RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

Bachelor of Science in Applied Sciences
First Year - Semester II Examination - Jan/Feb 2023

MAP 1203 - REAL ANALYSIS I

Time allowed: **Two (2) hours**

Answer ALL (04) questions

1. a) i. Define an upper bound and a lower bound of a non-empty set.
   ii. State the completeness axiom of the subset of real numbers.

(20 marks)

b) Let A and B be non-empty bounded subsets of $\mathbb{R}$. Show that the set $S = \{a + b \,|a \in A, b \in B\}$ is bounded above and that $\sup(A + B) = \sup A + \sup B$.

(35 marks)

c) Consider the set $A = \{x \in \mathbb{R} \,|x > 2\}$.
   i. What is a lower bound of A.
   ii. Let L be a lower bound of A such that $L > 2$ and let $y = \frac{L+2}{2}$. Show that $2 < y < L$.
   iii. Show that $y \in A$ and $L \leq y$. Hence, show that the infimum of A is 2.

(30 marks)

d) Find the Supremum and Infimum of the following sets, if they exists:
   i. $\{e^x \,|x \geq 0\}$.
   ii. $\left\{1 - \frac{1}{n} \,|n \in \mathbb{N}\right\}$.
   iii. $\left\{\frac{(-1)^n}{n+1} \,|n \in \mathbb{N}\right\}$.

(15 marks)

2. a) Using the $\epsilon - N$ definition, show the following:
   i. $\lim\limits_{n \to \infty} \left(1 - \dfrac{1}{2^n}\right) = 1$
   ii. $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$

(30 marks)

b) Prove that, every convergent sequence of real numbers is bounded.

(25 marks)

c) Is every bounded sequence of real numbers convergent? Justify your answer.

(15 marks)

d) Let $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2 + x_{n-1}}$ for $n > 1$. Then using mathematical induction prove that $0 \leq x_n \leq 2$ and that (x_n) is increasing for all $n \in \mathbb{N}$. Hence, discuss the convergence of (x_n) and find the limit of the sequence.

(30 marks)

3. a) Using $\epsilon - \delta$ definition prove the following limits:

   i. $\lim\limits_{x \to -1} \dfrac{3x - 1}{x + 3} = -2$

   ii. $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$

(30 marks)

b) Let $f(x)$ be a real valued function defined on some interval I containing a, except possibly at a. If $\lim\limits_{x \to a} f(x) = l_1$ and $\lim\limits_{x \to a} f(x) = l_2$, then prove that $l_1 = l_2$.

(20 marks)

c) Consider the function $f : \mathbb{R} \longrightarrow \{-1, 1\}$ given by,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

. Prove that $f(x)$ is discontinuous at every real number.

(20 marks)

d) Using the definition prove that,

   i. $f(x) = \dfrac{1}{x^2 - 1}$ is continuous at $x = 1$.

   ii. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

(30 marks)

4. a) Prove that every differentiable function is continuous. Is every continuous function differentiable? Justify your answer.

(25 marks)

b) Show that the following function is continuous at $x = 1$ for any real values of a,

where $f(x) = \begin{cases} ax + 1 & \text{if } x \geq 1 \\ x^2 + a & \text{if } x < 1 \end{cases}$. Find the condition for existence of derivative of $f(x)$ at $x = 1$.

(20 marks)

c) If $x > 0$, show that $x - \frac{x^2}{2} < \ln(1 + x) < x - \frac{x^2}{2(1+x)}$.

(30 marks)

d) State the L'Hospital Rule. Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - 2\cos x + e^x}{x \sin x}$.

(25 marks)

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