RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. Four Year Degree in Industrial Mathematics
Fourth Year - Semester I Examination – October/November 2017

MAT 4309 – COMBINATORICS

Time: Three (03) hours

Answer all (06) questions

1. a) Suppose that there are two goats on an island initially. The number of goats on the island doubles every year by natural reproduction, and some goats are either added or removed each year.
   i. Construct a recurrence relation for the number of goats on the island at the start of the n^{th} year, assuming that during each year extra 100 goats are put on the island.
   ii. Solve the recurrence relation obtained in part (i) to find the number of goats on the island at the start of the n^{th} year.
   iii. Construct a recurrence relation for the number of goats on the island at the start of the n^{th} year, assuming that n number of goats are removed during the n^{th} year for each $n \geq 1$.
   iv. Solve the recurrence relation obtained in part (iii) to find the number of goats on the island at the start of the n^{th} year.

   b) A bus driver pays all tolls, using only nickels (1 nickels=5 cents) and dimes (1dimes=1 cents), by throwing one coin at a time into the mechanical toll collector.
   i. Formulate the recurrence relation for the number of different ways the bus driver can pay a toll of n cents (when the order in which the coins are used matters).
   (*Hint: Let b_n denotes the number of ways of paying n cents*)
   ii. State the initial conditions.
   iii. In how many ways can the driver pay a toll of 45 cents?

2. a) The sequence $\{b_n\}$ is defined as follows:

$$b_n = \begin{cases} 0 & if\ n = 2k+1 \\ a_k & if\ n = 2k \end{cases}$$

   Find the generating function for b_n in terms of $a_k\ where\ k = 0,1,2,\ldots$.

   b) Solve the following recurrence relations using generating functions.

   i. $a_n = 3a_{n-1} + 2$ with the initial condition $a_0 = 1$ and $n \geq 1$.
   ii. $a_n = a_{n-1} + a_{n-2}$ with the initial condition $a_0 = 1\ and\ a_1 = 1$ and $n \geq 2$.

[P.T.O.

3.

a) State "Inclusion and Exclusion" principle for general case. For non negative integers $k_1, k_2, \ldots, k_r$ with $k_1 + k_2 + \ldots + k_r = n$ the multinomial coefficient, prove that

$$\binom{n}{k_1, k_2, \ldots, k_r} = \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-k_2-\ldots k_r-1}{k_r} = \frac{n!}{k_1!k_2!\ldots k_r!}.$$

b) How many permutations of the 26 letters which do not contain any of the sequences PUTNAM, EXAM, DEC, FIRST?

c) In how many ways can we arrange 21 books in shelves where each shelf has at least one, but at most 5?

d) Of how many arrangements does the word "GALENBINDUNUWEWA" have no consecutive pairs?

e) Determine the number of integers less than or equal to 5500 that are not divisible by 5, 8, 11 or 12.

4. a) Explain the steps clearly to select k objects from n objects in the following cases:

Case I: Order significant, repetition allowed
Case I: Order significant, repetition not allowed
Case II: Order not significant, repetition not allowed
Case IV: Order not significant, repetition allowed

b) How many solutions are there to the following equations?

i. $x_1 + x_2 + x_3 + x_4 = 23$ where x_i 's are positive integers such that $i = 1,2,3,4$ and $x_i \leq 7$

ii. $x_1 + x_2 + x_3 = 18$ where x_i 's are positive integers such that $i = 1,2,3$ and $x_1 \leq 7,\ x_2 \leq 5,\ x_3 \leq 3$.

5. a) Give a combinatorial interpretation of the coefficient of x^k in the expansion $(1 + x + x^2 + x^3 + \ldots)^n$ such that the power of x at most l and $l \leq k$.

Hence, find that the coefficient of x^7 in the expansion $(1 + x + x^2 + x^3 + \ldots)^8$.

b) Define a Latin Square and state the theorem for the number of Latin Squares of order n. Show that the numbers of $n \times n$ Latin squares are 1, 2, 12, 576 for $n = 1,2,3,4$ respectively.

6. a) Define a *Steiner Triple System*.

If (X, B) be a Steiner Triple System of order n, where $n > 0$ then prove that,

i. any element of X is contained in $(n-1)/2$ members of B.

ii. $|\mathrm{B}| = \dfrac{n(n-1)}{6}$.

iii. $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.

b) Construct $STS(7)$.

c) Construct the tournament schedule for 8 teams.

d) Using part b) and c) construct $STS(15)$.

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