QCQI Chapter 4 Exercises

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4.2: Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$\exp(iAx) = \cos(x)I + i\sin(x)A$$

Answer:

$$\exp(iAx) = \sum_{n=1}^{\infty} \frac{1}{n!}(iAx)^n \tag{1a}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n)!}(iAx)^{2n} + \frac{1}{(2n+1)!}(iAx)^{2n+1} \tag{1b}$$

$$= I\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}x^n + \frac{1}{(2n+1)!}(iAx)^{2n+1} \tag{1c}$$

$$= I\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}x^n + iA\frac{(-1)^n}{(2n+1)!}x^n \tag{1d}$$

$$= \cos(x)I + i\sin(x)A \tag{1e}$$

4.3: Show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_z(\pi/4)$
Answer:

$$\pi/8 \text{ gate} = exp(i\pi/8)\begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{bmatrix}$$

and

$$R_z(\pi/4) = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{bmatrix}$$

$exp(i\pi/8)$ is just a global phase factor dude.

4.4: Express the Hadamard gate H as a product of R_x and R_z rotations and $e^{i\varphi}$

Answer:

$$\text{Hadamard} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{2a}$$

$$R_z(\theta) = \begin{bmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{bmatrix} \tag{2b}$$

$$R_x(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix} \tag{2c}$$

$$R_z(-2\pi) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{2d}$$

$$R_x(\pi) = \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix} \tag{2e}$$

$$e^{i\cdot\frac{5\pi}{2}} \cdot R_x(\pi/2) \cdot R_z(\pi/2) \cdot R_x(\pi/2) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{2f}$$

The insight here is that this series of rotations is equivalent (within a global phase factor) to the Hadamard's more obvious set of rotations: 90 degrees around the y axis followed by 180 degrees around the x axis.

4.5 Prove that $(\hat{n}\cdot\sigma)^2 = I$ and use this to verify this equation:

$$R_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{n}\cdot\sigma/2) = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})(n_x X + n_y Y + n_z Z)$$

All quantum operators must be Hermitian and unitary, so then $R_{\hat{n}}^2$ must trivially be equal to I by:

$$R_{\hat{n}}^\dagger R_{\hat{n}} = I \text{ by being unitary} \tag{3a}$$

$$R_{\hat{n}}^\dagger = R_{\hat{n}} \text{ by being Hermitian} \tag{3b}$$

$$R_{\hat{n}}^2 = I \tag{3c}$$

4.6: I'm getting kinda tired so I'm not gonna write the full question for this one. It's so long dude. They want me to prove that $R_{\hat{n}}(\theta)$ rotates a Bloch vector around the $\hat{n}$ axis.

$$R_n(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z & -i\sin\frac{\theta}{2}(n_x - in_y)\\ -i\sin\frac{\theta}{2}(n_x + in_y) & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}n_z\end{bmatrix}$$

One observes that this matrix has the effect of changing the $|0\rangle\,|1\rangle$ basis vectors into

$$|0'\rangle = (\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z)\,|0\rangle - i\sin\frac{\theta}{2}(n_x + in_y)\,|1\rangle$$
$$|1'\rangle = (\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}n_z)\,|1\rangle - i\sin\frac{\theta}{2}(n_x - in_y)\,|0\rangle$$

My proof follows by proving these two vectors form an orthonormal basis that traces out a unit sphere in $\mathbb{R}^3$. In the interest of saving time, I didn't make it fully rigorous.

Both new vectors must be of norm one, because the rotation matrix is defined as

$$R_{\hat{n}}(\theta) = exp(-i\theta\hat{n} * \sigma/2)$$

Because the rotation operator is defined from a matrix exponential, and is Hermitian and unitary, it must have a basis of orthonormal vectors. Since the components of base vector are defined by a linear combination of four variables with only two equations and trigonometric functions, with an imaginary summand, their domain is in the unit circle of $\mathbb{C}$.

A vector of two complex numbers is equal in cardinality to $\mathbb{R}^4$, and since each component is in the unit circle of $\mathbb{C}$, the unit sphere is a subset of the values. There exists a bijection to the bloch sphere, by the axioms of set theory.

4.7 Show that $XYX = -Y$ and use this to prove that $XR_y(\theta)X = R_y(-\theta)$

$$XYX = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \tag{4a}$$
$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}-i & 0\\0 & i\end{bmatrix} \tag{4b}$$
$$= \begin{bmatrix}0 & i\\-i & 0\end{bmatrix} = -Y \tag{4c}$$
$$XR_y(\theta)X = X(\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y)X \tag{4d}$$
$$= \cos\frac{\theta}{2}XX - i\sin\frac{\theta}{2}XYX \tag{4e}$$
$$= \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}Y \tag{4f}$$
$$= R_y(-\theta) \tag{4g}$$

4.8 An arbitrary single qubit unitary operator can be written in the form

$$U = exp(i\alpha)R_{\hat{n}}(\theta)$$

for some real numbers α and θ, and a real three-dimensional unit vector $\hat{n}$.

1. Prove this fact. As we mentioned before, $R_{\hat{n}}(\theta)$ rotates a state vector to an arbitrary point on the Bloch Sphere - and there is a bijection between Bloch points and state vectors. Therefore, the rotation transformation can map any state vector to any state vector in a unitary fashion, within some global phase factor.

2. Find values for α and θ and $\hat{n}$ giving the Hadamard gate H.

$$Hadamard = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{5a}$$

$$= 1/\sqrt{(2)}(X+Z) \tag{5b}$$

$$e^{i\alpha}(\cos\frac{\theta}{2}I - (i\sin\frac{\theta}{2}(n_x X + n_z Z))) = 1/\sqrt{(2)}(X+Z) \tag{5c}$$

$$e^{i\alpha}(\cos\frac{\theta}{2}I - (i\sin\frac{\theta}{2}(n_x X + n_z Z))) = \begin{bmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z & -i\sin\frac{\theta}{2}n_x \\ -i\sin\frac{\theta}{2}n_x & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}n_z \end{bmatrix} \tag{5d}$$

$$\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z = -\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z \tag{5e}$$

$$\cos(\frac{\theta}{2}) = -\cos(\frac{\theta}{2}) \tag{5f}$$

$$\theta = \pi \tag{5g}$$

$$R_{\hat{n}}(\pi) = \begin{bmatrix} -in_z & -in_x \\ -in_x & in_z \end{bmatrix} \tag{5h}$$

$$n_x = n_z = \sqrt{2} \text{ - this makes the axis a unit vector} \tag{5i}$$

$$e^{i\frac{\pi}{4}}R_{(\sqrt{2},0,\sqrt{2})}(\pi) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = Hadamard \tag{5j}$$

$$\alpha = \frac{\pi}{4} \tag{5k}$$

$$\hat{n} = (\sqrt{2}, 0, \sqrt{2}) \tag{5l}$$

$$\theta = \pi \tag{5m}$$

3. Find values for α and θ and $\hat{n}$ giving the phase gate.

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \tag{6a}$$

$$= e^{-i\frac{\theta}{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \tag{6b}$$

$$R_z(\pi/2) = e^{-i\pi/4} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{6c}$$

$$e^{i\pi/4} R_z(\pi/2) = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S \tag{6d}$$

$$\tag{6e}$$