CS480/CS680 Problem Set 3 Model Solution

1. (a) (10 points)

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{r_x^2} & 0 & 0 & -\frac{c_x}{r_x^2} \\ 0 & \frac{1}{r_y^2} & 0 & -\frac{c_y}{r_y^2} \\ 0 & 0 & 0 & 0 \\ -\frac{c_x}{r_x^2} & -\frac{c_y}{r_y^2} & 0 & \frac{c_x^2}{r_x^2} + \frac{c_y^2}{r_y^2} - 1 \end{bmatrix}$$

(b) (10 points)

$$\mathbf{n}(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} f(x,y,z) \\ \frac{\partial}{\partial y} f(x,y,z) \\ \frac{\partial}{\partial z} f(x,y,z) \end{bmatrix} = \begin{bmatrix} \frac{2}{r_x^2}(x - c_x) \\ \frac{2}{r_y^2}(y - c_y) \\ 0 \end{bmatrix}$$

(c) (10 points)

$$\mathbf{S}(\theta, z) = \begin{bmatrix} r_x \cos\theta + c_x \\ r_y \sin\theta + c_y \\ z \\ 1 \end{bmatrix}, \text{where } -\pi \le \theta \le \pi \text{ and } z_{min} \le z \le z_{max}.$$

(d) (10 points)

$$\mathbf{n}(\theta, z) = \frac{\partial \mathbf{S}}{\partial \theta} \times \frac{\partial \mathbf{S}}{\partial z}$$

$$\frac{\partial \mathbf{S}}{\partial \theta} = \begin{bmatrix} -r_x \sin\theta \\ r_y \cos\theta \\ 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{S}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{n}(\theta, z) = \begin{bmatrix} r_y \cos\theta \\ r_x \sin\theta \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{n}}(\theta, z) = \frac{\mathbf{n}(\theta, z)}{|\mathbf{n}(\theta, z)|}$$

2. (15 points)

$$
\begin{aligned}
\mathbf{M}_{\text{affine}}\mathbf{P}(u) &= \mathbf{M}_{\text{affine}} \sum_{i=0}^{k} w_i \begin{bmatrix} x_{\mathbf{p}_i} \\ y_{\mathbf{p}_i} \\ z_{\mathbf{p}_i} \\ 1 \end{bmatrix} BEZ_{i,k}(u) \\
&= \sum_{i=0}^{k} w_i \, \mathbf{M}_{\text{affine}} \begin{bmatrix} x_{\mathbf{p}_i} \\ y_{\mathbf{p}_i} \\ z_{\mathbf{p}_i} \\ 1 \end{bmatrix} BEZ_{i,k}(u)
\end{aligned}
$$

3. (a) (10 points)

$$
\begin{aligned}
\begin{bmatrix} \mathbf{P}(0) \\ \mathbf{P}(1) \\ \mathbf{P}'(0) \\ \mathbf{P}'(1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}(1-t) & 0 & \frac{1}{2}(1-t) & 0 \\ 0 & -\frac{1}{2}(1-t) & 0 & \frac{1}{2}(1-t) \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} \\
&= \begin{bmatrix} (u^3 & u^2 & u & 1)|_{u=0} \\ (u^3 & u^2 & u & 1)|_{u=1} \\ (3u^2 & 2u & 1 & 0)|_{u=0} \\ (3u^2 & 2u & 1 & 0)|_{u=1} \end{bmatrix} M_C \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}(1-t) & 0 & \frac{1}{2}(1-t) & 0 \\ 0 & -\frac{1}{2}(1-t) & 0 & \frac{1}{2}(1-t) \end{bmatrix} &= \begin{bmatrix} (u^3 & u^2 & u & 1)|_{u=0} \\ (u^3 & u^2 & u & 1)|_{u=1} \\ (3u^2 & 2u & 1 & 0)|_{u=0} \\ (3u^2 & 2u & 1 & 0)|_{u=1} \end{bmatrix} M_C \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} M_C.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M_C &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}(1-t) & 0 & \frac{1}{2}(1-t) & 0 \\ 0 & -\frac{1}{2}(1-t) & 0 & \frac{1}{2}(1-t) \end{bmatrix} \\
&= \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{where } s = \frac{1}{2}(1-t).
\end{aligned}
$$

(b) (10 points)

$$
\begin{aligned}
\mathbf{B}(u) &= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} M_C \\
&= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{where } s = \frac{1}{2}(1-t)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{B_0}(u) &= -su^3 + 2su^2 - su \\
\mathbf{B_1}(u) &= (2-s)u^3 + (s-3)u^2 + 1 \\
\mathbf{B_2}(u) &= (s-2)u^3 + (3-2s)u^2 + su \\
\mathbf{B_3}(u) &= su^3 - su^2, \\
& \quad \text{where } s = \frac{1}{2}(1-t).
\end{aligned}
$$

(c) (10 points)

$$
\mathbf{P}'(u) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} \mathbf{M}_C \, \mathbf{M}_{geom}
$$

(d) (5 points)

Let segment 1 ($S1$) be followed by segment 2 ($S2$) *s.t.* $\mathbf{P}_{S1}(1) = \mathbf{P}_{S2}(0)$.

