RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B. Sc. (General) Degree in Applied Sciences
Second Year - Semester I Examination – September/ October 2019
MAP 2301 – ALGEBRA

Time allowed: Three (03) hours

Answer All (06) questions

1. a) Let A, B and C be any three sets. Prove the following set equations:
   - i. $A \cap B = A - (B - C)$,
   - ii. $(A \cap \overline{B}) \cup (\overline{A} \cap B) = (A \cup B) \cap (\overline{A \cap B})$.

   (30 marks)

   b) Let $X = \{x \in \mathbb{R} | x \leq a\} \cap \{x \in \mathbb{R} | min(x, a) \leq b\}, Y = \{x \in \mathbb{R} | x \leq min(a, b)\}$.

   Prove that $X = Y$. (25 marks)

   c) 100 students were asked which fast food restaurant they have been to this year. The results of the survey were as follows: 5 have been to all three 20 have been to both McDonald's and Pizza Hut 25 have been to both Subway and McDonald's 15 have been to both Pizza Hut and Subway 50 have been to McDonald's 40 have been to Pizza Hut, 45 have been to Subway.

   - i. Create a Venn diagram to model the information.
   - ii. How many did not go to any of the three?
   - iii. How many have been to McDonald's or Pizza Hut?
   - iv. How many have been to McDonald's or Pizza Hut, but not Subway?
   - v. How many have been to exactly two of the three?

   (25 marks)

   d) Let p, q and r be three prepositions.

   Construct the truth tables for the following propositional formulas:

i. $(p \vee q \to r) \vee p \vee q.$

ii. $(p \vee q) \wedge \big(p \to (r \wedge q)\big) \wedge \big(q \to (\neg r \wedge q)\big).$

(20 marks)

2. a) Define the following terms:
   i. Reflexive relation
   ii. Symmetric relation,
   iii. Anti-symmetric relation
   iv. Transitive relation
   v. Equivalence relation

(25 marks)

b) Le R be a relation defined on $\mathbb{Z}$. For all $x, y \in \mathbb{Z}$, xRy if $|x - y|$ is a multiple of 4.

Is R an equivalence relation?
Justify your answer. (15 marks)

c) Let $S = R - \{-\sqrt{7}\}$be a set. A relation R on the set S is defined by

for each $x, y \in S$,

$$xRy \text{ if } \frac{x}{y + \sqrt{7}} \in Q'.$$

Is S an equivalence relation?
Justify your answer. (20 marks)

d) Show that each of the following is a binary operation:

i. $a * b = \frac{a^2+b^2}{\sqrt[3]{5}}$, where $a, b \in \mathbb{R}$,

ii. $x * y = x^2 y$, where $x, y \in \mathbb{Z}$,

iii. $x * y = \frac{xy}{4}$, where $x, y \in \mathbb{Q}$,

iv. $x * y = \frac{e^{-\frac{1}{x}}}{3}$, where $x, y \in \mathbb{Q}$ and$x \neq 0$.

Further check which of the above i., ii., iii. and iv. are commutative and associative.

(40 marks)

3. a) Define the following types of functions:
   i. Well defined function,
   ii. One to one function,
   iii. Onto function,
   iv. Bijective function.

(20 marks)

b) Which of the following are Well defined functions, One to one functions and Onto functions? Justify your answers

i. $f: \mathbb{R} \to \{x \in \mathbb{R} | -1 < x < 1\}$, where$f(x) = \frac{x}{1+|x|}$.

ii. $f: R - \{2\} \to R$, where $f(x) = \frac{x^2-1}{x+2}$.

(40 marks)

c) Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$.

Consider the function $f: A \to B$ defined by

$$f(x) = \frac{x-2}{x-3}.$$

Is f bijective function? Justify your answer. (15marks)

d) Show that that the function $f: \mathbb{R}^+ \to [4, \infty)$ defined by $f(x) = x^2 + 4$ is invertible. Find the inverse of f. (25 marks)

4. a) Prove that *any two disjoint permutations are commute.* (15 marks)

b) Define an even permutation, odd permutation and transpositions. (30 marks)

Determine which of the followings are even permutation and odd permutation.

i. $(1\ \ 3\ \ 8\ \ 4)(2\ \ 5\ \ 3)(1\ \ 7\ \ 5\ \ 2)$

ii. $(1\ \ 5\ \ 9\ \ 7\ \ 4)(1\ \ 6\ \ 2)(4\ \ 3\ \ 9)$

c) Let $\alpha = (1\ \ 8\ \ 5)(2\ \ 7\ \ 3\ \ 6), \beta = (1\ \ 5\ \ 9\ \ 5)$ and $\gamma = (4\ \ 1\ \ 8)(2\ \ 3\ \ 6)$.

Find the orders of α, β and γ.

Determine the following permutations:

i. α^{-23},

ii. β^{-74},

iii. $\alpha^{2019}\beta^{10}\gamma^{25}$,

iv. $\alpha^{-999}\beta^{99}\gamma^{9}$.

(55 marks)

5. a) Define a group and an abelian group. (30 marks)

Let $G = \left\{A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| \, x, y \in \mathbb{R}, \det A \neq 0\right\}$ and let $\times$ denotes the ordinary matrix multiplication. Show that $(G,\times)$ is a group.

b) Let G be a group. Suppose $(ab)^i = a^i b^i$ forall $a, b \in G$ and three consecutive integers i. Then show that G is an abelian group. (30 marks)

c) Define a subgroup of group G. Let H, K be subgroups of the group G.

Prove that, HK is a subgroup of G if and only if $HK = KH$,

where $HK = \{hk | h \in H \text{ and } k \in K\}$ and $KH = \{kh | k \in K \text{ and } h \in H\}$.

(40 marks)

6. a) Show that the given two integers a and b, with $b > 0$, there exist unique integers q and r satisfying $a = qb + r, 0 \leq r < b$. (30 marks)

b) Prove that the linear Diophantine equation $x + by = c$ has a solution if and only if $d|c$, where $d = gcd(a, b)$.
Prove further that if x_0 and y_0 is a particular solution of this equation, then all other solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right)t, y = y_0 - \left(\frac{a}{d}\right)t,$$

where t is an integer. (40 marks)

c) Solve the linear Diophantine equation

$$169x + 52y = 1300.$$

(30 marks)

---END---