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RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. (General) Degree in Applied Sciences
First Year - Semester I Examination – May 2022
MAP 1301 – LINEAR ALGEBRA

Time: Three (03) hours

Answer <u>All</u> (06) questions

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1. a) Let V be the set of ordered triples of real numbers defines as $V = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$. Determine whether the set V is a vector space over the field $\mathbb{R}$ under the following vector addition $\oplus$ and scalar multiplication $\otimes$:

$$(x_1, x_2, 0) \oplus (y_1, y_2, 0) = (x_1 + y_1, x_2 + y_2, 0),$$
$$\alpha \otimes (x_1, x_2, 0) = (\alpha x_1, \alpha x_2, 0), \text{where } \alpha \in \mathbb{R}.$$

(45 Marks)

b) Define a subspace of a vector space.

Let W_1 and W_2 be subspaces of a vector space V over the field $\mathbb{F}$. Prove that

$W_1 + W_2 = \{x_1 + x_2 | x_1 \in W_1,\ x_2 \in W_2\}$ is a subspace of V over $\mathbb{F}$.

(35 Marks)

c) Determine which of the following are subspaces of the given vector spaces:

i. The set of vectors of the form $\{(a, b, 1) | a, b \in \mathbb{R}\} \subseteq \mathbb{R}^3$, where $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

ii. The set of matrices of the form $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc \neq 0,\ a, b, c, d \in \mathbb{R}\right\} \subseteq M_{22}$, where M_{22} is a vector space of all 2×2 matrices over $\mathbb{R}$.

(20 Marks)

2. a) Let $S = \{v_1, v_2, v_3, \ldots, v_k\}$ be a subset of a vector space V over a field F .

Explain the following briefly:

i. V is spanned by S .
ii. S is linearly independent.
iii. S is a basis for V .
iv. Dimension of V .

(20 Marks)

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b) Determine whether the given set of vectors spans the given vector space:

i. In $\mathbb{R}^3$: $S = \{(1,-1,2),(1,1,2),(0,0,2)\}$, where $\mathbb{R}^3$is a vector space over the field $\mathbb{R}$.

ii. In P_2 : $S = \{(1-x^2),(1-2x),(x+x^2)\}$, P_2 is the vector space over $\mathbb{R}$ such that $P_2 = \{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{R}\}$.

(20 Marks)

c) If u_1, u_2, u_3 are linearly independent vectors in a vector space V over the field $\mathbb{F}$, then prove that the set $\{u_1 - u_2, u_2 - u_3, u_3 - u_1\}$is also a linearly independent set in V .

(20 Marks)

d) Let W be the subset of P_3 with usual notations, where

$W = \{x^3 + 2x + 1, 2x^3 - 3x - 1, x^3 + 2x - 5, 2x^2 - 2x\}$. Find the dimension of the subspace spanned by W.

(Hint: Form a matrix and then reduce to row echelon form).

(40 Marks)

3. a) Using Gauss-Jordan elimination method, find the inverse of the following matrix A .

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}.$$

(40 Marks)

b) Fin all solutions of the following system of equations, depending on $a_1, a_2, a_3 \in \mathbb{R}$.

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= a_1 \\ 2x_1 - 5x_2 - 4x_3 &= a_2 \\ 4x_1 - 9x_2 - 8x_3 &= a_3 \end{aligned}$$

(60 Marks)

4. a) Define a rank of a matrix.

Using echelon form find the rank of the following matrix,

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

(40 Marks)

a) Define a linear transformation.

Show that the following mappings are linear transformations.

i. $T((x,y)) = (x+y, x)$, where$T: \mathbb{R}^2 \to \mathbb{R}^2$.

ii. $T((a,b,c)) = (2a - 3b + 4c)$, where$T: \mathbb{R}^3 \to \mathbb{R}$.

(40 Marks)

c) Find the kernel and image of the following linear transformations:

i. $T\big((x, y)\big) = (x - 2y, 2x + y)$, where $T\colon \mathbb{R}^2 \to \mathbb{R}^2$.

ii. $T\big((a, b, c)\big) = (a - c, a + b)$, where $T\colon \mathbb{R}^3 \to \mathbb{R}^2$.

(20 Marks)

5. a) State and prove the Rank-Nullity theorem.

(40 Marks)

b) For each of the following linear mappings T, find a basis and the dimension of their Kernal and Image spaces. Also verify the Rank-Nullity theorem.

i. $T\big((a, b)\big) = (a + b, a - b, b)$, where$T\colon \mathbb{R}^2 \to \mathbb{R}^3$.

ii. $T\big((x, y, z)\big) = (x + 2y, y - z, x + 2z)$, where$T\colon \mathbb{R}^3 \to \mathbb{R}^3$.

(60 Marks)

6. a) Define an inner product space.

Show that$\langle u, v\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ is an inner product on $\mathbb{C}^n$, where $u, v \in \mathbb{C}^n$ with

$u = (x_1, x_2, \ldots, x_n), v = (y_1, y_2, \ldots, y_n)$.

(35 Marks)

b) Let V be an inner product space over the complex field $\mathbb{C}$. Then, show that,

$|\langle x, y\rangle| \le \| x \| + \| y \|$ for all $x, y \in V$.

(30 Marks)

c) Define an eigenvalue of a matrix of order n.

Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ be a 3×3 matrix. Find eigenvalues and corresponding to eigenvectors of A .

(35 Marks)

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