Equilibria and Bifurcations of Reaction-Diffusion PDEs with Random Coefficients

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Abstract

This presentation addresses the study of the long-term qualitative behavior of reaction-diffusion partial differential equations (PDEs) depending on model parameters that are affected by uncertainty. In general, a wide range of model outcomes has to be considered, such as equilibria and their stability, possible destabilization mechanisms and corresponding bifurcation points, and the existence of solutions with specific spatio-temporal structures. The uncertainty in the model parameters is propagated to those outcomes in a highly nonlinear manner, thus requiring suitable uncertainty quantification (UQ) techniques.

As a case study, we consider the Klausmeier model, which describes the interaction between plants and water in semi-arid environments. For instance, the rainfall parameter significantly impacts the long-term behavior: high rainfall supports spatially-homogeneous equilibria with the land completely covered by vegetation, while lower rainfall supports also a range of spatially-organized solutions, with vegetation covering only small areas of land. From a modeling perspective, the organization of vegetation into so-called patterns can be interpreted as a way for the ecosystem to escape desertification and it is therefore of great relevance in applications. Mathematically, the emergence of patterns can be explained by the presence of a bifurcation point that allows for spatially-inhomogeneous and -homogeneous solutions to co-exist, leading to a so-called multi-stability region. We illustrate how to assess the propagation of uncertainty to the equilibria, bifurcation points and the multi-stability region, primarily based on a Monte Carlo sampling strategy.

However, addressing this type of UQ analysis requires a more comprehensive framework, both from the theoretical and numerical perspective. In [1] we move a first step in this direction and consider a prototypical PDE for bi-stability that exhibits pitchfork bifurcations, the Allen–Cahn equation. We introduce a random coefficient in the linear reaction part of the equation, thereby accounting for random, spatially-heterogeneous effects. Importantly, we assume a spatially constant, deterministic mean value of the random coefficient that we show to be in fact a bifurcation parameter in the Allen–Cahn equation with random coefficients, which then reads

$$\partial_t u = \Delta u + pu + g(\mathbf{x}, \omega)u - u^3, \qquad \mathbf{x} \in D,\ \omega \in \Omega,$$

where $p \in \mathbb{R}$ is the bifurcation parameter and $(\Omega, \mathcal{A}, \mathbb{P})$ a suitable probability space. To analyze the associated dynamical system, that is infinite-dimensional both in physical and probability space, we introduce a finite-dimensional parametrization of the probability space. The resulting framework allows to resort to bifurcation theory of deterministic dynamical systems. Thus, we show that the bifurcation points and bifurcation curves of equilibria become random objects. Finally, we propose the generalized polynomial chaos expansion based on sparse-grids collocation to efficiently approximate the statistical properties of the random bifurcation points and bifurcation curves.

References

[1] C. Kuehn, C. Piazzola, and E. Ullmann. Uncertainty quantification analysis of bifurcations of the Allen–Cahn equation with random coefficients. *Physica D: Nonlinear Phenomena*, 470, 134390, 2024.