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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences
Second Year – Semester II Examination– February/ March 2019

MAP 2204 – COMPLEX CALCULUS

Time: Two (02) hours

Answer all questions.

1. a) State and prove the **Triangle inequality** for complex numbers.

   Hence, show that $|z_1 + z_2| \geq ||z_1| - |z_2||$, where z_1 and z_2 are any two complex numbers.

   (40 marks)

   b) State **De Moivre's theorem**. Using the theorem,

   i. find all complex roots of the equation $z^2 - (\sqrt{3} + i) = 0$,

   ii. show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

   (60 marks)

2. a) Write down the **Cauchy Riemann equations** for $f(z) = u(x, y) + iv(x, y)$.

   Use Cauchy Riemann equations to prove that the function $f(z) = e^z$ for $z \in \mathbb{C}$ is differentiable everywhere.

   (30 marks)

   b) Let $u(x, y) = y/(x^2 + y^2)$

   i. Show that $u(x, y)$ is a harmonic function in $\mathbb{C}$.

   ii. Find a harmonic conjugate $v(x, y)$ such that $u(x, y) + iv(x, y)$ is analytic in $\mathbb{C}$.

   iii. Express $u(x, y) + iv(x, y)$ as a function of z, where $z = x + iy$.

   (50 marks)

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c) Show that, if $f'(z)=0$ everywhere in a domain D, then $f(z)$ is a constant throughout D.

(20 marks)

3. a) State **Cauchy's Integral formula.**

Evaluate $\int_C \frac{e^{az^2}dz}{z}$, where C is a positively oriented circle $|z|=1$.

Deduce that, $\int_0^{2\pi} e^{a\cos 2\theta}\cos(a\sin 2\theta)d\theta = 2\pi.$

(40 marks)

b) State **Modulus – Length inequality.**

Hence, show that $\left|\int_C \frac{z^{\frac{1}{2}}}{(z^2+1)}dz\right| \le \frac{\pi\sqrt{R}}{1-(1/R^2)}$, where C is the semi-circle with a radius of R, centered at the origin and $0\le\theta\le\pi$.

(40 marks)

c) Evaluate $\int_C \frac{e^{2z}dz}{z^4}$, where C is a positively oriented circle $|z|=1$.

(20 marks)

4. a) Find the Taylor series expansion for a function $f(z)=e^z$, if it is analytic in $|z|<\infty$.

Deduce the Maclaurin series expansion of the function $f(z)=\cos z$, given that f is analytic in $|z|<\infty$.

(30 marks)

b) Find the Laurent's series expansion of $f(z)=\frac{-1}{(z-1)(z-2)}$ in the annular region defined as $1<|z|<2$.

(30 marks)

c) State **Cauchy's Residue** theorem.

Hence, show that

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}d\theta = \frac{3\pi}{8} \text{ for } |z|<1.$$

(40 marks)

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