RAJARATA UNIVERSITY OF SRI LANKA
FACULTY OF APPLIED SCIENCES

B.Sc. General Degree in Applied Sciences
Second Year – Semester I Examination – October/ Nov. 2015

MAA 2201 – Mathematical Methods II

Answer **Four** Questions **only**. Time allowed: **Two Hours**

01. Cylindrical polar coordinates (R,ϕ,Z) of a point P are related to its Cartesian coordinates (x,y,z) by the position vector equation $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, where $x = R\cos\phi$, $y = R\sin\phi$, $z = Z$. Further relations $R = r\sin\theta$, $Z = r\cos\theta$ $(0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi)$ define Spherical polar coordinates (r,θ,ϕ) of the same point P.

(a)

i. Illustrate these relationships in a single diagram, when both θ and ϕ are acute angles.

ii. Find the set of unit vectors $(\underline{e}_R, \underline{e}_\phi, \underline{e}_Z)$ in the cylindrical polar system, in terms of (i,j,k) and show that the position vector $\underline{r} = R\underline{e}_R + Z\underline{e}_Z$. Hence express the unit vector $\underline{e}_r$ in the radial direction, in terms of $\underline{e}_R$ and $\underline{e}_Z$.

(b)

i. Identify the surfaces $r = a$, $R = b$ $(Z \ge 0)$ and $Z = c$, where $a > b > c > 0$

ii. Find the unit normal to these surfaces

iii. Identify the curves C_1 and C_2 of intersection of $r = a$ with the other two surfaces

(c) Hence show that the tangents $\underline{T}_1$ and $\underline{T}_2$ to the curves C_1 and C_2, for a given value of ϕ, are both parallel to the vector $\underline{e}_\phi$, and find a relationship connecting a,b,c when these two curves coincide.

[P.T.O.]

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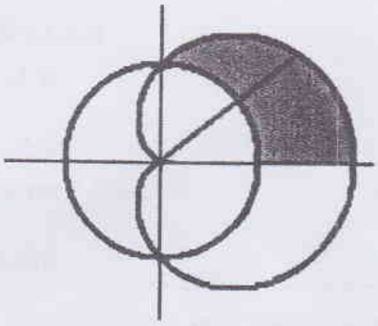
02.

(a) Verify Green's theorem in the plane for $\oint_C (2xy - x^2)dx + (x + y^2)dy$ where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

(b) Find the area of the region in the first quadrant that is outside the circle $r = a$ and inside the cardioid $r = a(1+\cos\theta)$ by double integrals using polar coordinates (r, θ).



03.

(a) V is the volume of the sector of the sphere defined by $0 \le r \le a$ and $\alpha \le \theta \le \pi$. Using Spherical Polar Coordinates show that $V = \frac{2}{3}a^3\pi(1+\cos\alpha)$.

(b) Evaluate the integral $J = \int_V \underline{r}dv$ over the same sector and hence show that the position vector of its centroid is $\frac{3}{8}a(\cos\alpha - 1)\underline{k}$.

(c) By using part (a), deduce the volume of the sector of the sphere defined by $0 \le r \le a$ and $0 \le \theta \le \alpha$.

04.

(a) Find the Laplace Transform of the convolution integral $\int_0^t u^{m-1}(t-u)^{n-1}du = F(t)$.

Hence find the $F(t)$ and deduce that the Beta function $\beta(m,n)$ defined by the integral $\int_0^1 u^{m-1}(t-u)^{n-1}du$ has the value $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(b) Using this results prove that,

i. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

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ii. $\displaystyle\int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+1}{2}\right)}$;Where p and q are constants.

(c) Find the Laplace Transform of $t^{-1/2}$ and deduce that $L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{1}{\sqrt{\pi t}}$

05.

(a) The Laplace Transform of $F(t)$ is defined by,

$$L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)dt \text{ ; Where } s \text{ is a parameter.}$$

With the usual notations derive the Laplace Transform of $\cos at$.

(b) Show that $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$, and deduce the Laplace Transform of $\frac{\sin^2 t}{t}$. Hence evaluate the integral $\int_0^{\infty}\left(\frac{e^{-3t}\sin^2 t}{t}\right)dt$.

(c) Using Laplace Transform method, solve the following differential equation for $Y(t)$:

$\frac{d^2Y}{dt^2} - 3\frac{dY}{dt} + 2Y = e^{3t}$, subject to the conditions $Y(0) = 1$ and $Y'(0) = 0$.