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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences
Second Year - Semester II Examination – September/ October 2020

MAP 2204 – COMPLEX CALCULUS

Time: Two (02) hours

Answer <u>All</u> (04) questions

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1. a) For two complex numbers z_1 and z_2, prove that $|z_1 + z_2| \geq \left||z_1| - |z_2|\right|$. **(25 marks)**

   b) If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $\left|\dfrac{1 - z_1\overline{z_2}}{z_1 - z_2}\right| < 1$.

   (25 marks)

   c) If $|z - 2 + i| \leq 2$, then find the greatest and least values of $|z|$. **(25 marks)**

   d) If $A, B, C,$ and D are the points z_1, z_2, z_3 *and* z_4 respectively, if $z_1 z_2 + z_3 z_4 = 0$ and $z_1 + z_2 = 0$, then show that the points $A, B, C,$ and D are concyclic.

   (Hint: Condition for four points to be concyclic, $\dfrac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}$ is purely real number)

   (25 marks)

2. a) Does $\lim\limits_{z \to 0}\left(\dfrac{z}{\bar{z}}\right)^2$ exist? Justify your answer. **(30 marks)**

   b) Solve the equation $\sinh z = i$. **(30 marks)**

   c) Find all the values of the following:

      i. $(1 - i)^{1+i}$

      ii. $i^{\sin i}$

   (Hint: When $z \neq 0$ and the exponent c is any complex number, the function z^c is defined by the means of the equation $z^c = e^{c \log z}$)

   (40 marks)

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3. a) Define an analytic function in the complex plane.

Prove the following statement,

"*If $f(z) = u(x, y) + iv(x, y)$ is a differentiable function in the neighborhood of $z = x + iy$ then the partial derivatives of u and v exist and satisfy $u_x = v_y$ and $v_x = -u_y$*".

(45 marks)

b) Is the converse of the above statement in part (a) true? Justify your answer.

(20 marks)

c) Let $f(z)$ and conjugate of $f(z)$ are ~~be an~~ analytic functions in a domain D. Prove that $f(z)$ is a constant function.

(15 marks)

d) Define a Harmonic function in a domain D. Let $u(x, y) = 2xy + 2x$. Show that $u(x, y)$ is harmonic and find the analytic function whose real part is $u(x, y)$. **(20 marks)**

4. a) State and prove the ML inequality.

Using ML inequality find the upper bound of $\left|\int_c \frac{(z^3+3)e^{iz}\log z}{(z^2-2)}\,dz\right|$, where

$C = \left\{z : z = 2e^{i\theta}, 0 \le \theta \le \frac{\pi}{3}\right\}$. **(45 marks)**

b) Sate Cauchy's theorem. **(05 marks)**

c) State Cauchy's Integral Formula.

Using Cauchy's Integral Formula, evaluate the integral $\int_c \frac{\cos 2\pi z}{(2z-1)(z-3)}\,dz$, where C is the circle with the center at the origin and radius 1. **(35 marks)**

d) Derive the Laurent series of the function $f(z) = \frac{1}{z^2(1-z)}$. **(15 marks)**