RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES, MIHINTALE

B. Sc. (General) Degree

Second Year – Semester II Examination – April/May 2015

MAP 2204 – Complex Calculus

Answer All Questions **Time allowed: Two hours**

1. (*a*) For any two complex numbers z_1 and z_2, show that $\left||z_1| - |z_2|\right| \le |z_1 + z_2|$.

   (*b*) Let z_1, z_2 be any two complex numbers and a, b are two real numbers such that $a^2 + b^2 \neq 0$. Prove that $|z_1|^2 + |z_2|^2 - \left|z_1^{\,2} + z_2^{\,2}\right| \le \dfrac{2|az_1 + bz_2|^2}{a^2 + b^2} \le |z_1|^2 + |z_2|^2 + \left|z_1^{\,2} + z_2^{\,2}\right|$

   (Hint: Take $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $a = r\cos\alpha$ *and* $b = r\sin\alpha$).

   (*c*) Define an open set.

   Is the following set S open? Justify your answer.

   $$S = \{z \in C \,|\, |z| < 1\} \cup \{z \in C \,|\, |z - 2i| < 1\}$$

2. (*a*) Define an Analytic Function.

   State and prove the **necessary** condition for the analytic function.

   (*b*) State the **sufficient** condition for the analytic function.

   (*c*) Suppose complex valued function satisfies the Cauchy-Riemann equation. Is this implies that function Analytic? Justify your answer.

3. (*a*) Let $f(z) = u + iv$ be analytic in region D.

   In each of the following cases, show that f is constant in D.

   (*i*) $\operatorname{Im} f(z) = \text{constant}$ (*ii*) $\left|\dfrac{f(z)}{2}\right| = \text{constant}$

   (*b*) Show that $u(x,y) = e^x x\cos y - e^x y\sin y$ is a harmonic function and find a real valued function $v(x,y)$ such that $u(x,y) + iv(x,y)$ is analytic. Express $u(x,y) + iv(x,y)$ as a function of z.

   (*c*) Solve the following equation

   $$e^z = 2i$$

4. (*a*) State and prove the *M-L* inequality. Let C denote the upper-half of th circle $|z| = R\ (R > 2)$, taken in the counter clock –wise direction. Show that,

$$\left|\int_c \frac{2z^2-1}{z^4-z^2-6}\,dz\right| \le \frac{\pi R(2R^2+1)}{(R^2-3)(R^2-2)}$$

(*b*) State and prove the Cauchy's theorem.

(*c*) $f(z) = \pi \exp(\pi \bar{z})$, and c is the boundary of the square with vertices at the point $0, 1, 1+i$ and i in the counter clock-wise direction.

(*d*) State Cauchy's Integral formula.

Evaluate $\int_c \frac{dz}{z^2+z+1}$ where C is the semi circle of radius 4 in the upper half plane and the line segment joining $(-4, 0)$ and $(4, 0)$.