RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (4 year) Degree in Applied Sciences
Fourth Year - Semester I Examination – October/November 2017

PHY4210 – ADVANCED QUANTUM MECHANICS

Time: Two (02) hours

Answer **all four** questions

All symbols have their usual meaning.

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(1) (a) Prove the following operator identities:

(i) $\left[\hat{A}, B\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]$ (05 marks)

(ii) $\left[\hat{A}, \hat{B}^{-1}\right] = -\hat{B}^{-1}\left[\hat{A}, \hat{B}\right]\hat{B}^{-1}$ (05 marks)

(b) If $\hat{A}$ and $\hat{B}$ are self-adjoint operators, then show that, $(\hat{A}\hat{B} - \hat{B}\hat{A})$ is self-adjoint, only if $\hat{A}$ and $\hat{B}$ commute with each other.

(07 marks)

(c) Suppose that $\hat{\alpha}$ is an operator representing a dynamical variable α and u is an eigenfunction of $\hat{\alpha}$ belonging to the eigenvalue a such that $\hat{\alpha}u = au$. Prove that the expectation values of $\langle\alpha\rangle$ and $\langle\alpha^n\rangle$ should be a and a^n respectively, where n is a positive integer.

(08 marks)

(2) (a) The Hamiltonian operator for the harmonic oscillator is $\hat{H} = \dfrac{\hat{p}^2}{2m} + \dfrac{1}{2}m\omega_0^2\hat{x}^2$ and the eigenvalue equation is $\hat{H}\psi_n = E_n\psi_n$. Show that the two operators $\hat{a}$ and $\hat{a}^\dagger$ defined by;

Contd.

$\hat{a} = \left(\frac{m\omega_0}{2\hbar}\right)^{1/2} \hat{x} + i\left(\frac{1}{2m\hbar\omega_0}\right)^{1/2} \hat{p}$, $\hat{a}^\dagger = \left(\frac{m\omega_0}{2\hbar}\right)^{1/2} \hat{x} - i\left(\frac{1}{2m\hbar\omega_0}\right)^{1/2} \hat{p}$ satisfy the following commutation rules.

i. $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ (05 marks)

ii. $\left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger$ (05 marks)

iii. $\left[\hat{a}^\dagger\hat{a}, \hat{a}\right] = -\hat{a}$ (05 marks)

(b) Let $\hat{a}_1, \hat{a}_1^\dagger$ and $\hat{a}_2, \hat{a}_2^\dagger$ be annihilation and creation operators for two independent harmonic oscillators, which satisfy the relations $[\hat{a}_k, \hat{a}_m] = 0 = \left[\hat{a}_k^\dagger, \hat{a}_m^\dagger\right]$ and $\left[\hat{a}_k, \hat{a}_m^\dagger\right] = \delta_{km}$, $(k, m = 1, 2)$. Let

$$\hat{J}_1 = \frac{1}{2}\left(\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2\right), \quad \hat{J}_2 = \frac{i}{2}\left(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2\right) \text{ and } \hat{J}_3 = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right).$$

Show that

i. $\left[\hat{J}_1, \hat{J}_2\right] = i\hat{J}_3$ (05 marks)

ii. $\left[\hat{J}_2, \hat{J}_3\right] = i\hat{J}_1$ (05 marks)

(3) A particle moves from the $+x$ direction towards the $-x$ direction in 1-D in the presence of an attractive potential $V(x)$ which is infinite for $x < 0$, is equal to the constant value $-V_0$ for $0 < x < a$ and is equal to 0 for $a < x$.

(a) Obtain the functional form of positive energy solutions $(E > 0)$ to the energy eigenvalue equation in the three regions of interest. (10 marks)

(b) What are the appropriate boundary conditions for this system at $x = 0$ and $x = a$? (06 marks)

(c) Applying boundary conditions, determine up to a single normalization constant A, the eigenstates of this system for positive energy solutions. For what energies, if any, are the solutions with $E > 0$ square normalizable? (09 marks)

Contd.

(4) (a) Briefly explain the situations where one can use perturbation theory in quantum mechanics.

(06 marks)

(b) Consider a particle of mass M confined to a 1-D potential box of length L. Use perturbation theory to calculate the effects of having a tilt to the box, represented by the addition of a linear potential $V_{\text{tilt}}(x) = \hbar\beta\left(\frac{x}{L} - \frac{1}{2}\right)$ to the potential of the box; $V_{\text{box}}(x) = 0$ for $0 < x < L$, and $V_{\text{box}}(x) = \infty$ elsewhere with β being a constant. Calculate the two lowest perturbed eigenstates to first-order and their corresponding eigenvalues to second-order.

You may use: $\frac{2\hbar\beta}{L}\int_0^L dx \sin\left(\frac{m\pi x}{L}\right)\left(\frac{x}{L} - \frac{1}{2}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{4\hbar\beta mn((-1)^{m+n} - 1)}{(m^2 - n^2)^2\pi^2}$.

(19 marks)

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