QCQI Chapter 4 Exercises

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4.2: Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$\exp(iAx) = \cos(x)I + i\sin(x)A$$

Answer:

$$\exp(iAx) = \sum_{n=1}^{\infty} \frac{1}{n!}(iAx)^n \tag{1a}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n)!}(iAx)^{2n} + \frac{1}{(2n+1)!}(iAx)^{2n+1} \tag{1b}$$

$$= I\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}x^n + \frac{1}{(2n+1)!}(iAx)^{2n+1} \tag{1c}$$

$$= I\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}x^n + iA\frac{(-1)^n}{(2n+1)!}x^n \tag{1d}$$

$$= \cos(x)I + i\sin(x)A \tag{1e}$$

4.3: Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_z(\pi/4)$
Answer:

$$\pi/8 \text{ gate} = exp(i\pi/8)\begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{bmatrix}$$

and

$$R_z(\pi/4) = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{bmatrix}$$

$exp(i\pi/8)$ is just a global phase factor dude.

4.4: Express the Hadamard gate H as a product of R_x and R_z rotations and $e^{i\varphi}$

Answer:

$$\text{Hadamard} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{2a}$$

$$R_z(\theta) = \begin{bmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{bmatrix} \tag{2b}$$

$$R_x(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix} \tag{2c}$$

$$R_z(-2\pi) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{2d}$$

$$R_x(\pi) = \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix} \tag{2e}$$

$$e^{i\cdot\frac{5\pi}{2}} \cdot R_x(\pi/2) \cdot R_z(\pi/2) \cdot R_x(\pi/2) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{2f}$$

The insight here is that this series of rotations is equivalent (within a global phase factor) to the Hadamard's more obvious set of rotations: 90 degrees around the y axis followed by 180 degrees around the x axis.

4.5 Prove that $(\hat{n}\cdot\sigma)^2 = I$ and use this to verify this equation:

$$R_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{n}\cdot\sigma/2) = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})(n_xX + n_yY + n_zZ)$$

Okay, proof by equational reasoning:

$\hat{n}$ is a unit vector so $n_z^2 + n_x^2 + n_y^2 = 1$.

$$(\hat{n}\cdot\sigma)^2 = (\begin{bmatrix}0 & n_x\\ n_x & 0\end{bmatrix} + \begin{bmatrix}0 & -in_y\\ -in_y & 0\end{bmatrix} + \begin{bmatrix}n_z & 0\\ 0 & -n_z\end{bmatrix})^2 \tag{3a}$$

$$= (\begin{bmatrix}n_z & n_x - in_y\\ n_x + in_y & n_z\end{bmatrix})^2 = \begin{bmatrix}n_z^2 + n_x^2 + n_y^2 & 2n_z(n_x - in_y)\\ 2n_z(n_x + in_y) & n_z^2 + n_x^2 + n_y^2\end{bmatrix} \tag{3b}$$

$$= \begin{bmatrix}1 & 2n_z(n_x - in_y)\\ 2n_z(n_x + in_y) & 1\end{bmatrix} \tag{3c}$$

Hermitian operators must be equal to their adjoint and so

$$2n_z(n_x + in_y) = 2n_z(n_x - in_y)$$

We'll call this term "scooby-doo".

The determinant of a rotation matrix must be 1, and so we have

$$1 - \text{scooby-doo}^2 = 1 \tag{4a}$$

$$-\text{scooby-doo}^2 = 0 \tag{4b}$$

$$\text{scooby-doo} = 0 \tag{4c}$$

and therefore

$$\begin{bmatrix} 1 & \text{scooby-doo} \\ \text{scooby-doo} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Having done this convuluted and scooby-dooful proof, I now realize that since all quantum operators must be Hermitian and unitary, then $R_{\hat{n}}^2$ must trivially be equal to I by:

$$R_{\hat{n}}^\dagger R_{\hat{n}} = I \text{ by being unitary} \tag{5a}$$

$$R_{\hat{n}}^\dagger = R_{\hat{n}} \text{ by being Hermitian} \tag{5b}$$

$$R_{\hat{n}}^2 = I \tag{5c}$$

4.6: I'm getting kinda tired so I'm not gonna write the full question for this one. It's so long dude. They want me to prove that $R_{\hat{n}}(\theta)$ rotates a Bloch vector around the $\hat{n}$ axis.

For a state vector $a|0\rangle + b|1\rangle$

$$R_n(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z & i\sin\frac{\theta}{2}(n_x - in_y) \\ i\sin\frac{\theta}{2}(n_x + in_y) & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}n_z \end{bmatrix} \tag{6a}$$

$$R_n\theta(a|0\rangle + b|1\rangle) = e^{-i\frac{\theta}{2}}n_z a|0\rangle + i\sin\frac{\theta}{2}(n_x - in_y)b|1\rangle + i\sin\frac{\theta}{2}(n_x + in_y)a|0\rangle + e^{i\frac{\theta}{2}}n_z b|1\rangle \tag{6b}$$

$$\frac{\partial R_n\theta(a|0\rangle + b|1\rangle)}{\partial a|0\rangle} = e^{-i\frac{\theta}{2}}n_z + i\sin\frac{\theta}{2}(n_x + in_y) \tag{6c}$$

$$\frac{\partial R_n\theta(a|0\rangle + b|1\rangle)}{\partial b|0\rangle} = e^{i\frac{\theta}{2}}n_z + i\sin\frac{\theta}{2}(n_x - in_y) \tag{6d}$$

$$\frac{\partial R_n\theta(a|0\rangle + b|1\rangle)}{\partial a|0\rangle} - \frac{\partial R_n\theta(a|0\rangle + b|1\rangle)}{\partial b|0\rangle} = 2i\sin\frac{\theta}{2}n_z - 2\sin\frac{\theta}{2}n_y \tag{6e}$$

which shows up in the polar coordinates basis (θ, φ) as

$$a = \cos\frac{\theta}{2} \text{ by definition of the Bloch vector} \tag{7a}$$

$$b = e^{i\varphi}\sin\frac{\theta}{2} \text{ by definition of the Bloch vector} \tag{7b}$$

$$\theta = 2\arccos a \tag{7c}$$

$$\varphi = -i\ln\frac{b}{\sin\frac{\theta}{2}} \tag{7d}$$