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RAJARATA UNIVERSITY OF SRI LANKA, MIHINTALE

FACULTY OF APPLIED SCIENCES, DEPARTMENT OF PHYSICAL SCIENCES

B.Sc. General Degree (First Year)

End Semester Examination - Semester II – February/March, 2013

MAP 1301 – LINEAR ALGEBRA

Proper Candidates [with Mid Semester Mmarks]: Time Allowed: **TWO HOURS**

Answer **FOUR** questions, selecting **two from each of the sections A and B.**

Other Candidates [without Mid Semester Marks]: Time Allowed: **THREE HOURS**

Answer **ALL SIX QUESTIONS.**

SECTION A

1. Find the values of the constants a , b so that the column vectors $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$ of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & 8 & a \\ 4 & -4 & b \\ 8 & 1 & 4 \end{pmatrix}$$

, are mutually perpendicular. **Assuming that** a and b **take those values,**

(i) verify that the row and column vectors of **A** are each of the same length and that $|\mathbf{A}| = -9^3$,

(ii) find the product of **A** and (its transpose) $\mathbf{A}^{\mathrm{T}}$, **as a scalar matrix**, and hence evaluate $|\mathbf{A}\,\mathbf{A}^T|$,

(iv) construct an orthogonal matrix as a multiple of **A**, and hence find the inverse of the matrix **A**.

Express the matrix equation $\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{d}}$, where $\underline{\mathbf{d}} = \begin{pmatrix} 5 \\ -1 \\ 10 \end{pmatrix}$, and $\underline{\mathbf{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an unknown column vector,

in **vector form** $x\,\underline{c}_1 + y\,\underline{c}_2 + z\,\underline{c}_3 = \underline{\mathbf{d}}$, take the scalar product of either side with each of the column vectors $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$, and **hence solve** this equation, for the components x, y, z of $\underline{\mathbf{x}}$.

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2. Given a matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 \\ 2 & 3 & 5 & 5 & 7 \\ 3 & 4 & 9 & 10 & 11 \\ 1 & 4 & 3 & 6 & 9 \end{pmatrix}$, applying elementary row operations,

convert it to an echelon matrix **B** and then to the **row – reduced echelon matrix C**, to be determined. Using matrices **B** and **C**

(i) find bases for the row space U and the column space V of the matrix **A** and the rank of **A**,

(ii) find a basis for the solution space W of the system of homogeneous equations represented by the matrix equation $\mathbf{AX} = \mathbf{O}$, and verify that $\text{rank}(\mathbf{A}) + \dim(W) = \dim(R^5)$.

If the matrix **A** defines a linear transformation T from $R^5 \to R^4$ such that $T(\mathbf{X}) = \mathbf{AX}$, **briefly explain** why the transformation T is neither one-to-one nor onto.

3. (a) Given an invertible $n \times n$ matrix **A** and an $n \times p$ matrix **B**, find the solution **X** of the equation $\mathbf{AX} = \mathbf{B}$, and show that this solution matrix is unique. What is the order of **X** ?

(b) Two matrices $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ -4 & 5 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{bmatrix} 11 & 25 \\ 12 & 17 \\ 4 & 1 \end{bmatrix}$ are given.

Using only elementary row operations on the 'augmented matrix' $[\mathbf{A} \vdots \mathbf{B}]$, **row – reduce** it to the form $[\mathbf{I}, \mathbf{X}]$, where $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, to find the solution matrix X, of the equation $\mathbf{AX} = \mathbf{B}$.

Deduce solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ to the systems of equations represented by $\mathbf{AX}_1 = \mathbf{B}_1$, $\mathbf{AX}_2 = \mathbf{B}_2$, where $\mathbf{B}_1 = \begin{bmatrix} 11 \\ 12 \\ 4 \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} 25 \\ 17 \\ 1 \end{bmatrix}$ are the two column vectors of the given matrix **B**.

Hence solve the following **two** systems of simultaneous equations, **separately:**

$$\begin{aligned} x \ + 2y + 3z &= a \\ 2x + 3y - z &= b \\ -4x + 5y - 3z &= c \end{aligned}\ , \text{(i) when } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \\ -3 \end{bmatrix}, \text{ and (ii) when } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ -7 \end{bmatrix}.$$

SECTION B

4. (a) Express the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as the product of three elementary matrices, and verify your answer.

Briefly explain why the matrix $\mathbf{B} = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$ cannot be expressed in the above manner.

(b) Find the characteristic polynomial of the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$. Hence

show that $\mathbf{M}$ is invertible, and derive the formula $2\mathbf{M}^{-1} = (\mathbf{I} + 2\mathbf{M}) - \mathbf{M}^2$ for the inverse of matrix $\mathbf{M}$, where $\mathbf{I}$ is the identity matrix of order three. **[Hint:** You may assume Cayley-Hamilton theorem].

Evaluate the matrix on the right-hand-side of the last equation above, and use your answer to verify that $(2\mathbf{M})(2\mathbf{M}^{-1}) = 4\mathbf{I}$.

Find the eigenvalues of the matrix $\mathbf{M}$ and verify that for any positive integer n eigenvalues of the matrix $\mathbf{M}^{2n}$ are given by λ^{2n}, where λ is an eigenvalue of $\mathbf{M}$.

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5. Find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$, and the corresponding unit eigenvectors.

Hence show that an orthogonal matrix **P** exists such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ is a diagonal matrix **D**, to be identified.

Show that the quadratic form $f(x, y) \equiv 5x^2 - 6xy + 5y^2$, may be written as $f(x, y) = \begin{bmatrix} x & y \end{bmatrix} \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$.

By means of the linear transformation $\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{P}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$, express the equation of the conic C, given by

$f(x, y) = 4$, in the standard form $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1$, where a and b are constants to be determined.

Identify the principal axes of the conic C and sketch it in the (x, y) –plane.

What is the effect of the linear transformation from the (x, y) system to the (X, Y) system, on the two coordinate axes Ox and Oy ?

6. Find the eigenvalues of the symmetric matrix **A** such that the following quadratic form

$f(x, y, z) \equiv 2x^2 + 4y^2 + 5z^2 - 4xz$ may be expressed as $f(x, y, z) \equiv \begin{bmatrix} x & y & z \end{bmatrix} \mathbf{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

and verify that the corresponding eigenvectors are mutually perpendicular.

Construct a symmetric, orthogonal matrix S which makes $\mathbf{S}^{\mathrm{T}}\mathbf{A}\,\mathbf{S}$ a diagonal matrix **D.**

Under the linear transformation $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{S}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, show that the coordinate axis Oy is mapped

into the OY- axis and the given quadratic form becomes $f(x, y, z) = aX^2 + bY^2 + cZ^2$.

Determine the values of constants a, b, c and find the Cartesian equations of the principal axes of the quadric surface $f(x, y, z) = 12$.