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RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences

Second Year-Semester II Examination – September/October 2020

MAP 2202 -REAL ANALYSIS II

Time: Two (2) hours.

Answer any four (4) questions.

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1. a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be infinite series of terms a_n and b_n.

State whether the following statements are **true** or **false**:

Justify your answer.

i. Let $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Let $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

iii. If (a_n) is a decreasing sequence with $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$, then

$\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent.

iv. If $a_n = \frac{1}{n^p}$ with $p > 1$ and if $b_n = \frac{1}{n^q}$ with $0 \leq q \leq 1$, for all $n \in \mathbb{N}$, then

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent.

(40 marks)

Contd.

b) Determine whether each of the following series is convergent or divergent:

i. $\sum_{n=1}^{\infty} \frac{2^n}{5^n+2^n}$,

ii. $\sum_{n=1}^{\infty} \frac{1}{5n-3}$,

iii. $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n^2}$,

iv. $\sum_{n=1}^{\infty} \frac{(-1)^n(n^3+3)}{4n^6+5n^3+5}$.

(60 marks)

2. a) Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$ for all $n = 1, 2, 3, \ldots$.

Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$.

If $l < 1$, then prove that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Deduce that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{n!}$ is convergent.

(45 marks)

b) Using the root test, show that the following series is convergent.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2n+7}{8n-2}\right)^n$$

(25 marks)

c) Using the integral test, determine whether the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)\ln(\ln(n))}$$

(30 marks)

3. a) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1.3 \ldots (2n-1)}{(n+1)!} x^{2n}.$$

(40 marks)

b) Find the power series expansion of

$$\frac{1}{1+x^2} \text{ for } |x| < 1.$$

Hence, find the power series expansion of $\tan^{-1} x$ for $|x| \le 1$.

(35 marks)

c) Let $f: [0, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Determine whether or not f is Riemann integrable on $[0, 1]$.

(25 marks)

4. a) Let $z = e^x \sin y, x = st^2, y = s^3 t$.

Using the chain rule for multivariable functions, find the partial derivative $\partial z/\partial s$ in terms of s and t.

(25 marks)

b) Find all critical points of the function $f(x, y) = x^4 + y^4 - 4xy$.

(25 marks)

Discuss the nature of each of the critical points.

(20 marks)

c) Suppose that the temperature T at a point (x, y, z) on the unit sphere $x^2 + y^2 + z^2 = 1$ is $T(x, y, z) = 30 + 5(x + z)$. Find, using the method of Lagrange multipliers, the extreme values of T.

(30 marks)

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5. a) Reversing the order of integration over the same region of integration, show that

$$\int_0^1 \int_y^1 \frac{y}{\sqrt{x^3+1}}\,dydx = \frac{1}{3}(\sqrt{2}-1).$$

(35 marks)

b) Expressing the area of the region enclosed by the cardioid

$$r = 2(1+\cos\theta),$$

in terms of an appropriate double integral, find its area.

(30 marks)

c) Using cylindrical coordinates, evaluate the triple integral

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\,dzdydx.$$

(35 marks)

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