\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$

   ii. $\lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}$.

[Question number 2 continues on next page...

c) Define a bounded sequence and a monotone sequence.

A sequence $\{S_n\}$ is defined as follows:

$$S_1 = a > 0,\ S_{n+1} = \sqrt{\frac{ab^2 + S_n^{\ 2}}{a+1}}, b > a, n \geq 1.$$ Show that $\{S_n\}$ is convergent and find its limit.

(*Hint: Every monotonically increasing bounded sequence is convergent*)

3. a) Define a limit of a function at a point.

   Using the above definition, prove that, $\lim_{x\to 0} x \sin\frac{1}{x} = 0$.

   b) Using the definition of a continues function at a point prove that

$$f(x) = \begin{cases} x^2 \cos\frac{1}{x}, & if\ x \neq 0 \\ 0 & ,\ if\ x = 0 \end{cases}$$ is continues at $x = 0$.

   c) Determine the constants a and b so that the function f defined below is continues everywhere:

$$f(x) = \begin{cases} 3 & ,\ if\ x \leq 2 \\ ax^2 + bx + 1 & ,\ if\ 2 < x \leq 3x \\ 7 - ax & ,\ if\ x > 3 \end{cases}$$

4. a) Show that the function $f(x)$ is not differentiable at $x = 1$, where

$$f(x) = \begin{cases} x^2 - 1 & ,\ if\ x \geq 1 \\ 1 - x & ,\ if\ x < 1 \end{cases}$$

   b) State the following theorems:

      i. Rolle's theorem.

      ii. Lagrange's Mean Value theorem.

      iii. Cauchy's Mean Value theorem.

   c) Show that $\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$ if $0 < u < v$ and deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

   (*Hint:* Use Lagrange's Mean Value theorem)

   d) State L'Hospital rule.

   Evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$.

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