From PINNs to Computer-Assisted Proofs for Fluid Dynamics

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Abstract

Physics-Informed Neural Networks (PINNs) have emerged as an alternative to traditional numerical methods for solving partial differential equations (PDEs) by leveraging the universal function approximation properties of deep neural networks and the recent advancements in automatic differentiation and optimization algorithms for neural networks [1, 2, 3]. Given a domain $\Omega \subseteq \mathbb{R}^n$ and a PDE of the form $\mathcal{F}u = f$ in Ω with boundary conditions $\mathcal{B}u = g$ in $\partial\Omega$, a PINN approximates the solution u using a neural network u_θ, where θ denotes the neural network parameters, $\mathcal{F}$ is a possibly nonlinear differential operator and $\mathcal{B}$ is a boundary value operator. The PINN is trained by minimizing a loss function given by the PDE residuals evaluated at a set of collocation points in Ω. The boundary conditions can be enforced either softly, by incorporating them into the loss function, or as hard constraints, by modifying the network architecture, which can improve accuracy and convergence by restricting the solution space.

Our research applies PINNs to the study of low regularity problems in fluid dynamics, with a particular focus on the incompressible 2D Euler equations. Specifically, we focus on vortex patches, which are a class of weak, non-smooth solutions of the incompressible 2D Euler equations, for which the vorticity is the characteristic function of a domain. A specific type of vortex patches are V-states, which rotate uniformly with constant angular velocity. Ellipses are a canonical example of V-states, and Deem and Zabusky provided numerical evidence supporting the existence of families of m-fold symmetric V-states for any integer $m \geq 3$ [4]. These V-states bifurcate from the disk taking the angular velocity as the bifurcation parameter. There are many theoretical results on V-states (e.g. [5, 6, 7] as a sample). Wu, Overman and Zabusky presented more numerical computations of V-states and found limiting V-states with corners for $3 \leq m \leq 6$ [8].

Based on the numerical evidence gathered, Overman proved that the tangent angle of the limiting V-states has a jump discontinuity of either $\pi/2$ (corner) or π (cusp) and conjectured that limiting V-states have right-angle corners rather than cusps [9]. The work in [10], as well as our own unpublished numerical results, have supported the conjecture numerically, whereas Pierrehumbert [11] had reported instead a cusp as the limiting state for translating V-states. Wang, Zhang and Zhou [12] provided a classification of the various potential singular points (corners or cusps), yet did not offer any explicit examples that verify any alternative, thereby leaving the conjecture unresolved.

We are currently addressing this problem by first obtaining an approximate solution using a PINN and then rigorously proving the existence of a nearby solution through a computer-assisted proof. Specifically, we invert the linear part of the equation onto the nonlinear component and employ interval arithmetic and fixed-point arguments, an approach that has proven successful in proving the existence of non-convex V-states [13]. Our PINN-based numerical approximation significantly improves over traditional methods, with a key factor being the integration of prior mathematical knowledge of the problem to effectively explore the solution space. Moreover, our results are consistent with the ones reported by Luzzato-Fegiz and Williamson [14], and the low residuals suggest that our numerical approximation is well-suited for the computer-assisted proof that we are currently developing.

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