If these two adjacent segments satisfy C1 continuity, then $\mathbf{P}'_{S1}(1) = \mathbf{P}'_{S2}(0)$ must hold.

$$
\begin{aligned}
\mathbf{P}'_{S1}(1) &= \left[\ (3u^2 \quad 2u \quad 1 \quad 0)|_{u=1}\ \right] M_C \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} \\
&= \left[\ 3 \quad 2 \quad 1 \quad 0\ \right] M_C \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} \\
&= \left[\ 0 \quad -s \quad 0 \quad s\ \right] \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} \\
\mathbf{P}'_{S2}(0) &= \left[\ (3u^2 \quad 2u \quad 1 \quad 0)|_{u=0}\ \right] M_C \begin{bmatrix} p_k \\ p_{k+1} \\ p_{k+2} \\ p_{k+3} \end{bmatrix} \\
&= \left[\ 0 \quad 0 \quad 1 \quad 0\ \right] M_C \begin{bmatrix} p_k \\ p_{k+1} \\ p_{k+2} \\ p_{k+3} \end{bmatrix} \\
&= \left[\ -s \quad 0 \quad s \quad 0\ \right] \begin{bmatrix} p_k \\ p_{k+1} \\ p_{k+2} \\ p_{k+3} \end{bmatrix}
\end{aligned}
$$

$\mathbf{P}'_{S1}(1) = \mathbf{P}'_{S2}(0)$, and therefore adjacent segments satisfy C1 continuity.

(e) (10 points)

Changing t only changes the scale factor in computing $\mathbf{P}'$. Therefore, it does not change the tangent direction at the endpoints, it only changes its magnitude.

$$
\begin{aligned}
\mathbf{P}'(0)|_{t=a} &= \frac{1}{2}(1-a)\,(\mathbf{p}_{k+1} - \mathbf{p}_{k-1}) \\
\mathbf{P}'(0)|_{t=b} &= \frac{1}{2}(1-b)\,(\mathbf{p}_{k+1} - \mathbf{p}_{k-1}) \\
\mathbf{P}'(1)|_{t=c} &= \frac{1}{2}(1-c)\,(\mathbf{p}_{k+2} - \mathbf{p}_{k}) \\
\mathbf{P}'(1)|_{t=d} &= \frac{1}{2}(1-d)\,(\mathbf{p}_{k+2} - \mathbf{p}_{k})
\end{aligned}
$$

and so, $\mathbf{P}'(0)|_{t=a} \propto \mathbf{P}'(0)|_{t=b}$ and $\mathbf{P}'(1)|_{t=c} \propto \mathbf{P}'(1)|_{t=d}$.

4. (CS680 only)

(a) (10 points)

$$\mathbf{S}(\phi,\theta) = \begin{bmatrix} (r_{axial} + r_x \cos^s \phi)\cos\theta \\ r_y \sin^s \phi \\ -(r_{axial} + r_x \cos^s \phi)\sin\theta \\ 1 \end{bmatrix}, \text{where } -\pi \le \phi \le \pi \text{ and } -\pi \le \theta \le \pi.$$

(b) (10 points)

$$\begin{aligned}
\mathbf{n}(\phi,\theta) &= \frac{\partial \mathbf{S}}{\partial \phi} \times \frac{\partial \mathbf{S}}{\partial \theta} \\
\frac{\partial \mathbf{S}}{\partial \phi} &= \begin{bmatrix} sr_x \cos\theta (cos\phi)^{s-1}(-sin\phi) \\ sr_y (\sin\phi)^{s-1}(cos\phi) \\ sr_x(-\sin\theta)(cos\phi)^{s-1}(-sin\phi) \end{bmatrix} \\
\frac{\partial \mathbf{S}}{\partial \theta} &= \begin{bmatrix} -(r_{axial} + r_x cos^s\phi)\sin\theta \\ 0 \\ -(r_{axial} + r_x cos^s\phi)\cos\theta \end{bmatrix} \\
\mathbf{n}(\phi,\theta) &= \begin{bmatrix} -sr_y(\sin\phi)^{s-1}(\cos\phi)(r_{axial} + r_x cos^s\phi)\cos\theta \\ sr_x(\cos\phi)^{s-1}(-\sin\phi)(r_{axial} + r_x cos^s\phi) \\ sr_y(\sin\phi)^{s-1}(\cos\phi)(r_{axial} + r_x cos^s\phi)\sin\theta \end{bmatrix} \\
\hat{\mathbf{n}}(\phi,\theta) &= \frac{\mathbf{n}(\phi,\theta)}{|\mathbf{n}(\phi,\theta)|}
\end{aligned}$$