RAJARATA UNIVERSITY OF SRI LANKA

FACULTY OF APPLIED SCIENCES

B.Sc. (General) Degree in Applied Sciences
First Year – Semester II Examination – November / December 2016

MAP1302 - DIFFRENTIAL EQUATIONS I

Time: Three (03) Hours.

Answer ALL questions.

01. (a) Population of mosquitoes in a certain area increases at a rate proportional to the current population and, in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially (at time $t = 0$), and predators (birds, etc.) eat 20,000 mosquitoes per day. Formulate a differential equation to determine the population of mosquitoes in the area at any time t. [Do not solve the equation] (40 marks)

(b) Newton's law of cooling states that the temperature of a hot liquid decreases at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. The coffee has a temperature of $200^0\,F$ when freshly poured into the cup, and one minute later it has cooled to $190^0\,F$ in a room at a constant temperature of $70^0\,F$. Form a differential equation to determine when the cup of coffee reaches a temperature of $150^0\,F$. (40 marks)

(c) A small ball is thrown vertically upwards with initial speed 20 ms^{-1} from the roof of a building 30 m high. Air resistance to its motion can be neglected and the motion is due to gravity alone.

(i) Formulate a differential equation for the speed of the ball and hence find the maximum height above the ground reached by the ball. (20 marks)

(ii) Assuming that the ball misses the building on its way down, find the time at which it hits the ground. (10 marks)

(iii) Plot the graph of velocity versus time, for the motion of the ball. (10 marks)

(d) A realistic model of a baseball in flight includes the effect of air resistance in addition to gravity. In this case, the equations of motion are

$$\frac{dv}{dt} = -kv \quad , \quad \frac{dw}{dt} = -g - kw \quad ;$$

where $v(t)$ = horizontal component of velocity, at time t

$w(t)$ = vertical component of velocity

k = coefficient of resistance per unit mass

and g = gravitational constant.

(i) Determine $v(t)$ and $w(t)$ in terms of initial speed u, its initial angle of elevation, θ and time t. (30 marks)

(ii) Let $x(t)$ and $y(t)$, respectively, be the horizontal and vertical displacements of the ball at time t. If $x(0) = 0$ and $y(0) = h$, find $x(t)$ and $y(t)$. (50 marks)

02. (a) Find a differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$ (20 marks)

(b) Find a differential equation of all circles through the origin. (30 marks)

(c) Find the function $y(x)$ which satisfies the equation, $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + \cos y}$. (30 marks)

(d) Solve the equation: $\dfrac{dy}{dx} = \dfrac{2x + 3y + 4}{4x + 6y + 5}$. (40 marks)

(e) Solve $\left\{ y\left(1 + \dfrac{1}{x}\right) + \cos y \right\} dx + (x + \log x - x\sin y)dy = 0$. (30 marks)

(f) Integrate $(1 + x^2)\left(\dfrac{dy}{dx}\right) + 2xy - 4x^2 = 0$.

Obtain an equation of the curve satisfying this equation and passing through the origin. (50 marks)

03. (a) Derive a method for solving Bernoulli's equation $\dfrac{dy}{dx} + Py = Qy^n$. (25 marks)

Hence solve $\left(\dfrac{dy}{dx}\right)(x^2y^3 + xy) = 1$. (30 marks)

(b) The operator $D = \frac{d}{dx}$ and $F(D)$ is a linear polynomial, with constant coefficients, in D.

Establish the following results:

(i) $F(D)\, e^{ax} = F(a) e^{ax}$ (30 marks)

(ii) $\frac{1}{F(D)} e^{ax} v(x) = e^{ax} \frac{1}{F(D+a)} v(x)$ (30 marks)

(iii) $\frac{1}{D-\alpha} f(x) = e^{\alpha x} \int f(x) e^{-\alpha x} dx$ (30 marks)

(c) Obtain the general solution for each of the following differential equations:

(i) $\frac{d^3 y}{dx^3} - 4\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$. (30 marks)

(ii) $(D^3 - 1)y = e^{2x} + 2e^x + 1$ (25 marks)

04. (a) Solve

(i) $(D^2 - 8D + 9)y = 40\sin 5x$. (40 marks)

(ii) $(D^2 - 4D + 4)y = x^2$. (40 marks)

(iii) $(D^2 + 2D + 1)y = x\cos x$. (50 marks)

(b) (i) Derive a method for solving Riccati's equation when one particular integral is known. (30 marks)

(ii) Show that $x^2 + 1$ is an integral of $y_1 = 2x - (x^2 + 1)y + y^2$. Hence find the general solution of the given equation. (40 marks)

05. (a) Explain a method for finding the solution of an equation given in the form $x = f(y, p)$, where $p = \frac{dy}{dx}$. Hence solve $x = y + p^2$. (30 marks)

(b) Solve, (i) $y = 2px + p^4 x^2$; for y. (30 marks)

(ii) $xy^2(p^2 + 2) = 2py^2 + x^3$; for p. (30 marks)

(c) (i) Define Clairaut's equation and identify the steps to solve such an equation. (30 marks)

(ii) Solve, $y = 3x + \log p$ (30 marks)

(iii) Solve $x^2 p^2 + yp(2x + y) + y^2 = 0$ by using the substitutions $y = u$, $xy = v$. (50 marks)

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