

Solving Generalized Lyapunov Equations with guarantees: application to the Reduction of Switched Linear Systems

Mattia Manucci, Benjamin Unger

Abstract

We deal with the efficient and certified approximation of the *generalized Lyapunov equation* (GLEs)

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^\top + \sum_{j=1}^M \left(\mathbf{N}_j \mathbf{X} \mathbf{N}_j^\top \right) + \mathbf{B}\mathbf{B}^\top = \mathbf{0}, \quad (1)$$

where $\mathbf{A}, \mathbf{N}_j \in \mathbb{R}^{n \times n}$, \mathbf{A} is Hurwitz, i.e., its spectrum is contained in the open left-half complex plane, and $\mathbf{B} \in \mathbb{R}^{n \times m}$ with $m \ll n$. GLEs with these features naturally arise in the context of *model order reduction* (MOR) of bilinear control systems [2] and linear parameter-varying systems as well as in the context of stochastic differential equations for stability analysis. For switched linear systems of the form

$$\Sigma_q \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_{q(t)} \mathbf{x}(t) + \mathbf{B}_{q(t)} \mathbf{u}(t), & \mathbf{x}(t_0) = \mathbf{0}, \\ \mathbf{y}(t) = \mathbf{C}_{q(t)} \mathbf{x}(t), \end{cases} \quad (2)$$

the authors of [3] introduced a balancing-based MOR method that requires the solution of certain GLEs. In (2), $q: \mathbb{R} \rightarrow \mathcal{J} := \{1, \dots, M\}$ is the external switching signal, which we assume to be an element of the set of allowed switching signals

$$\mathcal{S} := \{q: \mathbb{R} \rightarrow \mathcal{J} \mid q \text{ is right continuous with locally finite number of jumps}\}. \quad (3)$$

The symbols $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$ denote the *state*, the controlled *input*, and the measured *output*, respectively. The system matrices $\mathbf{A}_j \in \mathbb{R}^{n \times n}$, $\mathbf{B}_j \in \mathbb{R}^{n \times m}$, and $\mathbf{C}_j \in \mathbb{R}^{p \times n}$ correspond to the *ordinary differential equation* (ODE) active in mode $j \in \mathcal{J}$. Typically one refers to (2) as the *full-order model* (FOM). Sample applications of switched systems include robot manipulators, traffic management, automatic gear shifting, and power systems.

If (2) has to be evaluated repeatedly, for instance, in an optimal control context for different inputs, then a large dimension of the state variable renders this a computationally expensive task. In such scenarios, one can rely on MOR and replace (2) by the *reduced-order model* (ROM)

$$\tilde{\Sigma}_q \quad \begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}_{q(t)} \tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}_{q(t)} \mathbf{u}(t), & \tilde{\mathbf{x}}(t_0) = \mathbf{0}, \\ \tilde{\mathbf{y}}(t) = \tilde{\mathbf{C}}_{q(t)} \tilde{\mathbf{x}}(t), \end{cases} \quad (4)$$

with $\tilde{\mathbf{A}}_j \in \mathbb{R}^{r \times r}$, $\tilde{\mathbf{B}}_j \in \mathbb{R}^{r \times m}$, and $\tilde{\mathbf{C}}_j \in \mathbb{R}^{p \times r}$, and $r \ll n$. In many cases, see for instance [1], the reduced system matrices are obtained via Petrov–Galerkin projection, i.e., one constructs matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ and then defines

$$\tilde{\mathbf{A}}_j := \mathbf{W}^\top \mathbf{A}_j \mathbf{V}, \quad \tilde{\mathbf{B}}_j := \mathbf{W}^\top \mathbf{B}_j, \quad \tilde{\mathbf{C}}_j := \mathbf{C}_j \mathbf{V}. \quad (5)$$

The goal of MOR is thus to derive in a computationally efficient and robust way the matrices \mathbf{W}, \mathbf{V} such that the error $\mathbf{y} - \tilde{\mathbf{y}}$ is small in some given norm. One way to do so, originally presented in [3], is to solve opportune defined GLEs to obtain the projection matrices \mathbf{W}, \mathbf{V} and thus the reduced

system (5). Therefore, solving efficiently large-scale generalized Lyapunov equation becomes crucial for MOR. More in detail the MOR algorithm from [3] proceeds in two steps. First, we have to define the matrices $\mathbf{A} := \mathbf{A}_1$ and $\mathbf{N}_j := \mathbf{A}_j - \mathbf{A}_1$ for $j = 1, \dots, M$ and solve the GLEs

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^\top + \sum_{j=1}^M \left(\mathbf{N}_j \mathcal{P} \mathbf{N}_j^\top + \mathbf{B}_j \mathbf{B}_j^\top \right) = \mathbf{0}, \quad (6a)$$

$$\mathbf{A}^\top \mathcal{Q} + \mathcal{Q}\mathbf{A} + \sum_{j=1}^M \left(\mathbf{N}_j^\top \mathcal{Q} \mathbf{N}_j + \mathbf{C}_j^\top \mathbf{C}_j \right) = \mathbf{0}. \quad (6b)$$

Note that the matrix equations in (6) are of the form (1) by defining $\mathbf{B} := [\mathbf{B}_1, \dots, \mathbf{B}_M]$ for (6a) and $\mathbf{C} := [\mathbf{C}_1^\top, \dots, \mathbf{C}_M^\top]$, taking the transport on the other matrices for (6b). The symmetric and positive semi-definite solutions $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{n \times n}$ are referred to as the Gramians of (2). Second, let $\mathcal{P} = \mathbf{S}\mathbf{S}^\top$ and $\mathcal{Q} = \mathbf{R}\mathbf{R}^\top$ and compute the *singular value decomposition* (SVD) of the product of the Gramians factors

$$\mathbf{S}^\top \mathbf{R} = [\mathbf{U}_1, \mathbf{U}_2] \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{bmatrix} [\mathbf{V}_1, \mathbf{V}_2]^\top, \quad (7)$$

and the projection matrices \mathbf{V} and \mathbf{W} are obtained via

$$\mathbf{V} = \mathbf{S}\mathbf{U}_1\boldsymbol{\Sigma}_1^{-1/2} \quad \text{and} \quad \mathbf{W} = \mathbf{R}\mathbf{V}_1\boldsymbol{\Sigma}_1^{-1/2}. \quad (8)$$

This procedure is denoted as square-root balanced truncation (see [1, Sec. 7.3]). The use of the solutions of (6) as system Gramians is justified by [3, Thm. 3], where the authors show that the image of \mathcal{P} and \mathcal{Q} encode the reachability set and observability set of the switched system (2).

Our main contributions are the following: To deal with the large-scale setting, we apply the stationary algorithm from [4] in combination with a Krylov subspace projection framework to solve GLEs. We emphasize that this is a common strategy in the literature when dealing with GLEs. Our first contribution is the derivation of efficiently computable error estimates such that for any prescribed user tolerance `tol` an approximation $\tilde{\mathbf{X}}$ of (1) with guaranteed bound $\|\mathbf{X} - \tilde{\mathbf{X}}\|_2 \leq \text{tol}$ can be computed. Second, we show how the numerical error introduced in approximating (1) may deteriorate the quality and the stability of the ROM (4). This motivates us to propose a novel strategy that, by relying on the error certification provided by our algorithm, ensures stability and error certification of the MOR system. Finally, the results are validated through a synthetic example and a switched system arising from a parametric partial differential equation.

References

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