RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences
Second Year – Semester II Examination– February/March 2019

MAP 2202– REAL ANALYSIS II

Time: Two (02) hours

Answer **all** questions.

1. a) State each of the following tests for convergence and divergence of an infinite series.

   i. Ratio test

   ii. Root test

(20 marks)

b) Determine whether each of the following series is convergent or divergent:

i. $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n+1}$

ii. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

iii. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+2}{2^n+5}\right)$.

(60 marks)

c) Using the integral test, prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 0$, is:

i. convergent if $p > 1$ and

ii. divergent if $p \leq 1$.

(20 marks)

2. a) Find the radius of convergence and the interval of convergence of each of the following series:

i. $\sum_{n=0}^{\infty} \frac{(2n)!\, x^n}{(n!)^2}$

ii. $\sum_{n=0}^{\infty} \frac{nx^n}{(n+1)^2}$.

(50 marks)

b) Let $f(x) = \tan^{-1} x$.

i. Find the Maclaurin series expansion of $f(x)$ for $-1 \leq x \leq 1$.

ii. Deduce that, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

iii. Show that $\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1 + \frac{1}{3}\right) + \frac{x^6}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \ldots$ for $-1 \leq x \leq 1$.

(50 marks)

3. a) Let $f(x) = x^2, \forall\, x \in [0, a]$,where $a > 0$.

Prove that $f(x)$ is Riemann integrable on $[0, a]$ and $\int_0^a f(x)dx = \frac{a^3}{3}$.

(50 marks)

b) Let $f(x) = \begin{cases} 1\,; & if\ x\ is\ rational \\ -1\,; & if\ x\ is\ irrational \end{cases}$ is defined over [0,1].

By evaluating upper and lower Riemann integrals of $f(x)$, show that $f(x)$ is not Riemann integrable over [0,1].

(30 marks)

c) Is the following limit exist? Justify your answer.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2 + 3y^2}$$

(20 marks)

4. a) Define the **Gamma function** and the **Beta function**.

Hence evaluate $\int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}} dx$.

(30 marks)

b) Prove that $\tau(n)\tau(1-n) = \frac{\pi}{\sin n\pi}$ for $0 < n < 1$.

Hence show that $\int_0^2 x(8-x^3)^{1/3} dx = \frac{16\pi}{2\sqrt{3}}$.

(30 marks)

c) Define a **Metric space**$(\boldsymbol{X}, \boldsymbol{d})$.

Let R be a set of real numbers. Show that the function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x^2 - y^2|\ \forall\, x, y \in \mathbb{R}$ is pseudo metric on $\mathbb{R}$.

(40 marks)

END