On the stochastic approximation of the pth root of a stochastic matrix

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Abstract

The evolution of a discrete-time Markov chain, with a finite number n of states, is described in terms of an $n \times n$ matrix A, called transition matrix, whose (i, j)-th entry represents the probability to go from state i to state j in one unit of time. The matrix A is stochastic, i.e., it is nonnegative with sum of the entries on each row equal to 1.

Estimating transition probabilities at finer time scales requires finding pth roots of the stochastic matrix A. However, stochastic pth roots of a stochastic matrix do not always exist [2].

We propose two Riemannian optimization approaches to compute stochastic approximations of the pth root of a stochastic matrix. The first method searches within the manifold of positive stochastic matrices, while the second restricts the search to matrices sharing the stationary distribution with the original stochastic matrix A. This constraint, crucial for Markov chain embedding, is not addressed by existing constrained optimization methods. Numerical experiments demonstrate that our Riemannian methods are generally faster and more accurate, particularly the eigenvector-preserving approach, which yields superior accuracy even within a smaller search space. More details can be found in the paper [1].

[1] F. Durastante and B. Meini, Stochastic pth root approximation of a stochastic matrix: a Riemannian optimization approach, SIAM J. Matrix Anal. Appl. **45** (2024), no. 2, 875–904.

[2] N. J. Higham and L. Lin, On pth roots of stochastic matrices, Linear Algebra Appl. **435** (2011), no. 3, 448–463.