Imaginary number

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An **imaginary number**[note 1] is a complex number that can be written as a real number multiplied by the imaginary unit i,[note 2] which is defined by its property $i^2 = -1$.[1] The square of an imaginary number bi is $-b^2$. For example, $5i$ is an imaginary number, and its square is -25. Zero is considered to be both real and imaginary.[2]

... (repeats the pattern from blue area)
$i^{-3} = i$
$i^{-2} = -1$
$i^{-1} = -i$
$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$i^6 = -1$
$i^n = i^{\mathrm{n(mod\ 4)}}$

Originally coined in the 17th century as a derogatory term and regarded as fictitious or useless, the concept gained wide acceptance following the work of Leonhard Euler and Carl Friedrich Gauss.

An imaginary number bi can be added to a real number a to form a complex number of the form $a + bi$, where the real numbers a and b are called, respectively, the *real part* and the *imaginary part* of the complex number.[3][note 3] Some authors use the term **pure imaginary number** to denote what is called here an imaginary number, and *imaginary number* to denote any complex number with non-zero imaginary part.[4]

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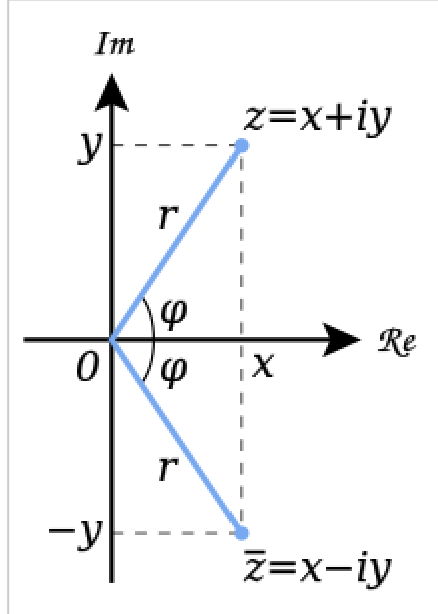
History

Although Greek mathematician and engineer Heron of Alexandria is noted as the first to have conceived these numbers,[5][6] Rafael Bombelli first set down the rules for multiplication of complex numbers in 1572. The concept had appeared in print earlier, for instance in work by Gerolamo Cardano. At the time imaginary numbers, as well as negative numbers, were poorly understood and regarded by some as fictitious or useless, much as zero once was. Many other mathematicians were slow to adopt the use of imaginary numbers, including René Descartes, who wrote about them in his *La Géométrie*, where the term *imaginary* was used and meant to be derogatory.[7][8] The use of imaginary numbers was not widely accepted until the work of Leonhard Euler (1707–1783) and Carl Friedrich Gauss (1777–1855). The geometric significance of complex numbers as points in a plane was first described by Caspar Wessel (1745–1818).[9]



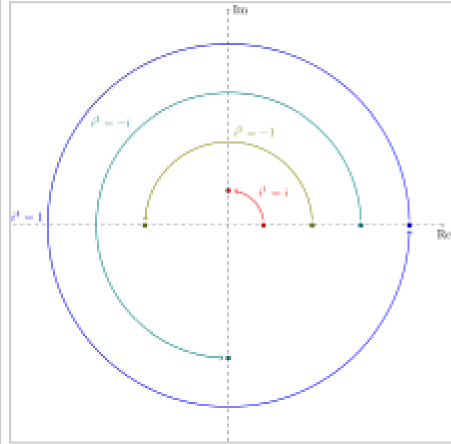
An illustration of the complex plane. The imaginary numbers are on the vertical coordinate axis.

In 1843 William Rowan Hamilton extended the idea of an axis of imaginary numbers in the plane to a four-dimensional space of quaternion imaginaries, in which three of the dimensions are analogous to the imaginary numbers in the complex field.

With the development of quotient rings of polynomial rings, the concept behind an imaginary number became more substantial, but then one also finds other imaginary numbers such as the j of tessarines which has a square of $+1$. This idea first surfaced with the articles by James Cockle beginning in 1848.[10]

Geometric interpretation

Geometrically, imaginary numbers are found on the vertical axis of the complex number plane, allowing them to be presented perpendicular to the real axis. One way of viewing imaginary numbers is to consider a standard number line, positively increasing in magnitude to the right, and negatively increasing in magnitude to the left. At 0 on this x-axis, a y-axis can be drawn with "positive" direction going up; "positive" imaginary numbers then increase in magnitude upwards, and "negative" imaginary numbers increase in magnitude downwards. This vertical axis is often called the "imaginary axis" and is denoted $i\mathbb{R}$, $\mathbb{I}$, or $\Im$.



90-degree rotations in the complex plane

In this representation, multiplication by -1 corresponds to a rotation of 180 degrees about the origin. Multiplication by i corresponds to a 90-degree rotation in the "positive" direction (i.e., counterclockwise), and the equation $i^2 = -1$ is interpreted as saying that if we apply two 90-degree rotations about the origin, the net result is a single 180-degree rotation. Note that a 90-degree rotation in the "negative" direction (i.e. clockwise) also satisfies this interpretation. This reflects the fact that $-i$ also solves the equation $x^2 = -1$. In general, multiplying by a complex number is the same as rotating around the origin by the complex number's argument, followed by a scaling by its magnitude.

