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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences
First Year - Semester II Examination – September/ October 2020

MAP 1203 – REAL ANALYSIS I

Time: Two (02) hours

Answer all (04) questions

1. a) Show that $\sqrt{5}+\sqrt{13}$ is an irrational number. **(25 marks)**

   b) Define the following terms for a set:
      i. Bounded,
      ii. Supremum,
      iii. Infimum.

   (15 marks)

   c) Find the Supremum, Infimum, Maximum and Minimum for each of the following sets if exist:
      i. $\left\{\dfrac{3n+1}{2n+1} \middle| n \in N\right\}$.
      ii. $\left\{2^{(-1)^n} \middle| n \in N\right\}$.
      iii. $\left\{1-\dfrac{1}{n} \middle| n \in N\right\}$.

   (45 marks)

   d) Show that $Sup\{r \in Q | r < a\} = a, \forall a \in R$. **(15 marks)**

2. a) Prove that '*Every convergent sequence is bounded*'.
   Is every bounded sequence convergent? Justify your answer.

   (40 marks)

   b) Using the definition, show that $\left\{\dfrac{1}{3^n} \middle| n \in N\right\}$ converges to 0. **(25 marks)**

   c) Show that the sequence $\{x_n\}$; $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$ for all $n \geq 1$ is monotonic, convergent and converges to $\sqrt{2}$. **(35 marks)**

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3. a) State the $\varepsilon - \delta$ definition for the limit of a function. Prove that, $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

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(~~25~~ marks)

b) Find the constants a and b such that the following function $f(x)$ has limit everywhere:

$$f(x) = \begin{cases} 3 & ;\ x \le 2 \\ ax^2 + bx + 1 & ;\ 2 < x < 3 \\ 7 - ax & ;\ x \ge 3 \end{cases}$$

(35 marks)

c) State the $\varepsilon - \delta$ definition for continuity of a function at a point. Using this definition, prove that $f(x)$ is continuous at $x = 1$, where

$$f(x) = \begin{cases} \dfrac{x^3 - 1}{x^2 - 1} & for\ x \ne 1 \\ \dfrac{3}{2} & for\ x = 1 \end{cases}.$$

(35 marks)

4. a) Define the derivative of a function at a point. Show that the following function $f(x)$ is continuous at $x = 1$, for all values of a, where

$$f(x) = \begin{cases} a^3 x + 1 & for\ x \ge 1 \\ x^2 + a^3 & for\ x < 1 \end{cases}.$$

Find the condition for the existence of the derivative at $x = 1$.

(35 marks)

b) State teach of the following theorems:

i. Rolle's Theorem.

ii. Lagrange Mean Value Theorem.

iii. Cauchy's Mean Value Theorem.

(15 marks)

c) Show that $\dfrac{a-b}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{a-b}{1+a^2}$ where $0 < a < b$.

Deduce that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$.

(30 marks)

d) State L'Hospital's rule.

Evaluate $\lim_{x\to 0} \dfrac{x\cot x - 1}{x^2}$.

(20 marks)

END