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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. (Four Year) Degree in Industrial Mathematics
Fourth Year - Semester I Examination – Jan/Feb 2021

MAT 4310 – COMPUTATIONAL MATHEMATICS

Time: Three (03) hours.

- Answer **all (05)** questions.
- Calculators will be provided.

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1. Consider the numerical solution of the initial value problem (IVP):

$$y'(x) = f\big(x, y(x)\big), \qquad a \le x \le b, \qquad y(a) = y_0,$$

using the improved Euler's method with step size h:

$$y_{j+1} = y_j + \frac{h}{2}\Big(f(x_j,\, y_j) + f\big(x_j + h,\, y_j + hf(x_j,\, y_j)\big)\Big),$$

where $x_j = a + jh,\ y_j = y(x_j)$ for $j = 0, 1, 2, \cdots$.

(a) i. Show that the improved Euler's method is consistent with the IVP.

ii. Show that the improved Euler's method is order 2 accuracy.

iii. Determine the interval of absolute stability of the improved Euler's method.

(50 marks)

(b) Consider the IVP; $y'(x) = 1 - x + 4y(x), \quad y(0) = 1.$

Using the improved Euler's method with $h = 0.025$, approximate the value of $y(0.075)$.

(50 marks)

2. Consider the initial value problem (IVP):

$$y'(x) = f\big(x, y(x)\big), \qquad a \le x \le b, \qquad y(a) = y_0.$$

(a) The two- step method:

$$y_{n+2} + b_1 y_{n+1} + b_0 y_n = hcf(x_{n+2}, y_{n+2}),$$

where b_0, b_1, and c are constants, is proposed to solve the IVP.

i. Determine b_0, b_1, and c such that the method is second order.

ii. Show that the method is consistent and stable.

iii. Is the method absolutely stable? Justify your answer.

(40 marks)

(b) i. Write down the 4-step Adams-Bashforth and the 3-step Adams-Moulton methods for the solving IVP.

ii. Write down Adam's predictor-corrector method with m corrections for solving the IVP.

iii. Consider the IVP: $y'(x) = x + y$, $y(1) = 2$.

Use Adam's predictor-corrector method with $m = 1$ and a step size of $h = 0.1$ to approximate $y(0.4)$.

(**Hint**: Use the fourth order Runge-Kutta method to find starting values)

(60 marks)

3. The linear system $Ax = b$ is to be solved iteratively by

$$x^{(k+1)} = Hx^{(k)} + b, \quad k = 0, 1, 2, \cdots.$$

(a) Let $A = \begin{pmatrix} 1 & \alpha \\ 2\alpha & 1 \end{pmatrix}$, where a ($\neq \pm 1/\sqrt{2}$) is real.

Find a necessary and sufficient condition on α for convergence of the Jacobi method.

(30 marks)

(b) Let $A = (a_{ij})$ be the $n \times n$ tridiagonal matrix with

$$a_{ij} = \begin{cases} 4 & \text{if } i = j, \\ -1 & \text{if } i = j+1 \text{ or } i = j-1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the Gauss-Seidel and Jacobi methods converge to the solution for any choice of initial starting vector $x^{(0)}$. **(30 marks)**

(c) Suppose that $A = \begin{pmatrix} 10 & 4 & -1 \\ 2 & 10 & 1 \\ 1 & -1 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ -19 \\ -2 \end{pmatrix}$.

Starting with initial guess $x^{(0)} = (0,0,0)^T$, perform 4 iterations of the Gauss-Seidel method. **(40 marks)**

4. Consider the linear system

$$Ax = b,$$

where $A = L + D + U$ is non-singular, L is strictly lower triangular, D is diagonal, and U is strictly upper triangular.

The successive over relaxation (SOR) iterative method with relaxation parameter $\sigma > 0$ has the form:

$$x^{(k+1)} = T_\sigma x^{(k)} + c, \quad k = 0, 1, \cdots,$$

where

$$c = \left(L + \frac{1}{\sigma} D\right)^{-1} b$$

and

$$T_\sigma = (\sigma L + D)^{-1}[(1-\sigma)D - \sigma U].$$

(a) Let $A = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix}$ and $x^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = b$.

Prove that the SOR method with $\sigma = 1$ does not converge, but that the SOR method with $\sigma = 1/2$ does converge. **(40 marks)**

(b) Let $A = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 5 & 2 \\ 5 & 4 & 10 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$.

i. Determine the iterative matrix T_σ.

ii. Establish the SOR method in matrix form.

iii. Determine the optimal relaxation parameter σ_0 and the rate of convergence.

iv. Taking $x^{(0)} = (0,0,0)^T$, perform three iterations of the SOR method for the solution of the system.

(60 marks)

5. The two-point boundary value problem (BVP) has the form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \qquad a \le x \le b,$$

with $y(a) = \alpha, y(b) = \beta$.

Let $x_j = a + jh$ with $h = (b-a)/(N+1)$ for all $j = 0, 1, \dots, N$.

(a) Using

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2)$$

and

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2),$$

establish the following scheme for the BVP:

$$a(x)y(x+h) + b(x)y(x) + c(x)y(x-h) = h^2 f(x) + O(h^2),$$

where the functions $a(x), b(x)$, and $c(x)$ are needed to be determined.

Hence, deduce a numerical method for the solution of the BVP at the interior grid points.

(30 marks)

(b) Using the numerical method in Part (a), approximate the solution of the following BVP with $N = 3$:

$$y''(x) + 2y'(x) + y(x) = 5x, \qquad y(0) = 0, \qquad y(1) = 0.$$

(70 marks)

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