62

# RAJARATA UNIVERSITY OF SRI LANKA
## FACULTY OF APPLIED SCIENCES

**Bachelor of Science in Industrial Mathematics**

**Fourth Year - Semester I Examination - July/August 2023**

**MAT 4310 - COMPUTATIONAL MATHEMATICS**

**Time: Three (03) hours**

---

- **Answer ALL (06) questions**
- **Calculators will be provided**

---

1. The improved Euler method to find an approximate solution $y_n$ at grid point $x_n$ to the exact solution $y(x_n)$ of the initial value problem: $y'(x) = f(x, y(x))$, $a \leq x \leq b$, $y(a) = y_0$ is given by:

$$y_{n+1} = y_n + \frac{h}{2}\left(f_n + g_n\right), \; n = 0, 1, 2, \cdots, N-1,$$

where $h = (b-a)/N$, $x_n = a + nh$, $f_n = f(x_n,\, y_n)$, and $g_n = f(x_{n+1},\, y_n + hf_n)$.

a) i. Show that the improved Euler method is consistent with the given differential equation.

ii. Show that the improved Euler method is order 2 accuracy.

iii. Applying the improved Euler method to the test equation $y'(x) = \lambda y(x)$, where $\lambda$ is a complex constant with $\Re(\lambda) < 0$ and $x \geq 0$, determine its region of absolute stability.

**(60 marks)**

Continued.

b) Use the improved Euler method with $h = 0.1$ to obtain the approximate value of $y(0.2)$ for the initial value problem:

$$y'(x) = x^2 + [y(x)]^2, \quad y(0) = 1.$$

**(40 marks)**

2. a) i. Briefly, explain each of the following multistep approaches for initial value problems of ordinary differential equations:

- Adams-Bashforth
- Adams-Moultan
- Predictor-Corrector

ii. Consider the following initial value problem:

$$y'(x) = xe^{3x} - 2y(x),\ y(0) = 0.$$

Use Adams fourth-order predictor-corrector method with step size $h = 0.2$ to obtain an approximation for $y(0.8)$.

Assume that: $y(0.2) \approx 0.026812801841426$, $y(0.4) \approx 0.150777835474151$, and $y(0.6) \approx 0.496019565629524$.

**(50 marks)**

b) A multistep method for the intial value problem: $y'(x) = f(x, y(x))$, $a \leq x \leq b$, $y(a) = y_0$ is given by:

$$y_{n+2} + 4y_{n+1} - 5y_n = h(4f_{n+1} + 2f_n),\ n = 0, 1, 2, \cdots,$$

where $h = (b-a)/N$, $x_n = a + nh$, $f_n = f(x_n, y_n)$, and $y_n$ is an approximation at grid point $x_n$ to the exact solution $y_n(x_n)$.

i. Determine the order of accuracy of the above method.

ii. Is the above method consistent with the given differential equations? Justify your answer.

iii. Determine whether or not this method is stable.

**(50 marks)**

3. Consider the linear system: $A\mathbf{x} = \mathbf{b}$ with $n$ equations and $n$ unknown variables, where $A = (a_{ij})$, $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, and $\mathbf{b} = (b_1, b_2, \cdots, b_n)^{\mathrm{T}}$.

   a) Prove that if the matrix $A$ is strictly diagonally dominant, then the Jacobi method is convergent for any choice of initial approximation $\mathbf{x}^{(0)}$ for $\mathbf{x}$.

   **(25 marks)**

   b) Consider the following linear system:

$$\begin{aligned} 4x_1 - x_2 \qquad &= 3, \\ -x_1 + 4x_2 - x_3 &= 2, \\ -x_2 + 4x_3 &= 3. \end{aligned}$$

      i. Deduce that the Jacobi method converges for the above system.
      ii. Find the rate of convergence of the Jacobi method.
      iii. Starting with the initial approximation $\mathbf{x}^0 = (0,0,0)^{\mathrm{T}}$ for the exact solution, perform three iterations of the Jacobi method.

   **(75 marks)**

4. Consider the following linear system:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= -1, \\ -x_1 + 3x_2 - x_3 &= 7, \\ x_1 - x_2 + 3x_3 &= -7 \end{aligned}$$

   a) i. Find the iteration matrix, $H$, of the Gauss-Seidel method for the above system.
      ii. Determine the spectral radius of $H$, and hence deduce that the Gauss-Seidel method is convergent to the exact solution of the system.
      iii. Find the rate of convergence of the Gauss-Seidel method.

   **(50 marks)**

   b) Starting with the initial approximations $\mathbf{x}^0 = (0,0,0)^{\mathrm{T}}$ to the exat solution of the foregoing system, compute the first three-iterations of the SOR method with the relation factor $\sigma = 1.25$.

   **(50 marks)**

5. a) Using the second order central difference approximations for $y'(x)$ and $y''(x)$, derive a finite difference equation to approximate the solution of the boundary value problem:

$$y''(x)+p(x)y'(x)+q(x)y(x)=f(x), \qquad y(a)=A, \quad y(b)=B,$$

at $N$ interior points uniformly distributed in the domain $[a,b]$.

**(35 marks)**

b) Use the above finite difference approximation with $N=3$ to approximate the solution of the boundary–value problem:

$$y''(x)+xy'(x)+y(x)=x, \qquad y(0)=1, \qquad y(1)=0.$$

**(65 marks)**

6. Consider the heat conduction equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad u=u(x,t), \quad 0<x<1, \quad 0<t<1,$$

with the initial and boundary conditions: $u(x,0)=s_0(x), \quad 0 \le x \le 1$, and $u(0,t)=0$, $u(1,t)=0, \quad 0 \le t \le 1$, respectively.

a) Derive the following standard finite difference schemes for the solution of the above equation:

i. The explicit scheme which is first order accurate in time ($t$) and second order accurate in space ($x$).

ii. The Crank-Nicholson scheme which is second order accurate both in time ($t$) and in space ($x$).

**(60 marks)**

b) Derive the stability criteria for the explicit scheme. **(40 marks)**

———————**END**———————