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RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES, MIHINTALE

B. Sc. (General) Degree

Second Year - End Semester Examination (Semester II), April / May 2016

MAP 2204 – Complex Calculus

Answer All Questions **Time allowed: Two hours**

1. (*a*) Prove that $|\mathrm{Re}\, z| \leq |z|$ where z is a complex number.

   (*b*) Given that $z \neq 0$ and $w \neq 0$ be two complex number, demonstrate that $|z+w| = |z| + |w|$ is true if and only if $w = tz$ for some $t > 0$.

   (*c*) Sketch the following sets and determine which are open? Which are closed? Which are neither open nor closed?

   (*i*) $\{z \in C : |z+3-2i| < 2\}$

   (*ii*) $\{z \in C : |z-1| \leq 1\} \cup \{z \in C : |z-1+i| < 1\}$

   (*iii*) $\{z \in C : 1 < |z-2| < 2\}$

2. (*a*) Define an analytic function in complex plane.
   If $f(z) = u(x,y) + iv(x,y)$ and f is differentiable at $z_0 \in C$, then prove that the partial derivatives of u and v exists at $z_0 \in C$ and satisfies the equations $u_x = v_y$ *and* $u_y = -v_x$.

   (*b*) Let z be a complex number and $f(z) = u(x,y) + iv(x,y)$ be analytic in C.
   If $\frac{u^2}{9} + \frac{v^2}{4} = 1$ then show that $f(z)$ is a constant function for all $z \in C$.

   (*c*) Let $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. Show that $u(x,y)$ is harmonic and find a harmonic conjugate $v(x,y)$.
   Express $u(x,y) + iv(x,y)$ as a function of z.

[P.T.O.

3. (*a*) Let $f(z)$ be a complex valued function such that $|f(z)| \leq M, \forall\, z \in C$ *and M is a positive number* and l is the length of curve c then prove that $\left|\int_c f(z)dz\right| \leq Ml$.

   Let $R > 2$ and C_R be the semi circle $z = Re^{i\theta}$; $\pi \leq \theta \leq 2\pi$ .Show that,

   $$\left|\int_{C_R} \frac{z+2}{(z^2+1)(z^2+4)}dz\right| \leq \frac{\pi R(R+2)}{(R^2-1)(R^2-4)}$$

   (*b*) State and prove the Cauchy's theorem.

   Evaluate $\int_c \frac{e^{\alpha z^2}}{z}dz, \alpha$ *is a real number* and c is the positively oriented circle $|z-2i| = 1$

4. (*a*) State Cauchy's integral formula and Cauchy's integral formula for derivatives.

   Let $f(z) = \frac{e^{2z}}{(z+1)^3(z-1)}$. Using Cauchy's integral formula or Cauchy's integral formula for derivatives,

   (*i*) Evaluate $\int_{c_1} f(z)dz$ where c_1 is the circle $|z+1| = \frac{1}{2}$, described in the counterclockwise direction.

   (*ii*) Evaluate $\int_{c_2} f(z)dz$ where c_2 is the circle $|z-1| = \frac{1}{2}$, described in the counterclockwise direction.

   (*b*) State the Taylor's theorem.

   Find Taylor series expansion for $\frac{1}{1-z}$ around 0 in $|z| < 1$.

   (*c*) Define an isolated singularity of a complex valued function.

   Let $f(z) = e^{\frac{1}{z}}$, classify the singularities of $f(z)$.