Square roots of negative numbers

Care must be used when working with imaginary numbers expressed as the principal values of the square roots of negative numbers. For example:[11]

$$6 = \sqrt{36} = \sqrt{(-4)(-9)} \neq \sqrt{-4}\sqrt{-9} = (2i)(3i) = 6i^2 = -6.$$

Sometimes this is written as:

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} \stackrel{\text{(fallacy)}}{=} \sqrt{(-1)(-1)} = \sqrt{1} = 1,$$

where the fallacy is that the rule $\sqrt{xy} = \sqrt{x}\sqrt{y}$ can fail when the variables are not suitably constrained, in this case when they are both negative.

See also

- Imaginary unit
- de Moivre's formula
- NaN (Not a number)
- Octonion

- Quaternion

Notes

1. This is the definition adopted in this article.
2. *j* is often used in Engineering contexts where *i* has other meanings (such as electrical current)
3. Both the real part and the imaginary part are defined as real numbers.

References

1. Uno Ingard, K. (1988). "Chapter 2". *Fundamentals of waves & oscillations*. Cambridge University Press. p. 38. ISBN 0-521-33957-X.
2. Sinha, K.C. *A Text Book of Mathematics XI*. Rastogi Publications. p. 11.2. ISBN 8171339123.
3. Aufmann, Richard; Barker, Vernon C.; Nation, Richard (2009). *College Algebra: Enhanced Edition* (6th ed.). Cengage Learning. p. 66. ISBN 1-4390-4379-5.
4. C.L. Johnston, J. Lazaris, *Plane Trigonometry: A New Approach*, Prentice Hall, 1991, p. 247.
5. Hargittai, István (1992). *Fivefold symmetry* (2nd ed.). World Scientific. p. 153. ISBN 981-02-0600-3.
6. Roy, Stephen Campbell (2007). *Complex numbers: lattice simulation and zeta function applications*. Horwood. p. 1. ISBN 1-904275-25-7.
7. Descartes, René, *Discourse de la Méthode* … (Leiden, (Netherlands): Jan Maire, 1637), appended book: *La Géométrie*, book three, p. 380. From page 380: (http://gallica.bnf.fr/ark:/12148/btv1b86069594/f464.item.zoom) "*Au reste tant les vrayes racines que les fausses ne sont pas tousjours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut bien tousjours en imaginer autant que jay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine, comme encore qu'on en puisse imaginer trois en celle cy,* $x^3 - 6xx + 13x - 10 = 0$*, il n'y en a toutefois qu'une reelle, qui est 2, & pour les deux autres, quoy qu'on les augmente, ou diminue, ou multiplie en la façon que je viens d'expliquer, on ne sçauroit les rendre autres qu'imaginaires*." (Moreover, the true roots as well as the false [roots] are not always real; but sometimes only imaginary [quantities]; that is to say, one can always imagine as many of them in each equation as I said; but there is sometimes no quantity that corresponds to what one imagines, just as although one can imagine three of them in this [equation], $x^3 - 6xx + 13x - 10 = 0$, only one of them however is real, which is 2, and regarding the other two, although one increase, or decrease, or multiply them in the manner that I just explained, one would not be able to make them other than imaginary [quantities].)
8. Martinez, Albert A. (2006), *Negative Math: How Mathematical Rules Can Be Positively Bent*, Princeton: Princeton University Press, ISBN 0-691-12309-8, discusses ambiguities of meaning in imaginary expressions in historical context.
9. Rozenfeld, Boris Abramovich (1988). "Chapter 10". *A history of non-euclidean geometry: evolution of the concept of a geometric space*. Springer. p. 382. ISBN 0-387-96458-4.
10. James Cockle (1848) "On Certain Functions Resembling Quaternions and on a New Imaginary in Algebra", London-Dublin-Edinburgh Philosophical Magazine, series 3, 33:435–9 and Cockle

(1849) "On a New Imaginary in Algebra", Philosophical Magazine 34:37–47

11. Nahin, Paul J. (2010). *An Imaginary Tale: The Story of "i" [the square root of minus one]*. Princeton University Press. p. 12. ISBN 978-1-4008-3029-9. Extract of page 12 (https://books.google.com/books?id=PflwJdPhBlEC&pg=PA12)

Bibliography

- Nahin, Paul (1998). *An Imaginary Tale: the Story of the Square Root of −1*. Princeton: Princeton University Press. ISBN 0-691-02795-1., explains many applications of imaginary expressions.

External links

- How can one show that imaginary numbers really do exist? (http://www.math.toronto.edu/mathnet/answers/imagexist.html) – an article that discusses the existence of imaginary numbers.
- In our time: Imaginary numbers (http://www.bbc.co.uk/programmes/b00tt6b2) Discussion of imaginary numbers on BBC Radio 4.
- 5Numbers programme 4 (http://www.bbc.co.uk/radio4/science/5numbers4.shtml) BBC Radio 4 programme
- Why Use Imaginary Numbers? (http://www2.dsu.nodak.edu/users/mberg/Imaginary/imaginary.htm) Basic Explanation and Uses of Imaginary Numbers

Categories: Complex numbers
