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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. (General)Degree in Applied Sciences
First Year - Semester I Examination – September/October 2019

MAP 1301 – LINEAR ALGEBRA

Time: Three (03) hours

- Answer **all (06)** questions
- The paper consists of four pages

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1. a) Let V be the set of all ordered pairs (a, b) of real numbers. Determine whether V is a vector space over $\mathbb{R}$ under the following operations defined for all the ordered pairs:

$$s(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2),$$
$$\alpha \otimes (a, b) = (0, \alpha b),$$

where $\alpha \in \mathbb{R}$. (40 marks)

b) Prove that W is a subspace of a vector space Vover the field $\mathbb{F}$ if and only if for all

$$\alpha, \beta \in \mathbb{F} \text{ and } \quad x, y \in W, \alpha x + \beta y \in W.$$ (30 marks)

c) Which of the following sets are subspaces of $\mathbb{R}^3$?

Justify your answer in each case.

i. $\{(0, x, x + 1) \in \mathbb{R}^3 : x \in \mathbb{R}\}$,

ii. $\{(a, b, c) \in \mathbb{R}^3 : a^2 = b^2\}$,

iii. $\{(a, b, c) \in \mathbb{R}^3 : a + 2b = 3c\}$.

(30 marks)

Contd.

2. a) Let $S = \{v_1, v_2, v_3, \dots, v_k\}$ be a subset of a vector space V over a field $\mathbb{F}$.

Explain the following briefly:

i. V is spanned by S.
ii. S is linearly independent.
iii. S is a basis for V.
iv. Dimension of V.

(20 marks)

b) Write the vector $\alpha = (1 \quad -4 \quad 5)$ as a linear combination of vectors

$$\{(1 \quad 0 \quad -2), (2 \quad -1 \quad 2), (5 \quad 1 \quad 3)\}.$$

(10 marks)

c) If x, y, z are linearly independent vectors in a vector space V over the field $\mathbb{F}$, then prove that,

i. $x + y, y + z, z + x$
ii. $x + y, x - y, x - 2y + z$

are also linearly independent.

(20 marks)

d) Find a basis for the subspace $U == \{(x, \quad y, \quad z, \quad t) \in \mathbb{R}^4 | 3x + y - 7t = 0\}$ of the vector space $\mathbb{R}^4$. What is the dimension of U? (20 marks)

e) Which of the followings are bases of $\mathbb{R}^3$?

i. $\{(2 \quad 4 \quad -3), (0 \quad 1 \quad 1), (0 \quad 1 \quad -1)\}$
ii. $\{(1 \quad 5 \quad -6), (2 \quad 1 \quad 8), (3 \quad -1 \quad 4), (2 \quad 1 \quad 1)\}$.

(30 marks)

3. a) Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ -1 & -3 & 4 & 2 \end{bmatrix}$$

(50 marks)

Contd.

b) Solve the following system of linear equations using row reduced form.

$$\begin{aligned} 2x - y + 3z &= 24 \\ 2y - z &= 14 \\ 7x - 5y \quad &= 6 \end{aligned}$$

(50 marks)

4. a) Define a linear transformation.

Which of the following mappings are linear transformations?

i. $T\big((x,y)\big) = (x+y, x)$, where $T: \mathbb{R}^2 \to \mathbb{R}^2$.

ii. $T\big((a,b,c)\big) = (3a - 2b + c, a - 3b - 2c)$, where $T: \mathbb{R}^3 \to \mathbb{R}^2$.

(40 marks)

b) Let V be the vector space of continuous functions $[a,b]$ with field $\mathbb{R}$ and let $T: V \to \mathbb{R}$ be defined by

$$T(f) = \int_a^b f(x)dx \text{ for all } f \in V.$$

Show that $T$is a linear transformation. (20 marks)

c) Define Kernal and Image of a linear transformation.

Let $T: V \to W$ be a linear transformation, where V and W are vector spaces over the field $\mathbb{F}$.

Show that the Kernal of T is a subspace of $V$and that the image of T is a subspace of W. (40 marks)

5. a) State the Rank-Nullity theorem. (10 marks)

b) Define an inner product space.

Let and $u = (u_1, u_2, \dots, u_n), v = (v_1, v_2, \dots, v_n)$, where $u_i, v_i \in \mathbb{C}$ for all $i = 1, 2, 3, \dots, n$.

Show that

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i \, \bar{v}_i$$

is an inner product on $\mathbb{C}^n$. (30 marks)

Contd.

c) Prove that $\langle x_1 + x_2, y_1 + y_2 \rangle = \langle x_1, y_1 \rangle + \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle + \langle x_2, y_2 \rangle$ on any inner product space. (20 marks)

d) Let V be an inner product space over the complex field $\mathbb{C}$.

Show that:

i. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{C}$ and $x \in V$.

ii. $|\langle x, y \rangle| \leq \|x\| + \|y\|$ for all $x, y \in V$.

(40 marks)

6. a) Define an eigenvalue of a matrix of order n.

Find eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & 1 & -1 \\ 3 & 1 & -2 \\ 5 & -3 & 4 \end{bmatrix}.$$

(60 marks)

b) Solve the following system of linear equations using Cramer's rule.

$$2x + 2y - z = 2$$
$$x - 3y + z = -28$$
$$-x + y \qquad = 14$$

(40 marks)

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