RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. (Four Year) Degree in Industrial Mathematics
Fourth Year - Semester I Examination – September/October 2019

MAT 4310 – COMPUTATIONAL MATHEMATICS

Time: Three (03) hours

- Answer all questions.
- This paper contains **FOUR** questions from **Page 1** to **Page 5**.
- Calculators will be provided.
- This is a closed book examination.
- This examination accounts for 60% of the module assessment.
- The marks assigned for each question and parts thereof are indicated in brackets.

1.
An initial value problem (IVP) is given by

$$\frac{dx}{dt} = t + x - 1 \, , x(0) = 1.$$

Solving the IVP, show that its exact solution, $x = x(t)$ is

$$x(t) = \exp(t) - t.$$ **(20 marks)**

It is required for a student to test the performance of **the classical fourth order Runge-Kutta method (RK4)** using the above IVP. An incomplete table which was prepared by him is exhibited by **Table 1** below:

Contd

Table 1: RK4 Method

t	x	Exact value	Absolute error
0.20			
0.40			
0.60			
0.80			
1.00			
1.20			
1.40			

i. Write down the classical **RK4** computational scheme for the above IVP. **(20 marks)**

ii. Copy **Table 1** given above to your answer booklet and **complete** it. Your results need to be rounded to **four** decimal places; and show all the necessary computational steps. Comment on your result. **(60 marks)**

2.

a) Write down **two advantages** of linear multi-step methods over one-step methods in the context of solving initial value problems for ordinary differential equations. **(10 marks)**
Consider the initial value problem (IVP)

$$x'(t) = f(t, x), \quad x(t_0) = \alpha.$$

Given a sequence of equally spaced mesh points (t_n) with step size h, the linear k-step method for the IVP is given by

$$\sum_{j=0}^{k} a_{k-j} x_{n-j} = h \sum_{j=0}^{k} b_{k-j} f(t_{n+j}, x_{n+j}),$$

where the coefficients a_j and b_j are each real constants for all $j = 0, 1, \dots, k$ and $a_k \neq 0, a_0^2 + b_0^2 \neq 0$.

i. Write down the first and second characteristic polynomials p and q of the above method, respectively. **(10 marks)**

ii. Write down the root condition. **(10 marks)**

iii. Define strong and weak stabilities of the method. **(10 marks)**

b) Fourth -order Adams-Bashforth explicit method(M_1) is given by

$$M_1 : x_{n+1} = x_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

i. Discuss the stability of M_1. **(20 marks)**

Contd.

ii. Is M_1 convergent?
Justify your answer. **(20 marks)**

iii. Consider the initial value problem (IVP)

$$x'(t) = -x(t) + 2\cos t, \quad x(0) = 1.$$

Approximate $x(2)$ using M_1 with **step size** $h = 0.25$ and the **starting values:**

$$\begin{aligned} x(0.25) &= 1.216316380965168, \\ x(0.5) &= 1.357008100494576, \\ x(0.75) &= 1.413327628897155. \end{aligned}$$

(40 marks)

Determine the absolute error if the exact solution is given by

$$x(t) = \cos t + \sin t.$$

(10 marks)

(Your results must be rounded to **five decimal** places)

iv. Fourth-order Adams-Moulton implicit method (M_2) is given by

$$M_2: x_{n+1} = x_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}).$$

Apply the Adams fourth-order predictor-corrector method with **step size** $h = 0.25$ and the starting values given in **Part iii** to approximate the solution of the IVP in Part iv at $t = 2$. Determine the absolute error. **(60 marks)**
(Your result must be rounded to **five decimal** places)

v. Comment on the results obtained in **Part iii** and **Part iv**. **(10 marks)**

3.

a) Write down **two advantages** of iterative methods over direct methods of solving systems of linear equations. **(10 marks)**

Let $S: A\mathbf{x} = \mathbf{b}$ be a linear system including n equations and n unknown variables x_i written in matrix form in which $A = (a_{i,j})$, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, $\mathbf{b} = (b_1, b_2, \dots, b_n)^T$.

Suppose further that $\mathbf{x}^{(k)} = \left(x_1{}^{(k)}, x_2{}^{(k)}, \dots, x_n{}^{(k)}\right)^T$ denotes the k^{th} approximation to the exact solution $\mathbf{x}$ for all $k = 0, 1, 2, \dots$.

i. Concerning S, write down the computational formula for each of the following iterative methods: **(30 marks)**

- Jacobian,
- Gauss Seidel,
- Successive Over-Rexation (SOR).

Contd.

ii. Prove **one** of the following theorems: **(25 marks)**

- If S is diagonally dominant, then the Jacobi method is convergent for any choice of initial approximation $\mathbf{x}^{(0)}$.
- If S is diagonally dominant, then the Gauss Seidel method is convergent for any choice of initial approximation $\mathbf{x}^{(0)}$.
- Let S be a positive definite system and let ω denote the relaxation parameter of SOR method. If $0 < \omega < 2$, then SOR method is convergent for any choice of initial approximation $\mathbf{x}^{(0)}$.

iii. Give an example for a linear system whose convergence occurs without diagonally dominant. **(05 marks)**

b) Use the **Gauss Seidel method** to obtain approximations to the exact solution of the following linear system starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^T$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 0.001.$$ **(60 marks)**

$$\begin{array}{rcrcrcrcr}
10x_1 & - & 2x_2 & + & 2x_3 & & & = & 6, \\
-x_1 & + & 11x_2 & - & x_3 & + & 3x_4 & = & 25, \\
2x_1 & - & x_2 & + & 10x_3 & - & x_4 & = & -11, \\
 & & 3x_2 & - & x_3 & + & 8x_4 & = & 15.
\end{array}$$

i. If the exact solution of the system is $\mathbf{x} = (1, 2, -1, 1)^T$, determine the relative absolute error with respect to l_∞-norm. **(10 marks)**

ii. Comment on your result. **(10 marks)**

4.

a) Determine the Crout factorization of the following matrix. **(50 marks)**

$$\begin{pmatrix} -1.92 & 1.3 & 0 & 0 \\ 0.7 & -1.92 & 1.3 & 0 \\ 0 & 0.7 & -1.92 & 1.3 \\ 0 & 0 & 0.7 & -1.92 \end{pmatrix}$$

b) Consider the following second order boundary value problem:

$$y'' + p(x)y' + q(x)y = f(x),$$

where $y(a) = \alpha, y(b) = \beta$.

Let $x_i = a + ih$ for all $i = 0, 1, \ldots, n$, where $h = (b - a)/n$.

Contd.

Employ appropriate finite difference approximations to discretise the preceding differential equation into the following finite difference scheme: **(50 marks)**

$$\left(1+\frac{h}{2}p_i\right)y_{i+1}+(-2+h^2q_i)y_i+\left(1-\frac{h}{2}p_i\right)y_{i-1}=h^2f_i,$$

where $y_i = y(x_i), p_i = p(x_i),\ q_i = q(x_i)$ and $f_i = f(x_i)$ for all $i = 0, 1, \dots, n$.

Approximate, using the above finite deference scheme, the solution of the following differential equation with $n = 5$. **(50 marks)**

$$y'' + 3y' + 2y = 4x^2, y(1) = 1, y(2) = 6$$

(Your solution must be rounded to **four** decimal places)

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