RAJARATA UNIVERSITY OF SRI LANKA, MIHINTALE

FACULTY OF APPLIED SCIENCES, DEPARTMENT OF PHYSICAL SCIENCES

B.Sc. General Degree (Third Year)

End Semester Examination – Semester I – February/March, 2013

MAT – 3319 FLUID DYNAMICS

Answer **ALL FIVE QUESTIONS** Time: **THREE HOURS**

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1. Show that Euler's equation of motion (to be assumed) of an inviscid homogeneous liquid of constant density ρ moving steadily under gravity can be expressed as $grad\left(\frac{p}{\rho}+\frac{1}{2}q^2+gz\right)=\underline{\mathbf{q}}\times curl\,\underline{\mathbf{q}}$.

If motion is irrotational as well, deduce Bernoulli's equation for pressure.

A rigid right circular cylinder C, of radius a stands with its axis vertical and its base attached to an infinite rigid horizontal plane Π whose equation is $z=0$. It is surrounded by an ocean of inviscid liquid of constant density ρ occupying the region $r \geq a$, also bounded below by the plane Π, and above by its free surface Σ open to the atmosphere at pressure p_0. The cylinder C extends above the free surface Σ and the two surfaces meet in a circle whose equations, in terms of **cylindrical polar coordinates** (r,θ,z), are $r=a,\ z=h$. Verify that the liquid velocity, $\underline{\mathbf{q}}=f(r)\underline{\mathbf{e}}_\theta$ satisfies the equation of continuity and the boundary conditions at C and Π.

Given further that the motion is irrotational everywhere in the liquid, and that $f(a)=a\omega$, where ω is a constant, determine the function $f(r)$. Using Bernoulli's equation, or otherwise, show that the equation of the free surface Σ is $z=h+\frac{a^2\omega^2}{2g}\left(1-\frac{a^2}{r^2}\right)$. Deduce that at large distances from the axis of the cylinder C, the surface Σ is nearly horizontal.

2. Two equal sources , each of strength m, are placed at the points $z = \pm a$ in the complex z-plane, and two equal sinks of the same strength m are placed at the points $z = \pm i\,a$ where a is a positive constant.

(i) Write down the complex potential for the given two-dimensional system of singularities when no rigid boundary is present, and derive the complex velocity, $u - iv = \dfrac{4ma^2 z}{z^4 - a^4}$. Locate stagnation point.

(ii) Find the velocity vector (in magnitude and direction) on the positive parts of both coordinate axes, distinguishing between the regions $0 < x, y < a$ and $x, y > a$.

(iii) Show that the velocity vector at a point on the **circular arc** $z = ae^{i\theta}$, $0 < \theta < \dfrac{\pi}{2}$, is of magnitude

$q = \dfrac{m}{a^2 \sin\theta\cos\theta}$, and find its direction.

Using symmetry of the system of singularities show further that both axes of coordinates and the circle $|z| = a$ are streamlines, and mark the direction of velocity along these streamlines.

3. A two-dimensional source and a sink of the same strength m are placed at the points A and B, represented in the complex z- plane, respectively by $z = \pm 2a$, outside a rigid circular boundary $|z| = a$.

Use the circle theorem of Milne-Thomson to show that $w = -m\log\left\{\dfrac{z-2a}{z+2a}\right\} - m\log\left\{\dfrac{z-a/2}{z+a/2}\right\}$

is the complex potential for the motion, and hence identify the points A', B' of the image singularities, together with their strengths.

Derive the complex velocity, locate the stagnation points C, C', and show that

(i) at any point P in the z- plane, fluid speed is given by $q = \dfrac{(5ma)(CP)(C'P)}{(AP)(A'P)(BP)(B'P)}$;

(ii) the portions of real axis outside the circular boundary form parts of the streamline $\psi = 0$, and mark the direction of velocity there;

(iii) the fluid velocity at a point $(0, y)$ on the imaginary axis is of magnitude $\dfrac{5ma\left(y^2 + a^2\right)}{\left(y^2 + 4a^2\right)\left(y^2 + a^2/4\right)}$,

and find the direction of velocity there.

4. Find the value of the constant C, in terms of known positive constants U and a such that the function $\Phi = \left(U r + \frac{C}{r^2}\right)\cos\theta,\ r \geq a$, where (r,θ,ω) denote spherical polar coordinates, is the velocity potential in the flow of an incompressible fluid past a fixed rigid sphere $r = a$.

[The the origin of coordinates ($r = 0$) is at the centre O of the sphere, and the axis $\theta = 0$ is in the positive direction of the unit vector $\underline{\mathbf{i}}$ along the axis of symmetry, Ox].

Find the radial and transverse components of velocity $\underline{\mathbf{q}}$ at any point P, in **any plane** ω = constant, through the axis of symmetry.

Deduce the **magnitude and direction** of $\underline{\mathbf{q}}$

(i) at points at infinity, and (ii) at any point $Q(a,\theta,\omega)$ on the surface S of the sphere.

Show that the pressure p at the point Q is given by $p = p_0 - \left(\frac{9\rho U^2}{8}\right)\sin^2\theta$, where p_0 is the pressure at a point of stagnation. Deduce that p takes its least value p_1 at points on a certain circle Γ, which has **to be identified**. Find the maximum value of U for which this motion, with no cavitation, is possible.

Show also that the **fluid thrust** $\underline{\mathbf{T}}$ on the portion S_0 of the sphere S between a point of stagnation and the circle Γ, may be evaluated as $\underline{\mathbf{T}} = \int_{S_0} p(-\underline{\mathbf{e}}_r)dS$, where $\underline{\mathbf{i}}.\underline{\mathbf{e}}_r = \cos\theta$, and the **area element** dS of the sphere $r = a$ together with the **appropriate limits of variables** in the surface integral are to be correctly stated.

Hence show that the fluid thrust $\underline{\mathbf{T}}$ is of magnitude $\pi a^2\left[p_1 + \frac{9\rho U^2}{16}\right]$, and find its direction.

5. A uniform solid sphere S of mass M, centre O and radius a, in an infinite homogeneous liquid at rest, is given an impulse $\underline{\mathbf{J}}$, and as a result gets a velocity $U\underline{\mathbf{i}}$. Show that the initial velocity potential ϕ_0 of the irrotational motion, in terms of spherical polar coordinates (r,θ,ω) is $\phi_0 = \frac{Ua^3}{2r^2}\cos\theta$, where the axis $\theta = 0$ is taken along unit vector $\underline{\mathbf{i}}$, in the direction of the positive Ox- axis.

Assuming the formula $P = \rho\phi_0$ for the **impulsive pressure** P at any point in the liquid, where ρ is its density, show that

(i) the initial impulsive thrust on the sphere S is $M'U(-\mathbf{i})/2$, where M' is the mass of liquid displaced by the sphere, and

(ii) the applied impulse $\underline{\mathbf{J}} = (M + M'/2)U\underline{\mathbf{i}}$.

Subsequently the solid sphere, acted upon by a constant force F, continues to move along the Ox-axis, with variable speed V. Assuming that $\phi = \dfrac{Va^3}{2r^2}\cos\theta$ is the velocity potential in this motion, express the liquid speed q in the form $\left(\dfrac{q}{V}\right)^2 = \left(\dfrac{a^3}{2r^3}\right)^2\{1 + 3\cos^2\theta\}$.

Hence evaluate the kinetic energy of the liquid and show further that

the total kinetic energy of the system (solid and liquid) is $\dfrac{1}{2}(M + M'/2)V^2$, and

$\dfrac{dV}{dt} = \dfrac{F}{M + M'/2}$.