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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. Four Year Degree in Industrial Mathematics
Fourth Year - Semester II Examination – February/March 2019
MAT 4309 – COMBINATORICS

Time: Three (03) hours

Answer all (06) questions

1. a) Suppose that there are initially four rabbits in a farm. The number of rabbits in the farm doubles in every year by natural reproduction, and some rabbits are either added or removed every year.

   i. Construct a recurrence relation for the number of rabbits in the farm at the start of the n^{th} year, assuming that during each year extra 72 rabbits are added to the farm.

   ii. Solve the recurrence relation obtained in part (i) to find the number of rabbits in the farm at the start of the n^{th} year.

   iii. Construct a recurrence relation for the number of rabbits in the farm at the start of the n^{th} year, assuming that n number of rabbits is removed during the n^{th} year for each $n \geq 3$.

   iv. Solve the recurrence relation obtained in part (iii) to find the number of rabbits in the farm at the start of the n^{th} year.

   b) The starting salary of new employee at an exciting new software company is $50,000 and it is promised that at the end of each year her salary will be doubled compared to salary drawn in the previous year, with an additional increment of $10,000 for each year she has been with the company.

   i. Construct a recurrence relation for her salary at the beginning of n^{th} year.

   ii. Solve this recurrence relation to find her salary for her n^{th} year of employment.

   c) In a raw there are n boxes to be coloured using red, blue and green so that no two boxes that are coloured green are consecutive. Construct the recurrence relation to find how many ways the above raw can be coloured.

2. a) The sequence $\{b_n\}$ is defined as follows:

   $$b_n = 2^n \; for \; n = 1, 2, 3, \ldots \text{ and } b_0 = 0.$$

   Find a closed form for the generating function b_n.

   b) Solve the following recurrence relations using generating functions:

   i. $h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$, where $n \geq 3$ with the initial conditions $h_0 = 1, h_1 = 1$ and $h_2 = -1$.

   ii. $a_n = 5a_{n-1} - 6a_{n-2}$ with the initial condition $a_0 = 1$ and $a_1 = -2$ and $n \geq 2$.

3. a) State the principle of "Inclusion and Exclusion" for the general case. For non-negative integers $k_1, k_2, \ldots, k_r$ *with* $k_1 + k_2 + \ldots + k_r = n$, prove that the multinomial coefficient,

$$\binom{n}{k_1, k_2, \ldots, k_r} = \binom{n}{k_1} \cdot \binom{n-k_1}{k_2} \cdots \binom{n-k_1-k_2-\ldots k_r-1}{k_r} = \frac{n!}{k_1! k_2! \ldots k_r!}.$$

   b) How many permutations of the 26 letters in the English alphabet which do not contain any of the sequences; MUTE, HOT, FLASH and BEAUTY?

   c) In how many ways 32 books could be arranged on 6 shelves where each shelf has at least one, but at most 6?

   d) Of how many arrangements does the word "ANKUNUKOLAPALESSA" have no consecutive pairs?

   e) Determine the number of integers less than or equal to 6425 that are not divisible by 3, 7, 9 or 13.

4. a) Explain the steps clearly to select k objects from n objects in each of the following cases:

   Case I: Order significant, repetition allowed
   Case II: Order significant, repetition not allowed
   Case III: Order not significant, repetition not allowed
   Case IV: Order not significant, repetition allowed

   b) How many solutions are there to each of the following equations?

   i. $x_1 + x_2 + x_3 + x_4 = 25$, where x_i's are positive integers such that $i = 1, 2, 3, 4$ and $x_i \leq 7$.

   ii. $x_1 + x_2 + x_3 + x_4 = 23$, where x_i's are integers such that $i = 1, 2, 3, 4$ and $1 \leq x_1 \leq 7$, $-1 \leq x_2 \leq 5$, $2 \leq x_3 \leq 7$ and $-2 \leq x_3 \leq 4$.

5. a) Find the coefficient of x^8 in the expansion of $(1 + x + x^2 + x^3 + x^4)^{18}$.

   b) Find the coefficient of $x_2^3 x_3^5 x_5^2$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$.

   c) Find the coefficient of $x_1^3 x_3^5 x_4^4$ in the expansion of $(x_1 + x_2 + x_3 + x_4)^{12}$.

6. a) Define a *Steiner Triple System* (STS) and state the condition for number elements in the *Steiner System*.

   b) Construct $STS(9)$ and $STS(13)$.

   c) Construct the tournament schedule for 10 teams and for 14 teams.

   d) Using part (b) and (c) construct $STS(19)$ and $STS(27)$.

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