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RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES

Bachelor of Science Honours in Applied Sciences /
B. Sc. (Joint Major) Degree in Chemistry and Physics

Fourth Year - Semester I Examination – July/August 2023

PHY 4210 - ADVANCED QUANTUM MECHANICS

Time: Two (02) hours

Answer **ALL four** questions

Unless otherwise specified all symbols have their usual meaning.

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1. a) Prove the following operator identities:

   i. $\left[\hat{A},\ \hat{B}+\hat{C}\right] = \left[\hat{A},\ \hat{B}\right] + \left[\hat{A},\ \hat{C}\right]$ **(05 marks)**

   ii. $\left[\hat{A}\hat{B},\ \hat{C}\right] = \hat{A}\left[\hat{B},\ \hat{C}\right] + \left[\hat{A},\ \hat{C}\right]\hat{B}$ **(05 marks)**

   b) If $\hat{L}$ is a non-Hermitian operator, then show that $\left(\hat{L}+\hat{L}^{\dagger}\right)$ is Hermitian.

   (07 marks)

   c) If two Hermitian operators $\hat{L}$ and $\hat{M}$ mutually commute, then show that they have common eigenfunctions.

   (08 marks)

2. a) The Hamiltonian operator for a harmonic oscillator is $\hat{H} = \dfrac{\hat{p}^2}{2m} + \dfrac{1}{2} m\,\omega_0^2\,\hat{x}^2$ and the eigenvalue equation is $\hat{H}\psi_n = E_n\psi_n$. Show that the two operators $\hat{a}$ and $\hat{a}^{\dagger}$ defined by;

Contd.

$$\hat{a} = \left(\frac{m\omega_0}{2\hbar}\right)^{1/2} \hat{x} + \mathrm{i}\left(\frac{1}{2m\hbar\omega_0}\right)^{1/2} \hat{p} \text{ and } \hat{a}^{\dagger} = \left(\frac{m\omega_0}{2\hbar}\right)^{1/2} \hat{x} - \mathrm{i}\left(\frac{1}{2m\hbar\omega_0}\right)^{1/2} \hat{p}$$

satisfy the following commutation rules.

i. $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$ **(05 marks)**

ii. $\left[\hat{a}^{\dagger}\hat{a}, \hat{a}^{\dagger}\right] = \hat{a}^{\dagger}$ **(05 marks)**

iii. $\left[\hat{a}^{\dagger}\hat{a}, \hat{a}\right] = -\hat{a}$ **(05 marks)**

b) Let $\hat{a}_1$, $\hat{a}_1^{\dagger}$ and $\hat{a}_2$, $\hat{a}_2^{\dagger}$ be annihilation and creation operators for two independent harmonic oscillators, which satisfy the relations $\left[\hat{a}_k, \hat{a}_m\right] = 0 = \left[\hat{a}_k^{\dagger}, \hat{a}_m^{\dagger}\right]$ and $\left[\hat{a}_k, \hat{a}_m^{\dagger}\right] = \delta_{km}$, $(k, m = 1, 2)$.

Let $\hat{J}_1 = \frac{1}{2}\left(\hat{a}_2^{\dagger}\hat{a}_1 + \hat{a}_1^{\dagger}\hat{a}_2\right)$, $\hat{J}_2 = \frac{\mathrm{i}}{2}\left(\hat{a}_2^{\dagger}\hat{a}_1 - \hat{a}_1^{\dagger}\hat{a}_2\right)$ and $\hat{J}_3 = \frac{1}{2}\left(\hat{a}_1^{\dagger}\hat{a}_1 - \hat{a}_2^{\dagger}\hat{a}_2\right)$.

Show that

i. $\left[\hat{J}_1, \hat{J}_2\right] = \mathrm{i}\hat{J}_3$ **(05 marks)**

ii. $\left[\hat{J}_2, \hat{J}_3\right] = \mathrm{i}\hat{J}_1$ **(05 marks)**

3. a) Briefly explain the situations where one can use approximation methods; the perturbation theory and the variation method, in quantum mechanics.

(10 marks)

b) Consider a particle of mass M confined to a 1-dimensional box of length L. Use perturbation theory to calculate the effects of adding a tilt to the box, represented by adding the linear potential

$$V_{\text{tilt}}(x) = \hbar\beta\left(\frac{x}{L} - \frac{1}{2}\right)$$

to the box potential,

$$V_{\text{box}}(x) = \begin{cases} 0; & 0 < x < L \\ \infty; & \text{elsewhere.} \end{cases}$$

Contd.

Determine the lowest perturbed eigenstates to first-order and their corresponding eigenvalues to second-order for the ground state of the system. The eigenfunctions and the eigenvalues for the unperturbed system are $\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ and $\frac{n^2\pi^2\hbar^2}{2ML^2}$ respectively.

You may use the matrix elements: $V_{mn} = \frac{2\hbar\beta}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\left(\frac{x}{L}-\frac{1}{2}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{4\hbar\beta mn((-1)^{m+n}-1)}{(m^2-n^2)^2\pi^2}$.

(15 marks)

4. A particle in 1-D is characterized by a state $|\psi\rangle$ with a wave function in the k representation given by $\psi(k) = Ae^{(-\gamma|k| + iak)}$, where γ and a are constants.

   a) Determine A, so that $|\psi\rangle$ is normalized to unity. **(10 marks)**

   b) What is the probability that a measurement will find the magnitude $|p|$ of the particle's momentum less than $\hbar q$? **(08 marks)**

   c) What is the probability density that a position measurement will find the particle at x? Hint: $\psi(x) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{ikx}\psi(k)dk$. **(07 marks)**

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