RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

Bachelor of Science in Applied Sciences

First Year - Semester II Examination – Jan/Feb 2023

MAP 1302 – DIFFERENTIAL EQUATIONS I

Time: Three (03) hours

Answer **All** questions.

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01. a) Briefly explain the following terminologies used in context of ordinary differential equations:

- Analytical solution
- Approximate solution
- General solution
- Particular solution
- Initial value problem

(10 marks)

b) Provide a complete description (order, degree, linear, nonlinear, homogeneous, non-homogeneous, ordinary, partial) of the differential equations given below.

i. $\frac{d\,f(x)}{dx} = 3\,f(x) + 5$

ii. $\frac{d^3f(x)}{dx^3} + f(x)\,\frac{d^2f(x)}{dx^2} = 0$

iii. $\frac{d^2\,f(x)}{dx^2} - 2\,f(x) = \ln x$

(05 marks)

c) i. Write down the general n^{th} order linear homogeneous differential equation.

ii. If $f_1(x)$ is a solution to part i), then show that for any arbitrary constant λ, the function $f_2(x) = \lambda\,f_1(x)$ is also a solution to the part i).

iii. Show that the property proved in ii); does not apply for the general non-homogeneous case.

(10 marks)

d) Suppose the functions $f_1(x), f_2(x), f_3(x), \dots$ are particular solutions to the general n^{th} order linear homogeneous differential equation. Show that any linear combination of $f_1(x), f_2(x), f_3(x), \dots$ is also a solution of the differential equation.

(05 marks)

02. Consider a first order differential equation that satisfies the initial conditions:

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$$\frac{df(x)}{dx} = g(x, f(x)), \text{ with } f(x_0) = f_0 \text{ and, } \left.\frac{d\,f(x)}{dx}\right|_{x=x_0} = g(x_0\,, f_0).$$

Note: The function $g(X, f(X))$, and the values correspond to the initial conditions are known.

a) Graphically, infer the given scenario for the unknown function, $f(x)$.
(05 marks)

b) In terms of the given information, write down an expression for the equation of the tangent line of the unknown function at x_0.
(05 marks)

c) Consider the point x_1, such that $x_1 = x_0 + h$, where h is a known constant. Compute y_1, the value of the tangent line to the curve $y = f(x)$ at x_1.
(05 marks)

d) Under which condition would y_1 become a reasonable approximation for $f(x_1)$?
(05 marks)

e) If y_1 is a good approximation for $f(x_1)$, write an expression for the equation of the tangent line of the unknown function at x_1 in terms of y_1.
(05 marks)

f) Write down the general formula for y_{n+1}, the approximated value to $f(x)$ at x_{n+1}.
(05 marks)

03. Consider the ordinary second order homogeneous linear differential equation with constant coefficients given by,

$$a\,\frac{d^2f(x)}{dx^2} + b\,\frac{df(x)}{dx} + c\,f(x) = 0.$$

a) Starting from the characteristic equation of the differential equation, establish the condition that constitutes the case of overlapping roots.
(05 marks)

b) What is the form of the particular solution $f_1(x)$ established from this condition? (Use the value for k, results from the characteristic equation)
(05 marks)

c) Is it possible to establish a general solution for the differential equation with the particular solution established in part b)? Explain.
(05 marks)

d) Consider the particular solution $f_1(x)$ of the form $f_1(x) = e^{kx}$ with the proper value for k attained under the condition of overlapping roots for the characteristic equation in part a). Suppose a second particular solution $f_2(x)$ is sought such that, $f_2(x) = v(x)f_1(x)$, where $v(x)$ is an unknown function. Establish a condition for $v(x)$, in order for $f_2(x)$ to be a solution to the differential equation.
(10 marks)

e) Find the general solution for second order homogeneous linear differential equation and show that it takes the form,

$$f(x) = C_1 e^{-\frac{b}{2a}x} + C_2\, x\, e^{-\frac{b}{2a}x} \text{ where } C_1 \text{and } C_2 \text{ are arbitrary constants.}$$
(05 marks)

04. a) The general non-homogeneous ordinary second order differential equation is given by,

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$$a(x)\,\frac{d^2 f(x)}{dx^2} + b(x)\,\frac{d\,f(x)}{dx} + c(x)f(x) = g(x).$$

Given the function $f_p(x)$ to be a particular solution to the differential equation, and the function $f_c(x)$ to be a general solution to the same differential equation for the situation $g(x) = 0$.

i. Show that the function $f(x) = f_p(x) + f_c(x)$ is a solution to the given differential equation.

ii. Can $f(x)$ [in part i)] be a general solution to the given differential equation? Explain.

(10 marks)

b) The homogeneous second order linear differential equation given by, $2x^2 \frac{d^2}{dx^2} f(x) + x\,\frac{d}{dx}\,f(x) - 3\,f(x) = 0$ has two particular solutions for $f_1(x)$ and $f_2(x)$ such that $f_1(x) = x^{-1}$ and $f_2(x) = x^{3/2}$. Show that $f_1(x)$ and $f_2(x)$ form a set of independent solutions.

(10 marks)

05. Consider a general homogeneous second order linear differential equation with non-constant coefficients

$$a(x)\frac{d^2 f(x)}{dx^2} + b(x)\frac{d\,f(x)}{dx} + c(x)f(x) = 0.$$

a) A function $f_1(x)$ is given to be a particular solution to the differential equation. We expect to know another particular solution for $f_2(x)$, such that $f_2(x) = v(x)f_1(x)$, where $v(x)$ is an unknown function. Show that for $f_2(x)$ to be a solution to the differential equation, the function $v(x)$ must be a solution to the second order differential equation,

$$P(x)\frac{d^2}{dx^2}\,v(x) + Q(x)\frac{d}{dx}\,v(x) = 0,$$

where $P(x) = p(x)f_1(x)$, and $Q(x) = \left(2p(x)\frac{d}{dx}f_1(x) + q(x)f_1(x)\right)$.

(10 marks)

b) Consider the differential equation $P(x)\frac{d^2}{dx^2}\,v(x) + Q(x)\frac{d}{dx}\,v(x) = 0$ in part a).

i. Show that through a substitution $w(x) = \frac{d}{dx}\,v(x)$, the given differential equation would transform into a first order differential equation.

ii. Obtain the general solution to the differential equation obtained in i).

iii. Hence, obtain $v(x)$ and $f_2(x)$ such that $f_2(x) = v(x)f_1(x)$.

iv. In terms of $f_1(x)$ and $f_2(x)$, write a general solution to the second order linear homogeneous differential equation with constant coefficients.

v. Comment on how these steps are applicable in the context of finding the solution for a general homogeneous second order linear differential equation with non-constant coefficients.

(20 marks)

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