A continuum limit for dense spatial networks

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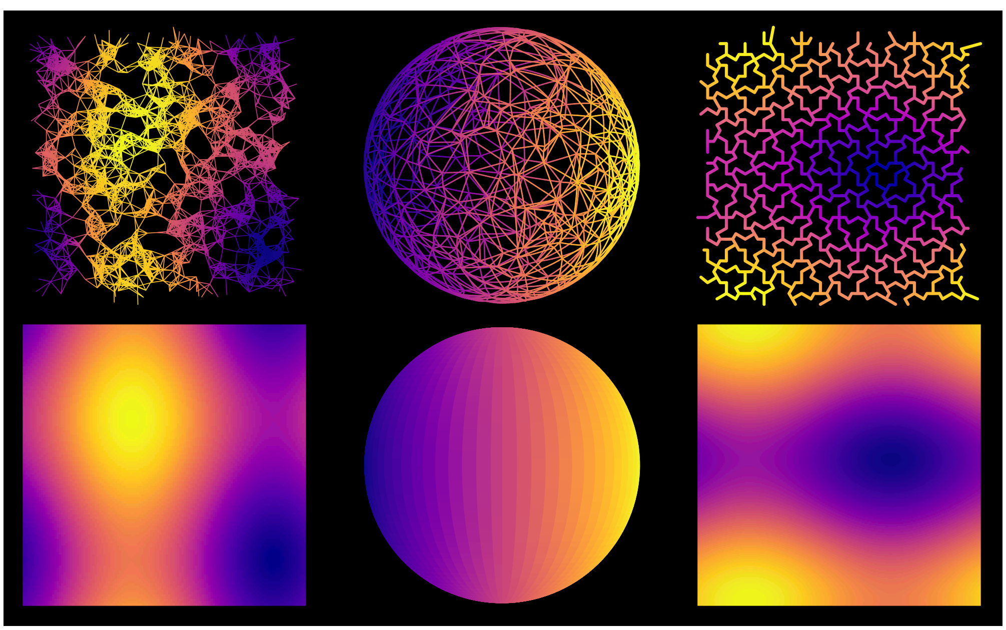
Abstract

Many physical systems—such as optical waveguide lattices and dense neuronal or vascular networks—can be modeled by *metric graphs*, where slender "wires" (edges) support wave or diffusion equations subject to Kirchhoff boundary conditions at the nodes. We propose a continuum-limit framework which replaces edgewise differential equations with a coarse-grained partial differential equation (PDE) defined on the continuous space occupied by the network [1]. The derivation naturally introduces an *edge-conductivity tensor*, an *edge-capacity function*, and a *vertex number density* to encode how each microscopic patch of the graph contributes to the macroscopic phenomena. These results have interesting similarities and differences with the Riemannian Laplace-Beltrami operator. We calculate all macroscopic parameters from first principles via a systematic discrete-to-continuous local homogenization, finding an anomalous effective embedding dimension resulting from a homogenized diffusivity. Numerical examples—including periodic lattices, random graphs, and quasiperiodic monotiles—demonstrate that each finite model converges to its corresponding PDE (posed on different manifolds like tori, disks, and spheres) in the limit of increasing vertex density. These high-density networks encode emergent material and functional properties. They reflect the ability of many real-world, space-filling networks to function simultaneously at multiple scales–both at the system-wide and local levels–using the continuum as a feature. We expect our results to be useful in modelling biological network growth [2], metamaterial design [3, 4], and open interesting avenues in the spectral theory of semidiscrete operators [5].



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