44

RAJARATA UNIVERSITY OF SRILANKA

FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree

Second Year-Semester II Examination-March/April 2014

MAP 2204 – COMPLEX CALCULUS

Answer **FOUR** Questions Only

Time Allowed: **Two hours**

1. (a) Use *De Moivre's* Theorem to show that

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right).$$

Hence find $\int_0^{\pi/2} \cos^5\theta \, d\theta$.

(b) Solve the equation $(z+1)^5 = z^5$ for $z \in C$.

(c) Find all complex values of z satisfying the equation $e^z = 4i$.

2. (a) Find the image of the half-plane $\mathrm{Re}(z) \le 2$ under the complex mapping $w = i\,z$ and represent the mapping graphically.

(b) Show that the vertical line $x = 1$ is mapped onto a parabola under the complex mapping $w = z^2$ and represent the mapping graphically.

(C) Under the transformation $w = z^2$ from z-plane to w-plane, identify the image of the parametric curve C described by $|z(t)| = 2, 0 \le t \le \pi$.

3. (a) Given that the functions $f(z)$ and $g(z)$ are analytic in a region containing the point z_0 and that $f(z_0)=g(z_0)=0$, but $g'(z_0)\neq 0$, derive a rule to find the limit of $\left\{\dfrac{f(z)}{g(z)}\right\}$, as $z \to z_0$.

(b) Using the above rule, repeatedly (if necessary), Show each of the following:

i. $\lim\limits_{z\to(-i)}\left\{\dfrac{(z^4-1)}{z^6+1}\right\}=-\dfrac{2}{3}$

ii. $\lim\limits_{z\to 0}\left\{\dfrac{(\sin z - z\cos z)}{z^3}\right\}=\dfrac{1}{3}$

(c) Show that $\int_0^{2\pi}\frac{\cos 3\varphi}{5-\cos 4\varphi}\,d\varphi=\frac{\pi}{12}$.

4. (a) An analytic function of $z=x+iy$, is given by $w=u+iv$, where each of *u* and *v* is a differentiable real function of *x* and *y*.

Show that $\dfrac{dw}{dz}$ can be expressed in either of the formulae $\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}$ or $\dfrac{\partial v}{\partial y}-i\dfrac{\partial u}{\partial y}$.

What deduction can be made from the above results?

(b)Find the real constants a and b so that the function given by $f(z)=3x-y+5+i(ax+by-3)$ is analytic.

(C) Show that the function $f(z)=3x^2y^2-6ix^2y^2$ is not analytic at any point but is differentiable along the coordinate axes.

5. (a) Evaluate $\oint_C \frac{1}{z}\,dz$, where C is the circle: $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$.

   (b) Using *Cauchy's Integral Formulas* evaluate

$$\oint_C \frac{z}{z^2+9}\,dz, \text{ where } C \text{ is the circle given by } |z-2i| = 4.$$

   (c)Let $f(x+iy) = u + iv = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right), & if \quad x,y \neq 0 \\ 0, & if\ x = y = 0. \end{cases}$

   Show that $f(x+iy)$ is satisfied Cauchy Riemann equations at $z = 0$,

   but not differentiable at $z = 0$.

6. (a) Express the function $f(z) = \frac{1+z}{1-z}$ in a Taylor Series centered at $z_0 = i$.

   (b) Find the Laurent series expansion for the functions:

   i. $f(z) = \frac{1}{(z-1)^2(z-3)}$, converges for $0 < |z-3| < 2$,

   ii. $f(z) = \frac{8z+1}{z(1-z)}$, converges for $0 < |z| < 1$,

   in the form $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty}\left(\frac{b_n}{z^n}\right)$.

   (c) Evaluate:

   iii. $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$, where the contour C is the circle $|z| = 2$

   iv. $\oint_C \frac{e^z}{z^4+5z^3}\,dz$,where the contour C is the circle $|z| = 2$.
